QUANTUM GROUPS LECTURE NOTES

Contents

1. Introduction and plan

This course is an introduction to *quantum groups*. I want to start out by giving a rough sense of what kind of objects quantum groups are, and of the various mathematical/physical contexts in which they appear.

Let's begin with the meaning of the word 'quantum'. In Heisenberg's formulation, a quantum mechanical system consists of the following ingredients:

- A family of $*$-algebras $A_\hbar$ (the 'algebra of observables') depending on a real parameter $\hbar$ (Planck's constant), whose fiber at $\hbar = 0$ is commutative.
- A Hilbert space representation $\mathcal{H}$ (the 'space of states') of the algebra $A_\hbar$;
- A special self-adjoint element $H \in A_\hbar$, called the *Hamiltonian*[1].

The canonical example is the quantum mechanics of a single particle in 1-dimensional space X with Euclidean coordinate x: we take $\mathcal{H} = L^2(X)$, and $H = \frac{1}{2}p^2 + V(x)$ where

$$p = -i\hbar \frac{d}{dx}$$

is the momentum operator, and $V(x)$ is a real valued function on X called the 'potential'. In this case the algebra of observables is generated (at least for a polynomial potential) by p and x, the operator of multiplication by the function $x : X \to \mathbb{R}$, and they satisfy the canonical commutation relation

$$[p, x] = -i\hbar.$$

The way that this setup produces real numbers we can measure is through taking *matrix elements* of the algebra $A_\hbar$ in its representation $\mathcal{H}$. Given a quantum state $\psi \in \mathcal{H}$ and a self-adjoint observable $\mathcal{O} \in A_\hbar$, the expectation value of the observable $\mathcal{O}$ is the real number

$$\langle \mathcal{O} \rangle = \langle \psi, \mathcal{O}\psi \rangle.$$

In the Heisenberg picture of time evolution, we think of the states remaining constant while the observables evolve in time according to

$$-i\hbar \frac{d\mathcal{O}}{dt} = [H, \mathcal{O}],$$

so that the evolution of observables can be written in terms of the unitary *time evolution operator* $U(t) = \exp(-iH/\hbar)$ as

$$\mathcal{O}(t) = U(t)^{-1} \mathcal{O} U(t).$$

Hence the expectation value of $\mathcal{O}$ evolves in time by

$$\langle \mathcal{O} \rangle_t = \langle \psi, U(t)^{-1} \mathcal{O} U(t) \psi \rangle = \langle \psi(t), \mathcal{O}_0 \psi(t) \rangle,$$

where $\psi(t) = U(t)\psi$ satisfies the Schrödinger equation

$$i\hbar \frac{d\psi}{dt} = H\psi(t).$$

In real life the Planck constant $\hbar$ is very small, and what we actually observe most of the time on the human scale is the *semiclassical limit* of the quantum system, where $\hbar \to 0$ and the algebra of observables becomes commutative. In the example above of the non-commutative algebra generated by p, x describing the QM of a 1-dimensional particle, we can think of its specialization at $\hbar = 0$ as the commutative algebra of functions on the classical mechanical *phase space* $T^*\mathbb{R}$ for the particle.

Although all commutators in $A_\hbar$ of course vanish at $\hbar = 0$, we can still remember their leading order term in $\hbar$:

$$\mathcal{O}_1\mathcal{O}_2 - \mathcal{O}_2\mathcal{O}_1 =: -i\hbar\{\mathcal{O}_1, \mathcal{O}_2\} + O(\hbar^2)$$

[1] Here we only consider the special case of 'time-independent' Hamiltonians

This equips the ring of functions on phase space (the 'classical observables') with the extra structure of a *Poisson bracket*. In the example above, the Poisson bracket for phase space is uniquely determined by

$$\{p, x\} = 1$$

together with the obvious skew-symmetry and bi-derivation properties inherited from those of the commutator: for instance, we have $\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + \{f_1, f_3\}f_2$. The time evolution of a classical observable f is then given by

$$\frac{df}{dt} = \{H, f\}.$$

Applying this with $f = p$, we recover the classical Newton second law for the trajectory of the particle in the form $\dot{p} = -V'(x)$.

Next we need to think about what we mean by a 'group'. From the point of view of algebraic geometry, the way we work with an (affine, algebraic) group G is via its coordinate ring $\mathbb{C}[G]$ of regular functions: the matrix coefficients of G in its finite dimensional representations. The commutative ring $\mathbb{C}[G]$ also has some extra structure, this time coming from the product operation $m : G \times G \to G$, which defines a ring map

$$m^* : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[G].$$

For example, if we consider the group B of upper triangular invertible 2×2 matrices then

$$\mathbb{C}[B] = \mathbb{C}[x_{11}^{\pm 1}, x_{22}^{\pm 1}, x_{12}]$$

with

$$m^*(x_{11}) = x_{11} \otimes x_{11}, \quad m^*(x_{12}) = x_{11} \otimes x_{12} + x_{12} \otimes x_{22}, \quad m^*(x_{22}) = x_{22} \otimes x_{22}.$$

The associativity of the group law translates into what's known as *co-associativity* for the *coproduct* m^*. We also get a special map $\epsilon : \mathbb{C}[G] \to \mathbb{C}$ given by evaluation at the identity element in G (called the *counit*), and a map $S : \mathbb{C}[G] \to \mathbb{C}[G]$ given by inversion (called the *antipode*). These structures satisfy a bunch of obvious compatibilities (e.g the one saying that the coproduct is a map of rings) and make the coordinate ring $\mathbb{C}[G]$ into what is called a *Hopf algebra.*

Unsurprisingly, one can arrive at the notion of quantum groups by thinking about combining these two concepts: the kind of objects we'll be interested in are $\hbar$-families of Hopf algebras $A_\hbar$ such that the fiber at $\hbar = 0$ is the commutative algebra of functions on a group.

Example 1.1. (A quantum analog of $\mathbb{C}[B]$) There is family of Hopf algebra structures on the same underlying vector space $\mathbb{C}[B] = \mathbb{C}[x_{11}^{\pm 1}, x_{22}^{\pm 1}, x_{12}]$, whose co-unit and coproduct are identical to those of $\mathbb{C}[B]$, but whose product is deformed to

$$x_{11}x_{12} = q x_{12}x_{11}, \quad x_{22}x_{12} = q^{-1}x_{22}x_{12}, \quad x_{11}x_{22} = x_{22}x_{11},$$

where we've re-parametrized our parameter $\hbar$ as

$$q = e^\hbar.$$

As we've seen, when we take the classical limit of such a 'quantum group' we get a Poisson bracket on the coordinate ring $\mathbb{C}[G]$ of matrix coefficients. The semiclassical shadow of the compatibility between products and coproducts built into the definition of a Hopf algebra is the fact that the multiplication map $G \times G \to G$ preserves Poisson brackets, and thus makes G into what's called a *Poisson-Lie group.*

We will be mostly interested in this when that G is a simple Lie group, for example $G = SL_N$. In this case, one can draw a parallel with a still more algebraic way to think about G – the group can be *reconstructed* from its rigid symmetric monoidal category of finite dimensional representations, together with a symmetric monoidal fiber functor. The quantum analogs of simple Lie groups we'll discuss can be thought of as deformations of this structure: they are equivalent to the data of a rigid *braided* monoidal category, with a braided monoidal fiber functor. The truly novel feature here is that the deformed categories are braided, but not symmetric monoidal: in other words, the double braiding isomorphisms $V \otimes W \to W \otimes V \to V \otimes W$ are not just the identity map, like they would be in the category of representations of a group. These deformed categories depend on the parameter $q \in \mathbb{C}^*$, and exhibit many interesting new phenomena when q is a root of unity.

The first goal for the course is to construct these braided tensor categories, and give a practical user's guide for how to work with them. Secondly, I hope to give a flavor of at least some of the applications of the theory to problems in other parts of math and physics:

- Applications to quantum integrable systems and lattice models in statistical mechanics, i.e. the 'quantum inverse scattering method'
- Applications to invariants of links and 3-manifolds, i.e. the Jones polynomial and Witten-Reshetikhin-Turaev invariants.
- Applications to representation theory: construction of crystal/canonical bases, applications to representation theory of algebraic groups in positive characteristic, ...

These applications gave an important impetus for the discovery of quantum groups in the first place. We definitely won't have time to cover all of them, but I hope we'll get to talk about a few in a reasonable amount of detail.

2. Poisson-Lie groups

2.1. **Poisson basics.** A *Poisson structure* on a manifold M is a Lie bracket

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which is also a bi-derivation on the ring of smooth functions on M: for functions f_1, f_2, f_3 we have

$$\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + \{f_1, f_3\}f_2.$$

The Poisson bracket determines, and is determined by, a bi-vector field $\pi \in \wedge^2 TM$ (the Poisson bivector) such that

$$\{f_1, f_2\} = \langle \pi, df_1 \otimes df_2\rangle.$$

A Poisson map $\phi : M \to N$ between Poisson manifolds is one satisfying

$$\{f_1 \circ \phi, f_2 \circ \phi\}_M = \{f_1, f_2\}_N \circ \phi$$

or equivalently $\phi_*(\pi_M) = \pi_N$. Given two Poisson manifolds M, N, we can equip $M \times N$ with a Poisson structure $\pi_{M\times N} = \pi_M + \pi_N$. The projections p_M, p_N to the factors are then Poisson maps, so that $\{p_M^* f, p_N^* h\}_{M\times N} = 0$.

The data of a Poisson bracket allows us to attach to any smooth function H on M a vector field X_H called the *Hamiltonian vector field* of H: this is just the derivation of $C^\infty(M)$ given by $X_H \cdot f = \{H, f\}$. Note that the Jacobi identity says that the map $C^\infty(M) \to \mathrm{Vect}(M)$ is a map of Lie algebras.

More geometrically, contraction with the Poisson tensor defines a bundle map $I : T^*M \to TM$, and the Hamiltonian vector X_H is the image of the differential of H under this map:

$$X_H = I(dH) = \langle \pi, dH \otimes - \rangle.$$

If $I : T^*M \to TM$ is non-degenerate, then we can apply the tensor square of its inverse to the Poisson bivector itself to get a non-degenerate 2-form ω on M such that

$$\omega(X_{f_1}, X_{f_2}) = \{f_1, f_2\}.$$

The Jacobi identity for the bracket implies that $d\omega = 0$: indeed, if we choose local coordinates x_i then the Hamiltonian vector fields X_{x_i} are locally linearly independent (but possibly non-commuting!) by the non-degeneracy of I, and we have

$$\begin{aligned} d\omega(X_{x_i}, X_{x_j}, X_{x_k}) &= X_{x_i}\omega(X_{x_j}, X_{x_k}) - X_{x_j}\omega(X_{x_i}, X_{x_k}) + X_{x_k}\omega(X_{x_i}, X_{x_j}) \\ &\quad - \omega([X_{x_i}, X_{x_j}], X_{x_k}) + \omega([X_{x_i}, X_{x_k}], X_{x_j}) - \omega([X_{x_j}, X_{x_k}], X_{x_i}) \\ &= \{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\} \\ &\quad - \omega(X_{\{x_i,x_j\}}, X_{x_k}) + \omega(X_{\{x_i,x_k\}}, X_{x_j}) - \omega(X_{\{x_j,x_k\}}, X_{x_i}) \\ &= 2\left(\{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\}\right) = 0. \end{aligned}$$

This calculation shows that a nondegenerate Poisson structure w is the same data as that of a symplectic form ω on M. In the symplectic case, the Hamiltonian vector field X_H is characterized by $dH = \langle \omega, X_H \otimes - \rangle$. When $M = T^*\mathbb{R}^n$ with canonical coordinates (p_i, x_i) and symplectic form $\omega = \sum_i dp_i \wedge dx_i$, we recover the Poisson bracket from classical mechanics:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} \right)$$

A interesting source of generally non-symplectic Poisson structures is the *Kirillov-Kostant* Poisson structure on the dual space to a Lie algebra. Indeed, we can regard $\mathfrak{g}$ as the space of linear functions on $\mathfrak{g}^*$, and then extend the bracket $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ from linear functions to all functions via the Leibniz rule. If we choose a basis in $\mathfrak{g}$ with structure constants $[x_i, x_j] = \sum c_{ij}^k x_k$, the Poisson bivector is given explicitly by

$$\pi(\xi) = \sum_{i,j,k} c_{ij}^k x_k(\xi) \partial_i \wedge \partial_j, \tag{2.1.1}$$

where we identify all tangent spaces with $\mathfrak{g}^*$ itself. Note that the Poisson-bivector respects the linear structure on the vector space $\mathfrak{g}^*$: we have $\pi(\xi + \eta) = \pi(\xi) + \pi(\eta)$.

If $x \in \mathfrak{g}$ is a linear function on $\mathfrak{g}^*$, its Hamiltonian vector field is the derivation which sends on another linear function y to $[x, y]$: $(X_x \cdot y)(\xi) = \langle \xi, [x, y] \rangle =: \mathrm{ad}_x^*(y)(\xi)$. In particular, the flow of its Hamiltonian vector field is given by the coadjoint action: $\Phi_{X_x}^t(\xi) = \mathrm{Ad}_{e^{tx}}^*(\xi)$.

We say that two points x, y in a Poisson manifold M are in the same *symplectic leaf* if they are connected by a piecewise smooth curve, each piece of which is an integral curve of a Hamiltonian vector field. More geometrically, the map $I : T^*M \to TM$ defines an *involutive distribution* on M: a family of subspaces of each tangent space which are locally spanned by a Lie subalgebra of vector fields (the involutivity follows from the fact that the map sending a function to its Hamiltonian vector field is a map of Lie algebras). Weinstein showed that this distribution integrates to define a foliation of M by immersed submanifolds (if the Poisson structure has constant rank, this follows from the Frobenius theorem, but the general case requires a Darboux-theorem for Poisson manifolds.) Setting $\omega_S(X_f, X_g) = \{f, g\}$ defines a symplectic form on each leaf S of this foliation, and for this reason its leaves are called the *symplectic leaves* of the Poisson manifold M.

Example 2.1. The symplectic leaves in the Poisson manifold $\mathfrak{g}^*$ are the orbits for the coadjoint action of a connected Lie group G with Lie algebra $\mathfrak{g}$: our calculation of Hamiltonian vector fields above shows that the symplectic leaf through ξ is contained in the coadjoint orbit through ξ, and since G is generated by the image of exp we deduce that any two points in the same coadjoint orbit lie in the same symplectic leaf.

Exercise 2.2. *Work out the symplectic leaves in $\mathfrak{su}_2^*$ and in the 3-dimensional Heisenberg Lie algebra with respect to the KKS Poisson structure.*

2.2. **Poisson-Lie groups and Lie bialgebras.** A Poisson-Lie group G is a Lie group equipped with a Poisson structure such that the multiplication $G \times G \to G$ is a Poisson map. Explicitly, this means that

$$\{f_1, f_2\}_G(gg') = \{f_1 \circ R_{g'}, f_2 \circ R_{g'}\}_G + \{f_1 \circ L_g, f_2 \circ L_g\}_G,$$

or at the level of bivectors,

$$\pi_{gg'} = (L_g)_*^{\otimes 2}(\pi_{g'}) + (R_{g'})_*^{\otimes 2}(\pi_g).$$

Plugging in $g = g' = 1$, we see that the Poisson bivector for a Poisson-Lie group must vanish at the identity element – in particular, a Poisson-Lie group cannot be symplectic.

Exercise 2.3. *Show that the inverse map $\iota : G \to G,\ g \mapsto g^{-1}$ in a Poisson Lie group is anti-Poisson: $\iota_* \pi_G = -\pi_G$.*

Note that the condition that $G \times G \to G$ be Poisson does not imply that the translation maps $L_g, R_g : G \to G$ are Poisson. Indeed, R_g and L_g will be Poisson precisely when $\pi_g = 0$.

Poisson-Lie groups form a category, where morphisms are Poisson group homomorphisms.

Example 2.4. Any Lie group G can be made into a Poisson-Lie group by giving it the zero Poisson structure.

Example 2.5. An example of a Poisson-Lie group with nontrivial Poisson structure is given by the additive group $(\mathfrak{g}^*, +)$ with the KKS Poisson structure. We'll soon see that there is a sense in which this example is actually 'dual' to the one above.

Exercise 2.6. *Show that if a Lie subgroup H is a Poisson submanifold of a Poisson-Lie group G, then the homogeneous space G/H inherits a Poisson structure such that the projection map from G is Poisson. Is there a weaker condition on the subgroup H under which the same is true of G/H?*

We'd like to give a local description of the Poisson structure on a Poisson-Lie group in a neighborhood of 1, analogous to the way that the Lie algebra locally describes its multiplication law.

We can use right translation to trivialize the tangent bundle TG, and thereby think about a Poisson structure on a Lie group as a map $\pi^R : G \to \mathfrak{g} \otimes \mathfrak{g} = T_1G \otimes T_1G$. The Poisson-Lie condition says that this map satisfies

$$\pi^R(gg') = (\mathrm{Ad}_g)^{\otimes 2}(\pi^R(g')) + \pi^R(g),$$

and thus π^R defines a 1-cocycle on G valued in the tensor square $\mathfrak{g} \otimes \mathfrak{g}$ of the adjoint representation. Its derivative at the identity gives a Lie algebra 1-cocycle $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$:

$$\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x).$$

The bracket can be computed as

$$\{f_1, f_2\}(g) = \langle \pi^R(g), R_g^*(df_1 \otimes df_2) \rangle.$$

Taking the derivative at the identity in the direction $x \in \mathfrak{g}$ and recalling $\pi^R(1) = 0$, we see that

$$d\{f_1, f_2\}(x) = \langle \delta(x), \xi_1 \otimes \xi_2 \rangle = \langle x, \delta^*(\xi_1 \otimes \xi_2) \rangle,$$

where $(df_i)_1 = \xi_i \in \mathfrak{g}^*$. Hence the map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ can be computed by choosing two functions f_i whose derivative at 1 is ξ_i, and taking the derivative of their bracket $\{f_1, f_2\}$ at 1, so we conclude that δ^* defines a Lie bracket on $\mathfrak{g}^*$.

A more algebraic way to think of this is as that, since $\pi^R(1) = 0$, the Poisson bracket maps the ideal I of (germs of) functions vanishing at $1 \in G$ to itself, thereby equipping I with a Lie algebra structure with respect to which I^2 is a Lie ideal. Hence we get a Lie algebra structure on the cotangent space I/I^2 to G at 1, given by $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$.

Summarizing, we see that the Lie algebra of a Poisson Lie group is a "Lie bialgebra", in the following sense:

Definition 2.7. A *Lie bialgebra* $\mathfrak{g}$ is a Lie algebra equipped with a 1-cocycle $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ valued in the tensor square of the adjoint representation, such that the dual map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket.

Example 2.8. Recall the Poisson-Lie group $(\mathfrak{g}^*, +)$, with the Kirillov-Kostant Poisson structure. From the linear form (2.1.1) of its Poisson tensor, we see that the co-bracket δ in this case is the dual of the Lie bracket in $\mathfrak{g}$.

Lie bialgebras form a category where morphisms are Lie algebra morphisms ϕ satisfying $\delta \circ \phi = \phi^{\otimes 2} \circ \delta$. We have the following analog of Lie's third theorem for Poisson-Lie groups:

Theorem 2.9. *Let G be a connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then every Lie bialgebra structure on $\mathfrak{g}$ integrates to a unique Poisson-Lie structure on G. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie bialgebras, then there is a unique morphism of Poisson-Lie groups $\Phi : G \to H$ with $d\Phi_e = \phi$.*

The key fact here is that for G simply connected, Lie algebra 1-cocycles valued in $\mathfrak{g} \otimes \mathfrak{g}$ can be uniquely integrated to ones for G. (This boils down to the existence and uniquness for a system of ODE on G, see the exercise below) Hence the we obtain from the Lie bialgebra structure a unique multiplicative bivector field on G. The left-hand-side of the Jacobi identity for the corresponding would-be Poisson bracket then defines a multiplicative tri-vector field with vanishing intrinsic derivative at the identity (by the Jacobi identity for δ^*), and hence is identically zero. This shows every Lie bialgebra can indeed be integrated. Moerover, Lie bialgebra maps can be lifted to Lie group ones by the usual Lie III theorem, and to see that the lift preserves the Poisson structure note that both $\Phi_*^{\otimes 2}\pi_G(g)$ and $\pi_H(\Phi(g))$ both define 1-cocycles for G valued in $\mathfrak{h}\otimes\mathfrak{h}$, whose associated Lie algebra cocycles agree. Hence by the uniqueness of lifting, the two coincide and Φ is Poisson.

Exercise 2.10. *For $x \in \mathfrak{g}$, consider the differential operator ∇_x on sections of $\wedge^2 TG$ given by*

$$(\nabla_x \cdot \phi)(g) = \frac{d}{dt}\phi(e^{tx}g)\big|_{t=0}.$$

If we fix a Lie cobracket δ, show that a Poisson bivector $\pi^R : G \to \wedge^2 TG$ integrating δ is a solution the inhomogeneous system of linear differential equations

$$\nabla_x \phi = \delta(x) + \mathrm{ad}_x^{\otimes 2} \cdot \phi, \quad x \in \mathfrak{g}$$

and that ∇ satisfies the flatness/consistency condition:

$$[\nabla_x, \nabla_y] = \nabla_{[x,y]}.$$

2.3. **Manin triples and Poisson-Lie duality.** There is another way to think about Lie bialgebras, which reveals a symmetry between the roles played by $\mathfrak{g}$ and $\mathfrak{g}^*$ in the definition. To formulate it, suppose that we have Lie algebra structures on both a finite dimensional vector space $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Then we can use the duality pairing to define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}\oplus\mathfrak{g}^*$ such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are both isotropic subspaces. We will reformulate the notion that the cobracket $\delta^* : \mathfrak{g}^* \to \mathfrak{g}^*\otimes\mathfrak{g}^*$ is a 1-cocycle in terms of the existence of a Lie algebra structure on $\mathfrak{g}\oplus\mathfrak{g}^*$ such that both $\mathfrak{g}, \mathfrak{g}^*$ are Lie subalgebras, and this form is $\mathfrak{g}\oplus\mathfrak{g}^*$-invariant.

Lemma 2.11. *In the setting above, there exists a (unique) Lie algebra structure on $\mathfrak{g}\oplus\mathfrak{g}^*$ such that the nondegenerate symmetric bilinear form $(\cdot,\cdot)$ is $\mathrm{ad}_{\mathfrak{g}\oplus\mathfrak{g}^*}$–invariant if and only if δ^* is a 1-cocycle, i.e. $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra.*

Proof. The point is that if a Lie algebra structure on $\mathfrak{g}\oplus\mathfrak{g}^*$ exists, then condition that this form be ad-invariant then uniquely determines the Lie bracket between $\mathfrak{g}$ and $\mathfrak{g}^*$: indeed choosing dual bases x_i, ξ_i in each, we can compute the $\mathfrak{g}^*$-component of $[\eta, y]$ as

$$\langle[\eta, y], x_i\rangle\xi_i = \sum\langle\eta, [y, x_i]\rangle\xi_i,$$

and the $\mathfrak{g}$-component is determined similarly. For this to actually define a Lie bracket, the Jacobi identity needs to hold, and one checks that this is equivalent to the cocycle condition on δ. $\square$

If we choose dual bases a_i, x_i in $\mathfrak{g}, \mathfrak{g}^*$ with respect to the form with structural constants

$$[a_i, a_j] = \sum_k a_{ij}^k a_k, \quad [x^i, x^j] = \sum_k \gamma_k^{ij} x_k$$

then the cross-brackets take the form

$$[a_j, x^k] = \sum_i c_{ij}^k x^i + \gamma_j^{ki} a_i. \tag{2.3.1}$$

A *Manin triple* is a Lie algebra $\mathfrak{d}$ equipped with an nondegenerate, invariant symmetric bilinear form $(\cdot,\cdot)$ and a pair of Lie subalgebras $\mathfrak{d}_+, \mathfrak{d}_-$ such that we have a vector space decomposition $\mathfrak{d} = \mathfrak{d}_+ \oplus \mathfrak{d}_-$, where both subspaces $\mathfrak{d}_\pm \subset \mathfrak{d}$ are isotropic with respect to the bilinear form. The discussion above shows that finite dimensional Lie bialgebras are in 1-1 correspondence with finite dimensional Manin triples.

With this under our belt, we can finally construct some more interesting examples of Poisson-Lie groups.

Example 2.12 (The standard Poisson-Lie structure on a complex semisimple Lie group)**.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ with Chevalley generators E_i, F_i, H_i corresponding to the nodes of the Dynkin diagram of $\mathfrak{g}$. We also write $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ for the standard Borel subalgebras with respect to this decomposition. For example, take $\mathfrak{g} = sl_2$ with standard generators

$$[E,F] = H, \quad [H,E] = 2E, \quad [H,F] = -2F.$$

and $\mathfrak{h} = \langle H \rangle, \mathfrak{n}_+ = \langle E \rangle, \mathfrak{n}_- = \langle F \rangle$. Then there is a unique Lie bialgebra structure on $\mathfrak{g}$ with cobracket determined by the following values on these generators:

$$\delta(H_i) = 0, \quad \delta(E_i) = \frac{d_i}{2} H_i \wedge E_i, \quad \delta(F_i) = \frac{d_i}{2} H_i \wedge F_i,$$

where $d_i a_{ij}$ is the symmetrized Cartan matrix of $\mathfrak{g}$. This Lie bialgebra comes from the following Manin triple.

Let $(\cdot,\cdot)$ be the invariant nondegenerate bilinear form on $\mathfrak{g}$ normalized so that

$$(E_i, F_i) = \frac{1}{d_i}, \quad (H_i, H_i) = \frac{2}{d_i}$$

where $d_i a_{ij}$ is the symmetrized Cartan matrix. Now consider the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ equipped with the pairing

$$\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1, x_2) - (y_1, y_2).$$

Now take $\mathfrak{d}_+$ to be the diagonally embedded copy of $\mathfrak{g}$, and $\mathfrak{d}_- \subset \mathfrak{b}_+ \oplus \mathfrak{b}_-$ to be the subset of pairs (x, y) such that the $\mathfrak{h}$-component of $x + y$ is zero. For example, when $\mathfrak{g} = sl_2$, $\mathfrak{d}_-$ is the 3-dimensional solvable Lie algebra generated by

$$(E, 0), \ (H, -H), \ (0, F).$$

Then since the $\mathfrak{n}_\pm$ are isotropic with respect to the Killing form, and pair to zero with $\mathfrak{h}$, this indeed defines a Manin triple and we obtain Lie bialgebra structures on $\mathfrak{g}$ and $\mathfrak{g}^* \simeq \mathfrak{d}_-$. The formulas mentioned above for the cobracket now follow easily by dualizing the bracket on $\mathfrak{p}_-$ using the form: for example, for any $(x_1, y_1), (x_2, y_2)$ in $\mathfrak{d}_-$ we have

$$\langle \delta(E_i), (x_1, y_1) \otimes (x_2, y_2) \rangle = \langle (E_i, E_i), [(x_1, y_1), (x_2, y_2)] \rangle$$

Using the fact that the root space basis satisfies $(E_\alpha, F_\beta) = 0$ for $\alpha \neq \beta$ and $(E_i, F_i) = d_i$, we see that indeed $\delta(E_i) = \frac{d_i}{2} H_i \wedge E_i$.

Remark 2.13. From the explicit form of δ, it's clear that the standard Poisson structure on a simple Lie algebra $\mathfrak{g}$ has the following properties:

- the subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$ are sub-Lie bialgebras of $\mathfrak{g}_{std}$;
- the simple root sl_2-triples $\mathfrak{sl}_2^{(\alpha_i)} \subset \mathfrak{g}$ are are sub-Lie bialgebras of $\mathfrak{g}_{std}$;
- we have $\delta|_{\mathfrak{h}} = 0$.

The last property implies that the left and right translation maps $L_h, R_h : G \to G$ by elements of H are Poisson maps, and hence H acts on the set of symplectic leaves in G_{std}.

Exercise 2.14 (Compact Lie groups)**.** *Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a complex simple Lie algebra, and recall its* Iwasawa decomposition*: as a real vector space, we have*

$$\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{h}_\mathbb{R} \oplus \mathfrak{n}),$$

where

$$\mathfrak{k} = \mathrm{span}\{E_\alpha - F_\alpha, \mathbf{i}(E_\alpha + F_\alpha), \mathbf{i}H_\alpha\}$$

is the Lie algebra of the compact real form of $\mathfrak{g}$. (For example, when $\mathfrak{g} = sl_n(\mathbb{C})$ we have $\mathfrak{k} = \mathfrak{su}_n$ and $\mathfrak{b}$ is the Lie algebra of traceless upper triangular matrices with real entries on the diagonal.) Show that the imaginary part of the Killing form makes $\mathfrak{g}$ into a Manin triple, and hence makes the compact Lie group K into a Poisson-Lie group.

From the reformulation via Manin triples, we also deduce the existence of *duality functor* $\mathfrak{g} \mapsto \mathfrak{g}^*$ on the category of Lie bialgebras – and hence by Theorem 2.9 that of a similar functor $G \mapsto G^*$ on the level of Poisson-Lie groups.

Example 2.15. Recall the Poisson-Lie group $(\mathfrak{g}^*, +)$, whose tangent Lie bialgebra is $\mathfrak{g}^*$), with trivial Lie bracket on $\mathfrak{g}^*$, and cobracket δ induced by the Lie bracket on $\mathfrak{g}$. Swapping the roles of $\mathfrak{g}, \mathfrak{g}^*$, we see that the dual Lie bialgebra $\mathfrak{g}$ has Lie bracket given by that of $\mathfrak{g}$, and trivial cobracket. Hence the corresponding Poisson-Lie group is the connected, simply connected Lie group G integrating $\mathfrak{g}$, equipped with the zero Poisson structure.

Example 2.16. The Poisson-Lie group dual to G with its standard Poisson-Lie structure is the solvable group

$$G^* = B_+ \times_H B_-,$$

where the maps in the fiber product are

$$B_\pm \to H \simeq B/[B,B], \; b \mapsto b^{\pm 1}.$$

2.4. **Coboundary Lie bialgebras and classical Yang-Baxter.** We will be interested in Lie bialgebras whose cobracket δ is not just a 1-cocycle, but in fact a 1-coboundary: there exists some $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta(x) = x \cdot r = \sum [x, a_i] \otimes b_i + a_i \otimes [x, b_i].$$

Of course this will always be the case if the corresponding cohomology group vanishes, which will happen for instance whenever G is a connected semisimple or compact Lie group.

A Poisson tensor defined by a Lie algebra coboundary $\delta = d(r)$ is easy to integrate: the Poisson bivector π^R on the corresponding simply connected Poisson-Lie group is given by

$$\pi^R(g) = \mathrm{Ad}_g(r) - r,$$

so that

$$\pi_g = L_g(r) - R_g(r).$$

Concretely, if we expand

$$r = \sum_i a_i \otimes x_i \in \mathfrak{g} \otimes \mathfrak{g}$$

this formula for the bivector gives the following bracket between functions:

$$\{f_1, f_2\}(g) = \frac{d}{dt}\bigg|_{t=0} \left(f_1(ge^{ta_i})f_2(ge^{tx_i}) - f_1(e^{ta_i}g)f_2(e^{tx_i}g)\right) \tag{2.4.1}$$

Example 2.17. The standard Lie bialgebra structure on sl_2 is coboundary, with

$$r = -\left(\frac{H \otimes H}{4} + E \otimes F\right).$$

Its image in the vector representation is the 4×4 matrix

$$r^{\mathbb{C}^2 \otimes \mathbb{C}^2} = -\frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The same is true of the standard structure on any simple Lie algebra: if $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the invariant element we used to define the Manin triple above and we write $\Omega = \Omega_{+,-} + \Omega_{0,0} + \Omega_{-,+}$ for its triangular decomposition, then we can take the element r to be

$$r = -\frac{1}{2}\Omega_{0,0} - \Omega_{+,-}. \tag{2.4.2}$$

Exercise 2.18. *Let $x_{11}, x_{12}, x_{21}, x_{22}$ be the functions on the standard Poisson-Lie $SL(2, \mathbb{C})$ given by the matrix elements of the vector representation. Show that*

$$\{x_{11}, x_{12}\} = \frac{1}{2}x_{11}x_{12}, \quad \{x_{11}, x_{21}\} = \frac{1}{2}x_{11}x_{21}, \quad \{x_{22}, x_{12}\} = -\frac{1}{2}x_{22}x_{12}, \quad \{x_{22}, x_{21}\} = -\frac{1}{2}x_{22}x_{21}$$

$$\{x_{12}, x_{21}\} = 0, \quad \{x_{11}, x_{22}\} = x_{12}x_{21}.$$

The requirement that δ^* define a Lie bracket (i.e is skew, and satisfies the Jacobi identity) corresponds to the following constraints on the element $r \in \mathfrak{g} \otimes \mathfrak{g}$: for skewness we need $r_{12} + r_{21}$ to be an invariant for the adjoint action, while (in the presence of skewness!) the Jacobi identity is equivalent to the requirement that

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} \tag{2.4.3}$$

The notation $[r_{12}, r_{13}]$ means that we form $r_{12} \otimes r_{13} \in \mathfrak{g}^{\otimes 4}$, and then apply the bracket operation on the indicated factors to get an element of $\mathfrak{g}^{\otimes 3}$:

$$[r_{12}, r_{13}] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j.$$

$$[r_{12}, r_{23}] = \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j,$$

$$[r_{13}, r_{23}] = \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].$$

Exercise 2.19. *Prove this by directly writing out the Jacobi identity for the bracket of $\mathfrak{g}^*$:*

$$\langle[\alpha,[\xi,\eta]],c\rangle + \cdots = 0.$$

(The only tricky point is that one needs to use the fact that $r + r_{21}$ is $\mathfrak{g}$-invariant (and the Jacobi identity in $\mathfrak{g}$) to eliminate all terms containing a tensor factor of the form $[[c,a_i],b_j]$)

The simplest way to ensure that (2.4.3) holds is to impose the *classical Yang-Baxter equation*

$$CYB(r) := [r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] = 0.$$

A Lie bialgebra whose cobracket comes from such a solution $r \in \mathfrak{g}\otimes\mathfrak{g}$ is called *quasitriangular*, and the element r is called its *classical r-matrix.*

Exercise 2.20. *If the r-matrix defining a coboundary Lie bialgebra satisfies the CYBE, then the map*

$$r : \mathfrak{g}^* \to \mathfrak{g}, \quad \eta \mapsto \langle \eta, r\otimes -\rangle$$

is a homomorphism of Lie algebras, where $\mathfrak{g}^*$ has the Lie algebra defined by δ^*.

Remark 2.21. Having the classical Yang-Baxter equation hold on the nose (i.e. not just up to an invariant in $\mathfrak{g}^{\otimes 3}$) is our reason for working with cocycles in $\mathfrak{g}\otimes\mathfrak{g}$ rather than in $\mathfrak{g}\wedge\mathfrak{g}$: the condition (called 'triangularity') for the CYBE to have a skew solution turns out to be too restrictive. A quasitriangular Lie bialgebra $(\mathfrak{g}, r)$ is called *factorizable* if the linear map

$$r + r_{21} : \mathfrak{g}^* \to \mathfrak{g}$$

is an isomorphism of vector spaces. So factorizable Lie bialgebras are sort of the opposite of triangular ones (for which $r + r_{21} = 0$), and it is the factorizable ones that we will mostly care about in the course.

Remark 2.22. Suppose that $\mathfrak{g}$ is a complex simple Lie algebra, and $r \in \mathfrak{g}\otimes\mathfrak{g}$ defines a quasitriangular Lie bialgebra structure on $\mathfrak{g}$, so r a solution of the classical YB equation. Then by Schur's lemma, the invariant $r_{12} + r_{21}$ must be a scalar multiple of the Killing form B. The skew-symmetrization of $\hat{r}$ then has the property that $CYB(\hat{r})$ is a multiple of the Maurer-Cartan form $\omega \in \wedge^3\mathfrak{g}$ defined by

$$\omega(x,y,z) = B([x,y],z),$$

which spans $H^3(\mathfrak{g},\mathbb{C}) \simeq \mathbb{C}$.

At this point, there are several natural questions to ask:

(1) Is there a systematic way to construct solutions of the classical Yang-Baxter equation?
(2) Is the standard Lie bialgebra structure on a simple Lie algebra, defined by the element r in (2.4.2), quasitriangular?
(3) If we fix a simple Lie algebra $\mathfrak{g}$, can one classify all solutions of the Yang-Baxter equation in $\mathfrak{g}\otimes\mathfrak{g}$?

An answer to questions 1 and 2 comes from the Drinfeld's construction of the *classical double* of a Lie bialgebra. Recall from the Manin triple formulation that for any Lie bialgebra $\mathfrak{g}$, the vector space $\mathfrak{g}\oplus\mathfrak{g}^*$ carries a unique Lie algebra structure such that the canonical symmetric bilinear form on $\mathfrak{g}\oplus\mathfrak{g}^*$ is invariant.

Proposition 2.23. *Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra, and $\mathfrak{g}^*$ its dual Lie bialgebra, and equip $\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g}^*$ with the canonical Lie algebra structure making $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ into a Manin triple. Then the canonical element*

$$r \in \mathfrak{g} \otimes \mathfrak{g}^* \subset (\mathfrak{g} \oplus \mathfrak{g}^*) \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)$$

satisfies the classical Yang-Baxter equation, and $r + r_{21}$ is identified with the canonical ad-invariant form $\langle , \rangle_{\mathfrak{d}}$. Hence r defines on $\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g}^*$ the structure of a factorizable quasitriangular Lie bialgebra. This Lie bialgebra structure has the property that the maps

$$\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \hookleftarrow \mathfrak{g}^*$$

are morphisms of Lie bialgebras.

Proof. That r satisfies the Yang-Baxter equation is immediate from the using formula (2.3.1) for the cross-brackets between $\mathfrak{g}$ and $\mathfrak{g}^*$ in the Lie algebra structure on $\mathfrak{d}$ to compute $[r_{12}, r_{23}]$. The skewness follows by observing that $r + r_{21}$ is the canonical symmetric invariant bilinear form on $\mathfrak{d}$; since this is nondegenerate we see that the Lie bialgebra structure is indeed factorizable. Similarly, one observes that for $a \in \mathfrak{g}$, the form of the cross-bracket is exactly what is needed to cancel the $\mathfrak{g}^*$ component in $\delta(a) = a \cdot r$. The Lie bialgebra structure is factorizable, since $r + r_{21}$ is the canonical symmetric invariant bilinear form on $\mathfrak{d}$. ☐

The following exercise is extremely important, and the key to deriving all the 'Leningrad' formulas for Poisson brackets in the double of a factorizable Lie bialgebra.

Exercise 2.24. *The standard Lie bialgebra structure on a simple Lie algebra $\mathfrak{g}$ is a Lie sub-bialgebra of the quasitriangular Lie bialgebra $\mathfrak{d}(\mathfrak{g}, \mathfrak{g}^*) \simeq \mathfrak{g} \oplus \mathfrak{g}$. Show that the r-matrix $r_{\mathfrak{d}}$ for $\mathfrak{d}$ is expressed in terms of the r-matrix r of $\mathfrak{g}$ as*

$$r_{\mathfrak{d}} = r_{14} + r_{24} - r_{31} - r_{32} \in (\mathfrak{g} \oplus \mathfrak{g}) \otimes (\mathfrak{g} \oplus \mathfrak{g}). \tag{2.4.4}$$

Here the confusing notations mean, for example with $\mathfrak{g} = sl_2$,

$$r_{14} = -\left(\frac{1}{4}(H, 0) \otimes (0, H) + (E, 0) \otimes (0, F)\right),$$

$$r_{32} = -\left(\frac{1}{4}(0, H) \otimes (H, 0) + (0, F) \otimes (E, 0)\right),$$

Example 2.25. The standard Lie bialgebra structure on $\mathfrak{g}$ is itself *almost* a double. Indeed, consider the trivial central extension $\mathfrak{g} \oplus \mathfrak{h}$ of $\mathfrak{g}$ by its Cartan subalgebra, equipped with the difference of the Killing form on $\mathfrak{g}$ and its restriction to $\mathfrak{h}$. Then we can embed the Borel subalgebras $\mathfrak{b}_{\pm}$ as isotropic subspaces of this extension, where the Cartan subalgebra in $\mathfrak{b}_+$ is embedded diagonally (i.e. as elements of the form (h, h)), and that from $\mathfrak{b}_-$ anti-diagonally (i.e. as elements $(h, -h)$). Using the root space basis we can easily write down dual basis in these subalgebras in $\mathfrak{g} \oplus \mathfrak{h}$, so we see that this construction gives $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_{\pm})$ the structure of a Manin triple, so that we obtain a triple of Lie bialgebras. Moreover, the Lie cobracket vanishes on the central summand $\mathfrak{h}$, so that the quotient $\mathfrak{g}$ inherits the structure of a Lie bialgebra, and this Lie bialgebra structure coincides with the standard one. In particular, we see that the dual Lie bialgebra to $\mathfrak{b}_+$ (with its standard LBA structure) is $\mathfrak{b}_-$, and that the double of $\mathfrak{b}_+$ is $\mathfrak{g} \oplus \mathfrak{h}$.

Exercise 2.26. *Describe the $\mathfrak{d}(\mathfrak{d}(\mathfrak{h}))$, the double of the double of an arbitrary Lie bialgebra $\mathfrak{h}$. Hint: the picture of $\mathfrak{d}(\mathfrak{d}(\mathfrak{h}))$ is similar to the description of the double of a standard simple Lie algebra $\mathfrak{g}$, for which $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}$.*

Remark 2.27. Recall the maps

$$r : \mathfrak{d} \to \mathfrak{d}, \quad d \mapsto \langle r, d \otimes - \rangle,$$

$$r_{21} : \mathfrak{d} \to \mathfrak{d}, \quad d \mapsto \langle r, - \otimes d \rangle.$$

Then r is the projection to the summand $\mathfrak{g}^*$ with respect to the splitting $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, and r_{21} the corresponding projection to $\mathfrak{g}$.

Example 2.28. Suppose that (G, r) is a coboundary Poisson Lie group and ρ^V, ρ^W are representations of G on vector spaces V, W. Then setting $r^{V;W} = (\rho^V \otimes \rho^W)(r) \in \mathrm{End}(V \otimes W)$, the matrix coefficients of these representations have Poisson brackets

$$\{\rho^V_{ij}, \rho^W_{kl}\} = \sum_a \rho^V_{ia}\rho^W_{kb} r^{V;W}_{aj;bl} - \rho^V_{aj}\rho^W_{bl} r^{V;W}_{ia;jb},$$

This can be expressed more succinctly in the 'Leningrad' notation as follows. The data of an algebraic representation V of G is the same as that of a comodule T^V for $\mathbb{C}[G]$:

$$T : V \to V \otimes \mathbb{C}[G], \quad \text{i.e} \quad T^V \in \mathrm{End}(V) \otimes \mathbb{C}[G].$$

Then we can write the formula above more invariantly as

$$\{T^V, T^W\} = [T^V \otimes T^W, r^{V;W}] \in \mathrm{End}(V \otimes W) \otimes \mathbb{C}[G],$$

where the left-hand-side means "form $T^V \otimes T^W \in \mathrm{End}(V \otimes W) \otimes \mathbb{C}[G]^{\otimes 2}$, and then apply the Poisson bracket $\mathbb{C}[G]^{\otimes 2} \to \mathbb{C}[G]$."

Exercise 2.29. *Show that the functions on the coboundary Poisson Lie group G defined by $H^V = \mathrm{tr}_V(g)$ as V ranges over all finite dimensional representations of G Poisson commute:*

$$\{H_V, H_W\} = 0.$$

In addition to the T^V, we can also form $\mathbb{C}[G^*]$-valued matrices $L^V_\pm$ by

$$L^V_+(b_+, b_-) = \rho^{V \otimes \mathbf{1}}(b_+, b_-), \quad L^V_-(b_+, b_-) = \rho^{\mathbf{1} \otimes V}(b_+, b_-)$$

where $\mathbf{1}$ is the trivial representation of G.

Exercise 2.30. *Use the embedding of Poisson Lie groups $G^* \subset D(G)$ and (2.4.4) to get the dual Leningrad formulas*

$$\{L^V_\pm, L^W_\pm\} = [L^V_\pm \otimes L^W_\pm, r^{V;W}], \quad \{L^V_+, L^W_-\} = [L^V_+ \otimes L^W_-, r^{V;W}].$$

Exercise 2.31. *Let e, f, k be the functions on SL_2^* given by the matrix elements of the tensor product of vector representations:*

$$\begin{bmatrix} k & k^{-1}e \\ 0 & k^{-1} \end{bmatrix}, \begin{bmatrix} k^{-1} & 0 \\ -kf & k \end{bmatrix}.$$

Show that their brackets are given by

$$\{k, e\} = \frac{1}{2}ke, \quad \{k, f\} = -\frac{1}{2}kf, \quad \{e, f\} = k^2 - k^{-2}$$

and that under the map $\Delta : \mathbb{C}[SL_2^*] \to \mathbb{C}[SL_2^*] \otimes \mathbb{C}[SL_2^*]$ *dual to the multiplication in* SL_2^* *we have*

$$\Delta(k) = k \otimes k, \quad \Delta(e) = e \otimes 1 + k^2 \otimes e, \quad \Delta(f) = f \otimes k^{-2} + 1 \otimes f.$$

2.5. **Dressing transformations and symplectic leaves in Poisson-Lie groups.** Recall the nice description of symplectic leaves in $\mathfrak{g}^*$ with its KKS structure: they are the orbits for the coadjoint action of a connected Lie group G with Lie algebra $\mathfrak{g}$. Recall also that $(\mathfrak{g}^*, +)$ and the group G with the zero Poisson structure are Poisson-Lie dual to one another. This picture actually generalizes to any such dual pair of Poisson-Lie groups in the following way.

We say that an action of a Poisson-Lie group G on a manifold M is *Poisson* if the map $G \times M \to M$ is Poisson. For example, the left and right translations of G (or any Poisson-Lie subgroup) on G are Poisson actions. Note that when the Poisson structure of G is nontrivial, the maps $g \cdot M \to M$ will *not* generally be Poisson.

Now let G be a Poisson-Lie group and let $D(G)$ be its double, so that we have Poisson-Lie subgroups

$$G \hookrightarrow D(G) \hookleftarrow G^*.$$

Since the Lie algebra of $D(G)$ is the direct sum of $\mathfrak{g}$ and $\mathfrak{g}^*$, the product map

$$G \times G^* \to D(G)$$

is a diffeomorphism onto a neighborhood of 1 in $D(G)$. Let's suppose for the moment that this map is a global diffeomorphism $G \times G^* \to D(G)$ (although this won't be true in many examples of interest, see below). Then we can identify G as a Poisson manifold with the homogeneous space $D(G)/G^*$. The latter carries a residual Poisson action of G^* by left translation, which under the identification $D(G)/G^* \simeq G$ gives a Poisson action of G^* on G called the *dressing action*, and the orbits of this action coincide with the symplectic leaves of G.

Indeed, let's compute the tangent space to the dressing orbit through $g \in G$. For $\xi \in \mathfrak{g}^*$, we have in $D(G)$

$$e^{t\xi} g = g e^{t\mathrm{Ad}_{g^{-1}}(\xi)}$$

Hence the corresponding tangent vector in G has the form ge^{tx} where x is the projection to $\mathfrak{g}$ of $\mathrm{Ad}_{g^{-1}}\xi \in \mathfrak{d}$. But recall that this projection is computed using the r-matrix as $x = \langle r, - \otimes \mathrm{Ad}_{g^{-1}}\xi\rangle$. In terms of our usual right-trivialization of TG, this is

$$\mathrm{Ad}_g\left(\langle r, - \otimes \mathrm{Ad}_{g^{-1}}\xi\rangle\right) = \langle \mathrm{Ad}_g^{\otimes 2}(r), - \otimes \xi\rangle = \langle \mathrm{Ad}_g^{\otimes 2}(r) - r, - \otimes \xi\rangle = \langle \pi^R(g), - \otimes \xi\rangle$$

Hence as ξ varies over all of $\mathfrak{g}^*$ we get the entire image of $\pi^R(g)$, so the tangent space to the orbit indeed coincides with the image of the bundle map $I : T^*G \to TG$.

Example 2.32. Recall that the double of a simple Lie group G is $G \times G$, with $G = G_\Delta$ is embedded diagonally and $G^* = B_+ \times_H B_-$ embedded in the natural way. Then the image of the multiplication map $G \times G^* \to G$ consists of elements of the form

$$(n_+ h g, n_- h^{-1} g), \quad g \in G, n_\pm \in N_\pm, h \in H.$$

In particular, G^*G is contained in the preimage of the open cell $B_+B_- \subset G$ under the projection

$$p_{G^*}: G\times G \to G = (G\times G)/G_\Delta,\ (g_1,g_2)\mapsto g_1g_2^{-1}$$

The homogeneous space $G \simeq (G\times G)/G_\Delta$ is equipped with a Poisson structure inherited from that of $D(G) = G\times G$. Let us denote this Poisson structure by π_{G_*}. As discussed above, the Poisson structure π_{G_*} has the property that the residual action of G by left translation on $D(G)$ is Poisson, and under the isomorphism $G \simeq (G\times G)/G_\Delta$, this action is given by the conjugation action of G on itself. This Poisson structure 'extends' the Poisson-Lie Poisson structure of the group G^* in the following sense: the restriction of the map p_{G^*} to $G^* \subset D(G)$ is a $2^{\mathrm{rk}(\mathfrak{g})}:1$ local diffeomorphism

$$m: G^* \to G = (G\times G)/G_\Delta, \quad (b_+,b_-)\longmapsto b_+b_-^{-1}$$

with $m_*\pi_{G^*} = \pi_{G_*}$. The symplectic leaves of $m(G^*)\subset G_*$ are given by the connected components of its intersection with the conjugacy classes in G; general symplectic leaves in G_* are components of intersections of the conjugacy classes with orbits for the inherited G^* action on $G = D(G)/G_\Delta$, which reads $(b_+,b_-)\cdot g = b_+gb_-^{-1}$.

On the other hand, to compute the dressing action of G^* on G we need to solve a factorization problem in $D(G) = G\times G$ of the form

$$(b_+g, b_-g) = (\tilde g\tilde b_+, \tilde g\tilde b_-) \in D(G) = G\times G,$$

so that g and its image under the dressing action lie in the same *double Bruhat cell*:

$$\tilde g \in B_+gB_+ \cap B_-gB_-.$$

It follows that the H-orbits of symplectic leaves in G with its standard Poisson-Lie structure are exactly the double Bruhat cells, and are thus labelled by pairs of Weyl group elements

$$G^{u,v} = B_+uB_+\cap B_-vB_-, \quad u,v\in W(G).$$

These examples are explained in detail in the papers

(1) *Poisson geometry of the Grothendieck resolution of a complex semisimple group* by Evens and Lu
(2) *On symplectic leaves and integrable systems in standard complex semisimple Poisson–Lie groups* by Kogan and Zelevinsky.

Exercise 2.33. *Describe the symplectic leaves of G, G_*, G^* completely explicitly when $G = SL(2,\mathbb{C})$.*

Not covered: Belavin-Drinfel'd classification, any infinite dimensional examples.

2.6. **Canonical coordinates on standard Poisson-Lie groups.** We now describe how to find canonical coordinates with respect to the standard Poisson-Lie structure on double Bruhat cells in a complex semisimple Lie group G. There are various flavors of such coordinates, but all come from the basic observation that the 2-dimensional Borel subgroup of each simple root $SL(2)$-triple is a Poisson-Lie subgroup, and it is very easy to put canonical coordinates on the $SL(2)$ Borel: indeed, we've seen that the coordinates g_{11}, g_{12} have brackets

$$\{g_{11},g_{12}\} = \frac{1}{2}g_{11}g_{12},$$

so that in terms of a Darboux pair $\{p,x\}=1$ we can put $g_{11}=e^p, g_{12}=e^{x/2}$.

So given any word $\mathbf{i} = (i_1, \dots, i_l)$ in the alphabet $\{1, \dots, r\} \sqcup \{-1, \dots, -r\}$ on the positive/negative simple roots, we get a Poisson multiplication map

$$H \times B_{i_1} \times \cdots \times B_{i_l} \to G.$$

By definition of the product Poisson structure, the coordinates from different factors just Poisson commute. This multiplication map factors through the quotient by the torus $\prod_j H_j^{k_j}$ where $k_j + 1$ is the number of letters $i_r \in \mathbf{i}$ with $|i_r| = j$, and this torus just scales the canonical coordinates on the factors described above.

Example 2.34. We use this method to put canonical coordinates on standard Poisson-Lie PGL_2. Consider the co-characters

$$H(x) = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$$

and set

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We consider the tori $\mathbb{C}^* \times \mathbb{C}^*$ inside the upper/lower Borels defined by

$$b_+ = H(x)EH(y), \quad b_- = H(w)FH(z).$$

Then we have $\{y, x\} = xy, \quad \{w, z\} = zw$. The fibers of the multiplication map

$$(\mathbb{C}^*)^4 \subset B_+ \times B_- \longrightarrow PGL_2, \quad (b_+, b_-) \mapsto b_+ b_-$$
$$(x, y, w, z) \mapsto H(x)EH(yw)FH(z)$$

are the orbits for the action of $\mathbb{C}^*$-action on $(\mathbb{C}^*)^4$ given by

$$(x, y), (w, z) \longmapsto (x, \lambda y), (\lambda^{-1} w, z), \quad \lambda \in \mathbb{C}^*.$$

Then corresponding to the word $\mathbf{i} = (1, -1)$ in the simple roots, we get a chart on the big double Bruhat cell in PGL_2 with coordinate functions $(x_1 := x,\ x_2 := yw,\ x_3 := z)$ given by

$$(\mathbb{C}^*)^3 \longrightarrow PGL_2^{w_0, w_0}, \quad (x_1, x_2, x_3) \mapsto H(x_1)EH(x_2)FH(x_3),$$

with the Poisson brackets

$$\{x_2, x_1\} = x_1 x_2, \quad \{x_2, x_3\} = x_2 x_3, \quad \{x_1, x_3\} = 0.$$

These coordinates are 'log-canonical', in the sense that the bracket of any two of them is proportional to their product. We can encode the coefficients of proportionality by the following quiver:

$$x_1 \longleftarrow x_2 \longrightarrow x_3$$

On the other hand, we can consider the chart on $PGL_2^{w_0, w_0}$ corresponding to the word $\mathbf{i}' = (-1, 1)$, so that we parametrize the group element as

$$(x', y', w', z') \mapsto H(w')FH(z')H(x')EH(y') = H(x_1')FH(x_2')EH(x_3').$$

where $(x_1' = w', x_2' = x'z', x_3' = y')$. This time their Poisson brackets

$$\{x_1', x_2'\} = x_1' x_2', \quad \{x_3', x_2'\} = x_2' x_3', \qquad \{x_1', x_3'\} = 0$$

are encoded by the quiver

$$x_1' \longrightarrow x_2' \longleftarrow x_3'$$

obtained by *mutating* the one above at the node labelled x_2. The following exercise describes the transition functions between the charts corresponding to words $\mathbf{i}, \mathbf{i}'$.

Exercise 2.35. *Show that the on the intersection of the images of the two tori above, the coordinates x_i' can be expressed as the* cluster mutation *of the ones x_i in direction 2:*

$$\begin{aligned} x_1' &= \frac{x_1 x_2}{1+x_2} \\ x_2' &= x_2^{-1} \\ x_3' &= \frac{x_3 x_2}{1+x_2} \end{aligned}$$

In general, if $\mathbf{i}$ is a shuffle of a reduced word for $(u,v) \in W \times W$, then applying the same construction gives a torus chart $(\mathbb{C}^*)^{l(u)+l(v)+\dim H}$ on the double Bruhat cell $G^{u,v}$. Transition maps between charts corresponding to different reduced words are again given by cluster mutations, and this makes the double Bruhat cell into a *cluster Poisson variety.*

Exercise 2.36. *Compute the quiver describing the cluster Poisson structure in the toric chart on $PGL_3^{w_0,w_0}$ corresponding to the double reduced word*

$$\mathbf{i} = (1,2,1,-2,-1,-2),$$

and generalize to PGL_n.

3. Hopf algebras

3.1. **Algebras.** Recall that an algebra over a field k is a k-vector space A with two additional pieces of structure: a multiplication map $m : A \otimes_k A \to A$, and a unit map $u : k \to A$ such that m is associative

$$\begin{tikzcd} A \otimes_k A \otimes_k A \arrow[r, "m\otimes id_A"] \arrow[d, "id_A \otimes m"'] & A \otimes_k A \arrow[d, "m"] \\ A \otimes_k A \arrow[r, "m"'] & A \end{tikzcd}$$

and the unit map is such that both composites

$$A \simeq k \otimes_k A \xrightarrow{u\otimes id_A} A \otimes A \xrightarrow{m} A$$

$$A \simeq A \otimes_k k \xrightarrow{id_A \otimes u} A \otimes A \xrightarrow{m} A$$

are the identity map id_A. We write $1_A = u(1)$ for the image of $1 \in k$ under the unit map.

Algebras form a category in the obvious way, with morphisms being k-linear maps $\phi : A \to B$ such that

$$\phi \circ m_A = m_B \circ (\phi \otimes \phi), \quad \phi(1_A) = 1_B.$$

If A and B are k-algebras, then we can give the the tensor product $A \otimes_k B$ an algebra structure by setting

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb').$$

Note that the algebra $A \otimes_k B$ comes with algebra maps $A, B \to A \otimes_k B$ whose images commute with each other.

Remark 3.1. In exactly the same way one can define the notion of an algebra object in any monoidal category $\mathcal{C}$, with the structural maps m, u being morphisms in $\mathcal{C}$. If $\mathcal{C}$ has the additional structure of a braiding, then the braided version of the tensor product construction described above gives the category of algebras in $\mathcal{C}$ a monoidal structure.

A (left)-module M over an algebra A is a vector space with an 'action-by-A' map $\alpha : A \otimes M \to M$ satisfying

$$\begin{tikzcd} A \otimes_k A \otimes_k M \arrow[r, "m\otimes id_M"] \arrow[d, "id_A\otimes\alpha"'] & A \otimes_k M \arrow[d, "\alpha"] \\ A \otimes_k M \arrow[r, "\alpha"'] & M \end{tikzcd}$$

and such that the unit in A induces the identity map on M under α. Although modules over an algebra form an abelian category – we can form kernels, cokernels and direct sums – note that there is no natural[2] A-module structure on the tensor product $M_1 \otimes_k M_2$ of two A-modules. Rather, the algebra which acts naturally on $M_1 \otimes_k M_2$ is the tensor product algebra $A \otimes_k A$: the action is given by

$$(a \otimes a') \cdot (m_1 \otimes m_2) = (a \cdot m_1) \otimes (a' \cdot m_2)$$

3.2. **Coalgebras.** The notion of a coalgebra is obtained by reversing all arrows in the definition above – if you like, by working in the opposite (symmetric monoidal) category Vect^{op}. So a coalgebra over k consists of a k-vector space C with a comultiplication $\Delta : C \to C \otimes_k C$ and a counit $\epsilon : C \to k$ such that Δ is *co-associative*: it satifies

$$\begin{tikzcd} C \otimes_k C \otimes_k C & C \otimes_k C \arrow[l, "\Delta\otimes id_C"'] \\ C \otimes_k C \arrow[u, "id_C\otimes\Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

and

$$(id_A \otimes \epsilon) \circ \Delta = id_A = (\epsilon \otimes id_A) \circ \Delta.$$

It is convenient to notate the coproduct as $\Delta(v) = \sum v^i_{(1)} \otimes v^i_{(2)}$, and often to even suppress the symbol $\sum$ and index i and write $\Delta(v) = v_{(1)} \otimes v_{(2)}$ with the summation implied; i.e. the right-hand-side is not generally a pure tensor, but a sum of such. This is called *Sweedler's notation.* The reason it is useful is that by the coassociativity we can write without ambiguity

$$(\Delta \otimes id_C)(\Delta(x)) = (\Delta \otimes id_C)(x_{(1)} \otimes x_{(2)}) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (1 \otimes \Delta)(x_{(1)} \otimes x_{(2)}),$$

et cetera.

A morphism of coalgebras is a linear map $\phi : C \to D$ satisfying

$$\Delta_D \circ \phi = (\phi \otimes \phi) \circ \Delta_C, \quad \epsilon_D \circ \phi = \epsilon_C.$$

[2]More precisely, we should maybe say that there is no such A-module structure which remembers information about the A-module structures of *both* M_1 and M_2. After all, one can always define $\alpha_{M_1 \otimes M_2} = \alpha_{M_1} \otimes \mathrm{id}_{M_2}$, or the other way around.

In Sweedler notation this means,

$$\phi(c)_{(1)} \otimes \phi(c)_{(2)} = \phi(c_{(1)}) \otimes \phi(c_{(2)}),$$

while the counit condition says

$$\epsilon(a_{(1)})a_{(2)} = \epsilon(a_{(2)})a_{(1)} = a.$$

Note that an algebra structure on a finite dimensional vector space A is the same as that of a coalgebra structure on its dual space $A^* = \mathrm{Hom}_k(A, k)$.

A (left) *comodule* V for a coalgebra C is a 'coassociative' map $\Phi : V \to C \otimes V$, (i.e. satisfying $(id_C \otimes \Phi) \circ \Phi = (\Delta \otimes id_V) \circ \Phi)$), and compatible with co-unit in the sense that $(\epsilon \otimes id_V) \circ \Phi = id_V$. We extend the Sweedler's notation to left comodules by writing

$$\Phi(v) = v_{(-1)} \otimes v_{(0)} \in C \otimes V$$

Similarly for a right comodule we write

$$\Phi(v) = v_{(0)} \otimes v_{(1)} \in V \otimes C.$$

Note that a coalgebra C is both a left and right comodule over itself, with $\Phi = \Delta$.

3.3. **Bi- and Hopf algebras.** A *bialgebra* over k is a vector space A which is simultaneously an algebra and a coalgebra, and whose co-algebra structure maps Δ, ϵ are morphisms of algebras. The latter condition is identical to the one that the algebra structure maps be morphisms of coalgebras. The point of a bialgebra is that we can pull back along the coproduct homomorphism $\Delta : A \to A \otimes A$ to give the tensor product $M \otimes N$ of two A-modules an A-module structure. In a similar way, we can form tensor products of A-comodules, using the multiplication of A

A bialgebra A is a *Hopf* algebra if it has an *antipode*: a linear map $S : A \to A$ satisfying

$$S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \epsilon(a)\mathbf{1}_A$$

for all a. If an antipode for A exists, then it is unique: the proof of this goes the same way as how you prove the uniqueness of inverses in a group. It is also an antihomomorphism of algebras: $S(ab) = S(b)S(a)$.

Indeed, the prototype example of a Hopf algebra is the group ring $k[G] = \bigoplus_{g \in G} X_g$ of a finite group. Recall its product is defined by $X_g X_h = X_{gh}$, so that X_e is the unit element. The coalgebra structure is defined with respect to the same basis, and extended linearly:

$$\Delta(X_g) = X_g \otimes X_g, \quad \epsilon(X_g) = 1,$$

and the antipode is $S(X_g) = X_{g^{-1}}$.

The point of the definition of a Hopf algebra is that the we can form tensor products and duals of its modules. Given a general associative algebra A, there is no natural, interesting action of A on the tensor product $V \otimes W$ of two A-modules. Rather, the algebra that naturally acts on such a tensor product is $A \otimes_k A$. But if we have a Hopf algebra, we can pull back along the coproduct $\Delta : A \to A \otimes A$ to make A act on $V \otimes W$. Similarly, we use the product of A to make the tensor product of A-comodules into an A-comodule.

For example, if V, W are two representations of a Lie algebra $\mathfrak{g}$, then $x \in \mathfrak{g}$ acts on $V \otimes W$ via

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

Similarly, the dual an A-module V^* is naturally a module over the opposite algebra A^{op}, i.e. a right-module for A But if A is Hopf, we can use the antipode $S : A \to A^{op}$ to turn V^* into a left A-module. For instance, the action of a group G on the dual V^* of one of its representations is given by

$$(g \cdot \phi)(v) = \phi(g^{-1}v).$$

An element a in a general Hopf algebra is called *grouplike* if a is invertible and $\Delta(a) = a \otimes a$. Note that this implies $S(a) = a^{-1}$ and $\epsilon(a) = 1$. The group like elements in any Hopf algebra form a group. In the case $A = k[G]$, the group of grouplike elements in A recovers G itself. Hence the group can be reconstructed from its group algebra – *but only if we take into account structure of the latter as a Hopf algebra.* For example, the group algebras of the quaternion group Q_8 and the dihedral group D_8 of order 8 are isomorphic as associative algebras, so looking at the algebra structure alone is not enough to reconstruct G.

Another example of a Hopf algebra comes from the enveloping algebra $U\mathfrak{g}$ of a Lie algebra. In the algebraic setup, $U\mathfrak{g}$ is defined to be the quotient of the tensor algebra $T\mathfrak{g}$ by the 2-sided ideal generated by the relations

$$xy - yx - [x, y] = 0, \quad x, y \in \mathfrak{g}.$$

The coalgebra structure is defined on the subspace $\mathfrak{g} \subset U\mathfrak{g}$ by

$$x \in \mathfrak{g} \implies \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0.$$

Since $\Delta(x)\Delta(y) - \Delta(y)\Delta(x) = \Delta([x, y])$, this definition extends uniquely to an algebra map $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$. For example, we have

$$\Delta(x^n) = \sum_k \binom{n}{k} x^k \otimes x^{n-k}.$$

The antipode is determined by $Sx = -x$ for $x \in \mathfrak{g}$. Morally, these formulas work because they come from writing $g = e^{tx}$ in the structure maps for the group algebra and taking first-order terms in t.

The classical Poincare-Birkhoff-Witt (PBW) theorem describes the size of $U\mathfrak{g}$ as a vector space. More precisely, $U\mathfrak{g}$ inherits a filtration (although not a grading) from the tensor algebra, and PBW asserts that the corresponding associated graded ring is isomorphic to the symmetric algebra $S\mathfrak{g}$ of polynomial functions on $\mathfrak{g}^*$. It follows from the following lemma that applies in slightly greater generality, and that we will also apply to quantum groups later on. We take $(X, <)$ to be a totally ordered set satisfying the descending chain condition (e.g. a basis for $\mathfrak{g}$ in the present example) and form the free associative algebra $k\langle X\rangle$ on X, extending the total order lexicographically to all monomials in the alphabet X. We will now build an algebra by quotienting $k\langle X\rangle$ by certain relations, one for each element r of a set R. Each relation must have the form $W_r = f_r$, where W_r is a monomial and $f_r = \sum_{m_s < W_r} c_s m_s$ is a linear combination of monomials less than W_r in the lexicographic order. We then define a *PBW monomial* to be a monomial containing no monomial W_r as a subword, and an element of $k\langle X\rangle$ to be *PBW ordered* if it is a sum of PBW monomials.

Lemma 3.2 (Bergman's Diamond Lemma)**.** *Suppose the set of generators X and relations R as above satisfy*

(1) Whenever monomials $W_r, W_{r'}$ can be factored as $W_r = AB, W_{r'} = BC$, then f_rC and $Af_{r'}$ can be reduced to the same PBW-ordered expression u. Schematically:

$$\begin{tikzcd} & ABC \arrow[dl] \arrow[dr] & \\ f_rC \arrow[dr] & & Af_{r'} \arrow[dl] \\ & u & \end{tikzcd}$$

(2) If $W_{r'} = AW_rC$, then Af_rC and $f_{r'}$ are equivalent to the same PBW-ordered expression.

Then the PBW monomials form a basis for the quotient of $k\langle X\rangle$ by the relations $W_r = f_r$.

In the case of $U\mathfrak{g}$, we have one W given by the 'out of order' monomial yx for each pair of distinct basis vectors ($x < y$), and $f_r = xy - [x, y]$, so condition (2) in the Diamond Lemma is actually void.

Exercise 3.3. *Check condition (1) in the Diamond Lemma for the standard presentation of $U\mathfrak{g}$.*

Note that the PBW theorem implies that as coalgebras we have $U\mathfrak{g} \simeq S\mathfrak{g}$, where the coalgebra structure on $S\mathfrak{g}$ comes from regarding it as the enveloping algebra of the abelian Lie algebra on the vector space $\mathfrak{g}$. The isomorphism is not canonical, and depends on a choice of basis of PBW monomials (i.e. normal ordering) for $U\mathfrak{g}$.

An element a in a general Hopf algebra is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. This implies $\epsilon(x) = 0$ and $Sx = -x$.

The space of primitive elements in A always forms a Lie algebra. In the case of $U\mathfrak{g}$, the primitive elements are exactly $\mathfrak{g}$ – this follows easily using the coalgebra isomorphism $U\mathfrak{g} \simeq S\mathfrak{g}$ we get from PBW.

Both kG and $U\mathfrak{g}$ are generally noncommutative as algebras but are *co-commutative* as coalgebras: we have $\Delta(a) = a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)}$, i.e $\Delta = \sigma \circ \Delta$ where $\sigma : A \otimes A \to A \otimes A, \sigma(a \otimes b) = b \otimes a$ is the flip of tensor factors.

We can dualize these to get commutative, non-co-commutative Hopf algebras. For example let $A = \mathcal{O}(SL(2))$ be the coordinate ring of $SL(2)$ (or more generally of any affine algebraic group G). It is a commutative algebra generated by $\{x_{ij}\}$ for $1 \leq i, j \leq 2$ satisfying the single relation $x_{11}x_{22} - x_{12}x_{21} = 1$. The coproduct comes from the map on coordinate rings induced by the group product $G \times G \to G$,

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$$

and this is not co-commutative. The co-unit is given by the identity element of G, thought of as a k-point.

Example 3.4. Here is an example of a non-commutative, non-commutative Hopf algebra over the ring $\mathbb{Q}(q)$ of rational functions in q, foreshadowed by Exercise 2.31. The Hopf algebra $U_q(sl_2)$ is generated by element E, F and a grouplike element K, with the relations

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = (q - q^{-1})(K - K^{-1}).$$

The coalgebra structure is

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(K) = K \otimes K, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

and $\epsilon(E) = 0 = \epsilon(F), \epsilon(K) = 1$.

Exercise 3.5. *Show that Δ descends from the tensor algebra on E, F, K, K^{-1} to give a well-defined algebra map from $U_q(sl_2)$ to its tensor square.*

It's easy to calculate the value of the antipode on the generators: $S(E) = -K^{-1}E$, $S(F) = -FK$. Note that in this case S is not an involution: instead, S^2 is the inner automorphism $a \mapsto K^{-1}aK$ which has infinite order: for example,

$$S^{2m}(E) = K^{-m}EK^m = q^{-2m}E.$$

Note that $U_q(sl_2)$ contains Hopf subalgebras $U_q(b_+)$ (generated by E, K) and $U_q(b_-)$ (generated by F, K. On the other hand, we do not really have an obvious q-analog of $U(n_-)$. The Hopf algebra $U_q(b_+)$ has a basis $\{K^nE^m\}_{n\in\mathbb{Z}, m\in\mathbb{Z}_{\geq 0}}$. The values of the coproduct on this basis are describe as follows.

Exercise 3.6. *Prove the q-binomial theorem: in the quantum plane algebra generated by x, y with relation $qxy = yx$, show that*

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m}_q x^m y^{n-m}$$

where

$$\binom{n}{m}_q = \frac{[n]_q!}{[n-m]_q![m]_q!}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q,$$

and

$$[n]_q := \frac{1-q^n}{1-q}.$$

Exercise 3.7. *Show that*

$$\Delta(E^n) = \sum_m \binom{n}{m}_{q^2} E^m K^{n-m} \otimes E^{n-m} \tag{3.3.1}$$

$$\Delta(F^n) = \sum_m \binom{n}{m}_{q^2} F^m \otimes F^{n-m} K^{-m}.$$

A general Hopf algebra A has some canonical representations on itself: the *left/right* regular representations, where A acts on itself by left and right multiplication (with the latter twisted by the antipode), and also the *adjoint representation* of A acting by

$$\mathrm{Ad}_a(b) := \sum a_{(1)} b S(a_{(2)}).$$

Exercise 3.8. *Show that the adjoint action makes A into a* module algebra *over itself, i.e. an algebra object in the category $A-$mod: its algebra structure maps are morphisms not just of coalgebras, but of Ad_A-modules.*

When $A = U\mathfrak{g}$, $x \in \mathfrak{g}$ acts by taking commutators: for any $a \in U\mathfrak{g}$,

$$\mathrm{Ad}_x(a) = xa - ax.$$

In particular, the subspace $\mathfrak{g}$ is a $U\mathfrak{g}$ submodule of the adjoint representation, since if $x, y \in \mathfrak{g}$ we have $\mathrm{Ad}_x(y) = xy - yx \in \mathfrak{g}$, so the subspace $\mathfrak{g}$ is preserved by the action of the generators of $U\mathfrak{g}$.

In the example $U_q(sl_2)$ considered above, the adjoint action of E, F, K is given by

$$E \triangleright a = Ea - KaK^{-1}E$$

$$F \triangleright a = FaK - aFK$$

$$K \triangleright a = KaK^{-1}$$

The adjoint action of K gives $U_q(sl_2)$ (and its subalgebras $U_q(b_\pm)$) an algebra grading by $\mathbb{Z}$.

Exercise 3.9. *Compute $\mathrm{Ad}_E(E^m)$, and conclude that the adjoint action of $U_q(sl_2)$ on itself is* not *locally finite: there are elements $c \in U_q(sl_2)$ for which the $U_q(sl_2)$-submodule generated by c under the adjoint action is infinite dimensional over $\mathbb{Q}(q)$. (Later we will characterize all locally ad-finite vectors in $U_q(g)$.)*

Definition 3.10. Suppose that A is Hopf algebra and X is an A-module algebra, i.e. an algebra object in $A - mod$. Then we get an algebra structure on the vector space $X \otimes_k A$, called the *smash product* of X with A:

$$(x \otimes a) \cdot (y \otimes b) := x(a_{(1)} \cdot y) \otimes a_{(2)}b, \qquad x, y \in X,\ a, b \in A.$$

3.4. **Duality.** If A, A' are two Hopf algebras over k, we say that a nondegenerate k-bilinear pairing $\langle \cdot, \cdot \rangle \colon A \otimes A' \to k$ is a *Hopf pairing* if it satisfies the following conditions:

$$\begin{aligned}
\langle a, xy \rangle &= \langle a_{(1)}, x \rangle \langle a_{(2)}, y \rangle \\
\langle ab, x \rangle &= \langle a, x_{(1)} \rangle \langle b, x_{(2)} \rangle \\
\langle 1_A, x \rangle &= \epsilon_{A'}(x) \\
\langle a, 1_{A'} \rangle &= \epsilon_A(a) \\
\langle Sa, x \rangle &= \langle a, S^{-1}x \rangle.
\end{aligned}$$

In this case we say that (A, A') form a *dual pair* of Hopf algebras with respect to $\langle \cdot, \cdot \rangle$.

Example 3.11. If G is an affine algebraic group, there is a Hopf pairing $U\mathfrak{g} \otimes \mathbb{C}[G] \to \mathbb{C}$ given by applying a left-invariant differential operator to a function and evaluating the identity element of G.

If V is a right A'-comodule, then we can give V an A-module structure via

$$a \cdot v := \langle a, v_{(1)} \rangle v_{(0)}.$$

Exercise 3.12. *As practice with Sweedler's notation, check that this does indeed define a left A-module.*

Since $\Phi(v) \in V \otimes A$ is a *finite* sum of pure tensors, we see that the corresponding left A-module is always *locally finite* in the sense of Exercise 3.9. Modules for A that come from A'-comodules in this way are sometimes called *integrable.*

If (A, A') is a dual pair, then A' is a left A-module algebra via the *left coregular representation*:

$$a \cdot x := \langle a, x_{(2)} \rangle x_{(1)}.$$

Example 3.13. The coregular action of $U\mathfrak{g}$ on $\mathcal{O}(G)$ reproduces the canonical action by left-invariant differential operators.

Definition 3.14. The *Heisenberg double* of a dual pair of Hopf algebras $H(A, A')$ is the smash product of the Hopf algebra A' with the module algebra given by its left coregular action on A. Explicitly, the product in $H(A, A')$ is

$$(x \otimes a) \cdot (y \otimes b) := \langle a_{(1)}, y_{(2)} \rangle \; xy_{(1)} \otimes a_{(2)}b, \qquad x, y \in A', \; a, b \in A.$$

Given a dual pair (A, A') of finite dimensional Hopf algebras, we can consider the canonical element

$$\Omega = \sum a_i \otimes x_i \in A \otimes A', \quad \langle a_i, x_j \rangle = \delta_{ij}$$

representing the identify map $A \to A$ (we can also do the same in cases that we have a topology on A, allowing the possibly infinite sum above to make sense as an element of a completion of $A \otimes A'$) Then we can encode the duality between A and A' in equations satisfied by the element Ω: for example, the first axiom in the definition of a Hopf pairing reads

$$(\Delta_A \otimes 1)(\Omega) = \Omega_{13}\Omega_{23}, \tag{3.4.1}$$

where

$$\Omega_{13}\Omega_{23} = \sum_{i,j} a_i \otimes a_j \otimes x_i x_j.$$

and we also have

$$(S_A \otimes 1)(\Omega) = \Omega^{-1}, \qquad (\epsilon_A \otimes 1)(\Omega) = 1_{A'} \qquad \text{et c.}$$

Exercise 3.15. *Show that the canonical element $\Omega \in A \otimes A'$, regarded as an element of the Heisenberg double $H(A, A')$, satisfies the* pentagon identity*:*

$$\Omega_{31}\Omega_{12} = \Omega_{12}\Omega_{23}\Omega_{31}.$$

Example 3.16. Consider the enveloping algebra $A = \mathbb{C}[x]$ of the 1-dimensional Lie algebra. As an algebra it is just the polynomial ring in 1 variable, and the coalgebra structure is determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$. Let A^* be the dual vector space to A, and let $\eta_k \in A^*$ be the functionals defined by $\langle \eta_k, x^j \rangle = \delta_{jk}$. We can put a Hopf structure on their span $A' \subset A^*$ by dualizing the Hopf structure on A. For example, we have

$$\langle \Delta(\eta_1), x^k \otimes x^l \rangle = \langle \eta_1, x^{k+l} \rangle = \delta_{1,k+l},$$

which shows that $\Delta(\eta_1) = \eta_1 \otimes 1 + 1 \otimes \eta_1$ so η_1 is primitive. As for the algebra side, we have

$$\langle \eta_1 \eta_m, x^n \rangle = \sum_k \binom{n}{k} \langle \eta_1 \otimes \eta_m, x^k \otimes x^{n-k} \rangle = n\delta_{n,m+1},$$

which shows that $\eta_1\eta_m = (m+1)\eta_{m+1}$ and hence $\eta_m = \frac{\eta_1^m}{m!}$. So putting $\eta = \eta_1$ we see that $A' = \mathbb{C}[\eta] \simeq A$ as a Hopf algebra. The canonical element in (the completion of) $A \otimes A'$ representing the identity map is given by the exponential function:

$$\Omega = \sum_{m\geq 0} x^m \otimes \eta_m = \sum_{m\geq 0} \frac{x^m \otimes \eta^m}{m!} = \exp\left(x \otimes \eta\right),$$

i.e. the integral kernel for the *Fourier transform.* In this case, the Heisenberg double is the standard Heisenberg algebra generated by a canonical pair η and x, with the relation

$$\eta x - x\eta = 1.$$

The group law for the exponential captures the Hopf pairing property (3.4.1):

$$\begin{aligned}(\Delta \otimes 1)(\Omega) &= \exp\left((x \otimes 1 + 1 \otimes x) \otimes \eta\right) \\ &= \exp\left(x \otimes 1 \otimes \eta\right) \cdot \exp\left(1 \otimes x \otimes \eta\right) \\ &= \Omega_{13}\Omega_{23}\end{aligned}$$

Example 3.17. Let's consider the following variant $U_\hbar(sl_2)$ of the Hopf algebra $U_q(sl_2)$, defined over the ring $\mathbb{C}[[\hbar]]$ of formal power series. We consider $U_\hbar(sl_2)$ as being generated by elements E, F, H, with relations $HE - EH = 2E, HF - FH = -2F$ and

$$EF - FE = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}.$$

In terms of the previous definition, the role of q is played by $e^\hbar$, and that of K by $e^{\hbar H}$, and the generators E, F have been rescaled by dividing by $q - q^{-1}$. With these substitutions the coalgebra structure is defined by the same formulas as in the $U_q(sl_2)$ case: for example,

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(E) = E \otimes 1 + e^{\hbar H} \otimes E.$$

Let's compute the dual of the Hopf subalgebra $U_\hbar(b_+)$ generated by H, E. As an algebra, $U_\hbar(b_+)$ is just the enveloping algebra of the Borel in sl_2, so it has a PBW basis $\{E^n H^m\}_{n,m\geq 0}$. Let's write $\eta_{a,m}$ for the dual basis of functionals $\langle \eta_{a,m}, E^k H^l\rangle = \delta_{a,k}\delta_{m,l}$. Let's also write $\overline{H} := \eta_{0,1}$ and $\overline{E} := \eta_{1,0}$. Then arguing as in the previous example, we compute that

$$\langle \eta_{a,m} \cdot \overline{H}, E^k H^l\rangle = \langle \eta_{a,m} \otimes \overline{H}, \Delta(E^k H^l)\rangle = \langle \eta_{a,m} \otimes \overline{H}, \Delta(E^k)\Delta(H^l)\rangle.$$

Using the formula (3.3.1) for $\Delta(E^k)$ we see that the only term in the latter that contributes is $E^k \otimes 1$, since we evaluate the functional $\overline{H}$ on the second factor. It follows as in the previous example that

$$\eta_{a,m} \cdot \overline{H} = (m+1)\eta_{a,m+1}.$$

In the same way we can compute the effect of multiplication by $\overline{E}$:

$$\begin{aligned}\langle \overline{E} \cdot \eta_{a,m}, E^k H^l\rangle &= \langle \overline{E} \otimes \eta_{a,m}, \Delta(E^k)\Delta(H^l)\rangle \\ &= [k]_{q^2}\langle \eta_{a,m}, E^{k-1}H^l\rangle,\end{aligned}$$

which shows that

$$\overline{E} \cdot \eta_{a,m} = [a+1]_{q^2}\eta_{a+1,m}.$$

Putting these two calculations together, we see that we can algebraically generate all the $\eta_{a.m}$ using $\overline{H}$ and $\overline{E}$: we have

$$\eta_{a,m} = \frac{\overline{E}^a \overline{H}^m}{[a]_{q^2}!m!}$$

Simiarly, the product $\overline{HE}$ is given by

$$\overline{H} \cdot \overline{E} = \eta_{1,1} + \hbar \eta_{1,0} = \overline{E} \cdot \overline{H} + \hbar \overline{E}. \tag{3.4.2}$$

The coproducts of these generators are also easy to compute: just as we did for the UEA of the rank 1 abelian Lie algebra, we find

$$\Delta(\overline{H}) = \overline{H} \otimes 1 + 1 \otimes \overline{H},$$

while

$$\langle \Delta \overline{E}, E^a H^r \otimes E^b H^s \rangle = \langle \overline{E}, E^a H^r E^b H^s \rangle = \delta_{s,0} \left(\delta_{a,1} \delta_{b,0} \delta_{r,0} + 2^r \delta_{a,0} \delta_{b,1} \right),$$

which shows that

$$\Delta(\overline{E}) = \overline{E} \otimes 1 + e^{2\overline{H}} \otimes \overline{E}. \tag{3.4.3}$$

So we see that the product and coproduct on the elements $\overline{E}, \overline{H}$ has a form very similar to that for the generators E, H of $U_q(b_+)$. As we will see later, the right parallel to draw is actually with the *co-opposite* Hopf algebra $U_q(b_-)^{cop}$ to the *opposite* quantum Borel subalgebra generated by F, H.

Indeed, if we extend scalars to $\mathbb{C}((\hbar))$ and introduce

$$F := \frac{\overline{E}}{q - q^{-1}}, \quad H_* := -\frac{2}{\hbar} \overline{H},$$

then equations (3.4.2) and (3.4.3) show that F and H_* generate a Hopf algebra isomorphic to $U_q(b_-)^{cop}$.

This time, the canonical element representing the Hopf pairing is given by

$$\begin{aligned}
\Omega &= \sum_{a,m \geq 0} E^a H^m \otimes \eta_{a,m} \\
&= \sum_{a,m} (-\hbar)^m (q - q^{-1})^a \frac{E^a H^m \otimes F^a (H_*/2)^m}{[a]_{q^2}!m!} \\
&= \exp_{q^2}((q - q^{-1}) E \otimes F) \exp\left(-\hbar \, \frac{H \otimes H_*}{2} \right) \\
&= \Psi_q(E \otimes F) \exp\left(-\hbar \, \frac{H \otimes H_*}{2} \right), \qquad (3.4.4)
\end{aligned}$$

where we have introduced the *q-exponential function*

$$\exp_{q^2}(z) := \sum_{n \geq 0} \frac{z^n}{[n]_{q^2}}, \tag{3.4.5}$$

and the *compact quantum dilogarithm*

$$\Psi_q(z) = \exp_{q^2}((q - q^{-1}) z) = \sum_{n \geq 0} \frac{(-q)^n z^n}{\prod_{k=1}^n (1 - q^{2k})}.$$

Note that

$$\Psi_q(q^2 z) = (1 + qz) \Psi_q(z),$$

so that the formal series Ψ_q has a product expansion

$$\Psi_q(z) = \prod_{n\geq 0} \frac{1}{1+q^{2n+1}z}.$$

Exercise 3.18. *Regarding the series above as an element in $\mathbb{Q}((q))[[z]]$, show that*

$$(1-q^2)\,\log\,\Psi_q(z) = \sum_{m\geq 1} \frac{(-qz)^m}{m[m]_{q^2}!},$$

so that the standard dilogarithm function $\mathrm{Li}_2(z)$ is the 'semiclassical action' $\Psi_q \sim e^{\mathrm{Li}_2(-z)/\hbar}$.

This time, the fact that canonical element Ω represents a Hopf pairing is reflected in the following properties of the quantum dilogarithm Ψ_q:

Exercise 3.19. *Show that for algebra elements u, v satisfying $q^2uv = vu$, we have*

$$\Psi_q(u+v) = \Psi_q(u)\Psi_q(v) \tag{3.4.6}$$

Exercise 3.20. *Derive the pentagon identity for Ψ_q:*

$$\Psi_q(v)\Psi_q(u) = \Psi_q(u)\Psi_q(quv)\Psi_q(v) \tag{3.4.7}$$

3.5. **Monoidal categories.** A monoidal category consists of the following data:

- A category $\mathcal{C}$ with a distinguished object $\mathbf{1}$ (called the monoidal unit) and a functor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$
- A natural isomorphism α (called the *associator*) between the two functors $\mathcal{C}\times\mathcal{C}\times\mathcal{C}\to\mathcal{C}$

$$\alpha_{x,y,z} : (x\otimes y)\otimes z \to x\otimes(y\otimes z),$$

  satisfying the *pentagon identity*.
- Natural isomorphisms $\mathbf{1}\otimes x \to x \simeq x\otimes\mathbf{1}$ (called left/right unitors) such that for all objects x, y we have commutative diagrams

$$\begin{tikzcd}
(x\otimes\mathbf{1})\otimes y \arrow[rr, "\alpha_{x,1,y}"] \arrow[dr] & & x\otimes(\mathbf{1}\otimes y) \arrow[dl] \\
& x\otimes y &
\end{tikzcd}$$

The prototypical example is the category Vect_k of vector spaces, with the monoidal structure given by the tensor product, the monoidal unit $\mathbf{1} = k$ given by the 1-dimensional vector space, the obvious associator $(u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$ and unitors given by the 'act by k' map.

As we've seen, another example of a monoidal category is the category $A - mod$ of modules over a Hopf algebra A, with the same associators as in Vect.

If $\mathcal{C}, \mathcal{D}$ are two monoidal categories, and $F : \mathcal{C}\to\mathcal{D}$ is a functor between them. Then if x, y are two objects of C we can form two objects of $\mathcal{D}$: $F(x\otimes_C y)$ and $F(x)\otimes_D F(y)$. To put it better, the functor F defines for us a pair of functors $\mathcal{C}\times\mathcal{C}\to\mathcal{D}$.

A *monoidal functor* consists of the following data:

- a functor $F : \mathcal{C}\to\mathcal{D}$.

- a natural isomorphism between the two functors described above $\mathcal{C}\times\mathcal{C}\to\mathcal{D}$ induced by F: i.e. natural in x, y isomorphisms

$$J_{x,y} : F(x)\otimes F(y)\to F(x\otimes y).$$

- an isomorphism $\epsilon : F(\mathbf{1}_C)\to\mathbf{1}_{\mathcal{D}}$.

These data must have the property that the two natural maps

$$(F(x)\otimes F(y))\otimes F(z)\to F(x\otimes(y\otimes z))$$

are equal, as are the two maps $\mathbf{1}\otimes F(x)\to F(\mathbf{1}\otimes x)$ and $F(x)\otimes\mathbf{1}\to F(x\otimes\mathbf{1})$.

If we drop the requirement that J and ϵ be isomorphisms, we get the notion of a 'lax' monoidal functor. The data of lax monoidal functor does not generally give us a natural transformation $F(x\otimes y)\to(x\otimes y)$, since we cannot invert the components of J. The notion of an *op-lax monoidal functor* is where we equip F with the extra structure of such a natural transformation $J' : F(x\otimes y)\to(x\otimes y)$ satisfying the obvious analogs of the axioms of a lax monoidal functor.

Exercise 3.21. *Write out the sketchy definitions above properly.*

Let C, D be monoidal categories and $L : C\to D$ a monoidal functor which has a right adjoint $R : D\to C$. The unit of the adjunction $\eta : 1\to RL$ defines a lax monoidal structure on R as follows:

$$\begin{aligned}R(d_1)\otimes R(d_2) &\to RL\,(R(d_1)\otimes R(d_2))\\ &\to R\,(LR(d_1)\otimes LR(d_2))\\ &\to R(d_1)\otimes R(d_2)\end{aligned}$$

where the first map is given by the (tensor square of the) unite $\eta : id_C\to RL$ of the adjunction, the second map is given by the (image under R of the) monoidal structure of L, and the last map comes from (the image under R of) the counit $\eta : LR\to 1_D$.

In the same way, the left adjoint of a monoidal functor naturally has an op-lax monoidal structure.

3.6. **Drinfel'd center and doubles of Hopf algebras.** Recall that the center $Z(R)$ of a ring R is the set of all elements $z\in R$ such that $zr = rz$ for all other elements $r\in R$. Note that by passing to the center, we gain an extra level of commutativity – even if R is non-commutative, its center $Z(R)$ is obviously a commutative ring. Note also that we can identify $Z(R)$ with the algebra $\mathrm{End}_{R\otimes R^{op}}(R)$ of endomorphisms of R commuting with the action of $R\otimes R^{op}$ on R by left and right multiplication. Indeed, such a map $\phi : R\to R$ is determined by its value $z = \phi(1)$ on the identity element, and the element $z = \phi(1)\in R$ must be central in R: indeed, for any $a\in R$

$$\phi(a) = \phi(a\cdot 1) = a\phi(1) = \phi(1\cdot a) = \phi(1)a.$$

One categorical level up, there is a notion of center that be defined for any monoidal category $\mathcal{C}$. The *Drinfel'd center* of $\mathcal{C}$ is a category $\mathcal{Z}(\mathcal{C})$ whose objects are tuples $(M, \{\sigma_N\})$, where σ_N is a system of isomorphisms $\{\sigma_N : M\otimes N\to N\otimes M\}_{N\in ob\mathcal{C}}$ which is natural in N: for any arrow

$f : N \to N'$ in $\mathcal{C}$ we have a commutative square

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\sigma_N} & N \otimes M \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle f\otimes 1} \\ M \otimes N' & \xrightarrow{\sigma_{N'}} & N \otimes M \end{array}$$

and which is compatible with the monoidal structure in the sense that the two ways to half-braid $M \otimes N_1 \otimes N_2$ are identical:

$$\sigma_{N_1 \otimes N_2} = (1 \otimes \sigma_{N_2}) \circ (\sigma_{N_1} \otimes 1)$$

A morphism from (M_1, σ) to (M_2, τ) in $\mathcal{Z}(\mathcal{C})$ is defined to be a morphism $\phi : M_1 \to M_2$ such that the diagram below commutes for each object N in $\mathcal{C}$:

$$\begin{array}{ccc} M_1 \otimes N & \xrightarrow{\sigma_N} & N \otimes M_1 \\ \downarrow{\scriptstyle \phi\otimes 1} & & \downarrow{\scriptstyle 1\otimes \phi} \\ M_2 \otimes N & \xrightarrow{\tau_N} & N \otimes M_2 \end{array}$$

The center $\mathcal{Z}(\mathcal{C})$ also has a monoidal structure defined on objects by

$$(M_1, \sigma) \otimes (M_2, \tau) = (M_1 \otimes M_2, (\sigma \otimes 1)(1 \otimes \tau)).$$

The analog of the fact that the center of any algebra is commutative is the fact that the center $\mathcal{Z}(\mathcal{C})$ of any monoidal category is *braided*: for two objects $(M_1, \sigma) \otimes (M_2, \tau)$, we define their braiding using $\sigma : M_1 \otimes M_2 \to M_2 \otimes M_1$.

Exercise 3.22. *Check that $\sigma : M_1 \otimes M_2 \to M_2 \otimes M_1$ satisifies the condition needed to define a morphism in $\mathcal{Z}(\mathcal{C})$.*

Exercise 3.23. *Show that the center is equivalent to the category* $\mathrm{End}_{\mathcal{C}-mod-\mathcal{C}^{op}}(\mathcal{C})$ *of* $(\mathcal{C}, \mathcal{C}^{op})$*-bimodule endofunctors of* $\mathcal{C}$*.*

When $\mathcal{C}$ is the category of modules over a Hopf algebra A, we can give a more concrete description of its center $\mathcal{Z}(\mathcal{C})$ as follows. Each object (M, σ) of the center clearly determines an A-module, namely M. On the other hand we can also associate a left A-*comodule* to M by using the half-brading $\sigma_M : M \otimes A \to A \otimes M$ for the regular representation A to define a coaction map

$$\delta : M \to A \otimes M, \qquad \delta(m) = \sigma_A \circ (m \otimes 1_A).$$

Since the coproduct $\Delta : A \to A \otimes A$ is a map of A-modules, we get a naturality square

$$\begin{array}{ccc} M \otimes A & \xrightarrow{\sigma_A} & A \otimes M \\ \downarrow{\scriptstyle 1\otimes \Delta} & & \downarrow{\scriptstyle \Delta\otimes 1} \\ M \otimes A \otimes A & \xrightarrow{\sigma_{A\otimes A}} & A \otimes A \otimes M \end{array}$$

which in view of $\sigma_{A \otimes A} = (1 \otimes \sigma_A)(\sigma_A \otimes 1)$ implies the coassiciativity of δ when we feed $m \otimes 1$ into the top left.

So an object (M, σ) of the Drinfeld center gives us an A-module and A-comodule structure on the same vector space M. Moreover, there is a compatibility condition between these two structures coming from the naturality of σ_A with respect to the A-module endomorphisms of the left regular representation, which are nothing but the endomorphisms coming from acting by *right* multiplication with elements of A by (as usual, look at the image of 1!) This tells us that $\sigma_A(m\otimes b) = m_{(-1)}b\otimes m_{(0)}$ and hence

$$\begin{aligned}\delta(am) = \sigma_A(am\otimes 1) &= \sigma_A(a_{(1)}m\otimes a_{(2)}Sa_{(3)})\\ &= (a_{(1)}\otimes a_{(2)})\cdot\sigma_A(m\otimes Sa_{(3)})\\ &= (a_{(1)}\otimes a_{(2)})\cdot m_{-1}Sa_{(3)}\otimes m_{(0)}\\ &= a_{(1)}m_{-1}Sa_{(3)}\otimes a_{(2)}m_{(0)}.\end{aligned}\tag{3.6.1}$$

A pair of module and comodule structures on M satisfying the compatibility condition (3.6.1) is a called a *Yetter-Drinfel'd module*, and the calculation above shows we have a functor from the center $\mathcal{Z}(A-mod)$ to the category of Yetter-Drinfel'd modules for A. In fact, using the fact that the action map $\alpha : A\boxtimes N\to N$ is a map of A modules where we let A act on $A\boxtimes N$ by $a\cdot(b\boxtimes n) := ab\boxtimes n$, it follows from naturality with respect to α that the Yetter-Drinfel'd module structure associated to the object (M,σ) completely determines the value of the half braiding on any other object N in $A - mod$: we have

$$\sigma_N(m\otimes n) = m_{(-1)}n\otimes m_{(0)}.\tag{3.6.2}$$

Exercise 3.24. *Show that if M is a Yetter-Drinfel'd module for a Hopf algebra A, the formula (3.6.2) defines an object (M,σ_N) of the center of $A - mod$. Hence the center is equivalent to the category of Yetter-Drinfel'd modules.*

Now suppose that $\langle,\rangle$ is a Hopf pairing of A with another Hopf algebra A'. (If A is finite dimensional, we can just take $A' = A^*$ with the canonical evaluation pairing.) Then we can turn the A-comodule structure on a Yetter-Drinfel'd module M into an A'_{op} module structure:

$$x\bullet m := \langle x, m_{(-1)}\rangle m_{(0)}$$

Let us further turn this into a left module for A' using the antipode:

$$x\triangleright m = \langle Sx, m_{(-1)}\rangle m_{(0)}$$

The Yetter-Drinfel'd condition (3.6.1) then becomes a condition relating the left actions of A and A' on M:

$$\begin{aligned}x\triangleright(a\cdot m) &= \langle Sx, a_{(1)}m_{(-1)}Sa_{(3)}\rangle a_{(2)}m_{(0)}\\ &= \langle Sx_{(3)}, a_{(1)}\rangle\langle Sx_{(3)}, Sa_{(3)}\rangle\langle Sx_{(2)}, m_{(-1)}\rangle a_{(2)}m_{(0)}\\ &= \left(\langle x_{(1)}, a_{(3)}\rangle\langle x_{(1)}, Sa_{(3)}\rangle a_{(2)}x_{(2)}\right)\triangleright m.\end{aligned}$$

Hence a Yetter-Drinfel'd module gives rise to a module over the *Drinfel'd double algebra* $D(A)$, which is the vector space $A\otimes_k A^*$ equipped with the product

$$(a\otimes x)(b\otimes y) := \langle x_{(1)}, b_{(3)}\rangle\langle x_{(3)}, Sb_{(1)}\rangle ab_{(2)}\otimes x_{(2)}y.$$

Derive R-matrix, define quasitriangular Hopf algebra, Yang-Baxter equation. Example of $U_q(sl_2)$ and its R-matrix. Dual double, quantum Leningrad formulas.

3.7. Reconstruction.

4. Deformations and quantized enveloping algebras

5. The quantum group $U_q(sl_2)$

6. Integral forms of $U_q(sl_2)$ and roots of unity

7. Quantum Weyl group and PBW bases for $U_q(g)$

8. R-matrix and quasitriangular structure of $U_q(g)$

9. Center of the quantum group

10. Tensor categories

11. Invariants of tangles and 3-manifolds

12. Crystals and canonical bases

13. Overflow

Quantum inverse scattering method (algebraic Bethe ansatz), quantum loop groups and quantum affine algebras, quantization of character varieties, KZ equation and Drinfel'd associators, quantum differential operators/flag varieties and localization