Discrete differential geometry in homotopy type theory

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Abstract

Higher inductive types can capture some concepts of differential geometry in two dimensions including connections, curvature, and vector fields. We define connections on higher inductive types. We then define tangent bundles and vector fields by looking at the special subclass of combinatorial manifolds, which are discrete in the sense of real cohesion[1], drawing inspiration from the field of discrete differential geometry. We prove the Gauss-Bonnet theorem and Poincaré-Hopf theorem for combinatorial manifolds.

> "It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, *The Future Does Not Compute*[2]

Notes so far from Mathieu and Steve

0.1 November 21, Mathieu and Steve

Constructors are present in the HIT, and the HIT is a type. The universe must be closed under inductive types.

Polygons of different cardinality will tangle with transport.

Greg suggests I could use an isomorphism from a combinatorial manifold to a HIT with just one 2-cell to capture "total curvature":

```
def S1:
base: S1
loop: base=base

def S2:
N: S2
surf: refl_N = refl_N

def torus:
b:torus
```

```
p,q:b=b
donut:p.q=q.p
```

The map $\mathcal{R}$ (realization) is a lossy map. $R(S2) : \mathcal{U}$.

"strict morphism of HITs": map constructors to constructors. $C_4 \to S^1$ is a strict map.

$\mathcal{R}(C_4)$ has an equivalence to $\mathcal{R}(S^1)$. Perhaps this is $\mathcal{R}$(the map $C_4 \to S^1$, the strict one). But $S^1 \to C_4$ is non-strict, uses generated data, uses data from $\mathcal{R}(C_4)$.

Claim: we cannot see the one-notch "90 degree" rotation of C_4 at the level of types.

0.1.1 Extension to next dimension as obstruction theory

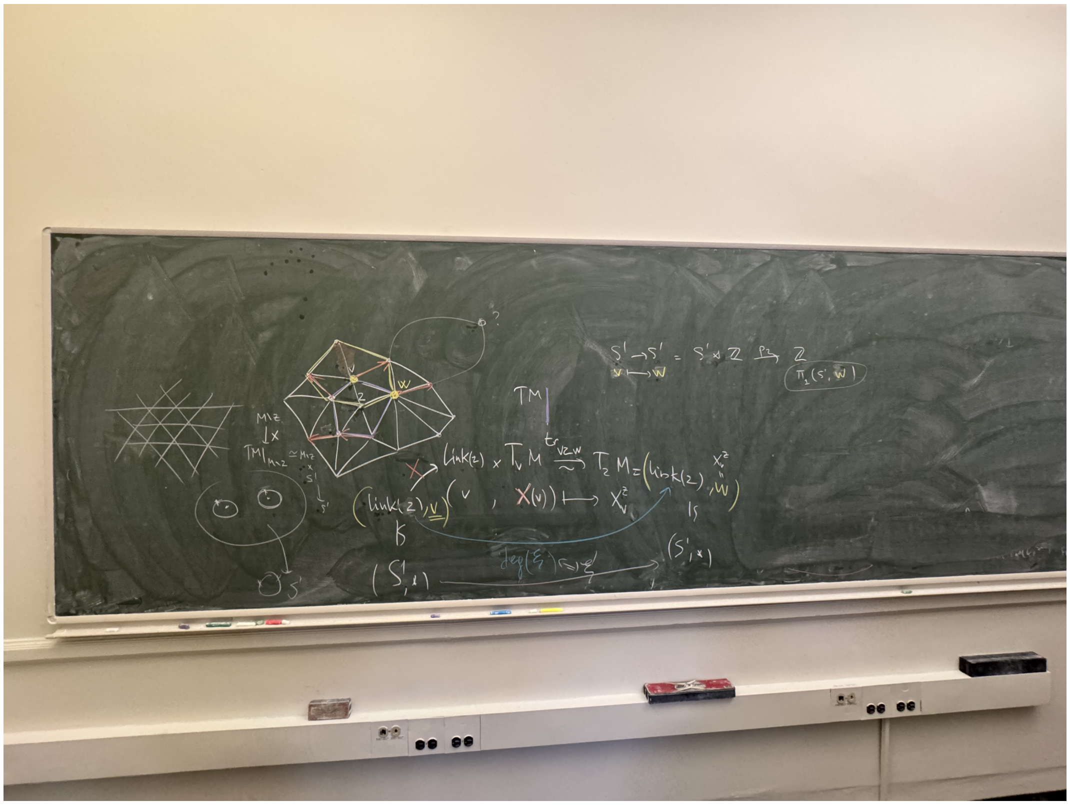
- Consider a type X with 1-skeleton X_1.
- HIT gives skeletal filtration
- functions naturally given as per-skeleton steps
- there are constraints/obstructions: obstruction to extend map T from vertices a, b to path $e : a = b$ is that $T(a)$ and $T(b)$ are in the same connected component.
- say a 2-cell F contracts a loop $L : F : L = \mathsf{refl}_v$ some vertex v
- we have a choice of v for each face F
- then $T(F)$ must be in same component as refl_v ($\mathsf{refl}_v : \pi_1 X$ with basepoint v)
- $T(F)$ is merely refl, and holonomy is merely refl, and connection on this face F is merely flat
- curvature is a function from 2-cells to $\pi_1(X, v)$

$$
\begin{aligned}
&T(v) : U \\
&T(L) : T(v) = T(v) \\
&T(F) : refl_{T(v)} = T(L)
\end{aligned}
$$

The class $||T(L)|| : \pi_1(U, T(v))$ is an obstruction to having $T(F)$, to extending T to F.

The hierarchy of maps extending to higher skeletons, which we call connections of dimension 1, 2, etc. This works for ANY HIT. The combinatorial manifold structure is just one special case that gives me a specific map T.

Then Mathieu drew on the board a patch of a combinatorial manifold with links of different vertex cardinalities. Then he wrote some maps that might help define the degree/index of a vector field.



0.2 November 27, Mathieu

An angle is: a path in S^1 from base, which is equiv to a path in Aut S^1 from the identity to another automorphism. (automorphism is primary: aut of tangent circle).

Viewed as automorphisms we can compose them.

Viewed as paths, multiply path 2 by the element at the endpoint of path 1.

Curvature maps a loop in the base to a path in automorphisms of the tangent circle at the base.

A vector field + a trivialization ("fiducial v.f." in the book by Needham) maps a loop in the base to the tangent circle at the basepoint. We can take the index/degree of this.

Given a loop $L : v = v$, v.f. F, take $F(L) : tr_L(F(v)) = F(v)$ (apd, dependent action on paths).

Type family $f : A- > U$, $\prod_{a:A} f\, a$ is the type of dependent functions. Let $X : \prod_{a:A} f\, a$. We can apply X to a path $p : a =_A b$. $X(p)$ is a path over p.

$$X(p) : tr_p(X(a)) = X(b)$$

$X(a), X(b)$ are terms of different types.

Greg reminded about horizontal projection interpretation of covariant derivative: covariant derivative of F along itself: lift F to F' horizontal on $TP(P \to M)$, take $[F', F']$

Tangent bundle $TM \to M$ w/ connection ω. Covariant derivative of F in the direction of X: $\nabla_X(F)(x)$.

1. Lift X and F to horizontal fields X', F' in TTM
2. Compute $[X', F']$ to TTM
3. Decompose $[X', F'] = [X, F]' + \omega([X', F'])$

Mathieu would rather use this formulation:

$$\nabla_X(F) = [X, F] + \omega(X)(F)$$

Now take $X = F$:

$$\nabla_F(F) = [F, F] + \omega(F)(F) = \omega(F)(F)$$

1 Overview

We will define

- combinatorial 2-manifolds
- circle bundles, and principal circle bundles of tangent bundles
- vector fields,

and then observe emerging from those definitions the presence of

- connections
- curvature
- the index of a vector field,

and prove

- the Gauss-Bonnet theorem
- and the Poincaré-Hopf theorem.

We will consider functions $M \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where M is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

We will see in Section 3.3 that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. We will construct a map $\mathsf{link} : M \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. Then we can consider the type of pointed mere circles $\mathrm{EM}_\bullet(\mathbb{Z}, 1) \overset{\text{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ as well as the first projection that forgets the point. This is a univalent fibration (univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe[3]). If we form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"'] \\ M \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z}, 1) \end{tikzcd}$$

then we have a bundle of mere circles, with total space given by the $\sum$-type construction. We will show that this is not a principal bundle, i.e. a bundle of torsors. Torsors are types with the additional structure of a group action. But if link satisfies an additional property (amounting to an orientation) then the pullback is a principal fibration, i.e. link factors through a map $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$, where $\mathrm{K}(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space.

We will argue that extending link to a function T on paths can be thought of as constructing a connection, notably one that is not necessarily flat (trivial). Moreover, lifting T to $T_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ can be thought of as a nonvanishing vector field. There will in general not be a total lift, just a partial function. The domain of $T_\bullet$ will have a boundary of circles, and the degree (winding number) on the disjoint union of these can be thought of as the index of $T_\bullet$. We can then examine the total curvature and the total index and prove that they are equal, and argue that they are equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds. This is similar to the classical proof of Hopf[4], presented in detail in Needham[5].

1.1 Future work

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[6][7] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

2 Torsors and principal bundles

The classical theory of principal bundles tells us to look for an appropriate classifying space of torsors to map into.

Definition 2.1. Let G be a group with identity element e (with the usual classical structure and properties). A G**-set** is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : ||X||_{-1} \times \prod_{x:X} \mathsf{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

then we call this data a G**-torsor**. Denote the type of G-torsors by BG.

If $(X, \phi), (Y, \psi) : BG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 2.1. There is a natural equivalence $(X =_{BG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for BG and we have a group isomorphism $\Omega BG \simeq G$.

Lemma 2.2. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 2.1. The pointed type $(BG, *)$ is a $\mathrm{K}(G, 1)$. □

In particular, to classify principal S^1-bundles we map into the space $\mathrm{K}(S^1, 1)$, a type of torsors of the circle. Since S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$, we have $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$.

2.1 Bundles of mere circles

We find it illuminating to look also at the slightly more general classifying space of $\mathrm{K}(\mathbb{Z}, 1)$-bundles, that is bundles whose fiber are equivalent to $\mathrm{K}(\mathbb{Z}, 1)$. We can understand very well when these are in fact bundles of circle torsors, which will in turn shed light on orientation in this setting.

We will follow Scoccola[8]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2.2. Let $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \overset{\text{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$. A $\mathrm{K}(G, n)$**-bundle** on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses two self-maps on the universe: suspension followed by $(n+1)$-truncation $||\Sigma||_{n+1}$ and forgetting a point $F_\bullet$ to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{||\Sigma||_{n+1}} \mathrm{EM}_{\bullet\bullet}(G, n+1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n+1)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 2.3. Given $f : M \to \mathrm{EM}(G, n)$, the **associated action of** M **on** G, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ ||\Sigma||_{n+1} \circ f$.

Theorem 2.1. (Scoccola[8] Proposition 2.39). A $\mathrm{K}(G, n)$ bundle $f : M \to \mathrm{EM}(G, n)$ is equivalent to a map in $M \to \mathrm{K}(G, n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Let's relate this to *orientation*. Note that the obstruction in the theorem is about a map into $\mathrm{EM}_\bullet(G, n+1)$ and further note that $\mathrm{EM}_\bullet(G, n) \simeq \mathrm{K}(\mathrm{Aut}\, G, 1)$ (independent of n). The theorem says that the data of a map into $\mathrm{EM}(G, n)$ factors into data about a map into $\mathrm{K}(G, n+1)$ and one into $\mathrm{K}(\mathrm{Aut}\, G, 1)$. Informally, $\mathrm{EM}(G, n)$ is a little too large to be a $K(G, n+1)$, as it includes data about automorphisms of G.

In the special case of $\mathrm{EM}(\mathbb{Z}, 1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z}, 1)$ is contractible. $\mathrm{Aut}\,\mathbb{Z}$ consists of the $\mathbb{Z}/2\mathbb{Z}$ worth of outer automorphisms given by multiplication by ± 1. If we look at the fiber sequence

$$\mathrm{K}(S^1, 1) \to \mathrm{BAut}\, S^1 \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z}, 1)$$

we see the automorphisms of the circle as an extension of the group of automorphisms that are homotopic to the identity (which are the torsorial actions) by the group that sends the loop in S^1 to its inverse. This is another way to see that a map $f : M \to \mathrm{BAut}\, S^1 \simeq \mathrm{EM}(\mathbb{Z}, 1)$ factors through $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$ if and only if the composition to $\mathrm{K}(\mathrm{Aut}\,\mathbb{Z}, 1)$ is trivial. This amounts to a choice of loop-direction for all the circles, and so deserves the name "f is *oriented*." In addition the map $\mathrm{BAut}\, S^1 \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z}, 1)$ deserves to be called the first Stiefel-Whitney class of f, and the requirement here is that it vanishes.

Note 2.1. Reinterpreting more of the theory of characteristic classes would be an enlightening future project. Defining a Chern class and Euler class in 2 dimensions is related to the goals of this note, but the full theory is about a family of invariants in different dimensions that have various relations between each other and satisfy other properties.

2.2 Pathovers in circle bundles

Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ and $P \overset{\text{def}}{=} \sum_{x:M} T(x)$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : T(a) = T(b)$. This is a path in the codomain, which in this case is a type of types. Type theory also provides a function called *transport*, denoted $\mathsf{tr}(p) : T(a) \to T(b)$ which acts on the fibers of P. $\mathsf{tr}(p)$ is a function, acting on the terms of the types $T(a)$ and $T(b)$, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathsf{tr}(p)(\alpha) =_{T(b)} \beta$ between $\alpha : T(a)$ and $\beta : T(b)$ in the fibers. We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

Lastly we want to recall that in the presence of a section $X : M \to P$ there is a dependent generalization of ap called apd: $\mathsf{apd}(X)(p) : \mathsf{tr}(p)(X(a)) = X(b)$ which is a pathover between the two values of the section over the basepoints of the path p.

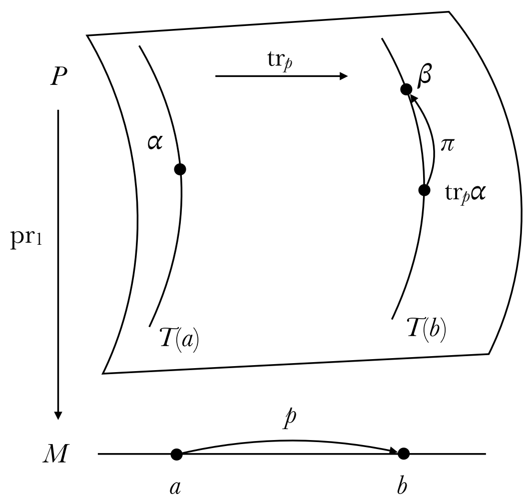


Figure 1: A path π over the path p in the base involves the transport function.

3 Combinatorial manifolds

3.1 Combinatorial manifolds

We will adapt to higher inductive types in a straightforward manner the classical construction of *combinatorial manifolds*. See for example the classic book by Kirby and Siebenmann[9]. These are a subclass of simplicial complexes.

Definition 3.1. An **abstract simplicial complex** M **of dimension** n consists of a set M_0 of vertices, and for each $0 < k \leq n$ a set M_k of subsets of M_0 of cardinality $k + 1$, such that any $(j + 1)$-element subset of M_k is an element of M_j. The elements of M_k are called k**-faces**. Denote by $\mathsf{SimpCompSet_n}$ the type of abstract simplicial complexes of dimension n (where the suffix Set reminds us that this is a type of sets).

Note that we don't require all subsets of M_0 to be included – that would make M an individual simplex. A simplicial complex is a family of simplices that are identified along various faces.

Definition 3.2. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $n - 1$-face containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

The link is all the neighboring vertices of v and the codimension 1 faces joining those to each other. See for example Figure 2.

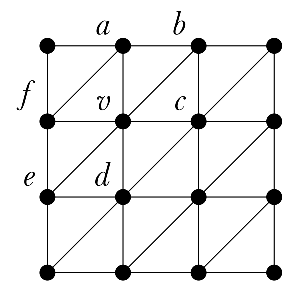


Figure 2: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

Definition 3.3. A **combinatorial manifold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a simplicial sphere of dimension $n - 1$ (i.e. its geometric realization is homeomorphic to an $n - 1$-sphere). Denote by $\mathsf{CombMfdSet_n}$ the type of combinatorial manifolds of dimension n (which the notation again reminds us are sets).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figures 3, 4, and 5 for some examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits a combinatorial triangulation[10]. So it appears reasonably well motivated to study this class of objects.

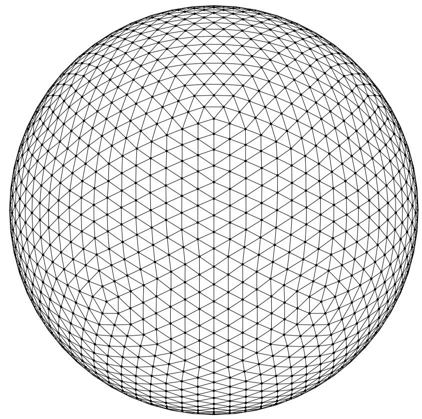


Figure 3: A combinatorial triangulation of a sphere, created with the tool `stripy`.

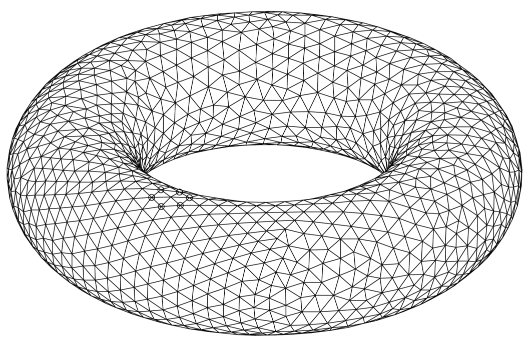


Figure 4: A torus with an interesting triangulation, from Wikipedia. The links have various vertex counts from 5-7. Clearly a constant value of 6 would also work. (By Ag2gaeh - Own work, CC BY-SA 3.0.)

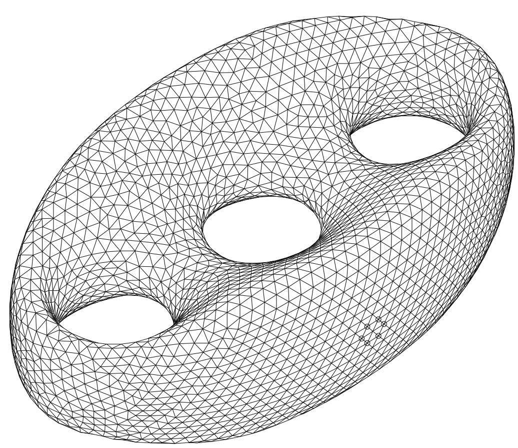


Figure 5: A 3-holed torus with triangulation, from Wikipedia. (By Ag2gaeh - Own work, CC BY-SA 3.0.)

3.2 Higher inductive combinatorial manifolds

We will convert a simplicial complex M of dimension at most 2 to a higher inductive type, in two steps.

Definition 3.4. Define $\mathsf{CombMfd}_2$ to be the type of **higher inductive constructors of combinatorial manifolds of dimension at most 2** and let $\mathcal{H} : \mathsf{CombMfdSet}_2 \to \mathsf{CombMfd}_2$ be a map from a combinatorial manifold to such a HIT following this method:

1. vertices: a function $\mathsf{v}_0 : M_0 \to \mathcal{H}(M)$ serving as the 0-dimensional constructors
2. edges: a function v_1 on 1-faces, sending $\{a, b\} \mapsto \mathsf{v}_0(a) = \mathsf{v}_0(b)$
3. 2-faces: a function v_2 on 2-faces, sending $\{a, b, c\} \mapsto \mathsf{refl}_a = \mathsf{v}_1(\{a, b\}) \cdot \mathsf{v}_1(\{b, c\}) \cdot \mathsf{v}_1(\{a, c\})^{-1}$.

We will assume there is a formal theory of such HITs, and that at least up to dimension 2 there are no obstructions to simply copying over the combinatorial data to the HIT constructors. For recent work on HITs see for example David Wärn's discussion of pushouts[11].

Definition 3.5. Denote by $\mathcal{R} : \mathsf{CombMfd}_2 \to \mathsf{Type}$ the process of generating a type from the HIT data (which we refer to as **realization**). Note that $\mathcal{R}(\mathcal{H}(M))$ is not in general a set, and may not even be 2-truncated for an arbitrary 2-dimensional combinatorial manifold $M : \mathsf{CombMfdSet}_2$.

We're making the distinction between $\mathcal{H}$ and $\mathcal{R}$ because we will mostly study functions and phenomena on $\mathcal{H}(M)$ for some simplicial complex M.

3.3 Polygons

We will now start looking at some examples, first by defining a type that is important both for the domain and the codomain of mere circles: a square.

Definition 3.6. The higher inductive type C_4 (where C stands for "circle"). See Figure 6.

$$\begin{aligned} C_4 &: \mathsf{Type} \\ c_1, c_2, c_3, c_4 &: C_4 \\ c_1c_2 &: c_1 = c_2 \\ c_2c_3 &: c_2 = c_3 \\ c_3c_4 &: c_3 = c_4 \\ c_4c_1 &: c_4 = c_1 \end{aligned}$$

The standard HoTT circle itself is a non-example of a combinatorial manifold since it lacks the second vertex of the edge:

Definition 3.7. The higher inductive type S^1 which we can also call C_1:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

$$
\begin{tikzcd}
c_1 \arrow[r, "c_1c_2"] & c_2 \arrow[d, "c_2c_3"] \\
c_4 \arrow[u, "c_4c_1"] & c_3 \arrow[l, "c_3c_4"]
\end{tikzcd}
$$

Figure 6: The HIT C_4.

Both of these are examples of a family of HIT data we will call Gon, the type of n-gon HITs for some natural number n, following the pattern of the HITs above. We'll see below that the realization of an n-gon is a mere circle, i.e. we have $\mathcal{R} : \mathsf{Gon} \to \mathrm{EM}(\mathbb{Z}, 1)$.

3.4 Adding and removing points from polygons

Given functions $\phi, \psi : A \to B$ between two arbitrary types we can form the type family of paths $\alpha : A \to \mathcal{U}, \alpha(a) \stackrel{\text{def}}{=} (\phi(a) =_B \psi(a))$. Transport in this family is given by concatenation as follows, where $p : a =_A a'$ and $q : \phi(a) = \psi(a)$ (see Figure 7):

$$\mathsf{tr}(p)(q) = \phi(p)^{-1} \cdot q \cdot \psi(p)$$

which gives a path in $\phi(a') = \psi(a')$ by connecting dots between the terms $\phi(a'), \phi(a), \psi(a), \psi(a')$. This relates a would-be homotopy $\phi \sim \psi$ specified at a single point, to a point at the end of a path. We will use this to help construct such homotopies.

$$
\begin{tikzcd}
\phi(a) \arrow[d, "q"'] & \phi(a') \arrow[l, "\phi(p)^{-1}"'] \\
\psi(a) \arrow[r, "\psi(p)"] & \psi(a') \\
a \arrow[r, "p"] & a'
\end{tikzcd}
$$

Figure 7: Transport along p in the fibers of a family of paths. The fiber over a is $\phi(a) = \psi(a)$ where $\phi, \psi : A \to B$.

Lemma 3.1. Let C_n be the polygon 1-dimensional HIT with n vertices. Then $C_2 \simeq C_1$ and in fact $C_n \simeq C_{n-1}$.

Proof. (Compare to [12] Lemma 6.5.1.) In the case of C_1 we will denote its constructors by base and loop. For C_2 we will denote the points by v_1, v_2 and the edges by ℓ_{12}, r_{21}. For C_3 and higher we will denote the points by $v_1, \ldots, v_n$ and the edges by $e_{i,j} : v_i = v_j$ where $j = i + 1$ except for $e_{n,1}$.

First we will define $f : C_2 \to C_1$ and $g : C_1 \to C_2$, then prove they are inverses.

$$
\begin{aligned}
f(v_1) = f(v_2) &= \mathsf{base} & \qquad g(\mathsf{base}) &= v_1 \\
f(\ell_{12}) &= \mathsf{loop} & g(\mathsf{loop}) &= \ell_{12} \cdot r_{21} \\
f(r_{21}) &= \mathsf{refl}_{\mathsf{base}}
\end{aligned}
$$

We need to show that $f \circ g \sim \mathrm{id}_{C_1}$ and $g \circ f \sim \mathrm{id}_{C_2}$. Think of f as sliding v_2 along r_{21} to coalesce with v_1. This may help understand why the unfortunately intricate proof is working.

We need terms $p : \prod_{a:C_1} f(g(a)) = a$ and $q : \prod_{a:C_2} g(f(a)) = a$. We will proceed by induction, defining appropriate paths on point constructors and then checking a condition on path constructors that confirms that the built-in transport of these type families respects the definition on points.

Looking first at $g \circ f$, which shrinks r_{21}, we have the following data to work with:

$$
\begin{aligned}
g(f(v_1)) = g(f(v_2)) &= v_1 \\
g(f(\ell_{12})) &= \ell_{12} \cdot r_{21} \\
g(f(r_{21})) &= \mathsf{refl}_{v_1}.
\end{aligned}
$$

We then need to supply a homotopy from this data to id_{C_2}, which consists of a section and pathovers over C_2:

$$
\begin{aligned}
p_1 &: g(f(v_1)) = v_1 \\
p_2 &: g(f(v_1)) = v_2 \\
H_\ell &: \mathsf{tr}(\ell_{12})(p_1) = p_2 \\
H_r &: \mathsf{tr}(r_{21})(p_2) = p_1.
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
p_1 &: v_1 = v_1 \\
p_2 &: v_1 = v_2 \\
H_\ell &: g(f(\ell_{12}))^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\
H_r &:= g(f(r_{21}))^{-1} \cdot p_2 \cdot r_{21} = p_1
\end{aligned}
$$

and then to

$$
\begin{aligned}
p_1 &: v_1 = v_1 \\
p_2 &: v_1 = v_2 \\
H_\ell &: (\ell_{12} \cdot r_{21})^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\
H_r &: \mathsf{refl}_{v_1} \cdot p_2 \cdot r_{21} = p_1
\end{aligned}
$$

To solve all of these constraints we can choose $p_1 \stackrel{\text{def}}{=} \mathsf{refl}_{v_1}$, which by consulting either H_ℓ or H_r requires that we take $p_2 \stackrel{\text{def}}{=} {r_{21}}^{-1}$.

Now examining $f \circ g$, we have

$$
\begin{aligned}
f(g(\mathsf{base})) &= \mathsf{base} \\
f(g(\mathsf{loop})) &= f(\ell_{12} \cdot r_{21}) = \mathsf{loop}
\end{aligned}
$$

and so we have an easy proof that this is the identity.

The proof of the more general case $C_n \simeq C_{n-1}$ is very similar. Take the maps $f : C_n \to C_{n-1}$, $g : C_{n-1} \to C_n$ to be

$$
\begin{aligned}
f(v_i) &= v_i \quad (i = 1, \ldots, n-1) & g(v_i) &= v_i & (i = 1, \ldots, n-1) \\
f(v_n) &= v_1 & g(e_{i,i+1}) &= e_{i,i+1} & (i = 1, \ldots, n-2) \\
f(e_{i,i+1}) &= e_{i,i+1} \quad (i = 1, \ldots, n-1) & g(e_{n-1,1}) &= e_{n-1,n} \cdot e_{n,1} & \\
f(e_{n-1,n}) &= e_{n-1,1} & & & \\
f(e_{n,1}) &= \mathsf{refl}_{v_1} & & &
\end{aligned}
$$

where f should be thought of as shrinking $e_{n,1}$ so that v_n coalesces into v_1.

The proof that $g \circ f \sim \mathrm{id}_{C_n}$ proceeds as follows: the composition is definitionally the identity except

$$
\begin{aligned}
g(f(v_n)) &= v_1 \\
g(f(e_{n-1,n})) &= e_{n-1,n} \cdot e_{n,1} \\
g(f(e_{n,1})) &= \mathsf{refl}_{v_1}.
\end{aligned}
$$

Guided by our previous experience we choose ${e_{n,1}}^{-1} : g(f(v_n)) = v_n$, and define the pathovers by transport.

The proof that $f \circ g \sim \mathrm{id}_{C_{n-1}}$ requires only noting that $f(g(e_{n-1,1})) = f(e_{n-1,n} \cdot e_{n,1}) = e_{n-1,1} \cdot \mathsf{refl}_{v_1} = e_{n-1,1}$. □

Corollary 3.1. All polygons are equivalent to S^1, i.e. we have a term in $\prod_{n:\mathbb{N}} ||C_n = S^1||$, and hence Gon is a subtype of $\mathrm{EM}(\mathbb{Z}, 1)$.

Proof. We can add $n-1$ points to S^1 and use Lemma 3.1. □

Definition 3.8. For $k : \mathbb{N}$ define $m_k : \mathsf{Gon} \to \mathsf{Gon}$ where $m_k : C_n \to C_{kn}$ adds k vertices between each of the original verticies of C_n.

With m_k we can start with a surface that has pentagonal plus hexagonal links, and apply m_6 to the pentagons and m_5 to the hexagons, and have a type family that is the tangent bundle (because it comes from the link) but where every fiber has 30 vertices.

3.5 The higher inductive type $\mathbb{O}$

We will create our first combinatorial surface, a 2-sphere. We will adopt the convention that a subscript indicates the dimension of a subskeleton of a complex. For instance, we have $\mathsf{base} : S^1_0$.

Definition 3.9. The HIT $\mathbb{O}_0$ is just 6 points, intended as the 0-skeleton of an octahedron, with vertices named after the colors on the faces of a famous Central European puzzle cube.

$$w, y, b, r, g, o : \mathbb{O}_0$$

Definition 3.10. The HIT $\mathbb{O}_1$ is the 1-skeleton of an octahedron.

$$\begin{array}{lll}
w, y, b, r, g, o : \mathbb{O}_1 & & yg : y = g \\
& wb : w = b & yo : y = o \\
& wr : w = r & br : b = r \\
& wg : w = g & rg : r = g \\
& wo : w = o & go : g = o \\
& yb : y = b & ob : o = b \\
& yr : y = r &
\end{array}$$

Definition 3.11. The HIT $\mathbb{O}$ is an octahedron:

$$\begin{array}{llll}
w, y, b, r, g, o : \mathbb{O} & & & \\
& wb : w = b & br : b = r & wbr : wb \cdot br \cdot wr^{-1} = \mathsf{refl}_w \\
& wr : w = r & rg : r = g & wrg : wr \cdot rg \cdot wg^{-1} = \mathsf{refl}_w \\
& wg : w = g & go : g = o & wgo : wg \cdot go \cdot wo^{-1} = \mathsf{refl}_w \\
& wo : w = o & ob : o = b & wob : wo \cdot ob \cdot wb^{-1} = \mathsf{refl}_w \\
& yb : y = b & & yrb : yr \cdot rb \cdot yb^{-1} = \mathsf{refl}_y \\
& yr : y = r & & ygr : yg \cdot gr \cdot yr^{-1} = \mathsf{refl}_y \\
& yg : y = g & & yog : yo \cdot og \cdot yg^{-1} = \mathsf{refl}_y \\
& yo : y = o & & ybo : yb \cdot bo \cdot yo^{-1} = \mathsf{refl}_y
\end{array}$$

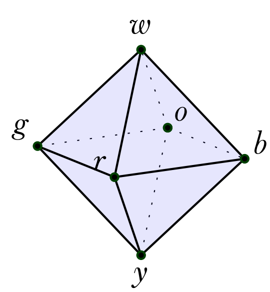


Figure 8: The HIT $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

We have obvious maps $\mathbb{O}_0 \xrightarrow{i_0} \mathbb{O}_1 \xrightarrow{i_1} \mathbb{O}$ that include each skeleton into the next-higher-dimensional skeleton.

4 Connections and vector fields

4.1 The function T

We will build up a map T out of $\mathbb{O}$ which is meant to be like the circle bundle of a tangent bundle. And so we will begin with the intrinsic data of the link at each point: taking the link of a vertex gives us a map from vertices to polygons.

Definition 4.1. $T_0 \overset{\text{def}}{=} \mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned} \mathsf{link}(w) &= brgo & \qquad \mathsf{link}(r) &= wbyg \\ \mathsf{link}(y) &= bogr & \qquad \mathsf{link}(g) &= wryo \\ \mathsf{link}(b) &= woyr & \qquad \mathsf{link}(o) &= wgyb \end{aligned}$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

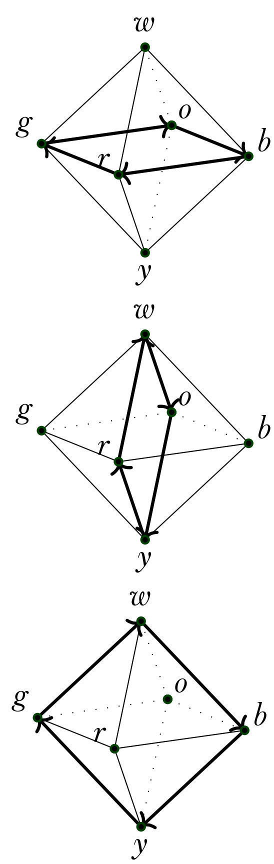


Figure 9: link for the vertices w, b and r.

To extend T_0 to a function T_1 on the 1-skeleton we have complete freedom. Defining a map by induction makes clear that the action on paths is its own structure. Two functions on the octahedron could agree on points but differ on edges. We are going to identify this 1-dimensional freedom with a connection:

Definition 4.2. A **connection** on a higher combinatorial manifold is an extension of a circle bundle from the 0-skeleton to the 1-skeleton.

Continuing the example, we will do something "tangent bundley", imagining how T_1 changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 4.3. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ (r, o fixed)
- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ (b, g fixed)
- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$
- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 9 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn.

Definition 4.4. The map $R : C_4 \to C_4$ rotates by one quarter turn, one "click":

- $R(c_1) = c_2$
- $R(c_2) = c_3$
- $R(c_3) = c_4$
- $R(c_4) = c_1$
- $R(c_1c_2) = c_2c_3$
- $R(c_2c_3) = c_3c_4$
- $R(c_3c_4) = c_4c_1$
- $R(c_4c_1) = c_1c_2$

Note by univalence the equivalence R gives a loop in the universe, a term of $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here *there is no additional freedom*.

Definition 4.5. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows:

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

where $H_R : R = \mathsf{refl}_{C_4}$ is the obvious homotopy given by composition with R^{-1}. Passing through univalence we get a 2-path between R and refl in the loop space $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

Definition 4.6. The **curvature of a connection** on a type family $T : \mathbb{M} \to \mathcal{U}$ at a vertex v of a 2-face f with boundary path p_f of a higher combinatorial manifold $\mathbb{M}$ is the automorphism $\mathsf{tr}(p_f)(Tv)$ together with a homotopy to id_{Tv}.

Note 4.1. We have defined a function on a cell by requiring it to correspond to the value on the boundary of that cell. This is familiar in classical differential topology, where it's called *the exterior derivative*. The duality of d and ∂ is recognizable in T_2, and we might say "curvature is the derivative of the connection."

4.2 T on concatenations of faces

Consider Figure 10 and the diamond on the left. Denote the clockwise triangular path around the left triangle by $bcab$, and around the right by $bdcb$. These are loops at b. Say that the tangent map T satisfies $T(bcab) = R$ where R is some rotation of the 4-sided polygon Tb, and similarly $T(bdcb) = R$ as well.

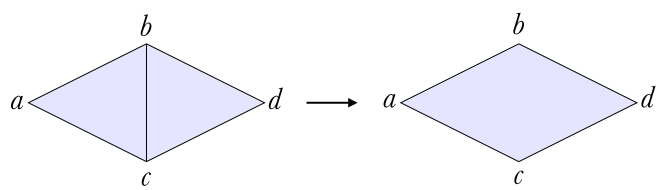


Figure 10: Concatenating the triangles bac and bdc gives the 4-gon $abdc$.

Since $bcab$ has a filler 2-cell, which we will call f_1, we know that $T(f_1) = H_R$ where $H_R : R = \mathsf{refl}$ is a path in $\mathrm{Aut}(Tb)$ from R to the identity. Similarly if f_2 is the filler for $bdcb$ then $T(f_2) = H_R$.

The two faces can be horizontally composed giving $f_1 \star f_2$, which fills the diamond $bdcab$. Then we also obtain $T(f_1 \star f_2) : R^2 = \mathrm{id}$.

In this way, given an explicit ordering of the faces (a "plan" for visiting all faces of the manifold), we can "sum" the curvature. For our ocatahedron where each triangle produces a rotation of R, we'd get R^8.

4.3 The torus

We can define a combinatorial torus as a similar HIT. This time each vertex will have six neighbors. So all the links will be merely equal to C_6 which is a hexagonal version of C_4. See Figure 11.

To help parse this figure, imagine instead Figure 12. We take this simple alternating-triangle pattern, then glue the left and right edges, then bend into Figure 11. The fact that each column in Figure 12 has four dots corresponds to the torus in Figure 11 having a square in front, diamonds in the middle, and a square in back.

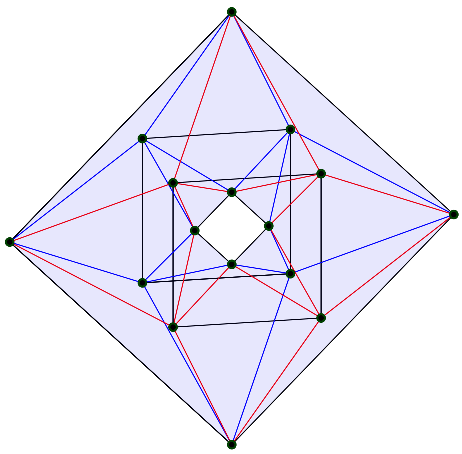


Figure 11: Torus embedded in 3-dimensional space. If you see color in your rendering then black lines trace four square-shaped paths, red ones connect the front square to the middle diamonds, and blue ones connect the back path to the middle ones.

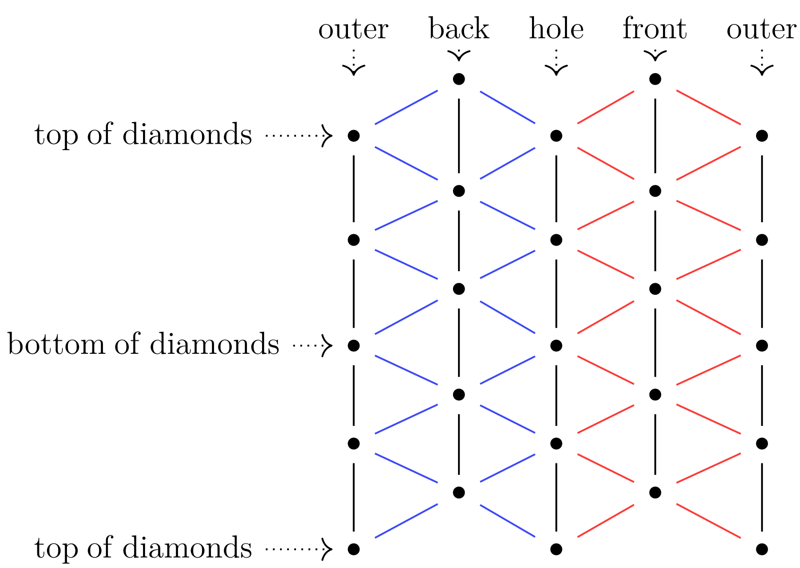


Figure 12: An inspiration for the torus. Identify the sides and then the top, definitionally, to get the actual torus.

This somewhat arbitrary and unfamiliar model of a torus has the helpful property that it is a combinatorial manifold that is somewhat minimal while still being representable by a donut shape. But the donut-shaped version suggests a very different connection than the flat model! Starting with the flat model, we can easily see how to define T_1 by sliding a link rigidly along the page to the link of some adjacent vertex. Then we can see that transport around any loop is the identity and so T_2 is always the identity (together with the homotopy $\mathsf{refl}_{\mathsf{id}}$ from the identity to itself). So if we imagine a way to visit every face like we did for the octahedron, starting and ending at some basepoint v, we expect to see no net rotation at all of Tv. Later we will call this "total curvature 0."

The donut-shaped torus suggests a different connection, one determined by the embedding in 3-space that we have represented. But the easiest way to think about that bundle and its connection and curvature is to wait until we have a proof of the Poincaré-Hopf theorem, so that we can instead compute with a vector field.

4.4 Existence of connections

How confident can we be that we can always define a connection on an arbitrary combinatorial manifold? Two things make the octahedron example special: the link is a 4-gon at every vertex, and every edge extends to a symmetry of the entire octahedron, embedded in 3-dimensional space. This imposed a coherence on the interactions of all the choices we made for the connection, which we can worry may not exist for more complex combinatorial data.

We know as a fact outside of HoTT that any combinatorial surface that has been realized as a triangulated surface embedded in 3-dimensional euclidean space can inherit the parallel transport entailed in the embedding. We could then approximate that data to arbitrary precision with enough subdivision of the fibers of T.

What would a proof inside of HoTT look like? We will leave this as an open question.

5 Vector fields

Definition 5.1. A **partial function** $f : A \to B$ is a function $f : A \to B + \star$, the disjoint union of B with the 1-element type.

If $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ is a bundle of mere circles, then a vector field should be a partial function $T_\bullet : \mathbb{M} \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ that lifts T. In other words, a pointing of *some* of the fibers. This aligns with the classical picture of a choice of nonzero vector at each point, except for some points where the vector field vanishes. So instead of having a notion corresponding to the full tangent vector space (one candidate for which would be the disk at each point, i.e. link plus its spokes and filler triangles) we are mapping some vertices to their circular fibers, and others to $\star$. This lets us continue to work with $\mathrm{EM}(\mathbb{Z}, 1)$ instead of a type of tangent spaces.

Figure 13 illustrates what removing the preimage of $\star$ looks like. The resulting type is no longer a combinatorial manifold, since it fails the condition about every point having a circular link.

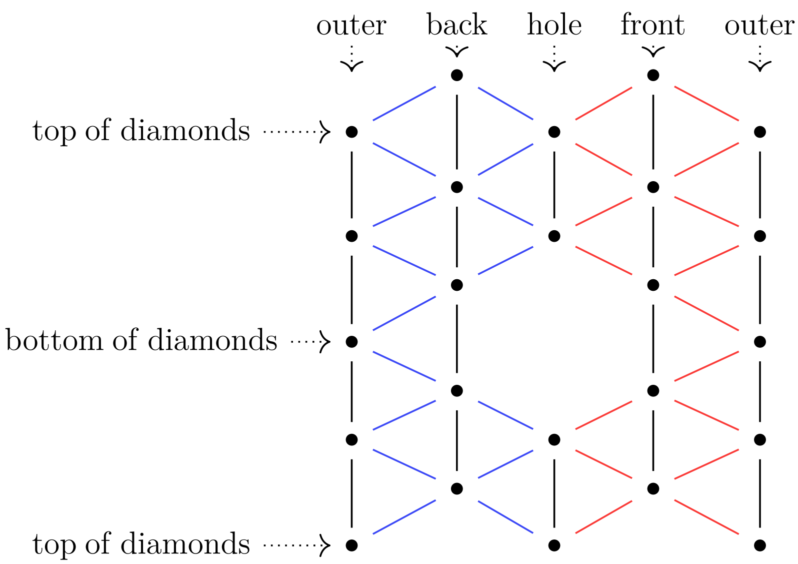


Figure 13: The flat torus with one vertex removed. This also removes the edges and faces containing that vertex.

Definition 5.2. Let $\mathbb{M} : \mathsf{CombMfd}_2$ be a combinatorial manifold and Z an isolated set of vertices. A **vector field** X **on** $\mathbb{M}$ **with zero set** Z is a partial section of P, i.e. a term $X : \prod_{x:\mathbb{M}\setminus Z} T(x)$ (and eliding the unique term of $Z \to \star$). The **exponential map** $\exp : P \to \mathbb{M}$ is the map sending points in a fiber to the corresponding point in the link of the base point: $\exp(x, y : \mathsf{link}(x)) = y$. In commutative diagram form we have:

$$\begin{tikzcd}
P \overset{\text{def}}{=} \sum_{C:T(\mathbb{M}\setminus Z)} C \arrow[r, "\overline{T}"] \arrow[d, shift left, "\mathrm{pr}_1"] \arrow[d, bend left, "\exp"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"] \\
\mathbb{M} \setminus Z \arrow[u, bend left, "X:\prod_{x:\mathbb{M}\setminus Z} Tx"] \arrow[ur, "T_\bullet"] \arrow[r, "T"] & \mathrm{EM}(\mathbb{Z}, 1)
\end{tikzcd}$$

where $T_\bullet = \overline{T} \circ X$. Note that exp is different from pr_1 since it spreads a fiber out onto the manifold. The composition $\exp \circ X$ is a map $\mathbb{M} \setminus Z \to \mathbb{M}$, and can be thought of as the flow of the vector field.

Let's see a few examples.

Definition 5.3. The **spinning vector field** X_{spin} on $\mathbb{O} \setminus \{w, y\}$ is given by the following data. We compose with exp to keep the notation directly in $\mathbb{O}$. See Figure 14

$$\begin{aligned}
\exp \circ X_{\text{spin}}(b) &= r\\
\exp \circ X_{\text{spin}}(r) &= g\\
\exp \circ X_{\text{spin}}(g) &= o\\
\exp \circ X_{\text{spin}}(o) &= b
\end{aligned}$$

We must also define pathovers and faceovers, i.e. $\mathsf{apd}(X_{\text{spin}}$. For example, $X_{\text{spin}}(b)$ is the point r in the link $woyr$. Transport along br takes the link of b to the link of r, mapping $r : Tb$ to $g : Tr$. This agrees with $X_{\text{spin}}(r)$ and so we will choose $X_{\text{spin}}(br) = \mathsf{refl}_g$ in Tr. We similarly choose refl pathovers for the other equatorial edges. And since we have deleted all the faces when removing the zeros, there are no faceovers.

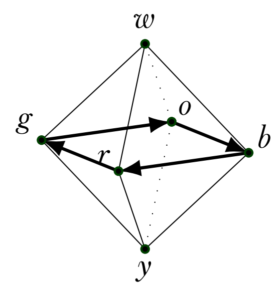


Figure 14: The vector field X_{spin} on $\mathbb{O}$, which circulates around the equator. w and y are zeros.

Definition 5.4. The **downward vector field** X_{down} on $\mathbb{O} \setminus \{w, y\}$ is given by the following data, where again we compose with exp to keep the notation directly in $\mathbb{O}$. See Figure 15

$$\begin{aligned}
\exp \circ X_{\text{spin}}(b) &= y\\
\exp \circ X_{\text{spin}}(r) &= y\\
\exp \circ X_{\text{spin}}(g) &= y\\
\exp \circ X_{\text{spin}}(o) &= y
\end{aligned}$$

We also need to select a pathover for each edge on the equator. Transport on all these edges takes y in one fiber to y in the next, so we choose the path refl_y in all four of these fibers. Again there are no faceovers to map.

5.1 Index of a vector field

Index should be an integer that computes a winding number "of the vector field" around a zero. We can compute an integer from a map by taking its *degree*, which is a construction we will assume

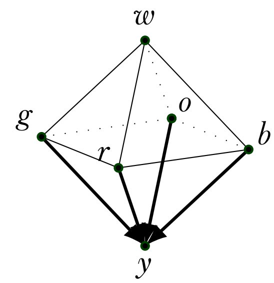


Figure 15: The vector field X_{down} on $\mathbb{O}$, which flows downward. w and y are zeros.

that we have, for example using [13], where they indeed require that degree agrees with winding number for maps $S^1 \to S^1$.

Definition 5.5. Let $\mathbb{M}$: $\mathsf{CombMfd}_2$ and let $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ be a bundle of circles given on $\mathbb{M}_0$ by link. Let $z : Z$ be an isolated zero and let $\mathsf{link}\, z$ be its polygonal link in $\mathbb{M}$, with a clockwise orientation, say with ordered vertices $\{l_{z1}, \ldots, l_{zn}\}$. We call the degree of the map $\mathsf{tr}(\mathsf{link}\, z) : Tl_{z1} = Tl_{z1}$ the **index of** X **at** z. It does not depend on which vertex we use.

Lemma 5.1. The index of X_{spin} at both y and at w is 1, and the same is true for X_{down}.

Proof. $\mathsf{apd}(X_{\text{spin}})(br) = \mathsf{refl}_{X_{\text{spin}}(r)}$ and similarly for the other edges and for X_{down}. So apd on the loop around the equator is the identity, which has index 1. □

If these vector fields both have index $+1$, what does index -1 look like? We can see an example on the torus with its downward vector field.

On the torus we can also consider both a spinning vector field and a downward vector field. Figure 16 shows one way to spin the torus, and in this case there are no zeros so the index is the degree of a map from the empty set, which is 0 (as it factors through a constant map).

Figure 17 shows a downward flow with four zeros. Although this is a picture of the flat torus, the vector field is derived from the shape of Figure 11 where we actually have a notion of up and down. We see at the position labeled (outer, top of diamonds), i.e. the top of the torus, an everywhere outward pointing vector field. At (outer, bottom of diamonds) we see an inward pointing vector field. But at (hole, top of diamonds), i.e. the top of the hole, we see something else. Imagine transport from the neighbor to the lower right to the neighbor below, then continuing clockwise around the link. If we assume that we define apd on these edges to be *counterclockwise* rotation, then transport around the whole link has degree -1. Similarly for the zero at (hole, bottom of diamonds). Adding these four indexes we again get 0.

Summarizing what we've seen in our examples, vector fields with isolated zeros have an index, and this index tracks with the total curvature, and the Euler characteristic.

5.2 Equality of total index and total curvature

Here we are inspired by the classical proof of Hopf[4], presented in detail in Needham[5].

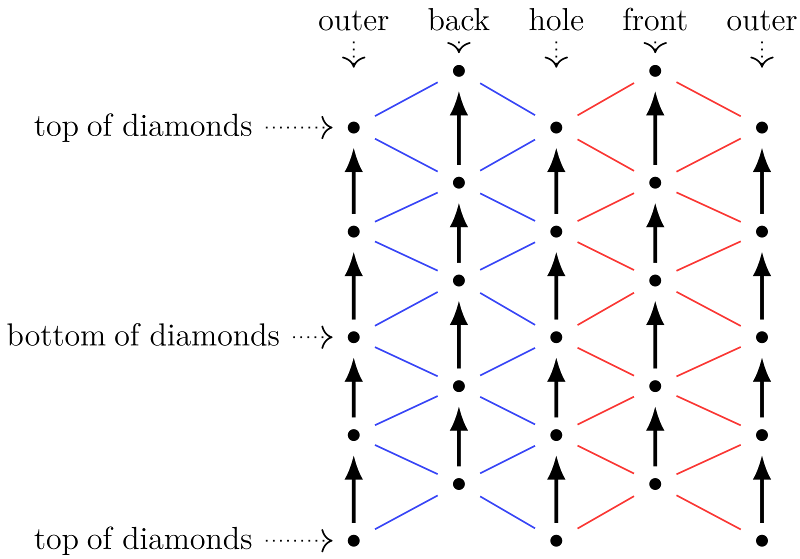


Figure 16: A vector field on the torus that spins and has no zeros.

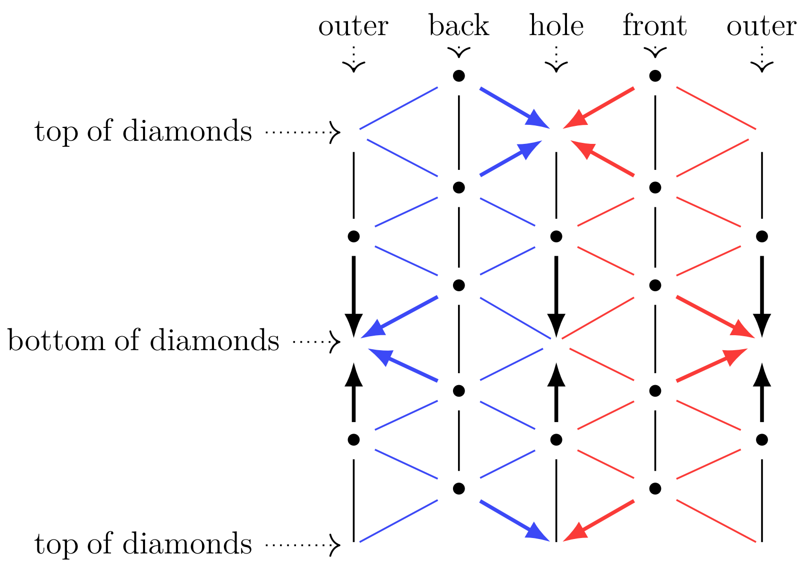


Figure 17: A vector field on the torus that flows downward, when viewed as a donut. The zeros are represented as missing dots. Every dot has one outgoing vector.

Definition 5.6. An **enumeration** of a principal bundle with connection and vector field on a higher combinatorial manifold consists of the following data:

- a family $T : \mathbb{M} \to \mathrm{K}(\mathbb{Z}, 2)$ on some higher combinatorial manifold
- a nonvanishing vector field $X : \mathbb{M} \setminus Z \to P$ with isolated zeros Z
- a total face of the replacement $\mathbb{M}_Z$ (Definition **??**), that is
  - a basepoint $a : \mathbb{M}_Z$
  - a term $f_{\mathbb{M}_Z} : \mathsf{refl}_a = \mathsf{refl}_a$ given by
    - an ordering of the face constructors $\{f_i\}$, with the sub-list of faces denoted $\{f_{zk}\}$ refers to the replacement faces at the zeros
    - a vertex v_i in each face
    - terms $a = v_i$ for each face

Note 5.1. An enumeration let us work both with $\mathbb{M} \setminus Z$ where the vector field is nonvanishing, while also having access to the disks that are missing from $\mathbb{M} \setminus Z$.

Lemma 5.2. The sub-list of faces $\{f_i\} - \{f_{zk}\}$ obtained by skipping the replacement faces at the zeros, is an ordering of face constructors for $\mathbb{M} \setminus Z$.

Proof. The algorithm that visits each face in order always starts and ends at a and so we can skip any faces we wish, to obtain an ordering of face constructors for the remaining union of faces. □

Note that on $\mathbb{M} \setminus Z$ the vector field X trivializes the bundle by mapping into the contractible type of pointed types over $\mathrm{K}(\mathbb{Z}, 2)$. So $X \simeq Y : \mathbb{M} \setminus Z \to (T\,a, a)$, the fixed pointed circle in the fiber of the basepoint a.

Lemma 5.3. The degree of Y is minus the total index of X.

Proof. The ordering of faces $\{f_i\} - \{f_{zk}\}$ provides an ordering of all the edges, say $\{e_l\}$. Each edge appears an even number of times in this list, traversed in opposite directions, except those bounding a replacement face. These replacement-bounding edges are traversed an odd number of times and can be concatenated to traverse the boundary counterclockwise. Concatenation of paths in S^1 is abelian, so $Y(\{e_l\})$ cancels except on the boundary of the replacement faces, which gives a map from the disjoint union of boundary circles to $T\,a$, with each boundary circle traversed in the counterclockwise direction. The orientation gives the minus sign. □

Consider now the total face $f_{\mathbb{M}_Z}$ of $\mathbb{M}_Z$ and its ordering of faces $\{f_i\}$. Y is only defined on some of these faces. We will define a new function on all the $\{f_i\}$.

Definition 5.7. The **coupling map** $C : \mathbb{M}_Z \to T\,a$ is defined to be Y on $\mathbb{M} \setminus Z$ and on the remaining faces it is defined by $C(f_i) \stackrel{\text{def}}{=} \mathsf{apd}(X)(\partial f_i)$ where $\partial f_i : v_i = v_i$ is the clockwise path around the face starting from the vertex supplied by the data of the total face.

Because apd uses both transport and the value of the vector field, it couples the connection with the vector field, hence the name. Of course in HoTT this coupling is built into the definition of apd, so that's another reminder that we aren't straying far from the foundations to find these geometrical concepts.

The fact that C is defined on all the faces, by using the value of the vector field only on the 1-skeleton of $\mathbb{M}_Z$ where it was always defined, lets us make the final part of the argument.

Lemma 5.4. $C : \cup_i\{f_i\} \to Ta$ is equal to a constant map.

Proof. The data of the total face provides an ordering of all the edges. Each edge appears an even number of times, traversed in opposite directions, including the edges in the replacement faces. Concatenation of paths in the polygon Ta is abelian, so the paths all cancel. □

C is similar to X and Y except that it is a total function. On any given edge it computes a path, that is, a homotopy from the function tr to the function X, which we can call "the difference between transport and the vector field on that edge." We have found a way to add all these homotopies together to get 0. We can call this total "the difference between the total index and the total curvature."

Recall now that we made some arbitrary choices in Definition 5.6 of an enumeration, and hence the function C. But since C is unconditionally constant, the space of extra data is contractible.

Corollary 5.1. The total index of a vector field with isolated zeros is independent of the vector field.

Corollary 5.2. The total curvature is an integer.

The last step is to link this value to the Euler characteristic.

5.3 Identification with Euler characteristic

Here we only point the way. Combinatorial manifolds are intuitive objects that connect directly to the classical definition of Euler characteristic. We can argue using Morse theory, the study of smooth real-valued functions on smooth manifolds and their singularities. Classically the gradient of a Morse function is a vector field that can be used to decompose the manifold into its *handlebody decomposition*. This would be an excellent story to pursue in future work.

Imagine starting with a classical 2-manifold of genus g that has been triangulated. Stand it upright with the holes forming a vertical sequence. Now install a vector field that points downward whenever possible. This vector field will have a zero at the top and bottom, and one at the top and bottom of each hole. The top and bottom will have zeros of (classical) index 1, and zeros in the holes will have index -1. We include some sketches in the case of a torus (Figure 18). This illustrates how we obtain the formula for genus g: $\chi(M) = 2 - 2g$. If we choose the triangulation so that the zeros are at vertices, we should be able to import that data into $\mathsf{CombMfd}_2$ and reproduce the computation using the tools in this note.

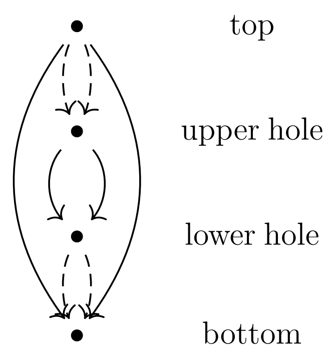


Figure 18: Schematic of the zeros of the downward flow of a torus.

Bibliography

[1] M. Shulman, "Brouwer's fixed-point theorem in real-cohesive homotopy type theory," 2017.

[2] S. Talbott, *The Future Does Not Compute: Transcending the Machines in Our Midst*. O'Reilly & Associates, 1995. [Online]. Available: https://books.google.com/books?id=KcXaAAAAMAAJ

[3] D. Christensen, "A characterization of univalent fibrations," https://jdc.math.uwo.ca/papers/univalence-ct2015.pdf, 2015, accessed: 2024–11-01.

[4] H. Hopf, "Differential geometry in the large: Seminar lectures, new york university, 1946 and stanford university, 1956," 1983. [Online]. Available: https://api.semanticscholar.org/CorpusID:117042538

[5] T. Needham, *Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts*. Princeton University Press, 2021. [Online]. Available: https://books.google.com/books?id=Mc0QEAAAQBAJ

[6] K. Crane, F. de Goes, M. Desbrun, and P. Schröder, "Digital geometry processing with discrete exterior calculus," in *ACM SIGGRAPH 2013 courses*, ser. SIGGRAPH '13. New York, NY, USA: ACM, 2013. [Online]. Available: https://www.cs.cmu.edu/~kmcrane/Projects/DDG/

[7] K. Crane, "Discrete connections for geometry processing," Master's thesis, California Institute of Technology, 2010. [Online]. Available: http://resolver.caltech.edu/CaltechTHESIS:05282010-102307125

[8] L. Scoccola, "Nilpotent types and fracture squares in homotopy type theory," *Mathematical Structures in Computer Science*, vol. 30, no. 5, p. 511–544, May 2020. [Online]. Available: http://dx.doi.org/10.1017/S0960129520000146

[9] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, ser. Annals of Mathematics Studies. Princeton University Press, 1977, no. 88, with notes by John Milnor and Michael Atiyah. MR:0645390. Zbl:0361.57004.

[10] J. H. C. Whitehead, "On C^1-complexes," *Annals of Mathematics*, pp. 809–824, 1940.

[11] D. Wärn, "Path spaces of pushouts," 2024. [Online]. Available: https://arxiv.org/abs/2402.12339

[12] Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[13] U. Buchholtz and K. H. (Favonia), "Cellular cohomology in homotopy type theory," *CoRR*, vol. abs/1802.02191, 2018. [Online]. Available: http://arxiv.org/abs/1802.02191