MASSIVE BLACK HOLE BINARIES

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Massive Black Hole Binaries

- Masses range from 10^4 to 10^8 Solar masses
- Depending on the mass can last from hours to months
- Detectable to very high redshifts
- Form during the merger of galaxies
- Rates of the order of 10 mergers per year
- Very high in Signal-to-Noise ratio

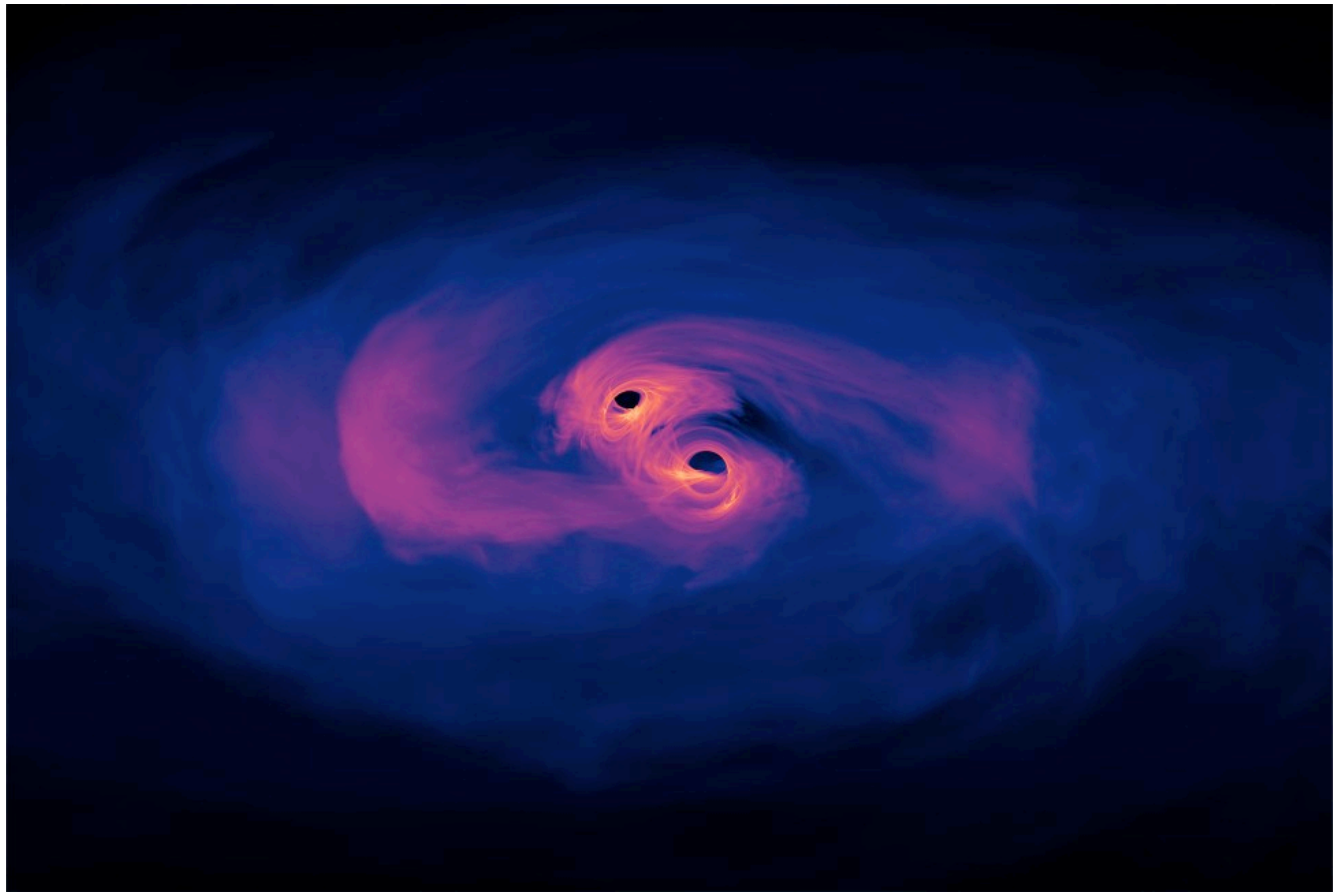


Image credit: NASA Goddard/Scott Noble

Parameter estimation

We are interested in the estimating the parameters of the signal given its parametrical model.

Bayes' formula for probability densities:

$$p(\boxed{\theta} \mid \boxed{x}) = \frac{p(x \mid \theta)p(\theta)}{p(x)}$$

parameters data

$$x = s(\theta) + n$$

Parameterisation

$p(\theta \mid x)$ – **posterior distribution**, probability density of parameter given data

$p(x \mid \theta)$ – **likelihood**, probability of measuring data given parameter

$p(\theta)$ — **prior** represents the knowledge on the parameters before performing the measurement

$p(x)$ — **evidence (marginal likelihood)**, probability of obtaining data over all possible values of the parameter. It can be also viewed as a normalisation factor such that the posterior is properly normalised.

Likelihood

Assuming that the noise is Gaussian we can write the likelihood in time domain in the following way

$$p(\boldsymbol{n}) = \frac{1}{\sqrt{2\pi \det(C)}} \exp\left[-\frac{1}{2}\boldsymbol{n}C^{-1}\boldsymbol{n}^{\mathrm{T}}\right]$$

Computing the full covariance might be very computationally complicated.

In the general case in the frequency domain the noise is uncorrelated between each frequency bin, therefore we often do the analysis in frequency domain and we can rewrite the likelihood as:

$$p(\boldsymbol{n}) = \prod_{j=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{n_j^2}{2\sigma_j^2}\right]$$

Likelihood

For the data model $x = s(\theta) + n$ It can be rewritten as:

$$p(\mathbf{x} \mid \theta) = \left(\prod_{j=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma_j} \right) \exp\left[-\frac{1}{2} \sum_{j=0}^{N-1} \frac{\left(x_j - s_j(\boldsymbol{\theta})\right)^2}{\sigma_j^2} \right]$$

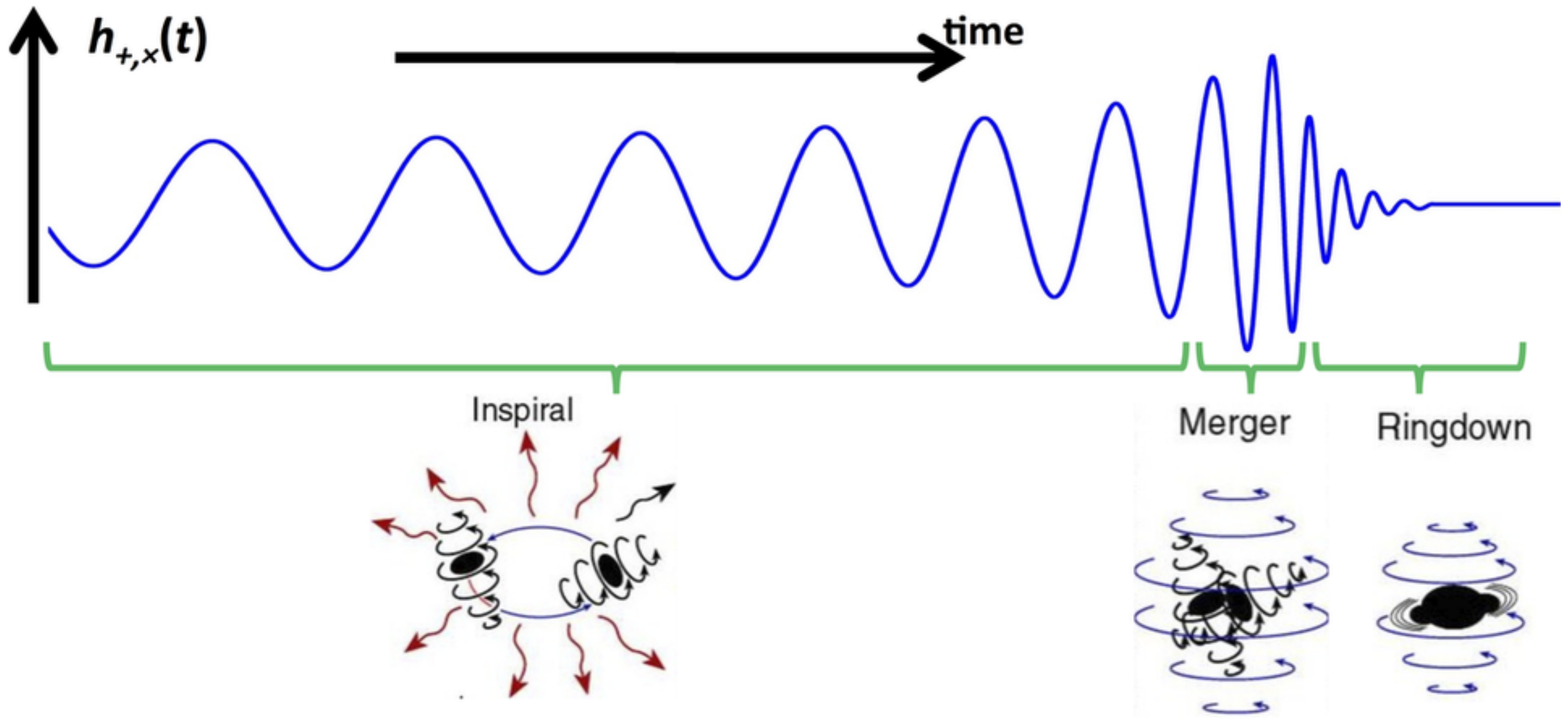
Parameterisation

m_1	Mass 1		
m_2	Mass 2	β	Ecliptic latitude
a_1	Spin 1	λ	Ecliptic longitude
a_2	Spin 2		
d	Distance	ϕ_0	Reference phase
i	Inclination	t_0	Reference time
ψ	Polarisation		

Phenomenological waveforms

Sascha Husa et al.
Frequency-domain gravitational waves from non-precessing black-hole binaries. I. New numerical waveforms and anatomy of the signal.
arXiv 1508.07250

Sebastian Khan et al.
Frequency-domain gravitational waves from non-precessing black-hole binaries. II. A phenomenological model for the advanced detector era.
arXiv 1508.07253



Post-Newtonian approximation for the inspiral part.

Numerical relativity simulations for the merger.

Ringdown can be though of
as a combination of damped sinusoids

Use analytic anzatz for Fourier domain amplitude and phase.

Phenomenological waveforms

The two polarisations of the waveform can be decomposed in the spherical harmonics.

$$h_+(t) - ih_\times(t) = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h_{\ell m}(t)\, {}_{-2}Y_{\ell m}(\iota_0, \varphi_0)$$

$$h_{lm} = A_{lm}(f) e^{i\phi_{lm}(f)}$$

Each multipole is separated into amplitude and phase

Higher harmonics

Lionel London et al.
First higher-multipole model of gravitational waves from spinning and coalescing black-hole binaries.
arXiv1708.00404

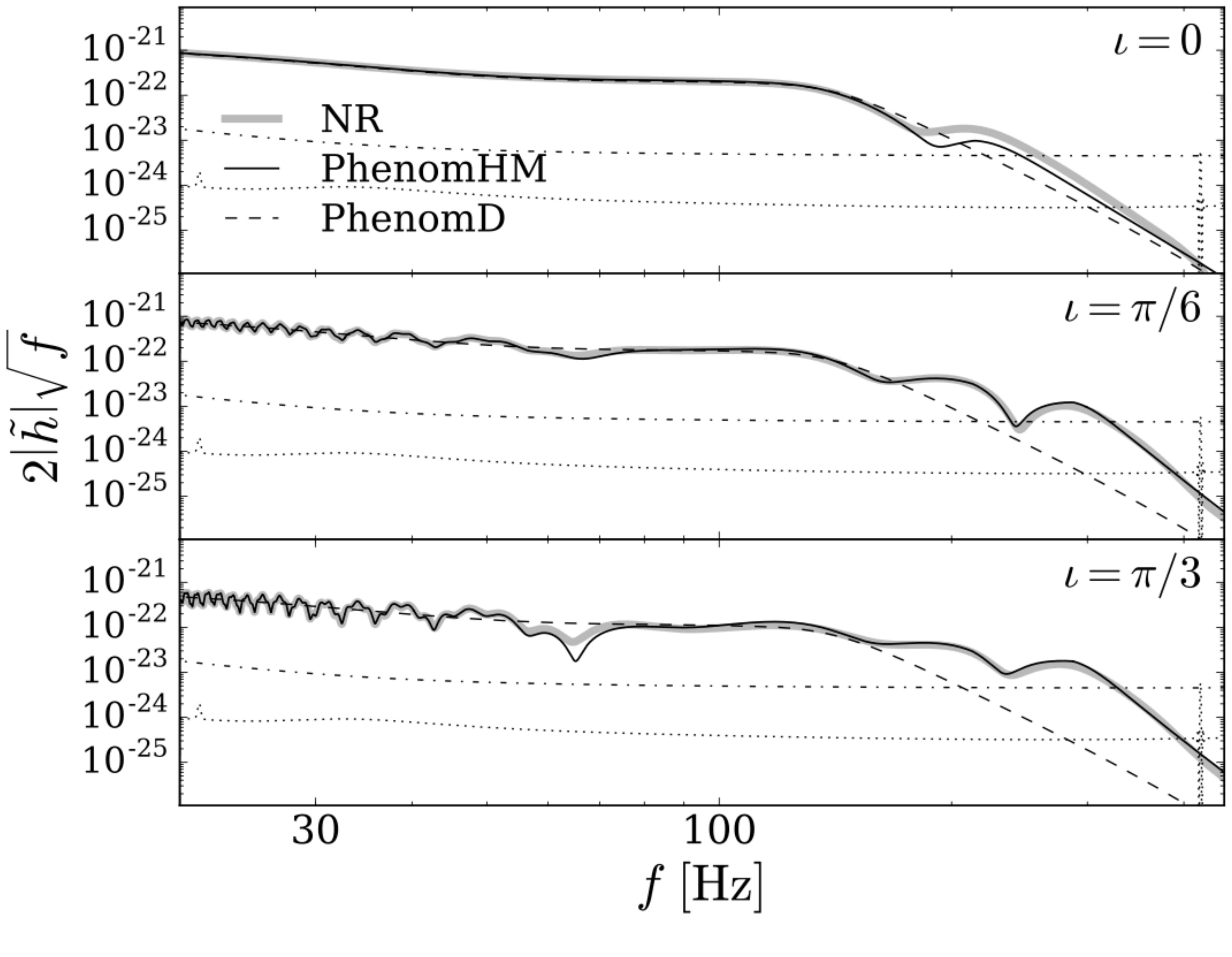
PhenomD implements dominant (2,2) harmonic (for non-precessing aligned spin waveform).

PhenomHM

$$(l, m) \in \{(2,2),(3,3),(4,4),(2,1),(3,2),(4,3)\}$$

Misaligned spins, taking into account precession of the waveforms.

Higher modes are important for:
- higher mass rations
- merger/ringdown
- breaking degeneracies in parameter space
- better sky localisation



FAST FREQUENCY RESPONSE

Sylvain Marsat and John Baker.
Fourier-domain modulations and delays of gravitational-wave signals.
arXiv 1806.10734

We can project the waveform on the detector arms and then apply Time Delay Interferometry.

$$y_{slr} = \frac{1}{2} \frac{1}{1 - \hat{k} \cdot n_l} n_l \cdot (h(t_s) - h(t_r)) \cdot n_l$$

This will be slow for parameter estimation.

Instead use **frequency domain response:** $\tilde{s}(f) \equiv \mathcal{T}(f) \tilde{h}(f)$

It depends on time (motion of LISA on the orbit) and on frequency (departure from the long wavelength approximation).

$$\mathcal{T}_{slr} = \frac{i\pi f L}{2} \mathrm{sinc}[\pi f L(1 - k \cdot n_l)] \exp[i\pi f(L + k \cdot (p_r + p_s))] n_l \cdot P \cdot n_l(t_f)$$

HETERODYNING

Neil Cornish.
Heterodyned Likelihood for Rapid Gravitational wave parameter estimation.
arXiv 2109.02728

Two steps of the finding the source parameters:
search and parameter estimation.

Heterodyned likelihood <— uses a reference waveform which has a high likelihood value
(i.e. parameters close to the true parameters)
to base band the likelihood calculation.

$$\bar{h} = \bar{A}(f)e^{i\bar{\Phi}(f)}$$ <- reference waveform

$$h = A(f)e^{i\Phi(f)}$$ <- another waveform close to it

Their ratio will be a slowly varying function:

$$\zeta(f) = \frac{h(f)}{\bar{h}(f)}$$

HETERODYNING

Likelihood: $$\ln L = (d \mid h) - \frac{1}{2}(h \mid h)$$

Can be rewritten as

$$(d \mid h) = 2\int \frac{d(f)h^*(f) + d^*(f)h(f)}{S(f)}df$$
$$= 2\int (\kappa(f)\zeta^*(f) + \kappa^*(f)\zeta(f))\,df$$

where

$\kappa(f) = d(f)\bar{h}^*(f)/S(f)$ is rapidly varying

$\zeta(f)$ is slowly varying

$$(h \mid h) = 2\int \frac{h(f)h^*(f) + h^*(f)h(f)}{S(f)}df$$
$$= 4\int df|\zeta(f)|^2\sigma(f)df$$

where

$\sigma^2(f) = |\bar{h}(f)|^2/S(f)$ is rapidly varying

$|\zeta(f)|^2$ is slowly varying

HETERODYNING

Zackay B et al.
Relative Binning and Fast Likelihood Evaluation for Gravitational Wave Parameter Estimation.
arXiv 1806.08792

Another perspective on the heterodyning technique.

Use linear interpolation inside the frequency bin.

$$r(f) = \frac{h(f)}{h_0(f)} = r_0(h,\ \mathrm{b}) + r_1(h,\ \mathrm{b})(f - f_{\mathrm{m}}(\mathrm{b})) + \cdots$$

f_m is the central frequency of the bin.

HETERODYNING

We can then rewrite the overlap which is the part of the likelihood

$$Z[d(f), h(f)] \equiv 4 \sum_f \frac{d(f)h^*(f)}{S_n(f)/T}$$

$$Z[d(f), h(f)] \approx \sum_{\mathrm{b}} (A_0(\mathrm{b}) r_0^*(h, \mathrm{b}) + A_1(\mathrm{b}) r_1^*(h, \mathrm{b}))$$

$$Z[h(f), h(f)] \approx \sum_{\mathrm{b}} \Big(B_0(\mathrm{b}) |r_0(h, \mathrm{b})|^2$$

$$+2B_1(\mathrm{b}) \mathfrak{Re}[r_0(h, \mathrm{b}) r_1^*(h, \mathrm{b})] \Big)$$

The rations r_0 and r_1 can be computed for a reduced frequency resolution.

HETERODYNING

Where the summary coefficients are computed at the maximal frequency resolution:

$$A_0(\mathrm{b}) = 4\sum_{f\in \mathrm{b}} \frac{d(f)h_0^*(f)}{S_n(f)/T}$$

$$A_1(\mathrm{b}) = 4\sum_{f\in \mathrm{b}} \frac{d(f)h_0^*(f)}{S_n(f)/T}(f - f_{\mathrm{m}}(\mathrm{b}))$$

$$B_0(\mathrm{b}) = 4\sum_{f\in \mathrm{b}} \frac{|h_0(f)|^2}{S_n(f)/T}$$

$$B_1(\mathrm{b}) = 4\sum_{f\in \mathrm{b}} \frac{|h_0(f)|^2}{S_n(f)/T}(f - f_{\mathrm{m}}(\mathrm{b})).$$

Good luck with tutorials!