

DRAFT: Discrete differential geometry in homotopy type theory

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Motivation

To use HoTT to study **connections** and **explain** their applicability to algebraic topology, via

- the Gauss-Bonnet theorem
- its vast generalization, Chern-Weil theory

Theorem (Gauss-Bonnet)

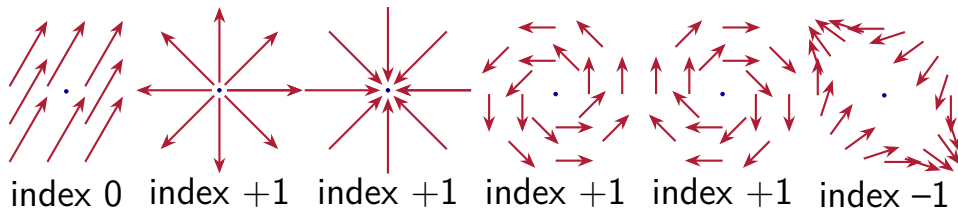
Let M be a compact 2-manifold without boundary, equipped with a Riemannian metric. Let K be the Gaussian curvature of M and let $\chi(M)$ be the Euler characteristic. Then

$$\frac{1}{2\pi} \int_M K \, dA = \chi(M).$$

Theorem (Poincaré-Hopf)

Let M be a compact smooth manifold without boundary. Let X be a vector field on M with isolated zeroes x_1, \dots, x_n . Then

$$\sum_{i=1}^n \text{index}_{x_i} = \chi(M).$$



Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

HoTT background

- ① Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-) Symmetry.
<https://github.com/UniMath/SymmetryBook>.
- ② Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023) Central H-spaces and banded types.
arXiv:2301.02636
- ③ Scoccola, L. (2020) Nilpotent types and fracture squares in homotopy type theory, MSCS 30(5). arXiv:1903.03245

Discrete manifolds in HoTT

- Recall the classical theory of **simplicial complexes**
- Define a **realization** procedure to construct types

Simplicial complexes

Definition

An **abstract simplicial complex** M of dimension n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \dots, M_n]$ consisting of

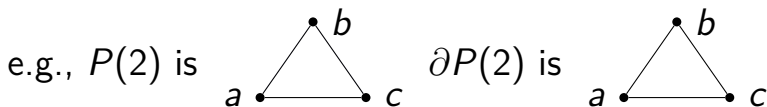
- a set M_0 of $(n + 1)$ vertices
- sets M_k of subsets of M_0 of cardinality $k + 1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j + 1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \dots, M_k]$ **the k -skeleton of M** .

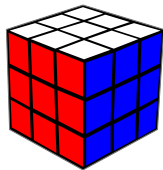
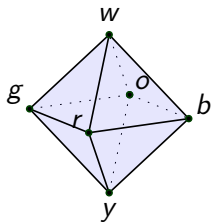
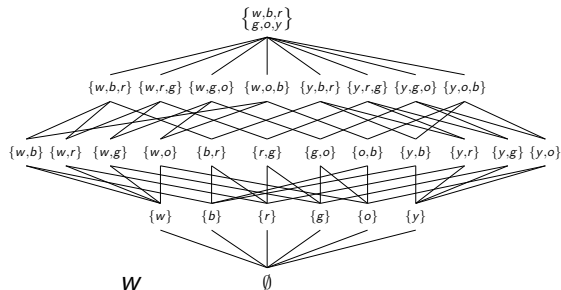
Simplicial complexes

Example

The **complete simplex of dimension n** , denoted $P(n)$, is the set $\{1, \dots, n+1\}$ and its power set. The $(n-1)$ -skeleton $P(n)_{\leq (n-1)}$ is denoted $\partial P(n)$ and will serve as a combinatorial $(n-1)$ -sphere.



Simplicial complexes



Here is a **Hasse diagram** of an abstract octahedron (vertices named for the colors on a Hungarian Cube)

Homotopy realization

We will **realize** simplicial complexes as pushouts.

The realization of a 0-dimensional complex M_0 is the set M_0 .

In particular the 0-sphere $\partial\Delta^1 \stackrel{\text{def}}{=} \partial P(1)$.

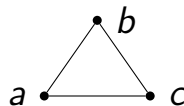
Homotopy realization

For a 1-dim complex $M \stackrel{\text{def}}{=} [M_0, M_1]$ form

$$\begin{array}{ccc}
 M_1 \times \partial\Delta^1 & \xrightarrow{\text{pr}_1} & M_1 \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *_{M_1} \\
 M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1
 \end{array}$$

$$\{\{w\}, \{g\}\} \leftarrow \{\{w, g\}\} \times \{0, 1\} \rightarrow \{\{w, g\}\}$$

Next construct a 1-sphere $\partial\Delta^2 \stackrel{\text{def}}{=} \triangle_{abc}$ as the realization of $\partial P(2)$:



$$\begin{array}{ccc} \partial P(2)_1 \times \partial\Delta^1 & \longrightarrow & \partial P(2)_1 \\ \downarrow & & \downarrow \\ \partial P(2)_0 & \longrightarrow & \partial\Delta^2 \end{array}$$

$$\begin{array}{ccc} \{\{a, b\}, \{b, c\}, \{c, a\}\} \times \{0, 1\} & \longrightarrow & \{\{a, b\}, \{b, c\}, \{c, a\}\} \\ \downarrow & & \downarrow \\ \{\{a\}, \{b\}, \{c\}\} & \longrightarrow & \partial\Delta^2 \end{array}$$

Homotopy realization

Then use $\partial\Delta^1, \partial\Delta^2$ to realize $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$.

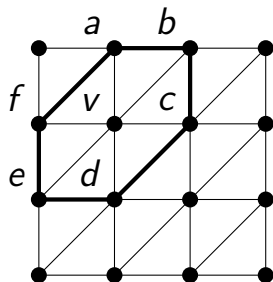
$$\begin{array}{ccccc}
 M_1 \times \partial\Delta^1 & \xrightarrow{\text{pr}_1} & M_1 & & \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *_{\mathbb{M}_1} & & \\
 M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
 & & \uparrow \mathbb{A}_1 & \searrow h_2 & \uparrow *_{\mathbb{M}_2} \\
 & & M_2 \times \partial\Delta^2 & \xrightarrow{\text{pr}_1} & M_2
 \end{array}$$

Homotopy realization

$*M_1, *M_2$ provide **hubs**. h_1, h_2 provide **spokes**.

$$\begin{array}{ccccc}
 M_1 \times \partial\Delta^1 & \xrightarrow{\text{pr}_1} & M_1 & & \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *M_1 & & \\
 M_0 = M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\
 & & \uparrow \mathbb{A}_1 & \searrow h_2 & \uparrow *M_2 \\
 & & M_2 \times \partial\Delta^2 & \xrightarrow{\text{pr}_1} & M_2
 \end{array}$$

Homotopy realization



The **link** of a vertex v in a 2-complex is the polygon of edges not containing v but whose union with v is a face.

This will be our model of the tangent space.

Theorem (Whitehead (1940))

*Every smooth n -manifold has a compatible structure of a **combinatorial manifold**: a simplicial complex of dimension n such that the link is a combinatorial $(n - 1)$ -sphere, i.e. its geometric realization is an $(n - 1)$ -sphere.*

<https://ncatlab.org/nlab/show/triangulation+theorem>

Definition

Let G be a group with identity element e . A G -set is a set X equipped with a homomorphism $\phi : (G, e) \rightarrow \text{Aut}(X)$. If we have

$$\text{is_torsor}(X, \phi) \stackrel{\text{def}}{=} ||X||_{-1} \times \prod_{x:X} \text{is_equiv}(\phi(-, x) : (G, e) \rightarrow (X, x))$$

we say (X, ϕ) is a G -torsor. Denote the type of G -torsors by BG .

Lemma

Point BG at G_{reg} , the G -torsor G acting on itself on the right. Then $\Omega_{G_{\text{reg}}} BG \simeq G$, so BG is a $K(G, 1)$.

- $S^1 : \mathcal{U}$ is not an $\text{Aut } S^1$ -torsor.
- It's a torsor for $(S^1 = S^1)_{(\text{id})}$, the identity component.
- This omits the flip, the reversal of **orientation**.
- See the Buchholtz et. al. H-spaces paper for more.

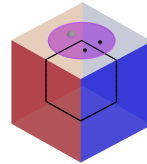
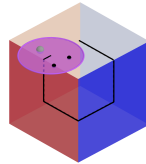
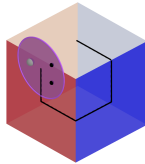
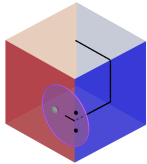
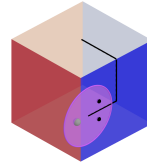
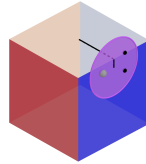
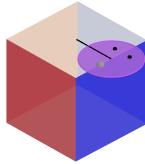
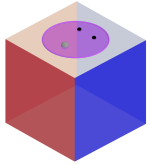
Definition

$$\mathrm{EM}(G, n) \stackrel{\mathrm{def}}{=} \mathrm{BAut}(K(G, n)) \stackrel{\mathrm{def}}{=} \sum_{Y: \mathcal{U}} \|Y \simeq K(G, n)\|_{-1}$$

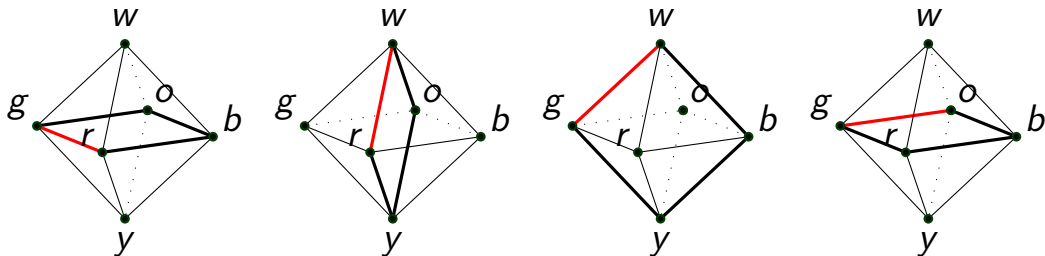
Definition

A $K(G, n)$ -bundle on a type M is a map $f : M \rightarrow \mathrm{EM}(G, n)$.

We further assume f factors through $K(G, n+1)$ and so is principal.



Extend **link** from vertices to edges of the octahedron, by imagining **tipping**:



We obtain $\text{tr}(\partial(wbr)) : Tw = Tw$.

Extend to the face wbr via **homotopy** $b(wbr) : \text{id} = \text{tr}(\partial(wbr))$.

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\iota_0} \dots \xrightarrow{\iota_{n-1}} \mathbb{M}_n$ is a cellular type and all the triangles commute in the diagram:

$$\begin{array}{ccccccc}
 \mathbb{M}_0 & \xrightarrow{\iota_0} & \mathbb{M}_1 & \xrightarrow{\iota_1} & \mathbb{M}_2 & \xrightarrow{\iota_2} & \dots \xrightarrow{\iota_{n-1}} \mathbb{M}_n \\
 & & \searrow f_0 & \searrow f_1 & \downarrow f_2 & & \nearrow f_n \\
 & & & & \mathcal{U} & &
 \end{array}$$

- The map f_k is a **k -bundle** on \mathbb{M} .
- The pair given by the map f_k and the proof $f_k \circ \iota_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a **k -connection on the $(k-1)$ -bundle f_{k-1}** .

Definition

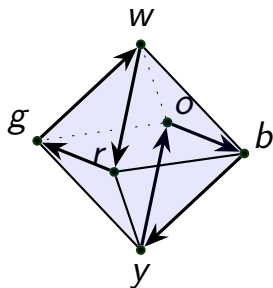
$$\begin{array}{ccc}
 M_k \times \partial\Delta^k & \xrightarrow{\text{pr}_1} & M_k \\
 \mathbb{A}_{k-1} \downarrow & \nearrow h_k & \downarrow *M_k \\
 \mathbb{M}_{k-1} & \xrightarrow{v_{k-1}} & \mathbb{M}_k \\
 & \searrow f_{k-1} & \downarrow f_k \\
 & & \mathcal{U}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \{m_k\} \times \partial\Delta^k & \xrightarrow{!} & \mathbf{1} \\
 \mathbb{A}_{k-1} \downarrow & \nwarrow b_k & \downarrow *M_k \\
 \mathbb{M}_{k-1} & \longrightarrow & \mathcal{U}
 \end{array}$$

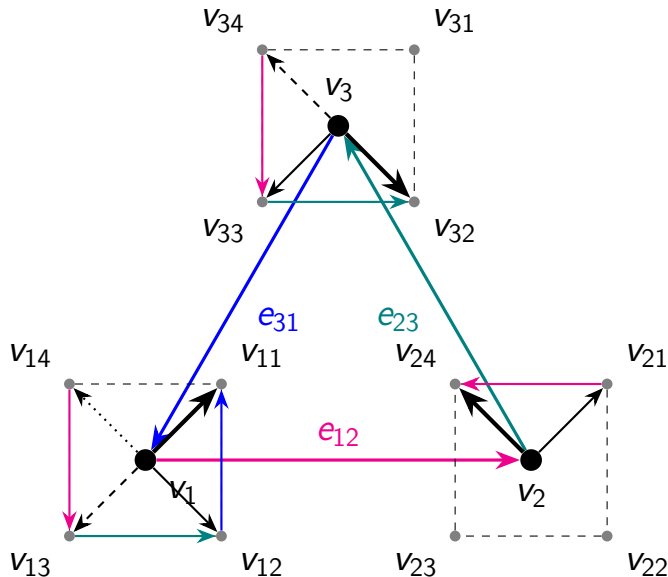
the filler b_k is called a **flatness structure** for the face m_k , and its ending path is called **curvature at the face m_k** .

Vector fields

Let $T : \mathbb{M}_2 \rightarrow K(\mathbb{Z}, 2)$ be an oriented tangent bundle on an oriented 2-dim cellular type

- A **vector field** is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It's a **nonvanishing** vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.





- $\partial F \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$
- We access pathovers asymmetrically:
 $X_{12} : T_{12}X_1 = T_2 X_2$
- $X(\partial F)$ is 3-sided inside a square
- To make a loop we cat with $b(\partial F)$

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad Tm = Tm$$

holonomy

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \quad \mathrm{id} =_{Tm=Tm} \mathrm{tr}(\partial F)$$

flatness

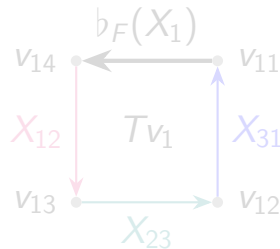
$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \quad \mathrm{tr}(\partial F)(X(m)) =_{Tm} X(m)$$

swirling

Definition

The index of the vector field X on the face F is the integer

$$I_F^X \stackrel{\mathrm{def}}{=} \Omega(\flat_F(X(m)) \cdot X_F) : \Omega(X(m) =_{Tm} X(m)).$$



$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad Tm = Tm$$

holonomy

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \quad \mathrm{id} =_{Tm=Tm} \mathrm{tr}(\partial F)$$

flatness

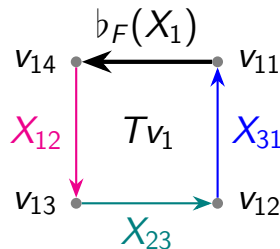
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How do we make these happen?

$$\sum_F \flat_F \iff \int_M K \, dA$$

$$\sum_F I_F^X \iff \sum_{i=1}^n \text{index}_{x_i}$$

$$? \iff \frac{1}{2\pi} \int_M K \, dA + \sum_{i=1}^n \text{index}_{x_i} = 0$$

Observation 1: Use the torsor structure. If we choose $m : \mathbb{M}$ then $T_m = T_m$ acts on all fibers. We can define subtraction $T_i \times T_i \rightarrow (T_m = T_m)$.

Observation 2: Use the vector field. Given $X_i : T_i$ we can form subtraction $-X_i : T_i \rightarrow (T_m = T_m)$. $X_{ij} - X_j : T_{ji} X_i - X_j =_{T_m = T_m} 0$.

Observation 3: Use ap of addition. We can add $\alpha : a =_{\mathbb{C}(4)} 0$ and $\beta : b =_{\mathbb{C}(4)} 0$ to form $\alpha + \beta : (a + b) =_{\mathbb{C}(4)} 0$.

Together these remove the dependency. We can compute b, l, X on each face independently and total them in $T_m = T_m$.

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Thank you.