

QUANTUM GROUPS LECTURE NOTES

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1. INTRODUCTION AND PLAN

This course is an introduction to *quantum groups*. I want to start out by giving a rough sense of what kind of objects quantum groups are, and of the various mathematical/physical contexts in which they appear.

Let's begin with the meaning of the word 'quantum'. In Heisenberg's formulation, a quantum mechanical system consists of the following ingredients:

- A family of $*$ -algebras A_{\hbar} (the 'algebra of observables') depending on a real parameter \hbar (Planck's constant), whose fiber at $\hbar = 0$ is commutative.
- A Hilbert space representation \mathcal{H} (the 'space of states') of the algebra A_{\hbar} ;
- A special self-adjoint element $H \in A_{\hbar}$, called the *Hamiltonian*¹.

¹Here we only consider the special case of 'time-independent' Hamiltonians

The canonical example is the quantum mechanics of a single particle in 1-dimensional space X with Euclidean coordinate x : we take $\mathcal{H} = L^2(X)$, and $H = \frac{1}{2}p^2 + V(x)$ where

$$p = -i\hbar \frac{d}{dx}$$

is the momentum operator, and $V(x)$ is a real valued function on X called the ‘potential’. In this case the algebra of observables is generated (at least for a polynomial potential) by p and x , the operator of multiplication by the function $x : X \rightarrow \mathbb{R}$, and they satisfy the canonical commutation relation

$$[p, x] = -i\hbar.$$

The way that this setup produces real numbers we can measure is through taking *matrix elements* of the algebra A_\hbar in its representation \mathcal{H} . Given a quantum state $\psi \in \mathcal{H}$ and a self-adjoint observable $\mathcal{O} \in A_\hbar$, the expectation value of the observable \mathcal{O} is the real number

$$\langle \mathcal{O} \rangle = \langle \psi, \mathcal{O}\psi \rangle.$$

In the Heisenberg picture of time evolution, we think of the states remaining constant while the observables evolve in time according to

$$-i\hbar \frac{d\mathcal{O}}{dt} = [H, \mathcal{O}],$$

so that the evolution of observables can be written in terms of the unitary *time evolution operator* $U(t) = \exp(-iHt/\hbar)$ as

$$\mathcal{O}(t) = U(t)^{-1} \mathcal{O} U(t).$$

Hence the expectation value of \mathcal{O} evolves in time by

$$\langle \mathcal{O} \rangle_t = \langle \psi, U(t)^{-1} \mathcal{O} U(t) \psi \rangle = \langle \psi(t), \mathcal{O}_0 \psi(t) \rangle,$$

where $\psi(t) = U(t)\psi$ satisfies the Schrödinger equation

$$i\hbar \frac{d\psi}{dt} = H\psi(t).$$

In real life the Planck constant \hbar is very small, and what we actually observe most of the time on the human scale is the *semiclassical limit* of the quantum system, where $\hbar \rightarrow 0$ and the algebra of observables becomes commutative. In the example above of the non-commutative algebra generated by p, x describing the QM of a 1-dimensional particle, we can think of its specialization at $\hbar = 0$ as the commutative algebra of functions on the classical mechanical *phase space* $T^*\mathbb{R}$ for the particle.

Although all commutators in A_\hbar of course vanish at $\hbar = 0$, we can still remember their leading order term in \hbar :

$$\mathcal{O}_1 \mathcal{O}_2 - \mathcal{O}_2 \mathcal{O}_1 =: -i\hbar \{ \mathcal{O}_1, \mathcal{O}_2 \} + O(\hbar^2)$$

This equips the ring of functions on phase space (the ‘classical observables’) with the extra structure of a *Poisson bracket*. In the example above, the Poisson bracket for phase space is uniquely determined by

$$\{p, x\} = 1$$

together with the obvious skew-symmetry and bi-derivation properties inherited from those of the commutator: for instance, we have $\{f_1 f_2, f_3\} = f_1 \{f_2, f_3\} + \{f_1, f_3\} f_2$. The time evolution of a classical observable f is then given by

$$\frac{df}{dt} = \{H, f\}.$$

Applying this with $f = p$, we recover the classical Newton second law for the trajectory of the particle in the form $\dot{p} = -V'(x)$.

Next we need to think about what we mean by a ‘group’. From the point of view of algebraic geometry, the way we work with an (affine, algebraic) group G is via its coordinate ring $\mathbb{C}[G]$ of regular functions: the matrix coefficients of G in its finite dimensional representations. The commutative ring $\mathbb{C}[G]$ also has some extra structure, this time coming from the product operation $m : G \times G \rightarrow G$, which defines a ring map

$$m^* : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[G].$$

For example, if we consider the group B of upper triangular invertible 2×2 matrices then

$$\mathbb{C}[B] = \mathbb{C}[x_{11}^{\pm 1}, x_{22}^{\pm 1}, x_{12}]$$

with

$$m^*(x_{11}) = x_{11} \otimes x_{11}, \quad m^*(x_{12}) = x_{11} \otimes x_{12} + x_{12} \otimes x_{22}, \quad m^*(x_{22}) = x_{22} \otimes x_{22}.$$

The associativity of the group law translates into what’s known as *co-associativity* for the *coproduct* m^* . We also get a special map $\epsilon : \mathbb{C}[G] \rightarrow \mathbb{C}$ given by evaluation at the identity element in G (called the *counit*), and a map $S : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$ given by inversion (called the *antipode*). These structures satisfy a bunch of obvious compatibilities (e.g the one saying that the coproduct is a map of rings) and make the coordinate ring $\mathbb{C}[G]$ into what is called a *Hopf algebra*.

Unsurprisingly, one can arrive at the notion of quantum groups by thinking about combining these two concepts: the kind of objects we’ll be interested in are \hbar -families of Hopf algebras A_\hbar such that the fiber at $\hbar = 0$ is the commutative algebra of functions on a group.

Example 1.1. (A quantum analog of $\mathbb{C}[B]$) There is family of Hopf algebra structures on the same underlying vector space $\mathbb{C}[B] = \mathbb{C}[x_{11}^{\pm 1}, x_{22}^{\pm 1}, x_{12}]$, whose co-unit and coproduct are identical to those of $\mathbb{C}[B]$, but whose product is deformed to

$$x_{11}x_{12} = qx_{12}x_{11}, \quad x_{22}x_{12} = q^{-1}x_{22}x_{12}, \quad x_{11}x_{22} = x_{22}x_{11},$$

where we’ve re-parametrized our parameter \hbar as

$$q = e^\hbar.$$

As we’ve seen, when we take the classical limit of such a ‘quantum group’ we get a Poisson bracket on the coordinate ring $\mathbb{C}[G]$ of matrix coefficients. The semiclassical shadow of the compatibility between products and coproducts built into the definition of a Hopf algebra is the fact that the multiplication map $G \times G \rightarrow G$ preserves Poisson brackets, and thus makes G into what’s called a *Poisson-Lie group*.

We will be mostly interested in this when that G is a simple Lie group, for example $G = SL_N$. In this case, one can draw a parallel with a still more algebraic way to think about G – the group can be *reconstructed* from its rigid symmetric monoidal category of finite dimensional representations,

together with a symmetric monoidal fiber functor. The quantum analogs of simple Lie groups we'll discuss can be thought of as deformations of this structure: they are equivalent to the data of a rigid *braided* monoidal category, with a braided monoidal fiber functor. The truly novel feature here is that the deformed categories are braided, but not symmetric monoidal: in other words, the double braiding isomorphisms $V \otimes W \rightarrow W \otimes V \rightarrow V \otimes W$ are not just the identity map, like they would be in the category of representations of a group. These deformed categories depend on the parameter $q \in \mathbb{C}^*$, and exhibit many interesting new phenomena when q is a root of unity.

The first goal for the course is to construct these braided tensor categories, and give a practical user's guide for how to work with them. Secondly, I hope to give a flavor of at least some of the applications of the theory to problems in other parts of math and physics:

- Applications to quantum integrable systems and lattice models in statistical mechanics, i.e. the 'quantum inverse scattering method'
- Applications to invariants of links and 3-manifolds, i.e. the Jones polynomial and Witten-Reshetikhin-Turaev invariants.
- Applications to representation theory: construction of crystal/canonical bases, applications to representation theory of algebraic groups in positive characteristic, ...

These applications gave an important impetus for the discovery of quantum groups in the first place. We definitely won't have time to cover all of them, but I hope we'll get to talk about a few in a reasonable amount of detail.

2. POISSON-LIE GROUPS

2.1. Poisson basics. A *Poisson structure* on a manifold M is a Lie bracket

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$$

which is also a bi-derivation on the ring of smooth functions on M : for functions f_1, f_2, f_3 we have

$$\{f_1 f_2, f_3\} = f_1 \{f_2, f_3\} + \{f_1, f_3\} f_2.$$

The Poisson bracket determines, and is determined by, a bi-vector field $\pi \in \wedge^2 TM$ (the Poisson bivector) such that

$$\{f_1, f_2\} = \langle \pi, df_1 \otimes df_2 \rangle.$$

A Poisson map $\phi : M \rightarrow N$ between Poisson manifolds is one satisfying

$$\{f_1 \circ \phi, f_2 \circ \phi\}_M = \{f_1, f_2\}_N \circ \phi$$

or equivalently $\phi_*(\pi_M) = \pi_N$. Given two Poisson manifolds M, N , we can equip $M \times N$ with a Poisson structure $\pi_{M \times N} = \pi_M + \pi_N$. The projections p_M, p_N to the factors are then Poisson maps, so that $\{p_M^* f, p_N^* h\}_{M \times N} = 0$.

The data of a Poisson bracket allows us to attach to any smooth function H on M a vector field X_H called the *Hamiltonian vector field* of H : this is just the derivation of $C^\infty(M)$ given by $X_H \cdot f = \{H, f\}$. Note that the Jacobi identity says that the map $C^\infty(M) \rightarrow \text{Vect}(M)$ is a map of Lie algebras.

More geometrically, contraction with the Poisson tensor defines a bundle map $I : T^*M \rightarrow TM$, and the Hamiltonian vector X_H is the image of the differential of H under this map:

$$X_H = I(dH) = \langle \pi, dH \otimes - \rangle.$$

If $I : T^*M \rightarrow TM$ is non-degenerate, then we can apply the tensor square of its inverse to the Poisson bivector itself to get a non-degenerate 2-form ω on M such that

$$\omega(X_{f_1}, X_{f_2}) = \{f_1, f_2\}.$$

The Jacobi identity for the bracket implies that $d\omega = 0$: indeed, if we choose local coordinates x_i then the Hamiltonian vector fields X_{x_i} are locally linearly independent (but possibly non-commuting!) by the non-degeneracy of I , and we have

$$\begin{aligned} d\omega(X_{x_i}, X_{x_j}, X_{x_k}) &= X_{x_i}\omega(X_{x_j}, X_{x_k}) - X_{x_j}\omega(X_{x_i}, X_{x_k}) + X_{x_k}\omega(X_{x_i}, X_{x_j}) \\ &\quad - \omega([X_{x_i}, X_{x_j}], X_{x_k}) + \omega([X_{x_i}, X_{x_k}], X_{x_j}) - \omega([X_{x_j}, X_{x_k}], X_{x_i}) \\ &= \{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\} \\ &\quad - \omega(X_{\{x_i, x_j\}}, X_{x_k}) + \omega(X_{\{x_i, x_k\}}, X_{x_j}) - \omega(X_{\{x_j, x_k\}}, X_{x_i}) \\ &= 2(\{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\}) = 0. \end{aligned}$$

This calculation shows that a nondegenerate Poisson structure w is the same data as that of a symplectic form ω on M . In the symplectic case, the Hamiltonian vector field X_H is characterized by $dH = \langle \omega, X_H \otimes - \rangle$. When $M = T^*\mathbb{R}^n$ with canonical coordinates (p_i, x_i) and symplectic form $\omega = \sum_i dp_i \wedge dx_i$, we recover the Poisson bracket from classical mechanics:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} \right)$$

A interesting source of generally non-symplectic Poisson structures is the *Kirillov-Kostant* Poisson structure on the dual space to a Lie algebra. Indeed, we can regard \mathfrak{g} as the space of linear functions on \mathfrak{g}^* , and then extend the bracket $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$ from linear functions to all functions via the Leibniz rule. If we choose a basis in \mathfrak{g} with structure constants $[x_i, x_j] = \sum c_{ij}^k x_k$, the Poisson bivector is given explicitly by

$$(2.1.1) \quad \pi(\xi) = \sum_{i,j,k} c_{ij}^k x_k(\xi) \partial_i \wedge \partial_j,$$

where we identify all tangent spaces with \mathfrak{g}^* itself. Note that the Poisson-bivector respects the linear structure on the vector space \mathfrak{g}^* : we have $\pi(\xi + \eta) = \pi(\xi) + \pi(\eta)$.

If $x \in \mathfrak{g}$ is a linear function on \mathfrak{g}^* , its Hamiltonian vector field is the derivation which sends on another linear function y to $[x, y]$: $(X_x \cdot y)(\xi) = \langle \xi, [x, y] \rangle =: \text{ad}_x^*(y)(\xi)$. In particular, the flow of its Hamiltonian vector field is given by the coadjoint action: $\Phi_{X_x}^t(\xi) = \text{Ad}_{e^{tx}}^*(\xi)$.

We say that two points x, y in a Poisson manifold M are in the same *symplectic leaf* if they are connected by a piecewise smooth curve, each piece of which is an integral curve of a Hamiltonian vector field. More geometrically, the map $I : T^*M \rightarrow TM$ defines an *involutive distribution* on M : a family of subspaces of each tangent space which are locally spanned by a Lie subalgebra of vector fields (the involutivity follows from the fact that the map sending a function to its Hamiltonian vector field is a map of Lie algebras). Weinstein showed that this distribution integrates

to define a foliation of M by immersed submanifolds (if the Poisson structure has constant rank, this follows from the Frobenius theorem, but the general case requires a Darboux-theorem for Poisson manifolds.) Setting $\omega_S(X_f, X_g) = \{f, g\}$ defines a symplectic form on each leaf S of this foliation, and for this reason its leaves are called the *symplectic leaves* of the Poisson manifold M .

Example 2.1. The symplectic leaves in the Poisson manifold \mathfrak{g}^* are the orbits for the coadjoint action of a connected Lie group G with Lie algebra \mathfrak{g} : our calculation of Hamiltonian vector fields above shows that the symplectic leaf through ξ is contained in the coadjoint orbit through ξ , and since G is generated by the image of \exp we deduce that any two points in the same coadjoint orbit lie in the same symplectic leaf.

Exercise 2.2. Work out the symplectic leaves in \mathfrak{su}_2^* and in the 3-dimensional Heisenberg Lie algebra with respect to the KKS Poisson structure.

2.2. Poisson-Lie groups and Lie bialgebras. A Poisson-Lie group G is a Lie group equipped with a Poisson structure such that the multiplication $G \times G \rightarrow G$ is a Poisson map. Explicitly, this means that

$$\{f_1, f_2\}_G(gg') = \{f_1 \circ R_{g'}, f_2 \circ R_{g'}\}_G + \{f_1 \circ L_g, f_2 \circ L_g\}_G,$$

or at the level of bivectors,

$$\pi_{gg'} = (L_g)_*^{\otimes 2}(\pi_{g'}) + (R_{g'})_*^{\otimes 2}(\pi_g).$$

Plugging in $g = g' = 1$, we see that the Poisson bivector for a Poisson-Lie group must vanish at the identity element – in particular, a Poisson-Lie group cannot be symplectic.

Exercise 2.3. Show that the inverse map $\iota : G \rightarrow G$, $g \mapsto g^{-1}$ in a Poisson Lie group is anti-Poisson: $\iota_*\pi_G = -\pi_G$.

Note that the condition that $G \times G \rightarrow G$ be Poisson does not imply that the translation maps $L_g, R_g : G \rightarrow G$ are Poisson. Indeed, R_g and L_g will be Poisson precisely when $\pi_g = 0$.

Poisson-Lie groups form a category, where morphisms are Poisson group homomorphisms.

Example 2.4. Any Lie group G can be made into a Poisson-Lie group by giving it the zero Poisson structure.

Example 2.5. An example of a Poisson-Lie group with nontrivial Poisson structure is given by the additive group $(\mathfrak{g}^*, +)$ with the KKS Poisson structure. We'll soon see that there is a sense in which this example is actually 'dual' to the one above.

Exercise 2.6. Show that if a Lie subgroup H is a Poisson submanifold of a Poisson-Lie group G , then the homogeneous space G/H inherits a Poisson structure such that the projection map from G is Poisson. Is there a weaker condition on the subgroup H under which the same is true of G/H ?

We'd like to give a local description of the Poisson structure on a Poisson-Lie group in a neighborhood of 1, analogous to the way that the Lie algebra locally describes its multiplication law.

We can use right translation to trivialize the tangent bundle TG , and thereby think about a Poisson structure on a Lie group as a map $\pi^R : G \rightarrow \mathfrak{g} \otimes \mathfrak{g} = T_1G \otimes T_1G$. The Poisson-Lie condition says that this map satisfies

$$\pi^R(gg') = (\text{Ad}_g)^{\otimes 2}(\pi^R(g')) + \pi^R(g),$$

and thus π^R defines a 1-cocycle on G valued in the tensor square $\mathfrak{g} \otimes \mathfrak{g}$ of the adjoint representation. Its derivative at the identity gives a Lie algebra 1-cocycle $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$:

$$\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x).$$

The bracket can be computed as

$$\{f_1, f_2\}(g) = \langle \pi^R(g), R_g^*(df_1 \otimes df_2) \rangle.$$

Taking the derivative at the identity in the direction $x \in \mathfrak{g}$ and recalling $\pi^R(1) = 0$, we see that

$$d\{f_1, f_2\}(x) = \langle \delta(x), \xi_1 \otimes \xi_2 \rangle = \langle x, \delta^*(\xi_1 \otimes \xi_2) \rangle,$$

where $(df_i)_1 = \xi_i \in \mathfrak{g}^*$. Hence the map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ can be computed by choosing two functions f_i whose derivative at 1 is ξ_i , and taking the derivative of their bracket $\{f_1, f_2\}$ at 1, so we conclude that δ^* defines a Lie bracket on \mathfrak{g}^* .

A more algebraic way to think of this is as that, since $\pi^R(1) = 0$, the Poisson bracket maps the ideal I of (germs of) functions vanishing at $1 \in G$ to itself, thereby equipping I with a Lie algebra structure with respect to which I^2 is a Lie ideal. Hence we get a Lie algebra structure on the cotangent space I/I^2 to G at 1, given by $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$.

Summarizing, we see that the Lie algebra of a Poisson Lie group is a “Lie bialgebra”, in the following sense:

Definition 2.7. A *Lie bialgebra* \mathfrak{g} is a Lie algebra equipped with a 1-cocycle $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ valued in the tensor square of the adjoint representation, such that the dual map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie bracket.

Example 2.8. Recall the Poisson-Lie group $(\mathfrak{g}^*, +)$, with the Kirillov-Kostant Poisson structure. From the linear form (2.1.1) of its Poisson tensor, we see that the co-bracket δ in this case is the dual of the Lie bracket in \mathfrak{g} .

Lie bialgebras form a category where morphisms are Lie algebra morphisms ϕ satisfying $\delta \circ \phi = \phi^{\otimes 2} \circ \delta$. We have the following analog of Lie’s third theorem for Poisson-Lie groups:

Theorem 2.9. *Let G be a connected and simply-connected Lie group with Lie algebra \mathfrak{g} . Then every Lie bialgebra structure on \mathfrak{g} integrates to a unique Poisson-Lie structure on G . If $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a morphism of Lie bialgebras, then there is a unique morphism of Poisson-Lie groups $\Phi : G \rightarrow H$ with $d\Phi_e = \phi$.*

The key fact here is that for G simply connected, Lie algebra 1-cocycles valued in $\mathfrak{g} \otimes \mathfrak{g}$ can be uniquely integrated to ones for G . (This boils down to the existence and uniqueness for a system of ODE on G , see the exercise below) Hence we obtain from the Lie bialgebra structure a unique multiplicative bivector field on G . The left-hand-side of the Jacobi identity for the corresponding would-be Poisson bracket then defines a multiplicative tri-vector field with vanishing intrinsic derivative at the identity (by the Jacobi identity for δ^*), and hence is identically zero. This shows every Lie bialgebra can indeed be integrated. Moreover, Lie bialgebra maps can be lifted to Lie group ones by the usual Lie III theorem, and to see that the lift preserves the Poisson structure note that both $\Phi_*^{\otimes 2} \pi_G(g)$ and $\pi_H(\Phi(g))$ both define 1-cocycles for G valued in $\mathfrak{h} \otimes \mathfrak{h}$, whose associated Lie algebra cocycles agree. Hence by the uniqueness of lifting, the two coincide and Φ is Poisson.

Exercise 2.10. For $x \in \mathfrak{g}$, consider the differential operator ∇_x on sections of $\wedge^2 TG$ given by

$$(\nabla_x \cdot \phi)(g) = \frac{d}{dt} \phi(e^{tx}g) \Big|_{t=0}.$$

If we fix a Lie cobracket δ , show that a Poisson bivector $\pi^R : G \rightarrow \wedge^2 TG$ integrating δ is a solution the inhomogeneous system of linear differential equations

$$\nabla_x \phi = \delta(x) + \text{ad}_x^{\otimes 2} \cdot \phi, \quad x \in \mathfrak{g}$$

and that ∇ satisfies the flatness/consistency condition:

$$[\nabla_x, \nabla_y] = \nabla_{[x, y]}.$$

2.3. Manin triples and Poisson-Lie duality. There is another way to think about Lie bialgebras, which reveals a symmetry between the roles played by \mathfrak{g} and \mathfrak{g}^* in the definition. To formulate it, suppose that we have Lie algebra structures on both a finite dimensional vector space \mathfrak{g} and its dual \mathfrak{g}^* . Then we can use the duality pairing to define a symmetric bilinear form (\cdot, \cdot) on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that \mathfrak{g} and \mathfrak{g}^* are both isotropic subspaces. We will reformulate the notion that the cobracket $\delta^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ is a 1-cocycle in terms of the existence of a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that both $\mathfrak{g}, \mathfrak{g}^*$ are Lie subalgebras, and this form is $\mathfrak{g} \oplus \mathfrak{g}^*$ -invariant.

Lemma 2.11. *In the setting above, there exists a (unique) Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that the nondegenerate symmetric bilinear form (\cdot, \cdot) is nondegenerate if and only if δ^* is a 1-cocycle, i.e. $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra.*

Proof. The point is that if a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ exists, then condition that this form be ad-invariant then uniquely determines the Lie bracket between \mathfrak{g} and \mathfrak{g}^* : indeed choosing dual bases x_i, ξ_i in each, we can compute the \mathfrak{g}^* -component of $[\eta, y]$ as

$$\langle [\eta, y], x_i \rangle \xi_i = \sum \langle \eta, [y, x_i] \rangle \xi_i,$$

and the \mathfrak{g} -component is determined similarly. For this to actually define a Lie bracket, the Jacobi identity needs to hold, and one checks that this is equivalent to the cocycle condition on δ . \square

If we choose dual bases a_i, x_i in $\mathfrak{g}, \mathfrak{g}^*$ with respect to the form with structural constants

$$[a_i, a_j] = \sum_k a_{ij}^k a_k, \quad [x^i, x^j] = \sum_k \gamma_k^{ij} x_k$$

then the cross-brackets take the form

$$(2.3.1) \quad [a_j, x^k] = \sum_i c_{ij}^k x^i + \gamma_j^{ki} a_i.$$

A *Manin triple* is a Lie algebra \mathfrak{d} equipped with an nondegenerate, invariant symmetric bilinear form (\cdot, \cdot) and a pair of Lie subalgebras $\mathfrak{d}_+, \mathfrak{d}_-$ such that we have a vector space decomposition $\mathfrak{d} = \mathfrak{d}_+ \oplus \mathfrak{d}_-$, where both subspaces $\mathfrak{d}_\pm \subset \mathfrak{d}$ are isotropic with respect to the bilinear form. The discussion above shows that finite dimensional Lie bialgebras are in 1-1 correspondence with finite dimensional Manin triples.

With this under our belt, we can finally construct some more interesting examples of Poisson-Lie groups.

Example 2.12 (The standard Poisson-Lie structure on a complex semisimple Lie group). Let \mathfrak{g} be a complex semisimple Lie algebra, and choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ with Chevalley generators E_i, F_i, H_i corresponding to the nodes of the Dynkin diagram of \mathfrak{g} . We also write $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ for the standard Borel subalgebras with respect to this decomposition. For example, take $\mathfrak{g} = \mathfrak{sl}_2$ with standard generators

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

and $\mathfrak{h} = \langle H \rangle, \mathfrak{n}_+ = \langle E \rangle, \mathfrak{n}_- = \langle F \rangle$. Then there is a unique Lie bialgebra structure on \mathfrak{g} with cobracket determined by the following values on these generators:

$$\delta(H_i) = 0, \quad \delta(E_i) = \frac{d_i}{2} H_i \wedge E_i, \quad \delta(F_i) = \frac{d_i}{2} H_i \wedge F_i,$$

where $d_i a_{ij}$ is the symmetrized Cartan matrix of \mathfrak{g} . This Lie bialgebra comes from the following Manin triple.

Let (\cdot, \cdot) be the invariant nondegenerate bilinear form on \mathfrak{g} normalized so that

$$(E_i, F_i) = \frac{1}{d_i}, \quad (H_i, H_i) = \frac{2}{d_i}$$

where $d_i a_{ij}$ is the symmetrized Cartan matrix. Now consider the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ equipped with the pairing

$$\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1, x_2) - (y_1, y_2).$$

Now take \mathfrak{d}_+ to be the diagonally embedded copy of \mathfrak{g} , and $\mathfrak{d}_- \subset \mathfrak{b}_+ \oplus \mathfrak{b}_-$ to be the subset of pairs (x, y) such that the \mathfrak{h} -component of $x + y$ is zero. For example, when $\mathfrak{g} = \mathfrak{sl}_2$, \mathfrak{d}_- is the 3-dimensional solvable Lie algebra generated by

$$(E, 0), (H, -H), (0, F).$$

Then since the \mathfrak{n}_\pm are isotropic with respect to the Killing form, and pair to zero with \mathfrak{h} , this indeed defines a Manin triple and we obtain Lie bialgebra structures on \mathfrak{g} and $\mathfrak{g}^* \simeq \mathfrak{d}_-$. The formulas mentioned above for the cobracket now follow easily by dualizing the bracket on \mathfrak{p}_- using the form: for example, for any $(x_1, y_1), (x_2, y_2)$ in \mathfrak{d}_- we have

$$\langle \delta(E_i), (x_1, y_1) \otimes (x_2, y_2) \rangle = \langle (E_i, E_i), [(x_1, y_1), (x_2, y_2)] \rangle$$

Using the fact that the root space basis satisfies $(E_\alpha, F_\beta) = 0$ for $\alpha \neq \beta$ and $(E_i, F_i) = d_i$, we see that indeed $\delta(E_i) = \frac{d_i}{2} H_i \wedge E_i$.

Remark 2.13. From the explicit form of δ , it's clear that the standard Poisson structure on a simple Lie algebra \mathfrak{g} has the following properties:

- the subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$ are sub-Lie bialgebras of \mathfrak{g}_{std} ;
- the simple root \mathfrak{sl}_2 -triples $\mathfrak{sl}_2^{(\alpha_i)} \subset \mathfrak{g}$ are sub-Lie bialgebras of \mathfrak{g}_{std} ;
- we have $\delta|_{\mathfrak{h}} = 0$.

The last property implies that the left and right translation maps $L_h, R_h : G \rightarrow G$ by elements of H are Poisson maps, and hence H acts on the set of symplectic leaves in G_{std} .

Exercise 2.14 (Compact Lie groups). Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a complex simple Lie algebra, and recall its Iwasawa decomposition: as a real vector space, we have

$$\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{n}),$$

where

$$\mathfrak{k} = \text{span}\{E_{\alpha} - F_{\alpha}, \mathbf{i}(E_{\alpha} + F_{\alpha}), \mathbf{i}H_{\alpha}\}$$

is the Lie algebra of the compact real form of \mathfrak{g} . (For example, when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ we have $\mathfrak{k} = \mathfrak{su}_n$ and \mathfrak{b} is the Lie algebra of traceless upper triangular matrices with real entries on the diagonal.) Show that the imaginary part of the Killing form makes \mathfrak{g} into a Manin triple, and hence makes the compact Lie group K into a Poisson-Lie group.

From the reformulation via Manin triples, we also deduce the existence of *duality functor* $\mathfrak{g} \mapsto \mathfrak{g}^*$ on the category of Lie bialgebras – and hence by Theorem 2.9 that of a similar functor $G \mapsto G^*$ on the level of Poisson-Lie groups.

Example 2.15. Recall the Poisson-Lie group $(\mathfrak{g}^*, +)$, whose tangent Lie bialgebra is \mathfrak{g}^* , with trivial Lie bracket on \mathfrak{g}^* , and cobracket δ induced by the Lie bracket on \mathfrak{g} . Swapping the roles of $\mathfrak{g}, \mathfrak{g}^*$, we see that the dual Lie bialgebra \mathfrak{g} has Lie bracket given by that of \mathfrak{g} , and trivial cobracket. Hence the corresponding Poisson-Lie group is the connected, simply connected Lie group G integrating \mathfrak{g} , equipped with the zero Poisson structure.

Example 2.16. The Poisson-Lie group dual to G with its standard Poisson-Lie structure is the solvable group

$$G^* = B_+ \times_H B_-,$$

where the maps in the fiber product are

$$B_{\pm} \rightarrow H \simeq B/[B, B], \quad b \mapsto b^{\pm 1}.$$

2.4. Coboundary Lie bialgebras and classical Yang-Baxter. We will be interested in Lie bialgebras whose cobracket δ is not just a 1-cocycle, but in fact a 1-coboundary: there exists some $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta(x) = x \cdot r = \sum [x, a_i] \otimes b_i + a_i \otimes [x, b_i].$$

Of course this will always be the case if the corresponding cohomology group vanishes, which indeed happens in our cases of interest when G is a connected semisimple or compact Lie group.

A Poisson tensor defined by a Lie algebra coboundary $\delta = d(r)$ is easy to integrate: the Poisson bivector π^R on the corresponding simply connected Poisson-Lie group is given by

$$\pi^R(g) = \text{Ad}_g(r) - r,$$

so that

$$\pi_g = L_g(r) - R_g(r).$$

Concretely, if we expand

$$r = \sum_i a_i \otimes x_i \in \mathfrak{g} \otimes \mathfrak{g}$$

this formula for the bivector gives the following bracket between functions:

$$(2.4.1) \quad \{f_1, f_2\}(g) = \frac{d}{dt} \Big|_{t=0} (f_1(ge^{ta_i})f_2(ge^{tx_i}) - f_1(e^{ta_i}g)f_2(e^{tx_i}g))$$

Example 2.17. The standard Lie bialgebra structure on sl_2 is coboundary, with

$$r = - \left(\frac{H \otimes H}{4} + E \otimes F \right).$$

Its image in the vector representation is the 4×4 matrix

$$r^{\mathbb{C}^2 \otimes \mathbb{C}^2} = -\frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The same is true of the standard structure on any simple Lie algebra: if $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the invariant element we used to define the Manin triple above and we write $\Omega = \Omega_{+,-} + \Omega_{0,0} + \Omega_{-,+}$ for its triangular decomposition, then we can take the element r to be

$$(2.4.2) \quad r = -\frac{1}{2}\Omega_{0,0} - \Omega_{+,-}.$$

Exercise 2.18. Let $x_{11}, x_{12}, x_{21}, x_{22}$ be the functions on the standard Poisson-Lie $SL(2, \mathbb{C})$ given by the matrix elements of the vector representation. Show that

$$\begin{aligned} \{x_{11}, x_{12}\} &= \frac{1}{2}x_{11}x_{12}, & \{x_{11}, x_{21}\} &= \frac{1}{2}x_{11}x_{21}, & \{x_{22}, x_{12}\} &= -\frac{1}{2}x_{22}x_{12}, & \{x_{22}, x_{21}\} &= -\frac{1}{2}x_{22}x_{21} \\ \{x_{12}, x_{21}\} &= 0, & \{x_{11}, x_{22}\} &= x_{12}x_{21}. \end{aligned}$$

The requirement that δ^* define a Lie bracket (i.e is skew, and satisfies the Jacobi identity) corresponds to the following constraints on the element $r \in \mathfrak{g} \otimes \mathfrak{g}$: for skewness we need $r_{12} + r_{21}$ to be an invariant for the adjoint action, while (in the presence of skewness!) the Jacobi identity is equivalent to the requirement that

$$(2.4.3) \quad [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$$

The notation $[r_{12}, r_{13}]$ means that we form $r_{12} \otimes r_{13} \in \mathfrak{g}^{\otimes 4}$, and then apply the bracket operation on the indicated factors to get an element of $\mathfrak{g}^{\otimes 3}$:

$$\begin{aligned} [r_{12}, r_{13}] &= \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j. \\ [r_{12}, r_{23}] &= \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j, \\ [r_{13}, r_{23}] &= \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]. \end{aligned}$$

Exercise 2.19. Prove this by directly writing out the Jacobi identity for the bracket of \mathfrak{g}^* :

$$\langle [\alpha, [\xi, \eta]], c \rangle + \cdots = 0.$$

(The only tricky point is that one needs to use the fact that $r + r_{21}$ is \mathfrak{g} -invariant (and the Jacobi identity in \mathfrak{g}) to eliminate all terms containing a tensor factor of the form $[[c, a_i], b_j]$)

The simplest way to ensure that (2.4.3) holds is to impose the *classical Yang-Baxter equation*

$$CYB(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

A Lie bialgebra whose cobracket comes from such a solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called *quasitriangular*, and the element r is called its *classical r -matrix*.

Remark 2.20. Having the classical Yang-Baxter equation hold on the nose (i.e. not just up to an invariant in $\mathfrak{g}^{\otimes 3}$) is our reason for working with cocycles in $\mathfrak{g} \otimes \mathfrak{g}$ rather than in $\mathfrak{g} \wedge \mathfrak{g}$: the condition (called ‘triangularity’) for the CYBE to have a skew solution turns out to be too restrictive.

Remark 2.21. Another reason to work with non-skew r -matrices is that if r satisfies the CYBE, then the map

$$r : \mathfrak{g}^* \rightarrow \mathfrak{g}, \quad \eta \mapsto \langle \eta, r \otimes - \rangle$$

is a homomorphism of Lie algebras, where \mathfrak{g}^* has the Lie algebra defined by δ^* . A quasitriangular Lie bialgebra (\mathfrak{g}, r) is called *factorizable* if the linear map

$$r + r_{21} : \mathfrak{g}^* \rightarrow \mathfrak{g}$$

is an isomorphism of vector spaces. So factorizable Lie bialgebras are sort of the opposite of triangular ones (for which $r + r_{21} = 0$), and it is the factorizable ones that we will mostly care about in the course.

Remark 2.22. Suppose that \mathfrak{g} is a complex simple Lie algebra, and $r \in \mathfrak{g} \otimes \mathfrak{g}$ defines a quasitriangular Lie bialgebra structure on \mathfrak{g} , so r a solution of the classical YB equation. Then by Schur’s lemma, the invariant $r_{12} + r_{21}$ must be a scalar multiple of the Killing form B . The skew-symmetrization of \hat{r} then has the property that $CYB(\hat{r})$ is a multiple of the Maurer-Cartan form $\omega \in \wedge^3 \mathfrak{g}$ defined by

$$\omega(x, y, z) = B([x, y], z),$$

which spans $H^3(\mathfrak{g}, \mathbb{C}) \simeq \mathbb{C}$.

At this point, there are several natural questions to ask:

- (1) Is there a systematic way to construct solutions of the classical Yang-Baxter equation?
- (2) Is the standard Lie bialgebra structure on a simple Lie algebra, defined by the element r in (2.4.2), quasitriangular?
- (3) If we fix a simple Lie algebra \mathfrak{g} , can one classify all solutions of the Yang-Baxter equation in $\mathfrak{g} \otimes \mathfrak{g}$?

An answer to questions 1 and 2 comes from the Drinfeld’s construction of the *classical double* of a Lie bialgebra. Recall from the Manin triple formulation that for any Lie bialgebra \mathfrak{g} , the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ carries a unique Lie algebra structure such that the canonical symmetric bilinear form on $\mathfrak{g} \oplus \mathfrak{g}^*$ is invariant.

Proposition 2.23. *Let \mathfrak{g} be a finite dimensional Lie bialgebra, and \mathfrak{g}^* its dual Lie bialgebra, and equip $\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g}^*$ with the canonical Lie algebra structure making $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ into a Manin triple. Then the canonical element*

$$r \in \mathfrak{g} \otimes \mathfrak{g}^* \subset (\mathfrak{g} \oplus \mathfrak{g}^*) \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)$$

satisfies the classical Yang-Baxter equation, and therefore defines on $\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g}^*$ the structure of a factorizable quasitriangular Lie bialgebra. This Lie bialgebra structure has the property that the maps

$$\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \hookleftarrow \mathfrak{g}^*$$

are morphisms of Lie bialgebras.

Proof. That r satisfies the Yang-Baxter equation is immediate from the using formula (2.3.1) for the cross-brackets between \mathfrak{g} and \mathfrak{g}^* in the Lie algebra structure on \mathfrak{d} to compute $[r_{12}, r_{23}]$. The skewness follows by observing that $r + r_{21}$ is the canonical symmetric invariant bilinear form on \mathfrak{d} ; since this is nondegenerate we see that the Lie bialgebra structure is indeed factorizable. Similarly, one observes that for $a \in \mathfrak{g}$, the form of the cross-bracket is exactly what is needed to cancel the \mathfrak{g}^* component in $\delta(a) = a \cdot r$. The Lie bialgebra structure is factorizable, since $r + r_{21}$ is the canonical symmetric invariant bilinear form on \mathfrak{d} . \square

The following exercise is extremely important, and the key to deriving all the ‘Leningrad’ formulas for Poisson brackets in the double of a factorizable Lie bialgebra.

Exercise 2.24. The standard Lie bialgebra structure on a simple Lie algebra \mathfrak{g} is a Lie sub-bialgebra of the quasitriangular Lie bialgebra $\mathfrak{d}(\mathfrak{g}, \mathfrak{g}^*) \simeq \mathfrak{g} \oplus \mathfrak{g}$. Show that the r -matrix $r_{\mathfrak{d}}$ for \mathfrak{d} is expressed in terms of the r -matrix r of \mathfrak{g} as

$$(2.4.4) \quad r_{\mathfrak{d}} = r_{14} + r_{24} - r_{31} - r_{32} \in (\mathfrak{g} \oplus \mathfrak{g}) \otimes (\mathfrak{g} \oplus \mathfrak{g}).$$

Here the confusing notations mean, for example with $\mathfrak{g} = \mathfrak{sl}_2$,

$$r_{14} = - \left(\frac{1}{4} (H, 0) \otimes (0, H) + (E, 0) \otimes (0, F) \right),$$

$$r_{32} = - \left(\frac{1}{4} (0, H) \otimes (H, 0) + (0, F) \otimes (E, 0) \right),$$

Example 2.25. The standard Lie bialgebra structure on \mathfrak{g} is itself *almost* a double. Indeed, consider the trivial central extension $\mathfrak{g} \oplus \mathfrak{h}$ of \mathfrak{g} by its Cartan subalgebra, equipped with the difference of the Killing form on \mathfrak{g} and its restriction to \mathfrak{h} . Then we can embed the Borel subalgebras \mathfrak{b}_{\pm} as isotropic subspaces of this extension, where the Cartan subalgebra in \mathfrak{b}_{+} is embedded diagonally (i.e. as elements of the form (h, h)), and that from \mathfrak{b}_{-} anti-diagonally (i.e. as elements $(h, -h)$). Using the root space basis we can easily write down dual basis in these subalgebras in $\mathfrak{g} \oplus \mathfrak{h}$, so we see that this construction gives $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_{\pm})$ the structure of a Manin triple, so that we obtain a triple of Lie bialgebras. Moreover, the Lie cobracket vanishes on the central summand \mathfrak{h} , so that the quotient \mathfrak{g} inherits the structure of a Lie bialgebra, and this Lie bialgebra structure coincides with the standard one. In particular, we see that the dual Lie bialgebra to \mathfrak{b}_{+} (with its standard LBA structure) is \mathfrak{b}_{-} , and that the double of \mathfrak{b}_{+} is $\mathfrak{g} \oplus \mathfrak{h}$.

Exercise 2.26. Describe the $\mathfrak{d}(\mathfrak{d}(\mathfrak{h}))$, the double of the double of an arbitrary Lie bialgebra \mathfrak{h} . Hint: the picture of $\mathfrak{d}(\mathfrak{d}(\mathfrak{h}))$ is similar to the description of the double of a standard simple Lie algebra \mathfrak{g} , for which $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}$.

Remark 2.27. Recall the maps

$$r : \mathfrak{d} \rightarrow \mathfrak{d}, \quad d \mapsto \langle r, d \otimes - \rangle,$$

$$r_{21} : \mathfrak{d} \rightarrow \mathfrak{d}, \quad d \mapsto \langle r, - \otimes d \rangle.$$

Then r is the projection to the summand \mathfrak{g}^* with respect to the splitting $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, and r_{21} the corresponding projection to \mathfrak{g} .

Example 2.28. Suppose that (G, r) is a coboundary Poisson Lie group and ρ^V, ρ^W are representations of G on vector spaces V, W . Then setting $r^{V;W} = (\rho^V \otimes \rho^W)(r) \in \text{End}(V \otimes W)$, the matrix coefficients of these representations have Poisson brackets

$$\{\rho_{ij}^V, \rho_{kl}^W\} = \sum_a \rho_{ia}^V \rho_{kb}^W r_{aj;bl}^{V;W} - \rho_{aj}^V \rho_{bl}^W r_{ia;jb}^{V;W},$$

This can be expressed more succinctly in the ‘Leningrad’ notation as follows. The data of an algebraic representation V of G is the same as that of a comodule T^V for $\mathbb{C}[G]$:

$$T : V \rightarrow V \otimes \mathbb{C}[G], \quad \text{i.e.} \quad T^V \in \text{End}(V) \otimes \mathbb{C}[G].$$

Then we can write the formula above more invariantly as

$$\{T^V, T^W\} = [T^V \otimes T^W, r^{V;W}] \in \text{End}(V \otimes W) \otimes \mathbb{C}[G],$$

where the left-hand-side means “form $T^V \otimes T^W \in \text{End}(V \otimes W) \otimes \mathbb{C}[G]^{\otimes 2}$, and then apply the Poisson bracket $\mathbb{C}[G]^{\otimes 2} \rightarrow \mathbb{C}[G]$.”

Exercise 2.29. Show that the functions on the coboundary Poisson Lie group G defined by $H^V = \text{tr}_V(g)$ as V ranges over all finite dimensional representations of G Poisson commute:

$$\{H_V, H_W\} = 0.$$

In addition to the T^V , we can also form $\mathbb{C}[G^*]$ -valued matrices L_{\pm}^V by

$$L_+^V(b_+, b_-) = \rho^{V \otimes \mathbf{1}}(b_+, b_-), \quad L_-^V(b_+, b_-) = \rho^{\mathbf{1} \otimes V}(b_+, b_-)$$

where $\mathbf{1}$ is the trivial representation of G .

Exercise 2.30. Use the embedding of Poisson Lie groups $G^* \subset D(G)$ and (2.4.4) to get the dual Leningrad formulas

$$\{L_{\pm}^V, L_{\pm}^W\} = [L_{\pm}^V \otimes L_{\pm}^W, r^{V;W}], \quad \{L_+^V, L_-^W\} = [L_+^V \otimes L_-^W, r^{V;W}].$$

Exercise 2.31. Let e, f, k be the functions on SL_2^* given by the matrix elements of the tensor product of vector representations:

$$\begin{bmatrix} k & e \\ 0 & k^{-1} \end{bmatrix}, \quad \begin{bmatrix} k^{-1} & 0 \\ f & k \end{bmatrix}.$$

Show that their brackets are given by

$$\{k, e\} = \frac{1}{2}ke, \quad \{k, f\} = -\frac{1}{2}kf, \quad \{e, f\} = k^2 - k^{-2}$$

2.5. Dressing transformations and symplectic leaves in Poisson-Lie groups. Recall the nice description of symplectic leaves in \mathfrak{g}^* with its KKS structure: they are the orbits for the coadjoint action of a connected Lie group G with Lie algebra \mathfrak{g} . Recall also that $(\mathfrak{g}^*, +)$ and the group G with the zero Poisson structure are Poisson-Lie dual to one another. This picture actually generalizes to any such dual pair of Poisson-Lie groups in the following way.

We say that an action of a Poisson-Lie group G on a manifold M is *Poisson* if the map $G \times M \rightarrow M$ is Poisson. For example, the left and right translations of G (or any Poisson-Lie subgroup) on G are Poisson actions. Note that when the Poisson structure of G is nontrivial, the maps $g \cdot M \rightarrow M$ will *not* generally be Poisson.

Now let G be a Poisson-Lie group and let $D(G)$ be its double, so that we have Poisson-Lie subgroups

$$G \hookrightarrow D(G) \hookleftarrow G^*.$$

Since the Lie algebra of $D(G)$ is the direct sum of \mathfrak{g} and \mathfrak{g}^* , the product map

$$G \times G^* \rightarrow D(G)$$

is a diffeomorphism onto a neighborhood of 1 in $D(G)$. Let's suppose for the moment that this map is a global diffeomorphism $G \times G^* \rightarrow D(G)$ (although this won't be true in many examples of interest, see below). Then we can identify G as a Poisson manifold with the homogeneous space $D(G)/G^*$. The latter carries a residual Poisson action of G^* by left translation, which under the identification $D(G)/G^* \simeq G$ gives a Poisson action of G^* on G called the *dressing action*, and the orbits of this action coincide with the symplectic leaves of G .

Indeed, let's compute the tangent space to the dressing orbit through $g \in G$. For $\xi \in G^*$, we have in $D(G)$

$$e^{t\xi}g = ge^{t\text{Ad}_{g^{-1}}(\xi)}$$

Hence the corresponding tangent vector in G has the form ge^{tx} where x is the projection to \mathfrak{g} of $\text{Ad}_{g^{-1}}\xi \in \mathfrak{d}$. But recall that this projection is computed using the r -matrix as $x = \langle r, - \otimes \text{Ad}_{g^{-1}}\xi \rangle$. In terms of our usual right-trivialization of TG , this is

$$\text{Ad}_g(\langle r, - \otimes \text{Ad}_{g^{-1}}\xi \rangle) = \langle \text{Ad}_g^{\otimes 2}(r), - \otimes \xi \rangle = \langle \text{Ad}_g^{\otimes 2}(r) - r, - \otimes \xi \rangle = \langle \pi^R(g), - \otimes \xi \rangle$$

Hence as ξ varies over all of \mathfrak{g}^* we get the entire image of $\pi^R(g)$, so the tangent space to the orbit indeed coincides with the image of the bundle map $I : T^*G \rightarrow TG$.

Example 2.32. Recall that the double of a simple Lie group G is $G \times G$, with $G = G_\Delta$ embedded diagonally and $G^* = B_+ \times_H B_-$ embedded in the natural way. Then the image of the multiplication map $G \times G^* \rightarrow G$ consists of elements of the form

$$(n_+hg, n_-h^{-1}g), \quad g \in G, n_\pm \in N_\pm, h \in H.$$

In particular, G^*G is contained in the preimage of the open cell $B_+B_- \subset G$ under the projection

$$p_{G^*} : G \times G \rightarrow G = (G \times G)/G_\Delta, \quad (g_1, g_2) \mapsto g_1g_2^{-1}$$

The homogeneous space $G \simeq (G \times G)/G_\Delta$ is equipped with a Poisson structure inherited from that of $D(G) = G \times G$. Let us denote this Poisson structure by π_{G_*} . As discussed above, the Poisson structure π_{G_*} has the property that the residual action of G by left translation on $D(G)$ is Poisson,

and under the isomorphism $G \simeq (G \times G)/G_\Delta$, this action is given by the conjugation action of G on itself. This Poisson structure ‘extends’ the Poisson-Lie Poisson structure of the group G^* in the following sense: the restriction of the map p_{G^*} to $G^* \subset D(G)$ is a $2^{\text{rk}(\mathfrak{g})} : 1$ local diffeomorphism

$$m : G^* \rightarrow G = (G \times G)/G_\Delta, \quad (b_+, b_-) \mapsto b_+ b_-^{-1}$$

with $m_* \pi_{G^*} = \pi_{G_*}$. The symplectic leaves of $m(G^*) \subset G_*$ are given by the connected components of its intersection with the conjugacy classes in G ; general symplectic leaves in G_* are components of intersections of the conjugacy classes with orbits for the inherited G^* action on $G = D(G)/G_\Delta$, which reads $(b_+, b_-) \cdot g = b_+ g b_-^{-1}$.

On the other hand, to compute the dressing action of G^* on G we need to solve a factorization problem in $D(G) = G \times G$ of the form

$$(b_+ g, b_- g) = (\tilde{g} \tilde{b}_+, \tilde{g} \tilde{b}_-) \in D(G) = G \times G,$$

so that g and its image under the dressing action lie in the same *double Bruhat cell*:

$$\tilde{g} \in B_+ g B_+ \cap B_- g B_-.$$

With a bit more work one can show that the H -orbits of symplectic leaves in G with its standard Poisson-Lie structure are exactly the double Bruhat cells, and are thus labelled by pairs of Weyl group elements

$$G^{u,v} = B_+ u B_+ \cap B_- v B_-, \quad u, v \in W(G).$$

These examples are explained in detail in the papers

- (1) *Poisson geometry of the Grothendieck resolution of a complex semisimple group* by Evens and Lu
- (2) *On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups* by Kogan and Zelevinsky.

Exercise 2.33. Describe the symplectic leaves of G, G_*, G^* completely explicitly when $G = SL(2, \mathbb{C})$.

Not covered: Belavin-Drinfel'd classification, any infinite dimensional examples.

2.6. Canonical coordinates on standard Poisson-Lie groups. We now describe how to find canonical coordinates with respect to the standard Poisson-Lie structure on double Bruhat cells in a complex semisimple Lie group G . There are various flavors of such coordinates, but all come from the basic observation that the 2-dimensional Borel subgroup of each simple root $SL(2)$ -triple is a Poisson-Lie subgroup, and it is very easy to put canonical coordinates on the $SL(2)$ Borel: indeed, we've seen that the coordinates g_{11}, g_{12} have brackets

$$\{g_{11}, g_{12}\} = \frac{1}{2} g_{11} g_{12},$$

so that in terms of a Darboux pair $\{p, x\} = 1$ we can put $g_{11} = e^p, g_{12} = e^{x/2}$.

So given any word $\mathbf{i} = (i_1, \dots, i_l)$ in the alphabet $\{1, \dots, r\} \sqcup \{-1, \dots, -r\}$ on the positive/negative simple roots, we get a Poisson multiplication map

$$H \times B_{i_1} \times \dots \times B_{i_l} \rightarrow G.$$

By definition of the product Poisson structure, the coordinates on each factor just Poisson commute. This map factors through the quotient by the torus $\prod_j H_j^{k_j}$ where $k_j + 1$ is the number of letters $i_r \in \mathbf{i}$ with $|i_r| = j$, and this torus just scales the canonical coordinates on the factors described above.

Example 2.34. We use this method to put canonical coordinates on standard Poisson-Lie PGL_2 . Consider the co-characters

$$H(x) = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$$

and set

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We consider the tori $\mathbb{C}^* \times \mathbb{C}^*$ inside the upper/lower Borels defined by

$$b_+ = H(x)EH(y), \quad b_- = H(w)FH(z).$$

Then we have $\{y, x\} = xy$, $\{w, z\} = zw$. The fibers of the multiplication map

$$\begin{aligned} (\mathbb{C}^*)^4 \subset B_+ \times B_- &\longrightarrow PGL_2, & (b_+, b_-) &\mapsto b_+ b_- \\ (x, y, w, z) &\mapsto H(x)EH(yw)FH(z) \end{aligned}$$

are the orbits for the action of \mathbb{C}^* -action on $(\mathbb{C}^*)^4$ given by

$$(x, y), (w, z) \longmapsto (x, \lambda y), (\lambda^{-1}w, z), \quad \lambda \in \mathbb{C}^*.$$

Then corresponding to the word $\mathbf{i} = (1, -1)$ in the simple roots, we get a chart on the big double Bruhat cell in PGL_2 with coordinate functions $(x_1 := x, x_2 := yw, x_3 := z)$ given by

$$(\mathbb{C}^*)^3 \longrightarrow PGL_2^{w_0, w_0}, \quad (x_1, x_2, x_3) \mapsto H(x_1)EH(x_2)FH(x_3),$$

with the Poisson brackets

$$\{x_2, x_1\} = x_1 x_2, \quad \{x_2, x_3\} = x_2 x_3, \quad \{x_1, x_3\} = 0.$$

These coordinates are ‘log-canonical’, in the sense that the bracket of any two of them is proportional to their product. We can encode the coefficients of proportionality by the following quiver:

$$x_1 \longleftarrow x_2 \longrightarrow x_3$$

On the other hand, we can consider the chart on PGL^{w_0, w_0} corresponding to the word $\mathbf{i}' = (-1, 1)$, so that we parametrize the group element as

$$(x', y', w', z') \mapsto H(w')FH(z')H(x')EH(y') = H(x'_1)FH(x'_2)EH(x'_3).$$

where $(x'_1 = w', x'_2 = x'z', x'_3 = y')$. This time their Poisson brackets

$$\{x'_1, x'_2\} = x'_1 x'_2, \quad \{x'_3, x'_2\} = x'_2 x'_3, \quad \{x'_1, x'_3\} = 0$$

are encoded by the quiver

$$x'_1 \longrightarrow x'_2 \longleftarrow x'_3$$

obtained by *mutating* the one above at the node labelled x_2 . The following exercise describes the transition functions between the charts corresponding to words \mathbf{i}, \mathbf{i}'

Exercise 2.35. Show that on the intersection of the images of the two tori above, the coordinates x'_i can be expressed as the cluster mutation of the ones x_i in direction 2:

$$\begin{aligned}x'_1 &= \frac{x_1 x_2}{1 + x_2} \\x'_2 &= x_2^{-1} \\x'_3 &= \frac{x_3 x_2}{1 + x_2}\end{aligned}$$

In general, if \mathbf{i} is a shuffle of a reduced word for $(u, v) \in W \times W$, then applying the same construction gives a torus chart $(\mathbb{C}^*)^{l(u)+l(v)+\dim H}$ on the double Bruhat cell $G^{u,v}$. Transition maps between charts corresponding to different reduced words are again given by cluster mutations, and this makes the double Bruhat cell into a *cluster Poisson variety*.

Exercise 2.36. Compute the quiver describing the cluster Poisson structure in the toric chart on $PGL_3^{w_0, w_0}$ corresponding to the double reduced word

$$\mathbf{i} = (1, 2, 1, -2, -1, -2),$$

and generalize to PGL_n .

3. HOPF ALGEBRAS

4. DEFORMATIONS AND QUANTIZED ENVELOPING ALGEBRAS

5. THE QUANTUM GROUP $U_q(sl_2)$

6. INTEGRAL FORMS OF $U_q(sl_2)$ AND ROOTS OF UNITY

7. QUANTUM WEYL GROUP AND PBW BASES FOR $U_q(g)$

8. R-MATRIX AND QUASITRIANGULAR STRUCTURE OF $U_q(g)$

9. CENTER OF THE QUANTUM GROUP

10. TENSOR CATEGORIES

11. INVARIANTS OF TANGLES AND 3-MANIFOLDS

12. CRYSTALS AND CANONICAL BASES

13. OVERFLOW

Quantum inverse scattering method (algebraic Bethe ansatz), quantum loop groups and quantum affine algebras, quantization of character varieties, KZ equation and Drinfel'd associators, localization