

# Connection Form Classifiers in Homotopy Type Theory

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## 1 David Jaz Myers' talk

David's informal talk was on October 15, 2021 at the MURI workshop at CMU.

The setting:

- Cohesive HoTT, localized at a real numbers object  $\mathbb{R}$ .
- The semantics are to be in an  $\infty$  version of the SDG Dubuc topos, which itself has a choice of  $\mathbb{R}$ , which we choose to coincide.
- This site has a special property with how the infinitesimal disks map in the topology – I believe the maps are the identity on the disk component.

References David mentions:

- Moerdijk and Reyes: Models for Infinitesimal Analysis [1]

- Bunge, Gago, San Luis: Synthetic Differential Topology [2]
- Yetter: On right adjoints to exponential functors [3]

Some facts we want to recall from SDG:

- $TX = \mathbb{D} \rightarrow X$
- $\mathbb{D}$  has only one crisp point. There is only one closed term of the type of the disk, which is 0.

## 1.1 Internalizing the amazing right adjoint

$\mathbb{D}$  is tiny, so externally it brings about the “amazing right adjoint”.

$$\mathrm{Hom}(X^{\mathbb{D}}, Y) \equiv \mathrm{Hom}(X, Y^{1/\mathbb{D}})$$

The object  $Y^{1/\mathbb{D}}$  then classifies maps from tangent vectors of  $X$  to  $Y$ . If we take  $Y = R$  then  $R^{1/\mathbb{D}}$  is a 1-form classifier!

How to internalize this? David cites David Yetter. Perhaps the 1987 paper “On Right Adjoints of Exponential Functors” cited at <https://ncatlab.org/nlab/show/amazing+right+adjoint>.

Our David proposes internalizing this adjunction by postulating an equivalence of function types, but he first soups them up into dependent function types:

- Postulate that for any type  $Y$  there is a type family  $\varepsilon : \mathbb{D} \vdash Y^{1/\varepsilon} \text{ Type}$
- Postulate  $\sigma : ((\varepsilon : \mathbb{D}) \rightarrow Y^{1/\varepsilon}) \rightarrow Y$  (the counit of this dependent version of the adjunction)

- Postulate that the function

$$((\varepsilon : \mathbb{D}) \rightarrow X_\varepsilon \rightarrow Y^{1/\varepsilon}) \rightarrow (((\varepsilon : \mathbb{D}) \rightarrow X_\varepsilon) \rightarrow Y)$$

sending  $f \mapsto [g \mapsto \sigma(\varepsilon \mapsto f_\varepsilon(g(\varepsilon)))]$  is  $\sharp$ -connected, i.e. under sharp this is an isomorphism.

The non-dependent version then looks like the adjunction:

$$\sharp(X \rightarrow Y^{1/\mathbb{D}}) = \sharp(X^{\mathbb{D}} \rightarrow Y)$$

David comments that this map is natural in  $X$  and partially natural in  $Y$  – there’s a restriction to mapping into  $Y$  “from crisp variables” (?) (To have something well-typed in the presence of sharp, I presume). Natural in this context means stable under pullback?

Classifying “linear forms”.  $X^{\mathbb{D}}$  always has an  $\mathbb{R}$ -action.

I realize in retrospect that I’m not clear on the definition of “linear form.” I think we are just talking about  $n$ -forms but those are smooth and not linear. Maybe they are linear in some other way.

Fact: For KL-vector spaces  $V, W$ ,  $f : V \rightarrow W$  is linear iff it is  $\mathbb{R}$ -equivariant. (KL is Kock-Lawvere).

Proof: (written on whiteboard, not transcribed yet), from Kock. It’s a trick with partial derivatives in the KL-vector spaces.

Therefore instead of constructing a classifier for linear 1-forms, we can classify  $\mathbb{R}$ -equivariant 1-forms. Kock has done this: the 1-form classifier is the equalizer

$$\Lambda^1(V) \rightarrow V^{1/\mathbb{D}} \rightrightarrows (V^{1/\mathbb{D}})^{\mathbb{R}}$$

where the two maps on the right are given by transposing

$$R^{\mathbb{D}} \times (V^{1/\mathbb{D}})^{\mathbb{D}} \rightarrow V$$

$$(r, v) \Rightarrow r(0)\sigma(v) \text{ or } \sigma(r(0)v)$$

We have stopped tracking crisp and sharp, but David believes that when we do, we will be unhappy with the limitations.

Cool facts from standard SDG:

- $d : R \rightarrow \Lambda^1(R)$  has as transpose  $v \mapsto v'(0)$
- If we have  $f : R \rightarrow R$  then  $df : R \rightarrow \Lambda^1$  ( $df$  means composition, but it's a slick link to the classical notation). Walking this back through transposes gives the derivative of  $f$  evaluated at the right basepoint.

Now add another axiom, the “principal of constancy”:  $f : R \rightarrow R, df = 0 \implies f$  constant.

Form this sequence of maps:

$$0 \rightarrow \ker d \rightarrow R \xrightarrow{d} \Lambda_{\text{closed}}^1 \rightarrow 0$$

and note that the principal of constancy tells us, via transposing (I think) that the backward map on the left, from  $R \rightarrow \ker d$  is constant. And so we can connect back to flat and conclude that  $\ker d = \flat R$ , which recreates that  $R/\flat R = \{\text{closed 1-form classifier}\}$ . Furthermore, by the good fibrations trick we can lift against maps and get the antiderivative  $F$  of  $f$  including a choice of value that

fixes a unique antiderivative:

$$\begin{array}{ccc}
 * & \xrightarrow{c} & \mathbb{R} \\
 0 \downarrow & \exists! F \nearrow & \downarrow d \\
 \mathbb{R} & \xrightarrow{f dx} & \Lambda^1_{\text{closed}}
 \end{array}$$

Mathieu had a discussion with David about what spaces are included in this  $\infty$ -topos. Certain cusps will not be included perhaps. David claimed that the cuspy spaces are microlinear.

Next David claimed that  $\int \Lambda^n_{\text{closed}}(R) = \flat B^n R$ . Check his modal fracture paper.

## 1.2 Principal bundles

Adjoint action of  $G$ . He cited the standard stuff from Buchholtz-van Doorn-Rijke. He previewed his Sunday talk by talking about  $BG$  being the type of *exemplars* of  $G$ . Let  $*$  be the base point of  $BG$ . The conjugation action is  $(x : BG) \times (x = x)$ .

The Lie algebra of  $G$  is  $T_{\text{id}}(* = *)$ .

The adjoint action on the Lie algebra is  $BG \rightarrow \text{Type}$  where  $E \mapsto T_{\text{id}}(E = E) := (v : \mathbb{D}) \rightarrow (E = E) \times (v(0) = \text{id})$ .

David assumes  $G$  is microlinear, which is a proposition. Hence  $E = E$  is microlinear.

Now we can form

$$B_{\nabla} G := (E : BG) \times \Lambda^1(T_{\text{id}}(E = E))$$

and this should be the classifying space of principal bundles with connection.

Let's reorganize the maps a little to see in another way that this is a connection.

Form the cospan of a principal bundle and projection from this new larger classifying space:

$$\begin{array}{ccc} X & \overset{E'}{\dashrightarrow} & B_{\nabla}G \\ & \searrow E \quad \swarrow p_1 & \\ & BG & \end{array}$$

What would it take to lift  $E$  to a map to  $E' : X \rightarrow B_{\nabla}G$ ? If we reorganize the data, we can view it as a fiberwise map over  $BG$ :

$$(e : BG) \rightarrow \text{fib}_E(e) \xrightarrow{\omega} \Lambda^1(T_{\text{id}}(E = E)).$$

And what is  $\text{fib}_E(e)$ ? It's  $(x : X) \times (E_x = e)$ . This is the total space of the bundle that  $E$  classifies. So the map  $\omega$  is a 1-form on the total space of the principal bundle, with values in the Lie algebra... perfect! (So long as taking  $\Lambda^1$  of a vector space corresponds to "with values in").

Problem: "this is a lie because  $E$  is not crisp!" In fact, to have  $E$  means we are in fact in  $\mathfrak{b}BG$  which is where we don't want to be, as we will only have flat connections.

## References

- [1] I. Moerdijk and G. Reyes, *Models for Smooth Infinitesimal Analysis*, ser. SpringerLink : Bücher. Springer New York,

2013. [Online]. Available: <https://books.google.com/books?id=UKjxBwAAQBAJ>

- [2] M. Bunge, F. Couso, A. Luis, and A. Fernández, *Synthetic Differential Topology*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, 2018. [Online]. Available: <https://books.google.com/books?id=7M1MDwAAQBAJ>
- [3] D. Yetter, “On right adjoints to exponential functors,” *Journal of Pure and Applied Algebra*, vol. 45, no. 3, pp. 287–304, 1987.