The Story of the Alternating Sign Matrix Conjecture

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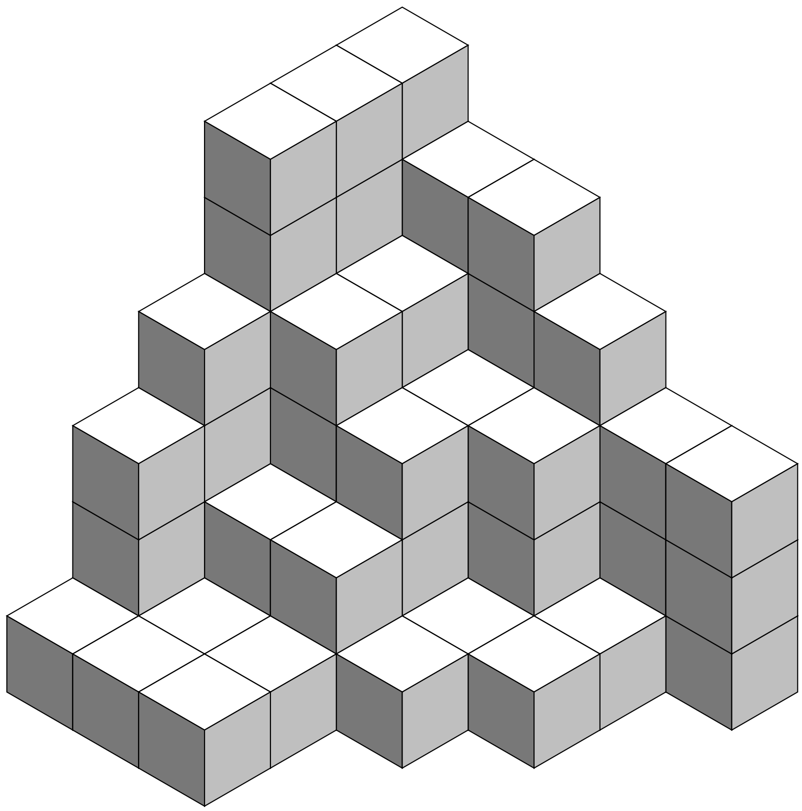
February 24, 2020



The following solution manual follows our two-term journey of learning about alternating sign matrices, their properties, and related (and some seemingly unrelated) concepts needed to go about proving how many there are. Some of these solutions include commentary about how to derive solutions logically, as sometimes this is the clearest way for a reader to develop the intuition on how to think about these problems.

Question 1.1.1

This chapter defines permutations as a one-to-one mapping from the set of $1, 2, 3, 4, ..., n$ to itself. The matrix of a permutation σ, S in row i has a 1 in the column $\sigma(i)$ and zeroes elsewhere in that row. We need to show that if $\vec{m} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$, $S\vec{m} = \begin{bmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{bmatrix}$.

Know that the i'th row of $S\vec{m}$ is equal to the sum of the entries of the i'th row of S multiplied by the corresponding rows of $\vec{m}$. Since, from above, the i'th row of S has a single nonzero entry of 1 in the $\Sigma(i)$'th column, the only term in the i'th row of $\vec{m}$ is $1\dot{\Sigma}(i)$.

Thus, $S\vec{m} = \begin{bmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{bmatrix}$.

Q.E.D

Question 1.1.5

We are instructed to find the largest possible subset of the alternating sign matrices of size $n \times n$ that includes the permutation matrices and is closed under multiplication.

We can determine first, that any alternating sign matrix without any -1 entries is a permutation matrix.

We also know from problem 1.1.10 that any alternating sign matrix must have only a single 1 in the top row.

Suppose a alternating sign matrix of size $n \times n$, A in this subset has a -1 entry in the ith row ($i \neq 1$). This subset must include the permutation matrix that's the identity except with a 1 in the ith column of the first row, and a 1 in the 1st column of the ith row, which we'll call P.

Then $P \times A$ will have a -1 in the i'th column of the 1st row, and so is not a alternating sign matrix.

Therefore all the entries of any alternating sign matrix in this subset must be non-negative, so the subset is the permutation matrices.

Question 1.1.10

Suppose, by way of contradiction, that some alternating sign matrix of size n did not have a single entry equal to 1 in the top row.

If it had 0 values the sum of the entries of the row would be ≤ 0, and so this is not possible.

Suppose then the top row had k entries equal to 1 with $k > 1$.

Then since the sum of the entries in the top row has to equal 1, there must be $k - 1$ entries in the top row equal to -1.

But since the sum of each entry of each column also has to equal 1, in each column beginning with -1, there must be at least one more 1 than -1 in the rows remaining.
But this is impossible without having two consecutive 1s, separated only by 0s. This is a contradiction of the definition of a alternating sign matrix.

Question 1.1.11

We are tasked with drawing up a pascal-style triangle for the matrices of symmetric groups.
Obviously, for matrices of size 1, there is only one possible such matrix.
For matrices of size two, there are two possibilities:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For matrices of size 3 there are a total of six:
With the first row 1 in the top left:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

With the first row 1 in the middle:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the first row 1 in the right:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

These examples demonstrate a pattern that will let us construct our triangle. For a permutation matrix of size n, after placing a 1 in a specific column of the first row, we know the rest of the first row and that column will be occupied by 0s. Discarding these, we are left with the possibilities for a permutation matrix of size $n-1$. Thus the possibilities for each of the n first-row-placement options of a permutation matrix of size n will be the sum of the options for a matrix of size $n-1$. This gives us the recursive formula: $s_0 = 1$, $s_2 = 1$, $s_n = n * s_{n-1}$, from whence we can construct this triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 2 & & 2 & & \\ & 6 & & 6 & & 6 & & 6 & \\ 24 & & 24 & & 24 & & 24 & & 24 \end{array}$$

and so on.

Question 1.2.5

within a box of dimensions r X r X r, elements within an orbit of 1 are all those such that their dimensions $(a, b, c) a, b, c \in Z, a, b, c < r$ remain unchanged when under action of elements of S_3. Obviously, this is all those of the form (a, a, a), of which there are r.

The elements within an orbit of 2 are those such that (a, b, c) can only be changed to one other distinct element under actions of S_3.
There such elements, if a,b,c are all the same you get the case above, if two are the same, there are two other distinct elements, if all three are different then there are 5 other distinct elements.

Elements with orbit of height 3 are all those of the form $(a, b, b) a \neq b$, of which there are $3r^2 - r$ total possibilities.

Elements with orbit of height 6 are all those of the form $(a, b, c) a \neq b \neq c$, of which there are $r^3 - 3r^2 + 2r$ total possibilities.

Question 1.2.6

The findings from the previous problem remain true up to orbits of size two, because the reduction from S_3 to C_3 doesn't change the relevant permutations (or lack thereof).
There are $r^3 - r$ orbits of height 3, all those of the form (a, b, b) and (a, b, c) under C_3.

Question 1.2.10

There are a total of 10 orbits within $B(3,3,3)$, and are enumerated below along with their sizes:

n	orbit
1	(1,1,1)
3	(1,2,1)
3	(2,2,1)
1	(2,2,2)
3	(2,3,2)
3	(3,3,2)
1	(3,3,3)
3	(3,3,1)
3	(3,1,1)
6	(1,2,3)

Thus $\prod_{\eta \in B(3,3,3)/S_3} \frac{1-q^{|\eta|(1+ht(\eta))}}{1-q^{|\eta|(1+ht(\eta))}} = \frac{(1-q^4)(1-q^7)(1-q^{10})(1-q^{18})^2(1-q^{24})(1-q^{27})(1-q^{42})}{(1-q^3)(1-q^6)(1-x^9)(1-x^{12})(1-x^{15})(1-x^{21})^2(1-x^{36})}$
Using Wolfram Alpha we can expand and simplify this:

In[∘]:= `ExpandNumerator[%8]`

In[2]:=

$$
\begin{aligned}
&(1 - x^{4} - x^{7} - x^{10} + x^{11} + x^{14} + x^{17} - 2 x^{18} - x^{21} + 2 x^{22} - x^{24} + 2 x^{25} - x^{27} + 3 x^{28} - 2 x^{29} + 2 x^{31} - \\
&\quad 2 x^{32} + 2 x^{34} - 3 x^{35} + x^{36} + x^{37} - 2 x^{38} + 2 x^{39} - x^{40} - 2 x^{41} + x^{42} - x^{43} - x^{44} + 3 x^{45} - 2 x^{46} + \\
&\quad x^{47} + x^{48} - 3 x^{49} + x^{50} + x^{51} - 3 x^{52} + 2 x^{53} - 3 x^{55} + 3 x^{56} - x^{57} - x^{58} + 3 x^{59} + x^{60} - x^{61} + 3 x^{62} - \\
&\quad 2 x^{63} - x^{64} + x^{65} - x^{66} + x^{68} - x^{69} - x^{70} + x^{71} - x^{72} + x^{73} + x^{77} - x^{78} + x^{79} - x^{80} - x^{81} + x^{82} - \\
&\quad x^{84} + x^{85} - x^{86} - 2 x^{87} + 3 x^{88} - x^{89} + x^{90} + 3 x^{91} - x^{92} - x^{93} + 3 x^{94} - 3 x^{95} + 2 x^{97} - 3 x^{98} + \\
&\quad x^{99} + x^{100} - 3 x^{101} + x^{102} + x^{103} - 2 x^{104} + 3 x^{105} - x^{106} - x^{107} + x^{108} - 2 x^{109} - x^{110} + 2 x^{111} - \\
&\quad 2 x^{112} + x^{113} + x^{114} - 3 x^{115} + 2 x^{116} - 2 x^{118} + 2 x^{119} - 2 x^{121} + 3 x^{122} - x^{123} + 2 x^{125} - x^{126} + \\
&\quad 2 x^{128} - x^{129} - 2 x^{132} + x^{133} + x^{136} + x^{139} - x^{140} - x^{143} - x^{146} + x^{150}) / \\
&(1 - q^{3} - x^{6} + q^{3} x^{6} - x^{9} + q^{3} x^{9} - x^{12} + q^{3} x^{12} + x^{18} - q^{3} x^{18} + x^{24} - q^{3} x^{24} + 2 x^{27} - 2 q^{3} x^{27} + \\
&\quad x^{30} - q^{3} x^{30} + x^{33} - q^{3} x^{33} - 2 x^{36} + 2 q^{3} x^{36} - 2 x^{39} + 2 q^{3} x^{39} - x^{42} + q^{3} x^{42} - x^{45} + q^{3} x^{45} + \\
&\quad x^{51} - q^{3} x^{51} + 2 x^{57} - 2 q^{3} x^{57} - 2 x^{63} + 2 q^{3} x^{63} - x^{69} + q^{3} x^{69} + x^{75} - q^{3} x^{75} + x^{78} - q^{3} x^{78} + \\
&\quad 2 x^{81} - 2 q^{3} x^{81} + 2 x^{84} - 2 q^{3} x^{84} - x^{87} + q^{3} x^{87} - x^{90} + q^{3} x^{90} - 2 x^{93} + 2 q^{3} x^{93} - x^{96} + q^{3} x^{96} - \\
&\quad x^{102} + q^{3} x^{102} + x^{108} - q^{3} x^{108} + x^{111} - q^{3} x^{111} + x^{114} - q^{3} x^{114} - x^{120} + q^{3} x^{120})
\end{aligned}
$$

`FullSimplify[%]`

In[4]:= `ExpandNumerator[%3]`

Out[4]=

$$
\begin{aligned}
&(-1 + x^{2} - x^{5} - x^{6} + x^{7} + x^{8} - x^{9} - 2 x^{12} + x^{13} + x^{14} - x^{15} + x^{16} - x^{17} - 2 x^{18} + 2 x^{19} + x^{20} - \\
&\quad 3 x^{21} + x^{22} + x^{23} - 2 x^{24} + 2 x^{25} - x^{26} - 2 x^{27} + 3 x^{28} - 3 x^{30} + 2 x^{31} + x^{32} - 3 x^{33} + 2 x^{34} - \\
&\quad 2 x^{36} + 3 x^{37} - x^{38} - 2 x^{39} + 3 x^{40} - 3 x^{42} + 2 x^{43} + x^{44} - 2 x^{45} + 2 x^{46} - x^{47} - x^{48} + 3 x^{49} - x^{50} - \\
&\quad 2 x^{51} + 2 x^{52} + x^{53} - x^{54} + x^{55} - x^{56} - x^{57} + 2 x^{58} + x^{61} - x^{62} - x^{63} + x^{64} + x^{65} - x^{68} + x^{70}) / \\
&((-1 + q^{3}) (1 + x + x^{2})^{2} (1 - x^{2} + x^{4}) (1 - x^{6} + x^{12}) (1 + (-1 + x) x (1 + x) (1 + x^{2}) (1 + (-1 + x) x^{2})) \\
&\quad (1 + (-1 + x) x (1 + x) (1 + (-1 + x) x) (1 + x + x^{7})))
\end{aligned}
$$

limit as x→-(-1)^(1/21) | series around x = -(-1)^(1/21) | plot | cancel | more...

In[5]:= `ExpandDenominator[%4]`

Out[5]=

$$
\begin{aligned}
&(-1 + x^{2} - x^{5} - x^{6} + x^{7} + x^{8} - x^{9} - 2 x^{12} + x^{13} + x^{14} - x^{15} + x^{16} - x^{17} - 2 x^{18} + 2 x^{19} + x^{20} - \\
&\quad 3 x^{21} + x^{22} + x^{23} - 2 x^{24} + 2 x^{25} - x^{26} - 2 x^{27} + 3 x^{28} - 3 x^{30} + 2 x^{31} + x^{32} - 3 x^{33} + 2 x^{34} - \\
&\quad 2 x^{36} + 3 x^{37} - x^{38} - 2 x^{39} + 3 x^{40} - 3 x^{42} + 2 x^{43} + x^{44} - 2 x^{45} + 2 x^{46} - x^{47} - x^{48} + 3 x^{49} - x^{50} - \\
&\quad 2 x^{51} + 2 x^{52} + x^{53} - x^{54} + x^{55} - x^{56} - x^{57} + 2 x^{58} + x^{61} - x^{62} - x^{63} + x^{64} + x^{65} - x^{68} + x^{70}) / \\
&(-1 + q^{3} + x^{2} - q^{3} x^{2} - x^{4} + q^{3} x^{4} - x^{5} + q^{3} x^{5} + x^{6} - q^{3} x^{6} - x^{8} + q^{3} x^{8} - x^{12} + q^{3} x^{12} - \\
&\quad x^{17} + q^{3} x^{17} - x^{18} + q^{3} x^{18} + x^{20} - q^{3} x^{20} - x^{22} + q^{3} x^{22} - x^{23} + q^{3} x^{23} - x^{28} + q^{3} x^{28} - \\
&\quad x^{32} + q^{3} x^{32} + x^{34} - q^{3} x^{34} - x^{35} + q^{3} x^{35} - x^{36} + q^{3} x^{36} + x^{38} - q^{3} x^{38} - x^{40} + q^{3} x^{40})
\end{aligned}
$$

As we can see this is not a polynomial. That it is not a polynomial means that this is not the correct formula for the totally symmetric plane partitions within $B(3, 3, 3)$ of size n because there will not be distinct coefficients for different powers of q representing different values of n.

Question 1.3.1

Consider the following pictures of two of the CSPPs:

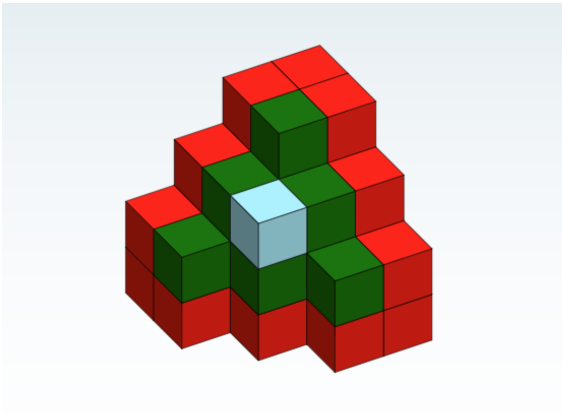


Figure 1: Part A

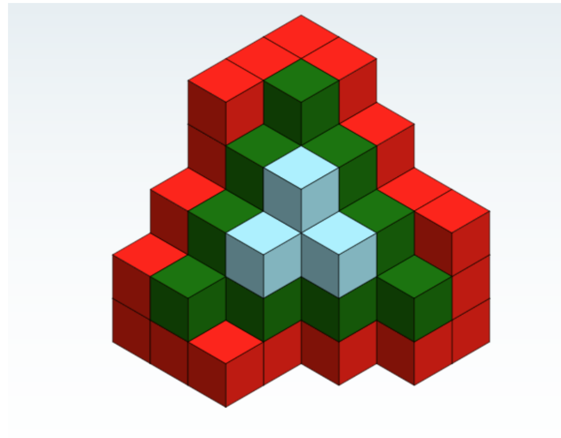


Figure 2: Part B

If we look at the section of each color, we get the following cyclically symmetric plane partitions:

Part A:
$$\begin{array}{cccc} 4 & 4 & 3 & 2 \\ 4 & 4 & 3 & 2 \\ 3 & 3 & 3 & \\ 2 & 2 & & \end{array}$$

Part B:
$$\begin{array}{ccccc} 5 & 5 & 4 & 3 & 3 \\ 5 & 5 & 4 & 3 & 2 \\ 5 & 4 & 4 & 3 & \\ 3 & 3 & 3 & & \\ 2 & 2 & 1 & & \end{array}$$

For the third part, we omit the picture due to the size of the plane partition.

The corresponding cyclically symmetric plane partition is:

$$\begin{array}{ccccccc} 7 & 7 & 7 & 4 & 3 & 3 & 3 \\ 7 & 6 & 6 & 4 & 3 & 3 & 1 \\ 7 & 6 & 6 & 3 & 3 & 3 & 1 \\ 4 & 4 & 4 & 3 & & & \\ 3 & 3 & 3 & & & & \\ 3 & 3 & 3 & & & & \\ 3 & 1 & 1 & & & & \end{array}$$

We leave it as an exercise to the reader to verify these are cyclically symmetric plane partitions. It is interesting to note that in the case the top row of a strict shifted plane partition is not i, j symmetric, then the resulting cyclically symmetric plane partition will not appear completely symmetric. This is shown by figures 1 and 2, where figure 2's top row is not entirely symmetric and the resulting CSPP looks a little off.

Question 1.3.2

For this question, we will not be using any visual tools, but rather what we know about how to get from strict shifted to cyclically symmetric plane partitions. We want to go backwards in this case, reconstructing the corresponding shells of the plane partitions.

a) $\begin{matrix} 3 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & \end{matrix}$

In this case, we see that the outer shell for this partition will be $3, 2, 2$, then the second layer will be $2, 1$, giving us the strict shifted plane partition of:

$\begin{matrix} 3 & 2 & 2 \\ & 2 & 1 \end{matrix}$

b) $\begin{matrix} 5 & 4 & 3 & 3 & 3 \\ 5 & 4 & 3 & 2 & \\ 5 & 3 & 3 & & \\ 2 & 2 & 1 & & \\ 1 & 1 & 1 & & \end{matrix}$

The outer shell for this strict shifted plane partition is defined by the bottom layer, $5, 4, 3, 3, 3$. The next shell is anything that is not in the first shell and has minimum coordinate 2. This includes three elements from row 2, two from row 3, and one from row 4, $3, 2, 1$, the last layer is seen as we have one point with minimum coordinate 3, namely $(3, 3, 3)$, this means the last row is 1, giving us a strict shifted plane partition of:

$\begin{matrix} 5 & 4 & 3 & 3 & 3 \\ & 3 & 2 & 1 & \\ & & 1 & & \end{matrix}$

c) With the previous two parts, we only provide a solution for this part:

$\begin{matrix} 7 & 4 & 4 & 4 & 3 & 1 & 1 \\ & 3 & 3 & 3 & & & \\ & & 2 & 2 & & & \\ & & & 1 & & & \end{matrix}$

Question 1.3.3

a) The best way to approach this problem is to think about the structure of how many elements will be in each shell and generalize that.

For the first partition, we can see that the outer shell will be constructed with a bottom layer of 11 elements, we only use 7 more of these elements with one of the edges, then the last edge will already have 7 blocks in it, meaning we only add another 4. This gives us 22 elements in our shell. The next layer has 4 elements on the bottom, but the second side already has two elements, so we add 2 more, from there, the last side has 3/4 elements, meaning we add 1, giving us 7 elements. The final layer has 1 element. The total for this is then 30 for this corresponding CSPP.

Lemma: The the total number of elements in shell i of a CSPP is equal to the 3(sum of row i - largest element in row i) + 1.

$$total_i = 3[\sum_{i=1}^{j} i - j] + 1$$

Proof: This is left as an exercise to the reader, although a simple explanation is as follows. While constructing the shells of a CSPP, think about the number of elements on one edge. We need to exclude the largest part from each edge otherwise it will be overcounted. The additional one is left over because we actually want to omit one element from the second and third edges but we omit three, meaning we need to add the one back on at the end.

b) Applying the lemma from above, we see that row one has 36 elements with largest part 7. This means we have a total of $3(36 - 7) + 1 = 88$ elements in this shell. The next row has 46 elements, the third has 19, while the fourth has 4. This gives us a total of 157 elements.

c) We use the above formula to calculate this to be 417.

Question 1.3.10

Find the fourteen descending plane partitions with parts less than or equal to 4 and exactly one part of size 4.

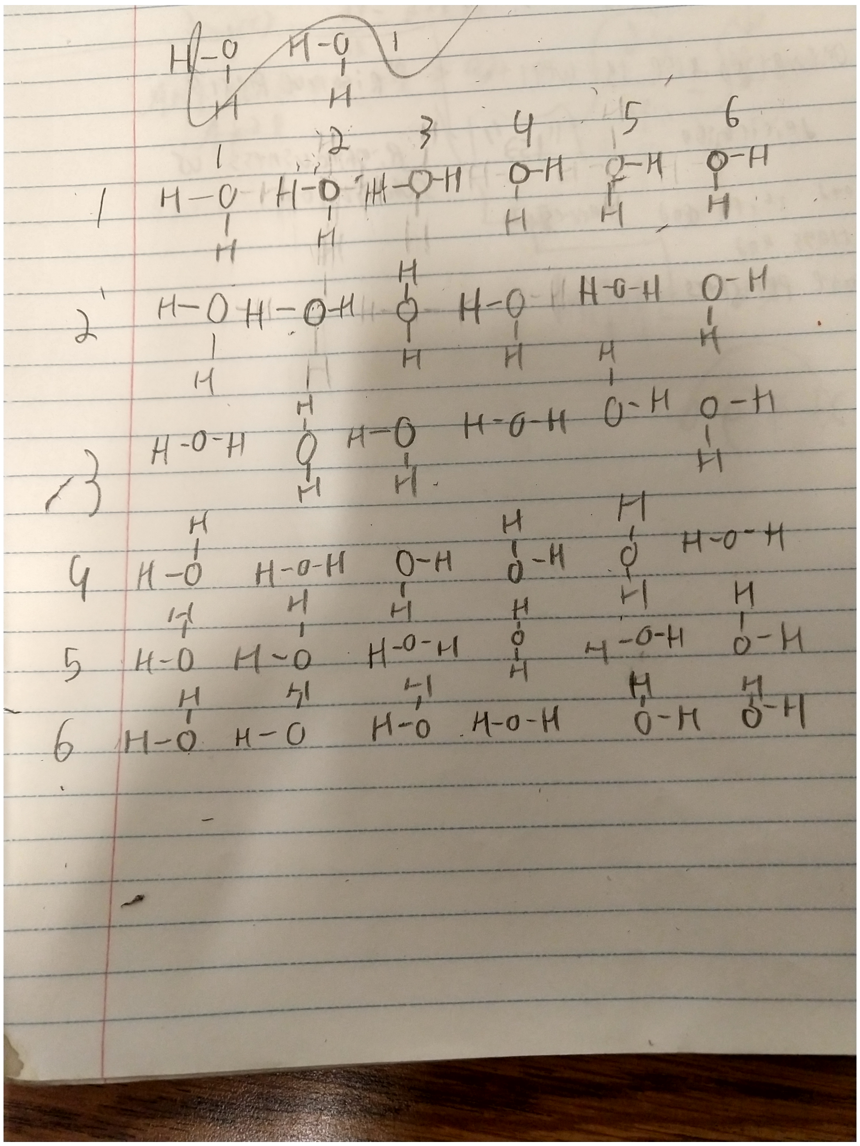
The fourteen are as follows:

```
4

4 4 ,  4 3  ,  4 2  ,  4 1

4 3 3  ,  4 3 2  , 4 3 1 , 4 2 1

 4 3      4 3 1      4 3 3     4 3 2     4 3 2 1
   2   ,    2     ,    2    ,    2    ,     2
```

This includes one partition of size one, four of size two, five of size three, three of size four, and one of size five.

Question 1.3.13

Below is the square ice configuration for this matrix:



Question 2.1.6

If we use dice in which one side has a 1, two sides have two, and three sides have 3, find the generating function for the number of ways rolling n using k dice.

Let's consider this combinatorially first. We want to think about how these dice are constructed. If we have a die with one of each number (1-6), then what our our options to sum to n using one die? Well, we only have have one way to get to each number, namely the face of the die that contains the corresponding number. This means the generating function for this simple example would be

$$q + q^2 + q^3 + q^4 + q^5 + q^6.$$

Now, if we modify this die to have the sides as above, how does our generating function change? We can see that there are different ways achieve these sums, but we can't actually sum to the numbers 4-6 with just one die. Our new generating function would be

$$q + 2q^2 + 3q^3.$$

The coefficients of each q^n correspond to the number of ways to sum to n.

Let's expand with this modified die into having more dice. How does our generating function differ? We suddenly have an product between each individual die, with each term being the generating function from above. For k dice, we get the product:

$$\prod_{n=1}^{k}(q + 2q^2 + 3q^3)$$

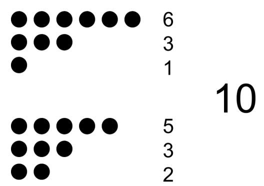
The expansion from this product gives us the coefficients that correspond to the amount of ways to sum to n using k dice. It follows that these generating functions are slightly modified, meaning that we don't actually have any $(1 + q...)$ terms, because if we use exactly k dice, the lower limit of the number we can sum to is exactly k.

Question 2.1.7

Write down all the partitions of 6. Write down all the partitions of 10 into distinct parts. Write down all the partitions of 10 into odd parts.

The partitions of 6 can be broken down into partitions with exactly k parts. The partition with exactly 1 part is 6. For two parts, we have 3 partitions, $5+1, 4+2, 3+3$. There are 3 partitions with exactly 3 parts, they are $4+1+1, 3+2+1, 2+2+2$. Four parts include $3+1+1+1, 2+2+1+1$. With 5 parts, we have $2+1+1+1+1$. Finally, there is only one partition with exactly 6 parts, namely $1+1+1+1+1+1$. This means there are a total of 11 partitions of 6.

These partitions can be represented as Ferrers graphs. The below picture denotes the partition of 10 (6,3,1) which corresponds to six elements in the first row, three in the second, and one in the third row.



10 - Distinct Parts (10)
10
9+1, 8+2, 7+3, 6+4
7+2+1, 6+3+1, 5+4+1, 5+3+2
4+3+2+1

10 - Odd Parts (10)
9+1, 7+3, 5+5
7+1+1+1, 5+3+1+1, 3+3+3+1
5+1+1+1+1+1,3+3+1+1+1+1
3+1+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1+1

Question 2.1.12

Proof: The polynomial form of the equation $(1-q)(1-q^2)(1-q^3)...$ will equal to $\sum_{k\in\mathbf{N}} aq^k, a \in \mathbf{Z}$. Let us begin by identifying the values of a, and r for each term.

Each term by value of k can be separated into sum of the the multiples of the $-q^i$s from some number of terms, and the 1s from some number of terms. For example, the term $k = 0$ can only be obtained by selecting only 1s from every term.

Since each of these multiples of q^is must add up to the same value of k, their numeracy must be equal to the number of partitions of k. The sign of each term of the sum will depend on the number of $-q^i$s, which can be rewritten $(-1)^1 q^i$. Thus for each value of k we have $aq^k = \sum_{\pi\in P(k)} (-1)^{|\pi|} q^k$ where $P(k)$ is the set of all partitions of k.

Since there will be one term in our final summation for each value of k, we have $\sum_{k\in\mathbf{N}} \sum_{\pi\in P(k)} (-1)^{|\pi|} q^k$ Which can be simplified as $\sum_{\pi\in\mathbf{P_D}} (-1)^{L(\pi))} q^{|\pi|}$.

Which is what we wanted to show.

Q.E.D

Question 2.1.13

Let P be the set of all partitions. Prove that

$$\frac{1}{(1-tq)(1-tq^2)(1-tq^3)...} = \sum_{\pi \in P} t^{L(\pi)} q^{|\pi|}$$

A combinatorial proof will suffice to show that how we construct a single partition results in the sum on the right side. To construct a single partition, we choose which parts we want to put in it. For instance, a partition of 25 might be $24+1$, resulting in our generating function having us select a tq^24 and a tq, with the rest being 1. This gives us a singular result on the right side which combines to be $t^2 \cdot q^25$. This is one single partition.

For all partitions, we want the ability to take any combination of items from the left. This means we can take any arbitrary collection of 1's and $tq^{i}{'s}$ from the left side. Each item we take contributes one for the length, the singular t in each term, and contributes q^i for the sum of a certain partition. The infinite product of all $(1-q^i)'s$ give us exactly the ability to pick any arbitrary combination of items on the left, meaning for any partition P, we can construct it by using the corresponding items from the left. This gives us the sum over all partitions $\sum_{\pi \in P} t^{L(\pi)} q^{|\pi|}$ on the right side, making this sum encompass all partitions.

Question 2.1.14

Evaluate the ModProd Function for various values of n and conjecture a summation formula.

Question 2.1.14

The following question tasks us with evaluating the ModProd method with various values of n and conjecturing a summation formula for the product of (1-q^i) for i>=1 and i not 2 (mod 4).

```
In[25]:= ModProd[l_, m_, n_] := Series[Product[Product[1 - q^(l[[i]] + m*t), {i, Length[l]}], {t, 0, n}], {q, 0, n}]

In[26]:= ModProd[{1, 3, 4}, 4, 10]
```

Out[26]= $1 - q - q^3 + q^6 + q^{10} + O[q]^{11}$

```
In[27]:= ModProd[{1, 3, 4}, 4, 50]
```

Out[27]= $1 - q - q^3 + q^6 + q^{10} - q^{15} - q^{21} + q^{28} + q^{36} - q^{45} + O[q]^{51}$

```
In[28]:= ModProd[{1, 3, 4}, 4, 100]
```

Out[28]= $1 - q - q^3 + q^6 + q^{10} - q^{15} - q^{21} + q^{28} + q^{36} - q^{45} - q^{55} + q^{66} + q^{78} - q^{91} + O[q]^{101}$

```
In[29]:= ModProd[{1, 3, 4}, 4, 250]
```

Out[29]= $1 - q - q^3 + q^6 + q^{10} - q^{15} - q^{21} + q^{28} + q^{36} - q^{45} - q^{55} + q^{66} + q^{78} - q^{91} - q^{105} + q^{120} + q^{136} - q^{153} - q^{171} + q^{190} + q^{210} - q^{231} + O[q]^{251}$

Given the evaluation for the ModProd method for n in {10,50,100,250}, the pattern reveals itself to the reader. If we look at each q^i, we can begin to see that the power on each term is a term corresponding to the sum of the first n integers and the sign of the term is based on the parity of the power.

The above output from the ModProd function gives the following conjectured formula:

$$\prod_{i\geq 1, i\neq 2(mod 4)} (1-q^i) = \sum_{k\geq 1} (-1)^{\frac{k(k-1)}{2}} q^{\frac{k(k-1)}{2}}$$

The following formula will produce the same results for any set of residue classes mod $4 \cdot k$ so long as we include all multiples of $1, 3, 4$ less than $4k$ within our set of residues.

Question 2.1.15

Find other sets of residue classes for other moduli for which you can conjecture summation formulae.

The following result is quite amazing, but also somewhat confusing. We are looking at the residue classes $[1, 2 \pmod 2]$ This is somewhat confusing as the residue class 2 (mod 2) is equal to 0 (mod 2), but the output differs within Mathematica.

The ModProd evaluation below looks at the evaluation of the terms of (1-q^j) where j is in the residue classes 1 and 2 (mod 2). The following result is quite astounding!

```
In[40]:= ModProd[{1, 2}, 2, 250]
```

Out[40]= $1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} - q^{77} + q^{92} + q^{100} - q^{117} - q^{126} + q^{145} + q^{155} - q^{176} - q^{187} + q^{210} + q^{222} - q^{247} + O[q]^{251}$

As a result of the above evaluation, we can actually conjecture a summation formula for $(1-q^j)$ where j is in $1, 2 \pmod 2$.

$$\prod_{j\geq 1,\ j\in 1,2 \pmod 2} (1-q^j) = 1 + \sum_{k\geq 1} (-1)^k [q^{\frac{k(3k-1)}{2}} + q^{\frac{(-k)((-3k)-1)}{2}}]$$

The above formula is the formula for the pentagonal numbers, which appears in this very section. Furthermore, any full set of residues in any modulus will produce the same result.

After exploring other sets of moduli and residue classes, an interesting pattern occurred. Looking at the residue $n \pmod{n}$ gave an extraordinary summation formula:

$$Q_n = 1 + \sum_{k=1}^{\infty} (-1)^k (q^{n[P_1]} + q^{n[P_{-1}]})$$

where P_k, P_{-k} are the pentagonal numbers corresponding to substituting $\pm k$ into the formula for the pentagonal numbers.

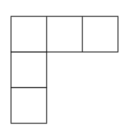
This means that for a given modulus n, the powers on each monomial will be a multiple of n and the corresponding pentagonal number, when we have only the least residue n.

Question 2.2.1

We need to prove that the number of self-conjugate partitions of n is the same as the number of partitions of n into distinct odd parts.

To do this we will need to find a bijection between the Ferrers diagrams of self-conjugate partitions and partitions of the same value with distinct odd parts.

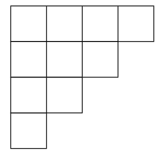
The characteristic of self-conjugate partitions in a Ferrers graph is that flipping the diagram across the main diagonal keeps the diagram the same. This is visible in the Ferrers graph of the partition of 5, $3, 1, 1$ below.



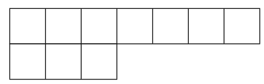
For each "layer" of such a diagram, the number of boxes in the l shape will be odd, since there will be equal numbers of vertical and horizontal boxes, and only one between them.

In addition, each of these values will be distinct since if it otherwise, the diagram would be a invalid partition (non weakly descending).

Thus by counting each of these layers we can produce a Ferrers diagram of odd distinct parts. This is demonstrated for the partition of 10 $4, 3, 2, 1$ below.



$\rightarrow$

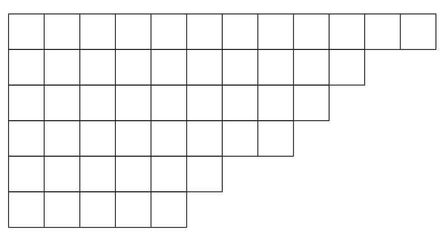


To complete this bijection with a function in the other direction, we can produce a "L" going from top to bottom for each value in a odd distinct ferrers diagram equal to $2k+1$ with the "wings" of size k.

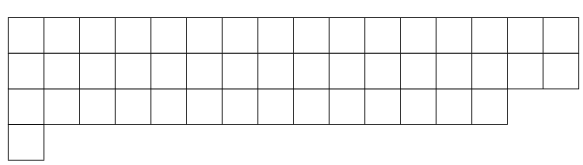
Therefore for any n the number of self conjugate partitions is the same as the number of partitions of n

Question 2.2.3

The partition $50 = 12 + 10 + 9 + 8 + 6 + 5$ has the Ferrers diagram:



We can construct the Ferrers diagram of the result via Sylvesters Bijection by creating a row of 5 boxes on the left, a row of six boxes on the right and curve with a column length of two, and a row length of seven on the left, a curve with a column length of 2 and a row length of eight on the right, a curve with a row length of 9 and a column length of four on the right. After de-centering this gives us the Ferrers diagram:



Which represent the partition 17+17+15+1.

Question 2.2.6

We can restate our bijection for finding a partition of odd parts to use the numerical partition – rewriting each number in the partition as its largest odd factor multiplied by the relevant power of 2, and then creating a new partition only using odd integers each the number of times of their power of 2 multiples added together. In this way, a partition of n made of k unique odd parts $o_1, o_2, ...o_k$ and is transformed by the given one to one correspondence into $(1+2+...)o_1+(1+...)o_2+...(1+...)o_k = 1*o_1+2*o_1+...+1*o_2+...+1*o_3+...1*o_k+....$

Using this new method, since the multiplied coefficient is always a power of 2 for each term, we can turn this new partition back into the original. Thus this is a unique bijection.

Second, we are tasked with proving that this is NOT the same as Sylvesters correspondence. To do this it is helpful to construct a numerical equation for Sylvester's bijection, a difficult task but not impossible.

The first distinct term resulting form Sylvester's bijection will be equal to the number of odd terms greater than or equal to 1, plus half the largest value minus one.

The second distinct term is equal to the number of odd terms in the original sequence greater than or equal to 3, plus $\frac{k-1}{2}-1$ where k is the last term of the sequence.

The nth distinct term of the sequence (if n is odd) is equal to the number of terms greater than or equal to n or greater plus $\frac{k-1}{2}$ where k is the $\lceil\frac{n}{2}$th largest term in the original sequence.

The nth distinct term of the sequence (if n is even) is equal to the number of terms greater than or equal to $n+1$ or greater plus $\frac{k-1}{2}-1$ where k is the $\frac{n}{2}$th largest term in the original sequence.

We can use a counter example to demonstrate that these are different, such as with the partition $50 = 17 + 17 + 15 + 1$ which when transformed by the sylvester bijection gives $50 = 12+10+9+8+6+5$ and when transformed by Glaisher's bijection gives $34+15+1$. Thus these two transformations are distinct.

Question 2.2.10

Let us begin by restating the Jacobi triple product identity:

$$\prod_{i=1}^{\infty}(1+xq^i)(1+x^{-1}q^{i-1})(1-q^i) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}x^n$$

Let us begin by substituting in q^4 for q (and moving from i to j) on the left.
This gives

$$\prod_{j=1}^{\infty}(1+xq^{4j})(1+x^{-1}q^{4j-4})(1-q^{4j}) = \sum_{n=-\infty}^{\infty} q^{2n(n+1)}x^n$$

Next we substitute in $x = -q^-1$ which gives

$$\prod_{j=1}^{\infty}(1-q^{4j-1})(1-q^{4j-3})(1-q^{4j}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}$$

Which is what we wanted to show.

Exercise 2.3.14.

Our first step is to try to transform each of these monotomic triangles into their respective matrices of 0s and 1s. These are below:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Which can then be transformed into Alternating Sign Matrices by either incrementing or decrementing according to the values in M:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Exercise 2.3.15.

The number of -1s in an alternating sign matrix is equal to the number of times in the corresponding Matrix-of-1s-and-0s that a "1" in place i, j is above a "0" in place $i, j+1$. Therefore the number of of -1s in an alternating sign matrix is equal to the number of times that a term in row k of a monotone pyramid is not in row $k+1$.

Exercise 2.3.16.

For a alternating sign matrix to be a permutation matrix, if there is a 1 in location (i, j) of the corresponding matrix of 0s and 1s, then there will be a 1 in location $(i, j + 1)$.
Therefore for such an ASM, any entry in row k of a monotone number triangle, must also be in row $k + 1$.

Exercise 2.4.7.

We can expand this matrix as below:

$$\begin{bmatrix} \frac{(x_1)!}{x_1!} & \frac{(x_1+1)!}{x_1!} & \frac{(x_1+2)!}{x_2!2} & \frac{(x_1+3)!}{x_2!3!} & \cdots & \frac{(x_1+n)!}{x_2!3!} \\ \frac{(x_2)!}{x_1!} & \frac{(x_2+1)!}{x_2!} & \frac{(x_2+2)!}{x_2!2} & \frac{(x_2+3)!}{x_2!3!} & \cdots & \frac{(x_2+n)!}{x_2!3!} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \end{bmatrix}$$

After simplifying, we see that the columns differ only in which variable x_i they use:

$$\begin{bmatrix} 1 & x_1+1 & \frac{(x_1+1)(x_1+2)}{2} & \frac{(x_1+1)(x_1+2)(x_1+3)}{3!} & \cdots & \frac{(x_1+1)(x_1+2)...(x_1+n-1)}{(n-1)!} \\ 1 & x_2+1 & \frac{(x_2+1)(x_2+2)}{2} & \frac{(x_2+1)(x_2+2)(x_2+3)}{3!} & \cdots & \frac{(x_2+1)(x_2+2)...(x_2+n-1)}{(n-1)!} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \end{bmatrix}$$

We note that for every row we can move $\frac{1}{1}, \frac{1}{2}, ..., \frac{1}{i}$ outside of the matrix, where i is the column-number. This results in the matrix below:

$$\frac{1}{2^{(n-2)}3^{(n-3)}4^{(n-4)}...(n-1)} \begin{bmatrix} 1 & x_1+1 & (x_1+1)(x_1+2) & ... & (x_1+1)(x_1+2)...(x_1+n-1) \\ 1 & x_2+1 & (x_2+1)(x_2+2) & ... & (x_2+1)(x_2+2)...(x_2+n-1) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

Each entry in this matrix A_{ij} will be a polynomial of x_j with degree $i - 1$.
We can remove all the degree 0 values from every column except for the first, by subtracting

the first column multiplied by the correct value from every other column.
This will leave our second column as $x_1, x_2, ..., x_n$.
Then we can remove all of the degree 1 values in a like manner by subtracting multiples of column 2, then all multiples of degree 2 using column 3, all the way up to column n-1.
This will not change the determinant of the matrix by properties of column operations, giving us:

$$\frac{1}{2^{(n-2)}3^{(n-3)}4^{(n-4)}...(n-1)} \cdot \begin{bmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & x_n^{n-1} \end{bmatrix}$$

Which from equation (2.25), we know to have determinant

$$\frac{1}{2^{(n-2)}3^{(n-3)}4^{(n-4)}...(n-1)} \prod_{1\leq i<j\leq n} (x_i - x_j).$$

Exercise 2.4.9

Let us begin by multiplying both sides of this equation by $\prod_{i,j=1}^{n}(1 - x_i x_j)$.

This gives us

$$\prod(1 - x_i y_j)det(1/(1 - x_i y_j)) = \prod_{1<i\leq j<n}.$$

We are given that the left side of this equation is an alternating polynomial in both the x and y variable, of degree n-1.
Then we know that

$$\frac{\prod(1 - x_i y_j)det(1/(1 - x_i y_j))}{\prod_{1<i\leq j<n}(x_i - x_j)(y_i - y_j)}$$

is a symmetric polynomial. But since the numerator is already an alternating polynomial, then this fraction comes out to 1.
Thus we have

$$\prod(1 - x_i y_j)\ \det(1/(1 - x_i y_j)) = \prod_{1<i\leq j<n}$$

which is what we wanted to show.

Q.E.D

Solutions

Exercise 3.1.1.

Find a one-to-one correspondence between partitions into at most m parts, each less than or equal to n, and partitions into at most n parts, each less than or equal to m. This proves combinatorially that

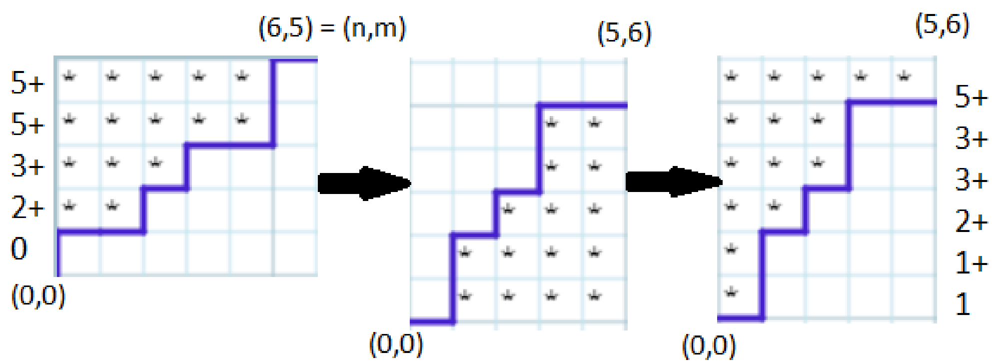
$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{bmatrix} m+n \\ n \end{bmatrix}.$$

So Proposition 3.1 says, the total number of partitions into at most m parts with each part less than or equal to n is equal to $\binom{m+n}{m}$. Furthermore, $\binom{m+n}{m} = f_{m,n}(1)$, where $f_{m,n}(q)$ $= \begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n \\ m \end{bmatrix}$ by notation.

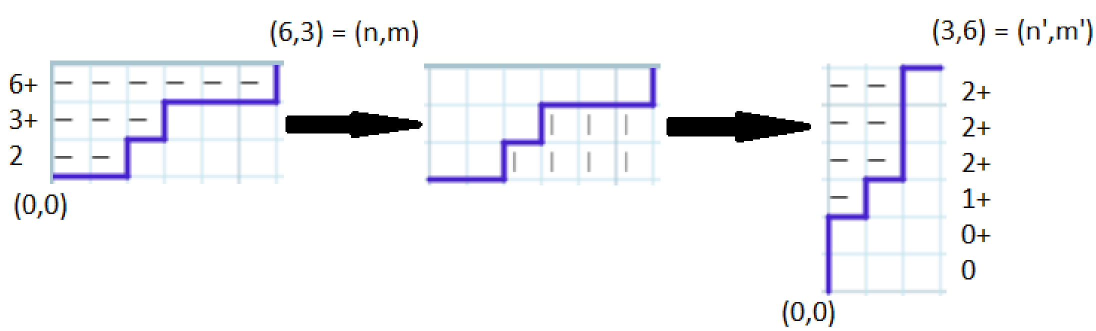
Also from the book: Lattice paths from (0,0) to (6,5) produce partitions with at most five parts, and each part will be less than or equal to six.

The bijection (1-1 correspondence) I can describe visually: Draw the parts in a partition as a lattice path, and then reflect everything across the line connecting (0,0) to (n,m). The new partition will then be evident.

Based on these two examples, there is a one-to-one correspondence.



For example, we can take 5+5+3+2 to 5+3+3+2+1+1, and



we can take 6+3+2 to 2+2+2+1.

This function in Python shows the corresponding partition for any given partition of at most m parts (each less than or equal to n). The corresponding partition is one into at most n parts, each less than or equal to m.

```
 # Exercise 3.1.1.
from array import *
print "One-to-one Correspondence Test\n"

def f(A,n,m):  # (path from 0,0 to n,m)
    Anew = [0] * n
    for i in range(n): Anew[i] = m
    for x in range(0, len(A)):
        # Given a part value, the arrayValues variable
        # shows how many and where
        if x == 0:
            temp = 0
            for k in range(0, x+1):
                if k != len(A)-1:
                    temp += A[k] - A[k+1]
                else:
                    temp += A[k]
            arrayValues = range(0, temp)

        elif x - 1 != len(A) - 1:
            temp = 0
            for k in range(0, x):
                if k != len(A)-1:
                    temp += A[k] - A[k+1]
                else:
                    temp += A[k]
            temp2 = 0

            for k in range(0, x+1):
                if k != len(A)-1:
                    temp2 += A[k] - A[k+1]
                else:
                    temp2 += A[k]

            arrayValues = range(temp, temp2)
```

```
        for i in reversed(range(A[x],n+1)):
            numberPartsLengthI = A.count(i)
            if i != 0:
                for w in arrayValues:
                    Anew[w] -= numberPartsLengthI
    print(sorted(Anew, reverse = True))

f([5, 3, 3, 2, 1, 1], n = 5, m = 6)
```

Question 3.1.2

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3.1.2. Verify equation (3.4).

Specifically, equation (3.4) states that $\prod_{i=1}^{m} \frac{1-q^{n+i}}{1-q^{i}} = \prod_{i=1}^{m} \frac{1-q^{n-1+i}}{1-q^{i}} + q^{n} \prod_{i=1}^{m-1} \frac{1-q^{n+i}}{1-q^{i}}$

$$\left(\frac{1-q^{n+1}}{1-q^{1}}\right)\left(\frac{1-q^{n+2}}{1-q^{2}}\right)\cdots\left(\frac{1-q^{n+m}}{1-q^{m}}\right) = \left(\frac{1-q^{n-1+1}}{1-q^{1}}\right)\left(\frac{1-q^{n-1+2}}{1-q^{2}}\right)\cdots\left(\frac{1-q^{n-1+m}}{1-q^{m}}\right) +$$

$$q^{n}\cdot\left(\frac{1-q^{n+1}}{1-q^{1}}\right)\left(\frac{1-q^{n+2}}{1-q^{2}}\right)\cdots\left(\frac{1-q^{n+m-1}}{1-q^{m-1}}\right)$$

expand

RHS

$$=\left(\frac{1-q^{n}}{1-q}\right)\left(\frac{1-q^{n+1}}{1-q^{2}}\right)\cdots\left(\frac{1-q^{n+m-2}}{1-q^{m-1}}\right)\left(\frac{1-q^{n+m-1}}{1-q^{m}}\right) +$$

$$q^{n}\left(\frac{1-q^{n+1}}{1-q}\right)\left(\frac{1-q^{n+2}}{1-q^{2}}\right)\cdots\left(\frac{1-q^{n+m-1}}{1-q^{m-1}}\right)$$

$$= \frac{(1-q^{n})(1-q^{n+1})\cdots(1-q^{n+m-2})(1-q^{n+m-1})}{(1-q)(1-q^{2})\cdots(1-q^{m-1})(1-q^{m})} + \frac{q^{n}(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})}{(1-q)(1-q^{2})\cdots(1-q^{m-1})}$$

$$= \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-2})(1-q^{n+m-1})}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)} + \frac{q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m)}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)}$$

$$= \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-2})(1-q^{n+m-1}) + q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m)}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)}$$

So we have established that LHS $= \dots =$ RHS :

$$\frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m})}{(1-q^1)(1-q^2)\cdots(1-q^m)} = \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-1}) + q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m)}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Question 3.1.3

That is, (eliminating the denominator) $\boxed{13}$

$$(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^{n+m}) = (1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-1}) + q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m).$$

Now divide both sides by the circled material:

$$\begin{aligned} 1-q^{n+m} &= (1-q^n)+q^n(1-q^m) \\ &= 1-q^n+q^n-q^{n+m} \\ &= 1-q^{n+m}. \quad \square \end{aligned}$$

Question 3.1.4

3.1.3. This exercise and those up to and including exercise 3.1.10 outline Gauss's evaluation of the Gaussian sum $G(\alpha) = \sum \alpha^{j^2}$ where α is a primitive kth root of unity such as $e^{2\pi i/k}$ and the summation is taken over all residue classes, j, modulo k, where k is odd. This evaluation was Gauss's original reason for defining Gaussian polynomials.

Define

$$f(q,m) = 1 - \begin{bmatrix} m \\ 1 \end{bmatrix} + \begin{bmatrix} m \\ 2 \end{bmatrix} - \cdots + (-1)^m \begin{bmatrix} m \\ m \end{bmatrix}.$$

Use the recursive formula, $\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j \end{bmatrix} + q^{m-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$, to prove that

$$f(q,m) = (1-q^{m-1}) f(q,m-2).$$

We're given the companion formula $\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \underline{\begin{bmatrix} m-1 \\ j \end{bmatrix}}$, so

$$\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \cdot \left(\begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1-j} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$ by the first recursive formula

$$= \underline{\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix}.$$ Now, the companion formula allows us to say that this $= \underline{\begin{bmatrix} m-2 \\ j-2 \end{bmatrix} + q^{j-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix}} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix}$.

Substituting this, $f(q,m) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}$

$$= \sum_{j=0}^{m} (-1)^j \cdot \left(\begin{bmatrix} m-2 \\ j-2 \end{bmatrix} + q^{j-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$

$$= \left(\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m-2 \\ j-2 \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^{j-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$

$$= \left(\sum_{j=0}^{m-2} (-1)^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \underbrace{\left(\sum_{j=0}^{m-2} (-1)^{j+1} q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \left(\sum_{j=0}^{m-2} (-1)^j q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right)}_{\text{as Matthew said, these cancel out.}} + \left(\sum_{j=0}^{m-2} (-1)^{j+1} q^{m-1} \begin{bmatrix} m-2 \\ j \end{bmatrix} \right)$$

$$= f(q,m-2) + 0 + (-1) q^{m-1} f(q,m-2)$$

$$= (1-q^{m-1}) f(q,m-2).$$ So $f(q,m) = (1-q^{m-1}) f(q,m-2)$.

Solutions

Question 3.1.5

3.1.4. Let k be odd. Show that $\dfrac{1-\alpha^{k-j}}{1-\alpha^{j}} = -\alpha^{-j}$.

Use this to prove that $\begin{bmatrix} k-1 \\ j \end{bmatrix}_\alpha = (-1)^j \alpha^{-j(j+1)/2}$,

and therefore $f(\alpha, k-1) = \displaystyle\sum_{j=0}^{k-1} \alpha^{-j(j+1)/2}$.

Let k be odd. $\alpha = e^{2\pi i h/k}$ because this is the general primitive kth root of unity.

$$\frac{1-\alpha^{k-j}}{1-\alpha^{j}} = \frac{1-(e^{2\pi i h/k})^{k-j}}{1-(e^{2\pi i h/k})^{j}} = \frac{1-e^{2\pi i h - 2\pi i h j/k}}{1-e^{2\pi i h j/k}}$$

$$= \frac{1-e^{-2\pi i h j/k}e^{2\pi i h}}{1-e^{2\pi i h j/k}} = \frac{-e^{-2\pi i h j/k}\cdot(e^{2\pi i h}-e^{2\pi i h j/k})}{1-e^{2\pi i h j/k}}$$

$$= \frac{-e^{-2\pi i h j/k}(1-e^{2\pi i h j/k})}{(1-e^{2\pi i h j/k})}$$ because the primitive 1st root of unity is always 1

$$= -(e^{2\pi i h/k})^{-j} = -\alpha^{-j}.$$

Question 3.1.6

3.1.4. Having shown $\frac{1-\alpha^{k-j}}{1-\alpha^j} = -\alpha^{-j}$,

$$\begin{bmatrix} k-1 \\ j \end{bmatrix}_\alpha = \begin{bmatrix} j+(k-1-j) \\ j \end{bmatrix}_\alpha = \frac{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{k-1})}{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^j)(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{k-1-j})}$$

$$= \left(\frac{1-\alpha^{k-1}}{1-\alpha^1}\right)\left(\frac{1-\alpha^{k-2}}{1-\alpha^2}\right)\cdots\left(\frac{1-\alpha^{k-j}}{1-\alpha^j}\right)\cdot\frac{(1-\alpha^{k-j-1})\cdots(1-\alpha^1)}{(1-\alpha)\cdots(1-\alpha^{k-1-j})} \qquad (3.1)$$

$$= (-\alpha^{-1})(-\alpha^{-2})\cdots(-\alpha^{-j})\cdot\frac{\cancel{(1-\alpha^{k-j-1})}(1-\alpha^{k-j-2})\cdots\cancel{(1-\alpha^1)}}{\cancel{(1-\alpha^{k-j-1})}\cdots(1-\alpha^2)\cancel{(1-\alpha)}}$$

based on the statement we showed

$= (-1)^j\alpha^{-j(j+1)/2}\cdot 1$, and therefore

$$F(\alpha,k-1) = \sum_{n=0}^{k-1}(-1)^n\begin{bmatrix} k-1 \\ n \end{bmatrix} = \sum_{j=0}^{k-1}(-1)^j(-1)^j\alpha^{-j(j+1)/2}$$

$$= \sum_{j=0}^{k-1}\alpha^{-j(j+1)/2}.$$

3.1.5. Use the fact that if k is odd and α is a primitive kth root of unity, then so is α^{-2} to prove that

$$f(\alpha^{-2}, k-1) = \alpha^{-[(k+1)/2]^2} \sum_{j=0}^{k-1} \alpha^{[j+(k+1)/2]^2}$$
$$= \alpha^{-[(k+1)/2]^2} G(\alpha).$$

Since α^{-2} is also a primitive kth root of unity, $f(\alpha^{-2}, k-1) = \sum_{j=0}^{k-1} (\alpha^{-2})^{-j(j+1)/2}$

$$= \sum_{j=0}^{k-1} \alpha^{2j(j+1)/2} = \sum_{j=0}^{k-1} \alpha^{j(j+1)} = \sum_{j=0}^{k-1} \alpha^{j^2+j} \cdot 1$$

$$= \sum_{j=0}^{k-1} \alpha^{j^2+j} e^{2\pi i jk}$$ because the only primitive 1st root of unity is equal to 1

$$= \sum_{j=0}^{k-1} \alpha^{j^2+j} (e^{2\pi i h/k})^{jk} = \sum_{j=0}^{k-1} \alpha^{j^2+j} \alpha^{jk} = \sum_{j=0}^{k-1} \alpha^{j^2+j(k+1)}$$

$$= \sum_{j=0}^{k-1} \alpha^{j^2 + \frac{j(k+1)}{2} + \frac{j(k+1)}{2} + \frac{(k+1)^2}{4} - \frac{(k+1)^2}{4}} = \sum_{j=0}^{k-1} \alpha^{\left(j+\frac{k+1}{2}\right)^2 - \left(\frac{k+1}{2}\right)^2}$$

$$= \sum_{j=0}^{k-1} \alpha^{[j+(k+1)/2]^2} \alpha^{-[(k+1)/2]^2} = \underline{\alpha^{-[(k+1)/2]^2} \sum_{j=0}^{k-1} \alpha^{[j+(k+1)/2]^2}}$$

Question 3.1.7

3.1.6. Use equation (3.7) to prove that
$$G(\alpha)=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)}). \quad (3.8)$$

From equation (3.7),
$$f(q,m)=\begin{cases}0, & \text{if } m \text{ is odd},\\ (1-q)(1-q^3)\cdots(1-q^{m-1}), & \text{if } m \text{ is even}.\end{cases}$$

Now, $f(\alpha^{-2},k-1)=\alpha^{-[(k+1)/2]^2}G(\alpha)$ based on exercise 3.1.5 and also
$$f(\alpha^{-2},k-1)=\begin{cases}0, & \text{if } k \text{ is even},\\ (1-\alpha^{-2})(1-(\alpha^{-2})^3)\cdots(1-(\alpha^{-2})^{k-2}), & \text{if } k \text{ is odd}.\end{cases}$$

So when k is odd, this becomes $(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})$. Since 3.1.5 $\Rightarrow$ $G(\alpha)=f(\alpha^{-2},k-1)\,\alpha^{[(k+1)/2]^2}$, $G(\alpha)=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\cdot\alpha^{[(k+1)/2]^2}$

$=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\alpha^1\alpha^3\cdots\alpha^k$ because $1+3+\dots+k=$ (the number of odd integers between 1 and k inclusive)$^2=\left(\frac{k+1}{2}\right)^2$.

Then $G(\alpha)=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\alpha^1\alpha^3\cdots\alpha^{k-2}\cdot(e^{2\pi ih/k})^k$, and because $\alpha^k=(e^{2\pi ih/k})^k=e^{2\pi ih/1}$, α^k is a 1st root of unity which is always equal to 1.

$$\Rightarrow G(\alpha)=\alpha^1(1-\alpha^{-2})\alpha^3(1-\alpha^{-6})\cdots\alpha^{k-2}(1-\alpha^{-2k+4})\cdot 1$$
$$=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)}). \quad \square$$

$\boxed{3.1.7.}$ Use the fact that $\alpha^{k-j} - \alpha^{-(k-j)} = -(\alpha^j - \alpha^{-j})$ to rewrite equation (3.8) as

$$G(\alpha) = (-1)^{(k-1)/2}(\alpha^2 - \alpha^{-2})(\alpha^4 - \alpha^{-4})\cdots(\alpha^{k-1} - \alpha^{-(k-1)}). \quad (3.9)$$

(3.8) says $G(\alpha) = (\alpha - \alpha^{-1})(\alpha^3 - \alpha^{-3})\cdots(\alpha^{k-2} - \alpha^{-(k-2)})$

$= (\alpha^{k-(k-1)} - \alpha^{-(k-(k-1))})(\alpha^{k-(k-3)} - \alpha^{-(k-(k-3))})$

$\cdots(\alpha^{k-2} - \alpha^{-(k-2)})$, which based on the fact

$= (-(\alpha^{k-1} - \alpha^{-(k-1)}))(-(\alpha^{k-3} - \alpha^{-(k-3)}))\cdots(-(\alpha^2 - \alpha^{-2}))$

$= (-1)^{\frac{k-1}{2}}(\alpha^{k-1} - \alpha^{-(k-1)})(\alpha^{k-3} - \alpha^{-(k-3)})\cdots(\alpha^2 - \alpha^{-2})$ because

(-1) is implicated $(k-1)/2$ times

$= (-1)^{(k-1)/2}(\alpha^2 - \alpha^{-2})(\alpha^4 - \alpha^{-4})\cdots(\alpha^{k-1} - \alpha^{-(k-1)})$. □

Question 3.1.8

3.1.8 Combine equations (3.8) and (3.9) to show that

$$G(\alpha)^2 = (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j}). \quad (3.10)$$

Show that
$$\prod_{j=1}^{k-1}(x-\alpha^{-2j}) = \frac{x^k-1}{x-1} = 1+x+x^2+\cdots+x^{k-1},$$
and therefore $G(\alpha)^2 = (-1)^{(k-1)/2}k$. (3.11)

For the first result, we can multiply (3.8)·(3.9): So $G(\alpha)^2$

$$= G(\alpha)\cdot G(\alpha) = (\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)}).$$
$$(-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)}).$$
$$= (-1)^{(k-1)/2}\cdot\alpha(1-\alpha^{-2})\alpha^3(1-\alpha^{-6})\cdots\alpha^{k-2}(1-\alpha^{-2(k-2)})\alpha^2(1-\alpha^{-4})\alpha^4(1-\alpha^{-8})$$
$$\cdots\alpha^{k-1}(1-\alpha^{-2(k-1)})$$
$$= \alpha\alpha^3\cdots\alpha^{k-2}\alpha^2\alpha^4\cdots\alpha^{k-1}(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})(1-\alpha^{-4})(1-\alpha^{-8})$$
$$\cdots(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$$
$$= \alpha\alpha^2\alpha^3\alpha^4\cdots\alpha^{k-2}\alpha^{k-1}(1-\alpha^{-2})(1-\alpha^{-4})(1-\alpha^{-6})(1-\alpha^{-8})\cdots(1-\alpha^{-2k+4})(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$$
$$= \alpha^{k(k-1)/2}(1-\alpha^{-2})(1-\alpha^{-4})\cdots(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$$
$$= \alpha^{k(k-1)/2}\cdot(1-\alpha^{-2})(1-\alpha^{-4})\cdots(1-\alpha^{-2(k-1)})(-1)^{(k-1)/2}$$
$$= (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j}).$$

For the second result, because given any primitive kth root of unity $\alpha = e^{2\pi i/k}$ with k odd, α^{-2} is also a primitive root of unity (from exercise 3.1.5.), and we know by definition that the roots of $x^k - 1$ are all kth roots of unity, $(\alpha^{-2})^j$, $0 \le j \le k-1$,

$$x^k - 1 = (x - (\alpha^{-2})^0)(x - (\alpha^{-2})^1) \dots (x - (\alpha^{-2})^{k-1})$$

$$= \prod_{j=0}^{k-1} (x - (\alpha^{-2})^j) = \prod_{j=0}^{k-1} (x - \alpha^{-2j}) = (x - \alpha^0) \prod_{j=1}^{k-1} (x - \alpha^{-2j})$$

$$\Rightarrow (x^k - 1)/(x-1) = \prod_{j=1}^{k-1} (x - \alpha^{-2j}).$$ Also,

$$\frac{x^k - 1}{x-1} = \frac{x + x^2 + x^3 + \dots + x^k - 1 - x - x^2 - \dots - x^{k-1}}{x-1} = \frac{(x-1)(1 + x + x^2 + \dots + x^{k-1})}{x-1}$$

$$= 1 + x + x^2 + \dots + x^{k-1}$$

And therefore, since $\alpha^{k(k-1)/2} = (e^{2\pi i/k})^{k(k-1)/2} = (e^{2\pi i})^{(k-1)/2}$

$$= (\cos(2\pi) + i\sin(2\pi))^{(k-1)/2} = (1+0)^{(k-1)/2} = 1,$$

$$G(\alpha)^2 = (-1)^{(k-1)/2} \alpha^{k(k-1)/2} \prod_{j=1}^{k-1} (1 - \alpha^{-2j})$$

$$= (-1)^{(k-1)/2} \cdot 1 \cdot (1^0 + 1^1 + 1^2 + \dots + 1^{k-1}) = (-1)^{(k-1)/2} \cdot (k)$$

$$= (-1)^{(k-1)/2} k.$$

Question 3.1.9

3.1.9. Use equation (3.9) to prove that

$$G(e^{2\pi i/k}) = (-1)^{(k-1)/2}\prod_{j=1}^{(k-1)/2}(e^{4\pi i j/k} - e^{-4\pi i j/k})$$

$$= (-1)^{(k-1)/2}(2i)^{(k-1)/2}\prod_{j=1}^{(k-1)/2}\sin\frac{4\pi j}{k}. \quad (3.12)$$

The first result is shown through substitution:

(3.9) $G(\alpha) = (-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)})$

$$\Rightarrow G(e^{2\pi i/k}) = (-1)^{(k-1)/2}((e^{2\pi i/k})^2-(e^{2\pi i/k})^{-2})\cdot((e^{2\pi i/k})^4-(e^{2\pi i/k})^{-4})\cdots((e^{2\pi i/k})^{k-1}-(e^{2\pi i/k})^{-(k-1)})$$

$$= (-1)^{(k-1)/2}\cdot(e^{4\pi i/k}-e^{-4\pi i/k})(e^{8\pi i/k}-e^{-8\pi i/k})\cdots(e^{2\pi i(k-1)/k}-e^{-2\pi i(k-1)/k})$$

$$= (-1)^{(k-1)/2}\cdot(e^{4\pi i\cdot 1/k}-e^{-4\pi i\cdot 1/k})(e^{4\pi i\cdot 2/k}-e^{-4\pi i\cdot 2/k})\cdots(e^{4\pi i(\frac{k-1}{2})/k}-e^{-4\pi i(\frac{k-1}{2})/k})$$

$$= (-1)^{(k-1)/2}\prod_{j=1}^{(k-1)/2}(e^{4\pi i j/k}-e^{-4\pi i j/k}).$$

Lastly, $e^{4\pi i j/k}-e^{-4\pi i j/k} = \left(\cos\left(\frac{4\pi j}{k}\right)+i\sin\left(\frac{4\pi j}{k}\right)\right) - \left(\cos\left(\frac{-4\pi j}{k}\right)+i\sin\left(\frac{-4\pi j}{k}\right)\right)$

based on Euler's identity, which $= \cos\left(\frac{4\pi j}{k}\right) - \cos\left(\frac{-4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right) - i\sin\left(\frac{-4\pi j}{k}\right)$

$= \cos\left(\frac{4\pi j}{k}\right) - \cos\left(\frac{4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right)$ because $\cos(-a)=\cos(a)$ and $\sin(-a) = -\sin(a)$.

This is equal to $2i\sin\left(\frac{4\pi j}{k}\right)$.

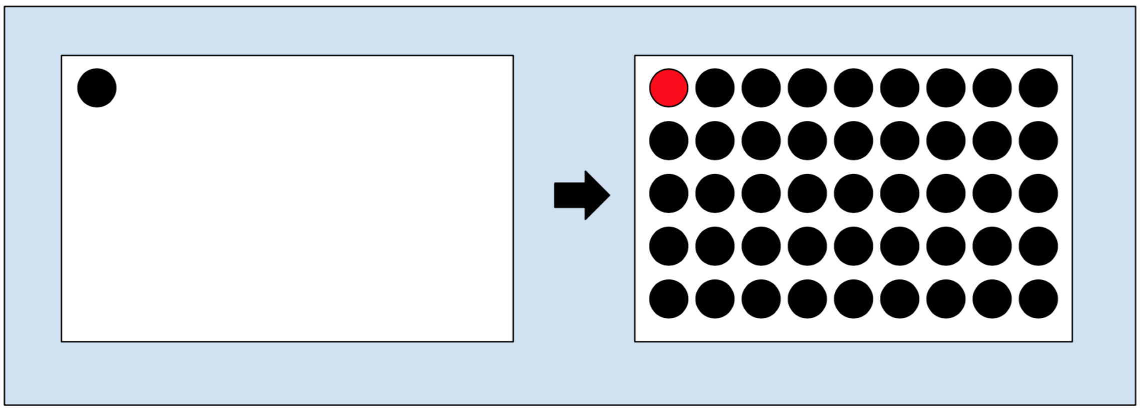
So substituting this, $(-1)^{(k-1)/2}\prod_{j=1}^{(k-1)/2}(e^{4\pi i j/k}-e^{-4\pi i j/k}) = (-1)^{(k-1)/2}\prod_{j=1}^{(k-1)/2} 2i\sin\left(\frac{4\pi j}{k}\right)$

$$= (-1)^{(k-1)/2}(2i)^{(k-1)/2}\prod_{j=1}^{(k-1)/2}\sin\frac{4\pi j}{k}. \quad \square$$

Question 3.1.12

Prove that in the Gaussian polynomial, $\binom{m+n}{n}$, the coefficient of q^i, $0 \le i \le mn/2$, will always be the same as the coefficient of q^{mn-i} by finding a one-to-one correspondence between partitions of i into at most m parts, each of which is less than or equal to n, and partitions of $mn-i$ into at most m parts less than or equal to n.

Proof. To interpret this combinatorially, we want to think about the definition for partitions into at most m parts less than or equal to n. The upper bound on the number of dots in a Ferrers graph will be mn. If we want to look at this for 0 dots in the graph, we only have one way to do that. In the same way, we only have one way for 1 dot as well. A natural one-to-one correspondence between these partitions and a partition of $mn - i$ is the complement of the Ferrers graph in the rectangle mn. For instance, the corresponding graph in $mn - i$ for $i = 1$ is the following:



The one-to-one correspondence is rather natural and the resulting argument stating that the number of partitions with at most m parts, each less than or equal to n will be symmetric, holds. This correspondence maps a partition with size i and transforms it into a partition with $mn - i$ parts. Each transformation is unique, as non-uniqueness would mean two different shapes can transform into the same ending shape, which is not possible geometrically. This holds for $0 \leq i \leq mn/2$. □ □

Question 3.1.19

Interpret the coefficient of x^i in the power series expansion of

$$\frac{1}{(xq;q)_n} = \frac{1}{(1 - xq)(1 - xq^2)...(1 - xq^n)}$$

For each power series expansion, we have a general expansion for $\frac{1}{(1-xq^n)} = (1 + xq^n + x^2q^{2n} + x^3q^{3n} + ...$ This means that when we choose an individual x^kq^{kn}, the coefficient on x^i indicates how many partitions we have with i parts.

When we look for power series from 1 to n, we are choosing partitions with largest part n. The corresponding coefficient on x^i is the number of ways we can construct partitions with i parts with largest part n. The following sum over binomial coefficients,

$$\sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

counts the number of partitions with i parts with largest part less than or equal to n, as we are choosing i parts from $n+i-1$ objects each time. For instance, if we have $n=2$, we can construct partitions with 2 parts, which are $1,1$, $2,1$ and $2,2$. This gives us three partitions, which is the same as $\binom{2+2-1}{2} = 3$.

Question 3.1.21

Merely by computing a few terms of the formula on the left, we see it will have the form $1 + x(q^1 + q^2 + ...) + x^2(q^3 + q^4 + ...)$
For any x^n, the multiplied powers of q must obviously start with $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, but since is is equal to the sum of n terms in the infinite sequence $q + q^2 + q^3 ...$ can have any value greater than that, and with a coefficient equal to the partitions of n with i parts.
This then gives us the equation

$$\sum_{i=0}^{\infty} q^{i(i+1)} 2x^i \prod_{j=0}^{i} \sum_{k=0}^{\infty} q^{kj}$$

Which we can further manipulate to

$$\sum_{i=0}^{\infty} \frac{q^{i(i+1)} 2x^i}{(q:q)_i}$$

Question 3.2.5

Prove that if $n \geq 2$, then

$$\sum_{\sigma \in S_n} (-1)^{I(\sigma)} = 0$$

We will use induction on n to prove this claim. For $n=2$, this is trivially true, as we have 12 and 21, which have inversion number of 0 and 1 respectively, meaning the sum over both is $(-1)^0 + (-1)^1 = 0$.

Assume the claim is true up to and including $n-1$, we want to show that for any resulting addition of n in any of n spots will cancel out with a addition of n in one of the spots.

Given positions $1, 2, ..., n$ to place the new element, the addition to our initial π will contribute $n-i$ inversions to each sequence. This gives a unique way to construct each new permutation. For instance, placing n at the first index will contribute $n-1$ to the inversion number, while the last index will contribute 0. These two are pairs as there is exactly the same amount of sequences which each placed spot of $n-i$ and $i-1$, meaning that they will cancel out and the resulting inversion number will be unchanged, giving the the sum over all permutations of S_n to still equal 0.

Question 3.2.13

Prove that $I(B) \geq 0$ for any alternating sign matrix B.

Consider the construction of any ASM. We know that the row and column sums must add up to 1. This means that any -1 in the matrix must have an element to the right of it which will contribute at least the same amount of inversion number as the -1, meaning that the inversion number of any ASM is bounded below by zero.

Question 3.3.1

Prove that

$$\lim_{r,s,t\to\infty} \prod_{(i,j,k)\in B(r,s,t)} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{l=1}^{\infty} \frac{1}{(1-q^l)^l}$$

We start out with the left hand side going to infinity. This means we have infinite space for our partitions. Let us rewrite this equation in the form $\prod_{l=1}^{\infty} \frac{(1-q^l)^{a_l}}{(1-q^l)^{b_l}}$ where a_l and b_l are equations of q and l.

We know that for any l, a_l is the number of partitions of three parts (i, j, and k) of $l+1$. b_l is the number of partitions of three parts (i, j, and k) of $l+2$.

The number of partitions of three parts of $l+1$ minus the number of partitions of three parts of $l+2$ is l by properties of partitions.

Therefore

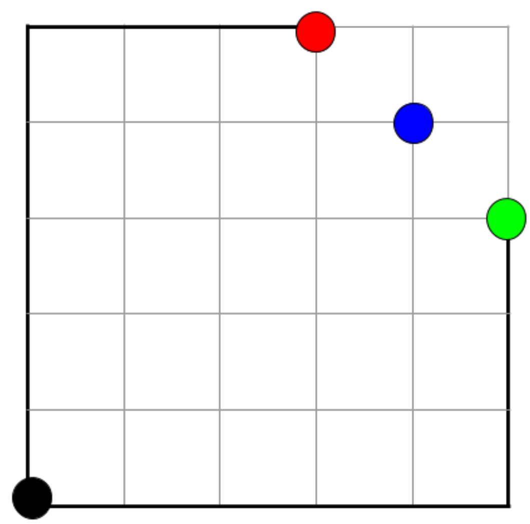
$$\lim_{r,s,t\to\infty} \prod_{(i,j,k)\in B(r,s,t)} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{j=l}^{\infty} \frac{1}{(1-q^l)^l}$$

which is what we wanted to show.
Q.E.D

Question 3.3.2

Prove that the chosen intersection point will always be an intersection of two consecutive paths.

Assume by way of contradiction that the chosen intersection point is not an intersection of two consecutive paths. This means that the rightmost and highest intersection point has the below design:



In the above image, our chosen intersection point was not the intersection of two consecutive paths. It is easy to see that no matter what, since we need a path to end at the blue dot, that lattice path MUST intersect at a point which is higher or to the right of the black dot, and therefore would be the new chosen intersection point according to the definition.

Question 3.4.1

The transposition of two elements in a permutation can be executed by performing an odd number of adjacent transpositions.

Proof. For this problem, we only look at the indices of a permutation rather than the number at a certain point. We will show that the optimal cost of swapping two elements with adjacent transpositions is $|2(i_2 - i_1)| - 1$, where i_2 and i_1 are the corresponding indices of the elements.

The proof that the optimal cost lies within how we swap adjacent elements. We need to move the element at i_2 to the place i_1. This has cost of $|i_2 - i_1|$. In the same way, we need to move i_1 to where i_2 was. This also has cost $|i_2 - i_1|$. At this point, we have total cost of adjacent swaps to be $|2(i_2 - i_1)|$. However, when i_2 is adjacent to i_1, we move them both with the cost of one swap, so we exclude one swap at the very end, giving us a total cost for a swap of two elements of $|2(i_2 - i_1)| - 1$, which is an odd number and never less than or equal to zero, as this problem is not well-defined for $i_2 = i_1$.

□

Question 3.4.5

For a matrix A of size n,

$$\det(A) = \sum_{\sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

.
This means for each of A_k we have $n!$ terms involving the bottom row of this matrix, half positive and half negative.
We can then write $det(A_k)$ as

$$\sum_{\sigma(i) \in S_n} a_{i1}^k \prod F$$

where F is a stand in for the other multiplied terms in each sequence, none of which will be in the bottom row (by the above equation), and which are consistent across A_k. Then $\sum_k^t det(A_k) = \sum_{\sigma(i) \in S_n} a_{i1}^k \prod F$ Which is equal to $\sum_{\sigma(i) \in S_n} \sum_k^t a_{i1}^k \prod F$ which gives us the determinant of the A matrix whose last row is

$$\sum_k^t (a_{n1}^k, a_{n2}^k, ..., a_{nn}^k).$$

Question 3.4.6

We know that $\binom{r-2-j}{j-1}$ is the number of lattice paths from (0,0) to (r-1,j-1).
From chapter 3.4, we know $\binom{r-2+j-k}{j-k}$ is the number of paths from (0,r-j) to (r-2, r-k).
We can shift this, so that starting from (0,0) this is the number of lattice paths to (r-2,j-k).

If we define each path as starting at (0,k), then each goes to (r-2,j).
Finally, we must adjust our indices with $r = r + 1$ and $j = j - 1$, from which we get the sum of paths with k vertical steps before the first horizontal step, ranging from 0 to r-1: $\sum_{k=0}^{r} \binom{r-2+j-k}{j-k}$.
Since the equation for the number of paths from (0,0) to (r-1,j-1) is composed of the number of paths with 0,1,2,...j vertical steps before the first horizontal step, then we have

$$\sum_{k=0}^{r} \binom{r-2+j-k}{j-k} = \binom{r-2-j}{j-1}.$$

Which is what we wanted to show.

Question 3.4.7

We know that $\binom{r-2-j}{j-1}$ is the number of lattice paths from (0,0) to (r-1,j-1).
From chapter 3.4, we know $\binom{r-2+j-k}{j-k}$ is the number of paths from (0,r-j) to (r-2, r-k).
We can shift this, so that starting from (0,0) this is the number of lattice paths to (r-2,j-k).
If we define each path as starting at (0,k), then each goes to (r-2,j).
Finally, we must adjust our indices with $r = r + 1$ and $j = j - 1$, from which we get the sum of paths with k vertical steps before the first horizontal step, ranging from 0 to r-1: $\sum_{k=0}^{r} \binom{r-2+j-k}{j-k}$.
Since the equation for the number of paths from (0,0) to (r-1,j-1) is composed of the number of paths with 0,1,2,...j vertical steps before the first horizontal step, then we have

$$\sum_{k=0}^{r} \binom{r-2+j-k}{j-k} = \binom{r-2-j}{j-1}$$

Which is what we wanted to show.

Question 3.4.9

Show that there is a one-to-one correspondence between descending plane partitions with row leaders $2 \leq a_1 < a_2 < ... < a_m \leq r$ and nests of non-intersecting lattice paths which the ith path goes from $(0, r - a_i)$ to $(a_i, r - 2)$.

The corresponding lattice nest is a nest of paths arranged along the y axis at $x = 0$. Each path starts exactly a_i steps down from the r. This means that a given path has $\binom{a_i + a_i - 2}{a_i}$ choices, where we choose the horizontal steps in this case. We define the steps a path takes to be defined by the row leader. At each spot, we must be at $r - a_{k,i}$ where k is the current index which we are at.

The descending aspect of these plane partitions is sufficient to show that no two paths can cross. If they did cross, this would mean that we have at some given point k, two rows in our partition have equal value. This is not possible within the constraints of descending plane partitions, therefore they must not cross. This is the one-to-one correspondence we find and very similar techniques related to proving theorem 3.6 can be used to express this as a determinant, although it is more complicated as our permutations are slightly different.

Question 3.5.1

Use Dodgson's algorithm to evaluate the determinant of:

$$\begin{pmatrix} 2 & 0 & 1 & 3 \\ -1 & 2 & 1 & -2 \\ 0 & -1 & 1 & 3 \\ 2 & 4 & -3 & 2 \end{pmatrix}.$$

Using Dodgson's, we get the following:

$$A = \begin{pmatrix} 2 & 0 & 1 & 3 \\ -1 & 2 & 1 & -2 \\ 0 & -1 & 1 & 3 \\ 2 & 4 & -3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A' = \begin{pmatrix} 4 & -2 & -5 \\ 1 & 3 & 5 \\ 2 & -1 & 11 \end{pmatrix} \qquad B' = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A'' = \begin{pmatrix} 7 & 5 \\ 7 & 38 \end{pmatrix} \qquad B'' = \begin{pmatrix} 3 \end{pmatrix}$$

$$A''' = \begin{pmatrix} 77 \end{pmatrix} \qquad B''' = (\ )$$

So, using Dodgson's algorithm to get A''' gives us a determinant of 77.

Question 3.5.4

Use the Desnanot-Jacobi adjoint matrix theorem to prove the Vandermonde formula by induction on number of variables.

Proceeding by induction on the number of variables, we set the base case to be the Vandermonde matrix with two variables x_1 and x_2. In this case, the Desanot-Jacobi theorem solved for $|M|$ comes out to $x_1 - x_2 = (1 \cdot x_1 - x_2 \cdot 1)/1$ which confirms the Vandermonde determinat.

Inductive step:
Now assume that the Vandermonde formula holds for up to n variables, prove it true for $n + 1$.

Using the same equation as above $|V_{n+1}| = |V_n|$(variables shifted by 1)$|V_n| - |V_n||V_n|$(variables shifted by 1)$/V_{n-2}$

Computed, this allows us to find the value of $|V_n| = \sum_{\sigma \in S_n} (-1)^{I(\sigma)}{}_{i=1}^{n} x_{\sigma(i)}^{n-1}$.

Question 3.5.6

3.5.6 Use the inductive definition of the λ-determinant to prove that

$$|x_j^{n-i}|_\lambda = \prod_{1\le i<j\le n}(x_i+\lambda x_j).$$

This definition is inherently inductive: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}_\lambda = a_{11}a_{22}+\lambda a_{12}a_{21}$

This is because we can use the idea of Dodgson's algorithm,

$$|M|\,|M_{1n}^{1n}| = |M_1^1|\,|M_n^n| - |M_n^1|\,|M_1^n|, \quad \text{(Theorem 3.12)}$$

to build the λ-determinant for an arbitrary square matrix.

As an example, the 3×3 λ-determinant of $|x_j^{n-i}|_\lambda$

$$= \begin{vmatrix} x_1^{3-1} & x_2^{3-1} & x_3^{3-1} \\ x_1^{3-2} & x_2^{3-2} & x_3^{3-2} \\ x_1^{3-3} & x_2^{3-3} & x_3^{3-3}\end{vmatrix}_\lambda$$

(as in the Vandermonde determinant) is, since for M_j^i we delete row i and column j,

$$[(x_1^2x_2+\lambda x_1x_2^2)(x_2+\lambda x_3) + \lambda(x_2^2x_3+\lambda x_2x_3^2)(x_1+\lambda x_2)]/x_2$$

$$=[x_1^2x_2^2+\lambda x_1^2x_2x_3+\lambda x_1x_2^3+\lambda^2x_1x_2^2x_3 + \lambda x_1x_2^2x_3+\lambda^2x_2^3x_3+\lambda^2x_1x_2x_3^2+\lambda^3x_2^2x_3^2]/x_2$$

$$= x_1^2x_2+\lambda x_1^2x_3+\lambda x_1x_2^2+\lambda^2x_1x_2x_3+\lambda x_1x_2x_3 + \lambda^2x_2^2x_3+\lambda^2x_1x_3^2+\lambda^3x_2x_3^2$$

$$=(x_1+\lambda x_2)\cdot[x_1x_2+\lambda x_1x_3+\lambda x_2x_3+\lambda^2x_3^2]$$

$$=(x_1+\lambda x_2)(x_1+\lambda x_3)(x_2+\lambda x_3) = \prod_{1\le i<j\le 3}(x_i+\lambda x_j).$$

(The determinant of (x_j^{n-i}) is an alternating polynomial when $\lambda=-1$ because transposing any two columns changes the sign of the determinant so Proposition 2.8 $\Rightarrow$ $\det(x_j^{n-i}) = c\prod_{1\le i<j\le n}(x_i-x_j)$ and $c=1$ because the coefficients are 1.)

3.5.6 $\quad\lfloor 2$

For the sake of induction, another example is $\begin{vmatrix} x_1^{2-1} & x_2^{2-1} \\ 1 & 1 \end{vmatrix}_\lambda$

$$= x_1 \cdot 1 + \lambda x_2 \cdot 1 = \prod_{1 \le i < j \le 2} (x_i + \lambda x_j).$$

With this true for $n=2$ and $n=3$, assume $|x_j^{n-i}|_\lambda = \prod_{1 \le i < j \le n} (x_i + \lambda x_j)$

Given $|M|_\lambda |M_{1\ n+1}^{1\ n+1}|_\lambda = |M_1^1|_\lambda\, |M_{n+1}^{n+1}|_\lambda + \lambda |M_{n+1}^{1}|_\lambda\, |M_1^{n+1}|_\lambda$,

for the $n+1 \times n+1$ matrix $M = x_j^{n+1-i}$

$$|M_1^1|_\lambda = \begin{vmatrix} x_2^{n-1} & x_3^{n-1} & \cdots & x_{n+1}^{n-1} \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_{n+1}^{n-2} \\ & & \vdots & \\ 1 & 1 & \cdots & 1 \end{vmatrix}_\lambda \text{ and } |M_{1\ n+1}^{1\ n+1}|_\lambda = \begin{vmatrix} x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_2 & x_3 & \cdots & x_n \end{vmatrix}_\lambda$$

$$|M_{n+1}^{n+1}|_\lambda = \begin{vmatrix} x_1^n & x_2^n & \cdots & x_n^n \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1 & x_2 & \cdots & x_n \end{vmatrix}_\lambda \text{ and } |M_{n+1}^{1}|_\lambda = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ & & \vdots & \\ 1 & 1 & \cdots & 1 \end{vmatrix}_\lambda$$

$$|M_1^{n+1}|_\lambda = \begin{vmatrix} x_2^n & x_3^n & \cdots & x_{n+1}^n \\ x_2^{n-1} & x_3^{n-1} & \cdots & x_{n+1}^{n} \\ & & \vdots & \\ x_2 & x_3 & \cdots & x_{n+1} \end{vmatrix}_\lambda$$

After pulling out some factors,

$$|M|_\lambda \cdot x_2 \cdots x_n \prod_{2 \le i < j \le n} (x_i + \lambda x_j) = \prod_{2 \le i \le j \le n+1} (x_i + \lambda x_j) \cdot x_1 \cdots x_n \cdot \prod_{1 \le i < j \le n} (x_i + \lambda x_j) + \lambda \prod_{1 \le i < j \le n} (x_i + \lambda x_j) \cdot x_2 \cdots x_{n+1} \prod_{2 \le i < j \le n+1} (x_i + \lambda x_j)$$

$$= x_1 (x_2 + \lambda x_{n+1}) \cdots (x_n + \lambda x_{n+1}) \prod_{1 \le i < j \le n} (x_i + \lambda x_j) + \lambda x_{n+1} (x_2 + \lambda x_{n+1}) \cdots (x_n + \lambda x_{n+1}) \prod_{1 \le i < j \le n} (x_i + \lambda x_j)$$

$$= \Big(\prod_{1 \le i < j \le n} (x_i + \lambda x_j) \Big) \cdot \big[x_1 (x_2 + \lambda x_{n+1}) \cdots (x_n + \lambda x_{n+1}) + \lambda x_{n+1} (x_2 + \lambda x_{n+1}) \cdots (x_n + \lambda x_{n+1}) \big]$$

$$= \Big[\Big(x_1 \prod_{2 \le i \le n} (x_i + \lambda x_{n+1}) \Big) + \lambda \Big(x_{n+1} \prod_{2 \le i \le n} (x_i + \lambda x_{n+1}) \Big) \Big] \cdot |x_j^{n-i}|_\lambda$$

$$= \Big[(x_1 + \lambda x_{n+1}) \prod_{2 \le i \le n} (x_i + \lambda x_{n+1}) \Big] |x_j^{n-i}|_\lambda = (x_1 + \lambda x_{n+1})(x_2 + \lambda x_{n+1}) \cdots$$

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$= (x_1+\lambda x_{n+1})(x_2+\lambda x_{n+1})\cdots(x_n+\lambda x_{n+1})\,|x_j^{n-i}|_\lambda$

$= (x_1+\lambda x_{n+1})\cdots(x_n+\lambda x_{n+1})\cdot \prod_{1\le i<j\le n}(x_i+\lambda x_j)$

based on the inductive hypothesis

$= \prod_{1\le i<j\le n+1}(x_i+\lambda x_j)$

$= |x_j^{n+1-i}|_\lambda$ which is then true for all n.

Question 3.5.10

Using equation (3.32), prove that

$$\sum_{B\in A_n} 2^{N(B)} = 2^{n(n-1)/2}$$

If we substitute $\lambda = 1$ we get $\prod_{1\le i<j\le n}(x_i + x_j)$. We let the term in the product to be $(x_i - (-1)x_j)$. This is now a form of a Vandermonde product with about which we know several properties. We know this is an alternating polynomial of degree $n(n-1)/2$ for this product.

If we go back to equation 3.32, we see that the $\lambda = 1$ gives us a right hand side of: $\sum_{B\in A_n} 2^{N(B)}$ then the product, which we know in this case has the degree which we want. This can be simplified to be a sum of the following form:

$$\sum_{B\in A_n} 2^{N(B)} = 2^{(n(n-1)/2)}.$$

A substitution-based rendering of this gives the following:

3.5.10. Using equation (3.32), prove that $\sum_{B\in A_n} 2^{N(B)} = 2^{n(n-1)/2}$.

Equation (3.32) says that $\prod_{1\le i<j\le n}(x_i+\lambda x_j) = \sum_{B\in A_n}\lambda^{I(B)}(1+\lambda^{-1})^{N(B)}$

$\prod_{i,j=1}^{n} x_j^{(n-i)B_{ij}}$.

This result (which is derived from the λ-determinant) in which B_{ij} is an entry of an ASM $(B_{ij})=B$ and $I(B)$ its inversion number suggests that for $x_i, x_j = 1$ and $\lambda = 1$,

$$\prod_{1\le i<j\le n}(2) = \sum_{B\in A_n} 2^{N(B)} \prod_{i,j=1}^{n} 1 \Rightarrow$$

$$\begin{pmatrix} \underset{i=1,j=2}{2} & \underset{i=1,j=3}{2} & \cdots & \underset{i=1,j=n}{2} \\ \underset{i=2,j=3}{2} & \underset{i=2,j=4}{2} & \cdots & \underset{i=2,j=n}{2} \\ & \vdots & & \\ \underset{i=n-1,j=n}{2} & & & \end{pmatrix} = \begin{pmatrix} 2^{n-1} \\ 2^{n-2} \\ \vdots \\ 2^{1} \end{pmatrix} = 2^{(n-1)+(n-2)+\dots+1}$$

$= 2^{(n-1)n/2}$. The sum of $1+2+\dots+n-1$: two cases

n is an even natural number: $\underbrace{1+2+\dots}+\frac{n}{2}+\underbrace{\dots+n-1}_{n-2 \text{ summands}}$

$$= (1+n-1)+(2+n-2)+\dots + \frac{n}{2} = \frac{n-2}{2}\cdot n + \frac{n}{2} = \frac{n(n-2)+n}{2} = \frac{n(n-1)}{2} \checkmark$$

n is odd: $1+2+\dots+n-1 = \underline{(1+n-1)}+\underline{(2+n-2)}+\dots$

$$= n\cdot\frac{n-1}{2} = \frac{n(n-1)}{2} \checkmark$$

Question 4.1.1

What are the partitions of 29 that immediately precede and follow $7+6+6+4+4+2$ in descending order?

The preceding partition can be found by figuring out which element can increase in the partition, which is furthest to the right. When we look at the partition, if we increase the last 4 to a 5, then we will have a partition with non-decreasing order. This means we have to push the 5 one more place over and create the smallest partition, which is namely the one with all $1's$ trailing. This gives us

$$7+6+6+5+1+1+1+1+1$$

.

The partition which follows is the partition where we decrease the furthest right element. This corresponds to a partition of

$$7+6+6+4+4+1+1$$

.

Question 4.1.2

Calculate $e_3(1,1,1,1,1)$. What is the value of e_n when there are m variables each set to 1? What is the value of $h_3(1,1,1,1,1)$? What is the value of h_n when there are m variables each set to 1?

If we calculate $e_3(1,1,1,1,1)$ we expand this symmetric function to be:

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + ...x_3x_4x_5$$

This corresponds to every single unique subset of size 3 from a set of size 5. This means that the sum is 10 and the corresponding value of e_n with m variables is $\binom{m}{n}$.

If we calculate $h_3(1,1,1,1,1)$ we realize this is equal to $m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$. The corresponding monomial symmetric functions in 5 variables is expanded to:

$$(x_1^3 + ... + x_5^3) + (x_1^2x_1 + x_1^2x_3 + ... + x_5^2x_4) + (x_1x_2x_3 + x_1x_2x_4 + ... + x_3x_4x_5)$$

This sum is equal to 35, while the sum for degree 2 with 5 variables is 15, while degree 1 is 5. This indicates that the degree has an impact on the total number. Similarly, if we have degree 3 with four variables, we have 20, meaning that the total number also scales with the number of variables we introduce.

I can't prove the following conjecture but have determined it true for up to $n = 5$, $m = 3$ that the number of these functions with m variables in degree n is $\binom{m+n-1}{m}$.

Question 4.1.5

Find the expansions of the five monomial symmetric functions of degree 4 in terms of the elementary symmetric functions.

The problem is best looked at from a bottom-up or top-down approach, depending in how you view these functions. We know that in degree four, $e_{(4)} = m_{(1,1,1,1)}$, therefore we can use that equality for the other polynomials as well.

We will be looking at the five partitions of four, $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ to represent the corresponding monomial symmetric functions in terms of elementary symmetric functions.

Given that we know $e_{(3,1)} = m_{(2,1,1)} + 4m_{(1,1,1,1)}$, we rearrange the equation to equal $e_{(3,1)} + 4m_{(1,1,1,1)} = m_{(2,1,1)}$, but we substitute in e_4 in the left-hand side, giving us a result of $m_{(2,1,1)} = e_{(3,1)} + 4e_4$.

We move down the equations in the book on pages 122-123, changing each equation which contains the singular term of a given monomial symmetric function and changing the corresponding equality into terms of only elementary symmetric functions.

Within $m_{(2,2)}$, we can notice that $2m_{(2,1,1)} + 6m_{(1,1,1,1)}$ is equivalent to $2e_{(3,1)} - 2m_{(1,1,1,1)} = 2e_{(3,1)} - 2e_4$. This gives us $m_{(2,2)} = e_{(2,2)} - 2e_{(3,1)} + 2e_4$ when we move terms to the other side.

The final two calculations are the most complicated, but the same process is repeated. By building on what we already know about the monomial symmetric functions in terms of elementary symmetric functions, it just becomes a game of substituting and being careful about the math. Below contains the five monomial symmetric functions in terms of elementary symmetric functions.

$$
\begin{aligned}
m_{(1,1,1,1)} &= e_{(4)} \\
m_{(2,1,1)} &= e_{(3,1)} - 4e_4 \\
m_{(2,2)} &= e_{(2,2)} - 2e_{(3,1)} + 2e_4 \\
m_{(3,1)} &= 4e_4 - e_{(3,1)} - 2e_{(2,2)} + e_{(2,1,1)} \\
m_{(4)} &= 4e_{(3,1)} + 2e_{(2,2)} + e_{(1,1,1,1)} - 4e_{(2,1,1)} - 4e_4
\end{aligned}
$$

Question 4.1.6

Prove that

$$h_r(1, q, q^2, ..., q^n) = \binom{n+r}{r}.$$

The key concept within this equality is the construction of our homogeneous symmetric function. When we input variables $x_1, ..., x_n$ we get the monomial symmetric functions. Instead of using these variables, we will look at the generating function by inputting q^is.

We are partitioning r individual terms with n choices of q which are not trivial (the 1 does not contribute). This means that when we choose for each term, we choose from this bucket of qs. This gives us a generating function of $\binom{n+r}{r}$ when we choose like that, and if we choose which q we want each time, we get $\binom{n+r}{n}$.

When looking back at exercise 4.1.2 this does agree with the conjectured formula, as we know $r = n+1$ number of variables meaning we do indeed get $\binom{n+r}{n}$.

Question 4.1.7

Prove that

$$e_r(1, q, q^2, ..., q^{n-1}) = q^{r(r-1)/2}\binom{n}{r}$$

We begin by referencing problem 4.1.2. It was proven that the value of e_3 with m variables set to 1 was $\binom{m}{r}$. We aren't using variables set to 1, but we have n variables in this case. On the RHS, we see that we have $q^{r(r-1)/2}$, this corresponds to the substitution of $q, ..., q^{n-1}$ instead of 1. We will always have this power of q within our generating function. Then, we know that the value of e_r is $\binom{m}{r}$ which in this case is $\binom{n}{r}$ as we have n options for our variables.

It turns out that the corresponding generating function is easily represented using this argument and that

$$e_r(1, q, q^2, ..., q^{n-1}) = q^{r(r-1)/2}\binom{n}{r}$$

.

Question 4.1.8

Prove that

$$e_n = \det(h_{1-i+j})_{i,j=1}^{n} \text{ and}$$
$$h_n = \det(e_{1-i+j})_{i,j=1}^{n}$$

We begin by referencing the given Jacobi-Trudi identities where these are represented as Schur functions. We have

$$s_\lambda(x_1, ..., x_n) = \det(h_{\lambda_i+j-i})_{i,j=1}^{k}.$$

This is for partitions into at most n parts.

Because we want $\lambda_i = 1$ for each i, our partition must be all 1s. This means we can rewrite the elementary symmetric functions as follows: $e_n = s_{(1,1,1,1,...,1)}(x_1, ..., x_n)$. This equality is easier to reason about when we think about the Schur function in terms of Young tableaux. If we have n rows for our partition and n elements to choose from, each element will appear exactly once, which corresponds to the most basic elementary function $e_n = \prod_{i=1}^{n} x_i = s_{\lambda=(1,1,1,1,...,1)}$. This can be generalized to have more than n variables, but this is still rather trivial to show as we will be adding more xs in every case.

Another identity that we can use in this case is the same Jacobi-Trudi identity to represent a Schur polynomial as a determinant over elementary polynomials. We have

$$s_\lambda(x_1, ..., x_n) = \det(e_{\lambda'_i+j-i})_{i,j=1}^{k}.$$

This is the conjugate partition we have from before. Because we used the partition of n 1s, the conjugate of this will be a partition of n with only n.

We want also to look at the matrices generated by this definition: $\begin{pmatrix} h_1 & h_2 & h_3 & ... & h_n \\ 1 & h_1 & ... & & h_{n-1} \\ 0 & 1 & h_1 & \dots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & ... & & 1 & h_1 \end{pmatrix}$

The other matrix is exactly the same although there are the elementary symmetric functions as entries rather than the homogeneous symmetric functions.

We can see that the maximal degree of these matrices must be n, which can be seen by the diagonal of 1's acting as a limiting factor for the degree of individual minors when we take the determinant. If we think of the semistandard tableaux for the elementary case, we can get similar factors within the tableau, but this is covered when we look at the determinant which will have the multiple of e_1s which gives us options to choose which factors to group together.

The corresponding result can be expressed in terms of lattice nests but is not presented here.

Question 4.2.2

Use Theorem 4.3 to find the coefficient of $x_1^2x_2^3x_3x_4^4x_5^3$ in the Schur function $s_{(5,4,3,1)}(x_1, x_2, x_3, x_4, x_5)$.

To find the coefficient, we want to find all the semistandard tableau of shape $(5, 4, 3, 1)$ with the corresponding $x's$.

Below are some of the tableau which satisfy the requirements. There are 16 in total.

1	1	2	2	2	1	1	2	2	4	1	1	2	2	5
3	4	4	5		2	4	4	5		2	4	4	4	
4	5	5			3	5	5			3	5	5		
5					4					4				
1	1	2	4	5	1	1	2	4	4	1	1	2	4	4
2	2	4	5		2	2	4	5		2	2	3	5	
3	4	5			3	4	5			4	4	5		
4					5					5				

Question 4.2.3

Use Theorem 4.3 to prove that s_λ equals m_λ plus a sum of monomial symmetric functions indexed by partitions smaller than . Use this fact to prove that the Schur functions of degree n form a basis for all homogeneous symmetric functions of degree n.

If we look at the largest partition, $\lambda = n$, we choose unique variables at each index, which gives us m_λ. However, we can also realize that in a semistandard tableaux, we can choose any combinations of variables such that we have increase down columns and weak increase across rows. This is equivalent to the statement that as we add rows to the partition by taking off elements from the first row (making a smaller partition), we are getting these different unique monomial terms of degree n. For instance, because these are sorted by row, we can take off the corresponding end of each row and move it down to get the same monomial.

As we go across each shape that is smaller than λ we know that we will have the entire Schur function and it must be symmetric.

Because homogeneous symmetric functions are sums of monomial symmetric functions and we have all shapes λ of size n, we know we have each monomial symmetric element. These form a basis for the homogeneous symmetric functions of degree n.

Question 4.2.7

Use equation 4.13 to prove that the generating function for column strict plane partitions in which each stack has odd height is given by

$$\prod_{i=1}^{\infty} \frac{1}{1-q^{2i-1}} \prod_{1\leq i<j} \frac{1}{1-q^{2i-2j-2}} = \prod_{n\geq 1} \frac{1}{(1-q^n)^{v(n)}}$$

, where $v(n) = 1$ if n is odd and $v(n) = \text{floor}(n/4)$ if n is even.

We begin by looking at equation 4.13 for when we have every number of stacks. We reindex to switch from every number to only odd by changing our ceiling function to only capture odds. When we want to index by increases over 1, we have to change the ceiling to a floor and divide by 2 as we are covering two steps with every single step now. We can't simply use this substitution but we need to change. If n is odd we don't have to worry and we can use the substitution. If n is even, we have to take the floor of n/4 as we are counting these stacks and that limits the number of stacks we can have in a given line such that we only take the odd stacks and can only use the floor of n for how many stacks we can have.

Question 5.2.1

Verify that if

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!},$$

then

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

We know that when we divide $A_{n,k}$ by $A_{n,k+1}$, the product term will cancel out as it does not involve k, leaving us with only the binomial coefficient as well as the factorials.

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n+k-2}{k-1}\frac{(2n-k-1)!}{(n-k)!}}{\binom{n+k-1}{k}\frac{(2n-k)!}{(n-k-1)!}} \tag{1}$$

$$= \frac{k!(n+k-2)!(n-1)!(n-k-1)!(2n-k-1)!}{(k-1)!(n+k-1)!(n-1)!(n-k-1)!(n-k)!(2n-k)!} \tag{2}$$

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{k(2n-k-1)}{(n+k-1)(n-k)} \tag{3}$$

In line (2), we can see that the corresponding factorials are grouped and we cancel out the desired products to get the final ratio.

Question 5.2.2

Prove that

$$\prod_{k=2}^{n} \frac{(k-1)!(3k-2)!}{(2k-2)!(2k-1)!} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

To begin this problem, observe that the term $(3k-2)!$ from the LHS cancels out with the $(3j+1)!$ term from the RHS. The RHS term starts with 1! and continues to $(3n-2)!$ while the LHS starts with 4! and continues to $(3n-2)!$. This leaves us with needing to prove the equivalence of the following:

$$\prod_{k=2}^{n} \frac{(k-1)!}{(2k-2)!(2k-1)!} = \prod_{j=0}^{n-1} \frac{1}{(n+j)!}$$

Looking at the LHS of this equation, the denominator expands as such, $(2)!(3)!...(2n-2)!(2n-1)!$, while the numerator expands as $(1)!(2)!...(n-1)!$. We can cancel out all terms from $(1)!$ to $(n-1)!$ from the top and bottom, leaving us with

$$\prod_{k=2}^{n} \frac{(k-1)!}{(2k-2)!(2k-1)!} = \frac{1}{(n)!(n+1)!...(2n-2)!(2n-1)!}$$

which we can rewrite as

$$\prod_{j=0}^{n-1} \frac{1}{(n+j)!},$$

equivalent to the RHS.

Question 5.2.4

Prove equation (5.8) by showing that each side counts the number of ways of choosing $a+b+1$ positions from a choice of $a+b+m+1$ positions.

$$\binom{a+b+m+1}{m} = \sum_{k=0}^{m} \binom{a+k}{k} \binom{b+m-k}{m-k}$$

The left hand side of the equation is relatively simple to show, as we have the identity $\binom{m+n}{n} = \binom{m+n}{m}$, which means we can replace the m with $a+b+1$, giving us the choice of $a+b+1$ positions from the entire $a+b+m+1$ positions.

The right hand side of the equation involves a counting argument while we iterate from $k=0$ to m for the sum. We have m objects to place into $a+b+m+1$ distinct position, where we see that there are $\binom{a+k}{k}$ ways to place the first k objects, and $\binom{b+m-k}{m-k}$ ways to place the final $m-k$ objects. We essentially pick a place to separate where we can put the first set of objects, and sum over the entirety of these divisions. The sum over all m gives us all the separations we need, but we have shown that we choose m positions in this case.

Choosing m positions to put objects in is the same as choosing the $a+b+1$ positions to have no objects in, therefore the sum on the right hand side still counts the number of was to choose $a+b+1$ positions from $a+b+m+1$ possible places. This shows that each side counts exactly what we want.

Question 5.2.9

Find a closed form for the value of

$$\sum_{k\geq 0}\binom{n}{k}\binom{2k}{k}(-1/4)^k$$

We begin solving a closed form for this value by factoring the $(-1/4)^k$ and rewriting the equation as a ratio between the $k+1$ and kth entry.

$$\begin{aligned}
\frac{A_{k+1}}{A_k} &= \frac{\binom{n}{k+1}\binom{2k+2}{k+1}(-1/4)^{k+1}}{\binom{n}{k}\binom{2k}{k}(-1/4)^k} \\
&= \frac{(n-k)(2k+1)(2k+2)}{(k+1)^3}\cdot(-1/4) \\
&= \frac{2(k+1)2(k+\frac{1}{2})(n-k)}{(k+1)^3} \\
&= (-1)\frac{(k+\frac{1}{2})(-1(k-n))}{(k+1)(k+1)} \\
&= \frac{(k+\frac{1}{2})(k-n)}{(k+1)(k+1)}
\end{aligned}$$

This is exactly the form for a 2F1;1 with parameters $\alpha_1=-n, \alpha_2=1/2, \gamma_1=1$. This is sufficient for a closed form solution for our summation.

Question 6.1.2

Prove the following special case of Conjecture 10: The number of permutations $\sigma \in S_n$ for which $\sigma(1) = k$ and $I(\sigma) = p$ is equal to the number of descending plane partitions in B(n,n,n) with exactly k - 1 parts of size n, no special parts (the entry in position (i,j) must be strictly greater than j-i), and a total of p parts.

As Eric said, we don't need to refer to non-intersecting lattice paths in order to do this. First, the definition of a descending plane partition. A descending plane partition is a strict shifted partition in which the number of parts in each row is strictly less than the largest part in that row and is greater than or equal to the largest part in the next row. (A strict shifted plane partition can be expressed as an arrangement of positive integers with each row indented, weak decrease across rows, and strict decrease down columns.)

Also, the definition of a special part is an entry that satisfies $a_{i,j} \leq j - i$), for any descending plane partition. For example,

$$\begin{array}{ccccccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots & \dots & \dots & \dots a_{1,r_1} \\ & a_{2,2} & a_{2,3} & \dots & \dots & \dots & a_{2,r_2} \\ & & & \vdots & & & \\ & & & a_{k,k} & \dots & a_{k,r_k} & \end{array}$$

as a generic descending plane partition has a direct correspondence (a bijection) with the inversion sequence $(a_1, a_2, \dots, a_{r_1})$, in which a_m = number of distinct $a_{m,j}$ such that $a_{m,j} = r_1$ + 2 - m.

This inversion sequence in turn is in bijection with $\sigma \in S_{r_1+1}$, for which a_m = number of inversions (and $a_1 + a_2 + \dots + a_{r_1} = I(\sigma)$) (m,j) such that $\sigma(j)$ = m, m = 1, 2, ..., r_1 and a_m = number of elements which are > m and to the left of m in σ.

Now, given $\sigma \in S_n$ where $\sigma(1)$ = k, there are a_k inversions (k,j) such that k is greater than 1, 2, ..., k - 1 which are to the right of k in σ).

So in the inversion sequence $(a_1, a_2, ..., a_{r_1})$, k - 1 = the number of distinct $a_{k,j}$ such that $a_{k,j} = r_1 + 2 - k$. Since there are 0 elements > k to the left of k in σ, $a_k = 0 \implies k - 1 = 0 \implies k = 1$.

a_k is the number of distinct $a_{k,j}$ such that $a_{k,j} = r_1 + 2 - k$ in the corresponding descending plane partition, so

a_k = k - 1 = number of parts of size $r_1 + 2 - k = r_1 + 2 - 1 = r_1 + 1 = n$.

The second condition is that $I(\sigma) = p \implies a_1 + a_2 + \dots + a_{r_1} = p \implies$ in the inversion sequence $(a_1, \dots, a_{r_1})$, $\sum_{a_{m,j}} = p \implies$ (b/c all $a_{m,j}$ are distinct) there are a total of p parts in the corresponding descending plane partition.

Suppose there is a special part in this partition. This means $\exists$ some $a_{m,j} \leq j - m \implies a_{m,j} = r_1 + 2 - m \leq j - m \implies j \geq r_1 + 2$ which does not occur, so in the inversion sequence, $a_m = 0$.

However, each a_m counts a unique set of distinct $a_{m,j}$; if this special part isn't counted in the inversion sequence $(a_1, a_2, ..., a_{r_1})$ then $a_1 + a_2 + \ldots + a_{r_1} = I(\sigma) \neq$ the total number of parts in the partition which equals p, a contradiction of what we are given. So there are no special parts.

Question 6.1.5

Prove that if an n × n alternating sign matrix with m -1s and inversion number equal to p is reflected over a vertical axis, it is transformed into an alternating sign matrix whose inversion number is n(n-1)/2 + m - p.

(page 88 of David Bressoud): Definition of the inversion number is the number of pairs of 1s in this matrix for which one of the 1s lies to the right and above the other.

We can calculate the inversion number by taking all pairs of matrix entries for which one of them lies to the right and above the other, multiplying each pair of entries together, and then adding up all of these products.

Example

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftarrow \text{Reflect over a vertical axis:} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The first matrix on the left has 7 pairs with product +1 and it has 2 pairs with product -1. Its inversion number is 5.

The second matrix has 11 pairs with product +1 and 4 pairs with product -1 so the inversion number = 7. (This = n(n-1)/2 + m - p, that is the dimension + the # of -1s - inversion of original ASM, = 5(5-1)/2 + 2 - 5 = 10 + 2 - 5 = 7 ✓).

$\frac{n(n-1)}{2}$ = 0, 1, 3, 6, 10, 15, 21 for n = 1, 2, ... which is the triangular number sequence. m - p is -3 in this example (this is -1 inv#) or m - p is -5 depending on how you look at it.

Aside from this example, we need to show that the total number of parts in a descending plane partition corresponds to the inversion number of an alternating sign matrix (This is Conjecture 10). We've already shown in Exercise 6.1.2 that there's a bijection between permutations $\sigma \in S_n$ with $I(\sigma)$ = p and descending plane partitions in B(n,n,n) with a total of p parts.

Additionally, Conjecture 10 suggests that there's a correspondence between $\sigma \in S_n$ with $I(\sigma) = p$ and ASMs with Inv # = p.

(This is described in Section 1.1 for ASMs with no -1s (no special parts)). Then we can apply the hidden symmetry from 6.1.

Question 6.1.6

Prove that the image of a descending plane partition under the reflection described on page 194 is always a descending plane partition.

We want to show

$$\begin{array}{l} a_{1,1}\ a_{1,2}\ a_{1,3}\ \dots\ \dots\ \dots\ \dots a_{1,r_1} \\ \quad a_{2,2}\ a_{2,3}\ \dots\ \dots\ \dots\ a_{2,r_2} \\ \qquad\qquad \vdots \\ \qquad\qquad a_{k,k}\ \dots\ a_{k,r_k} \end{array}$$

corresponds to another descending plane partition, under the following mapping:

$$\mathrm{b}_{i,j} = \begin{cases} j-i+1-\mathrm{a}_{i,j} & \text{if } a_{i,j} \text{ exists and } a_{i,j} \le j-i \\ j+1-\beta_{i,j} & \text{if } a_{i,j} \text{ is not defined (where } \beta_{i,j} = \|\{a_{x,i} \mid a_{x,i} \ge j+2-x\}\|) \\ \text{undefined} & \text{if } a_{i,j} > j-i. \end{cases}$$

and subsequent reflection across the southwest to northeast diagonal. This reflection, of course, takes the following form for this example:

$$\begin{array}{cccc} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ & b_{2,2} & b_{2,3} & b_{2,4} \\ & & b_{3,3} & b_{3,4} \\ & & & b_{4,4} \end{array} \quad \longleftrightarrow \quad \begin{array}{cccc} b_{4,4} & b_{3,4} & b_{2,4} & b_{1,4} \\ & b_{3,3} & b_{2,3} & b_{1,3} \\ & & b_{2,2} & b_{1,2} \\ & & & b_{1,1} \end{array}$$

As you can see, $b_{i,j} \to b_{5-j,5-i}$ and in this case, $5 = r_1 + 1$ (from a_{1,r_1}).

So to generalize, $b_{i,j} \to b_{r_1-j,r_1-i}$ under the reflection part.

As before,

$$\mathrm{b}_{r_1-j,r_1-i} = \begin{cases} j-i+1-\mathrm{a}_{i,j} & \text{if } a_{i,j} \text{ exists and } a_{i,j} \le j-i \\ j+1-\beta_{i,j} & \text{if } a_{i,j} \text{ is not defined } (\beta_{i,j} = \|\{a_{x,i} \mid a_{x,i} \ge j+2-x\}\|) \\ \text{undefined} & \text{if } a_{i,j} > j-i. \end{cases}$$

Given this definition of the complete mapping which has just been proved, you can see that it fulfills the qualities of being strict shifted:

Referring to the first and second aspects of the function ((i) $b_{r_1-j,r_1-i} = j-i+1-a_{i,j}$ and (ii) b_{r_1-j,r_1-i} = j + 1 - $\beta_{i,j}$),

- each row indented ✓by construction.
- weak decrease across rows: $b_{r_1-j,r_1-(i)} \ge b_{r_1-j,r_1-(i-1)}$:

i. j - i + 1 - $a_{i,j} \geq j-(i-1)+1-a_{i-1,j}$
-$a_{i,j} \geq 1-a_{i-1,j}$
$a_{i,j} \leq a_{i-1,j}-1$
Yes, b/c $a_{i,j} < a_{i-1,j}$ (strictly decreasing down rows in original)

ii. j + 1 - $\beta_{i,j} \geq j+1-\beta_{i-1,j}$
$\beta_{i,j} \leq \beta_{i-1,j}$
$\|\{a_{x,i} \mid a_{x,i} \geq j+2-x\}\|) \leq \|\{a_{x,i-1} \mid a_{x,i-1} \geq j+2-x\}\|)$
Yes; this is because $a_{x,i-1} \leq a_{x,i}$ (weak decrease across rows)

- strict decrease down columns: $b_{r_1-j,r_1-i} > b_{r_1-(j-1),r_1-i}$:

i. j - i + 1 - $a_{i,j} > (j-1)-i+1-a_{i,(j-1)}$
$-a_{i,j} > -1-a_{i,(j-1)}$
$a_{i,j} < 1+a_{i,j-1}$
$a_{i,j-1} > a_{i,j}-1$? Yes, b/c $a_{i,j} \geq a_{i,j-1}$ (weak decrease across rows)

ii. j + 1 - $\beta_{i,j} >$ (j - 1) + 1 - $\beta_{i,(j-1)}$
j + 1 - $\|\{a_{x,i} \mid a_{x,i} \geq j+2-x\}\| > j-\|\{a_{x,i} \mid a_{x,i} \geq j-1+2-x\}\|$
(the $a_{i,j}$ not defined case)
$\|\{a_{x,i} \mid a_{x,i} \geq j+2-x\}\|-1 < \|\{a_{x,i} \mid a_{x,i} \geq j+1-x\}\|$
Yes. $a_{x,i} = j+1-x$ is true ∀ j: take x = j + 1. $a_{j+1,i} =$ j + 1 - (j + 1) = 0 for some i, given that it's undefined at some point in that row.

Lastly, for our new (strict shifted) plane partition, the number of parts in each row is strictly less than the largest part in that row and is greater than or equal to the largest part in the next row:

We need to prove: # parts in each row < largest part in that row

parts in each row ≥ largest part in next row.

Looking at the mapping

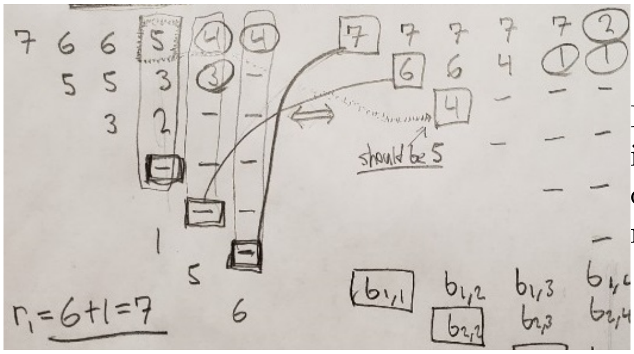
$$\mathrm{b}_{r_1-j,r_1-i} = \begin{cases} j-i+1-\mathrm{a}_{i,j} & \text{if } a_{i,j} \text{ exists and } \boxed{a_{i,j} \leq j-i} \quad \text{special part} \\ j+1-\beta_{i,j} & \text{if } a_{i,j} \text{ is not defined} \\ \text{undefined} & \text{if } \boxed{a_{i,j} > j-i} \quad \text{not a special part}. \end{cases}$$

we have

- For each row r_1-j, the largest part = $b_{r_1-j,r_1-j} = j+1-\beta_{i,j}$ only, because $a_{j,j} \leq j-j=0$ does not exist.
- So $\boxed{\mathrm{b}_{r_1-j,r_1-j}}$ = j + 1 - $\|\{a_{x,j} \mid a_{x,j} \geq j+2-x\}\|$.

I've included an image of an example from the book and how it translates into the partition based on the described mapping & reflection:

Solutions



It's pretty clear in this example that # parts in row $r_1 - j$, for j $\in \{1, 2, 3, 4, 5, 6\}$, is determined by # rows in column j - # non-special, defined parts.

So we have several observations:

1) That is, the # parts in row $r_1 - j$ of our translated partition is

rows in column j - # defined non-special parts in the original partition
= j - # parts $a_{i,j}$ which are > j - i.

2) Largest part in row $r_1 - j$ is j + 1 - # parts $a_{i,j}$ which are ≥ j + 2 - i

= j + 1 - # parts $a_{i,j}$ which are > j - i + 1,
which is clearly always the greater option.

3) The largest part in the next row, $r_1 - (j - 1)$, is by substitution

$(j-1) + 1 - \#$ parts $a_{i,j-1}$ which are > (j - 1) - i + 1, that is

$$j - \underbrace{\#\text{parts } a_{i,j-1} \text{ which are } > j - i.}_{\text{the previous column} \geq \text{the current column}}$$

Formally for column j, $a_{i,j-1} \geq a_{i,j}$ (weakly decreasing across rows).

For column j of our translated partition,

The largest part in the <u>next</u> $r_1 - (j - 1)$, j - # $a_{i,j-1}$ which $> j - i$

≤

parts in row $r_1 - j$, j - # $a_{i,j}$ which > j - i

<

the largest part in row $r_1 - j$, j - # $a_{i,j}$ which $> j - i + 1$.

So by definition the image is also a descending plane partition.

Question 6.1.7

Prove that the reflection of a descending plane partition takes a descending plane partition in which n appears j times to a descending plane partition in which n appears n - 1 - j times.

For this exercise, we don't need to look at the second step of our transformation.

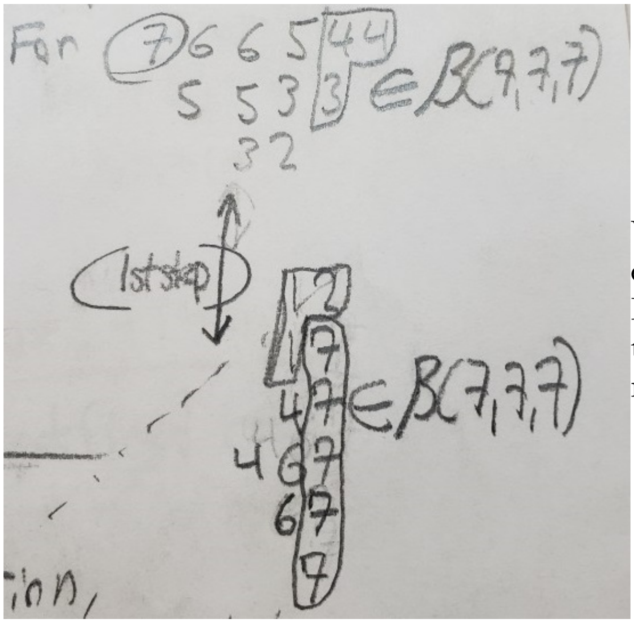
$$\mathrm{b}_{i,j} = \begin{cases} j-i+1-\mathrm{a}_{i,j} & \text{if } a_{i,j} \text{ exists and } \boxed{a_{i,j} \le j-i} \quad \text{special part} \\ j+1-\beta_{i,j} & \text{if } a_{i,j} \text{ is not defined} \\ \text{undefined} & \text{if } \boxed{a_{i,j} > j-i} \quad \text{not a special part}. \end{cases}$$

This is the first step of the transformation, where as stated earlier, $\beta_{i,j} = \|\{a_{x,i} \mid a_{x,i} \geq j+2-x\}\|$.

Let's suppose n appears j times. Then, n appears j = $j_1 + j_2$ times.

n	appears # times	in original	appears # times in reflected
7	1	7 - 1 - 1 = 5	5
6	2	6 - 1 - 2 = 3	2
5	3	5 - 1 - 3 = 1	0
4	2	4 - 1 - 2 = 1	2
3	3	3 - 1 - 3 = -1	0
2	1	2 - 1 - 1 = 0	1
1	0	1 - 1 - 1 = -1	2
	12		12

When I first attempted this problem, I assumed that n was referring to something other than what it is actually referring to. The truth is that n is only referring to the first part of the partition.



What you can see from this image is the example from the book worked out, in which I circled the value of n which is 7 and then translated it into the first step in the mapping.

n is only going to appear in the last column of the reflection, described by the b_{i,r_1} values. Otherwise, we would not have strict decrease down columns after the second step of our transformation (the true reflection across the diagonal) (# parts in a row < largest row part).

We already know that $b_{i,r_1} = b_{i,(n-1)}$ since β(n,n,n) is the smallest box.
The possible $b_{i,j}$ then are described by the following piecewise function:

$$\begin{cases} n = (n-1) - i + 1 - a_{i,(n-1)} \implies n = n - i - a_{i,(n-1)} \implies a_{i,(n-1)} = -i & ✗ \\ n = undefined & ✗ \\ n = (n-1) + 1 - \beta_{i,(n-1)} = n - \|\{a_{x,i} | a_{x,i} \geq (n-1) + 2 - x\}\| & \checkmark \iff \end{cases}$$

The first two can't happen. The first one implies that negative elements appear in the original partition, which isn't the case practically and more mathematically by definition since negative elements do not appear in a partition. The second one doesn't happen as well, because the dimension of an n × n × n box cannot be undefined.

Now, the third case is true by necessity since it's the third and last option, and it's true $\iff \beta_{i,(n-1)} = \|\{a_{x,i} | a_{x,i} \geq n + 1 - x\}\| = 0$, and so in the last column called $a_{x,(n-1)}$, the number of times that $a_{x,(n-1)} < n + 1 - x$ is equal to (n - 1) - j = # times that, for i all rows, $\beta_{i,(n-1)} = 0 = \|\{a_{x,i} | a_{x,i} > n - x,$ which is true iff $a_{x,i} = n\}\|$ iff $a_{x,i} \neq n$. ($a_{x,i}$ = n exactly j times).

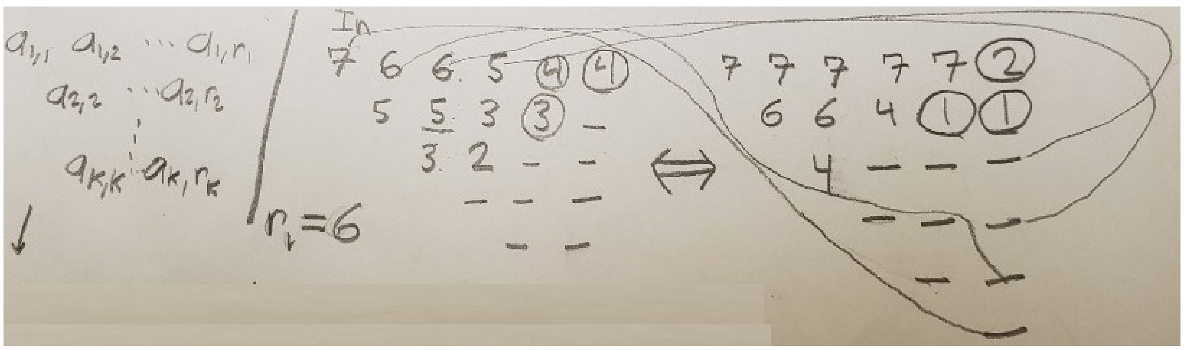
So $\beta_{i,(n-1)} = 0$ exactly $r_1 - j = (n-1) - j = n - 1 - j$ times = # times n appears in the reflection.

Question 6.1.8

Prove that applying the Mills, Robbins, and Rumsey reflection to a descending plane partition twice returns the original descending plane partition.

The complete reflection is given by the piecewise function

$$b_{r_1+1-j,r_1+1-i} = \begin{cases} j-i+1-a_{i,j} & \text{if } a_{i,j} \text{ exists and } \boxed{a_{i,j} \le j-i} \quad \text{special part} \\ j+1-\beta_{i,j} & \text{if } a_{i,j} \text{ is } \underline{\text{not defined}} \\ \text{undefined} & \text{if } \boxed{a_{i,j} > j-i} \quad \text{not a special part} \end{cases}.$$



For all non-special parts in the original partition, they are mapped to undefined b_{r_1+1-j,r_1+1-i}.

b_{r_1+1-j,r_1+1-i} is not defined $\implies c_{r_1+1-(r_1+1-i),r_1+1-(r_1+1-j)} = c_{i,j} = (r_1+1-i)+1-\beta_{r_1+1-j,r_1+1-i} = r_1+2-i-\beta_{r_1+1-j,r_1+1-i} = a_{i,j}$?

$6+2$ - 3 - $\beta_{6+1-4,6+1-3} = 5-\beta_{3,4} = 5-\#b_{x,3} s.t. b_{x,3} \ge 4+2-x = 6-x = 5-3 = 2 = a_{4,3}$.

So applying the reflection twice brings $a_{i,j}$ to $a_{j,i}$. For instance, for $a_{i,j} = a_{1,1}$: $b_{6+1-1,6+1-1} = b_{6,6} = \text{undefined} \to c_{6+1-6,6+1-6} = c_{1,1} = 6+1-\beta_{6,6} = 7-\#b_{x,6}$ s.t. $b_{x,6} \ge 6+2-x = 7$.

To generalize this,

- If $a_{i,j}$ exists and $a_{i,j} \le j-i$:

  $$b_{r_1+1-j,r_1+1-i} = j-i+1-a_{i,j}$$

  special parts $\to$ special parts $\to$ special parts, non-special parts $\to$ undefined $\to$ non-special parts and not defined $\to$ non-special parts $\to$ undefined.

  By reflecting over the diagonal we return to $a_{i,j}$ from $b_{j,i}$.

  That is, $a_{i,j} \mapsto b_{r_1+1-j,r_1+1-i} \mapsto c_{r_1+1-(r_1+1-i),r_1+1-(r_1+1-j)} = c_{i,j}$. We need to show that $c_{i,j} = a_{i,j}$. So

1) $a_{i,j}$ (is a special part) $\mapsto$ j - i + 1 - $a_{i,j} \mapsto (r_1+1-i)-(r_1+1-j)+1-b_{r_1+1-j,r_1+1-i} = r_1+1-i-r_1-1+j+1-b_{r_1+1-j,r_1+1-i} = j-i+1-\underline{b_{r_1+1-j,r_1+1-i}} = j-i+1-$

$\underline{(j - i + 1 - a_{i,j})} = a_{i,j}$

2) $a_{i,j}$ (is not a special part and exists) $\mapsto$ undefined $\mapsto (r_1 + 1 - i) + 1 - \beta_{r_1+1-j, r_1+1-i}$.

$r_1+1-i+1-$#oversized entries in column r_1+1-j of reflection $= r_1+1-i+1-(r_1+1-i+1-a_{i,j})$ because column r_1+1-j refers to the r_1-jth row or the original partition...

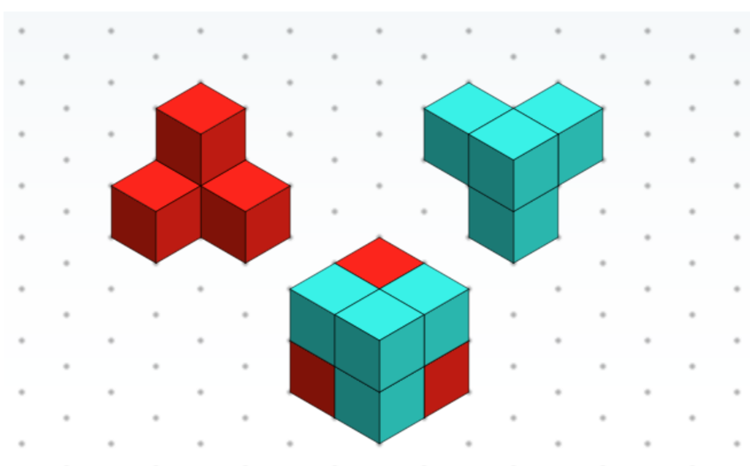
3) $a_{i,j}$ (is not defined) $\mapsto$ j + 1 - $\beta_{i,j} = b_{r_1+1-j, r_1+1-i}$ which is not a special part b/c it's $>$ j - i iff $i > \beta_{i,j} - 1 = |\{a_{x,i} | a_{x,i} \geq j + 2 - x\}| - 1 =$ at most, i - 1 because in column i there are i elements (at most) so it $\mapsto$ undefined $= a_{i,j}$.

Question 6.2.1

Show that any TSSCPP in β_{2n} must have a stack of height at least $n + 1$ at position (n, n).

Consider the requirement for a TSSCPP to have shells which are self-complementary and symmetric. The $n - th$ shell contains the 8-element cube from (n, n, n) to $(n + 1, n + 1, n + 1)$. If the element at $(n, n, n + 1)$ does not exist, then we do not have the self complementary requirement for the shells in a TSSCPP, therefore we must have the corresponding height of $n + 1$ at (n, n). The below picture provides a clear indication of the requirement with the nth shell having the $n + 1$ stack at (n, n), which uniquely defines a symmetric plane partition.

N-th Shell of a TSSCPP



Question 6.2.3

Fill in the missing values in the bird's-eye notation for the following $TSSCPP$ in $B(10, 10, 10)$ given the following lower-right quadrant:

$$\begin{array}{ccccc} 3 & 2 & 2 & 1 & ? \\ ? & 1 & 1 & 0 & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{array}$$

We know a few things about TSSCPPs. Each shell must be totally symmetric and self complementary, as well as the entire plane partition. The lower-right quadrant corresponds to the complement of the upper-left, meaning that each entry in the upper-left quadrant will be $2n - a_{i,j}$ for the corresponding entry in our lower right.

The following bird's-eye representation satisfies our initial condition.

$$\begin{array}{cccccccccc} 10 & 10 & 10 & 10 & 10 & 9 & 8 & 7 & 6 & 5 \\ 10 & 10 & 10 & 10 & 9 & 8 & 6 & 6 & 5 & 4 \\ 10 & 10 & 10 & 9 & 8 & 6 & 5 & 5 & 4 & 3 \\ 10 & 10 & 9 & 9 & 8 & 6 & 5 & 4 & 4 & 2 \\ 10 & 9 & 8 & 8 & 7 & 5 & 4 & 4 & 2 & 1 \\ 9 & 8 & 6 & 5 & 5 & 3 & 2 & 2 & 1 & 0 \\ 8 & 6 & 6 & 5 & 4 & 2 & 1 & 1 & 0 & 0 \\ 7 & 6 & 5 & 4 & 4 & 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 4 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Another creative way to visualize and build these objects is within Minecraft. Because this game is based on cubes (and so are our plane partitions), we can model TSSCPP's in a cool and fun way!

Question 6.2.4

Translate the TSSCPP of exercise 6.2.3 into a TSSCPP array and then into a nest of lattice paths.

We use the shells of the top-left to get our shells for the TSSCPP array.

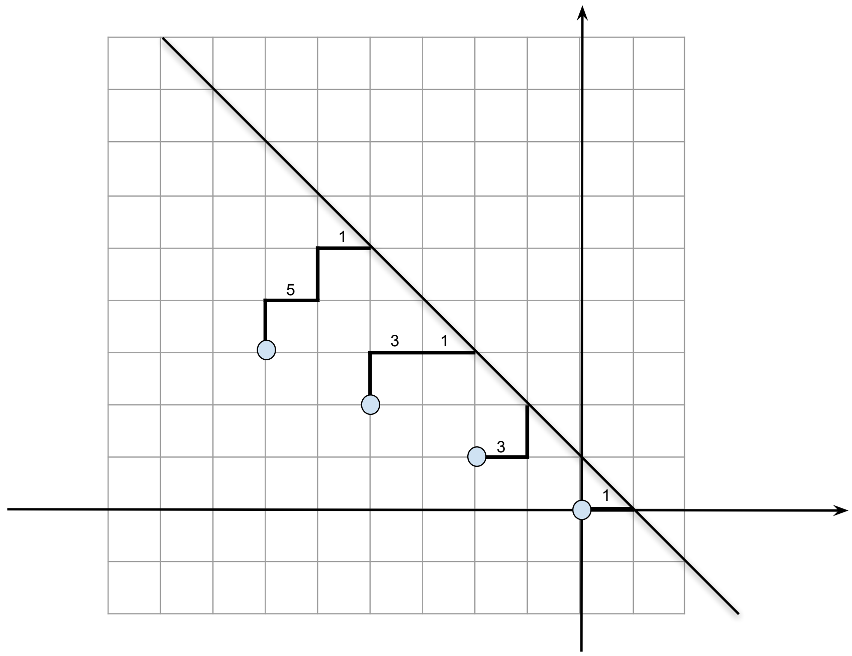
The first shell is 5,4,3,2,1. This corresponds to a row in the TSSCPP array of 9,5,1. The second shell, which starts at $(2,2)$ and tracks the elements of height 9 or greater, corresponds to a Ferrer's diagram of 4,3,3,1 and a corresponding TSSCPP row of 7,3,1.

The third, fourth, and fifth TSSCPP rows are 5,3; 3,1; and 1, respectively.

Arranging these according to the TSSCPP array rules gives us:

$$\begin{array}{ccccc} 9 & 5 & 1 & & \\ & 7 & 3 & 1 & \\ & & 5 & 3 & \\ & & & 3 & 1 \\ & & & & 1 \end{array}$$

The corresponding lattice paths are shown below.



Question 6.2.6

Prove that $|\mathcal{F}_{2n}| = (2n-1)|\mathcal{F}_{2n-2}|$, and therefore $|\mathcal{F}_{2n}| = 1 \cdot 3 \cdot 5 \cdots (2n-1)$

Consider an arbitrary perfect matching in F_{2n-1}. To go up to F_{2n}, we are adding the elements $2n-1$ and $2n$. The number of new perfect matchings of $\mathbf{F} \cup \{2n-1, 2n\}$ increases by $2n-2$, as we can have only new pairs which include $(2n, k)$, where $k \in \{1, ..., 2n-2\}$. For each one of these pairs, we will also have a pair which corresponds to $(a, 2n-1)$, where (a, k) was the pair for which we are taking k out of and pairing with $2n$. This covers every possible pair for both $2n$ and $2n-1$ matching with the original elements of $\mathbf{F}$. There is also the pair which corresponds to adding $(2n-1, 2n)$ to the perfect matching, giving us a total number of $(2n-1)$ more options per perfect matching.

This gives the total amount of perfect matchings in $\mathcal{F}_{2n}$ to be $(2n-1)|\mathcal{F}_{2n-2}$, which is exactly what we wanted to show.

Question 6.2.7

Prove that if n is odd, then any skew symmetric matrix must have determinant equal to zero.

A great tool from linear algebra tells us that the transpose of any symmetric matrix is itself. However, a skew symmetric matrix has a transpose which is the negation of itself.

$$A^T = -A$$

We also know a great identity which states that the determinants of A and A^T, which is that the determinants are equal. It is also true that $det(A) = -det(-A)$ if A is a square matrix with odd order.

When we combine these properties, we know that $det(A) = -det(A^T)$. This means that for an odd order skew-symmetric matrix, the determinant is equal to the negative of itself. The only number which is the negative of itself is zero, thus the determinant of any odd order skew-symmetric matrix must be zero.

Question 6.3.4

The second to last row of a size n monotomic triangle will consist of $n-1$ strictly increasing values between 1 and n inclusive. There are therefore n potential combinations.

The second farthest left column of a magog Triangle's values will be determined by the distance between the first and second left most lattice paths. Based on the starting points of TSSCPP lattice paths, there is a maximum distance of $n-1$ horizontal cubes between the two.

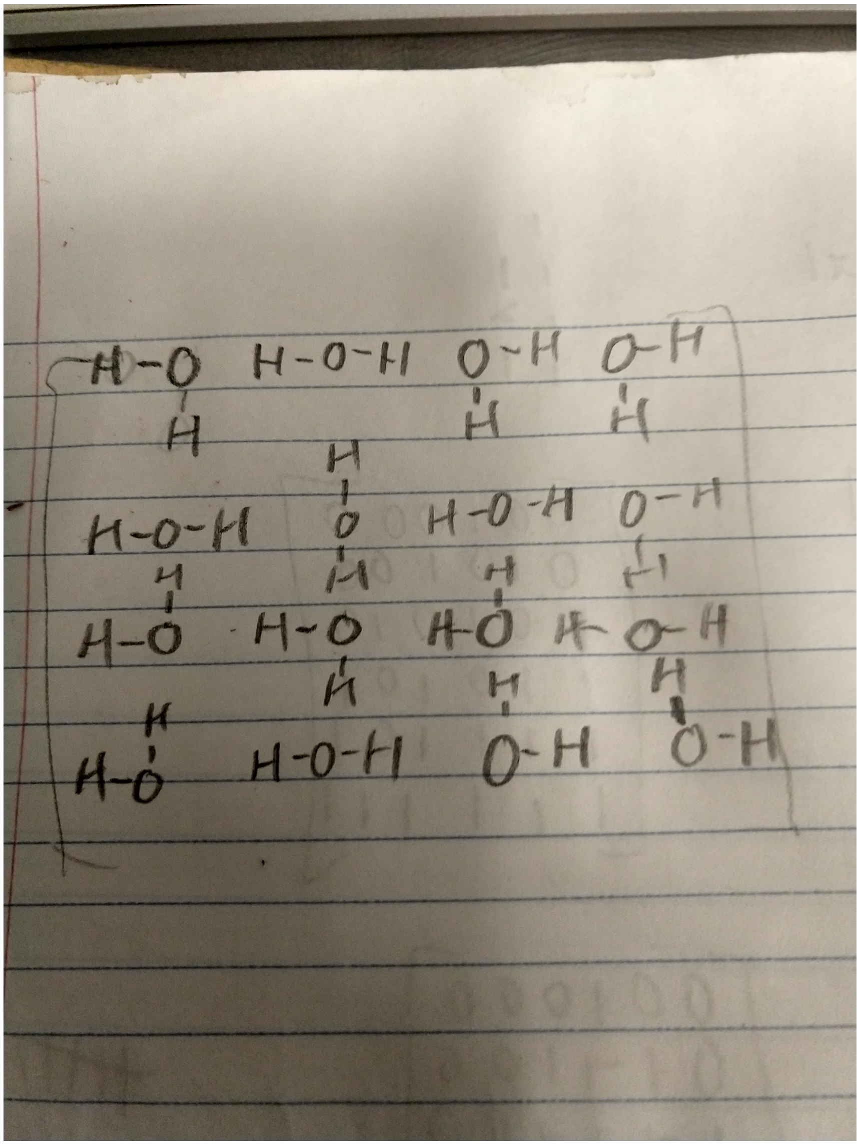
This means that there will be 0 or 1 cubes in the block form of the second leftmost column of the block version of the Magog triangle, with $n-1$ spaces to place them. This lees to n potential configurations.

Thus these values are equinumerous, which is what we wanted to show.

Q.E.D

Question 7.1.1

A drawing of the relevant square ice arrangement is attached:



Question 7.1.3

We know that in any square ice arrangement, the total number of edges directed up must be equal to the total number of edges directed down, and the total number of edges directed right must be equal to the total number of edges directed left.

Assume for the sake of contradicting that there was one more Southeast than Northwest oriented molecule (without loss of generality re vice versa and Southwest and Northeast oriented molecules).

Then we know that (the number right directed edges) - (the number left directed edges) = 1, and (number of down directed edges) - (number of up directed edges) = 1.

We cannot correct this imbalance with vertical or horizontal models, as neither of those changes the total comparison of edges. That leaves our only option to try to make it up with Southwest and Northeast molecules.

However, if there's n more Southwest than Northeast molecules then now right directed edges - left directed edges = 1-n, and (down directed edges) - (up directed edges) = 1+n.
There is now way with natural number values of n for $1 - n = 1 + n = 0$ to be true, and so we have a contradiction.

Thus there must be equal numbers of Southeast and Northwest oriented molecules, as well as Southwest and Northeast oriented molecules.