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Math 241
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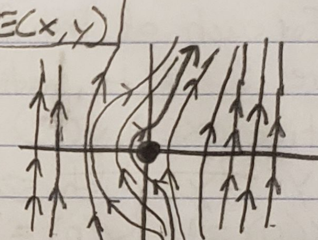
HW for February 22

p.172 24. $\frac{dx}{dt} = e^x - 1$, $\frac{dy}{dt} = 3x - xy$.

$e^0 - 1 = 3\cdot 0 - 0\cdot 0 = 0$ implies that $(0,0)$ is a stationary point.
The linear approximation then becomes $\frac{dx}{dt} = a(x-0) + b(y-0)$,
$\frac{dy}{dt} = c(x-0) + d(y-0)$.
where $a = \frac{\partial f}{\partial x}(0,0) = e^0 = 1$ and $b = \frac{\partial f}{\partial y}(0,0) = 0$ and $c = \frac{\partial g}{\partial x}(0,0)$
$= 3 - 0 = 3$ and $d = \frac{\partial g}{\partial y}(0,0) = -0 = 0$.
So $\underline{\frac{dx}{dt} = x, \frac{dy}{dt} = 3x}$ is the linear approximation.

48. $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = 12x^2$.
(a) Find a first integral.
$12x^2 dx - 2y\,dy = 0 \Rightarrow 12x^2 dx = 2y\,dy$.
$6x^2 dx = y\,dy \Rightarrow 6\cdot\int x^2 dx = \int y\,dy$
$6\cdot(\frac{1}{3}x^3 + C) = \frac{1}{2}y^2 \Rightarrow 2x^3 + 6C = \frac{1}{2}y^2$
$6C = \frac{1}{2}y^2 - 2x^3 \Rightarrow C = \boxed{\frac{1}{12}y^2 - \frac{1}{3}x^3 = E(x,y)}$
(b) Sketch the phase portrait.
((0,0) is the only stationary point)



54. Explain why it is not practical to draw a phase portrait for a nonautonomous system $\frac{dx}{dt} = f(t,x,y)$, $\frac{dy}{dt} = g(t,x,y)$.
The system would have too many variables so a phase portrait would need to be drawn for different values of t and this would require too many phase portraits.

61. (a) Show that if $(x(t), y(t))$ is a solution of the system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$, then $(x(-t), y(-t))$ is a solution of $\frac{dx}{dt} = -f(x,y)$, $\frac{dy}{dt} = -g(x,y)$.
So if $x'(t) = f(x,y)$, $-x'(t) = -f(x,y)$. And so
$\frac{d}{dt}(x(-t)) = x'(-t)\cdot\frac{d}{dt}(-t) = x'(-t)\cdot -1 = -x'(-t) =$
$-x'(t)$ by definition $= \underline{-f(x,y)}$. By the same logic,
$\frac{d}{dt}(y(-t)) = -g(x,y)$. This makes $(x(-t), y(-t))$ a solution of $x'(t) = -f(x,y)$, $y'(t) = -g(x,y)$.

p.172 61. (b) Show that the two systems in part (a) have the same trajectories, but that these are traversed in opposite directions.
The 1st system has a solution $x(t)$, $y(t)$ which corresponds to the solution $x(-t)$, $y(-t)$ in the 2nd system. So when t increases, $-t$ decreases which means that with respect to time the two systems are traversed in opposite directions.
(c) Explain why the system $\frac{dx}{dt} = -f(x,y)$, $\frac{dy}{dt} = -g(x,y)$ is said to be obtained by time reversal from the system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$.
They are traversed in opposite trajectories (which means that the directions of their phase portraits are opposite) and so the direction of time $-t$ is a reversal of t.

63. For each of the following pairs of initial conditions, use Figure 3.2.6 to predict what will happen as $t \to \infty$ to the solution $(x(t), y(t))$ of the system $\frac{dx}{dt} = x^2 - 2xy - x$, $\frac{dy}{dt} = xy + y^2 - 3y$ which satisfies those conditions.
(a) $x(0) = 4$, $y(0) = 3$.
The solution will approach $+\infty$ from the right side of the y-axis.
(b) $x(0) = 2$, $y(0) = 1.5$.
The solution approaches the origin $(0,0)$.

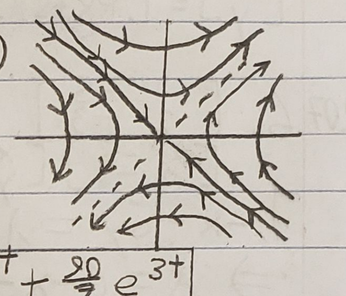
p.183 12. Show that the characteristic equation of the second-order differential equation obtained by elimination from the system
$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$
is $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$.
The first equation says $by = x' - ax \Rightarrow y = \frac{x' - ax}{b} = \frac{1}{b}x' - \frac{a}{b}x$
$\Rightarrow y' = \frac{1}{b}x'' - \frac{a}{b}x'$, so the second equation implies $\frac{1}{b}x'' - \frac{a}{b}x'$
$= cx + \frac{d}{b}x' - \frac{ad}{b}x \Rightarrow \frac{1}{b}x'' + \left(-\frac{a}{b} - \frac{d}{b}\right)x' + \left(\frac{ad}{b} - c\right)x = 0 \Rightarrow$
$x'' - (a+d)x' + (ad - bc)x = 0$. Say $x = e^{\lambda t}$. Then
$\lambda^2 e^{\lambda t} - (a+d)\lambda e^{\lambda t} + (ad - bc)e^{\lambda t} = 0$
$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$ is the characteristic equation.

p.197 15. $\frac{dx}{dt} = -5x+8y$, $\frac{dy}{dt} = 4x-y$, $x(0)=0$, $y(0)=15$.

(a) Solve the system or initial value problem using eigenvectors.

$-5x+8y = \lambda x$ and $4x-y=\lambda y \Rightarrow \lambda = -\frac{5x+8y}{x}$
$= \frac{4x-y}{y}$ so $y = -\frac{x}{2}$ and $\lambda = (4x+\frac{x}{2})/(-\frac{x}{2}) = \frac{8x+x}{-x}$
$= -\frac{9x}{x} = -9$ is one solution, and $y=x$ and $\lambda = \frac{4x-x}{x} = 3$
is another solution so $x(t) = Ce^{-9t}+De^{3t}$, $y(t) = -\frac{1}{2}\cdot$
$Ce^{-9t}+3De^{3t}$ is the general solution.

(Now (x,x) describes all stationary points)(b)



So $x(0)=0=C+D \Rightarrow C=-D$. Then
$y(0)=15=\frac{1}{2}D+3D \Rightarrow 30 = D+6D$
$\Rightarrow D = 30/7 \Rightarrow C=-30/7$.
So $\boxed{x(t) = -\frac{30}{7}e^{-9t}+\frac{30}{7}e^{3t},\ y(t) = \frac{15}{7}e^{-9t}+\frac{90}{7}e^{3t}}$.

22. Consider the system $\frac{dx}{dt} = ax+by$, $\frac{dy}{dt} = cx+dy$.

(a) Show that if $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions of the system, then so is any linear combination $C\cdot(x_1,y_1)+D\cdot(x_2,y_2) = (Cx_1+Dx_2, Cy_1+Dy_2)$, where C, D are constants.

$\frac{d}{dt}(Cx_1+Dx_2) = Cx_1'+Dx_2' = C\cdot(ax_1+by_1)+D\cdot$
$(ax_2+by_2) = a(Cx_1+Dx_2)+b\cdot(Cy_1+Dy_2)$.
And $\frac{d}{dt}(Cy_1+Dy_2) = Cy_1'+Dy_2' = C\cdot(cx_1+dy_1)+$
$D\cdot(cx_2+dy_2) = c\cdot(Cx_1+Dx_2)+d\cdot(Cy_1+Dy_2)$
which shows that (Cx_1+Dx_2, Cy_1+Dy_2) is a solution (for any C and D) by definition.

(b) Show that if $(\varepsilon_1, \varepsilon_2)$ is an eigenvector of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for the eigenvalue λ, then $e^{\lambda t}(\varepsilon_1,\varepsilon_2) = (e^{\lambda t}\varepsilon_1, e^{\lambda t}\varepsilon_2)$ is a solution of the system.

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \lambda\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \Rightarrow a\varepsilon_1+b\varepsilon_2 = \lambda\varepsilon_1$, $c\varepsilon_1+d\varepsilon_2 = \lambda\varepsilon_2$.
So when $(x,y) = (e^{\lambda t}\varepsilon_1, e^{\lambda t}\varepsilon_2)$, $\frac{dx}{dt} = \lambda\varepsilon_1 e^{\lambda t} = ae^{\lambda t}\varepsilon_1$
$+be^{\lambda t}\varepsilon_2$ and $\frac{dy}{dt} = \lambda e^{\lambda t}\varepsilon_2 = ce^{\lambda t}\varepsilon_1+de^{\lambda t}\varepsilon_2$ so by definition $(e^{\lambda t}\varepsilon_1, e^{\lambda t}\varepsilon_2)$ is a solution.

(c) Note that by part (b), if there are eigenvectors (P,R)

p.197 22. (c) and (Q, S) for the (distinct) eigenvalues λ_1 and λ_2, then $(e^{\lambda_1 t}\cdot P, e^{\lambda_1 t}\cdot R)$ and $(e^{\lambda_2 t}\cdot Q, e^{\lambda_2 t}\cdot S)$ will be solutions; hence by part (a),

$(Ce^{\lambda_1 t}\cdot P + De^{\lambda_2 t}\cdot Q,\ Ce^{\lambda_1 t}\cdot R + De^{\lambda_2 t}\,S)$ will be a solution. Check that this solution is the same as the general solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$ found in the text.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} PX+QY \\ RX+SY \end{bmatrix} = (Ce^{\lambda_1 t}P + De^{\lambda_1 t}Q,\ Ce^{\lambda_1 t}R + De^{\lambda_2 t}S)$.

p.207 6. $\begin{bmatrix} -1 & -5 \\ 2 & -3 \end{bmatrix}$. We want to find x, y, λ such that $\begin{bmatrix} -1 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\cdot\begin{bmatrix} x \\ y \end{bmatrix}$ which means that

$-x-5y = \lambda x$ and $2x-3y = \lambda y$.

$\Rightarrow \lambda = (-x-5y)/x = (2x-3y)/y$

$-x-5y = (2x^2-3xy)/y$

$-xy-5y^2 = 2x^2-3xy \Rightarrow -5y^2 = 2x^2-2xy$

$5y^2-2xy+2x^2 = 0 \Rightarrow y = [2x \pm \sqrt{(-2x)^2-4(5\cdot 2x^2)}]/10 = \frac{2}{10}x \pm \frac{1}{10}\cdot\sqrt{4x^2-40x^2} = \frac{1}{5}x \pm \frac{1}{10}\cdot(x\cdot 6i)$. So

$y = (\frac{1}{5} \pm \frac{3}{5}i)x$ and $\lambda = -2 \pm 3i$. So there are two sets of values. $\underline{\begin{bmatrix} x \\ (\frac{1}{5}+\frac{3}{5}i)x \end{bmatrix} \text{ and } \lambda = -2-3i}$, or $\underline{\begin{bmatrix} x \\ (\frac{1}{5}-\frac{3}{5}i)x \end{bmatrix} \text{ and } \lambda = -2+3i}$.

13. $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = -8x$.

$2y = \lambda x$ and $-8x = \lambda y \Rightarrow \lambda = \frac{2y}{x} = -\frac{8x}{y} \Rightarrow$

$\frac{2y}{x} = -\frac{8x}{y} \Rightarrow 2y^2 = -8x^2 \Rightarrow y^2 = -4x^2 \Rightarrow y = \sqrt{-4x^2} =$

$\pm x\sqrt{-4} = \pm 2ix$. So $y = 2ix$ and $\lambda = \frac{4ix}{x} = 4i$ or

$y = -2ix$ and $\lambda = -\frac{4ix}{x} = -4i$ are solutions.

$$\boxed{\begin{array}{l} x(t) = Ce^{4it} + De^{-4it},\quad y(t) = 2iCe^{4it} - 2iDe^{-4it}. \\ \text{or } x(t) = (C+D)\cos 4t + (C-D)\,i\sin 4t, \\ \quad y(t) = 2(C-D)i\cos 4t - 2\cdot(C+D)\sin 4t \text{ equivalently} \end{array}}$$

22. $\frac{dx}{dt} = 2x+2y$, $\frac{dy}{dt} = -x+4y$, $x(0) = 1$, $y(0) = 3$.

$2x+2y = \lambda x$ and $-x+4y = \lambda y \Rightarrow \lambda = \frac{2x+2y}{x} = \frac{-x+4y}{y}$ so

$2x+2y = (-x^2+4xy)/y \Rightarrow 2xy+2y^2 = -x^2+4xy \Rightarrow$

$2y^2-2xy+x^2 = 0 \Rightarrow y = (\frac{1}{2} \pm \frac{i}{2})x$ and $\lambda = 3 \pm i$ so

$x(t) = Ce^{(3+i)t} + De^{(3-i)t}$, $y(t) = (\frac{1}{2}+\frac{i}{2})Ce^{(3+i)t} + (\frac{1}{2}-\frac{i}{2})De^{(3-i)t}$.

p.207 22. This means $x = Ce^{3t}e^{it} + De^{3t}e^{-it}$, $y = (\frac{i}{2} + \frac{1}{2})Ce^{3t}e^{it} + (\frac{1}{2} - \frac{i}{2})De^{3t}e^{-it}$ $\Rightarrow$ $\boxed{x(t) = (C+D)e^{3t}\cos t + (C-D)i\sin t,}$

$\boxed{y(t) = (C - iD)e^{3t}\frac{1+i}{2}\cos t + (C+iD)e^{3t}\frac{i-1}{2}\sin t}$

26. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with real entries that has conjugate eigenvalues λ and $\bar{\lambda}$. Show that if $(\varepsilon_1, \varepsilon_2)$ is an eigenvector for λ, then $(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$ is an eigenvector for $\bar{\lambda}$.

So if $\lambda = \mu + iv$ and $\varepsilon_1 = e + if$, $\varepsilon_2 = g + ih$,

$\lambda\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix} = A \cdot \begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix} \Rightarrow (\mu + iv)(e + if) = ae + iaf + be + ibf$

$\mu e - vf + iev + i\mu f = ae + be + \underline{iaf + ibf}$. (1)

Now for $\bar{\lambda}\begin{bmatrix}\bar{\varepsilon}_1 \\ \bar{\varepsilon}_2\end{bmatrix} = A \cdot \begin{bmatrix}\bar{\varepsilon}_1 \\ \bar{\varepsilon}_2\end{bmatrix}$ (which we want to show),

$(\mu - iv)\cdot(e - if) = ae - iaf + be - ibf$

$\mu e - vf - (iev + i\mu f) = ae + be - (iaf + ibf)$ which is true (2) based on the previous equation. By similar logic the bottom rows also are equivalent so by definition $(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$ is an eigenvector for $\bar{\lambda}$.

p.229 23. Let $X = Ce^{\lambda_1 t}$, $Y = De^{\lambda_2 t}$ be the general solution, in the new coordinates, of a system with two distinct real eigenvalues λ_1 and λ_2 (see pp. 210-211).

(a) Show that the trajectories lie in the various quadrants of the X,Y-plane as follows: $C>0, D>0$: first quadrant; $C<0$, $D>0$: second quadrant; $C<0, D<0$: third quadrant; $C>0$, $D<0$: fourth quadrant.

These results follow because of the fact that $e^{\lambda t}$ is going to be positive, so the signs of C and D will determine whether a point is in quadrant 1 $(X+, Y+)$, quadrant 2 $(X-, Y+)$, quadrant 3 $(X-, Y-)$, or quadrant 4 $(X+, Y-)$.

(b) Show that the trajectories in the second quadrant can be found from those in the first quadrant by reflection in the Y-axis. All trajectories in Q_1 can be described by $Ce^{\lambda_1 t}$, $De^{\lambda_2 t}$ where $C, D > 0$. By definition, $-Ce^{\lambda_1 t}$, $De^{\lambda_2 t}$ will describe these in Q_2.

p.229 23. (c) Show that the trajectories in the third and fourth quadrants can be found from the others by reflection in the X-axis.

Say we have any $C, D > 0$. So $(Ce^{\lambda_1 t}, De^{\lambda_2 t})$ are in Q1 and $(-Ce^{\lambda_1 t}, De^{\lambda_2 t})$ describe all trajectories in Q2 as well. This means that by definition, we can reflect the set of all trajectories in Q1 (by negating Y) over the X-axis to get $(Ce^{\lambda_1 t}, -De^{\lambda_2 t})$ which describes all trajectories in Q4. Similarly, $(-Ce^{\lambda_1 t}, De^{\lambda_2 t}) \subset$ trajectories in Q2 and reflecting over $Y=0$ gives $(-Ce^{\lambda_1 t}, -De^{\lambda_2 t})$ which are the trajectories in Q3 (for C, D positive).