Connection Form Classifiers in Homotopy Type Theory

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1 David Jaz Myers' talk

David's informal talk was on October 15, 2021 at the MURI workshop at CMU.

The setting: cohesive HoTT, localized at a real numbers object R. The semantics are to be in an ∞ version of the SDG Dubuc topos, which itself has a choice of R, which we choose to coincide. The site has a special property with how the infinitesimal disks map in the topology – I believe the maps are the identity on the disk component.

References David mentions:

- Moerdijk and Reyes: Models for Infinitesimal Analysis [1]
- Bunge, Gago, San Luis: Synthetic Differential Topology [2]
- Yetter: On right adjoints to exponential functors [3]

Some facts we want to recall from SDG:

- $TX = \mathbb{D} \to X$
- foo

Externally speaking, $\mathbb{D}$ is tiny, so externally has the "amazing right adjoint".

$$\mathrm{Hom}(X^{\mathbb{D}}, Y) \equiv \mathrm{Hom}(X, Y^{1/\mathbb{D}})$$

The object $Y^{1/\mathbb{D}}$ then classifies maps from tangent vectors of X to Y. If we take $Y = R$ then $R^{1/\mathbb{D}}$ is a 1-form classifier!

How to internalize this? David cites David Yetter. Perhaps the 1987 paper "On Right Adjoints of Exponential Functors" cited at https://ncatlab.org/nlab/show/amazing+right+adjoint.

David proposes internalizing this adjunction by postulating an equivalence of function types, but he first soups them up into dependent function types:

* for any type Y there is a type family $\varepsilon : \mathbb{D} \vdash Y^{1/\varepsilon}$ Type * $\sigma : ((\varepsilon : \mathbb{D}) \to Y^{1/\varepsilon}) \to Y$ (the counit of this dependent version of the adjunction) * The function

$$((\varepsilon : \mathbb{D}) \to X_\varepsilon \to Y^{1/\varepsilon}) \to (((\varepsilon : \mathbb{D}) \to X_\varepsilon) \to Y)$$

sending $f \mapsto [g \mapsto \sigma(\varepsilon \mapsto f_\varepsilon(g(\varepsilon)))]$ is $\sharp$-connected, i.e. under sharp this is an isomorphism.

The non-dependent version then looks like the adjunction:

$$\sharp(X \to Y^{1/\mathbb{D}}) = \sharp(X^{\mathbb{D}} \to Y)$$

David comments that this map is natural in X and partially natural in Y – there's a restriction to mapping into Y "from crisp variables"(?) (To have something well-typed in the presence of sharp, I presume). Natural in this context means stable under pullback?

Classifying "linear forms". $X^{\mathbb{D}}$ always has an $\mathbb{R}$-action.

Fact: For KL-vector spaces V, W, $f : V \to W$ is linear iff it is $\mathbb{R}$-equivariant.

Proof: (written on whiteboard), from Kock. It's a trick with partial derivatives in the KL-vector spaces.

Therefore instead of constructing a classifier for "linear 1-forms", we can classify R-equivariant 1-forms. Kock has done this: the 1-form classifier is the equalizer

$$\Lambda^1(V) \to V^{1/\mathbb{D}} \rightrightarrows (V^{1/\mathbb{D}})^R$$

where the two maps on the right (excuse the use of a 2-dim arrow for two arrows) are given by transposing

$$R^{\mathbb{D}} \times (V^{1/\mathbb{D}})^{\mathbb{D}} \to V$$

$$(r, v) \Rightarrow r(0)\sigma(v) \text{ or } \sigma(r(0)v)$$

We have stopped tracking crisp and sharp, but David believes that when we do, we will be unhappy with the limitations.

Cool facts from standard SDG: - $d : R \to \Lambda^1(R)$ has as transpose $v \mapsto v'(0)$ - If we have $f : R \to R$ then $df : R \to \Lambda^1$ (df means composition, but it's a slick link to the classical notation). Walking this back through transposes gives the derivative of f evaluated at the right basepoint.

Now add another axiom, the "principal of constancy": $f : R \to R, df = 0 \implies f$ constant.

Form this sequence of maps:

$$0 \to \ker d \to R \xrightarrow{d} \Lambda^1_{\text{closed}} \to 0$$

and note that the principal of constancy tells us, via transposing (I think) that the backward map on the left, from $R \to \ker d$ is constant. And so we can connect back to flat and conclude that $\ker d = \flat R$, which recreates that $R/\flat R = \{$closed 1-form classifier$\}$. Furthermore, by the good fibrations trick we can lift against maps and get the antiderivative F of f including a choice of value that fixes a unique antiderivative:

$$\begin{tikzcd}
* \arrow[r, "c"] \arrow[d, "0"'] & \mathbb{R} \arrow[d, "d"] \\
\mathbb{R} \arrow[r, "f\,dx"'] \arrow[ur, dashed, "\exists! F"] & \Lambda^1_{\text{closed}}
\end{tikzcd}$$

Mathieu had a discussion with David about what spaces are included in this ∞-topos. Certain cusps will not be included perhaps. David claimed that the cuspy spaces are microlinear.

Next David claimed that $\int \Lambda^n_{\text{closed}}(R) = \flat B^n R$. Check his modal fracture paper.

2 Principal bundles

Adjoint action of G. He cited the standard stuff from Buchholtz-van Doorn-Rijke. He previewed his Sunday talk by talking about BG being the type of *exemplars* of G. Let $*$ be the base point of BG. The conjugation action is $(x : BG) \times x = x$. The adjoint action on the Lie algebra of G is $T_{\text{id}}(* = *)$.

Or is it, $BG \to \text{Type}$ where $E \mapsto T_{\text{id}}(E = E) := (v : \mathbb{D}) \to (E = E) \times (v(0) = \text{id})$.

David assumes G is microlinear, which is a proposition. Hence $E = E$ is microlinear.

Now we can form

$$(E : BG) \times \Lambda^1(T_{\text{id}}(E = E)) =: B_\nabla G$$

and this should be the classifying space of principal bundles with connection.

Let's reorganize the maps a little to see in another way that this is a connection.

Form the cospan of a principal bundle and projection from this new larger classifying space: $X \xrightarrow{E} BG \leftarrow B_\nabla G$. What would it take to lift E to a map to $E' : X \to B_\nabla G$? If we reorganize the data, we can view it as a fiberwise map over BG:

$$(e : BG) \to \text{fib}_E(e) \xrightarrow{\omega} \Lambda^1(T_{\text{id}}(E = E)).$$

And what is $\text{fib}_E(e)$? It's $(x : X) \times (E_x = e)$. This is the total space of the bundle that E classifies. So the map ω is a 1-form on the total space of the principal bundle, with values in the Lie algebra... perfect! (So long as taking Λ^1 of a vector space corresponds to "with values in").

Problem: "this is a lie because E is not crisp!" In fact, to have E means we are in fact in $\flat BG$ which is where we don't want to be, as we will only have flat connections.

Other note: the infinitesimal disk has only one crisp point. There is only one closed term of the type of the disk, which is 0.

References

[1] I. Moerdijk and G. Reyes, *Models for Smooth Infinitesimal Analysis*, ser. SpringerLink : Bücher. Springer New York, 2013. [Online]. Available: https://books.google.com/books?id=UKjxBwAAQBAJ

[2] M. Bunge, F. Couso, A. Luis, and A. Fernández, *Synthetic Differential Topology*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, 2018. [Online]. Available: https://books.google.com/books?id=7M1MDwAAQBAJ

[3] D. Yetter, "On right adjoints to exponential functors," *Journal of Pure and Applied Algebra*, vol. 45, no. 3, pp. 287–304, 1987.