3.1.4. Having shown $\dfrac{1-\alpha^{k-j}}{1-\alpha^{j}} = -\alpha^{-j}$,

$$\left[\begin{matrix} k-1 \\ j \end{matrix}\right]_\alpha = \left[\begin{matrix} j+(k-1-j) \\ j \end{matrix}\right]_\alpha = \frac{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{k-1})}{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^j)(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{k-1-j})}$$

$$= \left(\frac{1-\alpha^{k-1}}{1-\alpha^1}\right)\left(\frac{1-\alpha^{k-2}}{1-\alpha^2}\right)\cdots\left(\frac{1-\alpha^{k-j}}{1-\alpha^j}\right)\cdot\frac{(1-\alpha^{k-j-1})\cdots(1-\alpha^1)}{(1-\alpha)\cdots(1-\alpha^{k-1-j})}$$

$$= (-\alpha^{-1})(-\alpha^{-2})\cdots(-\alpha^{-j})\cdot\frac{\cancel{(1-\alpha^{k-j-1})}(1-\alpha^{k-j-2})\cdots\cancel{(1-\alpha^1)}}{\cancel{(1-\alpha^{k-j-1})}\cdots(1-\alpha^2)\cancel{(1-\alpha)}}$$

based on the statement we showed

$$= (-1)^j \alpha^{-j(j+1)/2}\cdot 1, \text{ and therefore}$$

$$f(\alpha, k-1) = \sum_{n=0}^{k-1}(-1)^n\left[\begin{matrix} k-1 \\ n \end{matrix}\right] = \sum_{j=0}^{k-1}(-1)^j(-1)^j\alpha^{-j(j+1)/2}$$

$$= \sum_{j=0}^{k-1}\alpha^{-j(j+1)/2}$$

3.1.5. Use the fact that if k is odd and α is a primitive kth root of unity, then so is α^{-2} to prove that

$$f(\alpha^{-2}, k-1) = \alpha^{-[(k+1)/2]^2}\sum_{j=0}^{k-1}\alpha^{[j+(k+1)/2]^2}$$

$$= \alpha^{-[(k+1)/2]^2}G(\alpha).$$

Since α^{-2} is also a primitive kth root of unity, $f(\alpha^{-2}, k-1) = \sum_{j=0}^{k-1}(\alpha^{-2})^{-j(j+1)/2}$

$$= \sum_{j=0}^{k-1}\alpha^{2j(j+1)/2} = \sum_{j=0}^{k-1}\alpha^{j(j+1)} = \sum_{j=0}^{k-1}\alpha^{j^2+j}\cdot 1$$

$$= \sum_{j=0}^{k-1}\alpha^{j^2+j}e^{2\pi i j k}$$ because the only primitive 1st root of unity is equal to 1

$$= \sum_{j=0}^{k-1}\alpha^{j^2+j}(e^{2\pi i/k})^{jk} = \sum_{j=0}^{k-1}\alpha^{j^2+j}\alpha^{jk} = \sum_{j=0}^{k-1}\alpha^{j^2+j(k+1)}$$

$$= \sum_{j=0}^{k-1}\alpha^{j^2+\frac{j(k+1)}{2}+\frac{j(k+1)}{2}+\frac{(k+1)^2}{4}-\frac{(k+1)^2}{4}} = \sum_{j=0}^{k-1}\alpha^{(j+\frac{k+1}{2})^2-(\frac{k+1}{2})^2}$$

$$= \sum_{j=0}^{k-1}\alpha^{[j+(k+1)/2]^2}\alpha^{-[(k+1)/2]^2} = \alpha^{-[(k+1)/2]^2}\sum_{j=0}^{k-1}\alpha^{[j+(k+1)/2]^2}$$