Formalising mathematics in Lean (Week 7 - Diamonds)

A GlaMS course

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Learning outcomes

- Identifying diamonds (i.e., type mismatch errors)
- Resolving diamonds

Example

- example [hE₁ : NormedAddCommGroup E]

```
example [hE1 : NormedAddCommGroup E]
  [hE2 : InnerProductSpace ℂ E] [Ring E] [Algebra ℂ E]
  (T : E →l[ℂ] E) [hE5 : FiniteDimensional ℂ E] :
   LinearMap.adjoint T = T
```

Example

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  example [hE₁ : NormedAddCommGroup E]
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  ```
  @LinearMap.adjoint ℂ E E Complex.instIsROrCComplex hE₁
    hE₁ hE₂ hE₂ hE₅ hE₅ T = T
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  ```
- No, this still gives the same error. We need to expand the error to see what's happening in order to fix it.
  ```

Identifying the cause

From the Infoview:

```
/- Error: application type mismatch
  LinearMap.adjoint T
argument
  T
has type
  @LinearMap ℂ ℂ _ _ _ E E
   NonUnitalNonAssocSemiring.toAddCommMonoid
   NonUnitalNonAssocSemiring.toAddCommMonoid Algebra.toModule
   Algebra.toModule : Type u_1
but is expected to have type
  @LinearMap ℂ ℂ _ _ _ E E AddCommGroup.toAddCommMonoid
   AddCommGroup.toAddCommMonoid NormedSpace.toModule
   NormedSpace.toModule : Type u_1 -/
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- What did we notice from the error?
  - `Algebra.toModule` is clashing with `NormedSpace.toModule`
  - `AddCommGroup.toAddCommMonoid` is clashing with `NonUnitalNonAssocSemiring.toAddCommMonoid`
- Let's look at the second case more closely...

Identifying the cause (3): looks like a diamond!

$$\begin{tikzcd}
 & \texttt{AddCommMonoid} & \\
\texttt{NUNASemiring} \arrow[ur] & & \texttt{ACG} \arrow[ul] \\
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- This is why we keep getting an application type mismatch error: because Lean keeps confusing `AddCommMonoid` induced by `NACG` with that induced by `Ring`, which are not known to be definitionally equal.

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Technically, this does fix the error. But, you are guaranteed to run into more problems later on!

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let f : E →ₗ[ℂ] E := sorry
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We're back to the same error...

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- So how do we fix this then?
- We need to resolve the diamonds!

Type inheritance diagrams

- Let `A`, `B`, and `C` be types.

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- If we also have `B` infers `C`, then we have the following diagram:

$$\begin{tikzcd} \mathtt{C} \\ \mathtt{B} \arrow[u] \\ \mathtt{A} \arrow[u] \end{tikzcd}$$

Diamonds

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- Say `A`, `B`, `C`, and `D` are types, then a *diamond* is when we have the following type inheritance diagram:

$$\begin{tikzcd}
& \mathtt{D} & \\
\mathtt{B} \arrow[ur] & & \mathtt{C} \arrow[ul] \\
& \mathtt{A} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

In other words, a diamond is when we have `D` being inferred by both `B` and `C` which are inferred by `A`.

Diamonds (2)

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- Transparent diamonds can be left alone, they won't raise any errors.
- Non-transparent diamonds, denoted by $\blacklozenge$, are the ones that will raise errors and are the ones that need to be addressed.
- We also say a diamond is *resolved* when there are no non-transparent sub-diamonds (i.e., a diamond within a diamond). In other words, the diamond is resolved when the error is resolved.

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- We need to use a new class $\texttt{E} = \texttt{C} + \texttt{B} \setminus \{\text{common traits of } \texttt{C} \text{ and } \texttt{B}\}$. Then E infers both B and C. So we get the following type inheritance diagram:

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- Now `D` created by `B` is equal to that created by `C`, because they are both created by `E`.

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- No.

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  In this case you would get,

$$
\begin{tikzcd}
 & \{\texttt{One A}\} & \\
B = \{\texttt{One A}, 3\} \arrow[ur] \arrow[r, Rightarrow] & 1_B = 1_C & C = \{\texttt{One A}, 2\} \arrow[ul] \arrow[l, Rightarrow] \\
 & \texttt{A} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

  Here, $\Rightarrow$ means that it uses B and C.

Example (3)

- The better solution is to simply have

$$\begin{tikzcd}
& \{\texttt{One A}\} & \\
\{\texttt{One A}, 3\} \arrow[ur] & \{\texttt{One A}, 2, 3\} \arrow[l] \arrow[r] & \{\texttt{One A}, 2\} \arrow[ul] \\
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Back to the first example

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\begin{tikzcd}
 & \texttt{AddCommMonoid} & \\
\texttt{NUNASemiring} \arrow[ur] \arrow[rr, phantom, "\Diamond_1"] & & \texttt{ACG} \arrow[ul] \\
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- In this example, there are technically two sub-diamonds since `Ring` implies `ACG`.

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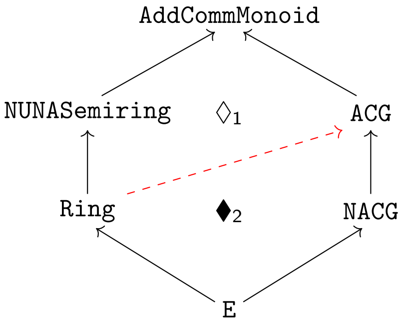

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- $\lozenge_1$ is transparent since `AddCommMonoid` created by `NUNASemiring` (created from `Ring`) is definitionally equal to that created by `ACG` (created from `Ring`).

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- $\lozenge_1$ is transparent since `AddCommMonoid` created by `NUNASemiring` (created from `Ring`) is definitionally equal to that created by `ACG` (created from `Ring`).
- So it suffices to address the second sub-diamond $\blacklozenge_2$.

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- Let's first take a look at how `NormedAddCommGroup` is defined:

```
class NormedAddCommGroup (E : Type*) extends Norm E,
    AddCommGroup E, MetricSpace E where
dist := fun x y => ‖x - y‖
/-- The distance function is induced by the norm. -/
dist_eq : ∀ x y, dist x y = ‖x - y‖ := by aesop
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- So we want to have the same class, but with `Ring` instead of `AddCommGroup`. This way `NormedAddCommGroup` would depend on `Ring`.

Resolving the diamond (2)

```
class NACGoR (E : Type*)
  extends Norm E, Ring E, MetricSpace E where
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- With this new class, we get `AddCommMonoid` via the ring structure, which is what we wanted.

Resolving the diamond (3)

We need to also create the instances that our new class induces, in particular `Ring` and `NormedAddCommGroup`.

This allows Lean to instantly see `NACGoR` as both a `Ring` and a `NormedAddCommGroup` (where `NormedAddCommGroup` is given by its `Ring` structure).

Resolving the diamond (4)

Our new type inheritance diagram would look like the following.

$$\begin{tikzcd}
 & \texttt{AddCommMonoid} & \\
\texttt{NUNASemiring} \arrow[ur] & & \texttt{ACG} \arrow[ul] \\
\texttt{Ring} \arrow[u] \arrow[urr, dashed] & \texttt{NACGoR} \arrow[l] \arrow[r] & \texttt{NACG} \arrow[u] \\
 & \texttt{E} \arrow[u] &
\end{tikzcd}$$

Resolving the diamond (5)

Now let's try that example again:

```
example [NACGoR E] [InnerProductSpace ℂ E] [Algebra ℂ E]
  [FiniteDimensional ℂ E] (T : E →ₗ[ℂ] E) :
  LinearMap.adjoint T = 0
```

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```

✓ It works!

Resolving the diamond (6)

- We now check that the instance `AddCommMonoid` created by the ring structure is definitionally equal to that created by `NACG`:

```
example [h : NACGoR E] :
  h.toAddCommMonoid =
    NormedAddCommGroup.toAddCommGroup.toAddCommMonoid :=
rfl
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- Thus we have resolved the first diamond $\Diamond$!
- But we still have potential issues left to address.

What now?

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- What if we wanted to do:

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example [NACGoR E] [InnerProductSpace ℂ E] [Algebra ℂ E]
  [FiniteDimensional ℂ E] (T : E →ₗ[ℂ] E) (x y : E) :
  ⟪T (x * y), T x⟫_ℂ
    = (LinearMap.adjoint (Algebra.linearMap ℂ E)) x
```

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```

- And we're back to the type mismatch error... **UGH**

Identifying the second error: another ♦

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- `Module` is created by
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  2. `Algebra` and `AddCommMonoid`

  And they are not definitionally equal.
- So we have another non-transparent diamond ♦ to resolve.
- Let's open the documentation of `Algebra` in `Mathlib` to see exactly how `Algebra` is defined. There's actually a whole section on implementation:
  `Mathlib.Algebra.Algebra.Basic#Implementation-notes`.

Implementation notes in `Mathlib.Algebra.Algebra.Basic`

- Implementation notes:
  "There are two ways to talk about an `R`-algebra `A` when `A` is a semiring:

  1.
  ```
  variable [CommSemiring R] [Semiring A]
  variable [Algebra R A]
  ```
  2.
  ```
  variable [CommSemiring R] [Semiring A]
  variable [Module R A] [SMulCommClass R A A]
    [IsScalarTower R A A]
  ```

Implementation notes in `Mathlib.Algebra.Algebra.Basic` (2)

- This means we can replace `[Algebra ℂ E]` with `[SMulCommClass ℂ E E] [IsScalarTower ℂ E E]`, because `[Module ℂ E]` is given by the inner product space, which then resolves our diamond.

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- Although, this comes with a small caveat:

  "Typeclass search does not know that the second approach implies the first, but this can be shown with:

```
example {R A : Type*} [CommSemiring R] [Semiring A]
  [Module R A] [SMulCommClass R A A] [IsScalarTower R A
  A] : Algebra R A :=
Algebra.ofModule smul_mul_assoc mul_smul_comm
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  ```

  — `Mathlib.Algebra.Algebra.Basic#Implementation-notes`
- Let's attribute the above example as a local instance in our file, so that the second approach does imply the first.

Resolving the second diamond

- Now we have no errors and unresolved diamonds remaining!

[1] In Lean 3, the proof is `by ext; refl`, which means the diamond would remain unresolved - this is another example of how Lean 4 is more powerful.

Resolving the second diamond

- Now we have no errors and unresolved diamonds remaining!
- `Module` created by the algebra is definitionally equal[1] to that created by the inner product space:

```
example [NACGoR E] [h : InnerProductSpace ℂ E]
  [SMulCommClass ℂ E E] [IsScalarTower ℂ E E] :
  h.toModule = Algebra.toModule :=
rfl
```

[1]In Lean 3, the proof is `by ext; refl`, which means the diamond would remain unresolved - this is another example of how Lean 4 is more powerful.