

DRAFT: Discrete differential geometry in homotopy type theory

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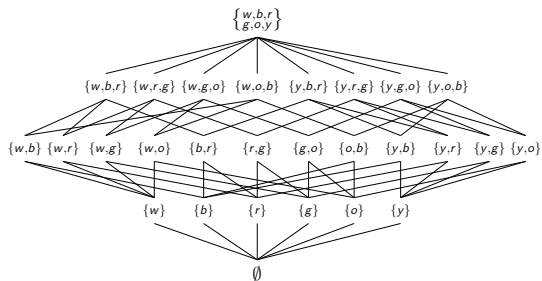
Motivation

The motivation is to provide a deeper explanation for Chern-Weil theory by finding connections and curvature in everyday principal bundles.

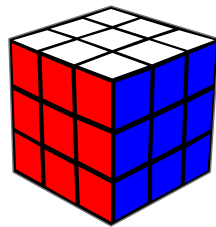
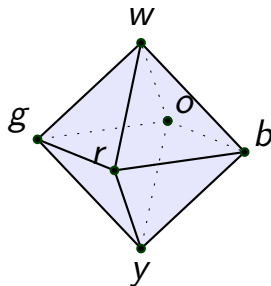
- Construct a type of manifolds
- Construct a classifying space of principal bundles
- Identify connections and curvature
- Put these to use in 2-d to prove that total curvature is an integer

- The classical theory of **simplicial complexes**
- A **realization** functor via higher inductive types (pushouts)

Simplicial complexes



A **Hasse diagram** presents a poset of simplicial complex inclusions (named for the colors on a Hungarian Cube)



Higher realization

$$\begin{array}{ccc} M_1 \times \partial\Delta^1 & \xrightarrow{\text{pr}_1} & M_1 \\ \mathbb{A}_0 \downarrow & \nearrow h_1 \lrcorner & \downarrow * \mathbb{M}_1 \\ M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1 \end{array}$$

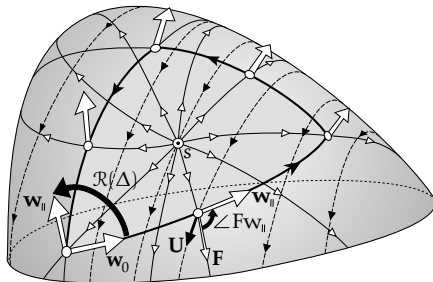
Higher realization

$$\begin{array}{ccccc}
 & & M_2 \times \partial\Delta^2 & \xrightarrow{\text{pr}_1} & M_2 \\
 & & \downarrow \mathbb{A}_1 & \nearrow h_2 & \downarrow *M_2 \\
 M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\
 \uparrow \mathbb{A}_0 & & \nearrow h_1 & \uparrow *M_1 & \\
 M_1 \times \partial\Delta^1 & \xrightarrow{\text{pr}_1} & M_1 & &
 \end{array}$$

Torsors

$$EM(\mathbb{Z}, 1), K(\mathbb{Z}, 2)$$

Classical proof



[26.2] The difference $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ can be found by summing over the edges K_j the change $\Phi(K_j)$ in the illustrated angle $\angle Fw_{||}$, i.e., the rotation of $w_{||}$ relative to F .

- The classical proof is discrete-flavored.
- “ $\angle Fw_{||}$ ” looked a lot like a pathover.
- Hopf’s Φ is defined on edges, not loops. We imitated that too.

Figure: from Tristan Needham, *Visual Differential Geometry and Forms*