1 SUBJECT NOTEBOOK

80 Sheets • College Ruled

Combinatorial
Theory 333

Sep 14 Fri. Math 333

LaTeX: Overleaf

$$\int_0^\infty e^{-x}\,dx = 1$$

Will help w/ LaTeX.

Pictures \vskip \vspace

Graphs and Multigraphs

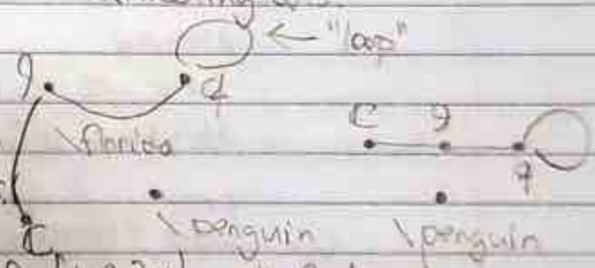
Idea: want to study several areas of combinatorics, one of which is graph theory.

Def$^{\text{n}}$: a multigraph is a set V of vertices.
A set E of edges, and a function f from E to the set of subsets of V of size 1 and 2.

example: $V = \{9, 4, \backslash\text{penguin}, \mathbb{C}\}$ ← \{\mathbb C\}
$E = \{\backslash\text{florida}, Q, ku\}$
$f(\backslash\text{florida}) = \{9, 4\}$
$f(Q) = 4$
$f(ku) = \{9, \mathbb{C}\}$

Usually represent vertices with dots, edges with curves connecting dots.

a graph is a multigraph with no loops, and no multiple edges connecting same vertices.



example: V = set of subsets of $\{1,2,3,4\}$ with 2 elements
two vertices are adjacent (connected by an edge) whenever their intersection has size 1.

Can you draw this graph so that it does not have any edge crossings?

example: V = set of people who have appeared in a theatrical release movie.

Two vertices are adjacent if they have appeared in the same movie.

example: V = set people who are authors of published math papers.
two vertices are connected if they have been authors on the same paper.
Paul Erdos number.

The Fourfold Way, Continued

Recall we want to count the ways to chose k elements from a set of size n.

We have a couple assumptions, leading to 4 cases.

① Order matters, repetition allowed
n^k

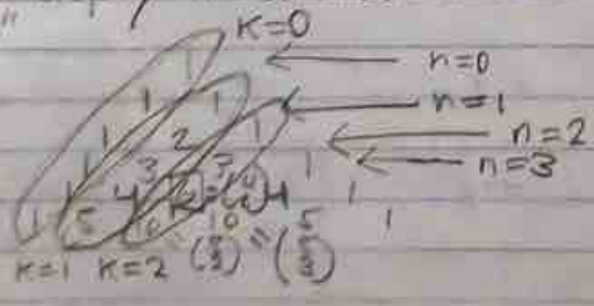
② Order matters, repetition not allowed: $\frac{n!}{(n-k)!}$
k-permutations or permutations.

③ Order does not matter, repetition not allowed.
These are subsets of size k of our set of size n.
Notation of Terminology:

$\binom{n}{k}$ = number of subsets of size k of a set of size n.

read "n choose k". These #s are the "binomial coefficients".

usually display the binomial coefficients in "Pascal's Triangle"



Proposition: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
for all n, k with $0 \leq k \leq n$.
Proof: How many k-permutations of $[n]$ are there?
$\frac{n!}{(n-k)!}$ by last meeting's work.
We can construct the k-permutations by
① choosing a subset of size k — $\binom{n}{k}$ ways
② put the k elements in order — $k!$ ways

So $\frac{n!}{(n-k)!} = \binom{n}{k} k!$

so $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\square$

Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
Proof: deferred. $\square$

④ order does not matter
repetition is allowed
we should think about how many of each object we choose.
So we can order the elements of A and illustrate each choice of k objects as

$\underbrace{***}_{\text{3 copies of 1st object}}|\underbrace{|}_{\text{0 copies of second object}}\underbrace{****}_{\text{4 copies of 3rd object}}|\underbrace{**}_{\text{2 copies of 4th object}}$

k stars $= k$ objects.
$n-1$ bars (separators) to make n bins.

we choose positions for the bars, then place stars:
$\binom{k+n-1}{n-1}$ ways or $\binom{k+n-1}{k}$ ways.

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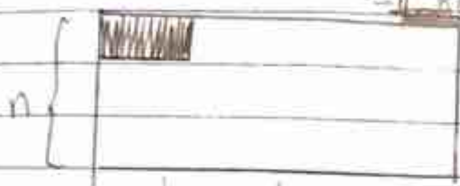
Combinatorial Theory 333
Professor Eric Egge
Homework for Wednesday, September 19

Chapter 1:

G1. How many tilings of an $m \times n$ board with dominoes and monominoes use exactly one domino?

$(m-1)n + (n-1)m = mn - n + nm - m$

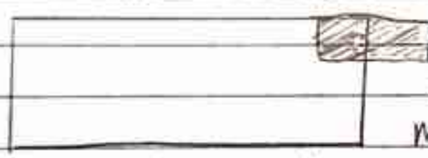
$= \boxed{2nm - n - m}$



This question can best be rephrased as the equivalent: How many different ways can a single domino be placed on the board?

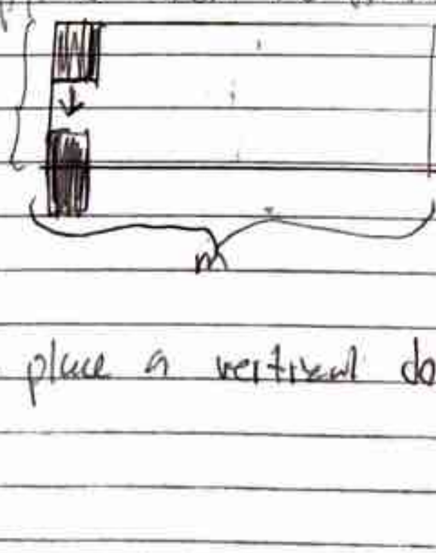
m

There are n rows in which to place a horizontal domino, and since the domino is 2 blocks long we cannot place the left half in the rightmost square;



So instead of m spaces we have m-1 spaces for the domino to move horizontally.

The same logic applies when the domino is oriented vertically:



So $(m-1) \cdot n$ ways to place a horizontal domino

and $(n-1)m$ ways to place a vertical domino exist.

G8. Back when Slimpy's burgers was open in Ann Arbor, Michigan, customers could design their own burger. First, they would choose the burger size: small, medium, large, or humongous. Then they would choose as many cheeses as they wanted from among American, cheddar, provolone, swiss, monterey jack, and colby. Then they would choose a bun type: white, wheat, bagel, or sesame seed. Then they would choose as many grill items as they wanted from among onions, onion rings, fries, macaroni and cheese, pineapple, mushrooms, tomatoes, bacon, and peppers. Finally they would choose as many condiments as they wanted from among ketchup, mustard, bbq sauce, sweet and sour sauce, szechwan sauce, lettuce, tomato, relish, mayo, blue cheese dressing, sriracha sauce and guacamole. How many different burgers could a customer design?

① Choices: 1) Burger size: small, medium, large, or humongous.
2) Cheeses: American, cheddar, provolone, swiss, montrey jack, and colby.
3) Bun type: white, wheat, bagel, or sesame seed.
4) Grill items: Onions, onion rings, fries, macaroni and cheese, pineapple, mushrooms, tomatoes, bacon, and peppers.
5) Condiments: Ketchup, mustard, bbq sauce, sweet and sour sauce, szechwan sauce, lettuce, tomato, relish, mayo, blue cheese dressing, sriracha sauce, and guacamole.

G17. How many paths in the diagram in Figure 15 start at some A, end at Z, contain each letter of the alphabet exactly once, and traverse the letters in alphabetical order? Diagonal moves are not allowed.

A path in a multigraph is a walk with no repeated vertices.

A walk in a multigraph is an alternating sequence $v_1, e_1, v_2, \ldots, e_{n-1}, v_n$ of vertices and edges such that the endpoints of e_j are v_j and v_{j+1} for $1 \leq j \leq n-1$. (Chapter 2)

Each A has a path that first goes to the bottom of the triangle and then, having reached the bottom, turns right if it is on the left half of the triangle

or

left if it is on the right half of the triangle, until it reaches the only Z that is available.

Since this is the only Z, these are the only possible paths.

There is one such path for each A, and there are $25 \cdot 2 + 1 = 51$ As, so $\boxed{51 \text{ paths}}$

6.18. Suppose n is a nonnegative integer.
(a) Give an algebraic proof that

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}.$$

$\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$, so

$$\binom{n}{2} = \frac{n!}{2!\cdot(n-2)!},$$

$$\binom{\binom{n}{2}}{2} = \frac{\left(\frac{n!}{2!(n-2)!}\right)!}{2!\left(\frac{n!}{2!(n-2)!}-2\right)!}$$

$$= \frac{\left(\frac{n!}{2!(n-2)!}\right)\left(\frac{n!}{2!(n-2)!}-1\right)}{2!} = \frac{\frac{n(n-1)}{2!}\cdot\left(\frac{n(n-1)}{2!}-1\right)}{2!}$$

$$= \frac{\frac{n(n-1)}{2}\cdot\left(\frac{n(n-1)-2}{2}\right)}{2} = \frac{n(n-1)\cdot(n(n-1)-2)}{8}$$

$$= \frac{(n^2-n)\cdot(n^2-n-2)}{8} = \frac{n^4-n^3-2n^2-n^3+n^2+2n}{8}$$

$$= \boxed{\frac{n^4-2n^3-n^2+2n}{8}} \checkmark$$

And

$$3\binom{n+1}{4} = 3\cdot\frac{(n+1)!}{4!\cdot(n+1-4)!}$$

$$= \frac{(n+1)!}{8(n-3)!} = \frac{(n+1)(n)(n-1)(n-2)}{8}$$

$$= \boxed{\frac{n^4-2n^3-n^2+2n}{8}} \checkmark.$$

$$\frac{(n^2+n)(n^2-3n+2)}{8}$$

$$= n^4+n^3-3n^3-3n^2+2n^2+2n$$

642. At a certain small college in the upper Midwest, 29 different academic majors are offered, including Mathematics, Biology, Religion, Philosophy, Music, History, Economics, and Psychology. Unfortunately, the college has fallen on hard times, and the current sophomore class has only seventeen students (who are about to declare a major).

(a) In how many different ways can the seventeen sophomores declare a major?

Each sophomore is unique and so repetitions of majors are allowed.

So there are $\boxed{29^{17} \text{ ways.}}$

(b) In order to keep as many departments active as possible, the seventeen sophomores agree that no two of them will declare the same major. How many ways are there for these seventeen students to declare a major with this new restriction?

They must all choose different majors (no repetition allowed). So there are

$29 \cdot 28 \cdot 27 \cdot 26 \cdot 25$
$\cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$
$\cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15$
$\cdot 14 \cdot 13$ or $\frac{29!}{12!} = nPr(29,17)$. This assumes

that order matters, repetition not allowed. In essence, Student 1 picking a Biology major is not the same event as Student 15 selecting a Biology major

Chapter [illegible]:

[illegible]. We say (temporarily) that a multigraph G with vertex set V is partition-connected whenever there is no partition of V into nonempty sets V_1 and V_2 such that every edge of G has both endpoints in the same set. Show a multigraph is connected if and only if it is partition-connected.

A multigraph G is an ordered pair (V, E) of sets together with a function f from E to the set of subsets of V of size one or two.

We say a multigraph G is connected whenever, for every pair of vertices u and v, there is a walk from u to v.

A walk can contain repeated vertices.

A walk in a multigraph is an alternating sequence $v_1, e_1, v_2, \dots, e_{n-1}, v_n$ of vertices and edges such that the endpoints of e_j are v_j and v_{j+1} for $1 \le j \le n-1$.

Say a graph is not connected. Then $\exists$ a vertex v_1 s.t. there is no walk from v_1 to another vertex in G, call it v_2. So we could have

, for example. Then,

$\exists$ a partition P of V into $\langle \underbrace{\langle \text{all vertices connected to } v_1 \rangle}_{\text{not empty b/c contains } v_1}, \underbrace{\langle \text{all vertices connected to } v_2 \rangle}_{\text{not empty b/c contains } v_2} \rangle$.

Since v_1, v_2 are not connected, there's no walk from v_1 to v_2 so no edge connecting v_1 & connected vertices to v_2. So no edge with one vertex in the 1st set and the other in the 2nd set $\Rightarrow$ all edges have vertices in the same set $\Rightarrow$ G is not partition-connected. $\therefore$ $\neg$connected $\Rightarrow$ $\neg$partition-connected.

ii2. Assume a multigraph G is not partition-connected.

"not P iff not Q"
means
"P iff Q"

Then $\exists$ a partition P of V into nonempty sets V_1 and V_2 s.t. every edge of G has both endpoints in the same set

$\Leftrightarrow$ no edge has one endpoint $\in V_1$ and another $\in V_2$

$\Rightarrow$ for all $v \in V_1$, $w \in V_2$, $\nexists$ an alternating sequence v, e, w of vertices and edges s.t. the endpoints of e are v and w (Ch. 2.1).

$\Rightarrow$ there is no walk from v to w

$\Rightarrow$ B/C $v, w \in G$, G is not connected

$\therefore$ not partition-connected $\Rightarrow \neg$ connected.

$\therefore$ $\neg$ partition-connected iff $\neg$ connected

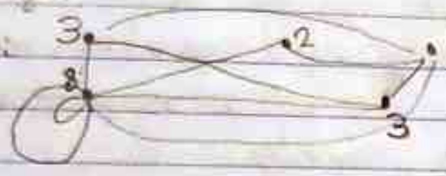
$\therefore$ Partition-Connected iff Connected

Multigraphs, continued: Recall we defined a multigraph to be a set of vertices together with a multiset of edges. (intuitively)

Multiset: repeats allowed. connecting pairs of vertices or vertices to themselves.

We defined a graph (simple graph?) to be a multigraph with no loops or multiple edges.

example:



Def^{n} The degree of a vertex is the number of times that vertex is an endpoint of an edge

Note: loops contribute 2 to the degree of the vertex they're attached to

First theorem of graph theory. If G is a multigraph with finitely many vertices, finitely many edges, then the number of vertices of odd degree is even.

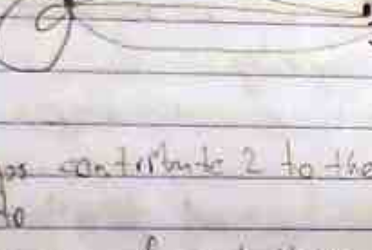
Proof: Suppose the degrees of the vertices are $d_1, d_2, \ldots, d_n$.

Then $d_1 + d_2 + \cdots + d_n = 2(\#\text{edges})$ even

if an odd # of these are odd then the sum is odd, so the sum is even.

Historically, first result of graph theory is Konigsberg bridge problem.

Kaliningrad



Question: Can you find a path crossing each bridge once and starting / ending in same place?

~1730s

1707 - 1783

1707 - 1726 Basel, Switzerland

1726 - 1741 St. Petersburg, Russia

1741 - 1766 Berlin

1766 - 1783 St. Petersburg

Euler's idea: make a multigraph
vertices = landmasses
edges = bridges.

Terminology: a walk in a multigraph is a sequence
vertex, edge, vertex, edge, ..., edge, vertex
so v_1, e_1, v_2 means e_1 connects v_1, v_2

a trail is a walk with no repeated edges

a circuit is a ~~walk~~ trail which begins and ends at same vertex

an Eulerian circuit is a circuit ~~with no repeated edges.~~
which uses every edge

an Eulerian trail is a trail which uses every edge.

Idea: if we have an Eulerian circuit then the degrees of vertices must all be even.

The converse is also true!

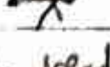
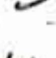
Theorem: if G is a connected, finite multigraph, G has an Eulerian circuit iff all vertices have even degree

Proof: ($\Rightarrow$) each time we enter a vertex across an edge, we must leave across a different edge, contributing 2 to the degree.

All edges are in the circuit so each degree is a sum of 2, so even.

($\Leftarrow$) We argue by induction on # of edges

base case: 0 edges
in this case, only 1 vertex, easy to construct an Eulerian circuit

1 edge   has an Eulerian circuit

same odd degree vertices

Suppose G has $n \geq 2$ edges, all vertices have even degree, G is connected and result holds for any graph with all even degrees and fewer edges than G.

Pick a vertex

By repeatedly choosing new edges, we get a

which might not include all vertices, or all edges.

Remove edges we have used. We get a multigraph that might be disconnected. Pick any connected piece.

All vertices have even degree because we removed 0 or 2 edges from each.

By induction each connected component has an Eulerian circuit.

Make an Eulerian circuit for all of G by v_1, unused Eulerian circuits incident to v_1, move to v_2, unused Eulerian circuits incident to v_2, repeat.

More on Binomial Coefficients and Counting

Recall: we saw the # of ways to choose k elements from a set of size n is

		repeats allowed	
		yes	no
order matters	yes	n^k	$\frac{n!}{(n-k)!}$
	no	$\binom{n+k-1}{k}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proposition: For $n \geq 1$, $1 \leq k \leq n-1$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof: could use fact that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

another approach: $\binom{n}{k}$ = # of subsets of $\{1, \ldots, n\}$ of size k

$\binom{n}{k}$ = # sets including 1 + # sets not including 1

$= \binom{n-1}{k-1} + \binom{n-1}{k}$

← Choose k-1 elements of {2, ..., n}

choose k elements from {2, ..., n}

□

example: DNA is a chain of letters, each of which is either A C G T

adenine cytosine guanine thymine

How many DNA chains of length 10 are there?

4^{10}

How many DNA chains consist of exactly

7 A 12 T

8 C 11 G ?

① Choose positions for A → $\binom{38}{7}$

② then choose positions for C $\binom{31}{8}$

③ then choose positions for T $\binom{23}{12}$

④ place Gs

Total $= \binom{38}{7}\binom{31}{8}\binom{23}{12} = \frac{38! \, \cancel{31!} \, \cancel{23!}}{7! \, \cancel{31!} \, 8! \, 12! \, 11! \, \cancel{23!}}$

$= \frac{38!}{7! \, 8! \, 12! \, 11!}$

PIE Principle of Inclusion-Exclusion

Recall: The addition principle says if $A_1 \cap A_2 = \emptyset$ (and A_1, A_2 are finite) then $|A_1 \cup A_2| = |A_1| + |A_2|$

Similarly, if $A_1, \dots, A_n$ have $A_j \cap A_k = \emptyset$ for all $j \neq k$

then $|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$

Question: Can we find $|A_1 \cup \cdots \cup A_n|$ in terms of something useful even when $A_j \cap A_k \neq \emptyset$ for some $j \neq k$?

Idea: try small cases. $n=2$

$$|A_1 \cup A_2| = \underbrace{|A_1| + |A_2|}_{\text{counts anything in both sets twice}} - |A_1 \cap A_2|$$

$n=3$ $\quad |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$
$- |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$ does not count anything in all 3 sets.

$n=4$ $\quad |A_1 \cup A_2 \cup A_3 \cup A_4| = \underbrace{|A_1| + |A_2| + |A_3| + |A_4|}_{+4} - |A_1 \cap A_2| -$
$|A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4|$ $\quad -6$
$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|$ $\quad +4$
$- |A_1 \cap A_2 \cap A_3 \cap A_4|$ $\quad -1$

PIE For any finite sets $A_1, \dots, A_n$,

$$|A_1 \cup \cdots \cup A_n| = \sum_{\substack{Y \subseteq \{1,2,\dots,n\} \\ Y \neq \emptyset}} (-1)^{|Y|-1} \left| \bigcap_{j \in Y} A_j \right|$$

The Hats Problem

Classic Probability Problem: n people, each with their own hat
(all distinguishable) (all hats are distinguishable)

If we randomly give each person a hat from the n hats, what is the probability that no person gets their own hat?

answer: $\dfrac{\#\text{ ways no one gets their own hat}}{\text{total \# ways to give out hats}}$

\# permutations of $1, \dots, n$ $= n!$. We can think of

... as counting certain permutations

<u>Def^n</u> for any $\pi \in S_n$, a fixed point j is a number with $\pi(j) = j$

<u>Def^n</u> a derangement is a permutation with no fixed pts.
d_n = \# of derangements in S_n.

n	0	1	2	3	4	5	6	7
d_n	1	0	1	2	9	44	265	1854
				-1	-1	-1	$+1$	-1

123 132 213 231 321 312

To get our answer, we need a formula for d_n.

to start, $d_n = n! - \#$ perms with a fixed pt.

$A_1 =$ set of perms in which 1 is a fixed pt

$A_2 =$ set of permutations in which 2 is a fixed pt

In general, A_j is the set of perms in which j is a fixed point:

We know $d_n = n! - |A_1 \cup \cdots \cup A_n|$

$$= n! - \underbrace{|A_1|}_{(n-1)!} - \underbrace{|A_2|}_{(n-1)!} - \cdots - \underbrace{|A_n|}_{(n-1)!}$$
$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots$$
$$- |A_1 \cap A_2 \cap A_3| - \cdots$$
$$+ |A_1 \cap A_2 \cap A_3 \cap A_4| + \cdots$$

There are n terms $|A_j|$ and $|A_j| = (n-1)!$ for all j so

$$d_n = n! - n! + \underbrace{|A_1 \cap A_2|}_{(n-2)!} + \underbrace{|A_1 \cap A_3|}_{(n-2)!} - |A_1 \cap A_2 \cap A_3| - \cdots$$

Each $|A_j \cap A_k|$ has size $(n-2)!$. There are $\binom{n}{2}$ of these terms.

$$d_n = n! - n! + \binom{n}{2}(n-2)! - |A_1 \cap A_2 \cap A_3| - \cdots$$
$$= n! - n! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \binom{n}{4}(n-4)! - \cdots$$
$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)!$$
$$= \sum_{j=0}^{n} (-1)^j \frac{n!}{j!\cancel{(n-j)!}} \cancel{(n-j)!}$$
$$= n! \boxed{\sum_{j=0}^{n} \frac{(-1)^j}{j!}} \approx n! \frac{1}{e}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Conjecture: $d_n = n d_{n-1} + (-1)^n$. Can we use the formula to check this?

The Binomial Theorem

Recall: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. display in Pascal's Triangle.

1 1
1 2 1
1 3 3 1
1 4 6 4 1

Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$\binom{n}{k} = \binom{n}{n-k}$ $\qquad$ $\binom{n}{k}$ is the # of subsets of $[n]$ of size k.

We call the #s the "binomial coefficients" because $\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$

$(x+y)^2 = x^2 + 2xy + y^2$ $\qquad$ The Binomial Theorem

Also $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Proof: We argue by induction on n.

when $n=0$ the claim is that $(x+y)^0 = 1$, which is true.

when $n=1$ the claim is $(x+y)^1 = \binom{1}{0}x + \binom{1}{1}y = x+y$ which is also true.

Assume $n \geq 2$ and $(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$

Multiply by $(x+y)$: $(x+y)^n = (x+y) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$

$$= x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} + y \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$$

$$\sum_{k=0}^{n} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n} \binom{n-1}{k} x^k y^{n-k}$$

ok because new term is $\binom{n-1}{n} \cdots = 0$

re-index: $j = k+1$, replace j with k

$$= \sum_{k=0}^{n} \binom{n-1}{k-1} x^k y^{n-k} + \sum_{k=0}^{n} \binom{n-1}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$\binom{n}{k}$

Binomial Coefficient Identities

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

But there are many other identities and relationships involving binomial coefficients.

① add entries in each row

row	0	1	2	3	4	5	6	7
sum of entries	1	2	4	8	16	32	64	128

Proposition: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Proof: set $x=y=1$ in binomial theorem. □

Algebraic

Combinatorial Proof: How many subsets does $[n]$ have?

answer 1: 2 choices, in or out, for each $j \in [n]$ so 2^n subsets total.

answer 2: count subsets by their sizes:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$$

size 0 subsets, size 1 subsets

answers must be the same so $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. □

② take alternating sum of entries in each row

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} \cdots$$

row	0	1	2	3	4	5	6	7
alt sum	1	0	0	0	0	0	0	0

Proposition: For $n \geq 1$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

algebraic proof: set $x=-1$, $y=1$ in binomial theorem. □

combinatorial proof: It's good enough to prove

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$$

We want to show the # of subsets of $P([n])$ of even size is same as # of odd size.

We prove this by giving a bijection.

Say x is a subset.

if $1 \in X$, then remove 1.

if $1 \notin X$, then put 1 into X.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

This is a function

even size sets $\longrightarrow$ odd size sets

odd size sets $\longrightarrow$ even size sets

Even better, this fun is its own inverse, so it's a bijection.
(it's an "involution": $f \circ f =$ identity).

③ take sums of entries in each row with coefficients $0, 1, 2, \ldots$:

row	0	1	2	3	4	5	6	7
sum	0	1	4	12	32	80	192	448?

$2 \cdot 2^1 \quad 3 \cdot 2^2 \quad 4 \cdot 2^3 \quad 5 \cdot 2^4 \quad 6 \cdot 2^5$

Proposition: For all $n \geq 0$, $\longrightarrow$ reminds me of a derivative.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Algebraic Proof: The binomial theorem says

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Set $y=1$: $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

differentiate:

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

set $x=1$ to get

$$n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k. \quad \square$$

Combinatorial Proof: In a set of n people, how many ways are there to choose a team (subset) and a member of the team (element of the subset) to be the captain?

answer 1: choose a captain (n ways)
then choose rest of team (2^{n-1} ways)
for a total of $n2^{n-1}$ ways.

answer 2: Suppose we choose a team of size k. There are $\binom{n}{k}$ teams, and k choices of captain for each team so there are $k\binom{n}{k}$ ways. In total we have $\sum_{k=0}^{n} k\binom{n}{k}$ ways.

(c) add the squares of the entries in each row

row	0	1	2	3	4	5	6	7
sum	1	2	6	20	70	252	1324	3432

Proposition. For $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Pigeonhole Principle

The Pigeonhole Principle says
if you classify each of n objects into k classes and $n > k$ then at least one class contains two or more objects.

example: n people in a room, various handshakes happen.
no one shakes their own hand.
Show there are 2 people who shook hands with the same number of people.

Proof: classify people by # of people they shook hands with.
Smallest possible # is 0.
largest possible # is n-1.

if someone shook 0 hands then noone shook their hand, and classes are $0,1,2,\dots,n-2$ $n-1$ classes

n people

so some class has 2 people. if everyone shook at least 1 hand then classes are $1,2,\dots,n-1 \to$ $n-1$ classes, n people so some class has 2 people.

example: Choose 1001 integers from among $1,2,\dots,2000$. Show among your integers there are two a, b, such that a divides b.

Each # in our set factors as $2^j \cdot$ odd.

(ex: $28 = 2^2 \cdot 7$ $50 = 2^1 \cdot 25$

$36 = 2^2 \cdot 9$

$400 = 2^4 \cdot 25$)

Let's classify #s in our set by the odd # in this factorization. We have 1000 classes, & 1001 numbers. Two numbers must be in same class. Say they are $2^j x$ ← x is odd

$2^k x$

if $j<k$ then $2^j x$ divides $2^k x$

if $k<j$ then $2^k x$ divides $2^j x$

Chapter 1 G5. How many sets of ten cards from a standard deck of 52 cards contain at least one card from each suit?

There are 4 suits.

Hearts	Spades	Clubs	Diamonds	
7	1	1	1	$\binom{13}{7}\binom{13}{1}^3$
6	2	1	1	
6	1	2	1	
6	1	1	2	$\binom{13}{6}\cdot\binom{13}{1}^2\binom{13}{2}$
5	3	1	1	
5	1	3	1	
5	1	1	3	$\binom{13}{5}\cdot\binom{13}{1}^2\binom{13}{3}$
5	2	2	1	
5	2	1	2	
5	1	2	2	$\binom{13}{5}\cdot\binom{13}{2}^2\cdot\binom{13}{1}$

$\binom{13}{1}^4$ Choosing one card from each suit.

$10-4=6$ cards left to choose,

and $52-4=48$ cards to chose from

Each card is unique.

$\binom{48}{6}$

$\binom{13}{1}^4\binom{48}{6}$

G10. Use our formula for d_n to show that for all $n \geq 2$ we have $d_n = (n-1)(d_{n-1} + d_{n-2})$.

$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$

G10. $\boxed{d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!}$ Prop. 1.28

So
$$d_{n-1} = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-1-k)!$$
$$= \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} (n-1-k)!$$
$$= \sum_{k=0}^{n-2} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} (n-1-k)! + (-1)^{n-1} \cdot \frac{(n-1)!}{(n-1)!(n-1-(n-1))!} \cdot (n-1-(n-1))!$$
$$= \sum_{k=0}^{n-2} (-1)^k \frac{(n-1)\cdot(n-2)!}{k!(n-2-k)!} (n-2-k)! + (-1)^{n-1} \cdot \frac{1}{1\cdot 0!} \cdot 0!$$
$$= (n-1) \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} (n-2-k)! + (-1)^{n-1}$$
$$= (n-1)\cdot d_{n-2} + (-1)^{n-1}$$
So $\underline{d_{n-1} = (n-1) d_{n-2} + (-1)^{n-1}}$.

Now, $d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} (n-k)!$
$$= \sum_{k=0}^{n-2} (-1)^k \frac{n!}{k!(n-k)!} (n-k)! + (-1)^{n-1} \frac{n!}{(n-1)!(n-(n-1))!} \cdot (n-(n-1))! + (-1)^n \frac{n!}{n!(n-n)!} (n-n)!$$
$$= \sum_{k=0}^{n-2} (-1)^k \frac{(n-2)!\,(n-1)(n)}{k!(n-2-k)!} (n-2-k)! + (-1)^{n-1} n + (-1)^n \cdot 1$$
$$= n(n-1) \sum_{k=0}^{n-2} (-1)^k \frac{(n-2)!}{k!(n-2-k)!} (n-2-k)! + n(-1)^{n-1} + (-1)^n$$
$$= n(n-1) d_{n-2} + n(-1)^{n-1} + (-1)^{n-1}\cdot(-1)$$
$$= n(n-1) d_{n-2} + (-1)^{n-1}\cdot(n-1)$$
$$= (n-1)(n\, d_{n-2} + (-1)^{n-1})$$
$$= (n-1)(d_{n-2} + (n-1)d_{n-2} + (-1)^{n-1}).$$ Substitute
$$= (n-1)(d_{n-2} + d_{n-1}).$$

G31. How many ways are there for n people to sit on the floor in a circle?

How can n people sit in a row?

$\underline{n}\ \underline{n-1}\ \cdots\ \underline{2}\ \underline{1} = n!$

$n!$ ways to sit in a row.

The circle is exactly like a row except

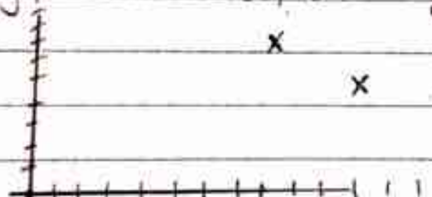
$p_1 p_2 p_3$ is the same as $p_2 p_3 p_1$, $p_3 p_1 p_2$.

$\underline{p_1} p_2 \cdots p_n$ is $\underline{p_2} p_3 \cdots p_n p_1$ is ... is $\underline{p_n p_1} \cdots p_{n-1}$.

There are n ways to select the first person in a row, yet this first person does not matter in the case of a circle. So

$n!/n = \underline{(n-1)!}$ ways.

W6. How many lattice paths from (0,0) to (13,9) do not pass through either (8,7) or (11,5)? no overlap. see graph.



lattice paths from (0,0) to (8,7): $\binom{8+7}{8}$

there are $\binom{n+m}{m}$ lattice paths from (0,0) to (m,n).

paths from (8,7) to (13,9): $\binom{13+9-8-7}{13-8}$

from (0,0) to (11,5): $\binom{11+5}{11}$

(11,5) to (13,9): $\binom{13+9-11-5}{13-11}$

So we exclude $\left[\binom{8+7}{8}+\binom{13+9-8-7}{13-8}+\binom{11+5}{11}+\binom{13+9-11-5}{13-11}\right]$

= 10839 lattice paths

from $\binom{13+9}{13} = 497420$.

$497420 - 10839 = \boxed{486581}$

Chapter 2:

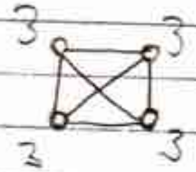
G8. Suppose G is a (simple) graph with no Eulerian circuit. Is it always possible to create a graph with an Eulerian circuit by adding edges to G? Explain.

i.e, if E has no loops and no repeated elements.

simple graph ⊆ multigraph. So

Prop. 2.4.

G has an Eulerian circuit iff every vertex has even degree.



Counterexample:

These graphs are undirected.
We cannot repeat edges so there are no more edges to add ;
It is not always possible.

G11.

A circuit is a trail which begins and ends at the same vertex.

A trail is a walk in which no edge occurs more than once.

Every vertex has even degree. ⟵ Prop. 2.4.
iff G has an Eulerian circuit.

Chapter 4. G20. (a) Evaluate (by hand) the following sums, and write each answer in the form $\frac{m}{n}$.

$\frac{1}{1}+\frac{2}{2}+\frac{1}{3} = 1+1+\frac{1}{3} = \frac{3+3+1}{3} = \underline{\frac{7}{3}}$

$\frac{1}{1}+\frac{3}{2}+\frac{3}{3}+\frac{1}{4} = 1+1+\frac{3}{2}+\frac{1}{4} = 2+\frac{6+1}{4} = \frac{8+7}{4} = \underline{\frac{15}{4}}$

$\frac{1}{1}+\frac{4}{2}+\frac{6}{3}+\frac{4}{4}+\frac{1}{5} = 1+2+2+1+\frac{1}{5} = 6+\frac{1}{5} = \underline{\frac{31}{5}}$

Make a general conjecture based on your answers.

$\frac{7}{3}, \frac{15}{4}, \frac{31}{5} \dots \qquad \frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \frac{15}{4}, \frac{31}{5}, \dots$

$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \frac{a_4}{b_4}, \frac{a_5}{b_5}, \dots$

$\underline{a_n = a_{n-1} + 2^{b_{n-1}}}$

where $\frac{a_n}{b_n}$ is the sum of

$\frac{m_1}{n_1} + \cdots + \frac{m_n}{n_n}$

	row
1	0
1 1	1
1 2 1	2
1 3 3 1	
1 4 6 4 1	

where m_x is the xth element of the $(n-1)$th row of Pascal's Δ.

(b) Use integration and the binomial Theorem (with $y=1$) to prove your conjecture.

Binomial Theorem: $\forall n \geq 0$ we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Set $y=1$: $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$$\int (x+1)^n = \int \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \int \binom{n}{k} x^k \, dx$$

$$= \sum_{k=0}^{n} \int \frac{n!}{k!(n-k)!} x^k dx = \sum n! \int \frac{x^k}{k!(n-k)!} dx$$

$$= \sum n! \cdot \frac{1}{k!(n-k)!} \int x^k dx = n! \sum \frac{1}{k!(n-k)!} \frac{1}{k+1} x^{k+1}$$

$$= n! \times \sum_{k=0}^{n} \frac{x^{k}}{(k+1)! \cdot (n-k)!} \quad \text{when } x=1, \text{ this is}$$

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{1}{(k+1)} = \underline{\sum \binom{n}{k} \frac{1}{k+1}}$$

Monday

Ordinary Generating Functions

clothesline

Binomial Thm $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

polynomial whose coefficients are nth row of Pascal's Δ factor to $(x+1)^n$

don't care about convergence for ogf

Defn ogf for sequence of numbers or finite sets.

ogf $1,1,1,\dots$ is $1+x+x^2+\dots = \frac{1}{1-x}$

Power Series w/o Convergence

any power series w/ constant term 0.

F $1+x+x^2+\dots = \frac{1}{1-x}$ (geom. series) /

F2. We can substitute to get new series (w/in reason)

$1+x^2+(x^2)^2+\dots = \frac{1}{1-x^2}$

$\frac{1}{1-x-x^2}$ we can try to find coef. of x^n.

$1+x+2x^2+3x^3+5x^4+8x^5+\dots$

F3. we can take derivatives if $f(x)=a_0+a_1x+a_2x^2+\dots$

then $f'(x) =$

$\frac{1}{1-x} = 1+x+x^2+\dots$

$\frac{1}{(1-x)^2} = 1+2x+3x^2+\dots$

$\frac{2}{(1-x)^3} = 2+6x+12x^2+\dots$

$\frac{1}{(1-x)^3} = 1+3x+6x^2+10x^3+\dots$

these coef's appear in Pascal's Δ.

F4: We can integrate if $f(x)=a_0+a_1x+a_2x^2+\dots$

then $\int f(x)dx = C+a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\dots$

for some constant c.

ex ogf for $\tan^{-1}(x)$.

ex ogf to $0,1,0,0,-1,0,0,1,0,0,-1,0,0,1,\dots$

$x \quad -x^4 \quad +x^7 \quad -x^{10} \quad +x^{13}$

$= x(1-x^3+x^6-x^9+x^{12}-\dots)$

$= \frac{x}{1+x^3}$.

Binomial Coef. Identities, continued:

$\sum_{k=0}^{n} \binom{n}{k} = 2^n$ $\quad\quad$ $\sum k\binom{n}{k} = n2^{n-1}$

$n \ge 1$ $\sum (-1)^k \binom{n}{k} = 0$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \leftarrow \text{conjecture.}$$

algebraic proof:

$\binom{2n}{n}$ is the coef. of x^n in $(1+x)^{2n}$.

We also know $(1+x)^{2n} = (1+x)^n(1+x)^n$

We have " $= \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right)$

$= \left(1+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n}x^n\right)\left(1+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n}x^n\right)$

the coef. of x^n is $\binom{n}{n}\binom{n}{0}+\binom{n}{n-1}\binom{n}{1}+\binom{n}{n-2}\binom{n}{2}+\binom{n}{n-3}\binom{n}{3}+\dots+\binom{n}{2}\binom{n}{n-2}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{0}\binom{n}{n}$

$= \sum_{k=0}^{n} \underbrace{\binom{n}{n-k}}_{\binom{n}{k}}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}^2$

Both expressions for coef. of x^n in $(1+x)^{2n}$ must be equal so $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

combinatorial proof: Suppose we have n students living in Watson, n students living in Myers. How many ways are there to choose n of our 2n students?

ans 1: $\binom{2n}{n}$ ways

ans 2: k = # of students chosen who live in Watson

$0 \le k \le n$

total is $\sum_{k=0}^{n} \underbrace{\binom{n}{k}}^{\text{\# ways to choose Watson students}} \underbrace{\binom{n}{n-k}}_{\text{\# ways to choose Myers students}} = \sum_{k=0}^{n}\binom{n}{k}^2$

$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2$?

n	alt sum
0	1
1	0
2	$-2 = 1-4+1$
3	$0 = 1+9-9-1$
4	6
5	0
6	-20
7	0
8	70

Conjecture: $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$

proof?

HW for Sep 26

Chapter 1. 64.

{1} {1}   1 1   1 2

{1, 2} {1, 2}   ~~2~~ 2   2 2

$1 \le i < j \le n \Rightarrow f(i) \le f(j)$ Monotone non-decreasing

$1 \to 1$ $\quad 1 \to 1$ $\quad 1 \to 2$ $\quad$ **3**

$2 \to 1$ $\quad 2 \to 2$ $\quad 2 \to 2$

$n=3$: $1 \to 1$ $\quad 1 \to 2$ $\quad 1 \to 1$ $\quad 1 \to 1$ $\quad 1 \to 1$

$2 \to 1$ $\quad 2 \to 2$ $\quad 2 \to 1$ $\quad 2 \to 1$ $\quad 2 \to 2$

$3 \to 1$ $\quad 3 \to 2$ $\quad 3 \to 3$ $\quad 3 \to 2$ $\quad 3 \to 3$

$$\underbrace{\begin{matrix}1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 2 & 2 & 3\\ 1 & 2 & 3 & 2 & 3 & 3\end{matrix}}_{6} \quad \underbrace{\begin{matrix}2 & 2 & 2\\ 2 & 2 & 3\\ 2 & 3 & 3\end{matrix}}_{3} \quad \underbrace{\begin{matrix}3\\ 3\\ 3\end{matrix}}_{1}$$

10 ways

order 1, 2, ..., n with repetition allowed but must be increasing.

1 1 2

1 2 2   1

2 1   2

3

~~A For~~

$$\binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(n+k-1-n)!}$$

$$\frac{(2n-1)!}{n!(n-1)!} = \binom{2n-1}{n}$$

$$\frac{(2n-1)!}{n!(2n-1-n)!}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{1}{1} = 1$ $\quad \binom{3}{2} = 3$

$\binom{5}{3}$

1.64. We can see that a sequence that isn't in increasing

How many nondecreasing functions are there from [n] to [n]?

We have to place n domain points into k range objects.

However, each su

$\binom{n-1}{k-1}$

d_1, d_2 d_3

~~A sequence~~ The set of increasing functions are the set of ways to rearrange the range R so that repetition is allowed but they must be increasing. By their definition they are the set of non-decreasing values for the ranges.

D:	R
1	1
2	2
3	3
⋮	⋮
n	n

We select subsets of the domain [n] thinking that we need to select k of them. (It does not matter which of [n] we select

Our subsets are unique b/c they are in increasing order. If they were not increasing then we know that reordering them could always produce an increasing result so it would be nonunique. So we are selecting k-subsets (the property of a set does not include permutations) of [n] where repetition is allowed: $\binom{n+k-1}{k}$. (In our case, $\binom{2n-1}{n}$) b/c k=n for us.

1.6.38. MISSISSIPPI has 1 M
4 Is
So we have to order them 4 Ss
such that those groups 2 Ps
are all together. So MISP
MIPS

$$nPr(4,4) = \frac{4!}{(4-4)!} = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 12 \cdot 2 = \boxed{24}$$

G40. $d_n = n d_{n-1} + (-1)^n$.

$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$ (Prop. 1.28)

$\Rightarrow d_{n+1} = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (n+1-k)! = \sum_{k=0}^{n+1} \left((-1)^k \frac{(n+1)!}{k!(n+1-k)!} (n+1-k)! \right)$

$= \sum_{k=0}^{n+1} (-1)^k \cdot (n+1) \cdot \frac{n!}{k!(n-k)!} (n-k)!$

$= (n+1) \cdot \sum_{k=0}^{n} \left[(-1)^k \cdot \binom{n}{k} (n-k)! \right] + (-1)^{n+1}$

So $d_{n+1} = (n+1) d_n + (-1)^{n+1} \quad \forall n \geq 0$

$\Rightarrow d_n = n d_{n-1} + (-1)^n$.

W2. $\int_0^\infty (x-1)^n e^{-x} \, dx$ Find and prove a formula.

Let S_n denote $\int_0^\infty (x-1)^n e^{-x} \, dx$.

$S_1 = \int_0^\infty (x-1)^1 e^{-x} \, dx \approx 0$.

$S_2 = \int_0^\infty (x-1)^2 e^{-x} \, dx \approx 1$.

$S_0 = \int_0^\infty (x-1)^0 e^{-x} \, dx = 1$.

So $S = \langle 1, 0, 1, \ldots \rangle$, the 1st 3 derangement #s.

We want to show that $S_n = n S_{n-1} + (-1)^n$ which

$\Rightarrow S_n = d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$ (Prop. 1.28).

$\int u \, dv = uv - \int v \, du$

$\int_0^\infty (x-1)^n e^{-x}\,dx$ $\qquad$ $\int_0^\infty (x-1)(x-1)^n e^{-x}\,dx$

$\int u\,dv = uv - \int v\,du$

$u=(x-1)$ $\quad$ $dv=(x-1)^n e^{-x}\,dx$

$du=1$ $\quad$ $v= \int (x-1)^n e^{-x}\,dx$

$(x-1)\int (x-1)^n e^{-x}\,dx$

$- \iint (x-1)^n e^{-x}\,dx\,dx$

$\int (x-1)^{n+1} e^{-x}\,dx = (-1)^{n+1} + (n+1)\int (x-1)^n e^{-x}\,dx$

$= \int (x-1)^n (x-1) e^{-x}\,dx$

$(n+1)(x-1)^n = \frac{d}{dx}\left((x-1)^{n+1}\right)$

$(-1)^{n+1} + \int (n+1)(x-1)^n e^{-x}\,dx$

$(-1)^{n+1} + \int \frac{d}{dx}\left((x-1)^{n+1}\right) e^{-x}\,dx$

$(x-1)^{n+1} = x(x-1)^n -$

$\int_0^\infty (x-1)^n e^{-x}\,dx = (-1)^{n+1}\cdot n\int_0^\infty (x-1)^{n-1} e^{-x}\,dx$

$\int (x-1)^n e^{-x}\,dx = (-1)^n + n\int (x-1)^{n-1} e^{-x}\,dx$

Chapter 3. G1. $M = \{a, b, c, d, e, f, g, h, i\}$.

$a \cdot b =$ an integer^2

$x \in M \ni 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} = x$

$2^3 = 8$ different parity cases for the exponents e_1, e_2, e_3.

9 elements in M means that classifying 9 elements into 8 different parity cases means that $\exists$ a pair w/ the same parity cases for the exponents. (Pigeonhole Prop 3.1)

So

$$2^{e_1} 3^{e_2} 5^{e_3} \cdot 2^{e_4} 3^{e_5} 5^{e_6}$$

$$= 2^{e_1+e_4} 3^{e_2+e_5} 5^{e_3+e_6}$$

is a perfect square b/c its square root

$$2^{\frac{e_1+e_4}{2}} 3^{\frac{e_2+e_5}{2}} 5^{\frac{e_3+e_6}{2}}$$ is an integer

b/c the exponents are integers b/c e_1, e_4 & e_2, e_5 & e_3, e_6 ...

each pair has the same parity (even or odd)

and odd + odd = even & even + even = even

and an even #/2 is an integer

So the exponents are all integer $\Rightarrow$ the sqrt is an integer $\Rightarrow$ the # is a perfect square.

Chapter 3. G6a. 10 people
None older than 60
Each at least 1.

Two groups of different people
the sum of whose ages was the same.

Excluding the empty set group, we have
$\sum_{k=1}^{10} \binom{10}{k} = 1023$ unique groups
$60 \cdot 10 = 600$ different possible total age sums
for a group of people
So $\exists$ 2 groups where the sum of ages is the same. If they overlap remove the common people from each group. Since these groups may overlap ~~too~~ but they are unique (~~so~~ are not identical) we can say that the remaining two groups are not empty.

G6b. $\sum_{k=1}^{9} \binom{9}{k} =$

$\binom{n-j}{j-1} =$

Chapter 4: 68 (a) $\sum_{j=1}^{n} \binom{n-j}{j-1}$ for $1 \le n \le 7$.

$^{n}S_n$

n	S_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13.

S_n is the nth Fibonacci number.

(b) $S_3 = S_2 + S_1$ Base case

Assume $S_n = S_{n-1} + S_{n-2}$.

$S_n = \sum_{j=1}^{n+1} \binom{n+1-j}{j-1}$

$= \sum_{j=1}^{n+1} \binom{n-j}{j-1} + \sum_{j=1}^{n+1} \binom{n-j}{j-2}$

$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{1}{n-1} + \underline{\binom{0}{n}}$

$\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + ($

$\binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2}$

$\binom{0}{0}$

b/c $\binom{1}{0}$ $\binom{1}{1}$

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

So $= \sum_{j=1}^{n+1} \binom{n-j}{j-1} + \sum_{j=0}^{n} \binom{n-j}{j-1} = S_{n+1} + S_n$

$\sum_{j=1}^{n+1} \binom{n-j}{j-2} = \sum_{j=0}^{n} \binom{n-j}{j-1}$

68. (c) $\sum_{j=1}^{n} \binom{n-j}{j-1}$

$$= \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1, 1, 2, 3, 5, 8, 13, 21

8, 7, 6, 5, 4, 3, 2, 1

Ch. 4

G13. Suppose we have a set containing ~~m apples and n oranges~~ people wearing red shirts

n oranges. Thus we want to find the total number of distinct subsets that we can take from this larger set in a basket.

ppl wearing black shirts.

For any basket containing k items we can select $\binom{m}{j}$ apples and $\binom{n}{k-j}$ oranges remain to be chosen. This is for a basket of size k w/ j apples. But we can select any number of apples; j can be 0 up to k.

So we have to add # ways to pick 0 apples and k oranges ... + # ways to pick k apples and 0 oranges.

So there are $\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$ ways to select

any sized (0 to $m+n$) subset of people.

Of course each person is unique so we can also select $\binom{m+n}{k}$ since $m+n =$ ~~total~~ # of people.

G8. C dots & bars.

Either 1 or 2 dots between bars

k instances of 2 dots

$n-2k$ instances of 1 dot remaining

$n-k$ instances total.

ways to order 2-dot groups (k of them) from total instances: $\binom{n-k}{k}$

need to find all ways for all k

$$\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=1}^{n+1} \binom{n+1-k}{k-1}$$

We can either reach the $n-2$ th dot and have a 2 dot group or reach the $n-1$ th dot and have a 1 dot group.

Erdős-Szekeres Thm: An Application of PHP

Generalized PHP: $n > mk$

$\exists$ a class w/ at least $m+1$ objects

$n > k$ " 2 objects

Proof. Suppose not. If each class has m or fewer objects, we can have at most mk objects. But $n > mk$. $\square$

Long increasing or long decreasing subsequences exist for large enough permutations.

Proof ES Thm: Write out π. Above each entry write length of longest decreasing subsequence ending at that entry.

Objects Classes

Running example: $\{13, 7, 6, 4\}$

$\pi = 1, 2, 3, 5, 8, 13, 7, 6, 12, 4, 11, 9, 10$

$m = 4$ 1 1 1 1 1 1 2 3 2 4 3 4 4

$k = 3$ $\{8, 7\}$ $\{13, 12, 11, 10\}$

$n = 13$

Back to proof of ES Theorem. If we ever have a $k+1$ or larger above any number, then we have a decreasing subsequence of length $k+1$.

So largest # above any entry is a k.

Since $n > mk$ and we have k possible numbers above entries, there is some number appearing above at least $m+1$ entries.

Notice if a, b have same number above them and a is left of b then $a < b$

because if $a > b$ then we have a decreasing subsequence ending at a which we can extend to get a longer decreasing subsequence ending at b.

So entries w/ the repeated label are an increasing subsequence of length $m+1$ or more. $\square$

Addition & Multiplication of Generating Functions:

Recall: the generating function for a sequence $a_0, a_1, \ldots$ is $a_0 + a_1 x + a_2 x^2 + \ldots$

The gf for a seq $A_0, A_1, \ldots$ of finite sets is $|A_0| + |A_1| x + |A_2| x^2 + \ldots$

Moral: the addition and multiplication principles for gfs work just as they do for counting problems.

ex Find ogf for # ways to make a set of n tiles using either all red tiles or all blue tiles but may only have an even # of blue tiles.

n	0	1	2	3	4	5
# ways	2	1	2	1	2	1

(no tiles) or is it 0 2?

1 red tile

$$ogf = 2 + x + 2x^2 + x^3 + 2x^4 + x^5 + \ldots$$
$$= 1 + x + x^2 + x^3 + x^4 +$$
$$+ 1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x} + \frac{1}{1-x^2}$$

red blue

ogf for sets of only red tiles

ogf for sets of only blue tiles, even #

Addition Principle for ogf: If $A_0, A_1, \ldots,$
$B_0, B_1, \ldots$

are finite sets, ogf for As is $A(x)$

ogf for Bs is $B(x)$ and $A_j \cap B_j = \emptyset$

for all j then ogf for $A_0 \cup B_0, A_1 \cup B_1, \ldots$ is $A(x) + B(x)$.

ex addition principle for ogfs:

Find ogf for Fibonacci #s.

(We conjectured it's $\frac{1}{1-x-x^2}$)

Recall combinatorial interpretation of Fib. #s:

Recall: F_n = # tilings of a $1 \times n$ board with □, ▭

Set $F(x) = \sum_{n=0}^{\infty} F_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$

$= \sum_{n=0}^{\infty} (\# \text{ tilings of } 1 \times n \text{ w/ } □, ▭)\, x^n$

Three sets of tilings

① empty tiling

ogf $= 1 + 0x + 0x^2 + \cdots = \boxed{1}$? where used 2 earlier?

② tilings that end w/ □

$\underbrace{\qquad\qquad}_{n-1}$ □

n	0	1	2	3	4
#	0	1	1	2	3

□

□□

▭□

□□□

ogf $= x + x^2 + 2x^3 + \cdots$

$= \boxed{F(x)\ x}$

③ tilings end w/ ▭

ogf $= x^2 F(x)$

So by addition principle: ogf $= 1 + xF(x) + x^2 F(x) = F(x)$

Solve for F: $F(x) = \frac{1}{1-x-x^2}$

example: Find ogf for # of sets & any # of red tiles and an even # of blue tiles.

n	0	1	2	3	4
# of sets	1	1	2	2	3

rrr
rbb = brb = bbr ←
bbrr
bbbb

In general, if a_n = # of ways to only have red,) n tiles total
b_n = # of ways to only have blue)

then our answer:

for n tiles = $a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \cdots + a_0 b_n$

Notice: $\left(\sum_n a_n x^n\right)\left(\sum_n b_n x^n\right) = \sum \begin{matrix} a_n b_0 \\ + \\ a_{n-1} b_1 \\ + \\ a_{n-2} b_2 \\ \vdots \end{matrix} \; x^n$

So our answer is $\frac{1}{1-x} \cdot \frac{1}{1-x^2}$

ways to construct a set of only red tiles

ways using only blue

Multiplication Principle: Suppose $A(x)$ is ogf for $A_0, \ldots$
$B(x)$ is ogf for $B_0, \ldots$

Let C_n be set of ordered pairs (a_j, b_{n-j})
where $a_j \in A_j$, $b_{n-j} \in B_{n-j}$

then the ogf for $C_0, C_1, \ldots$ is $A(x)B(x)$.

example: Find ogf for # ways to make n cents using only pennies (1 cent each), nickels (5 cents each) and dimes (10 cents each)

C_{10}: d, s
n, p 4 ways
n, n
p

If $P(x)$ is ogf for # ways to make n cents in pennies
$N(x)$ is ogf for # ways to make n cents in nickels
$D(x)$ is ogf for # ways to make n cents in dimes
[illegible] difference between 2 pennies
then our ogf is $P(x)\,N(x)\,D(x)$

$P(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$

$N(x) = 1 + x^5 + x^{10} + \dots = \frac{1}{1-x^5}$

0 cases # ways to make 5 cents, with nickels

$D(x) = 1 + x^{10} + x^{20} + \dots = \frac{1}{1-x^{10}}$

our ogf is $\frac{1}{1-x}\,\frac{1}{1-x^5}\,\frac{1}{1-x^{10}}$

Fri. Sep 28.

Finding and Using an OGF

Recall: if $A(x)$, $B(x)$ are ogfs for $A_0, A_1, A_2, \ldots$ / $B_0, B_1, B_2, \ldots$

then ① if $A_j \cap B_j = \emptyset$ for all j then $A(x) + B(x)$ is the ogf for $A_0 \cup B_0, A_1 \cup B_1, \ldots$

② the ogf for (a_j, b_{n-j}), where $a_j \in A_j$, $b_{n-j} \in B_{n-j}$ is $A(x)B(x)$.

Def^{n} a composition of a nonnegative integer n is a sequence of positive integers which sum to n.

By convention, 0 has one composition.

Note: the compositions 2,1 and 1,2 are different (order matters!).

example: the compositions of 3 are

1,1,1
1,2
2,1
3

the compositions of 4 are

4	2,1,1
3,1	1,2,1
2,2	1,1,2
1,3	1,1,1,1

n	0	1	2	3	4	5	6	7
# compositions of n	1	1	2	4	8	16	32	64

Conjecture: If $n \geq 1$ then the # of compositions of n is 2^{n-1}.

Idea: Find a bijection from compositions of n to set of subsets of $[n-1]$.

Second Idea: use ordinary generating functions.

ogf for compositions = ogf compositions with 0 parts $\leftarrow 1$

+ ogf for compositions with exactly 1 part $\leftarrow x+x^2+x^3+\dots = \frac{x}{1-x}$

+ ogf compositions with exactly 2 parts $\leftarrow \left(\frac{x}{1-x}\right)^2$

+ $\left(\frac{x}{1-x}\right)^3 + \dots$

$\left(\frac{x}{1-x}\right)^2$: ogf for 1st part $\frac{x}{1-x}$, ogf for 2nd part $\frac{x}{1-x}$

$$= 1 + \frac{x}{1-x} + \left(\frac{x}{1-x}\right)^2 + \left(\frac{x}{1-x}\right)^3 + \left(\frac{x}{1-x}\right)^4 + \dots$$

$$= \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x} = \frac{1-2x+x}{1-2x}$$

$$= \frac{1-2x}{1-2x} + \frac{x}{1-2x}$$

$$= 1 + \frac{x}{1-2x}$$

$$1 + x(1+2x+4x^2+8x^3+16x^4+\dots)$$
$$= 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \dots + 2^{n-1}x^n + 2^n x^{n+1} + \dots$$

Conclusion: # of compositions of n is 2^{n-1}.

Find ogfs from recurrence relations

Recall: We found the ogf for the Fibonacci #s: it's $\frac{1}{1-x-x^2}$

We got it by using the fact that it satisfies

$$F(x) = 1 + xF(x) + x^2F(x)$$

Alternative method: we know $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

set $F(x) = \sum_{n=0}^{\infty} F_n x^n$

multiply the relation by x^n, sum over all $n \geq 2$
all n for which relation holds.

$$F_n x^n = F_{n-1}x^n + F_{n-2}x^n$$

$$\sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1}x^n + \sum_{n=2}^{\infty} F_{n-2}x^n$$

Rewrite each sum in terms of $F(x)$

$$F(x) \underset{n=0 \text{ in } F(x)}{-1} \underset{n=1 \text{ in } F(x)}{-x} = x\underbrace{\sum_{n=2}^{\infty} F_{n-1}x^{n-1}}_{\substack{F_1x + F_2x^2 + F_3x^3 \\ n=2 \quad n=3 \quad n=4}} + x^2 \underbrace{\sum_{n=2}^{\infty} F_{n-2}x^{n-2}}_{\substack{F_0 + F_1x + \\ n=2}}$$

$$F(x) - 1 - x = x(F(x) - 1) + x^2F(x)$$

solve for $F(x)$ to get $F(x) = \frac{1}{1-x-x^2}$
$= \frac{1}{1-(x+x^2)}$

Now we want to write $\frac{1}{1-x-x^2}$ as a sum of 2 or more geometric series: $\frac{1}{1-\phi x} + \frac{1}{1-\beta x}$

How can we write $\frac{1}{1-x-x^2}$ this way?

We can check that if $\tau = \frac{1+\sqrt{5}}{2}$
$\bar{\tau} = \frac{1-\sqrt{5}}{2}$

then $\frac{1}{1-x-x^2} = \frac{\frac{\tau}{\sqrt{5}}}{1-\tau x} - \frac{\frac{\bar{\tau}}{\sqrt{5}}}{1-\bar{\tau}x}$

$$= \frac{\tau}{\sqrt{5}}(1 + \tau x + \tau^2x^2 + \ldots) - \frac{\bar{\tau}}{\sqrt{5}}(1 + \bar{\tau}x + \bar{\tau}^2x^2 + \ldots)$$

coefficient of x^n is $F_n = \dfrac{\tau^{n+1}}{\sqrt{5}} - \dfrac{\bar{\tau}^{n+1}}{\sqrt{5}}$

$$= \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{1.6}^{n+1} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{-.6}^{n+1}$$

Observation: ① For n large, F_n is integer nearest

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

② If we get this formula by some other (possibly nonrigorous) method, we can prove it works by induction.

example: Find the # of sequences of n 0s, 1s, 2s in which no 0s are adjacent.

n	0	1	2	3	4	5	6	7
# seq.	1	3	8	22	60	164	448	1224

a_n = # of seqs of length n = # end in 0 + # end in 1 + # end in 2

$$\underbrace{\overbrace{\ldots\ldots}^{n-1}\,2}_{n \text{ entries}}$$

$$\overbrace{\ldots\ldots}^{n-2}\,2\,0 \qquad \underbrace{\ldots\ldots 2}_{n-1}\,0$$

$a_n = 2a_{n-1} + 2a_{n-2}$, $n \geq 2$.

Using same technique, we find $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \frac{1+x}{1-2x-2x^2}$

$$\frac{1}{(1+2x)(1-3x)} = \frac{A}{1+2x} + \frac{B}{1-3x}$$

$$= \frac{A}{1-(-2x)} + \frac{B}{1-(3x)}$$

$r_1 = -2,\ r_2 = 3$

$a_n = A\,(-2)^n + B \cdot 3^n$

$A(1-3x) + B(1+2x) = 1$

set $x = -\frac{1}{2}$:

$A\left(1+\frac{3}{2}\right) + B\left(1-\frac{2}{2}\right) = 1$

$A\left(\frac{5}{2}\right) = 1$

$A = 2/5$

set $x = \frac{1}{3}$:

$A\left(1-\frac{3}{3}\right) + B\left(1+\frac{2}{3}\right) = 1$

$B(5/3) = 1$

$B = 3/5$

$$a_n = \frac{2}{5}(-2)^n + \frac{3}{5} \cdot 3^n$$

$$= \frac{2 \cdot (-2)^n + 3 \cdot 3^n}{5}$$

$$= \frac{-(-2)^{n+1} + 3^{n+1}}{5}$$

LHRRCCs

Linear Homogeneous Recurrence Relations with Constant Coefficients.

Recall: ogf for Fibonacci #s (method of computing them)

1. multiply recurrence relation by x^n

$$F_n x^n = F_{n-1}x^n + F_{n-2}x^n$$

2. sum over all $n \geq 2$
   (n for which relation holds)

$$\sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1}x^n + \sum_{n=2}^{\infty} F_{n-2}x^n$$

3. set $F(x) = \sum_{n=0}^{\infty} F_n x^n$
   write all sums in terms of $F(x)$

$$F(x) - 1 - x = x(F(x) - 1) + x^2 F(x)$$

4. Solve (algebraically) for $F(x)$

$$F(x) = \frac{1}{1-x-x^2}$$

Idea: we can apply this method to any sequence satisfying a recurrence relation of form

$$a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \dots + \beta_k a_{n-k} \quad (n \geq k)$$

where $\beta_1, \dots, \beta_k$ are constants. (not dependent on n.)

coefficient is not constant

non LHRCCs: $a_n = n a_{n-1}$

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + a_2 a_{n-3} + \dots + a_{n-1} a_0$$

n	0	1	2	3	4	5	6	7
a_n	1	1	2	5	14	42	132	429

$a_0 a_0$ $a_0 a_1 + a_1 a_0$

constant coefficients, homogeneous not linear.

$$\underbrace{a_n = 2a_{n-1} + 3}_{\text{not a LHRCC}}$$

so $a_n = 2a_{n-1} + 3$ is not a LHRRCC but if $a_0, a_1, \dots$ satisfies it then gives us a LHRRCC.

$a_{n-1} = 2a_{n-2} + 3$ and $a_n - a_{n-1} = 2a_{n-1} - 2a_{n-2}$

So $a_n = 3a_{n-1} - 2a_{n-2}$

$a_n = 2a_{n-1} + 3^n$

$-3(a_{n-1} = 2a_{n-2} + 3^{n-1})$

Try $a_n = 2a_{n-1} + n^5$ .. can you find a LHRRCC this sequence satisfies?

The technique we used to get gf for F_n works for any sequence satisfying a LHRRCC:

If $a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \dots + \beta_k a_{n-k}$ $\quad n \geq k$

then

$a_n x^n = \beta_1 a_{n-1} x^n + \beta_2 a_{n-2} x^n + \dots + \beta_k a_{n-k} x^n$

sum for $n \geq k$

$$\sum_{n\geq k}^{\infty} a_n x^n = \sum_{n\geq k}^{\infty} \beta_1 a_{n-1} x^n + \dots + \sum_{n\geq k}^{\infty} \beta_k a_{n-k} x^n$$

Set $A(x) = \sum_{n=0}^{\infty} a_n x^n$

we have

$A(x) - (\text{terms of degree} < k) = \beta_1 x (A(x) - \text{terms of degree} < k) + \beta_2 x^2 (A(x) - \text{small terms}) + \beta_3 x^3 (A(x) - \text{small}) + \dots + \beta_k x^k (A(x) - \text{small})$

small terms: degree $< k$

When we solve for $A(x)$ we get

$(1 - \beta_1 x - \beta_2 x^2 - \dots - \beta_k x^k) A(x) = \text{degree less than } k.$

So

$$A(x) = \frac{\text{degree less than } k}{\boxed{1 - \beta_1 x - \beta_2 x^2 \dots - \beta_k x^k}}$$

① we get a rational function! $\left(\frac{\text{poly}}{\text{poly}}\right)$

② denominator tells us the recurrence relation!

Theorem: Suppose $k, \beta_1, \dots, \beta_k$ are constants. Then the following are equivalent for a sequence $a_0, a_1, \dots$

1) $a_0, \dots$ satisfies $a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \dots + \beta_k a_{n-k}$ for $n \geq k$

2) The ogf for $a_0, \dots$ has the form $\dfrac{P(x)}{1-\beta_1 x - \beta_2 x^2 - \dots - \beta_k x^k}$ where degree of P is at most $k-1$.

Proof: we already showed $(1) \Rightarrow (2)$ so suppose ogf for $a_0, \dots$ is $\dfrac{P(x)}{1-\beta_1 x - \dots - \beta_k x^k}$

So $\dfrac{P(x)}{1-\beta_1 x - \dots - \beta_k x^k} = \sum_{n=0}^{\infty} a_n x^n$

$$P(x) = (1-\beta_1 x - \beta_2 x^2 - \dots - \beta_k x^k)\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

If $n \geq k$ then coefficient of x^n in $P(x)$ is 0 (b/c no terms higher than $k-1$)

The coefficient of x^n on right is

$$a_n - \beta_1 a_{n-1} - \beta_2 a_{n-2} - \beta_3 a_{n-3} - \dots - \beta_k a_{n-k}$$

$$1 \cdot a_n x^n \quad -\beta_1 x \cdot a_{n-1} x^{n-1} \quad -\beta_2 x^2 \cdot a_{n-2} x^{n-2}$$

$= 0$

So $a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \dots + \beta_k a_{n-k}$

Example: Suppose $a_0 = 2$, $a_1 = 3$ (5), $a_2 = 1$ (10),
and ogf for $a_0, a_1, \ldots$ is $\frac{(x+2)}{1-2x+x^3}$.

probably wrong: inconsistent with a_0, a_1, a_2

Find a_3. $\quad \beta_1 = 2 \quad \beta_2 = 0 \quad \beta_3 = -1$

$$a_3 = 2a_2 - a_0 = 2 - 2 = 0$$

$$a_4 = 2a_3 - a_1 = 0 - 3 = -3$$

$$\frac{1}{1-2x+x^3}$$

$$\frac{x+2}{1-2x+x^3} = \frac{x}{1-2x+x^3} + \frac{2}{1-2x+x^3}$$

$$\frac{x+2}{1-2x+x^3} = 2 + 5x + 10x^2 + \cdots$$

Maclaurin series

Newton's Binomial Theorem

Recall: we showed for any nonnegative integer n,

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

we also saw

$$\underbrace{\sum_{k=0} \binom{-1}{k} x^k \overset{?}{=} (1-x)^{-1}}_{\text{my wish}} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

we also saw

$$\sum_{k=0?!}^{!!} \binom{-2}{k} x^k \overset{?}{=} \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

$$\sum_{k=0} \binom{-3}{k} x^k = \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$$

coefficients are diagonals in Pascal's Δ!

But what do $\binom{-2}{k}, \binom{-3}{k}$ mean?

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)\cancel{(n-k)!}}{k!\cancel{(n-k)!}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$ defined for all n. (any complex # n).

Take this as to be new definition of $\binom{n}{k}$

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

α any real #

$$f(x) = (1+x)^{\alpha}$$

@ 0

$$f'(x) = \alpha(1+x)^{\alpha-1} \qquad \alpha$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \qquad \alpha(\alpha-1)$$

$$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \qquad \alpha(\alpha-1)(\alpha-2)$$

$$= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

people are distinguishable, questions,

Solving LHRRCCs.

Recall: A sequence $a_0, a_1, \ldots$ satisfies a LHRRCC whenever there are constants $k, B_1, \ldots, B_k$ such that if $n \geq k$ then

$$a_n = B_1 a_{n-1} + B_2 a_{n-2} + \cdots + B_k a_{n-k}$$

We proved $a_0, a_1, \ldots$ satisfies

$$a_n = B_1 a_{n-1} + B_2 a_{n-2} + \cdots + B_k a_{n-k} \quad n \geq k$$

iff

$$\sum_{n=0}^{\infty} a_n x^n = \frac{P(x)}{1 - B_1 x - B_2 x^2 - \cdots - B_k x^k}$$

where $P(x)$ has degree less than k

What about $\frac{x^2}{1-x-x^2}$?

we can do long division to get

$$\frac{x^2}{1-x-x^2} = \cancel{\frac{-1-x+x^2+x+1}{1-x-x^2}}$$

$$= -1 + \frac{x+1}{1-x-x^2}$$

Idea: If we have a rational ogf, and we can factor the denominator, then we can use partial fractions and geometric series to get a formula for a_n.

$$\frac{P(x)}{1-B_1x-B_2x^2-\cdots-B_kx^k} = \frac{P(x)}{(1-d_1x)(1-d_2x)\cdots(1-d_kx)}$$

$$= \frac{A_1}{1-d_1x} + \frac{A_2}{1-d_2x} + \cdots + \frac{A_k}{1-d_kx}$$

$$= A_1 \sum_{n=0}^{\infty} d_1^n x^n + A_2 \sum_{n=0}^{\infty} d_2^n x^n + \cdots + A_k \sum_{n=0}^{\infty} d_k^n x^n$$

So $a_n =$ coef. of x^n $= A_1 d_1^n + A_2 d_2^n + \cdots + A_k d_k^n$.

Now that we know the form of the answer, we can bypass ogf.

example: Suppose $a_0 = 1$, $a_1 = 6$

3 and for $n \geq 2$ we have

2 $$a_n = 5a_{n-1} - 6a_{n-2}$$

1 Look for constants r so $r^n = a_n$ satisfies

6 $$a_n = 5a_{n-1} + 6a_{n-2}.$$

$$r^n = 5r^{n-1} - 6r^{n-2}$$

$$r^n - 5r^{n-1} - 6r^{n-2} = 0$$
$$r^{n-2}(\underline{r^2-5r-6}) = 0$$
$$r^{n-2}(r-3)(r-2) = 0$$
$$r = 3 \text{ or } r = 2$$

Terminology: r^2-5r+6 is the "characteristic polynomial" for the recurrence and its roots are the "characteristic values".

We need constants a, b with $a_n = a2^n + b3^n$.

In particular, $a_0 = a + b = 1$

$$a_1 = 2a + 3b = 6$$
$$2a + 2b = 2$$
$$b = 4$$
$$a = -3$$

Conclusion: $a_n = -3 \cdot 2^n + 4 \cdot 3^n$.

example: a_n = total # of parts in all compositions of n.
Find a formula for a_n.

Recall: a <u>composition</u> of n is a way of writing n as a sum of positive integers.
Order matters: $2+1$, $1+2$ are different compositions of 3.

# of compositions of n is 2^{n-1}.

n	1	2	3	4	5	6
a_n	1	3	8	20	48	112

$1+1$
2

Idea: Find a recurrence relation for a_n and solve it.

a_n = # of parts in all compositions of n.
= # of parts in comps of n whose last part is 1
+ # of parts in comps of n that do not end in 1.

$$= \underbrace{a_{n-1}}_{\text{parts already there}} + \underbrace{2^{n-2}}_{\text{1s we add}} + a_{n-1}$$

$= \quad m \quad n-1$

→ build these by starting with any composition of $n-1$, and adding 1 to last part.

So $a_n = 2a_{n-1} + 2^{n-2}$

check: $n=3$

$$a_3 = 2a_2 + 2 = 2\cdot 3 + 2 = 8$$

$n=4$ $\quad a_4 = 2a_3 + 2^2 = 16 + 4 = 20$

We have $a_n = 2a_{n-1} + 2^{n-2}$

$$2\left(a_{n-1} = 2a_{n-2} + 2^{n-3}\right)$$

$$2a_{n-1} = 4a_{n-2} + 2^{n-2}$$

subtract: $a_n - 2a_{n-1} = -4a_{n-2} + 2a_{n-1}$

$$a_n = 4a_{n-1} - 4a_{n-2}.$$

Look for solutions of form $a_n = r^n$.

Characteristic polynomial is $r^2 - 4r + 4$ which has one root: $r = 2$.

Look for a constant a with $a_n = a2^n$.

$$a_2 = a\cdot 4 = 3$$

$$a = \frac{3}{4}$$

$$a_3 = a\cdot 8 = 8$$

$$a = 1$$

$$a_4 = 16a = 20$$

$$a = \frac{20}{16} = \frac{5}{4}$$

No solution of our form!!!

If we use ogfs, we find $\sum_{n=0}^{\infty} a_n x^n = \frac{x-x^2}{1-4x+4x^2} = \frac{x-x^2}{(1-2x)^2}$

$$= x\left(\underbrace{\frac{1}{1-2x}}_{a2^n} + \frac{x}{(1-2x)^2}\right)$$

$$\frac{x}{(1-2x)^2} = x(1-2x)^{-2}$$

$$= x\sum_{k=0}^{\infty}\binom{-2}{k}(-2x)^k$$

$$\binom{-2}{k} = \left[(-2)(-3)(-4)\cdot(-2-k-1)\right]/k!$$

B/C there are k factors,

$$= \left[(-1)^k\, 2\cdot 3\cdot 4 \cdots (k+1)\right]/k! = (-1)^k (k+1)$$

and

$$x \sum_{k=0}^{\infty} \binom{-2}{k} (-2x)^k = x \sum_{k=0}^{\infty} (k+1) 2^k x^k$$

Form of $a_n = a2^n + b(n+1)2^n$

$= (\text{linear function of } n)\, 2^n.$

Look for constants c, d with

$a_n = (cn + d) 2^n$

$a_1 = (c+d)\cdot 2 = 1$

$a_2 = (2c+d)\cdot 4 = 3$

$2c + 2d = 1$ $\quad$ $4c = 1$

$8c + 4d = 3$ $\quad$ $c = \frac{1}{4}$ $\quad$ $d = \frac{1}{4}$

$4c + 4d = 2$ $\quad$ $\boxed{a_n = (n+1)2^{n-2}} = \left(\frac{n+1}{4}\right)2^n$

check small values: $a_1 = 2\cdot\frac{1}{2} = 1$

$a_2 = 3\cdot 1 = 3$

$a_3 = 4\cdot 2 = 8$

$a_4 = 5\cdot 4 = 20$

Moral: if characteristic polynomial has repeated root, r, repeated k times, then look for solutions of form $(a_1 n^{k-1} + a_2 n^{k-2} + \cdots + a_{k-1} n + a_k) r^n$

<u>OEIS</u>

Monday October 8 2018

Catalan Numbers Recall: we understand sequences whose ogfs are rational functions: they satisfy LHRRCCs and have solutions of form $p_1(n)r_1^n + p_2(n)r_2^n + \dots + p_k(n)r_k^n$.

we look at a couple of sequences whose ogfs are not rational. first is sequence of Catalan numbers.

Def$^{\underline{n}}$ a Catalan path of length n is a lattice path using only North $(0,1)$ and East steps which end at (n,n) and do not pass below $y=x$. $(1,0)$

$n=0$ $\emptyset$ $n=1$ $n=2$

$n=3$ ... C_n = # of Catalan paths of length n

the nth Catalan number.

alternative: paths like where we stay above x-axis. Dyck paths

n	0	1	2	3	4	5	6	7
C_n	1	1	2	5	14	42	132	429

we want today to get ① a recurrence relation for C_n.
② a formula for C_n.

The recurrence relation: use the "first-return decomposition"

every Catalan path eventually returns to $y=x$.

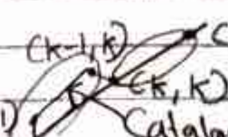
classify paths of length n according to the point (k,k) where they first return to $y=x$.

$k=1$ $k=2$

How many paths of length n first return at (k,k)?

$$\left(C_n = \sum_{k=1}^{n} \#\text{ of paths with } k\text{th return at } (k,k) \right)$$

typical first return at (k,k):



path of length $n-k$

C_{n-k} of these

Catalan path from $(0,1)$ to $(k-1,k)$. It has length $k-1$.

One N and one E are removed.

Conclusion: There are $C_{k-1}C_{n-k}$ paths of length n with first return (k,k), and $C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$

$n=3$: $C_3 = C_0C_2 + C_1C_1 + C_2C_0$
$= 1\cdot 2 + 1\cdot 1 + 2\cdot 1$

$n=4$: $C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1$
$= 14.$

Finding a Formula for C_n

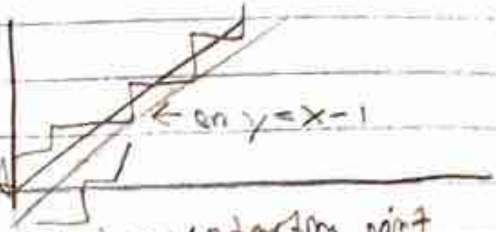
Idea C_n = total # of paths from $\binom{0}{0}$ to $\binom{n}{n}$ (allowing crossings of line $y=x$)

− # paths which pass below $y=x$, "bad paths"

From HW this is $\binom{2n}{n}$

Use the "reflection principle"

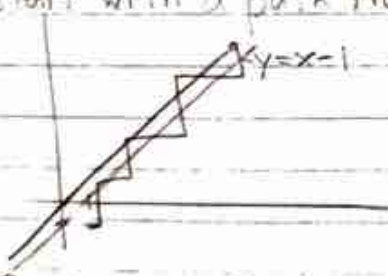
① Find 1st point where path crosses below $y=x$ — it's followed by an E



② Reflect the part of the path up to and including this step, over $y=x-1$

③ We now have a path from $(1,-1)$ to (n,n). Claim: this is a bijection from set of bad paths to set of all paths from $(1,-1)$ to (n,n).

To prove the claim, we describe the inverse of our function. Start with a path from $(1,-1)$ to (n,n)



① $(1,-1)$, (n,n) are on opposite sides of $y=x-1$, so path crosses $y=x-1$. Find the first N that touches $y=x-1$. ~~crossing~~

② Reflect the part of the path up to and including that north step.

③ We get a "bad path" from $(0,0)$ to (n,n)

Observation: The E step identified in 1st function is reflection of N step identified in 2nd function.

So the two functions reflect the same initial segment over the same line, so they're inverses.

Now the # of paths from $(1,-1)$ to (n,n) is $\binom{n-1+n+1}{n-1} = \binom{2n}{n-1}$ ($n-1$ E, $n+1$ N)

So $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \cdots = \binom{2n+1}{n}\frac{1}{2n+1}$

Combinatorial Interpretations of C_n.

Suppose π, σ are permutations. We say π contains σ whenever π has a subsequence of same length as σ and with same relative order.

If π does not contain σ, then we say π avoids σ.

example: 32145 contains 12

41523 contains 132

41523 contains 213

41523 contains 231

41523 contains 312

41523 avoids 321

Rodica Simion Frank Schmidt

Notation: $S_n(\sigma)$ = set of permutations of length n which avoid σ

$Av_n(\sigma)$ = set of permutations of length n which avoid σ

sigma

Game: Find $|S_n(\sigma)|$ for all n, all σ.

Open Question: Find a formula for $|S_n(1324)|$

What about $|S_n(132)|$?

Conjecture: $|S_n(132)| = C_n$.

123

213

⋮

$n=1$: 1

$n=2$: 12, 21

$n=3$: 123, 213, 231, 312, 321

$n=4$: 14 of them

$n=5$: 42 of them

OGF for the Catalan Numbers

Recall: C_n is the number of Catalan paths of length n. A Catalan path is a lattice path using only North $(0,1)$ and East $(1,0)$ steps, starting at $(0,0)$ ending at (n,n) (for length n), never passing below $y=x$.

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \dots + C_{n-2}C_1 + C_{n-1}C_0$$
$$= \sum_{k=1}^{n} C_{k-1}C_{n-k}.$$
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

We want a simple closed form for $\sum_{n=0}^{\infty} C_n x^n$.

Idea: use our formula $C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n$

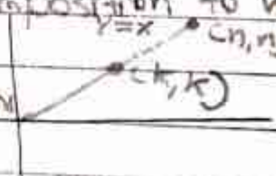
multiply by x, differentiate: $xC(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}$

$C(x) + xC'(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$

next: Find $f(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$ and solve resulting first-order differential equation.

Idea 2: Guess the correct answer, check it works

Idea 3: use the first-return decomposition to write an equation involving $C(x)$



In terms of generating functions

$$C(x) = 1 + C(x)\,C(x)\,x$$

empty path; choose path between 1st N and E of 1st return; choose an ending path; for initial N and E at 1st return

We have $C = 1 + xC^2$

$0 = 1 - C + xC^2$

by quadratic formula $C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$

$C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$

so $C(0) = 1$. we can check $\lim_{x\to 0^+}\frac{1+\sqrt{1-4x}}{2x} \neq 1$

$\lim_{x\to 0^+}\frac{1-\sqrt{1-4x}}{2x} = 1$

$$(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n$$

Moral: $\sum_{n=0}^{\infty} C_n x^n = \frac{1-\overbrace{\sqrt{1-4x}}}{2x}$

Notice: we can use binomial thm to check this right to left.

By complex analysis magic, this says $C_n \approx 4^n$.

Combinatorial Interpretations of the Catalan Numbers

Def^n If π, θ are permutations we say π contains θ if π has a subsequence in same relative order as θ.

if π does not contain θ, we say π avoids θ.

$S_n(\theta_1, \dots, \theta_k) = Av_n(\theta_1, \dots, \theta_k)$ is the set of all perms of length n which avoid all of $\theta_1, \dots, \theta_k$.

example: Find $Av_4(321)$ and $Av_4(132, 123)$

	4321	4312	4231	4213	4123	4132
321	✓	✓	✓	✓	X	✓
132, 123	X	X	X	X	✓	✓

$Av_4(321)$

1234 2134 3124 4123
1243 2143 3142
1324 2314 3412
1342 2341
1423 2413

n	0	1	2	3	4
#	1	1	2	5	14

$Av_4(132, 123)$

3214 4321
3241 4312
3412 4213
3421 4231 ...

n	0	1	2	3	4
#	1	1	2	4	8 ...

$Av_4(132, 123, 213)$

3412 4321
$\underbrace{3421}_{2}$ 4312
$\underbrace{4231}_{3}$

n	0	1	2	3	4
#	1	1	2	3	5

Theorem: For all $n \geq 0$, $|Av_n(321)| = C_n$.

Proof: we give a bijection between $Av_n(321)$ and Catalan paths of length n.

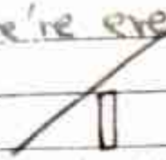
running example: 9 1 2 3 4 5 6 7 ~~8~~ 10

2 1 4 3 6 5 8 7 9

1 6 2 7 3 8 4 9 5 214537869

5 6 7 1 2 3 4 8 9

To construct $f(\pi)$ for $\pi \in Av_n(321)$, make a bar graph of π... Project each bar to the right.
Resulting outline is the Catalan path $f(\pi)$.
There is a building of height n so path ends at (n,n).
If we're ever below the line $y=x$, then we have

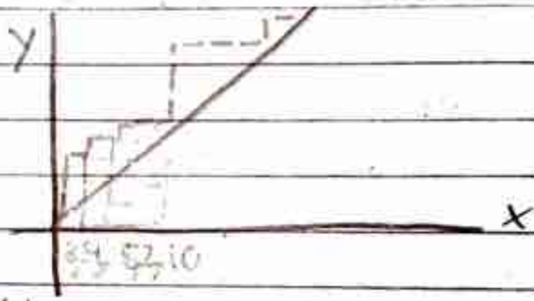


all entries in kth position and to its left are at most $k-1$.

Then we don't have enough numbers for these positions.

How do we invert the map?

Given a Catalan path



In canyons, must always use smallest available #.

If we don't, we get a 321 pattern.

shadow caster not smallest available smallest available

Lemma: π avoids 321 iff we can partition the entries into two disjoint increasing subsequences.

Brief Proof: ($\Leftarrow$) Any subseq of length 3 has at least 2 entries in an increasing sequence, so not a 321

Keep track of left to right maxima: entries larger than all entries to their left

one subsequence is these entries, other is all other entries.

high = 88
mean = 71.4
median = 71

72-88 A
60-71 B
48-59 C

Wednesday October 17

Partitions, Part I

Recall: a partition of a positive integer n is a sequence $\lambda_1, \lambda_2, \ldots$ of nonnegative integers such that
$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$
and $\lambda_1 + \lambda_2 + \lambda_3 + \ldots = n$

The terms $\lambda_1, \lambda_2, \ldots$ are the parts of λ, we usually don't write the 0s.

convention: parts are just nonzero terms

Notation: If λ is a partition of n, we write $|\lambda| = n$
$(|\lambda| = \lambda_1 + \lambda_2 + \ldots)$
or $\lambda \vdash n$

\vdash

also $\ell(\lambda) = \#$ of (nonzero) parts of λ.

$p(n) = \#$ of partitions of n.

One Goal: Find a way to compute $p(n)$ for large n.

n	0	1	2	3	4	5	6	7
$p(n)$	1	1	2	3	5	7	11	15

convention

What's the ogf for this sequence?

Proposition: $\sum_{n=0}^{\infty} p(n)x^n = \binom{\text{use}}{\text{only 1s}}\binom{\text{use only}}{\text{2s}}\binom{\text{use only}}{\text{3s}}\binom{\text{use only}}{\text{4s}}\binom{\text{use only}}{\text{5s}}(\ldots$

$= (1+x+x^2+\ldots)(1+x^2+x^4+\ldots) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\cdots$

$= \prod_{j=1}^{\infty} \frac{1}{1-x^j}$

We can get variations: if we allow at most 3 1s and any number of any other part then ogf is $(1+x+x^2+x^3)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots$

if we only allow an even # of 1s, any number of any other part: $\underbrace{\left(\frac{1}{1-x^2}\right)}_{\text{1s}} \underbrace{\prod_{j=2}^{\infty} \frac{1}{1-x^j}}_{\text{all other parts}}$

Counting and Linear Algebra

Recall: 4.G38 (?)

Find a formula for $\overbrace{\text{\# of sequences of letters}}^{a_n}$ of length n in {MATHS} with no consecutive As or Ss.

Method One:

$a_n =$ (# of sequences ending in different letters) + (# ending in same letter) $= 4a_{n-1}$

append one of 4 letters not at end

choose a shorter sequence

$+ 3a_{n-2}$

append MM TT HH

choose a shorter sequence

Method ~~Two~~ Five: $v_n =$ # sequences ending A

$w_n =$ # sequences ending S

$x_n =$ # sequences ending M

$y_n =$ T

$z_n =$ # H

$v_n = \quad w_{n-1} + x_{n-1} + y_{n-1} + z_{n-1}$

$w_n = v_{n-1} \quad + x_{n-1} + y_{n-1} + z_{n-1}$

$x_n = v_{n-1} + w_{n-1} + x_{n-1} + y_{n-1} + z_{n-1}$

$y_n = v_{n-1} + w_{n-1} + x_{n-1} + y_{n-1} + z_{n-1}$

$z_n = v_{n-1} + w_{n-1} + x_{n-1} + y_{n-1} + z_{n-1}$

In terms of matrices and vectors

$$\begin{pmatrix} v_n \\ w_n \\ x_n \\ y_n \\ z_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} v_{n-1} \\ w_{n-1} \\ x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix} = A^2 \begin{pmatrix} v_{n-2} \\ w_{n-2} \\ x_{n-2} \\ y_{n-2} \\ z_{n-2} \end{pmatrix}$$

$$= A^3 \begin{pmatrix} v_{n-3} \\ \vdots \\ z_{n-3} \end{pmatrix} = A^{n-1} \begin{pmatrix} v_1 \\ w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = \underbrace{A^{n-1}}_{PDP^{-1}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

If the eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ then each of x_n, y_n, z_n, v_n, w_n is a linear combination of $\lambda_1^n, \lambda_2^n, \lambda_3^n, \lambda_4^n, \lambda_5^n$.

Picture of A:

$$\begin{array}{c|c} & \begin{matrix} A & S & M & T & H \end{matrix} \\ \begin{matrix} A \\ S \\ M \\ T \\ H \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Each of the ones corresponds to a particular directed edge.

The entries in the matrix A are 1 if we can go from row label to column label in 1 step

0 if we cannot

Markov Chain

Catalan Objects

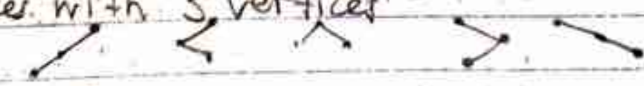
Recall: We know C_n is the # of Catalan paths of length n. We showed $|S_n(321)| = C_n$

you showed $|S_n(123)| = C_n$

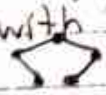
Fact: $|S_n(132)| = |S_n(213)| = |S_n(231)| = |S_n(312)| = C_n$. other objects counted by C_n include

① binary trees with n vertices.

Start at root vertex, each vertex has 0, 1, or 2 children. left children differ from right children. binary trees with 3 vertices:



- one binary tree with 5 vertices is

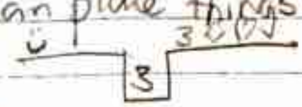


To show # binary trees with n vertices is C_n, argue by induction on n, show the #s satisfy same recurrence as catalan #s.

$$\#\,\text{trees}_{n\text{ verts}} = \sum_{k=0}^{n-1} \begin{pmatrix} \#\text{ binary trees} \\ k \text{ vertices} \\ (\text{left}) \end{pmatrix} \begin{pmatrix} \#\text{ binary trees} \\ n-1-k \text{ vertices} \\ (\text{right subtree}) \end{pmatrix}$$

② stack-sortable permutations of length n.

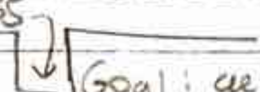
a stack is an object you can place things on, or remove things from top



putting on stack = pushing.

taking top item off = popping.

We can sort some permutations using one pass through a stack 6174235



Goal: get 1234567

617423 push 3: then 3,5 are out of order

5 pop 5: then 6,7 after 5

123 — stack-sortable.

N N E E N E

132 — stack-sortable push push pop pop push pop

213 — stack-sortable push pop push push pop pop

231 "

not 312

N N N E E E

321 " push push push pop pop pop

So we have 5 permutations of 3 that are stack-sortable.

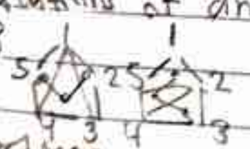
Proof #1: show stack sortable is equivalent to avoiding 312

Proof #2: bijection to Catalan paths.

push = N pop = E □

⑰ triangulations of an n+2-gon: subdivide the n+2 gon into Δs using non intersecting diagonals between vertices.

n=3



Proof: bijection to binary trees. □

Friday October 9

Ferrers Diagrams of Partitions (Young)

Recall: a <u>partition</u> of a positive integer n is a sequence $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ of nonnegative integers with $\lambda_1 + \lambda_2 + \ldots = n$. The <u>parts</u> of a partition are the nonzero terms in the partition. $p(n) = \#$ of partitions of n.

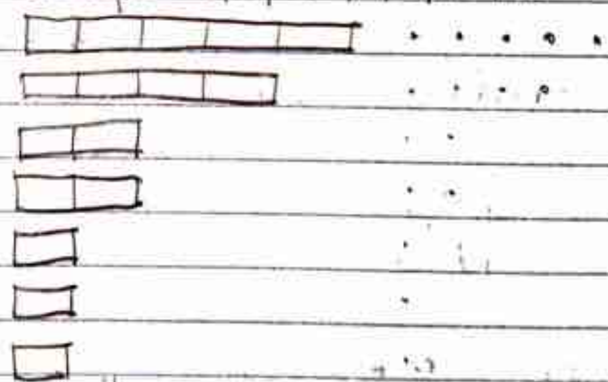
Goal: Compute $p(n)$ for large n.

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \Bigg(\sum_{\substack{0 \leq m < k \\ \gcd(m,k)=1}} e^{\pi i \left(s(m,k) - \frac{2nm}{k}\right)} \Bigg) \frac{d}{dx}\left[\frac{1}{\sqrt{x-\frac{1}{24}}} \sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)} \right)\right]\Bigg|_{x=n}.$$

No good formula for $p(n)$ is known. But we can draw pictures of partitions to get a recurrence relation.

Def^{n} The Ferrers diagram of a particular λ has
λ_1 $|x|$ boxes in top row
λ_2 $|x|$ boxes in 2nd row
λ_j $|x|$ boxes in jth row
all rows left justified. example $5,4,2,2,1,1,1$ has

Ferrers diagram



French Style "English style"

To get our recurrence relation, set $p_k(n) = \#$ of partitions of n with exactly k parts.

observation: $p_k(n)$ is also $\#$ partitions of n with largest part is k.

Proof: The <u>conjugate</u> of a partition λ, written λ', is the partition whose Ferrers diagram is the reflection of the Ferrers diagram of λ.

ex. If $\lambda = 5,4,3,2,4,1,1$ then $\lambda' = 7,4,2,2,1$

Since $(\lambda')' = \lambda$, the conjugate map is a bijection from partitions with largest part k to partitions with exactly k parts.

Def^n we say λ is self-conjugate whenever $\lambda' = \lambda$.
ex. 5, 4, 3, 2, 1 is self-conjugate.
3, 1, 1 is also self-conjugate.

we can display $p_k(n)$ in a triangle
$\quad \quad \quad \quad$ $k=0$

1 ← n=0

0 1 ← n=1

0 1 1 ← n=2

0 1 1 1

0 1 $\boxed{2}$ 1 1

0 1 2 2 1 1

k=1 k=2 k=3 k=4 k=#parts k=largest part

3,1 2,2

2,2 2,1,1

Proposition

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1)$$

$$p_3(5) = p_3(2) + p_2(4)$$

$$2 = 0 + 2$$

Proof: $p_k(n) =$ (partitions of n with last part 1) + (with last part 2 or more)
$= p_{k-1}(n-1) + p_k(n-k)$

k rows  partitions of n-1 with k-1 parts

Proof: 6 $\square$

to build these 
pick a partition of $\square$ n-k, put
in a new 1st column of length k $\square$

Observation: we can the recurrence to find
$p_k(n)$ for many large of k, n. This helps us find $p(n)$
b/c $p(n) = \sum_{k=0}^{n} p_k(n)$.

More Catalan objects

Recall: C_n is the number of Catalan paths of length n.

is the number of stack-sortable permutations of length n, the # of binary trees with n vertices

$= $ triangulations of an $n+2$-gon

$= |S_n(321)| = |S_n(123)| = |S_n(132)| = |S_n(231)|$

$= |S_n(213)| = |S_n(312)|$

$=$ parenthesis sequences $=$ # of noncrossing partitions of $[n]$.

To see the last set is counted by C_n. Def^{n} A partition (or set partition) of a set A is a collection of nonempty subsets of A $B_1, \ldots, B_k$ with 1) $B_1 \cup \cdots \cup B_k = A$.
2) $B_j \cap B_m = \emptyset$ if $j \neq m$. B_j is a "block" of A.

example: set partitions of $[3]$: $\{1\}, \{2\}, \{3\} = 1/2/3$

12/3 13/2 23/1 123

set partitions of $[4]$

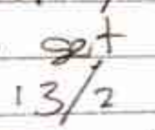
1234	12/34	1/2/3/4
123/4	13/24	
124/3	14/23	
134/2	12/3/4	
234/1	13/2/4	
	14/3/2	
	23/1/4	
	24/1/3	
	34/1/2	

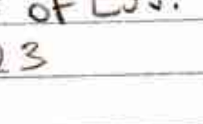
the nth Bell number, B_n, is the # of set partitions of $[n]$

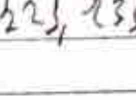
$B_3 = 5$

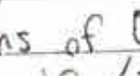
$B_4 = 15$

diagram of a set partition: put $1, 2, \ldots, n$ around a circle, draw polygons whose corners are block elts.

1234: 

14/235 

13/24: 

12/3/4: 

a noncrossing partition is a partition whose diagram has no crossing. equivalently, there are no $a < b < c < d$ where a, c in one block b, d in another block

Proposition: # noncrossing partitions of [n] is C_n.
Outline of Proof: For any NC partition write #s in each block in increasing order, write blocks so smallest elements are in decreasing order. Erase dividers.

23/1/4 The resulting permutation is in $S_n(132)$
4/23/1 and the map is a bijection. □
4231

Friday, October 26

Rogers-Ramanujan Identities

One of the people who wrote to Ramanujan was Hardy. Ramanujan found that Rogers had proved in a math journal what he had already proved earlier.

Recall: we proved that

\# partitions of n with only odd parts
= \# partitions of n w/ distinct parts

We want to discover other results of form

\# of partitions of n with all parts in the set { .... } = \# of partitions of n with property ___.

To find/conjecture results like this, pick a property X, try to find a set on left that works.

As an example, consider partitions of n in which any 2 parts differ by at least 2. (if instead "differ by at least 1" then we're talking about partitions with distinct parts.)

n

partitions in which part differ by at least 2

number

n	0	1	2	3	4	5	6
p	∅	1	2	3, ~~2,1~~	4, 3,1	5, 4,1	6, 5,1, 4,2
number	1	1	1	1	2	2	3

7	8	9	10	11	12
7, 6,1, 5,2					
3	4	5	6	7	9

Start w/ smallest n, put parts in our set until we see a pattern.

the set = {1,

n	0	1	2	3	4	5	6	7
partitions of n using only elts of the set	$\emptyset$	1	1^2	1^3	1^4; 4	1^5; 4,1	1^6; $4,1^2$; 6	1^7; $4,1^3$; 6,1
#	1	1	1	1	2	2	3	3

Do not include 2 as a possible part, because already have 1 partition of 2.

8	9	10	11	12
1^8; 4^2; $4,1^4$; $6,1^2$	1^9; $4^2,1$; $4,1^5$; $6,1^3$; 9			
4	5	6	7	9

The set = $\{1, 4, 6, 9, 11, 14, 16, 19, 21,$

Conjecture: The number of partitions of n in which no two parts differ by 0 or 1 is the same as the number of partitions of n in which every part is $\pm 1 \bmod 5$.

Open Problem: Find a simple bijection.

Idea: There should be an identity relating generating functions for these sets.

ogf for partitions of n in which all parts are $\pm 1 \bmod 5$ is $\frac{1}{1-x} \prod_{j\geq 1} \frac{1}{(1-x^{5j-1})(1-x^{5j+1})}$.

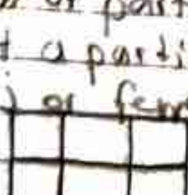
We'll write the ogf for partitions of n as a sum

$1 + \sum_{j\geq 1} x^{1+3+5+\cdots+(2j-1)}$ (gf for partitions w/ at most j parts)

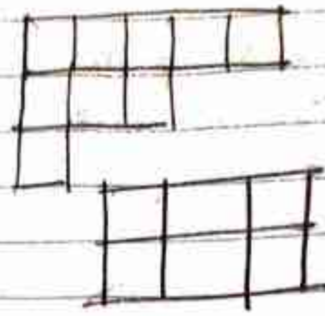
j = # of parts in the partition

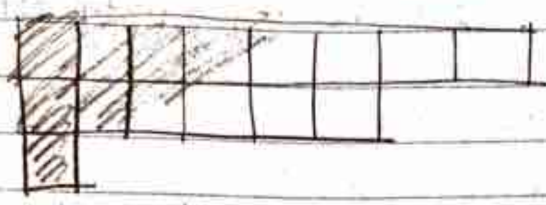
To construct a partition like this, choose any partition with j or fewer parts.

$j = 4$



add 1 to jth part
3 to $j-1$th part
5 to $j-2$th part
⋮
$2j-1$ 1st part





Observation: these boxes must be in any partition with j parts, if no 2 parts differ by 0 or 1.

So ogf $= 1 + \sum_{j=1}^{\infty} x^{j^2} \underbrace{\frac{1}{(1-x)(1-x^2)\cdots(1-x^j)}}$

take conjugate of a partition with all parts in $\{1, 2, \ldots, j\}$
to get a partition that parts

A Recurrence for $p(n)$

Recall: we showed $\prod_{j=1}^{\infty}(1-x^j) = 1 + \sum_{n=1}^{\infty}(-1)^n\left[x^{\frac{n(3n-1)}{2}} + x^{\frac{n(3n+1)}{2}}\right]$

"pentagonal numbers"

$= 1 \underbrace{- x - x^2}_{n=1} + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \ldots$

Because $\prod_{j=1}^{\infty}(1-x^j)$ is the denominator for ogf for $\{p(n)\}$ we get

Proposition: For all $n \geq 1$

$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots$ where $p(k) = 0$ if $k < 0$.

Proof. We know $\sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty}\frac{1}{1-x^j}$

so $\left[\sum_{n=0}^{\infty} p(n)x^n\right]\prod_{j=1}^{\infty}(1-x^j) = 1$

so $\left[\sum_{k=0}^{\infty} p(k)x^k\right]\left[1 + \sum_{j=1}^{\infty}(-1)^j\left[x^{\frac{j(3j-1)}{2}} + x^{\frac{j(3j+1)}{2}}\right]\right] = 1$

The coefficient of x^n on each side is:

$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \ldots$

on right (assuming $n \geq 1$)

to get x^n on left above, need $x^k x^{\text{pent}\#} = x^n$ so $n = k + \text{pent}\#$

$k = n - \text{pent}\#$

Now solve for $p(n)$. $\square$

Planar Graphs and the 4-Color Theorem

The 4-color theorem addresses a problem of map-coloring.
We have a bounded region in the plane, partitioned into finitely many subregions (called countries). Countries are "simply connected"

all one piece. no holes in any country

We say countries are adjacent if they share an interval along their boundaries.

Problem: assign a color to each country so no two adjacent countries get same color.

We want to do this with as few colors as possible.

In fact, we want to know for what n it's possible to color every map with n or fewer colors.

4-color Theorem: Every map in the plane can be "properly" colored with 4 or fewer colors.

adjacent countries get different colors.

Idea: Turn theorem into a statement about graphs.

Each country in our map is a vertex, 2 countries are adjacent in the graph whenever they are adjacent on map.

Terminology: a coloring of a graph is a function from set of vertices to a set of colors.

a proper coloring is a coloring in which no two adjacent vertices are assigned the same color.

a graph is planar whenever it can be drawn in the plane with no edge crossings.

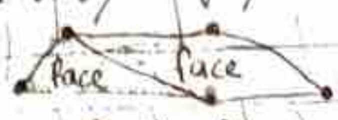
a plane graph is a graph drawn in the plane with no edge crossings.

Four Color Theorem for Graphs: If G is planar graph, then G has a proper coloring with 4 or fewer colors.

Questions: example of a nonplanar graph?
how do we tell if a given graph is planar?

To answer these, need a tool: Euler's formula:
If G is a plane graph connected, finite with at least 1 vertex then #vertices − #edges + #faces = 2.
A face in a plane graph is a maximal connected region bounded by edges, vertices.

$5-6+3=2$



Proof: argue by induction on # faces.

ex

faces = 1

In this case, argue by induction on # vertices.

~~$v=0 \Rightarrow e=0$~~ $v=1 \Rightarrow e=0$ so

~~$f=1$~~ $v-e+f=1-0+1=2$ ✓

suppose $v=$ # vertices ≥ 2. Then we must have a vertex of degree 1. [start at vertex, walk to adjacent vertex, repeat, which we can do because degree ≥ 2 for each vertex if no vertex of degree 1].

Remove that vertex and its incident edge.

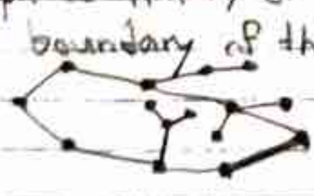
New graph is planar so $(v-1)-(e-1)+f=2$

so $v-e+f=2$.

Now suppose we have at least 2 faces.

The graph is finite, so at least one face is finite.

Look at boundary of that face:



Pick an edge on boundary separating our face from another face.

Remove that edge.

vertices in new G is V

edges in new G is $e-1$.

faces in new G is $f-1$.

By induction $v-(e-1)+(f-1)=2$

so $v-e+f=2$.

Recall: The complete graph on n vertices written K_n, is the graph on n vertices in which all pairs of vertices are adjacent.

K_1: K_2:  K_3: 

K_4:   K_5: 

Claim: K_5 is not planar.

Proof: Suppose we draw K_5 in plane with no edge crossings. Then $v-e+f=2$

so $5-10+f=2$

$f=7$

$$\frac{3f}{2} \le e$$

$$\frac{21}{2} \le 10$$

$$10.5 \le 10$$

a contradiction so K_5 not planar. □

Wednesday October 31

The Five Color Theorem

Recall: A coloring of a graph is an assignment of colors to its vertices.

A proper coloring of a graph is a coloring in which any pair of adjacent vertices have different colors.

A graph is planar if it can be drawn in the plane with no edge crossings. We saw K_1, K_2, K_3, K_4 (complete graphs on 1, 2, 3, 4 vertices) are planar. We showed if $n \geq 5$ then K_n is not planar. Main tool is Euler's formula: if we draw a in the plane with no crossings, then

$$v - e + f = 2.$$

assume no loops, no multiple edges

The Five Color Theorem: If G is a planar graph then G has a proper coloring using at most 5 colors.

Lemma: If G is a planar graph then G has a vertex of degree at most 5.

Proof: Assume each vertex has degree at least 6, estimate # of edges in 2 ways.

① $e \geq \frac{6v}{2}$ because we get at least 6 edges at each vertex, but each edge appears at 2 vertices.

So $e \geq 3v$.

② We can also count edges around each face. We have no loops, no multiple edges, so each face has at least 3 edges, but each edge is counted twice, because it bounds 2 faces.

So $e \geq \frac{3f}{2}$

Rearranging, we now know $v \leq \frac{1}{3}e$ and $f \leq \frac{2}{3}e$. By Euler's Formula, $2 = v - e + f \leq \frac{1}{3}e - e + \frac{2}{3}e = 0$

This is a contradiction. $\square$

Proof of Five Color Theorem: We argue by induction on v = # vertices in G.

If $v \leq 5$ then color each vertex with a different color and the resulting coloring will be proper.

Assume $v \geq 6$.

Let x be a vertex of degree 5 or less.
Remove x, and all edges incident with x.
Properly color rest of graph with 5 colors.
Put x and its edges back in graph.
If degree of x is 4 or less then some color is not used among neighbors. Color x with that color.

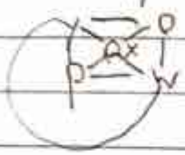
If x has degree 5 and all 5 neighbors have different colors (if not, then there is an unused color among neighbors, so give x that color) then we have

For any colors a, b, an ab-Kempe chain ~~containing a vertex~~ from y to z is a sequence of adjacent vertices alternating in color a, b, a, b, ... from y to z.
If there is no green-purple Kempe chain connecting G and P then there is a connected graph of only green + purple vertices including G, not including P. Switch green, purple in that graph. Color x green.

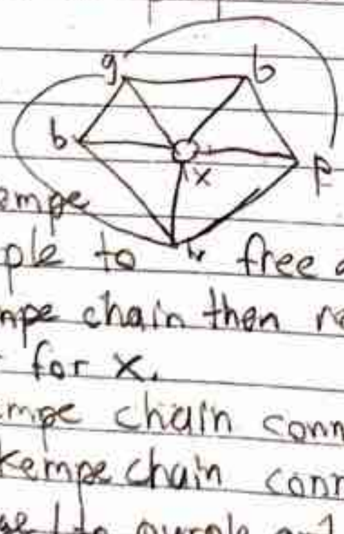
If there is a green-purple Kempe chain connecting G and P.
Any yellow-blue Kempe chain connecting Y and B crosses the green-purple Kempe chain at a vertex. That can't happen: that vertex would be two different colors.
Now we can recolor blue neighbor yellow, color x blue.

Proof of 4-color Theorem: Argue by induction on v.
If $v \leq 4$, use a different color on each vertex.
Assume $v \leq 5$, pick vertex x of degree 5 or less.
Remove x, properly color rest, put x back.

If neighbors do not use all 4 colors, then we have a color for x.
If degree is 4, all neighbors have different colors, then



If no yellow-white kempe chain, recolor yellow or white vertex to free a color for x.
If there is a yellow-white kempe chain, then any orange-purple chain would cross it, so no orange-purple chain.
So recolor the orange or the purple vertex to free a color for x.
If degree is 5 then

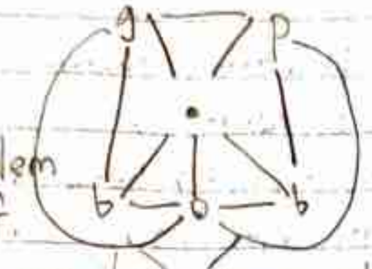


If no green-purple kempe chain, then recolor green or purple to free a color for x.
If no green-white kempe chain then recolor green or white to free a color for x.
Now no blue-purple kempe chain connecting 1 with purple, no blue-white kempe chain connecting 3 with white vertex. Change 1 to purple and 3 to white. Color x blue. □

Monday

4-color theorem.

History of 4-color Problem



1880: Kempe publishes "our" proof.

1889: Heawood gives a counterexample showing Kempe's argument fails. (these kempe chains can cross.)

1976: Appel and Hakken proved the 4-color theorem
How?

Outline of Proof. ① Assume statement is false, and G is a minimal counterexample.
fewest vertices ② Show that any such counter example contains at least one of some list of configurations.

In our case:

③ Show that every configuration can be "reduced" meaning replaced with a smaller one preserving non-4-colorability.

1976: 1936 configurations
1983: 1258
1996: 633

How can we show at least one configuration on a list must appear? Proposition: Any planar graph with all vertices of degree 5 or more contains either $5 — 5$ or $5 — 6$
degrees

Proof: Notation: For any vertex x, $d(x) =$ degree of x. "charge" of x is $6 - d(x)$

You can use Euler's formula to show that in all of the faces are triangles the total charge is 12

$$\sum_{x \in G} (6 - d(x)) = 12.$$

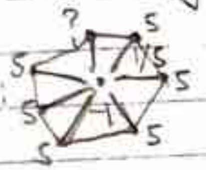
positive charge

All vertices with positive charge share their charge equally among neighbors.

Total charge is positive so some vertices have positive charge after discharging process.

If newly positive vertex has degree 5, it has a neighbor of degree 5 giving it charge: $5 \text{ --- } 5$

If newly positive vertex has degree 6 then it has a degree 5 neighbor giving it charge: $6 \text{ --- } 5$

degree 7: 

degree 8:

-2

need 10 neighbors to get positive charge!

<u>q-Binomial Coefficients</u> Defn For any $\pi \in S_n$, an <u>inversion</u> (j,k) is an ordered pair with $j<k$ and $\pi(j) > \pi(k)$.

Notation: the inversion number of π, $inv(\pi)$, is the number of inversions in π

examples: 4, 3, 2. inversions are (1,2)
(1,3)
(1,4)
(3,4)

inversion number $inv(4132) = 4$.

inversion number $inv(1234) = 0$

largest inversion # in S_4 is $\binom{4}{2} = 6$.

$inv(4321) = 6$

I care about ogf for inversion #s: $\sum_{\pi \in S_n} q^{inv(\pi)}$

n=2

$12 : q^0$
$21 : q^1$) $1+q$

$1+2q+2q^2+q^3$

n=3			
123 0	213 1	3	321 : (1,2), (2,3), (1,3)
132 1	231 2	2	312
$1+q$	$q+q^2$	q^2+q^3	

$\underline{n=4}$ 2134 (1,2) 1 $q+2q^2+2q^3+q^4$

2143 (1,2), (3,4) 2

2341 (1,4), (1,2) 3

2314 (1,2), (1,3) 2

2413 3

2431 4

1234 0 $1+2q+2q^2+q^3$

1243 1

1324 1 $\underline{3} = q^2+2q^3+2q^4+q^5$

1342 2

1432 3 $\underline{4} = q^3+2q^4+2q^5+q^6$

1423 2

$$\sum_{\pi\in S_4} q^{inv(\pi)} = 1+3q+5q^2+6q^3+5q^4+3q^5+q^6$$

Notice: $\sum_{\pi\in S_3} q^{inv(\pi)} = (1+q+q^2)\sum_{\pi\in S_2} q^{inv(\pi)}$

$$\sum_{\pi\in S_4} q^{inv(\pi)} = (1+q+q^2+q^3)(1+q+q^2)(1+q)$$

Definition: For any $n\geq 1$, set $[n]_q = 1+q+q^2+\cdots+q^{n-1}$
"q-integer" or "q-analogue of n"

Proposition: For $n\geq 1$, $\sum_{\pi\in S_n} q^{inv(\pi)} = [n]_q! =$

$[n]_q[n-1]_q\cdots[2]_q[1]_q$.

Proof: By induction on n. We checked this for $n=1,2,3,4$. If $n\geq 5$ then $\sum_{\pi\in S_n} q^{inv(\pi)} = \sum_{j=1}^{n}\sum_{\substack{\pi\in S_n\\ \pi(1)=j}} q^{inv(\pi)}$

$$= \sum_{j=1}^{n} q^{j-1} \sum_{\substack{\pi\in S_n\\ j=\text{1st entry}}} q^{\underbrace{inv(\pi)-(j-1)}}$$

inversions using $\pi(1)=j$ — inversions in π': all entries right of j.

$$= \sum_{j=1}^{n} q^{j-1} \sum_{\pi\in S_{n-1}} q^{inv(\pi)} = [n]_q([n-1]_q!)$$

Def^{n} a descent in a permutation π is a number j with $\pi(j) > \pi(j+1)$

the major index, written $maj(\pi)$ is the sum of the descents.

		descents		$maj(\pi)$
example:	123	$\emptyset$	six permutations in S_3	
	132	$\langle 2 \rangle$		2
	213	$\langle 1 \rangle$		1
	231	$\langle 2 \rangle$		2
	312	$\langle 1 \rangle$		1
	321	$\langle 1,2 \rangle$		3

Wed Nov 7

q-binomial coefficients

Recall: An inversion in a permutation $\pi \in S_n$ is an ordered pair (j,k) with $j<k$ and $\pi(j) > \pi(k)$.

We write $\mathrm{Inv}(\pi)$ to denote the number of inversions in π.

We showed $\sum_{\pi \in S_n} q^{\mathrm{Inv}(\pi)} = [n]_q [n-1]_q \cdots [2]_q [1]_q = [n]_q!$

where $[k]_q = 1+q+q^2+\cdots+q^{k-1}$.

If we set $q=1$ in this result, we get # of permutations in $S_n = n(n-1)\ldots 2\cdot 1$.

Def^n $Z_{m,n}$ = set of sequences of m 0s, n 1s

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{\pi \in Z_{m,n}} q^{\mathrm{Inv}(\pi)}$$

(Notice: $|Z_{m,n}| = \binom{m+n}{n}$)

example: $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$

Scratch work:

π	0011	0101	1001	0110	1010	1100
$\mathrm{Inv}(\pi)$	0	1	2	2	3	4

q-Pascal Triangle

$$\begin{array}{lccccccccc l}
n=0 & & & & & 1 & & & & & m=0 \\
n=1 & & & & 1 & & 1 & & & & m=1 \\
n=2 & & & 1 & & 1+q & & 1 & & & m=2 \\
n=3 & & 1 & & 1+q+q^2 & & 1+q+q^2 & & 1 & & m=3 \\
n=4 & 1 & & 1+q+q^2+q^3 & & \boxed{1+q+2q^2+q^3+q^4} & & 1+q+q^2+q^3 & & 1 & m=4
\end{array}$$

$m=n=1$: 10 inv = 1; 01 inv = 0

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_q \quad \text{(choose)}$$

Notice: It's not true that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$

But, Theorem ~~Conjecture~~: $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q + q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$

Proof: By definition, $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{\pi \in Z_{m,n}} q^{\mathrm{Inv}(\pi)}$

$$= \sum_{\substack{\pi \in Z_{m,n} \\ \text{last entry } 0}} q^{\mathrm{Inv}(\pi)} + \sum_{\substack{\pi \in Z_{m,n} \\ \text{last entry } 1}} q^{\mathrm{Inv}(\pi)}$$

$$= \qquad + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$$

To get 2nd term, remove 1 from end of each π, since that 1 is involved in no inversions, this doesn't change the inversion number.

So both $\_\_\,1$ and $\_\_$ contribute same q^{inv}.

To get 1st term, notice ending 0 is involved in an inversion for each 1 to its left so it's involved in n inversions.

So the 1st term is $q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$.

So we know $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q + q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$.

Conclusion: Both $\begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q = q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$

$= \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q + q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$

and $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$.

We also know $\binom{m+n}{n} = \frac{(m+n)!}{n!\,m!}$

Best case scenario: $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q!\,[n]_q!}$

example: $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q!}{[2]_q!\,[2]_q!} = \frac{[4]_q [3]_q}{1\,[2]_q}$

$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{1+q}$

$= \frac{\cancel{(1+q)}(1+q^2)(1+q+q^2)}{\cancel{1+q}}$

$= 1+q+q^2+q^3+q^4 \qquad = 1+q+2q^2+q^3+q^4$ ☺

Proof: we argue by induction on $m+n$ using one of our q-Pascal identities.

□

Proof: We give a bijective proof of $[m]_q!\,[n]_q! \begin{bmatrix} m+n \\ n \end{bmatrix}_q = [m+n]_q!$

Our bijection will take an ordered triple (α, β, γ) where $\alpha \in S_m$, $\beta \in S_n$, $\gamma \in Z_{m,n}$, and return a permutation $\pi \in S_{m+n}$ such that $inv(\pi) = inv(\alpha) + inv(\beta) + inv(\gamma)$.

1

Given $(\overset{m}{\alpha}, \overset{n}{\beta}, \overset{m\ 0s}{\underset{n\ 1s}{\gamma}})$ put entries of α in place of 0s in γ and put the entries of β in place of 1s, and add m to each of them.

ex: $\alpha = 312$ maps to $\pi = \underset{0}{3}\,\underset{1}{7}\,\underset{1}{4}\,\underset{0}{1}\,\underset{0}{2}\,\underset{1}{8}\,\underset{1}{5}\,\underset{1}{6}$

$\beta = 41523$

$\gamma = 01100111$ After adding m to the entries of β, all entries of β are greater than all entries of α, so inversions in γ correspond to inversions in π.

All other inversions in π are within α or β, so there are $\mathrm{inv}(\alpha) + \mathrm{inv}(\beta)$ other inversions.

To construct the inverse of our function, suppose $\pi \in S_{m+n}$.

Then α is the permutation inside π consisting of $1, 2, \dots, m$.

Replace these with 0s. And β is the permutation we get by subtracting m from all entries greater than m. Replace these with 1s.

Fibinomials

Reindex Fibonacci numbers: $f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2} \quad n \geq 2$

Now define $[n]_F! = f_n f_{n-1} f_{n-2} \cdots f_2 f_1$

and $\begin{bmatrix} n \\ k \end{bmatrix}_F = \dfrac{[n]_F!}{[k]_F!\,[n-k]_F!}$

Fib-Pascal triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 6 & & 3 & & 1 & & \\
 & 1 & & 5 & & 15 & & 15 & & 5 & & 1 & \\
1 & & 8 & & 40 & & 60 & & 40 & & 8 & & 1
\end{array}$$

Monday

q-binomial theorems   Recall: we showed if x, y are variables which do not commute, but instead satisfy $yx = qxy$ for some q which does commute with x and y then
$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

Commutative q-binomial theorem
$$(1+x)(1+qx)(1+q^2x)(1+q^3x)\cdots(1+q^{n-1}x)$$
$$= \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

smallest exponent on q we can get for coefficient of x^k is $0+1+2+\dots+(k-1) = \binom{k}{2}$

Proof: We claim coefficient of x^k on left is $q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$.

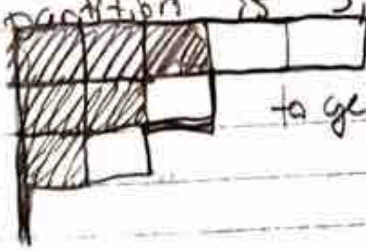
To get x^k on left, we choose k terms of terms
$$\underbrace{q^{j_1}x, q^{j_2}x, \dots, q^{j_k}x}_{k \text{ terms}}$$
and multiply them. We can assume $j_1 < j_2 < j_3 < \dots < j_k$

We notice $j_k, j_{k-1}, \dots, j_2, j_1$ is a partition with k distinct parts, one of which might be 0.

Subtract 0 from last part, 1 from 2nd to last part, 2 from 3rd to last part, etc.

ex: if our partition is $5,3,2,0$ $(k=4$

then we



remove orange boxes to get $2,1,1$

We get all partitions with at most k parts and largest part at most $(n-1)-(k-1)=n-k$.

This is all partitions in a $k\times(n-k)$ box.

We have removed $0+1+2+\cdots+(k-1)$ boxes, this is our $q^{\binom{k}{2}}$

By GO, the ogf for partitions in a $k\times(n-k)$ box is $\begin{bmatrix} n \\ k \end{bmatrix}_q$ so our coefficient is $q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q$

example: $n=5$

$k=3$

$$(1+x)(1+qx)(1+q^2x)(1+q^3x)(1+q^4x)$$

2,1,0 $\rightsquigarrow$ 0,0,0		4,3,0	2,2,0
3,1,0 $\rightsquigarrow$ 1,0,0		3,2,1	1,1,1
4,1,0 $\rightsquigarrow$ 2,0,0		4,2,1	2,1,1
3,2,0 $\rightsquigarrow$ 1,1,0		4,3,1	2,2,1
4,2,0 $\rightsquigarrow$ 2,1,0		4,3,2	2,2,2

Fibinomials

Recall: we defined $\begin{bmatrix} n \\ k \end{bmatrix}_F = \dfrac{[n]_F!}{[k]_F!\,[n-k]_F!}$

where $[j]_F! = f_j f_{j-1} f_{j-2}\cdots f_2 f_1$

and $f_0=0$ and $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$.

$f_1=1$

We showed $\begin{bmatrix} m+n \\ m \end{bmatrix}_F = f_{n+1}\begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_F + f_{m-1}\begin{bmatrix} m+n-1 \\ m \end{bmatrix}_F$.

We want a combinatorial interpretation for $\begin{bmatrix} m+n \\ n \end{bmatrix}_F$.

Notation: $B_{m,n}$ = set of partitions in an $m\times n$ box.

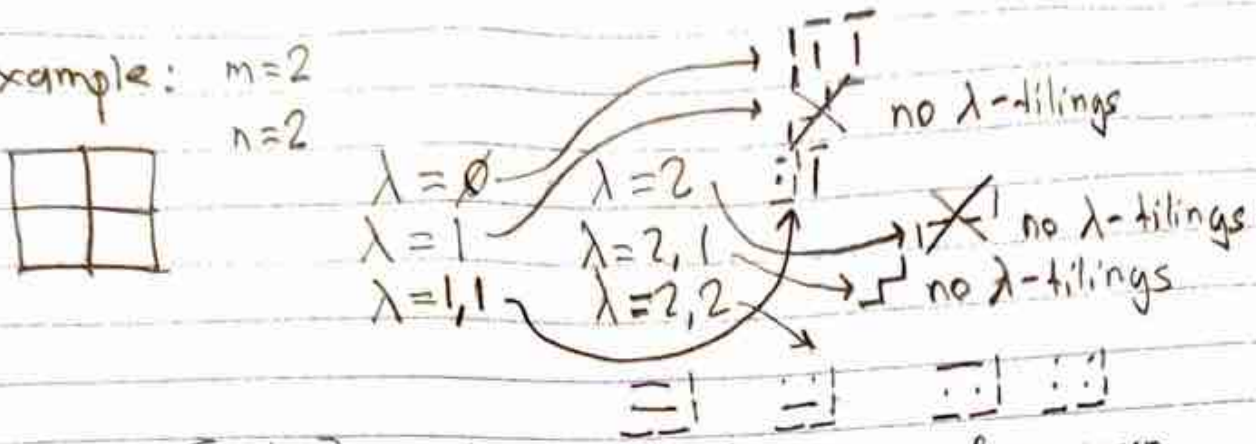
Def$^{\underline{n}}$ Suppose $\lambda\in B_{m,n}$. A λ-tiling of the $m\times n$ box is a tiling of the whole box with ◫, □ such that

① all dominoes inside λ are ◫

all dominoes outside λ are ⊟

② if a column ~~contains~~ any □ outside λ then its bottom tile is ⊟. is nonempty

example: $m=2$
$n=2$



Theorem: $\begin{bmatrix} n+m \\ m \end{bmatrix}_F$ is the number λ-tilings of an $m \times n$ box over all λ.

Proof: by induction on $m+n$, showing these #s of λ-tilings satisfy same recurrence as $\begin{bmatrix} m+n \\ m \end{bmatrix}_F$.

Def$^{\underline{n}}$ The FibCatalan # CF_n is $CF_n = \frac{1}{f_{n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix}_F$

n	0	1	2	3	4	5	6
CF_n	1	1	3	20	364	17017	2097018

Fact: For any n, q, r, $\frac{f_r}{f_{nq+r}} \begin{bmatrix} nq+r \\ n \end{bmatrix}_F$ is an integer.

Open Question: What does CF_n count?