1

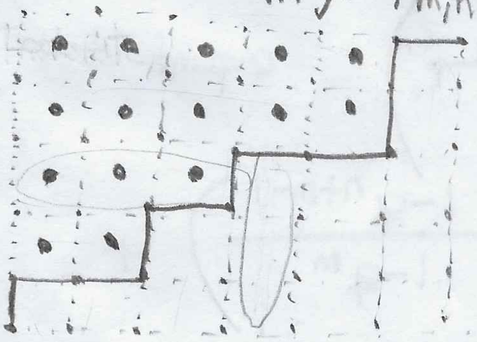
Exercises

3.1.1. Find a one-to-one correspondence (bijection) between partitions into at most m parts, each less than or equal to n, and partitions into at most n parts, each less than or equal to m. This proves combinatorially that

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{bmatrix} m+n \\ n \end{bmatrix}.$$

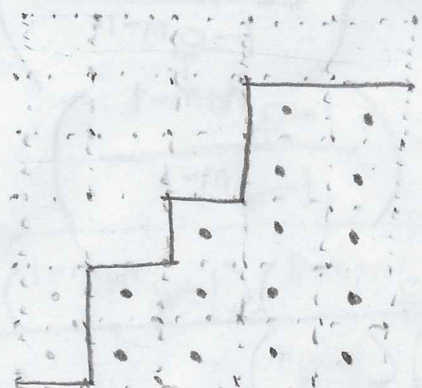
So Proposition 3.1 says, the total number of partitions into at most m parts with each part less than or equal to n is equal to $\binom{m+n}{m}$.

Furthermore, $\binom{m+n}{m} = f_{m,n}(1)$, where $f_{m,n}(q) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n \\ m \end{bmatrix}$ by notation.



From the book: Lattice paths from $(0,0)$ to $(6,5)$ produce partitions with at most five parts, and each part will be less than or equal to six.

$5+5+3+2 = 15$



$5+3+3+2+1+1 = 15$
(this isn't always the case)

Set $l = m$. // m is the height.
For each part x in our original partition:
For integers i s.t. $x < i \leq n$:
$l -=$ # of parts of length i.

$(n,m) = (6,3)$

Bijection: Given a list $p_{old} = [p_1, p_2, \ldots]$ of parts:
Store $p_{new} = p_{old}$
For each part $p_i \in p_{new}$:
Set int $l = m$. // m is the height of the original graph //
For i in $(x+1):n$:
$l -=$ # of parts of length i in p_{old}.
Set $p_i = l$
Return p_{new}.

$6+3+2$
$\rightarrow 2+2+2+1+0+0$

3.1.2. Verify equation (3.4).

Specifically, equation (3.4) states that $\prod_{i=1}^{m} \frac{1-q^{n+i}}{1-q^i} = \prod_{i=1}^{m} \frac{1-q^{n-1+i}}{1-q^i} + q^n \prod_{i=1}^{m-1} \frac{1-q^{n+i}}{1-q^i}$

$$\left(\frac{1-q^{n+1}}{1-q^1}\right)\left(\frac{1-q^{n+2}}{1-q^2}\right)\cdots\left(\frac{1-q^{n+m}}{1-q^m}\right) = \left(\frac{1-q^{n-1+1}}{1-q^1}\right)\left(\frac{1-q^{n-1+2}}{1-q^2}\right)\cdots\left(\frac{1-q^{n-1+m}}{1-q^m}\right) +$$

$$q^n \cdot \left(\frac{1-q^{n+1}}{1-q^1}\right)\left(\frac{1-q^{n+2}}{1-q^2}\right)\cdots\left(\frac{1-q^{n+m-1}}{1-q^{m-1}}\right)$$ expand

RHS

$$= \left(\frac{1-q^n}{1-q}\right)\left(\frac{1-q^{n+1}}{1-q^2}\right)\cdots\left(\frac{1-q^{n+m-2}}{1-q^{m-1}}\right)\left(\frac{1-q^{n+m-1}}{1-q^m}\right) +$$

$$q^n\left(\frac{1-q^{n+1}}{1-q}\right)\left(\frac{1-q^{n+2}}{1-q^2}\right)\cdots\left(\frac{1-q^{n+m-1}}{1-q^{m-1}}\right)$$

$$= \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-2})(1-q^{n+m-1})}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)} + \frac{q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})}{(1-q)(1-q^2)\cdots(1-q^{m-1})}$$

$$= \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-2})(1-q^{n+m-1})}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)} + \frac{q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m)}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)}$$

$$= \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-2})(1-q^{n+m-1}) + q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m)}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)}$$

So we have established that LHS = ... = RHS :

$$\frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m})}{(1-q^1)(1-q^2)\cdots(1-q^m)} = \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-1}) + q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m)}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

That is, (eliminating the terominator)

$$(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^{n+m}) = (1-q^n)(1-q^{n+1})\cdots(1-q^{n+m-1}) + q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m-1})(1-q^m).$$

Now divide both sides by the circled material:

$$1-q^{n+m} = (1-q^n)+q^n(1-q^m)$$
$$= 1-q^n+q^n-q^{n+m}$$
$$= 1-q^{n+m}. \quad \square$$

3.1.3 This exercise and those up to and including exercise 3.1.10 outline Gauss's evaluation of the Gaussian sum $G(\alpha)=\sum \alpha^{j^2}$ where α is a primitive kth root of unity such as $e^{2\pi i/k}$ and the summation is taken over all residue classes, j, modulo k, where k is odd. This evaluation was Gauss's original reason for defining Gaussian polynomials.

Define
$$f(q,m) = 1-\begin{bmatrix} m \\ 1 \end{bmatrix}+\begin{bmatrix} m \\ 2 \end{bmatrix}-\cdots+(-1)^m\begin{bmatrix} m \\ m \end{bmatrix}.$$

Use the recursive formula, $\begin{bmatrix} m \\ j \end{bmatrix}=\begin{bmatrix} m-1 \\ j \end{bmatrix}+q^{m-j}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$, to prove that
$$f(q,m)=(1-q^{m-1})f(q,m-2).$$

Now prove by induction that
$$f(q,m)=\begin{cases} 0, & \text{if } m \text{ is odd}, \\ (1-q)(1-q^3)\cdots(1-q^{m-1}), & \text{if } m \text{ is even}. \end{cases}$$
(3.7).

We are proving

$$f(q,m) = 1-\begin{bmatrix} m \\ 1 \end{bmatrix}+\begin{bmatrix} m \\ 2 \end{bmatrix}-\cdots+(-1)^m\begin{bmatrix} m \\ m \end{bmatrix}$$ substitute

$$= 1-\begin{bmatrix} m-1 \\ 1 \end{bmatrix}-q^{m-1}\begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}+\begin{bmatrix} m-1 \\ 2 \end{bmatrix}+q^{m-2}\begin{bmatrix} m-1 \\ 2-1 \end{bmatrix}+\cdots+(-1)^m\left(\begin{bmatrix} m-1 \\ m \end{bmatrix}+\begin{bmatrix} m-1 \\ m-1 \end{bmatrix}\cdot q^{m-m}\right)$$ recursive formula

$$= 1-\begin{bmatrix} m-1 \\ 1 \end{bmatrix}+\begin{bmatrix} m-1 \\ 2 \end{bmatrix}-\cdots+(-1)^m\begin{bmatrix} m-1 \\ m \end{bmatrix} - q^{m-1}\begin{bmatrix} m-1 \\ 0 \end{bmatrix}+q^{m-1}q^{-1}\begin{bmatrix} m-1 \\ 1 \end{bmatrix}+\cdots+(-1)^m\cdot q^{m-1}q^{1-m}\begin{bmatrix} m-1 \\ m-1 \end{bmatrix}.$$

$$= (1-q^{m-1}).$$

$$\sum_{n=0}^{m}(-1)^m\begin{bmatrix} m \\ n \end{bmatrix} = \sum_{n=0}^{m}(-1)^m\left(\begin{bmatrix} m-1 \\ n \end{bmatrix}+q^{m-n}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right)$$
$$= \sum_{n=0}^{m}(-1)^m\begin{bmatrix} m-1 \\ n \end{bmatrix} + \sum_{n=0}^{m}(-1)^m q^{m-n}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}$$
$$= \sum_{n=0}^{m}(-1)^m\left(\begin{bmatrix} m-2 \\ n \end{bmatrix}+q^{m-n}\begin{bmatrix} m-2 \\ n-1 \end{bmatrix}\right)+$$

3.1.3. This exercise and those up to and including exercise 3.1.10 outline Gauss's evaluation of the Gaussian sum $G(\alpha) = \sum \alpha^{j^2}$ where α is a primitive kth root of unity such as $e^{2\pi i/k}$ and the summation is taken over all residue classes, j, modulo k, where k is odd. This evaluation was Gauss's original reason for defining Gaussian polynomials.

Define
$$f(q,m) = 1 - \begin{bmatrix} m \\ 1 \end{bmatrix} + \begin{bmatrix} m \\ 2 \end{bmatrix} - \cdots + (-1)^m \begin{bmatrix} m \\ m \end{bmatrix}.$$

Use the recursive formula, $\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j \end{bmatrix} + q^{m-j}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$, to prove that
$$f(q,m) = (1-q^{m-1}) f(q,m-2).$$

We're given the companion formula $\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^{j}\begin{bmatrix} m-1 \\ j \end{bmatrix}$, so

$$\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \cdot \left(\begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1-j} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$ by the first recursive formula

$$= \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix}.$$ Now, the companion formula allows us to say that this $= \begin{bmatrix} m-2 \\ j-2 \end{bmatrix} + q^{j-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix}$.

Substituting this, $f(q,m) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}$

$$= \sum_{j=0}^{m} (-1)^j \cdot \left(\begin{bmatrix} m-2 \\ j-2 \end{bmatrix} + q^{j-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$

$$= \left(\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m-2 \\ j-2 \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^{j-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$

$$= \left(\sum_{j=0}^{m-2} (-1)^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \underbrace{\left(\sum_{j=0}^{m-2} (-1)^{j+1} q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \left(\sum_{j=0}^{m-2} (-1)^j q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right)}_{\text{as Matthew said, these cancel out.}} + \left(\sum_{j=0}^{m-2} (-1)^{j+1} q^{m-1} \begin{bmatrix} m-2 \\ j \end{bmatrix} \right)$$

$$= f(q,m-2) + 0 + (-1) q^{m-1} f(q,m-2)$$

$$= (1-q^{m-1}) f(q,m-2).$$ So $f(q,m) = (1-q^{m-1}) f(q,m-2)$.

3.1.3. Given that $f(q,m) = (1-q^{m-1})f(q,m-2)$,

Let $m=1$. $f(q,m) = 1 - \begin{bmatrix}1\\1\end{bmatrix} = 1-1 = 0$ in that case.

$f(q,m) = (1-q^{m-1})\cdot f(q,m-2) = (1-q^{m-1})\cdot 0 = 0$ if we accept the inductive hypothesis,

so $\underline{f(q,m) = 0 \text{ if } m \text{ is odd}}$.

Let $m=2$. $f(q,2) = 1 - \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix} = 1 - \begin{bmatrix}2\\1\end{bmatrix} + 1 = -\begin{bmatrix}2\\1\end{bmatrix} + 2$

$-\begin{bmatrix}2\\1\end{bmatrix} = -\left(\begin{bmatrix}2-1\\1\end{bmatrix} + q^{2-1}\begin{bmatrix}2-1\\1-1\end{bmatrix}\right) = -\begin{bmatrix}1\\1\end{bmatrix} - q\begin{bmatrix}1\\0\end{bmatrix} = -1-q$, so

$f(q,2) = -1-q+2 = (1-q)$.

$f(q,m) = (1-q^{m-1})\cdot f(q,m-2) = (1-q^{m-1})\cdot\left((1-q)(1-q^3)\cdots(1-q^{m-2-1})\right)$

by the inductive hypothesis, so

$f(q,m) = (1-q)(1-q^3)\cdots(1-q^{m-1})$, if m is even.

3.1.4. Let k be odd. Show that $\dfrac{1-\alpha^{k-j}}{1-\alpha^j} = -\alpha^{-j}$.

Use this to prove that $\begin{bmatrix}k-1\\j\end{bmatrix}_\alpha = (-1)^j\alpha^{-j(j+1)/2}$,

and therefore $f(\alpha, k-1) = \sum_{j=0}^{k-1}\alpha^{-j(j+1)/2}$.

Let k be odd. $\alpha = e^{2\pi i h/k}$ because this is the general primitive kth root of unity.

$$\frac{1-\alpha^{k-j}}{1-\alpha^j} = \frac{1-(e^{2\pi i h/k})^{k-j}}{1-(e^{2\pi i h/k})^j} = \frac{1-e^{2\pi i h - 2\pi i h j/k}}{1-e^{2\pi i h j/k}}$$

$$= \frac{1-e^{-2\pi i h j/k}e^{2\pi i h}}{1-e^{2\pi i h j/k}} = \frac{-e^{-2\pi i h j/k}\cdot(e^{2\pi i h} - e^{2\pi i h j/k})}{1-e^{2\pi i h j/k}}$$

$$= \frac{-e^{-2\pi i h j/k}(1-e^{2\pi i h j/k})}{(1-e^{2\pi i h j/k})}$$ because the primitive 1st root of unity is always 1

$$= -(e^{2\pi i h/k})^{-j} = \underline{-\alpha^{-j}}.$$

3.1.4. Having shown $\dfrac{1-\alpha^{k-j}}{1-\alpha^{j}} = -\alpha^{-j}$,

$$\left[{k-1 \atop j}\right]_\alpha = \left[{j+(k-1-j) \atop j}\right]_\alpha = \frac{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{k-1})}{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{j})(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^{k-1-j})} \qquad (3.1)$$

$$= \left(\frac{1-\alpha^{k-1}}{1-\alpha^{1}}\right)\left(\frac{1-\alpha^{k-2}}{1-\alpha^{2}}\right)\cdots\left(\frac{1-\alpha^{k-j}}{1-\alpha^{j}}\right)\cdot\frac{(1-\alpha^{k-j-1})\cdots(1-\alpha^{1})}{(1-\alpha)\cdots(1-\alpha^{k-1-j})}$$

$$= (-\alpha^{-1})(-\alpha^{-2})\cdots(-\alpha^{-j})\cdot\cancel{\frac{(1-\alpha^{k-j-1})(1-\alpha^{k-j-2})\cdots(1-\alpha^{1})}{(1-\alpha^{k-j-1})\cdots(1-\alpha^{2})(1-\alpha)}}$$

based on the statement we showed

$= (-1)^j\alpha^{-j(j+1)/2}\cdot 1$, and therefore

$$f(\alpha,k-1) = \sum_{n=0}^{k-1}(-1)^n\left[{k-1 \atop n}\right] = \sum_{j=0}^{k-1}(-1)^j(-1)^j\alpha^{-j(j+1)/2}$$

$$= \sum_{j=0}^{k-1}\alpha^{-j(j+1)/2}$$

3.1.5. Use the fact that if k is odd and α is a primitive kth root of unity, then so is α^{-2} to prove that

$$f(\alpha^{-2},k-1) = \alpha^{-[(k+1)/2]^2}\sum_{j=0}^{k-1}\alpha^{[j+(k+1)/2]^2}$$

$$= \alpha^{-[(k+1)/2]^2}G(\alpha).$$

Since α^{-2} is also a primitive kth root of unity, $f(\alpha^{-2},k-1) = \sum_{j=0}^{k-1}(\alpha^{-2})^{-j(j+1)/2}$

$$= \sum_{j=0}^{k-1}\alpha^{2j(j+1)/2} = \sum_{j=0}^{k-1}\alpha^{j(j+1)} = \sum_{j=0}^{k-1}\alpha^{j^2+j}\cdot 1$$

$= \sum_{j=0}^{k-1}\alpha^{j^2+j}e^{2\pi i jk}$ because the only primitive 1st root of unity is equal to 1

$$= \sum_{j=0}^{k-1}\alpha^{j^2+j}(e^{2\pi i/k})^{jk} = \sum_{j=0}^{k-1}\alpha^{j^2+j}\alpha^{jk} = \sum_{j=0}^{k-1}\alpha^{j^2+j(k+1)}$$

$$= \sum_{j=0}^{k-1}\alpha^{j^2+\frac{j(k+1)}{2}+\frac{j(k+1)}{2}+\frac{(k+1)^2}{4}-\frac{(k+1)^2}{4}} = \sum_{j=0}^{k-1}\alpha^{(j+\frac{k+1}{2})^2-(\frac{k+1}{2})^2}$$

$$= \sum_{j=0}^{k-1}\alpha^{[j+(k+1)/2]^2}\alpha^{-[(k+1)/2]^2} = \alpha^{-[(k+1)/2]^2}\sum_{j=0}^{k-1}\alpha^{[j+(k+1)/2]^2}$$

3.1.6. Use equation (3.7) to prove that

$$G(\alpha)=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)}). \quad (3.8)$$

From equation (3.7),

$$f(q,m)=\begin{cases}0, & \text{if } m \text{ is odd},\\ (1-q)(1-q^3)\cdots(1-q^{m-1}), & \text{if } m \text{ is even}.\end{cases}$$

Now, $f(\alpha^{-2},k-1)=\alpha^{-[(k+1)/2]^2}G(\alpha)$ based on exercise 3.1.5 and also

$$f(\alpha^{-2},k-1)=\begin{cases}0, & \text{if } k \text{ is even},\\ (1-\alpha^{-2})(1-(\alpha^{-2})^3)\cdots(1-(\alpha^{-2})^{k-2}), & \text{if } k \text{ is odd}.\end{cases}$$

So <u>when k is odd</u>, this becomes $(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})$. Since 3.1.5 $\Rightarrow$ $G(\alpha)=f(\alpha^{-2},k-1)\,\alpha^{[(k+1)/2]^2}$, $G(\alpha)=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\,\alpha^{[(k+1)/2]^2}$

$=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\alpha^1\alpha^3\cdots\alpha^k$ because $1+3+\cdots+k=$ (the number of odd integers between 1 and k inclusive$)^2=\left(\frac{k+1}{2}\right)^2$

Then $G(\alpha)=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\alpha^1\alpha^3\cdots\alpha^{k-2}\cdot(e^{2\pi i h/k})^k$,

and because $\alpha^k=(e^{2\pi i h/k})^k=e^{2\pi i h/1}$, α^k is a 1st root of unity which is always equal to 1.

$\Rightarrow G(\alpha)=\alpha^1(1-\alpha^{-2})\alpha^3(1-\alpha^{-6})\cdots\alpha^{k-2}(1-\alpha^{-2k+4})\cdot 1$

$=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)})$. $\square$

3.1.7. Use the fact that $\alpha^{k-j}-\alpha^{-(k-j)}=-(\alpha^j-\alpha^{-j})$ to rewrite equation (3.8) as

$$G(\alpha)=(-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)}). \quad (3.9)$$

(3.8) says $G(\alpha)=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)})$

$=(\alpha^{k-(k-1)}-\alpha^{-(k-(k-1))})(\alpha^{k-(k-3)}-\alpha^{-(k-(k-3))})$

$\cdots(\alpha^{k-2}-\alpha^{-(k-2)})$, which based on the fact

$=(-(\alpha^{k-1}-\alpha^{-(k-1)}))(-(\alpha^{k-3}-\alpha^{-(k-3)}))\cdots(-(\alpha^2-\alpha^{-2}))$

$=(-1)^{\frac{k-1}{2}}(\alpha^{k-1}-\alpha^{-(k-1)})(\alpha^{k-3}-\alpha^{-(k-3)})\cdots(\alpha^2-\alpha^{-2})$ because

(-1) is implicated $(k-1)/2$ times

$=(-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)})$. $\square$

3.1.8 Combine equations (3.8) and (3.9) to show that

$$G(\alpha)^2 = (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j}). \quad (3.10)$$

Show that $\prod_{j=1}^{k-1}(x-\alpha^{-2j}) = \frac{x^k-1}{x-1} = 1+x+x^2+\cdots+x^{k-1}$,

and therefore $G(\alpha)^2 = (-1)^{(k-1)/2}k$. (3.11)

For the first result, we can multiply (3.8)·(3.9): So $G(\alpha)^2$

$= G(\alpha)\cdot G(\alpha) = (\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)})\cdot$
$(-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)})$.

$= (-1)^{(k-1)/2}\cdot\alpha(1-\alpha^{-2})\alpha^3(1-\alpha^{-6})\cdots\alpha^{k-2}(1-\alpha^{-2(k-2)})\alpha^2(1-\alpha^{-4})\alpha^4(1-\alpha^{-8})$
$\cdots\alpha^{k-1}(1-\alpha^{-2(k-1)})$

$= \alpha\alpha^3\cdots\alpha^{k-2}\alpha^2\alpha^4\cdots\alpha^{k-1}(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})(1-\alpha^{-4})(1-\alpha^{-8})$
$\cdots(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$

$= \alpha\alpha^2\alpha^3\alpha^4\cdots\alpha^{k-2}\alpha^{k-1}(1-\alpha^{-2})(1-\alpha^{-4})(1-\alpha^{-6})(1-\alpha^{-8})\cdots(1-\alpha^{-2k+4})(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$

$= \alpha^{k(k-1)/2}(1-\alpha^{-2})(1-\alpha^{-4})\cdots(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$

$= \alpha^{k(k-1)/2}\cdot(1-\alpha^{-2})(1-\alpha^{-4})\cdots(1-\alpha^{-2(k-1)})(-1)^{(k-1)/2}$

$= (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j})$.

For the second result, because given any primitive kth root of unity $\alpha = e^{2\pi i/k}$ with k odd, α^{-2} is also a primitive root of unity (from exercise 3.1.5.), and we know by definition that the roots of x^k-1 are all kth roots of unity, $(\alpha^{-2})^j$, $0\le j\le k-1$,

$$x^k-1 = (x-(\alpha^{-2})^0)(x-(\alpha^{-2})^1)\cdots(x-(\alpha^{-2})^{k-1})$$

$$= \prod_{j=0}^{k-1}(x-(\alpha^{-2})^j) = \prod_{j=0}^{k-1}(x-\alpha^{-2j}) = (x-\alpha^0)\prod_{j=1}^{k-1}(x-\alpha^{-2j})$$

$\Rightarrow (x^k-1)/(x-1) = \prod_{j=1}^{k-1}(x-\alpha^{-2j})$. Also,

$$\frac{x^k-1}{x-1} = \frac{x+x^2+x^3+\cdots+x^k-1-x-x^2-\cdots-x^{k-1}}{x-1} = \frac{(x-1)(1+x+x^2+\cdots+x^{k-1})}{x-1}$$

$= 1+x+x^2+\cdots+x^{k-1}$

And therefore, since $\alpha^{k(k-1)/2} = (e^{2\pi i/k})^{k(k-1)/2} = (e^{2\pi i})^{(k-1)/2}$
$= (\cos(2\pi)+i\sin(2\pi))^{(k-1)/2} = (1+0)^{(k-1)/2} = 1$,

$G(\alpha)^2 = (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j})$

$= (-1)^{(k-1)/2}\cdot 1\cdot(1^0+1^1+1^2+\cdots+1^{k-1}) = (-1)^{(k-1)/2}\cdot(k)$

$= (-1)^{(k-1)/2}k$.

3.1.9. Use equation (3.9) to prove that

$$G(e^{2\pi i/k}) = (-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} (e^{4\pi i j/k} - e^{-4\pi i j/k})$$

$$= (-1)^{(k-1)/2} (2i)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} \sin \frac{4\pi j}{k}. \quad (3.12)$$

The first result is shown through substitution:

$$(3.9)\ G(\alpha) = (-1)^{(k-1)/2} (\alpha^2 - \alpha^{-2})(\alpha^4 - \alpha^{-4}) \cdots (\alpha^{k-1} - \alpha^{-(k-1)})$$

$$\Rightarrow G(e^{2\pi i/k}) = (-1)^{(k-1)/2} ((e^{2\pi i/k})^2 - (e^{2\pi i/k})^{-2}) \cdot ((e^{2\pi i/k})^4 - (e^{2\pi i/k})^{-4}) \cdots ((e^{2\pi i/k})^{k-1} - (e^{2\pi i/k})^{-(k-1)})$$

$$= (-1)^{(k-1)/2} \cdot (e^{4\pi i/k} - e^{-4\pi i/k})(e^{8\pi i/k} - e^{-8\pi i/k}) \cdots (e^{2\pi i(k-1)/k} - e^{-2\pi i(k-1)/k})$$

$$= (-1)^{(k-1)/2} \cdot (e^{4\pi i \cdot 1/k} - e^{-4\pi i \cdot 1/k})(e^{4\pi i \cdot 2/k} - e^{-4\pi i \cdot 2/k}) \cdots (e^{4\pi i (\frac{k-1}{2})/k} - e^{-4\pi i (\frac{k-1}{2})/k})$$

$$= (-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} (e^{4\pi i j/k} - e^{-4\pi i j/k}).$$

Lastly, $e^{4\pi i j/k} - e^{-4\pi i j/k} = \left(\cos\left(\frac{4\pi j}{k}\right) + i \sin\left(\frac{4\pi j}{k}\right)\right) - \left(\cos\left(\frac{-4\pi j}{k}\right) + i \sin\left(\frac{-4\pi j}{k}\right)\right)$ based on Euler's identity, which $= \cos\left(\frac{4\pi j}{k}\right) - \cos\left(\frac{-4\pi j}{k}\right) + i \sin\left(\frac{4\pi j}{k}\right) - i \sin\left(\frac{-4\pi j}{k}\right)$

$= \cos\left(\frac{4\pi j}{k}\right) - \cos\left(\frac{4\pi j}{k}\right) + i \sin\left(\frac{4\pi j}{k}\right) + i \sin\left(\frac{4\pi j}{k}\right)$ because $\cos(-a) = \cos(a)$ and $\sin(-a) = -\sin(a)$.

This is equal to $2i \sin\left(\frac{4\pi j}{k}\right)$.

So substituting this, $(-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} (e^{4\pi i j/k} - e^{-4\pi i j/k}) = (-1)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} 2i \sin\left(\frac{4\pi j}{k}\right)$

$$= (-1)^{(k-1)/2} (2i)^{(k-1)/2} \prod_{j=1}^{(k-1)/2} \sin \frac{4\pi j}{k}. \quad \square$$

3.1.10. Prove that $\prod_{j=1}^{(k-1)/2} \sin(4\pi j/k)$ is positive for $k \equiv \pm 1 \pmod 8$ and negative for $k \equiv \pm 1 \pmod 8$. Combine this with equations (3.11) and (3.12) to prove that

$$G(e^{2\pi i/k}) = i^{[(k-1)/2]^2} \sqrt{k}. \quad (3.13)$$