Dean Gladish
Math 241
Professor Krusemeyer
HW for Monday, January 28

p. 72 31. Consider the initial value problem $-xy + (1-x^2)\frac{dy}{dx} = 0$, $y(0) = 2$.

(a) Show that the equation is separable, and solve the initial value problem this way.

So $(1-x^2)\frac{dy}{dx} = xy \Rightarrow \frac{1-x^2}{x}\frac{dy}{dx} = y$

$\frac{1}{y}\frac{dy}{dx} = \frac{x}{1-x^2} \Rightarrow \int \frac{1}{y}dy = \int \frac{x}{1-x^2}dx \Rightarrow \log(y) = -\frac{1}{2}\log(1-x^2)$ $+C$. We can say $\log(1-x^2)$ b/c $1-x^2 = 1-0^2 = 1$ is positive (given $y(0)=2$).

So $y(x) = e^{-\frac{1}{2}\log(1-x^2)+C} = e^C \cdot e^{\log((1-x^2)^{-1/2})}$.

$y(x) = C\cdot(1-x^2)^{-1/2}$ (the new $C = e^{\text{old } C}$)

$= \frac{C}{\sqrt{1-x^2}}$.

Given $y(0) = 2 = \frac{C}{\sqrt{1-0^2}} \Rightarrow C = 2$,

$\boxed{y(x) = \frac{2}{\sqrt{1-x^2}}}$.

(b) Show that there is an integrating factor of the form $\mu = \mu(x)$, and solve the initial value problem this way.

Say $\mu = \mu(x)$. Then $\mu(x)M(x,y) + \mu(x)N(x,y)\frac{dy}{dx} = 0$ is exact $\Leftrightarrow \frac{\partial}{\partial y}(-xy\mu(x)) = \frac{\partial}{\partial x}((1-x^2)\cdot\mu(x))$

$-x\mu(x) + (-xy)\cdot 0 = -2x\mu(x) + (1-x^2)\mu'(x)$

$-x\mu(x) = -2x\mu(x) + (1-x^2)\frac{d\mu}{dx}$

$x\mu(x) = (1-x^2)\frac{d\mu}{dx}$

$\int \frac{x}{1-x^2}dx = \int \frac{1}{\mu(x)}d\mu \Rightarrow C - \frac{1}{2}\log(1-x^2) = \log(\mu(x))$

$\boxed{\mu(x) = \frac{1}{\sqrt{1-x^2}}}$ when $C = 1$. (this works b/c $1-x^2 = 1-0^2 > 0$)

Now, $\int \mu(x)M(x,y)dx = E(x,y) = \int \frac{-xy}{\sqrt{1-x^2}}dx = \sqrt{1-x^2}\,y + C$.

Since $\frac{\partial}{\partial y}(\sqrt{1-x^2}\,y + C) = \sqrt{1-x^2} = \sqrt{1-x^2}\cdot\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$

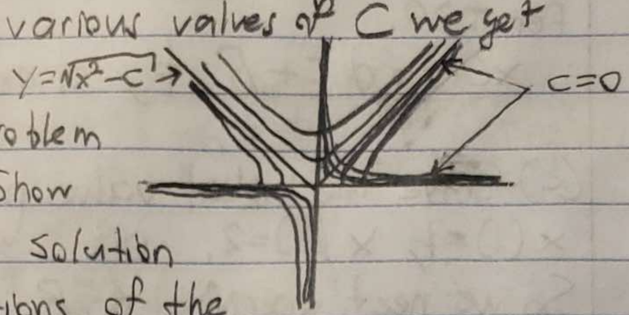
$= \frac{1-x^2}{\sqrt{1-x^2}}$

$= \mu(x)\cdot N(x,y) = \mu(x)\cdot(1-x^2)$,

C can be 0. So $\boxed{E(x,y) = \sqrt{1-x^2}\,y = C}$ is a solution (that is, $y(x) = \frac{C}{\sqrt{1-x^2}}$.) where C is arbitrary constant.

(c) What would have changed in part (a), and in part (b), if the initial condition had been $y(2) = 3$?

p.72 66. $y=\sqrt{x^2-c}$ as the level curves, and if we plot these and the orthogonal trajectories for various values of C we get



p.94 7. Consider the initial value problem
$x''=x^{1/3}$, $x(0)=0$, $x'(0)=0$. Show that in addition to the obvious solution $x=0$, there are several solutions of the form $x=Ct^3$ (for which values of C?). Does this contradict Theorem 2.2.1 (p. 86)?

Given $\frac{d^2x}{dt^2}=f(t,x,v)=x^{1/3}$, $x(0)=0$, $v(0)=0$ we know that having multiple solutions (instead of a unique solution) $\boxed{\text{does not contradict Theorem 2.2.1}}$ b/c the theorem only tells us something if $\frac{\partial f}{\partial x}$ (and also f, $\frac{\partial f}{\partial v}$) is defined and continuous in a rectangular box containing the point (t_0,x_0,v_0) in its interior.

In our case, $\frac{\partial f}{\partial x}=\frac{1}{3}\cdot x^{-2/3}=\frac{1}{3x^{2/3}}$ which is not defined at $(0,0,0)=(t_0,x_0,v_0)$, which is contained in all described rectangular boxes. So it doesn't contradict Theorem 2.2.1.

To show multiple solutions exist,
Suppose $x=Ct^3 \Rightarrow x'=3Ct^2$ and $x''=6Ct$.
Given $x''=x^{1/3}$, $6Ct=(Ct^3)^{1/3}$
$$216C^3t^3=Ct^3$$
$$216C^2=1$$
$C^2=\frac{1}{216}$ so $\boxed{C=\frac{1}{6\sqrt{6}}}$. This is a value of C that allows $x=\frac{1}{6\sqrt{6}}t^3$ to satisfy the conditions.

p.94 16. (a) Show that $x_1(t)=t$ and $x_2(t)=\frac{1}{t}$ are solutions to the equation $t^2x''+tx'-x=0$.

$$t^2x_1''+tx_1'-x_1=t^2\frac{d^2}{dt^2}(t)+t\cdot\frac{d}{dt}(t)-t$$
$$=t^2\cdot 0+t\cdot 1-t=0.$$
$$t^2x_2''+tx_2'-x_2=t^2\frac{d^2}{dt^2}\left(\frac{1}{t}\right)+t\cdot\frac{d}{dt}\left(\frac{1}{t}\right)-\frac{1}{t}$$
$$=t^2\cdot 2t^{-3}+t\cdot -t^{-2}-\frac{1}{t}$$
$$=2t^{-1}-t^{-1}-t^{-1}=0.$$

p.72 31. (c) In part (a), we now have $y(2)=3$ which implies that $1-x^2$ becomes negative instead of positive making $\log(1-x^2)$ impractical.
So we would separate the equation and instead of doing $\int \frac{1}{y}dy = \int \frac{x}{1-x^2}dx$, we calculate $\int -\frac{1}{y}dy = \int \frac{x}{x^2-1}dx$

$\Rightarrow -\log(y) = \frac{1}{2}\log(x^2-1) + C$

$e^{-\log(y)} = e^{\log((x^2-1)^{1/2})+C}$

$\frac{1}{y} = e^c \cdot (x^2-1)^{1/2} \Rightarrow \frac{1}{y} = C\cdot\sqrt{x^2-1}$ (new $C = e^{\text{old } C}$)

$y(x) = \frac{1}{C\sqrt{x^2-1}}$.

$y(2)=3$ So $3 = \frac{1}{C\sqrt{4-1}} \Rightarrow C3\sqrt{3}=1 \Rightarrow C = \frac{1}{3\sqrt{3}}$. So

$\boxed{y(x) = \frac{3\sqrt{3}}{\sqrt{x^2-1}}}$ (instead of $\frac{2}{\sqrt{1-x^2}}$).

In part (b), $1-x^2 = 1-4^2 < 0$ given $y(2)=3$ now,
so instead of $\int \frac{x}{1-x^2}dx = \int \frac{1}{\mu(x)}d\mu$ we should find
$\int \frac{x}{x^2-1}dx = \int -\frac{1}{\mu(x)}d\mu$ (to avoid $\log(1-x^2)$).
So $\int x(x^2-1)^{-1/2}dx = \int -\frac{1}{\mu(x)}d\mu$

$C + \frac{1}{2}\log(x^2-1) = -\log(\mu(x))$

$e^c e^{\log((x^2-1)^{1/2})} = e^{\log(\mu(x)^{-1})} \Rightarrow \underline{C}\cdot(x^2-1)^{1/2} = \frac{1}{\mu(x)}$

(new $C = e^{\text{old } C}$) $\Rightarrow \mu(x) = \frac{1}{C\sqrt{x^2-1}}$. Let $C=1$, then
$\boxed{\mu(x) = \frac{1}{\sqrt{x^2-1}}}$ is an integrating factor (which is different from $\mu(x) = \frac{1}{\sqrt{1-x^2}}$).
Additionally, $E(x,y)$ changes to $\boxed{-\sqrt{x^2-1}\,y = C = E(x,y)}$ (instead of $\sqrt{1-x^2}\,y = C$) which results in the new $y(x)$ shown in part (a).

66. $x^2-y^2=c$. The orthogonal trajectories are the solution curves of the differential equation $\frac{dy}{dx} = \frac{\partial E/\partial x}{\partial E/\partial x}$.
$\frac{\partial E}{\partial y} = -2y$ and $\frac{\partial E}{\partial x} = 2x$ so solving $\frac{dy}{dx} = \frac{-2y}{2x} \Rightarrow -\frac{1}{2}\int\frac{1}{y}dy = \frac{1}{2}\int\frac{1}{x}dx$

$\Rightarrow -\frac{1}{2}\log(y) = \frac{1}{2}\log(x) + C \Rightarrow e^{-\frac{1}{2}\log(y)} = e^{\frac{1}{2}\log(x)+C}$

$e^{\log(y^{-1/2})} = e^{\log(x^{1/2})}e^c \Rightarrow y^{-1/2} = e^c x^{1/2}$

$\sqrt{y} = \frac{1}{e^c\sqrt{x}} \Rightarrow \boxed{y(x) = \frac{1}{e^{2c}x}}$ are the orthogonal trajectories.

We are given $x^2-y^2=c \Rightarrow -y^2 = c-x^2 \Rightarrow \boxed{y=\sqrt{x^2-c}}$

p.106 18. $t^2x'' - 6tx' + 10x = 0$, $x(1) = 0$, $x'(1) = 1$.
(One solution is $x = t^2$.)
We want to look for solutions of the form $x = t^\lambda$, so
assume there are more; $t^2 \frac{d^2}{dt^2}(t^\lambda) - 6t\frac{d}{dt}(t^\lambda) + 10t^\lambda = 0$
$\Rightarrow t^2 \cdot \lambda(\lambda-1)t^{\lambda-2} - 6t\lambda t^{\lambda-1} + 10t^\lambda = 0$.
So when $t = 1$, $\lambda(\lambda-1) - 6\lambda + 10 = 0$
$\lambda^2 - \lambda - 6\lambda + 10 = 0 \Rightarrow \lambda^2 - 7\lambda + 10 = 0$
$\Rightarrow (\lambda-5)(\lambda-2) = 0$.
So $\lambda_1 = 5$, $\lambda_2 = 2$. So the other basic solution is $x = t^5$.
Then the general solution is $x(t) = \alpha t^2 + \beta t^5$
($x'(t) = 2\alpha t + 5\beta t^4$ and $x''(t) = 2\alpha + 20\beta t^3$.)
$x(1) = 0$ so $0 = \alpha + \beta \Rightarrow \alpha = -\beta$
$x'(1) = 1 = 2\alpha + 5\beta = -2\beta + 5\beta = 3\beta$
$\beta = 1/3$ and so $\alpha = -1/3$.
Then $\boxed{x(t) = -\frac{1}{3}t^2 + \frac{1}{3}t^5.}$

19. $t^2x'' - 6tx' + 10x = 0$, $x(0) = 0$, $x'(0) = 1$.
(One solution is $x = t^2$.)
As in the previous question (18) we get
$x(t) = \alpha t^2 + \beta t^5$. However, $x'(t) = 2\alpha t + 5\beta t^4$
will be 0 when $t = 0$ so there are no α, β such that
$x'(0) = 1$. So $\boxed{\text{no solution}}$ exists.

p.110 4. $z = e^{(-2+5i)t} = e^{-2t}e^{5it}$.
By Euler's formula $e^{5it} = \cos 5t + i\sin 5t$.
So $\mathrm{Re}(z) = e^{-2t}\cos 5t$, $\mathrm{Im}(z) = e^{-2t}\sin 5t$.

10. $x'' + 2x' + 10x = 0$.
The characteristic equation $\lambda^2 + 2\lambda + 10 = 0$ has solutions
$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$. So $x = e^{-t}e^{3it}$.
Euler's formula says $e^{3it} = \cos 3t + i\sin 3t$ so
$\boxed{x(t) = e^{-t}(\alpha\cos 3t + \beta\sin 3t)}$ and we don't do the
other case because α, β are arbitrary.

p.94 16. (b) Find the general solution to this differential equation for $t > 0$.

$x(t) = \alpha t + \beta \cdot \frac{1}{t}$, α and β are arbitrary constants.

(c) Solve the initial value problem $t^2 x'' + t x' - x = 0$, $x(1) = 1$, $x'(1) = 2$.

So we need specific α, β. $x'(t) = \alpha - \beta t^{-2}$ and $x''(t) = 2\beta t^{-3}$, so

given $x(1) = 1 = \alpha + \beta$, $\alpha = 1 - \beta$.

$x'(1) = \alpha - \beta \cdot 1^{-2} = 2 \Rightarrow$ substituting $\alpha = 1 - \beta$,

$x'(1) = 1 - \beta - \beta = 2$

$-2\beta = 1$

$\beta = -\frac{1}{2}$ so $\alpha = 1 - -\frac{1}{2} = \frac{3}{2}$.

$x(t) = \frac{3}{2} t - \frac{1}{2} \cdot \frac{1}{t}$

$\boxed{x(t) = \frac{3t}{2} - \frac{1}{2t}}$

p.106 16. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$, $y(5) = 6$, $y'(5) = 0$.

The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$, which gives the double root $\lambda = -2$.

So the basic solutions are e^{-2x} and xe^{-2x} (found using reduction of order).

This means that the general solution is $y(x) = \alpha e^{-2x} + \beta x e^{-2x}$.

$\Rightarrow y'(x) = -2\alpha e^{-2x} - 2\beta x e^{-2x}$.

Given that $y(5) = 6 = \alpha e^{-10} + \beta \cdot 5e^{-10} \Rightarrow \alpha e^{-10} = 6 - \beta \cdot 5e^{-10}$,

$\alpha = 6e^{10} - 5\beta$. So $y(x) = (6e^{10} - 5\beta) \cdot e^{-2x} + \beta x e^{-2x}$

$= 6e^{10-2x} - 5\beta e^{-2x} + \beta x e^{-2x} = 6e^{10-2x} + (x-5)\beta e^{-2x}$.

$\Rightarrow y'(x) = -12e^{10-2x} + 1 \cdot \beta e^{-2x} + (x-5) \cdot -2\beta e^{-2x}$

If $y'(5) = 0 = -12e^0 + \beta e^{-10} + 0$, then

$0 = -12 + \beta e^{-10} \Rightarrow \beta e^{-10} = 12 \Rightarrow \beta = 12e^{10}$.

(So $\alpha = 6e^{10} - 60 \cdot e^{10} = -54e^{10}$) and then

$y(x) = 6e^{10-2x} + (x-5) \cdot 12e^{10} \cdot e^{-2x}$

$y(x) = 6e^{10-2x} + 12(x-5)e^{10-2x} = \boxed{e^{10-2x} \cdot (12x - 54) = y(x)}$

p.117 4. $x'' - 5x' + 6x = 3\sin t - \cos t$.

Differentiating $\sin t, \cos t$ yields $\cos t, -\sin t$ so we will look for a solution of the form $x(t) = A\sin t + B\cos t$,

So $x'(t) = A\cos t - B\sin t$

$x''(t) = -A\sin t - B\cos t$.

Given that $x'' - 5x' + 6x = 3\sin t - \cos t$,

$x'' - 5x' + 6x = (-A\sin t - B\cos t) - 5\cdot(A\cos t - B\sin t)$
$+ 6\cdot(A\sin t + B\cos t)$
$= -A\sin t - B\cos t - 5A\cos t + 5B\sin t + 6A\sin t + 6B\cos t$
$(-A + 5B + 6A)\sin t + (-B - 5A + 6B)\cos t$
$(5A + 5B)\sin t + (-5A + 5B)\cos t = 3\sin t - \cos t$.

So $\begin{cases} 5A + 5B = 3 \\ -5A + 5B = -1 \end{cases}$ $\Rightarrow 5A = 3 - 5B$
So $5B - 3 + 5B = -1$
$\Rightarrow 10B = 2$, and so $B = 1/5 \Rightarrow 5A = 3 - 1 = 2$
$A = 2/5$.

So a solution is $\boxed{x(t) = \frac{2}{5}\sin t + \frac{1}{5}\cos t}$.

(The general solution is the sum of this and all solutions of the homogeneous equation.)

20. $x'' - 4x' + 3x = t + 1$, $x(0) = 0$, $x'(0) = 0$.

First we look for a solution of the form $At + B$ and get

$x(t) = At + B$, $x'(t) = A$, $x''(t) = 0$.

So $0 - 4A + (3At + 3B) = t + 1$ for all t.

When $t = 0$, $-4A + 3B = 1$ which implies that since

$-4A + 3B + 3At = 1 + t$, $3At = t$

So $3A = 1 \Rightarrow A = 1/3 \Rightarrow B = (1 + \frac{4}{3})/3 = (\frac{7}{3})/3 = \frac{7}{9}$.

So $x(t) = \frac{1}{3}t + \frac{7}{9}$ is a solution to the inhomogeneous equation but will not solve the initial values so we need to add the solutions of the homogeneous equation

$x'' - 4x' + 3x = 0$.

The characteristic equation $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow$

$(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 1$.

p.110 19. $x'' + 6x' + 10x = 0$, $x(0) = 1$, $x'(0) = 3$.

The characteristic equation $\lambda^2 + 6\lambda + 10 = 0$ is not easily factored so use the quadratic formula to obtain the basic solutions.

$(-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}) / (2 \cdot 1) = (-6 \pm \sqrt{36 - 40}) / 2 =$

$(-6 \pm \sqrt{-4}) / 2 = (-6 \pm 2i) / 2$.

So $x_1(t) = e^{-3+i}$ and $x_2(t) = e^{-3-i}$. Then

$x(t) = (\alpha e^{-3}) e^{it} + (\beta e^{-3}) e^{-it}$.

By Euler's formula we get $e^{it} = \cos t + i \sin t$

and $e^{-it} = \cos t - i \sin t$, so

$x(t) = \alpha e^{-3}(\cos t + i \sin t) + \beta e^{-3}(\cos t - i \sin t)$

$= \alpha e^{-3} \cos t + \alpha e^{-3} i \sin t + \beta e^{-3} \cos t - \beta e^{-3} i \sin t$

$= (\alpha + \beta) e^{-3} \cos t + (\alpha - \beta) e^{-3} i \sin t$

$\Rightarrow x'(t) = -(\alpha + \beta) e^{-3} \sin t + (\alpha - \beta) e^{-3} i \cos t$.

So $x'(0) = 0 + (\alpha - \beta) e^{-3} i = 3$

$(e^3 - \beta - \beta) e^{-3} i = 3 \quad \Rightarrow (1 - 2\beta e^{-3}) i = 3$

$-2\beta e^{-3} = \frac{3}{i} - 1 \quad \Rightarrow \beta = \left(\frac{1}{2} - \frac{3}{2i}\right) e^3$

$\therefore \alpha = e^3 \cdot \left(1 + \frac{3}{2i} - \frac{1}{2}\right) = e^3 \cdot \left(\frac{1}{2} + \frac{3}{2i}\right)$

So $\boxed{x(t) = \left(\frac{1}{2} + \frac{3}{2i}\right)(\cos t + i \sin t) + \left(\frac{1}{2} - \frac{3}{2i}\right)(\cos t - i \sin t)}$

28. (See p. 104.) Show that for any complex number $\lambda = \mu + i\nu$, the derivative of the complex-valued function $z(t) = e^{\lambda t}$ is $\lambda e^{\lambda t}$

Let $z(t) = e^{\lambda t} = e^{(\mu + i\nu)t} = e^{\mu t} e^{i(\nu t)}$. Since

$e^{ix} = \cos x + i \sin x$ (this is Euler's formula),

$z(t) = e^{\mu t} \cdot (\cos(\nu t) + i \sin(\nu t))$. So

$\frac{dz}{dt} = \mu e^{\mu t} \cdot (\cos(\nu t) + i \sin(\nu t)) + e^{\mu t} \cdot (-\sin(\nu t) \cdot \nu + i \cos(\nu t) \cdot \nu)$

(product rule).

$= \mu e^{\mu t} \cos(\nu t) + \mu e^{\mu t} i \sin(\nu t) - e^{\mu t} \nu \sin(\nu t) + e^{\mu t} \nu i \cos(\nu t)$

$= (\mu e^{\mu t} + i\nu e^{\mu t}) \cdot (\cos(\nu t) + i \sin(\nu t)) = (\mu + i\nu) e^{\mu t} (\cos(\nu t) + i \sin(\nu t))$

$= \lambda \cdot e^{\mu t} \cdot (\cos(\nu t) + i \sin(\nu t))$

$= \lambda e^{\lambda t}$.

p.117 20. So we get basic solutions $x_1 = e^{3t}$ and $x_2 = e^t$ which make $x(t) = \alpha e^{3t} + \beta e^t$ the solution to the homogeneous equation.

So adding this to the first $x(t)$ gives $x(t) = \frac{1}{3}t + \frac{7}{9} + \alpha e^{3t} + \beta e^t$. To solve for α, β we have $x(0) = 0 = 0 + \frac{7}{9} + \alpha + \beta \Rightarrow \alpha = -\beta - \frac{7}{9}$.

Since $x'(t) = \frac{1}{3} + 3\alpha e^{3t} + \beta e^t$,

$$x'(0) = 0 = \frac{1}{3} + 3\alpha + \beta$$
$$= \frac{1}{3} + (-3\beta - \frac{21}{9}) + \beta$$
$$= \frac{1}{3} - 3\beta - \frac{21}{9} + \beta$$
$$= \frac{3-21}{9} - 2\beta$$

So $2\beta = -\frac{18}{9} = -2$

$\beta = -1 \Rightarrow \alpha = -(-1) - \frac{7}{9} = \frac{2}{9}$.

Then $\boxed{x(t) = \frac{1}{3}t + \frac{7}{9} + \frac{2}{9}e^{3t} - e^t}$.