The Story of the Alternating Sign Matrix Conjecture

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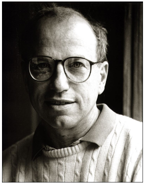
What is an Alternating Sign Matrix?



David Robbins (photo courtesy of Ken Robbins - AMS)

Definition

An **Alternating Sign Matrix (ASM)** is a square matrix with the following properties:

- All entries in the matrix are in the set $\{-1, 0, 1\}$
- Each row and column must sum to 1
- Non-zero entries in each row and column must alternate in sign

What is an Alternating Sign Matrix?

Alternating Sign Matrices of Size 3

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

What is an Alternating Sign Matrix?

- ASM of size 5:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- **Fact:** An ASM's outer edge can only have a single 1 in each respective column and row.

Counting alternating sign matrices

We construct a new matrix, M, given an ASM A:

$$M_{i,j} = \sum_{r=1}^{i} A_{r,j} \qquad \text{column sum up to row } i$$

A correspondence for size 5

$$\overset{\textbf{A}}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}} \Leftrightarrow \overset{\textbf{M}}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}$$

Monotone Triangle

From the matrix M, we can create a **monotone triangle**, where the entries in the ith row correspond to the columns containing 1s in our matrix M.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Leftrightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 3 & & & \\ & & 1 & & 2 & & 4 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Fact: There is a one-to-one correspondence between monotone triangles of size n and alternating sign matrices.

Another Triangular Phenomenon

The ASM Pascal Triangle

$A_{n,k}$ corresponds to the number of ASMs with a 1 at the top of the kth column

n											
1						1					
2					1		1				
3				2		3		2			
4			7		14		14		7		
5		42		105		135		105		42	
6	429		1287		2002		2002		1287		429

How many ASM's are there?

n	n!	A_n
1	1	1
2	2	2
3	6	7
4	24	42
5	120	429
6	720	7436
$\vdots$	$\vdots$	$\vdots$

- The number of ASM's A_n grows faster than $n!$

The Refined ASM Conjecture

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

The Big Idea

The ASM Conjecture

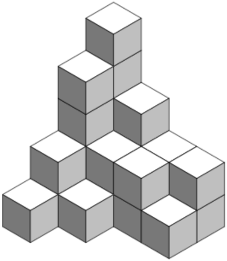
The total number of size n alternating sign matrices is

$$A_n = A_{n+1,1} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

A Different Set of Objects

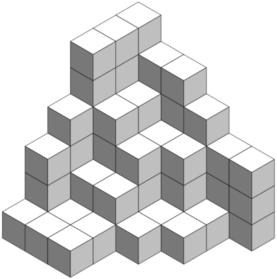
Definition

A **Plane Partition** in the space $\mathcal{B}(r, s, t)$, a box of size $r \times s \times t$ is an arrangement of stacks of unit cubes in a corner where gravity goes down and towards the walls.



A Plane Partition in $\mathcal{B}(5, 5, 5)$

Counting Plane Partitions



A Plane Partition of 75 in $\mathcal{B}(7, 6, 6)$.

Bird's Eye Representation of the the Plane Partition of 75

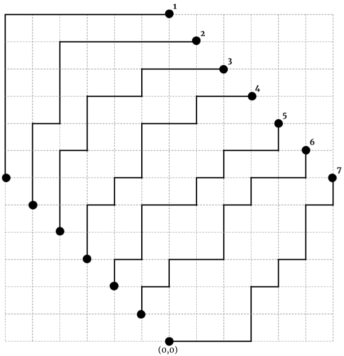
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

Representing Plane Partitions as Nests of Lattice Paths

In $\mathcal{B}(7,6,6)$, we look at $r = 7$ partitions. We transform into a lattice nest by outlining the entries greater than $r - i$ for the ith nest.

$$\begin{array}{cccccc}
6 & \|5 & 5 & 4 & 3 & 3 \\
6 & \|4 & 3 & 3 & 1 & \\
6 & \|4 & 3 & 1 & 1 & \\
\|4 & 2 & 2 & 1 & & \\
\|3 & 1 & 1 & & & \\
\|1 & 1 & 1 & & &
\end{array}$$

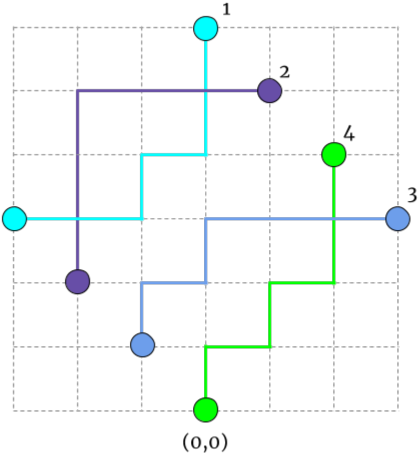
$\Leftrightarrow$



Lattice Nest Limitations

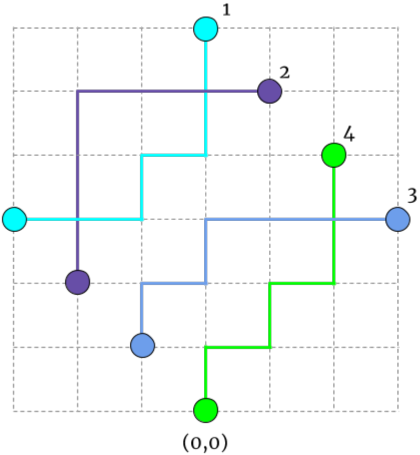
- Our goal is to count the non-intersecting lattice nests
- Counting the number of lattice paths would be easy, if they were allowed to cross
- For r paths given $\mathcal{B}(r, s, t)$, the total would be $\binom{s+t}{s}^r$

Plane Partition Proof - Lattice Nest



- The nest on the left corresponds to the permutation 1243.
- Switching the tails of paths 3 and 4 to cancel with a separate nest

Plane Partition Proof - Lattice Nest



How do we cancel intersecting lattice nests?

We pair lattice nests by selecting the rightmost, highest intersection between two lattice paths.

Once we have a given intersection point, we switch the paths such that they do not cross. The result of this switch gives a new nest with permutation number 1234 but same shape.

How many paths are there for a permutation?

For a given permutation of n, the number of nests is

$$\prod_{i=1}^{n} \binom{t+s}{s-i+\sigma(i)}$$

Given a set number of paths r, we count the total number using a sum over all possible permutations of r.

$$\sum_{\sigma \in S_i} (-1)^{I(\sigma)} \prod_{i=1}^{r} \binom{t+s}{s-i+\sigma(i)}$$

Determinant Machinery

Using machinery helping with determinant evaluations, our sum becomes:

$$\sum_{\sigma \in S_i} (-1)^{I(\sigma)} \prod_{i=1}^{r} \binom{t+s}{s-i+\sigma(i)} = det\left(\binom{t+s}{s-i+j}\right)_{i,j=1}^{r}$$

A Different Type of Plane Partition

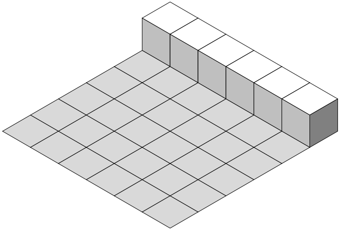
The same techniques used in the previous proof can be applied to different types of plane partitions.

Definition

Cyclic Symmetry in a plane partition means:

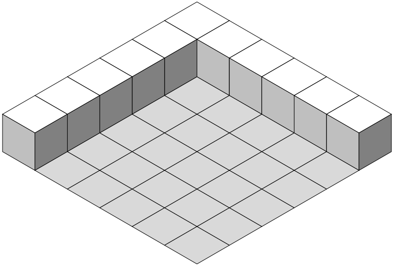
$$(i, j, k) \in P \Leftrightarrow (j, k, i) \in P \Leftrightarrow (k, i, j) \in P$$

A Cyclic Way of Plane Partitioning



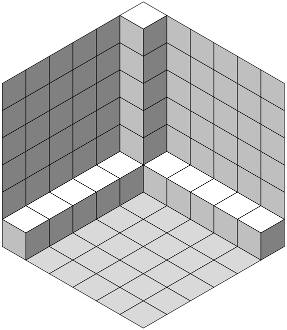
Constructing a Cyclically Symmetric Plane Partition

A Cyclic Way of Plane Partitioning



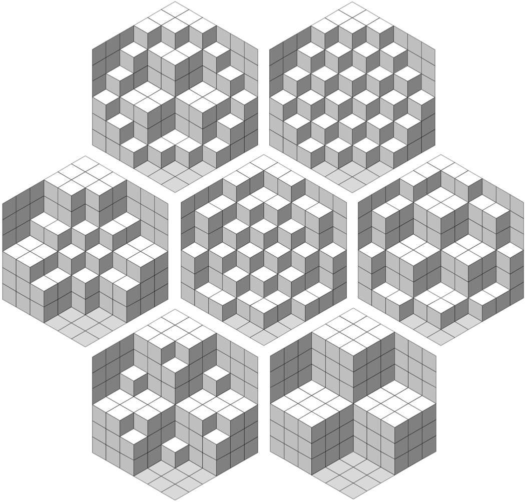
Constructing a Cyclically Symmetric Plane Partition

A Cyclic Way of Plane Partitioning



Constructing a Cyclically Symmetric Plane Partition

Totally Symmetric Self-Complementary Plane Partitions



TSSCPPs in $\mathcal{B}(6, 6, 6)$

A Hidden Symmetry

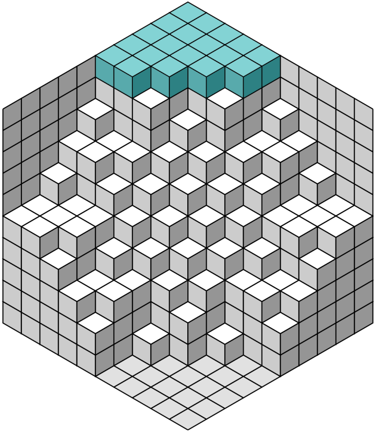
David Robbins' Interesting Discovery

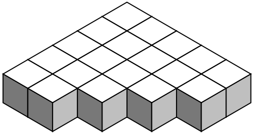
n	TSSCPP $\in \mathcal{B}(2n, 2n, 2n)$
1	1
2	2
3	7
4	42
5	429
$\vdots$	$\vdots$

TSSCPP Conjecture

The number of **totally symmetric self-complementary** plane partitions that fit inside $\mathcal{B}(2n, 2n, 2n)$ is equal to the number of *nxn* alternating sign matrices.

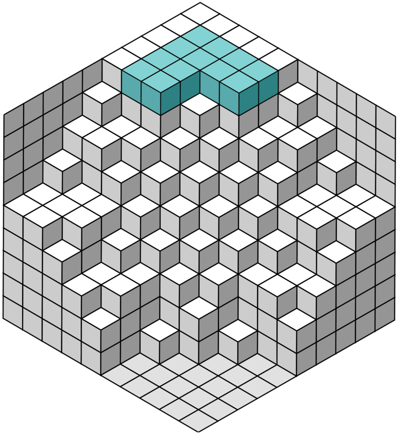
Separate Representations

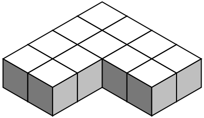




Outer Shell: 9 7 3

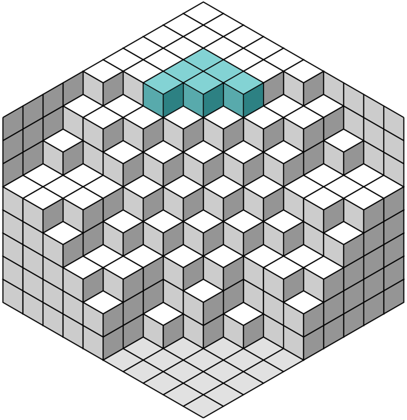
Separate Representations

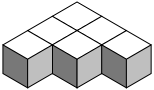




Second Shell: 7 5

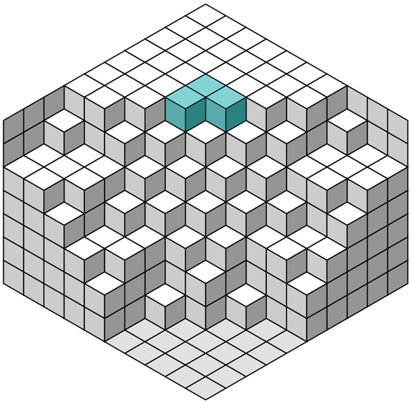
Separate Representations

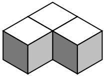




Third Shell: 5 1

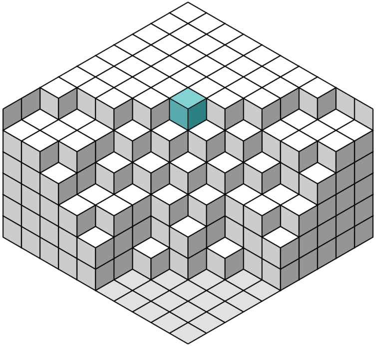
Separate Representations

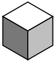




Fourth Shell: 3

Separate Representations





Fifth Shell: 1

A TSSCPP Array

Partitions
$9, 7, 3$
7,5
5,1
3
1

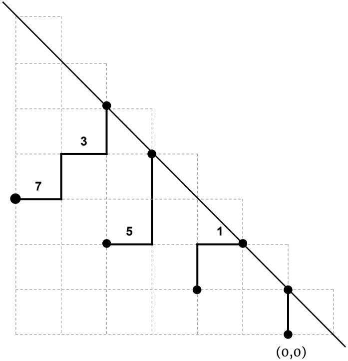
A TSSCPP Array of Order 5				
9	7	3		
	7	5		
		5	1	
			3	
				1

A TSSCPP Lattice Nest

9	7	3		
	7	5		
		5	1	
			3	
				1

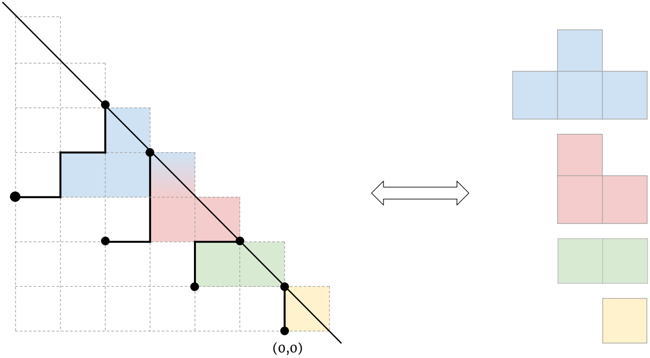
$\Leftrightarrow$

Lattice Nest from the TSSCPP Array



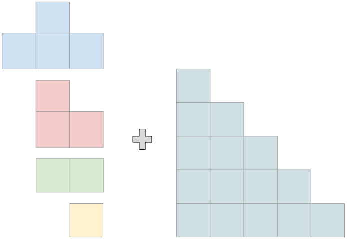
TSSCPP Nest into Gog and Magog Land

We create **shapes** from our lattice nest. These correspond to the lattice squares to the right of our paths.



Transforming Shapes into a Magog Triangle

We stack the shapes from top to bottom, aligned with the bottom left of a half-triangle of the same size as the TSSCPP array.



$$\Leftrightarrow \quad \begin{array}{ccccc} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 3 & \\ 1 & 1 & 2 & 4 & 5 \end{array}$$

A Familiar Triangle

Looking back at a monotone triangle will help us:

$$
\begin{array}{ccccccccc}
 & & & & 4 & & & & \\
 & & & 2 & & 4 & & & \\
 & & 2 & & 3 & & 4 & & \\
 & 1 & & 2 & & 3 & & 5 & \\
1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Gog and Magog Trapezoids

Taking the bottom k rows of a magog triangle gives a (n, k)-magog trapezoid, while the first k northwest rows of a monotone triangle gives us a (n, k)-gog trapezoid

A $(5, 3)$-magog trapezoid

$$\begin{array}{ccccc}
1 & 1 & 1 & & \\
1 & 1 & 2 & 4 & \\
1 & 1 & 3 & 4 & 5
\end{array}$$

A $(5, 3)$-gog trapezoid

$$\begin{array}{ccccccc}
 & & & & 4 & & \\
 & & & 2 & & 4 & \\
 & & 2 & & 3 & & 4 \\
 & 1 & & 2 & & 3 & \\
1 & & 2 & & 3 & &
\end{array}$$

A Big Conclusion!

In 1994, Doron Zielberger published a proof to conclude that TSSCPPs in $\mathcal{B}(2n, 2n, 2n)$ count the number of $n \times n$ alternating sign matrices.

Magog-Gog Trapezoids

For $1 \leq k \leq n$, the number of (n, k)-magog trapezoids is equal to the number of (n, k)-gog trapezoids.

Thank you!

Thank you all for coming! :)

We would also like to acknowledge and thank our advisor, Eric Egge, for his invaluable help regarding our studies of ASMs.