Time Spent: 3 hours

Collaborators and Resources: Talked with Nathaniel MacArthur-Warner about different scenarios

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Bonus Problem

Part (a)

Suppose you know that the unique ball is **heavier** than the other n - 1 identical balls. Give a strategy to identify the unique ball in the fewest uses of the scale possible. (It only requires two uses of the scale for $n = 9$.)

So for this problem we determined that by splitting the group of balls into three equally sized groups we would be able to identify the unique ball in the most efficient (based on least number of comparisons) way. Below is my algorithm which describes the most efficient way to make comparisons with three main base cases . . .

Given {1,2, ..., n} steel balls, all equal in size and weight except for one ball which appears identical but is *heavier* than the other n - 1 identical balls

1. If n is evenly divisible (no remainder) by 3
   (a) Place balls 1, 2, ... $\frac{n}{3}$ on one end of the scale, and place balls $\frac{n}{3} + 1$, $\frac{n}{3} + 2$, ..., $\frac{2 \cdot n}{3}$ on the other end of the scale.
   (b) If the scale determines they are equal in weight, then we know that the last 1/3 of the n balls contains the unique one and can discard the rest.
   (c) If the scale determines that they are unequal in weight, select the heavier group (also 1/3 of the n balls) to be our new set of steel balls and discard the rest. We know that it's the heavier group because in this case we're given that the unique ball is heavier.

   (d) Repeat this process indefinitely until there is one ball left in question, and then we will have the unique one

2. If n divided by 3 has a remainder of 1, then set aside the remaining ball and then repeat the same process outlined above until we are left with one ball. Except this time, we have to compare this ball to the remainder, and select the one which is heavier.

3. Finally, if n divided by 3 has a remainder of 2 we do the same thing, setting aside the 2 remaining balls and then repeating the process outlined above until we are left with one ball. We then concatenate this ball with the two remainders and then do one more iteration of the original strategy to determine which one is unique.

Claim 1. The algorithm will correctly work for any number n of balls.

Proof. The reason that this is the case is that each time we make a comparison between two thirds of the total number of balls (which is revised recursively), we notice that either they have the same weight (so the other third has the unique ball) or they don't (so the heavier third on the scale has the unique ball). Either way, we are able to reduce the size of the set of n balls by 3 each time, successfully keeping a group with the unique ball under consideration until we reach the base case wherein we are comparing three balls. Again, we compare ball 1 with ball 2 and if they're the same then ball 3 is the unique one. Otherwise, (ex. if ball 1 is heavier then it's the unique ball) we are still able to determine which ball is the heavier one without making any more comparisons than necessary.

An example of the above scenario for 9 balls would be the following. We can demonstrate that it only requires two uses of the scale. Label the balls 1,2, ..., 9. For the purposes of the example, the heavier ball could be the 3rd one. So place balls 1,2,3 on one side of the scale and balls 4,5,6 on the other side of the scale. The 1,2,3 group sinks, so we know that since the unique ball is heavier it is in the 1,2,3 group. We then compare 1 with 2, and since they have equal weight we know the (uniquely) heavier ball is the one labeled 3. □

Part (b)

Suppose the unique ball could be either heavier or lighter than the others but you do not know which. For $n = 13$, give a strategy to find the unique ball with only three uses of the scale.

So for this problem, I realized that if the unique ball had to be found with only three comparisons then relying only on the current comparison to make judgments as in the previous problem wouldn't work. It wouldn't work unless we included prior knowledge at each step and found contradictions based on the fact that there's only one unique ball which can't be both heavier and lighter at the same time. By observing the behavior of a group of balls we were able to determine the characteristic of the unique ball based on several logical conclusions. Nathaniel and I also talked about how since we don't know whether the ball we are looking for is lighter or heavier, we had to eliminate certain balls by contradiction in order to reduce the number of steps needed. So below is my algorithm (strategy) in plain-text ...

Given 13 steel balls {1,2,3,4,5,6,7,8,9,10,11,12,13}, each of which are equal in size and equal in weight except for one which looks identical to the others but weighs a slightly different amount. We don't know whether the unique ball is heavier or lighter than the others.

1. Place balls 1,2,3,4 on one side of the scale and balls 5,6,7,8 on the other side of the scale.
2. If the two sides are equal in weight
    (a) Then we know that the unique ball is one of 9,10,11,12,13.
    (b) Place 9,10,11 on one side of the scale and 1,2,3 (non-unique) on the other side.
        i. If they're equal then we know the unique ball is 12 or 13.
            A. Place ball 1 (which we know to be non-unique) on one side of the scale and place ball 12 on the other side.
            B. If they're equal, then the unique ball is 13
            C. If they're not equal, then the unique ball is 12

   ii. If they're not equal then we know the unique ball is one of 9,10,11
      A. Weigh 9 against 10. If they're equal then the unique ball is 11. Otherwise, since we know the previous weighing sank (or rose) then we choose the ball out of 9 and 10 which sinks (or rises) and declare this to be the unique ball.
   iii. The previous weighing sank (or rose)

3. If the two sides are unequal in weight then we know the unique ball is in 1,2,3,4 or 5,6,7,8.

   (a) Place balls 1,2,5 on one side of the scale and place 3,4,6 on the other side of the scale.

   (b) If the scale is equal after this then we know the unique ball is 7 or 8.

      i. For the third weighing, place ball 1 (non-unique) with ball 7
      ii. If they're balanced then the unique ball is 8.
      iii. Otherwise, the unique ball is 7.

   (c) If the scale is unequal, that is 1,2,5 rises (or sinks) and 3,4,6 sinks (or rises)

      i. Then the unique ball is in 1,2,5 or 3,4,6 so we eliminate the balls out of 1,2,5 and 3,4,6 which did not behave consistently.
      ii. For example that is, 1,2,5 rose (or sank) in the 2nd weighing and 1,2,3,4 sank (or rose) in the 1st weighing so it can't be 1 (since the unique ball can't be lighter and heavier)
      iii. By the same reasoning it can't be 2
      iv. It can't be 6 because 3,4,6 sank (or rose) in the 2nd weighing and rose (or sank) in the 1st weighing.

v. So the unique ball is in 5,3,4.

vi. If it's 3 or 4 then it's heavier (or lighter) because the group 3,4,6 sank (or rose)

vii. So place 3 on one side of the scale and 4 on the other side. If they're unequal pick the heavier one and that's the unique one. If they're equal then pick 6 and that's the unique one.

Claim 2. The strategy works for the $n = 13$ case with only three uses of the scale.

Proof. Based on the construction of the algorithm, we know that it covers all contingencies and thus is able to correctly make a judgment based on the rules of logic. An example of this strategy would be the following. Let's say we have balls 1,2, ..., 13 and ball 5 happens to be the unique ball and it happens to be lighter.

So as described, we compare balls 1,2,3,4 with balls 5,6,7,8 and determine that the 1,2,3,4 group sinks while the 5,6,7,8 group rises. We can thus associate the elements in 1,2,3,4 (as possible candidates) with the characteristic of sinking (rising for the other group).

However at this point we don't know which group contains the ball. So we compare balls 1,2,5 with 3,4,6 and see that the 1,2,5 group rises while the 3,4,6 group falls. The ball can't be 1 because 1 fell (1st use) and then rose (2nd use). It can't be 2 either because 2 fell (1st use) then rose (2nd use). It can't be 6 because 6 rose and then fell. So we've reduced the pool of balls to 5,3,4. Comparing 3 and 4 (which both fell in the 2nd comparison), we find that they are equal in weight so the unique ball has to be 5. Furthermore we can look at the result of the 1st comparison and determine that since the 5,6,7,8 group rose then 5 must be lighter as well. □