2) Derivation of Motion Tracking Equation :

a) → **Brightness Constancy Assumption**:- The brightness constancy assumption state that the intensity of a pixel remains constant overtime.

$$I(x,y,t) = I(x+u, y+v, t+\Delta t) \quad \text{--- (1)}$$

where:

- $I(x,y,t)$ is the intensity of the pixel at position (x,y) in frame t.
- u & v are the horizontal and vertical components of the motion vector, representing the displacement of the pixel.
- Δt is the time difference between consecutive frames.

→ **Taylor Series Expansion**: We can expand $I(x+u, y+v, t+\Delta t)$ using a first-order taylor series expansion.

$$I(x+u, y+v, t+\Delta t) \simeq I(x,y,t) + \frac{dI}{dx}u + \frac{dI}{dy}v + \frac{dI}{dt}\Delta t.$$

where:

$\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ are the spatial gradient of the image intensity in the x and y directions.

$\frac{\partial I}{\partial t}$ → temporal gradient, (change of intensity over time).

→ **Optical Flow Equation**: By equating the two expression.

for $I(x,y,t)$ ⇒ By (1) & (2)

$$I(x,y,t) + \frac{dI}{dx}u + \frac{\partial I_v}{\partial y} + \frac{\partial I}{\partial t}\Delta t = I(x,y,t)$$

Rearranging terms and dividing it by 'Δt'

$$\Rightarrow \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial u} v + \frac{\partial I}{\partial t} \frac{\Delta t}{\Delta t} = I(x, y, t)$$

$$u \cdot I_x + v \cdot I_y + I_t = 0$$

where

$I_x : \partial I / \partial x$ (spatial gradient of the image intensity in the x direction)

$I_y : \partial I / \partial y$ (spatial gradient of the image intensity in the y direction)

$I_z : \partial I / \partial t$ (temporal gradient)

$\boxed{I_x u + I_y v + I_t = 0}$ → This equation represents the constrint on the motion of pixels between consecutive ~~fr~~ frames, Known as the optical flow equation.

b) Procedure for performing Lucas-Kande Algorithm for motion tracking when the motion is known to be affine:

$$u(x, y) = a' * x + b' * y + c_1$$

$$v(x, y) = a'_2 * x + b_2 * y + c_2$$

1. Affine Motion Model: This model describes motion as an affine transformation. This model includes. translation, rotation, scaling and skewing. The affine motion model is represented as:

$$u(x, y) = a_1 x + b_1 y + c_1$$

$$v(x, y) = a_2 x + b_2 y + c_2$$

Here (u, v) are the horizontal and vertical components of the motion vector at position (x, y) and (a_1, b_1, c_1) and (a_2, b_2, c_2) are the affine motion parameters.

2.) Lucas-Kanade Equations: For affine motion, we have two equations ~~d~~ derived from the optical flow equation.

$I_x a_1 + I_y b_1 + I_t = 0$ $I_{y_1}, I_{x_1} \rightarrow$ Spatial gradients of images intensity in the y and x direction.

$I_y a_2 + I_y b_2 + I_t = 0$ $I_t \rightarrow$ temporal gradient.

3) Least squares Solution: To estimate the affine motion parameters, we solve the system of the equations using Least squares. we formulate a system of Linear Equations for a neighborhood of pixels.

$$A \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = -b.$$

A: matrix of spatial gradients (concatenated I_x and I_y gradients) for all pixels in the neighborhood

b is the vector of temporal gradients for all the pixels in the neighborhood.

$$A = \begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \\ I_{xm} & I_{ym} \end{bmatrix}$$

we solve this overdetermined system using least squares to estimate the affine motion parameter a_1, b_1, a_2 and b_2.

4) Motion Estimation: After obtaining the motion parameters a_1, b_1, a_2, b_2 using Least squares, we can compute the motion vectors $u(x,y)$ and $v(x,y)$ for each pixel using the affine motion model.

$u(x,y) = a_1 x + b_1 y + c_1$

$v(x,y) = a_2 x + b_2 y + c_2$

Here (x,y) represents the co-ordinates of each pixel in the image and (c_1, c_2) are the translation components of the affine transformation. To estimate c_1 & c_2, we ~~can~~ can use the ~~ca~~ centroid or the mean of the displacement vectors $u(x,y)$ and $v(x,y)$ over ROI. Least squares method.

$$I_x a_1 + I_y b_1 + I_t = 0$$
$$I_x a_2 + I_y b_2 + I_t = 0$$

$Ax = b$. where $x = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}$ ∴ b- temporal gradients.

$$\begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \\ \vdots & \vdots \\ I_{xm} & I_{ym} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = - \begin{bmatrix} I_{t_1} \\ I_{t_2} \\ \vdots \\ I_{tm} \end{bmatrix}$$

Solving for the motion parameters:

To obtain the least squares solution, we minimize the error between the left & right sides of the equation.

$$\min_x \| Ax + b \|^2$$

The soln can be found using pseudo-inverse of A:

$$x = (A^T A)^{-1} A^T b.$$

substituting A & b with respective values.

$$\begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = \left[\begin{pmatrix} I_{x_1}, I_{x_2} \cdots I_{xm} \\ I_{y_1}, I_{y_2} \cdots I_{ym} \end{pmatrix} \begin{bmatrix} I_{x_1} \\ I_{y_1} \\ \vdots \\ I_{xm} \\ I_{ym} \end{bmatrix} \right] \begin{pmatrix} I_{x_1} I_{x_2} \cdots I_{xm} \\ I_{y_1} I_{y_2} \cdots I_{ym} \end{pmatrix} \begin{bmatrix} -I_t \\ -I_{t_2} \\ \vdots \\ -I_{tm} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \\ \vdots & \vdots \\ I_{xm} & I_{ym} \end{bmatrix}^T \begin{pmatrix} I_{x_1}, I_{x_2} \cdots \cdots I_{xm} \\ I_{y_1}, I_{y_2} \cdots I_{ym} \end{pmatrix}$$

$$= \begin{bmatrix} I_{x_1} \\ I_{x_2} \\ \vdots \\ I_{x_m} \end{bmatrix} (I_{x_1}\, I_{x_2} \cdots I_{x_m}) + \begin{pmatrix} I_{y_1} \\ I_{y_2} \\ \vdots \\ I_{y_m} \end{pmatrix} (I_{y_1}\, I_{y_2} \cdots I_{y_m})$$

$$= \begin{pmatrix} \Sigma_i I^2_{x_i} & \Sigma_i I_{x_i} I_{y_i} \\ \Sigma I_{x_i} I_{y_i}, & \Sigma_i I^2_{y_i} \end{pmatrix}$$

Now compute $A^T b = \begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \\ \vdots & \vdots \\ I_{x_m} & I_{y_m} \end{bmatrix}^T \begin{bmatrix} -I_{t_1} \\ -I_{t_2} \\ -I_{t_m} \end{bmatrix}$

$$= \begin{bmatrix} I_{x_1} \\ I_{x_2} \\ I_{x_m} \end{bmatrix} (-I_{t_1}) + \begin{bmatrix} I_{y_1} \\ I_{y_2} \\ I_{y_m} \end{bmatrix} (-I_{t_2})$$

$$= \begin{pmatrix} \Sigma I_{x_i}(-I_{t_i}) \\ \Sigma I_{y_i}(-I_{t_i}) \end{pmatrix}$$

Lets consider $G = A^T A$ and $h = A^T b$.

Now that we have both. $G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$$x = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = G^{-1} h = \frac{1}{g_{11} g_{22} - g_{12} g_{21}} \begin{pmatrix} g_{22} & -g_{12} \\ g_{21} & g_{11} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$x = \frac{1}{g_{11} g_{22} - g_{12} g_{21}} \begin{bmatrix} g_{22} h_1 - g_{12} h_2 \\ -g_{21} h_1 + g_{11} h_2 \end{bmatrix}$$

$$x = \frac{1}{\Sigma_i I^2_{x_i}\, \Sigma I^2_{y_i} - (\Sigma I_{x_i} I_{y_i})^2} \begin{bmatrix} (\Sigma I^2_{y_i}) \Sigma_i I_{x_i}(-I_{t_i}) - (\Sigma I_{x_i} I_{y_i})(\Sigma_i I_{y_i}(-I_{t_i}) \\ -(\Sigma I^2_{x_i}) \Sigma I_{x_i}(-I_{t_i}) + \Sigma I_{x_i} I_{y_i}\, \Sigma I_{y_i}(-I_{t_i}) \end{bmatrix}$$

$\therefore$

$$a_1 = \frac{\Sigma I^2_{y_i} \, \Sigma I_{x_i} (-I_{t_i}) - \Sigma I_{x_i} I_{y_i} \, \Sigma I_{y_i} (-I_{t_i})}{D}$$

$$b_1 = \frac{-\Sigma I_{x_i} I_{y_i} \, \Sigma I_{x_i} (-I_{t_i}) + \Sigma I^2_{x_i} \, \Sigma I_{y_i} (-I_{t_i})}{D}$$

$$a_2 = \frac{\Sigma I^2_{y_i} \, \Sigma I_{x_i} (-I_{t_i}) - \Sigma I_{x_i} I_{y_i} \, \Sigma I_{y_i} (-I_{t_i})}{D}$$

$$b_2 = \frac{\Sigma I^2_{x_i} \, \Sigma I_{y_i} (-I_{t_i}) - \Sigma I_{x_i} I_{y_i} \, \Sigma I_{x_i} (-I_{t_i})}{D}.$$