6.1.7. Prove that the reflection of a descending plane partition takes a descending plane partition in which n appears j times to a descending plane partition in which n appears $n-1-j$ times.

For this exercise, we don't need to look at the second step of our transformation.

$$b_{ij} = \begin{cases} j-i+1-a_{ij}, & \text{if } a_{ij} \text{ exists and } a_{ij} \le j-i, \quad \text{special part} \\ j+1-B_{ij}, & \text{if } a_{ij} \text{ is not defined}, \quad \text{not a special part} \\ \text{undefined} & \text{if } a_{ij} > j-i, \end{cases}$$

First step

where $B_{ij} = \|\{a_{xi} \mid a_{xi} \ge j+2-x\}\|$.

Let's suppose n appears j times. Then, n appears $j = j_1 + j_2$ times.

n	appears # times in original		appears # times in reflected
7	1	$7-1-1=5$	5
6	2	$6-1-2=3$	2
5	3	$5-1-3=1$	0
4	2	$4-1-2=1$	2
3	3	$3-1-3=-1$	0
2	1	$2-1-1=0$	1
1	0	$1-1-1=-1$	2
	12		12

For
7 6 6 5 $\boxed{4\ 4}$
5 5 3 $\boxed{3}$ $\in B(7,7,7)$
3 2

1st step

$\boxed{1\ 2}$
1 7
4 7 $\in B(7,7,7)$
4 6 7
6 7
7

n is only going to appear in the last column of the reflection, described by the b_{i,n_1} values. Otherwise, we would not have strict decrease down columns after the second step of our transformation (the true reflection across the diagonal).

$b_{i,n_1} = b_{i,(n-1)}$ since $B(n,n,n)$ is the smallest box (# parts in a row < largest row part)

possible b_{ij} are

$$\begin{cases} n = (n-1)-i+1-a_{i,(n-1)} \Rightarrow n = n-i-a_{i,(n-1)} \Rightarrow a_{i,(n-1)} = -i \quad \times \\ n = \text{undefined} \quad \times \\ n = (n-1)+1-B_{i,(n-1)} = n - \|\{a_{xi} \mid a_{xi} \ge (n-1)+2-x\}\| \Leftrightarrow \end{cases}$$

$B_{i,(n-1)} = \|\{a_{xi} \mid a_{xi} \ge n+1-x\}\| = 0$, and so in the last column called $a_{x,(n-1)}$, the number of times that $a_{x,(n-1)} < n+1-x$ is equal to $(n-1)-j$ = # times that, for i all rows,

$B_{i,(n-1)} = 0 = \|\{a_{xi} \mid a_{xi} > n-x$, which is true iff $a_{xi} = n\}\|$ iff $a_{xi} \ne n$. ($a_{xi} = n$ exactly j times)

So $B_{i,(n-1)} = 0$ exactly $n_1 - j = (n-1)-j = n-1-j$ times = # times n appears in the reflection.