That is, (eliminating the denominator)

$$(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^{n+m}) = (1-q^{n})(1-q^{n+1})\cdots(1-q^{n+m-1}) + q^{n}(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^{m}),$$

Now divide both sides by the circled material:

$$\begin{aligned} 1-q^{n+m} &= (1-q^{n}) + q^{n}(1-q^{m}) \\ &= 1-q^{n}+q^{n}-q^{n+m} \\ &= 1-q^{n+m}. \quad \square \end{aligned}$$

3.1.3 This exercise and those up to and including exercise 3.1.10 outline Gauss's evaluation of the Gaussian sum $G(\alpha)=\sum \alpha^{j^2}$ where α is a <u>primitive kth root of unity</u> such as $e^{2\pi i/k}$ and the summation is taken over all residue classes, j, modulo k, where k is odd. This evaluation was Gauss's original reason for defining Gaussian polynomials.

Define
$$f(q,m) = 1-\begin{bmatrix} m \\ 1 \end{bmatrix}+\begin{bmatrix} m \\ 2 \end{bmatrix}-\cdots+(-1)^m\begin{bmatrix} m \\ m \end{bmatrix}.$$

Use the recursive formula, $\begin{bmatrix} m \\ j \end{bmatrix}=\begin{bmatrix} m-1 \\ j \end{bmatrix}+q^{m-j}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$, to prove that
$$f(q,m)=(1-q^{m-1})f(q,m-2).$$

Now prove by induction that
$$f(q,m)=\begin{cases} 0, & \text{if } m \text{ is odd,} \\ (1-q)(1-q^{3})\cdots(1-q^{m-1}), & \text{if } m \text{ is even.} \end{cases} \quad (3.7).$$

We are proving

$$f(q,m)=1-\begin{bmatrix} m \\ 1 \end{bmatrix}+\begin{bmatrix} m \\ 2 \end{bmatrix}-\cdots+(-1)^m\begin{bmatrix} m \\ m \end{bmatrix} \quad \text{substitute}$$

$$= 1-\begin{bmatrix} m-1 \\ 1 \end{bmatrix}-q^{m-1}\begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}+\begin{bmatrix} m-1 \\ 2 \end{bmatrix}+q^{m-2}\begin{bmatrix} m-1 \\ 2-1 \end{bmatrix}+\cdots+(-1)^m\left(\begin{bmatrix} m-1 \\ m \end{bmatrix}+\begin{bmatrix} m-1 \\ m-1 \end{bmatrix}\cdot q^{m-m}\right) \quad \text{recursive formula}$$

$$= 1-\begin{bmatrix} m-1 \\ 1 \end{bmatrix}+\begin{bmatrix} m-1 \\ 2 \end{bmatrix}-\cdots+(-1)^m\begin{bmatrix} m-1 \\ m \end{bmatrix} - q^{m-1}\begin{bmatrix} m-1 \\ 0 \end{bmatrix}+q^{m-1}q^{-1}\begin{bmatrix} m-1 \\ 1 \end{bmatrix}+\cdots+(-1)^m\cdot q^{m-1}q^{1-m}\begin{bmatrix} m-1 \\ m-1 \end{bmatrix},$$

$$=(1-q^{m-1}).$$

$$\begin{aligned} \sum_{n=0}^{m}(-1)^m\begin{bmatrix} m \\ n \end{bmatrix} &= \sum_{n=0}^{m}(-1)^m\left(\begin{bmatrix} m-1 \\ n \end{bmatrix}+q^{m-n}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right) \\ &= \sum_{n=0}^{m}(-1)^m\begin{bmatrix} m-1 \\ n \end{bmatrix}+\sum_{n=0}^{m}(-1)^m q^{m-n}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \\ &= \sum_{n=0}^{m}(-1)^m\left(\begin{bmatrix} m-2 \\ n \end{bmatrix}+q^{m-n}\begin{bmatrix} m-2 \\ n-1 \end{bmatrix}\right)+ \end{aligned}$$