Discrete differential geometry in homotopy type theory

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Abstract

Higher inductive types can capture some concepts of differential geometry in two dimensions including connections, curvature, and vector fields. We define connections on higher inductive types. We then define tangent bundles and vector fields by looking at the special subclass of combinatorial manifolds, which are discrete in the sense of real cohesion[1], drawing inspiration from the field of discrete differential geometry. We prove the Gauss-Bonnet theorem and Poincaré-Hopf theorem for combinatorial manifolds.

> "It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, *The Future Does Not Compute*[2]

Changelist

Notes so far from Mathieu and Steve

0.1 January 24, Mathieu

Slicing a type over $\mathrm{EM}(\mathbb{Z}, 1)$ gives us the type of types together with a map to $\mathrm{EM}(\mathbb{Z}, 1)$.

0.2 November 21, Mathieu and Steve

Constructors are present in the HIT, and the HIT is a type. The universe must be closed under inductive types.

Polygons of different cardinality will tangle with transport.

Greg suggests I could use an isomorphism from a combinatorial manifold to a HIT with just one 2-cell to capture "total curvature":

```
def S^1:
base: S^1
loop: base=base
```

```
def S^2:
N: S^2
surf: refl_N = refl_N

def torus:
b:torus
p,q:b=b
donut:p.q=q.p
```

The map $\mathcal{R}$ (realization) is a lossy map. $\mathcal{R}(S^2) : \mathcal{U}$.

"strict morphism of HITs": map constructors to constructors. $C_4 \to S^1$ is a strict map.

$\mathcal{R}(C_4)$ has an equivalence to $\mathcal{R}(S^1)$. Perhaps this is $\mathcal{R}$(the map $C_4 \to S^1$, the strict one). But $S^1 \to C_4$ is non-strict, uses generated data, uses data from $\mathcal{R}(C_4)$.

Claim: we cannot see the one-notch "90 degree" rotation of C_4 at the level of types.

0.2.1 Extension to next dimension as obstruction theory

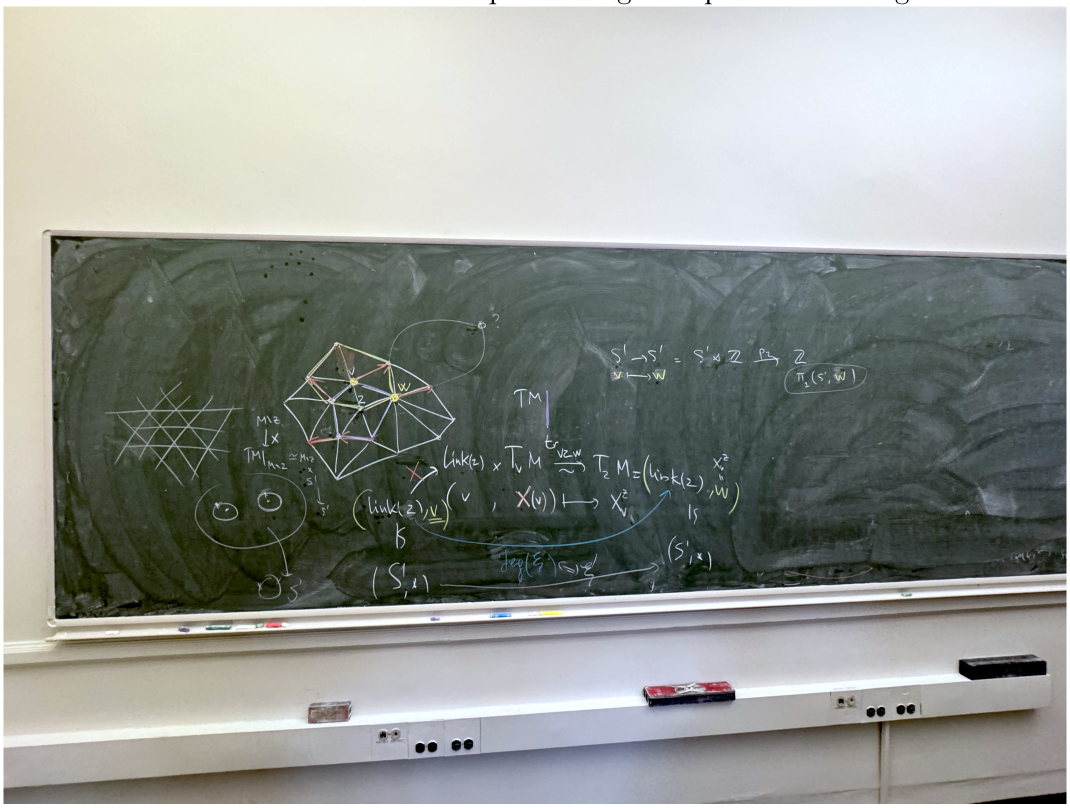
- Consider a type X with 1-skeleton X_1.
- HIT gives skeletal filtration
- functions naturally given as per-skeleton steps
- there are constraints/obstructions: obstruction to extend map T from vertices a, b to path $e : a = b$ is that $T(a)$ and $T(b)$ are in the same connected component.
- say a 2-cell F contracts a loop L, i.e. $F : L = \mathsf{refl}_v$ some vertex v
- we have a choice of v for each face F
- then $T(F)$ must be in same component as refl_v ($\mathsf{refl}_v : \pi_1 X$ with basepoint v)
- $T(F)$ is merely refl, and holonomy is merely refl, and connection on this face F is merely flat
- curvature is a function from 2-cells to $\pi_1(X, v)$

$$
\begin{aligned}
&T(v) : U \\
&T(L) : T(v) = T(v) \\
&T(F) : \mathsf{refl}_{T(v)} = T(L)
\end{aligned}
$$

The class $||T(L)|| : \pi_1(U, T(v))$ is an obstruction to having $T(F)$, to extending T to F.

The hierarchy of maps extending to higher skeletons, which we call connections of dimension 1, 2, etc. This works for ANY HIT. The combinatorial manifold structure is just one special case that gives me a specific map T.

Then Mathieu drew on the board a patch of a combinatorial manifold with links of different vertex cardinalities. Then he wrote some maps that might help define the degree/index of a vector field.



0.3 November 27, Mathieu

An angle is: a path in S^1 from `base`, which is equiv to a path in Aut S^1 from the identity to another automorphism. (automorphism is primary: aut of tangent circle).

Viewed as automorphisms we can compose them.

Viewed as paths, multiply path 2 by the element at the endpoint of path 1.

Curvature maps a loop in the base to a path in automorphisms of the tangent circle at the base.

A vector field + a trivialization ("fiducial v.f." in the book by Needham) maps a loop in the base to the tangent circle at the basepoint. We can take the index/degree of this.

Given a loop $L : v = v$, v.f. F, take $F(L) : tr_L(F(v)) = F(v)$ (`apd`, dependent action on paths).

Type family $f : A \to U$, $\prod_{a:A} f\, a$ is the type of dependent functions. Let $X : \prod_{a:A} f\, a$. We can apply X to a path $p : a =_A b$. $X(p)$ is a path over p.

$$X(p) : tr_p(X(a)) = X(b)$$

$X(a)$, $X(b)$ are terms of different types.

Greg reminded about horizontal projection interpretation of covariant derivative: covariant derivative of F along itself: lift F to F' horizontal on $TP(P \to M)$, take $[F', F']$

Tangent bundle $TM \to M$ w/ connection ω. Covariant derivative of F in the direction of X: $\nabla_X(F)(x)$.

1. Lift X and F to horizontal fields X', F' in TTM
2. Compute $[X', F']$ to TTM
3. Decompose $[X', F'] = [X, F]' + \omega([X', F'])$

Mathieu would rather use this formulation:

$$\nabla_X(F) = [X, F] + \omega(X)(F)$$

Now take $X = F$:

$$\nabla_F(F) = [F, F] + \omega(F)(F) = \omega(F)(F)$$

1 Overview

We will define

- combinatorial 2-manifolds
- circle bundles, and principal circle bundles of tangent bundles
- vector fields,

and then observe emerging from those definitions the presence of

- connections
- curvature
- the index of a vector field,

and prove

- the Gauss-Bonnet theorem
- and the Poincaré-Hopf theorem.

We will consider functions $M \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where M is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

We will see in Section 3.1 that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. We will construct a map $\mathsf{link} : M \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. Then we can consider the type of pointed mere circles $\mathrm{EM}_\bullet(\mathbb{Z}, 1) \overset{\text{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ as well as the first projection that forgets the point. This is a univalent fibration (univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe[3]). If we form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"'] \\ M \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z}, 1) \end{tikzcd}$$

then we have a bundle of mere circles, with total space given by the $\sum$-type construction. We will show that this is not a principal bundle, i.e. a bundle of torsors. Torsors are types with the additional structure of a group action. But if link satisfies an additional property (amounting to an orientation) then the pullback is a principal fibration, i.e. link factors through a map $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$, where $\mathrm{K}(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space.

We will argue that extending link to a function T on paths can be thought of as constructing a connection, notably one that is not necessarily flat (trivial). Moreover, lifting T to $T_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ can be thought of as a nonvanishing vector field. There will in general not be a total lift, just a partial function. The domain of $T_\bullet$ will have a boundary of circles, and the degree (winding number) on the disjoint union of these can be thought of as the index of $T_\bullet$. We can then examine the total curvature and the total index and prove that they are equal, and argue that they are equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds. This is similar to the classical proof of Hopf[4], presented in detail in Needham[5].

1.1 Future work

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[6][7] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

2 Torsors and principal bundles

The classical theory of principal bundles tells us to look for an appropriate classifying space of torsors to map into.

Definition 2.1. Let G be a group with identity element e (with the usual classical structure and properties). A G**-set** is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : \|X\|_{-1} \times \prod_{x:X} \mathsf{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

then we call this data a G**-torsor**. Denote the type of G-torsors by BG.

If $(X, \phi), (Y, \psi) : BG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 2.1. There is a natural equivalence $(X =_{BG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for BG and we have a group isomorphism $\Omega BG \simeq G$.

Lemma 2.2. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 2.1. The pointed type $(BG, *)$ is a $\mathrm{K}(G, 1)$. □

In particular, to classify principal S^1-bundles we map into the space $\mathrm{K}(S^1, 1)$, a type of torsors of the circle. Since S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$, we have $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$.

2.1 Bundles of mere circles

We find it illuminating to look also at the slightly more general classifying space of $\mathrm{K}(\mathbb{Z}, 1)$-bundles, that is bundles whose fiber are equivalent to $\mathrm{K}(\mathbb{Z}, 1)$. We can understand very well when these are in fact bundles of circle torsors, which will in turn shed light on orientation in this setting.

We will follow Scoccola[8]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2.2. Let $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \overset{\text{def}}{=} \sum_{Y:\mathcal{U}} \|Y \simeq \mathrm{K}(G, n)\|_{-1}$. A $\mathrm{K}(G, n)$**-bundle** on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses two self-maps on the universe: suspension followed by $(n+1)$-truncation $\|\Sigma\|_{n+1}$ and forgetting a point $F_\bullet$ to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{\|\Sigma\|_{n+1}} \mathrm{EM}_{\bullet\bullet}(G, n+1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n+1)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 2.3. Given $f : M \to \mathrm{EM}(G, n)$, the **associated action of** M **on** G, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ \|\Sigma\|_{n+1} \circ f$.

Theorem 2.1. (Scoccola[8] Proposition 2.39). A $\mathrm{K}(G, n)$ bundle $f : M \to \mathrm{EM}(G, n)$ is equivalent to a map in $M \to \mathrm{K}(G, n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Let's relate this to *orientation*. Note that the obstruction in the theorem is about a map into $\mathrm{EM}_\bullet(G, n+1)$ and further note that $\mathrm{EM}_\bullet(G, n) \simeq \mathrm{K}(\mathrm{Aut}\, G, 1)$ (independent of n). The theorem says that the data of a map into $\mathrm{EM}(G, n)$ factors into data about a map into $\mathrm{K}(G, n+1)$ and one into $\mathrm{K}(\mathrm{Aut}\, G, 1)$. Informally, $\mathrm{EM}(G, n)$ is a little too large to be a $K(G, n+1)$, as it includes data about automorphisms of G.

In the special case of $\mathrm{EM}(\mathbb{Z}, 1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ is contractible. $\mathrm{Aut}\, \mathbb{Z}$ consists of the $\mathbb{Z}/2\mathbb{Z}$ worth of outer automorphisms given by multiplication by ± 1. If we look at the fiber sequence

$$\mathrm{K}(S^1, 1) \to \mathrm{BAut}\, S^1 \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$$

we see the automorphisms of the circle as an extension of the group of automorphisms that are homotopic to the identity (which are the torsorial actions) by the group that sends the loop in S^1 to its inverse. This is another way to see that a map $f : M \to \mathrm{BAut}\, S^1 \simeq \mathrm{EM}(\mathbb{Z}, 1)$ factors through $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$ if and only if the composition to $\mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ is trivial. This amounts to a choice of loop-direction for all the circles, and so deserves the name "f is *oriented*." In addition the map $\mathrm{BAut}\, S^1 \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ deserves to be called the first Stiefel-Whitney class of f, and the requirement here is that it vanishes.

Note 2.1. Reinterpreting more of the theory of characteristic classes would be an enlightening future project. Defining a Chern class and Euler class in 2 dimensions is related to the goals of this note, but the full theory is about a family of invariants in different dimensions that have various relations between each other and satisfy other properties.

2.2 Pathovers in circle bundles

Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ and $P \overset{\text{def}}{=} \sum_{x:M} T(x)$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : T(a) = T(b)$. This is a path in the codomain, which in this case is a type of types. Type theory also provides a function called *transport*, denoted $\mathsf{tr}(p) : T(a) \to T(b)$ which acts on the fibers of P. $\mathsf{tr}(p)$ is a function, acting on the terms of the types $T(a)$ and $T(b)$, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathsf{tr}(p)(\alpha) =_{T(b)} \beta$ between $\alpha : T(a)$ and $\beta : T(b)$ in the fibers. We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

Lastly we want to recall that in the presence of a section $X : M \to P$ there is a dependent generalization of ap called apd: $\mathsf{apd}(X)(p) : \mathsf{tr}(p)(X(a)) = X(b)$ which is a pathover between the two values of the section over the basepoints of the path p.

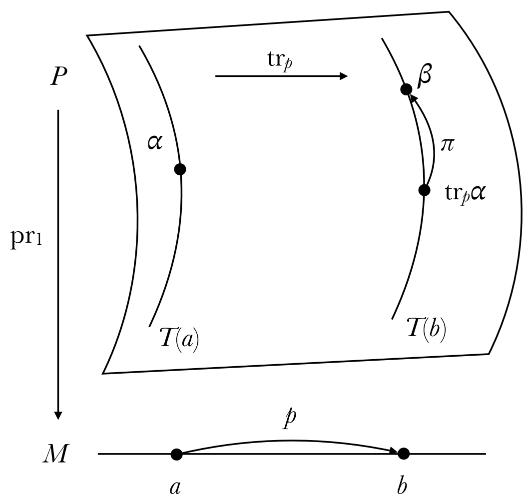


Figure 1: A path π over the path p in the base involves the transport function.

3 Combinatorial manifolds

We will adapt to higher inductive types in a straightforward manner the classical construction of *combinatorial manifolds*. See for example the classic book by Kirby and Siebenmann[9]. These are a subclass of simplicial complexes.

3.1 Polygons

We will now start looking at some examples, first by defining a type that is important both for the domain and the codomain of mere circles: a square.

Definition 3.1. The higher inductive type C_4 (where C stands for "circle"). See Figure 2.

$$\begin{aligned} C_4 &: \mathsf{Type} \\ c_1, c_2, c_3, c_4 &: C_4 \\ c_1c_2 &: c_1 = c_2 \\ c_2c_3 &: c_2 = c_3 \\ c_3c_4 &: c_3 = c_4 \\ c_4c_1 &: c_4 = c_1 \end{aligned}$$

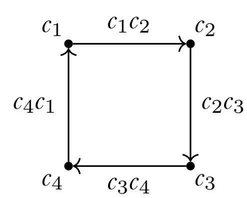


Figure 2: The HIT C_4.

The standard HoTT circle itself is a non-example of a combinatorial manifold since it lacks the second vertex of the edge:

Definition 3.2. The higher inductive type S^1 which we can also call C_1:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

Both of these are examples of a family of HIT data we will call Gon, the set of n-gon HITs for some natural number n, following the pattern of the HITs above. We'll see below that the realization of an n-gon is a mere circle, i.e. we have $\mathcal{R} : \mathsf{Gon} \to \mathrm{EM}(\mathbb{Z}, 1)$.

3.2 Adding and removing points from polygons

Recall that given functions $\phi, \psi : A \to B$ between two arbitrary types we can form a type family of paths $\alpha : A \to \mathcal{U}$ by $\alpha(a) \stackrel{\text{def}}{=} (\phi(a) =_B \psi(a))$. Transport in this family is given by concatenation as follows, where $p : a =_A a'$ and $q : \phi(a) = \psi(a)$ (see Figure 3):

$$\mathsf{tr}(p)(q) = \phi(p)^{-1} \cdot q \cdot \psi(p)$$

which gives a path in $\phi(a') = \psi(a')$ by connecting dots between the terms $\phi(a')$, $\phi(a)$, $\psi(a)$, $\psi(a')$. This relates a would-be homotopy $\phi \sim \psi$ specified at a single point, to a point at the end of a path. We will use this to help construct such homotopies.

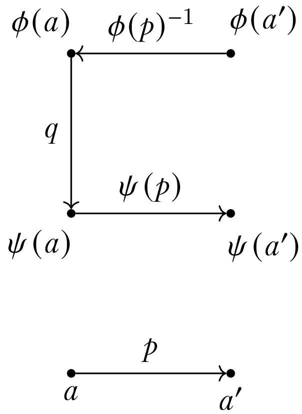


Figure 3: Transport along p in the fibers of a family of paths. The fiber over a is $\phi(a) = \psi(a)$ where $\phi, \psi : A \to B$.

Lemma 3.1. Let C_n be the polygon 1-dimensional HIT with n vertices. Then $C_2 \simeq C_1$ and in fact $C_n \simeq C_{n-1}$.

Proof. (Compare to [10] Lemma 6.5.1.) In the case of C_1 we will denote its constructors by base and loop. For C_2 we will denote the points by v_1, v_2 and the edges by ℓ_{12}, r_{21}. For C_3 and higher we will denote the points by $v_1, \ldots, v_n$ and the edges by $e_{i,j} : v_i = v_j$ where $j = i + 1$ except for $e_{n,1}$.

First we will define $f : C_2 \to C_1$ and $g : C_1 \to C_2$, then prove they are inverses.

$$\begin{aligned} f(v_1) = f(v_2) &= \mathsf{base} \\ f(\ell_{12}) &= \mathsf{loop} \\ f(r_{21}) &= \mathsf{refl}_{\mathsf{base}} \end{aligned} \qquad\qquad \begin{aligned} g(\mathsf{base}) &= v_1 \\ g(\mathsf{loop}) &= \ell_{12} \cdot r_{21} \end{aligned}$$

We need to show that $f \circ g \sim \mathrm{id}_{C_1}$ and $g \circ f \sim \mathrm{id}_{C_2}$. Think of f as sliding v_2 along r_{21} to coalesce with v_1. This may help understand why the unfortunately intricate proof is working.

We need terms $p : \prod_{a:C_1} f(g(a)) = a$ and $q : \prod_{a:C_2} g(f(a)) = a$. We will proceed by induction, defining appropriate paths on point constructors and then checking a condition on path constructors that confirms that the built-in transport of these type families respects the definition on points.

Looking first at $g \circ f$, which shrinks r_{21}, we have the following data to work with:

$$\begin{aligned} g(f(v_1)) = g(f(v_2)) &= v_1 \\ g(f(\ell_{12})) &= \ell_{12} \cdot r_{21} \\ g(f(r_{21})) &= \mathsf{refl}_{v_1}. \end{aligned}$$

We then need to supply a homotopy from this data to id_{C_2}, which consists of a section and pathovers over C_2:

$$\begin{aligned} p_1 &: g(f(v_1)) = v_1 \\ p_2 &: g(f(v_1)) = v_2 \\ H_\ell &: \mathsf{tr}(\ell_{12})(p_1) = p_2 \\ H_r &: \mathsf{tr}(r_{21})(p_2) = p_1. \end{aligned}$$

which simplifies to

$$\begin{aligned} p_1 &: v_1 = v_1 \\ p_2 &: v_1 = v_2 \\ H_\ell &: g(f(\ell_{12}))^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\ H_r &:= g(f(r_{21}))^{-1} \cdot p_2 \cdot r_{21} = p_1 \end{aligned}$$

and then to

$$\begin{aligned} p_1 &: v_1 = v_1 \\ p_2 &: v_1 = v_2 \\ H_\ell &: (\ell_{12} \cdot r_{21})^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\ H_r &: \mathsf{refl}_{v_1} \cdot p_2 \cdot r_{21} = p_1 \end{aligned}$$

To solve all of these constraints we can choose $p_1 \overset{\text{def}}{=} \mathsf{refl}_{v_1}$, which by consulting either H_ℓ or H_r requires that we take $p_2 \overset{\text{def}}{=} {r_{21}}^{-1}$.

Now examining $f \circ g$, we have

$$\begin{aligned} &f(g(\mathsf{base})) = \mathsf{base} \\ &f(g(\mathsf{loop})) = f(\ell_{12} \cdot r_{21}) = \mathsf{loop} \end{aligned}$$

and so we have an easy proof that this is the identity.

The proof of the more general case $C_n \simeq C_{n-1}$ is very similar. Take the maps $f : C_n \to C_{n-1}$, $g : C_{n-1} \to C_n$ to be

$$\begin{aligned} f(v_i) &= v_i \quad (i = 1, \ldots, n-1) & g(v_i) &= v_i & (i = 1, \ldots, n-1) \\ f(v_n) &= v_1 & g(e_{i,i+1}) &= e_{i,i+1} & (i = 1, \ldots, n-2) \\ f(e_{i,i+1}) &= e_{i,i+1} \quad (i = 1, \ldots, n-1) & g(e_{n-1,1}) &= e_{n-1,n} \cdot e_{n,1} & \\ f(e_{n-1,n}) &= e_{n-1,1} & & & \\ f(e_{n,1}) &= \mathsf{refl}_{v_1} & & & \end{aligned}$$

where f should be thought of as shrinking $e_{n,1}$ so that v_n coalesces into v_1.

The proof that $g \circ f \sim \mathrm{id}_{C_n}$ proceeds as follows: the composition is definitionally the identity except

$$\begin{aligned} g(f(v_n)) &= v_1 \\ g(f(e_{n-1,n})) &= e_{n-1,n} \cdot e_{n,1} \\ g(f(e_{n,1})) &= \mathsf{refl}_{v_1}. \end{aligned}$$

Guided by our previous experience we choose ${e_{n,1}}^{-1} : g(f(v_n)) = v_n$, and define the pathovers by transport.

The proof that $f \circ g \sim \mathrm{id}_{C_{n-1}}$ requires only noting that $f(g(e_{n-1,1})) = f(e_{n-1,n} \cdot e_{n,1}) = e_{n-1,1} \cdot \mathsf{refl}_{v_1} = e_{n-1,1}$. ◻

Corollary 3.1. All polygons are equivalent to S^1, i.e. we have a term in $\prod_{n:\mathbb{N}} ||C_n = S^1||$, and hence Gon is a subtype of $\mathrm{EM}(\mathbb{Z}, 1)$.

Proof. We can add $n - 1$ points to S^1 and use Lemma 3.1. ◻

Definition 3.3. For $k : \mathbb{N}$ define $m_k : \mathsf{Gon} \to \mathsf{Gon}$ where $m_k : C_n \to C_{kn}$ adds k vertices between each of the original verticies of C_n.

With m_k we can start with a collection of pentagons and hexagons and make the collection homogeneous: by applying m_6 to the pentagons and m_5 to the hexagons we obtain a collection of 30-gons. This will be useful when we work more with the link function.

3.3 Abstract simplicial complexes

Definition 3.4. An **abstract simplicial complex M of dimension** n consists of a set M_0 of vertices, and for each $0 < k \leq n$ a set M_k of subsets of M_0 of cardinality $k + 1$, such that any $(j + 1)$-element subset of M_k is an element of M_j. The elements of M_k are called **k-faces**. Denote by $\mathsf{SimpCompSet_n}$ the type of abstract simplicial complexes of dimension n (where the suffix Set reminds us that this is a type of sets). M is automatically equipped with a chain of inclusions $M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_n = M$ where each $M_k : \mathsf{SimpCompSet_k}$ is a complex in its own right. We call M_k the **k-skeleton** of M.

Definition 3.5. Let Δ^n be the standard n-simplex in $\mathbb{R}^n$ given by $\{x_1, \ldots, x_n | \sum_i x_i \leq 1\}$. Let $M : \mathsf{SimpCompSet_n}$. The **geometric realization** $|M| : \mathsf{Top}$ **of** M in the category of topological spaces is given inductively as follows: $|M_0| = M_0$, and given $|M_{k-1}|$ we form $|M_k|$ by the pushout

$$\begin{tikzcd} M_k \times \partial\Delta^k \arrow[r, "\text{attach}"] \arrow[d, hook, "\text{id}\times\text{incl}"'] & {|M_{k-1}|} \arrow[d, "i_k"] \\ M_k \times \Delta^k \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & {|M_k|} \end{tikzcd}$$

which attaches each k-simplex by taking the convex hull of the appropriate $k + 1$ points in $|M_{k-1}|$. The collection of vertical maps on the right gives a sequence of inclusion maps of skeleta $|M_0| \xrightarrow{i_1} |M_1| \xrightarrow{i_2} \cdots \xrightarrow{i_n} |M_n| = |M|$.

Definition 3.6. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $n-1$-face containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

The link is all the neighboring vertices of v and the codimension 1 faces joining those to each other. See for example Figure 4.

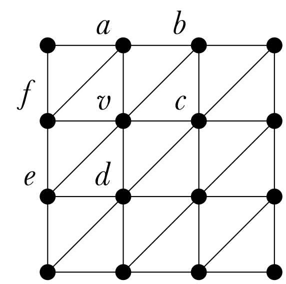


Figure 4: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

Definition 3.7. A **combinatorial manifold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a simplicial sphere of dimension $n-1$ (i.e. its geometric realization is homeomorphic to an $n-1$-sphere). Denote by $\mathsf{CombMfdSet_n}$ the type of combinatorial manifolds of dimension n (which the notation again reminds us are sets).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figures 5, 6, and 7 for some examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits a combinatorial triangulation[11]. So it appears reasonably well motivated to study this class of objects.

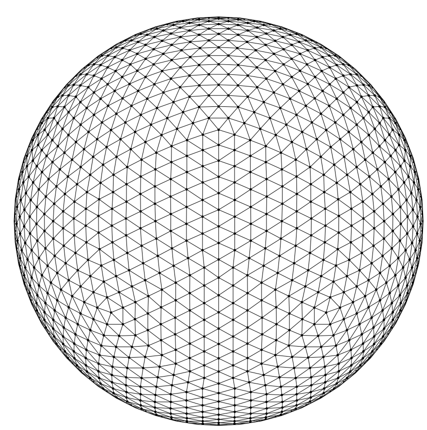


Figure 5: A combinatorial triangulation of a sphere, created with the tool `stripy`.

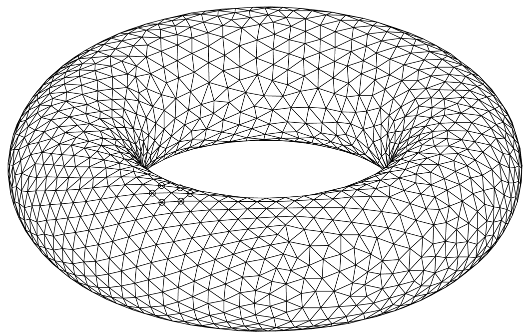


Figure 6: A torus with an interesting triangulation, from Wikipedia. The links have various vertex counts from 5-7. Clearly a constant value of 6 would also work. (By Ag2gaeh - Own work, CC BY-SA 3.0.)

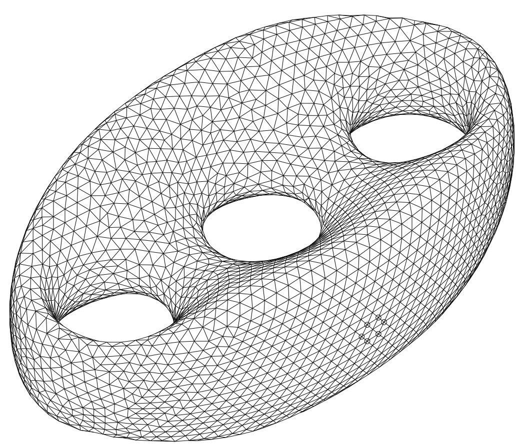


Figure 7: A 3-holed torus with triangulation, from Wikipedia. (By Ag2gaeh - Own work, CC BY-SA 3.0.)

3.4 Higher inductive combinatorial manifolds

Instead of $|M|$: Top we can use the simplicial complex to obtain $\mathbb{M}$: Type by forming a homotopy pushout:

Definition 3.8. Define the function **realization** $\mathcal{R}$: $\mathsf{SimpCompSet_n} \to \mathsf{Type}$ by forming a homotopy pushout of the data, for example in dimension 2:

$$
\begin{tikzcd}
 & E_2 \times C_3 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\Searrow h_2"] & M_2 \arrow[d] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times S^0 \arrow[u, "\partial_0"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\Nearrow h_1"] & M_1 \arrow[u] &
\end{tikzcd}
$$

The types C_3 and X_1 are 1-types, X_2 is a 2-type, and the rest are sets. The map ∂_1 maps each pair (e, S^0) to the pair of points this edge connects. Similarly, ∂_2 maps each (f, C_3) to the triangle that this face bounds. The h_i are the proofs of commutativity, and the two squares are also both homotopy pushouts. Note that the pushouts could be re-expressed as HIT constructors.

Using the chain of skeleton inclusions gives us more, it gives us a map to *presented types*:

Definition 3.9. Let Type^n denote the universe of n-types. Denote by $\mathsf{Type}^{[n]}$ the type of **presented n-types** consisting of types and maps $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$ where X_i : Type^i is an i-type.

We will import some of von Raumer[12], where he proves that given $\mathbb{M}$: $\mathsf{Type}^{[2]}$ we obtain a double groupoid.

3.5 The higher inductive type $\mathbb{O}$

We will create our first combinatorial surface, a 2-sphere. We will adopt the convention that a subscript indicates the dimension of a subskeleton of a complex. For instance, we have $\mathsf{base} : S^1_0$.

Definition 3.10. The HIT $\mathbb{O}_0$ is just 6 points, intended as the 0-skeleton of an octahedron, with vertices named after the colors on the faces of a famous Central European puzzle cube.

$$w, y, b, r, g, o : \mathbb{O}_0$$

Definition 3.11. The HIT $\mathbb{O}_1$ is the 1-skeleton of an octahedron.

$$
\begin{aligned}
w, y, b, r, g, o &: \mathbb{O}_1 & yg &: y = g \\
wb &: w = b & yo &: y = o \\
wr &: w = r & br &: b = r \\
wg &: w = g & rg &: r = g \\
wo &: w = o & go &: g = o \\
yb &: y = b & ob &: o = b \\
yr &: y = r & &
\end{aligned}
$$

Definition 3.12. The HIT $\mathbb{O}$ is an octahedron:

$$
\begin{array}{llll}
w, y, b, r, g, o : \mathbb{O} & & & \\
 & wb : w = b & br : b = r & wbr : wb \cdot br \cdot wr^{-1} = \mathsf{refl}_w \\
 & wr : w = r & rg : r = g & wrg : wr \cdot rg \cdot wg^{-1} = \mathsf{refl}_w \\
 & wg : w = g & go : g = o & wgo : wg \cdot go \cdot wo^{-1} = \mathsf{refl}_w \\
 & wo : w = o & ob : o = b & wob : wo \cdot ob \cdot wb^{-1} = \mathsf{refl}_w \\
 & yb : y = b & & yrb : yr \cdot rb \cdot yb^{-1} = \mathsf{refl}_y \\
 & yr : y = r & & ygr : yg \cdot gr \cdot yr^{-1} = \mathsf{refl}_y \\
 & yg : y = g & & yog : yo \cdot og \cdot yg^{-1} = \mathsf{refl}_y \\
 & yo : y = o & & ybo : yb \cdot bo \cdot yo^{-1} = \mathsf{refl}_y
\end{array}
$$

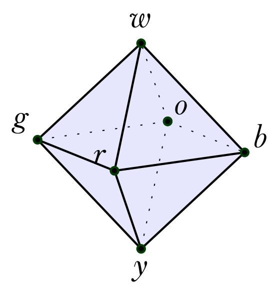


Figure 8: The HIT $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

We have obvious maps $\mathbb{O}_0 \xrightarrow{i_0} \mathbb{O}_1 \xrightarrow{i_1} \mathbb{O}$ that include each skeleton into the next-higher-dimensional skeleton by the obvious inclusion of constructors.

4 Connections and curvature

4.1 The function T

We will build up a map T out of $\mathbb{O}$ which is meant to be like the circle bundle of a tangent bundle. And so we will begin with the intrinsic data of the link at each point: taking the link of a vertex gives us a map from vertices to polygons.

Definition 4.1. $T_0 \stackrel{\text{def}}{=} \mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned} \mathsf{link}(w) &= brgo & \qquad \mathsf{link}(r) &= wbyg \\ \mathsf{link}(y) &= bogr & \qquad \mathsf{link}(g) &= wryo \\ \mathsf{link}(b) &= woyr & \qquad \mathsf{link}(o) &= wgyb \end{aligned}$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

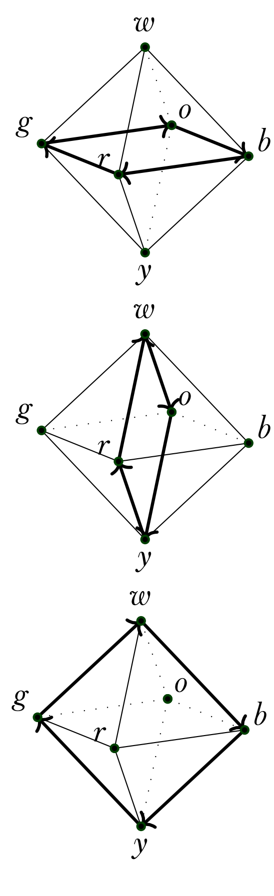


Figure 9: link for the vertices w, b and r.

To extend T_0 to a function T_1 on the 1-skeleton we have complete freedom. Defining a map by induction makes clear that the action on paths is its own structure. Two functions on the octahedron could agree on points but differ on edges. We are going to identify this 1-dimensional freedom with a connection:

Definition 4.2. A **connection** on a higher combinatorial manifold is an extension of a circle bundle from the 0-skeleton to the 1-skeleton.

Continuing the example, we will do something "tangent bundley", imagining how T_1 changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 4.3. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ (r, o fixed)
- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ (b, g fixed)
- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$
- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 9 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn.

Definition 4.4. The map $R : C_4 \to C_4$ rotates by one quarter turn, one "click":

- $R(c_1) = c_2$
- $R(c_2) = c_3$
- $R(c_3) = c_4$
- $R(c_4) = c_1$
- $R(c_1c_2) = c_2c_3$
- $R(c_2c_3) = c_3c_4$
- $R(c_3c_4) = c_4c_1$
- $R(c_4c_1) = c_1c_2$

Note by univalence the equivalence R gives a loop in the universe, a term of $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here *there is no additional freedom*.

Definition 4.5. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows:

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

where $H_R : R = \mathsf{refl}_{C_4}$ is the obvious homotopy given by composition with R^{-1}. Passing through univalence we get a 2-path between R and refl in the loop space $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

Definition 4.6. The **curvature of a connection** on a type family $T : \mathbb{M} \to \mathcal{U}$ at a vertex v of a 2-face f with boundary path p_f of a higher combinatorial manifold $\mathbb{M}$ is the automorphism $\mathsf{tr}(p_f)(Tv)$ together with a homotopy to id_{Tv}.

Note 4.1. We have defined a function on a cell by requiring it to correspond to the value on the boundary of that cell. This is familiar in classical differential topology, where it's called *the exterior derivative*. The duality of d and ∂ is recognizable in T_2, and we might say "curvature is the derivative of the connection."

4.2 T on concatenations of faces

Consider Figure 10 and the diamond on the left. Denote the clockwise triangular path around the left triangle by $bcab$, and around the right by $bdcb$. These are loops at b. Say that the tangent map T satisfies $T(bcab) = R$ where R is some rotation of the 4-sided polygon Tb, and similarly $T(bdcb) = R$ as well.

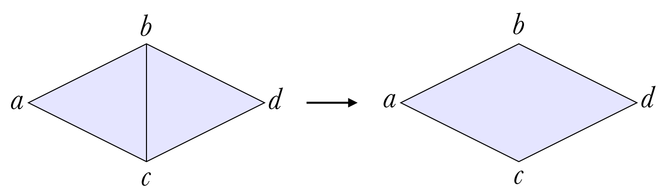


Figure 10: Concatenating the triangles bac and bdc gives the 4-gon $abdc$.

Since $bcab$ has a filler 2-cell, which we will call f_1, we know that $T(f_1) = H_R$ where $H_R : R = \mathsf{refl}$ is a path in $\mathrm{Aut}(Tb)$ from R to the identity. Similarly if f_2 is the filler for $bdcb$ then $T(f_2) = H_R$.

The two faces can be horizontally composed giving $f_1 \star f_2$, which fills the diamond $bdcab$. Then we also obtain $T(f_1 \star f_2) : R^2 = \mathrm{id}$.

In this way, given an explicit ordering of the faces (a "plan" for visiting all faces of the manifold), we can "sum" the curvature. For our ocatahedron where each triangle produces a rotation of R, we'd get R^8.

4.3 The torus

We can define a combinatorial torus as a similar HIT. This time each vertex will have six neighbors. So all the links will be merely equal to C_6 which is a hexagonal version of C_4. See Figure 11.

To help parse this figure, imagine instead Figure 12. We take this simple alternating-triangle pattern, then glue the left and right edges, then bend into Figure 11. The fact that each column in Figure 12 has four dots corresponds to the torus in Figure 11 having a square in front, diamonds in the middle, and a square in back.

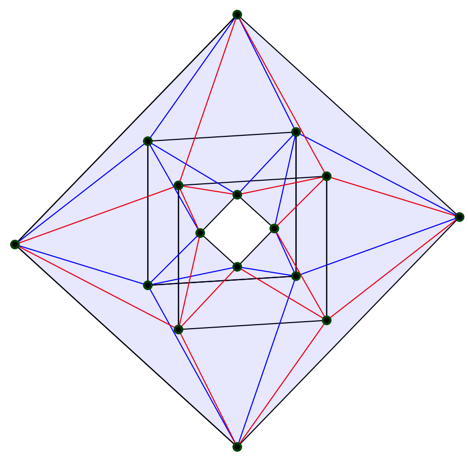


Figure 11: Torus embedded in 3-dimensional space. If you see color in your rendering then black lines trace four square-shaped paths, red ones connect the front square to the middle diamonds, and blue ones connect the back path to the middle ones.

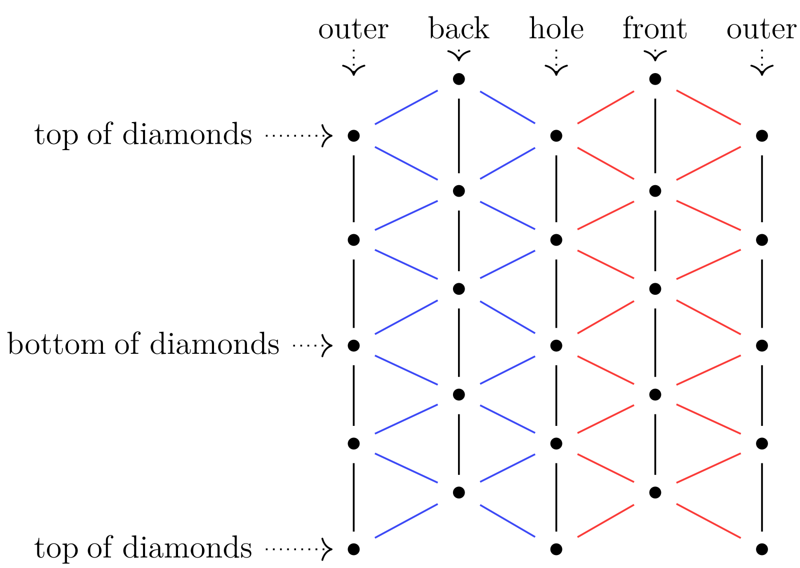


Figure 12: An inspiration for the torus. Identify the sides and then the top, definitionally, to get the actual torus.

This somewhat arbitrary and unfamiliar model of a torus has the helpful property that it is a combinatorial manifold that is somewhat minimal while still being representable by a donut shape. But the donut-shaped version suggests a very different connection than the flat model! Starting with the flat model, we can easily see how to define T_1 by sliding a link rigidly along the page to the link of some adjacent vertex. Then we can see that transport around any loop is the identity and so T_2 is always the identity (together with the homotopy $\mathsf{refl}_{\mathrm{id}}$ from the identity to itself). So if we imagine a way to visit every face like we did for the octahedron, starting and ending at some basepoint v, we expect to see no net rotation at all of Tv. Later we will call this "total curvature 0."

The donut-shaped torus suggests a different connection, one determined by the embedding in 3-space that we have represented. But the easiest way to think about that bundle and its connection and curvature is to wait until we have a proof of the Poincaré-Hopf theorem, so that we can instead compute with a vector field.

4.4 Existence of connections

How confident can we be that we can always define a connection on an arbitrary combinatorial manifold? Two things make the octahedron example special: the link is a 4-gon at every vertex, and every edge extends to a symmetry of the entire octahedron, embedded in 3-dimensional space. This imposed a coherence on the interactions of all the choices we made for the connection, which we can worry may not exist for more complex combinatorial data.

We know as a fact outside of HoTT that any combinatorial surface that has been realized as a triangulated surface embedded in 3-dimensional euclidean space can inherit the parallel transport entailed in the embedding. We could then approximate that data to arbitrary precision with enough subdivision of the fibers of T.

What would a proof inside of HoTT look like? We will leave this as an open question.

5 Vector fields

Definition 5.1. A **partial function** $f : A \to B$ is a function $f : A \to B + \mathbb{1}$, the disjoint union of B with the 1-element type.

If $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ is a bundle of mere circles, then a vector field should be a partial function $T_\bullet : \mathbb{M} \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ that lifts T. In other words, a pointing of *some* of the fibers. This aligns with the classical picture of a choice of nonzero vector at each point, except for some points where the vector field vanishes.

Definition 5.2. Let $\mathbb{M} : \mathsf{CombMfd}_2$ be a combinatorial manifold and Z an isolated set of vertices. A **vector field** X **on** $\mathbb{M}$ **(with zero set** Z**)** is a partial section of P, i.e. a term $X : \prod_{x:\mathbb{M}\setminus Z} T(x)$ (and eliding the unique term of $Z \to \mathbb{1}$).

$$\begin{tikzcd}
P \overset{\mathrm{def}}{=} \sum_{C:T(\mathbb{M}\setminus Z)} C \arrow[r, "\overline{T}"] \arrow[d, shift left, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"] \\
\mathbb{M} \setminus Z \arrow[u, bend left, "X:\prod_{x:\mathbb{M}\setminus Z} Tx"] \arrow[ur, "T_\bullet"] \arrow[r, "T"] & \mathrm{EM}(\mathbb{Z}, 1)
\end{tikzcd}$$

where $T_\bullet = \overline{T} \circ X$.

The lift $T_\bullet$ equips the fibers of T with points, and provides pointings to the transport maps. (The action of X on paths provides the proofs of pointedness.)

The goal now is to recompute the total curvature in the context of some vector field X and obtain three values:

1. The total curvature, e.g. in the case of $\mathbb{O}$ the value R^8.
2. The total winding of X, which will take the proof of pointedness around all faces in the domain of X. Being a loop in a circle, this is an integer.
3. A pointed version of curvature, which couples the winding of X to the transport, which is a total function, and which produces a contractible pointed map, i.e. 0.

Once we have these, we will unpack how they provide a proof in HoTT of the Poincaré-Hopf theorem and the Gauss-Bonnet theorem.

5.1 Index of a vector field

5.2 Equality of total index and total curvature

5.3 Identification with Euler characteristic

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