<u>3.1.6.</u> Use equation (3.7) to prove that

$$G(\alpha)=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)}). \quad (3.8)$$

From equation (3.7),

$$f(q,m)=\begin{cases}0, & \text{if } m \text{ is odd,}\\ (1-q)(1-q^3)\cdots(1-q^{m-1}), & \text{if } m \text{ is even.}\end{cases}$$

Now, $f(\alpha^{-2},k-1)=\alpha^{-[(k+1)/2]^2}G(\alpha)$ based on exercise 3.1.5 and also

$$f(\alpha^{-2},k-1)=\begin{cases}0, & \text{if } k \text{ is even,}\\ (1-\alpha^{-2})(1-(\alpha^{-2})^3)\cdots(1-(\alpha^{-2})^{k-2}), & \text{if } k \text{ is odd.}\end{cases}$$

So <u>when k is odd</u>, this becomes $(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})$. Since 3.1.5 $\Rightarrow$ $G(\alpha)=f(\alpha^{-2},k-1)\,\alpha^{[(k+1)/2]^2}$, $G(\alpha)=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\,\alpha^{[(k+1)/2]^2}$

$=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\alpha^1\alpha^3\cdots\alpha^k$ because $1+3+\cdots+k=$ (the number of odd integers between 1 and k inclusive)$^2=\left(\frac{k+1}{2}\right)^2$

Then $G(\alpha)=(1-\alpha^{-2})(1-\alpha^{-6})\cdots(1-\alpha^{-2k+4})\alpha^1\alpha^3\cdots\alpha^{k-2}\cdot(e^{2\pi ih/k})^k$,

and because $\alpha^k=(e^{2\pi ih/k})^k=e^{2\pi ih/1}$, α^k is a 1st root of unity which is always equal to 1.

$$\Rightarrow G(\alpha)=\alpha^1(1-\alpha^{-2})\alpha^3(1-\alpha^{-6})\cdots\alpha^{k-2}(1-\alpha^{-2k+4})\cdot 1$$

$$=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)}). \ \square$$

<u>3.1.7.</u> Use the fact that $\alpha^{k-j}-\alpha^{-(k-j)}=-(\alpha^j-\alpha^{-j})$ to rewrite equation (3.8) as

$$G(\alpha)=(-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)}). \quad (3.9)$$

(3.8) says $G(\alpha)=(\alpha-\alpha^{-1})(\alpha^3-\alpha^{-3})\cdots(\alpha^{k-2}-\alpha^{-(k-2)})$

$=(\alpha^{k-(k-1)}-\alpha^{-(k-(k-1))})(\alpha^{k-(k-3)}-\alpha^{-(k-(k-3))})\cdots(\alpha^{k-2}-\alpha^{-(k-2)})$, which based on the fact

$=(-(\alpha^{k-1}-\alpha^{-(k-1)}))(-(\alpha^{k-3}-\alpha^{-(k-3)}))\cdots(-(\alpha^2-\alpha^{-2}))$

$=(-1)^{\frac{k-1}{2}}(\alpha^{k-1}-\alpha^{-(k-1)})(\alpha^{k-3}-\alpha^{-(k-3)})\cdots(\alpha^2-\alpha^{-2})$ because (-1) is implicated $(k-1)/2$ times

$=(-1)^{(k-1)/2}(\alpha^2-\alpha^{-2})(\alpha^4-\alpha^{-4})\cdots(\alpha^{k-1}-\alpha^{-(k-1)}). \ \square$