3.1.9. Use equation (3.9) to prove that

$$G(e^{2\pi i/k}) = (-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} (e^{4\pi i j/k} - e^{-4\pi i j/k})$$
$$= (-1)^{(k-1)/2} (2i)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} \sin \frac{4\pi j}{k}. \quad (3.12)$$

The first result is shown through substitution:

$$(3.9)\ G(\alpha) = (-1)^{(k-1)/2} (\alpha^2 - \alpha^{-2})(\alpha^4 - \alpha^{-4}) \cdots (\alpha^{k-1} - \alpha^{-(k-1)})$$
$$\Rightarrow G(e^{2\pi i/k}) = (-1)^{(k-1)/2} ((e^{2\pi i/k})^2 - (e^{2\pi i/k})^{-2}) \cdot ((e^{2\pi i/k})^4 - (e^{2\pi i/k})^{-4}) \cdots ((e^{2\pi i/k})^{k-1} - (e^{2\pi i/k})^{-(k-1)})$$
$$= (-1)^{(k-1)/2} \cdot (e^{4\pi i/k} - e^{-4\pi i/k})(e^{8\pi i/k} - e^{-8\pi i/k}) \cdots (e^{2\pi i(k-1)/k} - e^{-2\pi i(k-1)/k})$$
$$= (-1)^{(k-1)/2} \cdot (e^{4\pi i \cdot 1/k} - e^{-4\pi i \cdot 1/k})(e^{4\pi i \cdot 2/k} - e^{-4\pi i \cdot 2/k}) \cdots (e^{4\pi i (\frac{k-1}{2})/k} - e^{-4\pi i (\frac{k-1}{2})/k})$$
$$= (-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} (e^{4\pi i j/k} - e^{-4\pi i j/k}).$$

Lastly, $e^{4\pi i j/k} - e^{-4\pi i j/k} = \left(\cos\left(\frac{4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right)\right) - \left(\cos\left(\frac{-4\pi j}{k}\right) + i\sin\left(\frac{-4\pi j}{k}\right)\right)$

based on Euler's identity, which $= \cos\left(\frac{4\pi j}{k}\right) - \cos\left(\frac{-4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right) - i\sin\left(\frac{-4\pi j}{k}\right)$

$= \cos\left(\frac{4\pi j}{k}\right) - \cos\left(\frac{4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right) + i\sin\left(\frac{4\pi j}{k}\right)$ because $\cos(-a) = \cos(a)$ and $\sin(-a) = -\sin(a)$.

This is equal to $2i\sin\left(\frac{4\pi j}{k}\right)$.

So substituting this, $(-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} (e^{4\pi i j/k} - e^{-4\pi i j/k}) = (-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} 2i\sin\left(\frac{4\pi j}{k}\right)$

$$= (-1)^{(k-1)/2} (2i)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} \sin \frac{4\pi j}{k}. \quad \square$$

3.1.10. Prove that $\prod_{j=1}^{(k-1)/2} \sin(4\pi j/k)$ is positive for $k \equiv \pm 1 \pmod 8$ and negative for $k \equiv \pm 1 \pmod 8$. Combine this with equations (3.11) and (3.12) to prove that

$$G(e^{2\pi i/k}) = i^{[(k-1)/2]^2} \sqrt{k}. \quad (3.13)$$