3.1.3. This exercise and those up to and including exercise 3.1.10 outline Gauss's evaluation of the Gaussian sum $G(\alpha) = \sum \alpha^{j^2}$ where α is a primitive kth root of unity such as $e^{2\pi i/k}$ and the summation is taken over all residue classes, j, modulo k, where k is odd. This evaluation was Gauss's original reason for defining Gaussian polynomials.

Define
$$f(q,m) = 1 - \begin{bmatrix} m \\ 1 \end{bmatrix} + \begin{bmatrix} m \\ 2 \end{bmatrix} - \cdots + (-1)^m \begin{bmatrix} m \\ m \end{bmatrix}.$$

Use the recursive formula, $\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j \end{bmatrix} + q^{m-j}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$, to prove that
$$f(q,m) = (1-q^{m-1})\, f(q,m-2).$$

We're given the companion formula $\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \underline{\begin{bmatrix} m-1 \\ j \end{bmatrix}}$, so

$$\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \cdot \underline{\left(\begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1-j}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)}$$ by the first recursive formula

$$= \underline{\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix}.$$ Now, the companion formula allows us to say that this $= \underline{\begin{bmatrix} m-2 \\ j-2 \end{bmatrix} + q^{j-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix}} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix}$.

Substituting this, $f(q,m) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}$

$$= \sum_{j=0}^{m} (-1)^j \cdot \left(\begin{bmatrix} m-2 \\ j-2 \end{bmatrix} + q^{j-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} + q^{m-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$

$$= \left(\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m-2 \\ j-2 \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^{j-1}\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \left(\sum_{j=0}^{m} (-1)^j q^{m-1} \begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \right)$$

$$= \left(\sum_{j=0}^{m-2} (-1)^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \underbrace{\left(\sum_{j=0}^{m-2} (-1)^{j+1} q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right) + \left(\sum_{j=0}^{m-2} (-1)^j q^j \begin{bmatrix} m-2 \\ j \end{bmatrix} \right)}_{\text{as Matthew said, these cancel out.}} + \left(\sum_{j=0}^{m-2} (-1)^{j+1} q^{m-1} \begin{bmatrix} m-2 \\ j \end{bmatrix} \right)$$

$$= f(q,m-2) + 0 + (-1) q^{m-1} f(q,m-2)$$

$$= (1-q^{m-1})\, f(q,m-2).$$ So $f(q,m) = (1-q^{m-1})\, f(q,m-2)$.