3.1.8 Combine equations (3.8) and (3.9) to show that

$$G(\alpha)^2 = (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j}). \qquad (3.10)$$

Show that $\prod_{j=1}^{k-1}(x-\alpha^{-2j}) = \frac{x^k-1}{x-1} = 1+x+x^2+\cdots+x^{k-1}$,

and therefore $G(\alpha)^2 = (-1)^{(k-1)/2}k$. (3.11)

For the first result, we can multiply (3.8)·(3.9): So $G(\alpha)^2$

$$= G(\alpha)\cdot G(\alpha) = (\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)})\cdot (-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)}).$$

$$= (-1)^{(k-1)/2}\cdot\alpha(1-\alpha^{-2})\alpha^3(1-\alpha^{-6})\cdots\alpha^{k-2}(1-\alpha^{-2(k-2)})\alpha^2(1-\alpha^{-4})\alpha^4(1-\alpha^{-8})\cdots\alpha^{k-1}(1-\alpha^{-2(k-1)})$$

$$= \alpha\alpha^3\cdots\alpha^{k-2}\alpha^2\alpha^4\cdots\alpha^{k-1}(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})(1-\alpha^{-4})(1-\alpha^{-8})\cdots(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$$

$$= \alpha\alpha^2\alpha^3\alpha^4\cdots\alpha^{k-2}\alpha^{k-1}(1-\alpha^{-2})(1-\alpha^{-4})(1-\alpha^{-6})(1-\alpha^{-8})\cdots(1-\alpha^{-2k+4})(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$$

$$= \alpha^{k(k-1)/2}(1-\alpha^{-2})(1-\alpha^{-4})\cdots(1-\alpha^{-2k+2})(-1)^{(k-1)/2}$$

$$= \alpha^{k(k-1)/2}\cdot(1-\alpha^{-2})(1-\alpha^{-4})\cdots(1-\alpha^{-2(k-1)})(-1)^{(k-1)/2}$$

$$= (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j}).$$

For the second result, because given any primitive kth root of unity $\alpha = e^{2\pi i/k}$ with k odd, α^{-2} is also a primitive root of unity (from exercise 3.1.5.), and we know by definition that the roots of x^k-1 are all kth roots of unity, $(\alpha^{-2})^j$, $0\le j\le k-1$,

$$x^k-1 = (x-(\alpha^{-2})^0)(x-(\alpha^{-2})^1)\cdots(x-(\alpha^{-2})^{k-1})$$

$$= \prod_{j=0}^{k-1}(x-(\alpha^{-2})^j) = \prod_{j=0}^{k-1}(x-\alpha^{-2j}) = (x-\alpha^0)\prod_{j=1}^{k-1}(x-\alpha^{-2j})$$

$$\Rightarrow (x^k-1)/(x-1) = \prod_{j=1}^{k-1}(x-\alpha^{-2j}). \text{ Also,}$$

$$\frac{x^k-1}{x-1} = \frac{x+x^2+x^3+\cdots+x^k-1-x-x^2-\cdots-x^{k-1}}{x-1} = \frac{(x-1)(1+x+x^2+\cdots+x^{k-1})}{x-1}$$

$$= 1+x+x^2+\cdots+x^{k-1}$$

And therefore, since $\alpha^{k(k-1)/2} = (e^{2\pi i/k})^{k(k-1)/2} = (e^{2\pi i})^{(k-1)/2}$

$= (\cos(2\pi)+i\sin(2\pi))^{(k-1)/2} = (1+0)^{(k-1)/2} = 1$,

$$G(\alpha)^2 = (-1)^{(k-1)/2}\alpha^{k(k-1)/2}\prod_{j=1}^{k-1}(1-\alpha^{-2j})$$

$$= (-1)^{(k-1)/2}\cdot 1\cdot(1^0+1^1+1^2+\cdots+1^{k-1}) = (-1)^{(k-1)/2}\cdot(k)$$

$$= (-1)^{(k-1)/2}k.$$