What Shape Cools Fastest?

Heat Transfer Analysis of Square vs. Circular Rods

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1 Introduction

Heat transfer from a hot rod to a cooler ambient environment occurs primarily through convection at the rods surface as well as conduction within the rod. This report compares two aluminum rods—one with a circular cross section and one with a square cross section—to determine which geometry cools faster. Both rods have the same surface area (A_s) and length (L), ensuring a fair comparison of their heat dissipation characteristics. We use the Lumped Capacitance Method (LCM), justified by calculating the Biot number.

2 Thermal Properties and Parameters

The rod material is aluminum, with the following properties:

- **Thermal Conductivity:** $k = 237\ \mathrm{W/(m \cdot K)}$
- **Density:** $\rho = 2702\ \mathrm{kg/m^3}$
- **Specific Heat Capacity:** $c_p = 903\ \mathrm{J/(kg \cdot K)}$
- **Thermal Diffusivity:** $\alpha = \frac{k}{\rho c_p} \approx 9.75 \times 10^{-5}\ \mathrm{m^2/s}$

The rods are initially at $T_i = 100°\mathrm{C}$ and are exposed to still air at $T_\infty = 20°\mathrm{C}$. We can take the convective heat transfer coefficient $h = 10\ \mathrm{W/m^2{\cdot}K}$ for free convection. This value falls between the values of $2 - 25\ \mathrm{W/m^2{\cdot}K}$ given in Table 1.1. (While radiation could be considered at higher temperatures, it is often negligible compared to convection at near-ambient conditions, so we focus primarily on convective cooling.)

3 Rod Geometry and Surface Area Matching

3.1 Circular Rod

- **Length:** $L = 6\,\mathrm{in} = 0.1524\,\mathrm{m}$
- **Diameter:** $D = 9.525\,\mathrm{mm} = 0.009525\,\mathrm{m}$
- **Surface Area (excluding ends):** $A_s = 0.00455\,\mathrm{m^2}$. This A_s corresponds to the cylindrical lateral surface area πDL if the rod ends are excluded.

3.2 Square Rod

To keep the same lateral surface area, we set $4wL = A_s$, giving

$$w = \frac{A_s}{4L} = 7.475\,\text{mm}. \tag{1}$$

Hence, the square rod has cross-sectional side $w = 0.007475\,\text{m}$ and the same length $L = 0.1524\,\text{m}$.

4 Heat Transfer Modes and Assumptions

Modes of heat transfer:

- **Conduction:** Inside the rod, heat flows from the hot interior to the surface.
- **Convection:** At the surface, heat is transferred from the rod to the air.
- **Radiation (neglected in this analysis):** At temperatures near 100°C we can say convection dominates.

Key Assumptions:

- The rod properties are homogeneous and isotropic.
- Temperature within the rod is spatially uniform (justified if Bi < 0.1). This allows use of the Lumped Capacitance Method.
- The convective heat transfer coefficient h is constant over the rod's surface.
- The rods are cooled only by convection from their lateral surfaces, the ends of the rod are perfectly insulated and their surface areas are excluded.

5 Biot Number Analysis

The Biot number (Bi) measures the ratio of internal conduction resistance to external convection resistance:

$$\text{Bi} = \frac{hL_c}{k}, \quad \text{where} \quad L_c = \frac{V}{A_s}. \tag{2}$$

For the LCM to be valid, we require Bi < 0.1, indicating that conduction inside the rod is rapid compared to surface convection. As an example for the circular rod:

- $V_{\text{circle}} \approx 0.0000109\,\text{m}^3, \quad A_s = 0.00455\,\text{m}^2.$
- $L_{c,\text{circle}} = \frac{V_{\text{circle}}}{A_s} \approx 0.00238125\,\text{m}.$
- $\text{Bi} = \frac{hL_{c,\text{circle}}}{k} = \frac{10\,\text{W}/(\text{m}^2\,\text{K}) \times 2.38 \times 10^{-3}\,\text{m}}{237\,\text{W}/(\text{m}\,\text{K})} \approx 1.0 \times 10^{-4} \ll 0.1.$

Analogous calculations for the square rod give a similarly small Biot number, validating LCM.

6 Lumped Capacitance Cooling: Time Constant and Cooling Rate

6.1 General Lumped Capacitance Formulation

Under LCM, the rod's temperature $T(t)$ satisfies:

$$\frac{dT}{dt} = -\frac{hA_s}{\rho c_p V}\,[T(t) - T_\infty], \tag{3}$$

whose solution is an exponential decay:

$$T(t) = T_\infty + [T_i - T_\infty]\exp\!\Big(-\frac{hA_s}{\rho V_c} t\Big), \tag{4}$$

where the time constant τ is:

$$\tau = \frac{\rho c_p V}{hA_s}. \tag{5}$$

A smaller τ indicates faster cooling.

6.2 Numerical Values for Each Geometry

We can calculate V, m, and τ for each shape:

Circular Rod

- Volume: $V_{\text{circle}} = \pi\left(\frac{D}{2}\right)^2 L \approx 1.087 \times 10^{-5}\,\text{m}^3$.
- Mass: $m_{\text{circle}} = \rho\, V_{\text{circle}} \approx 0.02936\,\text{kg}$.
- Time constant: $\tau_{\text{circle}} = \frac{m_{\text{circle}} c_p}{hA_s} \approx 581\,\text{s} \approx 9.7\,\text{min}$.

Square Rod

- Volume: $V_{\text{square}} = w^2\, L \approx 8.52 \times 10^{-6}\,\text{m}^3$.
- Mass: $m_{\text{square}} = \rho\, V_{\text{square}} \approx 0.0230\,\text{kg}$.
- Time constant: $\tau_{\text{square}} = \frac{m_{\text{square}} c_p}{hA_s} \approx 454\,\text{s} \approx 7.6\,\text{min}$.

6.3 Heat Transfer Rate

Since we are primarily dealing with convection, the heat transfer rate at time t is:

$$q(t) = hA_s\big[T(t) - T_\infty\big]. \tag{6}$$

Initially, $q(0) = hA_s\big[T_i - T_\infty\big]$. Both rods have the same A_s, so they begin with the same $q(0)$. However, the rod with lower thermal mass ($\rho c_p V$) cools more quickly, reducing $\big[T(t) - T_\infty\big]$ faster.

7 Discussion of Results

From the computed time constants, the square rod ($\tau_{\text{square}} \approx 454\,\text{s}$) cools faster than the circular rod ($\tau_{\text{circle}} \approx 581\,\text{s}$). This is a reduction of about 22% in time constant. Though both rods have the *same* lateral surface area, the square rod has a smaller cross-sectional area and hence a smaller volume (and mass). With a lower thermal mass per unit surface area, the square is shown to cool more quickly.