

# Discrete differential geometry in homotopy type theory

Greg Langmead

Carnegie Mellon University

April 2025

# Summary

# Summary

This work brings to HoTT

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature = total index

# Classical $\rightarrow$ HoTT

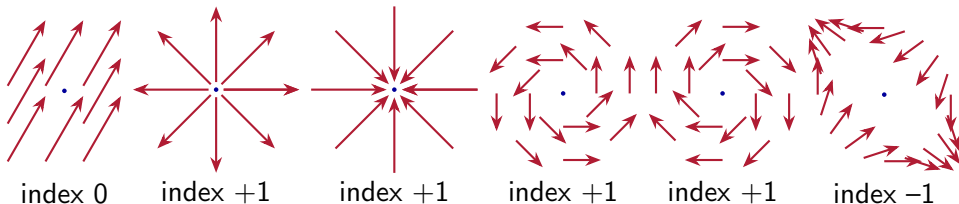
Let  $M$  be a smooth, oriented 2-manifold without boundary,  $F_A$  the curvature of a connection  $A$  on the tangent bundle, and  $X$  a vector field with isolated zeroes  $x_1, \dots, x_n$ .

$$\begin{array}{ccc} \frac{1}{2\pi} \int_M F_A = \sum_{i=1}^n \text{index}_X(x_i) = \chi(M) & & \\ \downarrow & & \downarrow \\ \sum_{\text{faces } F} b_F = & \Longleftrightarrow & \sum_{\text{faces } F} L_F^X \end{array}$$

## Classical index

Near an isolated zero there are only three possibilities: index 0, 1,  $-1$ .

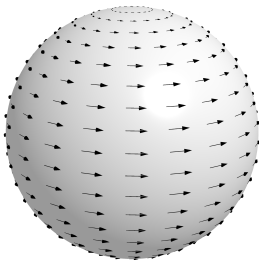
Index is the winding number of the field as you move clockwise around the zero.



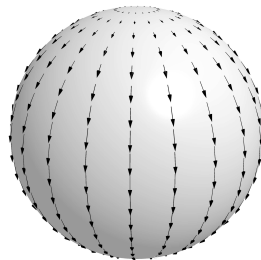
# Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index  $+1$  at each pole  $= 2$



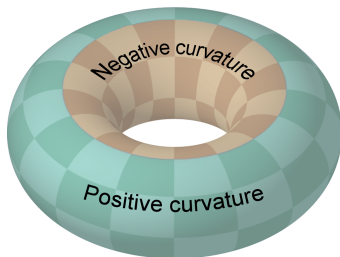
Height: index  $+1$  at each pole  $= 2$

# Gauss-Bonnet theorem

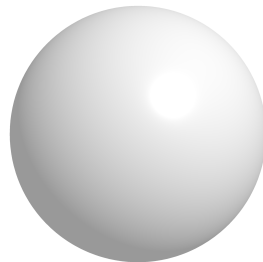
Total curvature divided by  $2\pi$  is the Euler characteristic.

Curvature in 2D is a function  $F_A : M \rightarrow \mathbb{R}$ .

$\int_M F_A$  sums the values at every point.



Positive and negative curvature cancel: **0**



Constant curvature 1, area  $4\pi$ : **2**

# Plan

- Combinatorial manifolds
- Torsors and classifying maps
- Connections and curvature
- Vector fields
- Main theorem



# HoTT background

- ① **Symmetry**,  
Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)  
<https://github.com/UniMath/SymmetryBook>.
- ② **Central H-spaces and banded types**,  
Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)  
arXiv:2301.02636
- ③ **Nilpotent types and fracture squares in homotopy type theory**,  
Scoccola, L. (2020)  
MSCS 30(5). arXiv:1903.03245

## Combinatorial manifolds

# Manifolds in HoTT

- Recall the classical theory of **simplicial complexes**
- Define a **realization** procedure to construct types

# Simplicial complexes

## Definition

An **abstract simplicial complex**  $M$  of **dimension**  $n$  is an ordered list of sets

$M \stackrel{\text{def}}{=} [M_0, \dots, M_n]$  consisting of

- a set  $M_0$  of vertices
- sets  $M_k$  of subsets of  $M_0$  of cardinality  $k + 1$
- downward closed: if  $F \in M_k$  and  $G \subseteq F$ ,  $|G| = j + 1$  then  $G \in M_j$

We call the truncated list

$M_{\leq k} \stackrel{\text{def}}{=} [M_0, \dots, M_k]$  **the  $k$ -skeleton of  $M$ .**

# Simplicial complexes

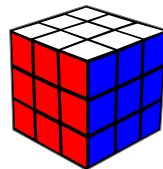
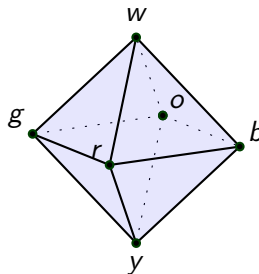
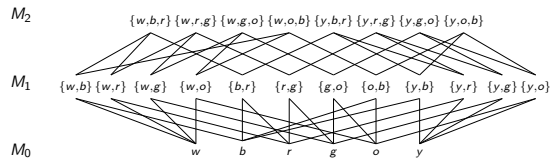
## Definition

An **abstract simplicial complex**  $M$  of **dimension**  $n$  is an ordered list of sets  $M \stackrel{\text{def}}{=} [M_0, \dots, M_n]$  consisting of

- a set  $M_0$  of vertices
- sets  $M_k$  of subsets of  $M_0$  of cardinality  $k + 1$
- downward closed: if  $F \in M_k$  and  $G \subseteq F$ ,  $|G| = j + 1$  then  $G \in M_j$

We call the truncated list

$M_{\leq k} \stackrel{\text{def}}{=} [M_0, \dots, M_k]$  the  **$k$ -skeleton** of  $M$ .

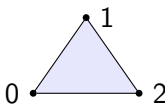
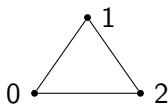


# Simplicial complexes

## Example

The **complete simplex of dimension  $n$** , denoted  $\Delta(n)$ , is the set  $\{0, \dots, n\}$  and its power set. The  $(n-1)$ -skeleton  $\Delta(n)_{\leq (n-1)}$  is denoted  $\partial\Delta(n)$  and will serve as a combinatorial  $(n-1)$ -sphere.

$\Delta(1)$  is visually  $0 \bullet \text{---} \bullet 1$ ,  $\partial\Delta(1)$  is visually  $0 \bullet \quad \bullet 1$ ,

$\Delta(2)$  is visually ,  $\partial\Delta(2)$  is visually 

## Homotopy realization: dimension 0

We will **realize** simplicial complexes by means of **a sequence of pushouts**.

Base case: the realization  $\mathbb{M}$  of a 0-dimensional complex  $M$  is  $M_0$ .

In particular the 0-sphere  $\partial\Delta(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$ .

# Homotopy realization: dimension 1

For a 1-dim complex  $M \stackrel{\text{def}}{=} [M_0, M_1]$  the realization is given by

$$\begin{array}{ccc}
 M_1 \times \partial\Delta(1) & \xrightarrow{\text{pr}_1} & M_1 \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *_{M_1} \\
 M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1
 \end{array}$$



# Homotopy realization: dimension 1

For example the simplicial 1-sphere  $\partial\Delta(2) \stackrel{\text{def}}{=} \begin{array}{c} \bullet 1 \\ \diagup \quad \diagdown \\ 0 \bullet \quad \bullet 2 \end{array}$  is given by

$$\begin{array}{ccc} \partial\Delta(2)_1 \times \partial\Delta(1) & \longrightarrow & \partial\Delta(2)_1 \\ \downarrow & \nearrow h_1 & \downarrow \\ \partial\Delta(2)_0 & \longrightarrow & \partial\Delta(2) \end{array}$$

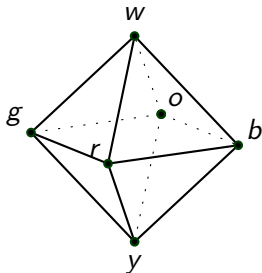
i.e.

$$\begin{array}{ccc} \{\{0,1\}, \{1,2\}, \{2,0\}\} \times \{0,1\} & \longrightarrow & \{\{0,1\}, \{1,2\}, \{2,0\}\} \\ \downarrow & \nearrow h_1 & \downarrow \\ \{0,1,2\} & \longrightarrow & \partial\Delta(2) \end{array}$$

# Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron  $\mathbb{O}$ :

$$\begin{array}{ccc}
 \{\{w, g\}, \dots\} \times \{0, 1\} & \longrightarrow & \{\{w, g\}, \dots\} \\
 \downarrow & \nearrow h_1 & \downarrow \\
 \{w, g, \dots\} & \xrightarrow{\quad \perp \quad} & \mathbb{O}_1
 \end{array}$$



## Homotopy realization: dimension 2

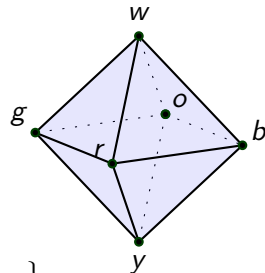
To realize  $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$  use  $\partial\Delta(1), \partial\Delta(2)$ :

$$\begin{array}{ccccc}
 M_1 \times \partial\Delta(1) & \xrightarrow{\text{pr}_1} & M_1 & & \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *M_1 & & \\
 M_0 = \mathbb{M}_0 & \xrightarrow{\quad} & \mathbb{M}_1 & \xrightarrow{\quad} & \mathbb{M}_2 \\
 & & \uparrow \mathbb{A}_1 & \searrow h_2 & \uparrow *M_2 \\
 & & M_2 \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & M_2
 \end{array}$$

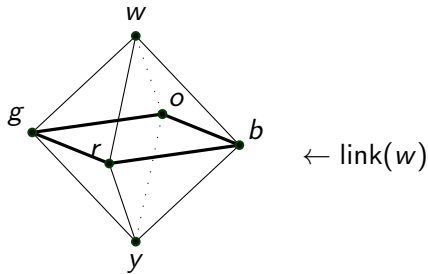
## Homotopy realization: dimension 2

The full octahedron  $\mathbb{O}$ :

$$\begin{array}{ccccc}
 \{\{w, g\}, \dots\} \times \{0, 1\} & \xrightarrow{\text{pr}_1} & \{\{w, g\}, \dots\} & & \\
 \downarrow & \nearrow h_1 & \downarrow & & \\
 \{w, g, \dots\} & \xrightarrow{\quad} & \mathbb{O}_1 & \xrightarrow{\quad} & \mathbb{O}_2 \\
 & & \uparrow & \searrow h_2 & \uparrow \\
 & & \{\{w, g, r\}, \dots\} \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & \{\{w, g, r\}, \dots\}
 \end{array}$$



## Homotopy realization: dimension 2



The **link** of a vertex  $w$  in a 2-complex is: the sets not containing  $w$  but whose union with  $w$  is a face.

A **combinatorial manifold** is a simplicial complex all of whose links are\* simplicial spheres.

This will be our model of the **tangent space**.

\*the (classical) geometric realization is homeomorphic to a sphere

# Combinatorial manifolds $\leftrightarrow$ smooth manifolds

## Theorem (Whitehead (1940))

*Every smooth  $n$ -manifold has a compatible structure of a **combinatorial manifold**: a simplicial complex of dimension  $n$  such that the link is a combinatorial  $(n - 1)$ -sphere, i.e. its geometric realization is an  $(n - 1)$ -sphere.*

<https://ncatlab.org/nlab/show/triangulation+theorem>

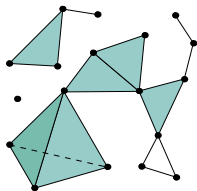
# Combinatorial manifolds $\leftrightarrow$ smooth manifolds

## Theorem (Whitehead (1940))

*Every smooth  $n$ -manifold has a compatible structure of a **combinatorial manifold**: a simplicial complex of dimension  $n$  such that the link is a combinatorial  $(n - 1)$ -sphere, i.e. its geometric realization is an  $(n - 1)$ -sphere.*

<https://ncatlab.org/nlab/show/triangulation+theorem>

Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:



# Torsors



What type families  $\mathbb{M} \rightarrow \mathcal{U}$  will we consider? Families of **torsors**, also called **principal bundles**.

# Torsors

Let  $G$  be a (higher) group.

# Torsors

Let  $G$  be a (higher) group.

## Definition

- A **right  $G$ -object** is a type  $X$  equipped with a homomorphism  $\phi : G^{\text{op}} \rightarrow \text{Aut}(X)$ .

# Torsors

Let  $G$  be a (higher) group.

## Definition

- A **right  $G$ -object** is a type  $X$  equipped with a homomorphism  $\phi : G^{\text{op}} \rightarrow \text{Aut}(X)$ .
- $X$  is furthermore a  **$G$ -torsor** if it is inhabited and the map  $(\text{pr}_1, \phi) : X \times G \rightarrow X \times X$  is an equivalence.

# Torsors

Let  $G$  be a (higher) group.

## Definition

- A **right  $G$ -object** is a type  $X$  equipped with a homomorphism  $\phi : G^{\text{op}} \rightarrow \text{Aut}(X)$ .
- $X$  is furthermore a  **$G$ -torsor** if it is inhabited and the map  $(\text{pr}_1, \phi) : X \times G \rightarrow X \times X$  is an equivalence.
- The inverse is  $(\text{pr}_1, s)$  where  $s : X \times X \rightarrow G$  is called **subtraction** (when  $G$  is commutative).

# Torsors

Let  $G$  be a (higher) group.

## Definition

- A **right  $G$ -object** is a type  $X$  equipped with a homomorphism  $\phi : G^{\text{op}} \rightarrow \text{Aut}(X)$ .
- $X$  is furthermore a  **$G$ -torsor** if it is inhabited and the map  $(\text{pr}_1, \phi) : X \times G \rightarrow X \times X$  is an equivalence.
- The inverse is  $(\text{pr}_1, s)$  where  $s : X \times X \rightarrow G$  is called **subtraction** (when  $G$  is commutative).
- Let  **$BG$**  be the type of  $G$ -torsors.

# Torsors

Let  $G$  be a (higher) group.

## Definition

- A **right  $G$ -object** is a type  $X$  equipped with a homomorphism  $\phi : G^{\text{op}} \rightarrow \text{Aut}(X)$ .
- $X$  is furthermore a  **$G$ -torsor** if it is inhabited and the map  $(\text{pr}_1, \phi) : X \times G \rightarrow X \times X$  is an equivalence.
- The inverse is  $(\text{pr}_1, s)$  where  $s : X \times X \rightarrow G$  is called **subtraction** (when  $G$  is commutative).
- Let  **$BG$**  be the type of  $G$ -torsors.
- Let  $G_{\text{reg}}$  be the  $G$ -torsor consisting of  $G$  acting on itself on the right.

# Facts



# Facts

- ①  $\Omega(BG, G_{\text{reg}}) \simeq G$  and composition of loops corresponds to multiplication in  $G$ .

# Facts

- ①  $\Omega(BG, G_{\text{reg}}) \simeq G$  and composition of loops corresponds to multiplication in  $G$ .
- ②  $BG$  is connected.

# Facts

- ①  $\Omega(BG, G_{\text{reg}}) \simeq G$  and composition of loops corresponds to multiplication in  $G$ .
- ②  $BG$  is connected.
- ③  $1 \ \& \ 2 \implies BG$  is a  $K(G, 1)$ .

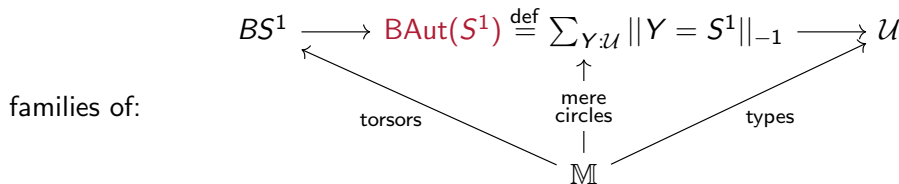
# Facts

- ①  $\Omega(BG, G_{\text{reg}}) \simeq G$  and composition of loops corresponds to multiplication in  $G$ .
- ②  $BG$  is connected.
- ③ 1 & 2  $\implies BG$  is a  $K(G, 1)$ .

See the Buchholtz et. al. H-spaces paper for more.

## How to map into $BS^1$

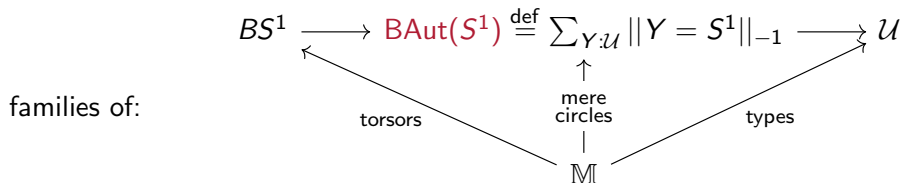
To construct maps into  $BS^1$  we **lift** a family of **mere circles**.



We will assume we have such a lift when we need it. (Remark: the lift is a choice of **orientation**.)

## How to map into $BS^1$

To construct maps into  $BS^1$  we **lift** a family of **mere circles**.



We will assume we have such a lift when we need it. (Remark: the lift is a choice of **orientation**.)

Other names:

- $\text{BAut}(S^1) = BO(2) = \text{EM}(\mathbb{Z}, 1)$  (where  $\text{EM}(G, n) \stackrel{\text{def}}{=} \text{BAut}(K(G, n))$ )
- $BS^1 = BSO(2) = K(\mathbb{Z}, 2)$

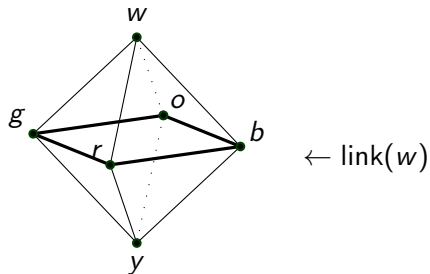
## Connections and curvature

# Connections

Connections are extensions of a bundle to higher skeleta.



## Recall link



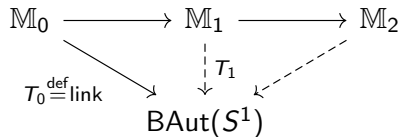
The **link** of a vertex  $w$  in a 2-complex is: the sets not containing  $w$  but whose union with  $w$  is a face.

Define the **tangent bundle** on a combinatorial manifold to be

$$T_0 \stackrel{\text{def}}{=} \text{link} : \mathbb{M}_0 \rightarrow \text{BAut}(S^1).$$

# Connections on the tangent bundle

An extension  $T_1$  of  $T_0$  to  $\mathbb{M}_1$  is called **a connection on the tangent bundle**.

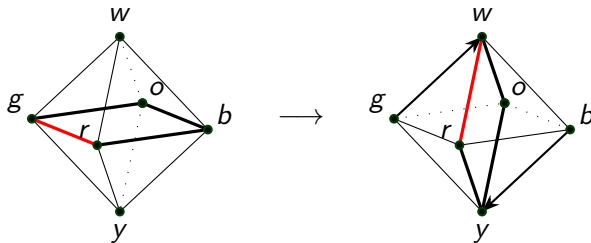


## $T_1 : \mathbb{M}_1 \rightarrow \text{BAut}(S^1)$ extending link

We will define  $T_1$  on the edge  $wb$ , so we need a term

$$T_1(wb) : \text{link}(w) =_{\text{BAut}(S^1)} \text{link}(b).$$

We imagine tipping:

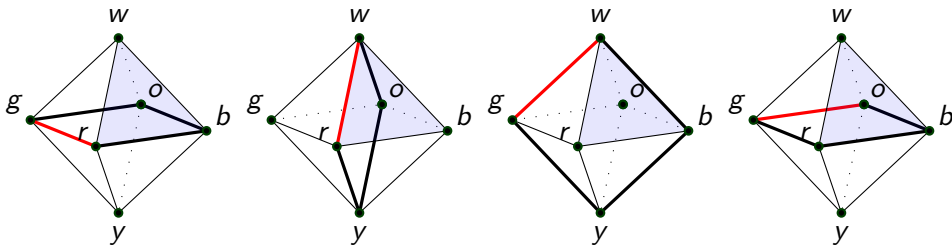


$$T_1(g : \text{link}(w)) \stackrel{\text{def}}{=} w : \text{link}(b), \dots$$

Use this method to define  $T_1$  on every edge.

## $T_1 : \mathbb{M}_1 \rightarrow \text{BAut}(S^1)$ extending link

Denote the path  $wb \cdot br \cdot rw$  by  $\partial(wbr)$ . Consider  $T_1(\partial(wbr))$ :



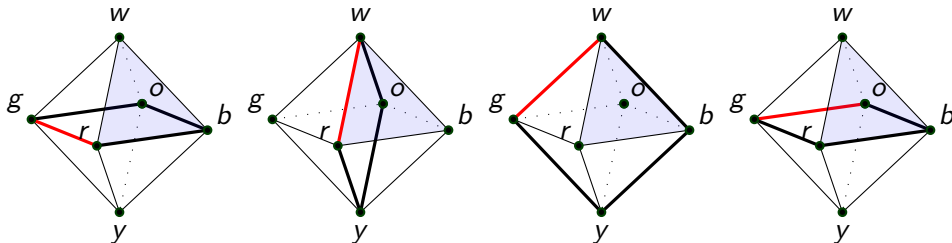
We come back rotated by  $1/4$  turn. Call this rotation  $R : \text{link}(w) =_{\text{BAut}(S^1)} \text{link}(w)$ .



## Extending $T_1$ to a face

Let  $H_{wbr} : \text{refl}_w =_{w=\mathbb{M}w} \partial(wbr)$  be the filler homotopy of the face.

$T_2$  must live in  $T_1(\text{refl}_w) =_{(\text{link}(w)=_{\text{BAut}(S^1)} \text{link}(w))} T_1(\partial(wbr)) = R$

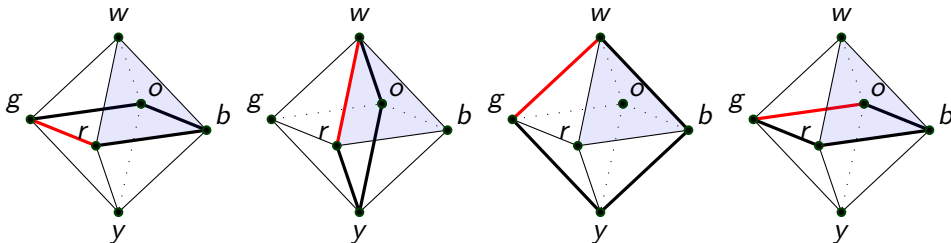


## Extending $T_1$ to a face

Let  $H_{wbr} : \text{refl}_w =_{w=\mathbb{M}w} \partial(wbr)$  be the filler homotopy of the face.

$T_2$  must live in  $T_1(\text{refl}_w) =_{(\text{link}(w)=_{\text{BAut}(S^1)} \text{link}(w))} T_1(\partial(wbr)) = R$

$T_2$  must be a homotopy  $H_R : \text{id} = R$  between automorphisms of  $\text{link}(w)$ .



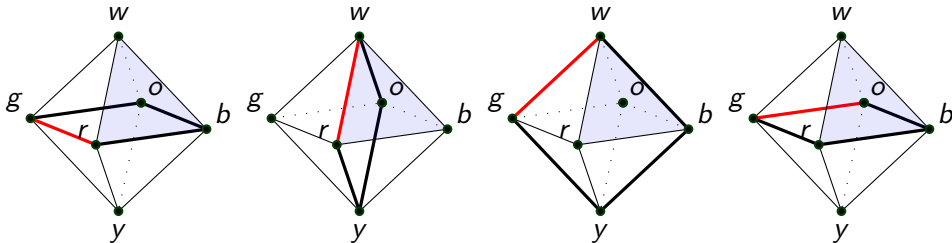
## Extending $T_1$ to a face

Let  $H_{wbr} : \text{refl}_w =_{w=\mathbb{M} w} \partial(wbr)$  be the filler homotopy of the face.

$T_2$  must live in  $T_1(\text{refl}_w) =_{(\text{link}(w)=_{\text{BAut}(S^1)} \text{link}(w))} T_1(\partial(wbr)) = R$

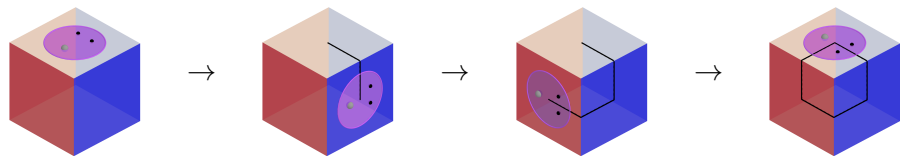
$T_2$  must be a homotopy  $H_R : \text{id} = R$  between automorphisms of  $\text{link}(w)$ .

For example, a path  $H_R(g) : g = Rg = o$ . Choose  $go$ .





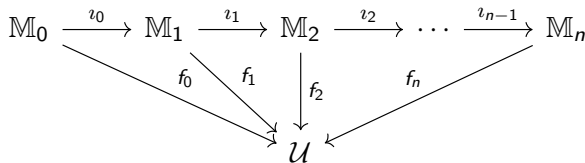
# Original inspiration



# The definition of a connection

## Definition

If  $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\iota_0} \dots \xrightarrow{\iota_{n-1}} \mathbb{M}_n$  is the realization of a combinatorial manifold and all the triangles commute in the diagram:



- The map  $f_k$  is a  **$k$ -bundle** on  $\mathbb{M}$ .
- The pair given by the map  $f_k$  and the proof  $f_k \circ \iota_{k-1} = f_{k-1}$ , i.e. that  $f_k$  extends  $f_{k-1}$  is called a  **$k$ -connection on the  $(k-1)$ -bundle  $f_{k-1}$** .

# The definition of curvature

## Definition (cont.)

An extension consists of  $M_2$ -many extensions to faces:

$$\begin{array}{ccc}
 M_2 \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & M_2 \\
 \mathbb{A}_1 \downarrow & \nearrow h_2 & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathbb{M}_2 \\
 & \searrow T_1 & \downarrow T_2 \\
 & & \mathcal{U}
 \end{array}$$

Here's the outer square for a single face  $F$ :

$$\begin{array}{ccc}
 \{F\} \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & \{F\} \\
 \mathbb{A}_1 \downarrow & \nwarrow b_F & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathcal{U}
 \end{array}$$

# The definition of curvature

## Definition (cont.)

An extension consists of  $M_2$ -many extensions to faces:

$$\begin{array}{ccc}
 M_2 \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & M_2 \\
 \mathbb{A}_1 \downarrow & \nearrow h_2 & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathbb{M}_2 \\
 & \searrow T_1 & \downarrow T_2 \\
 & & \mathcal{U}
 \end{array}$$

Here's the outer square for a single face  $F$ :

$$\begin{array}{ccc}
 \{F\} \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & \{F\} \\
 \mathbb{A}_1 \downarrow & \nwarrow b_F & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathcal{U}
 \end{array}$$

$T_1(\partial(F))$  is the curvature at the face  $F$  and the filler  $b_F : \text{id} = T_1(\partial F)$  is called a flatness structure for the face  $F$ .

# The definition of curvature

## Definition (cont.)

An extension consists of  $M_2$ -many extensions to faces:

$$\begin{array}{ccc}
 M_2 \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & M_2 \\
 \mathbb{A}_1 \downarrow & \nearrow h_2 & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathbb{M}_2 \\
 & \searrow T_1 & \downarrow T_2 \\
 & & \mathcal{U}
 \end{array}$$

Here's the outer square for a single face  $F$ :

$$\begin{array}{ccc}
 \{F\} \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & \{F\} \\
 \mathbb{A}_1 \downarrow & \nwarrow b_F & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathcal{U}
 \end{array}$$

$T_1(\partial(F))$  is the curvature at the face  $F$  and the filler  $b_F : \text{id} = T_1(\partial F)$  is called a flatness structure for the face  $F$ .

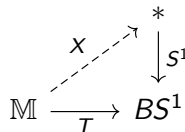
The distinction between the path  $b_F$  and the endpoint  $T_1(\partial(F))$  is small enough to be confusing.

## Vector fields

## Vector fields

Let  $T : \mathbb{M} \rightarrow BS^1$  be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of  $T$ , so that's not a good definition.

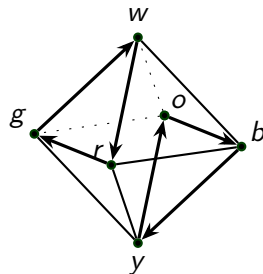
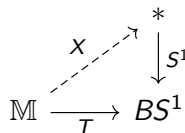


## Vector fields

Let  $T : \mathbb{M} \rightarrow BS^1$  be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of  $T$ , so that's not a good definition.

Our solution:





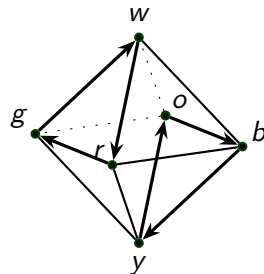
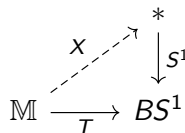
## Vector fields

Let  $T : \mathbb{M} \rightarrow BS^1$  be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of  $T$ , so that's not a good definition.

Our solution:

- A **vector field** is a term  $X : \prod_{m:\mathbb{M}_1} Tm$ .



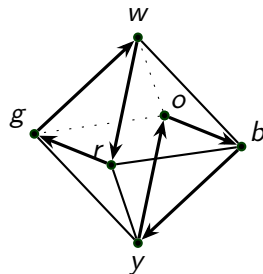
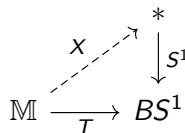
## Vector fields

Let  $T : \mathbb{M} \rightarrow BS^1$  be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of  $T$ , so that's not a good definition.

Our solution:

- A **vector field** is a term  $X : \prod_{m:\mathbb{M}_1} Tm$ .
- It models a classical **nonvanishing** vector field on the 1-skeleton.



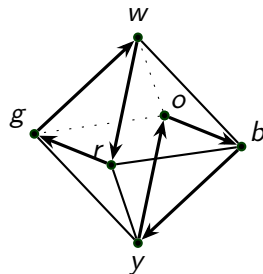
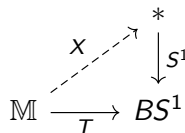
## Vector fields

Let  $T : \mathbb{M} \rightarrow BS^1$  be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

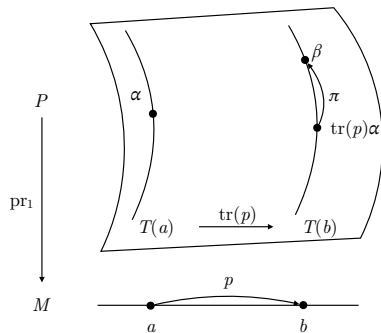
- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of  $T$ , so that's not a good definition.

Our solution:

- A **vector field** is a term  $X : \prod_{m:\mathbb{M}_1} Tm$ .
- It models a classical **nonvanishing** vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

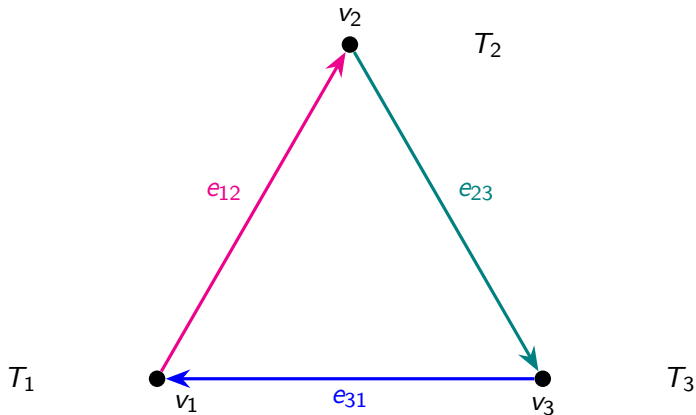


## Reminder: pathovers

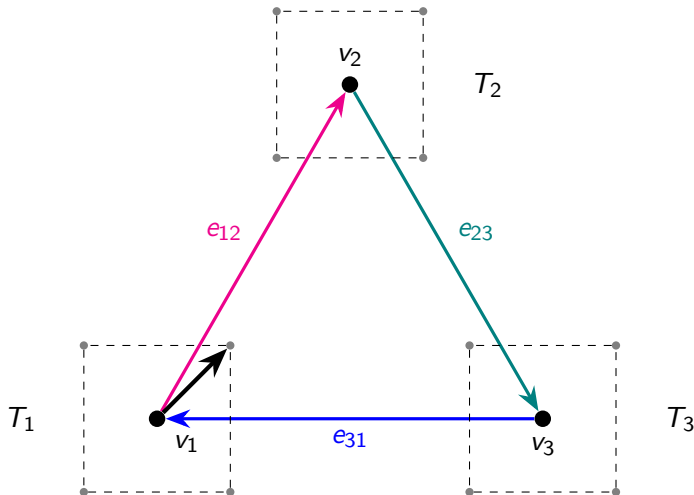


- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display  $\pi$ , the path over  $p$ .
- Dependent functions map paths to pathovers:  
 $\text{apd}(X)(p) : \text{tr}_p(X(a)) = X(b)$  (simply denoted  $X(p)$ ).

Next goal: define the index of a vector field on a face by computing  $X(\partial F)$  around a face.



An example of **swirling** and **index** at this face.



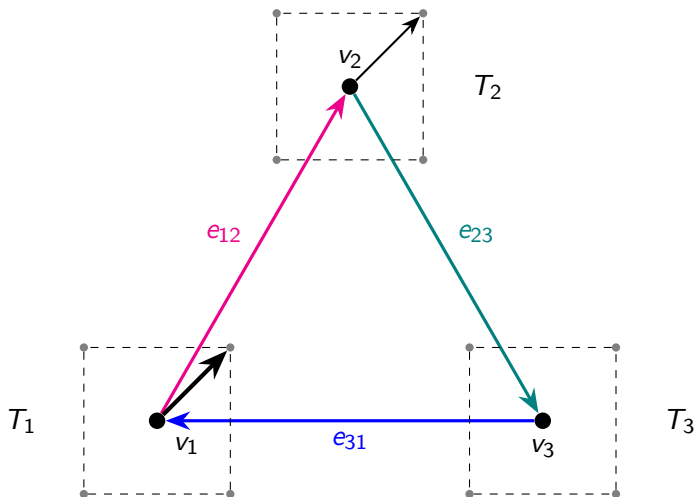
An example of **swirling** and **index** at this face.

- Denote by  $X_1$  this vector  $X(v_1) : T_1$ .

•

•

•



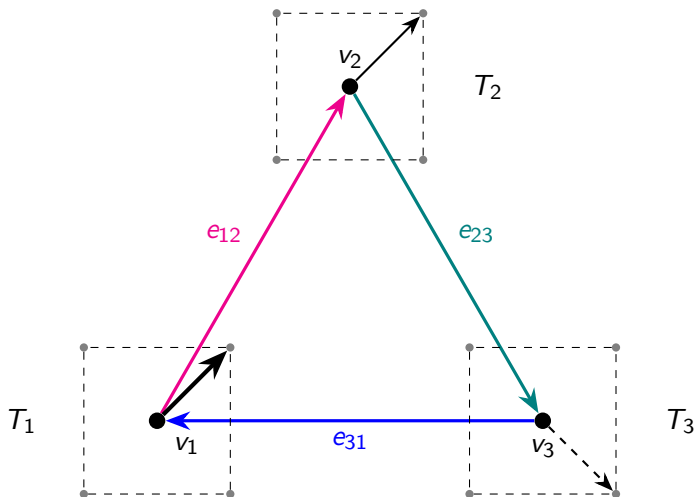
An example of **swirling** and **index** at this face.

- Denote by  $X_1$  this vector  $X(v_1) : T_1$ .
- Say  $T_{21}$  is trivial. Denote the transported vector as thinner.

•

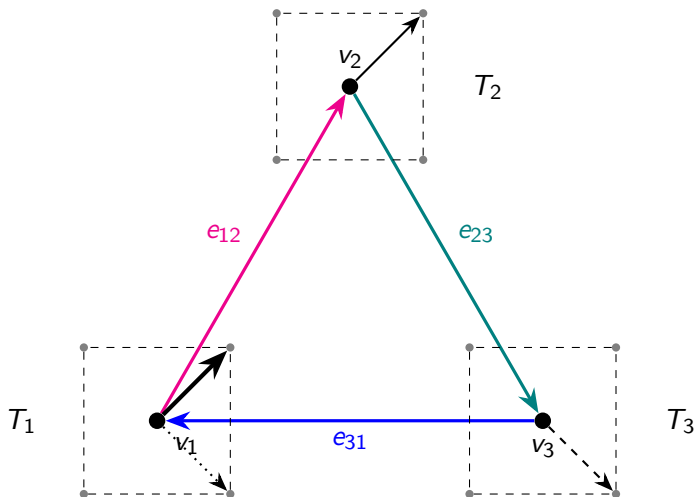
•





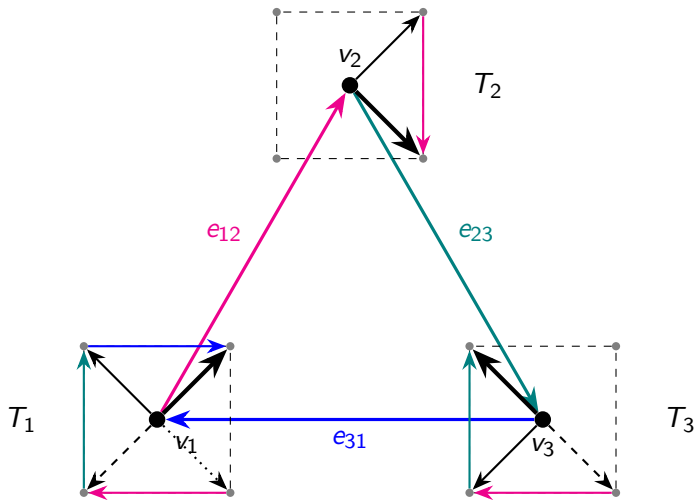
An example of **swirling** and **index** at this face.

- Denote by  $X_1$  this vector  $X(v_1) : T_1$ .
- Say  $T_{21}$  is trivial. Denote the transported vector as thinner.
- Say  $T_{32}$  rotates clockwise. Denote the twice-transported vector as dashed.
-

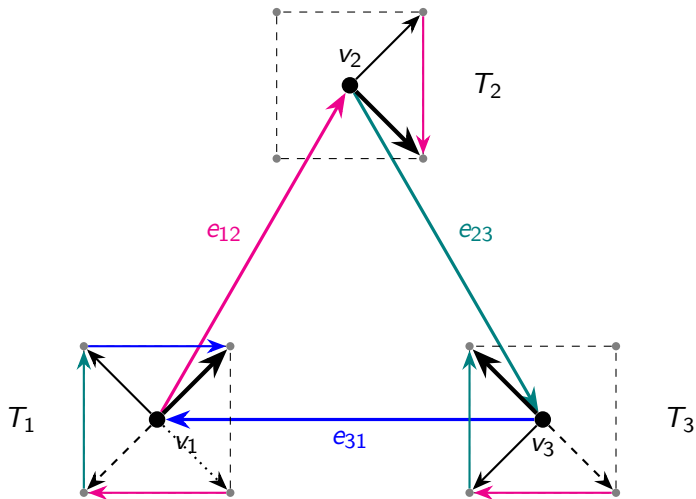


An example of **swirling** and **index** at this face.

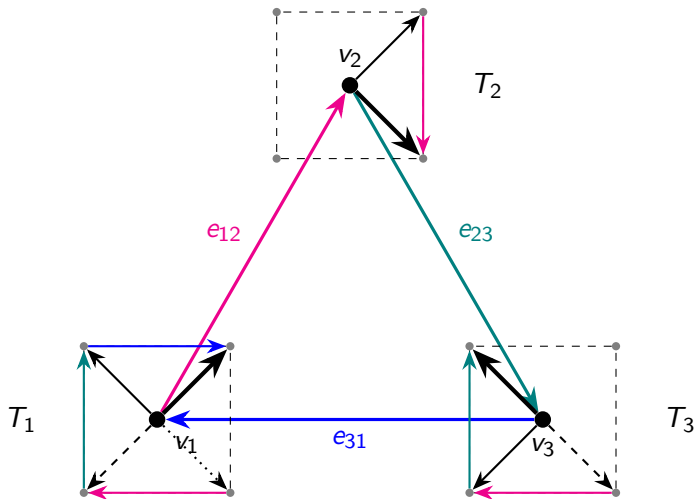
- Denote by  $X_1$  this vector  $X(v_1) : T_1$ .
- Say  $T_{21}$  is trivial. Denote the transported vector as thinner.
- Say  $T_{32}$  rotates clockwise. Denote the twice-transported vector as dashed.
- Say  $T_{13}$  is trivial. The thrice-transported vector is dotted.



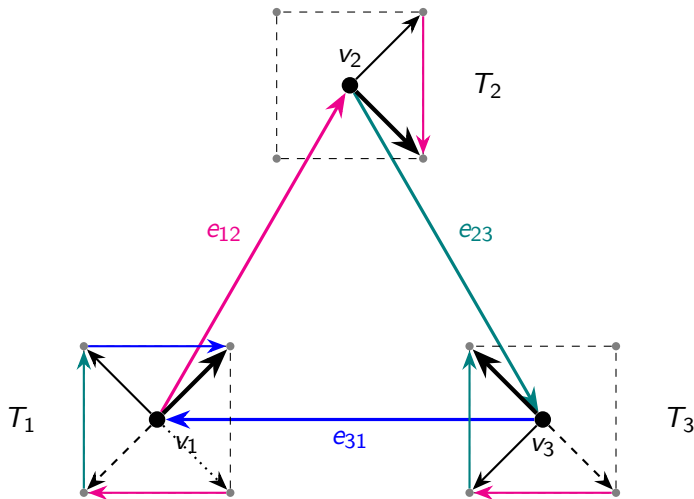
- $X$  on  $e_{12}$  is red, etc.



- $X$  on  $e_{12}$  is red, etc.
- We translated all pathover data to the end of the loop.



- $X$  on  $e_{12}$  is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)



- $X$  on  $e_{12}$  is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)
- The total pathover  $X(\partial F)$  is called **the swirling  $X_F$**  of  $X$  at the face  $F$ .

## Symbolic version

$$T_1 \xrightarrow{T_{21}} T_2 \xrightarrow{T_{32}} T_3 \xrightarrow{T_{13}} T_1$$

$$\begin{array}{ccccccc}
 & & & & T_{13} T_{32} T_{21} X_1 & & \\
 & & & & T_{13} T_{32} X_{21} : \parallel & & \\
 & & T_{32} T_{21} X_1 & & T_{13} T_{32} X_2 & & \\
 & & T_{32} X_{21} : \parallel & & T_{13} X_{32} : \parallel & & \\
 & & T_{32} X_2 & & T_{13} X_3 & & \\
 & & X_{32} : \parallel & & X_{13} : \parallel & & \\
 & & X_3 & & X_1 & & \\
 & T_{21} X_1 & & & & & \\
 X_{21} : \parallel & & & & & & \\
 X_1 & X_2 & & & & & 
 \end{array}$$

# Index

$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \quad \text{curvature}$

$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$

$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$



# Index

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$b_F \stackrel{\mathrm{def}}{=} b(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

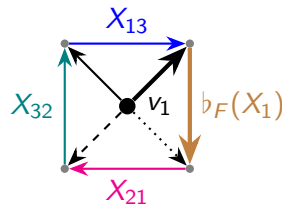
$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

## Definition

The **flattened swirling** of the vector field  $X$  on the face  $F$  is the loop

$$L_F^X \stackrel{\mathrm{def}}{=} b_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The **index** of the vector field  $X$  on the face  $F$  is the integer  $I_F^X$  such that  $\mathrm{loop}^{I_F^X} =_{S^1} (L_F^X) - X_1$ .



# Index

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$b_F \stackrel{\mathrm{def}}{=} b(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

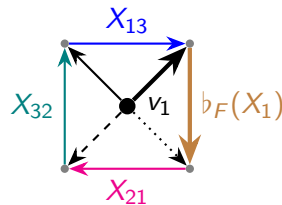
(Recall that  $T_1$  being an  $S^1$ -torsor means we can use subtraction to obtain an equivalence  $s(-, X_1) : T_1 \xrightarrow{x \mapsto x - X_1} S^1$ .)

## Definition

The **flattened swirling** of the vector field  $X$  on the face  $F$  is the loop

$$L_F^X \stackrel{\mathrm{def}}{=} b_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The **index** of the vector field  $X$  on the face  $F$  is the integer  $I_F^X$  such that  $\mathrm{loop}_{S^1}^{I_F^X} =_{S^1} (L_F^X) - X_1$ .



# Main theorem

## Simplifying swirling

Swirling involves concatenating dependent paths. Can we simplify that?

# Pay off all our assumptions 1: torsor structure, vector field

$$T_1$$

$$\begin{array}{c}
 T_{13} T_{32} T_{21} X_1 \\
 T_{13} T_{32} X_{21}: \parallel \\
 T_{13} T_{32} X_2 \\
 T_{13} X_{32}: \parallel \\
 T_{13} X_3 \\
 X_{13}: \parallel \\
 X_1
 \end{array}$$

# Pay off all our assumptions 1: torsor structure, vector field

 $T_1$ 

- Def:  $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$  (trivialization on 0-skeleton).

 $T_{13} T_{32} T_{21} X_1$ 
 $T_{13} T_{32} X_{21} : \parallel$ 
 $T_{13} T_{32} X_2$ 
 $T_{13} X_{32} : \parallel$ 
 $T_{13} X_3$ 
 $X_{13} : \parallel$ 
 $X_1$

# Pay off all our assumptions 1: torsor structure, vector field

 $T_1$ 

- Def:  $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$  (trivialization on 0-skeleton).
- Def:  $\rho_{ji} \stackrel{\text{def}}{=} \alpha_j(T_{ji}(X_i))$  is the rotation of  $T_{ji}$ .

 $T_{13} T_{32} T_{21} X_1$ 
 $T_{13} T_{32} X_{21} : \parallel$ 
 $T_{13} T_{32} X_2$ 
 $T_{13} X_{32} : \parallel$ 
 $T_{13} X_3$ 
 $X_{13} : \parallel$ 
 $X_1$ 

$$\begin{array}{ccc}
 T_i & \xrightarrow{T_{ji}} & T_j \\
 \text{base} \mapsto X_i \left( \begin{array}{c} \nearrow \alpha_i \downarrow \\ \searrow \end{array} \right. & & \left. \downarrow \alpha_j \nearrow \right) \text{base} \mapsto X_j \\
 S^1 & \xrightarrow{(-) + \rho_{ji}} & S^1
 \end{array}$$

# Pay off all our assumptions 1: torsor structure, vector field

 $T_1$ 

- Def:  $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$  (trivialization on 0-skeleton).
- Def:  $\rho_{ji} \stackrel{\text{def}}{=} \alpha_j(T_{ji}(X_i))$  is the rotation of  $T_{ji}$ .

 $T_{13} T_{32} T_{21} X_1$ 
 $T_{13} T_{32} X_{21} : \parallel$ 
 $T_{13} T_{32} X_2$ 
 $T_{13} X_{32} : \parallel$ 
 $T_{13} X_3$ 
 $X_{13} : \parallel$ 
 $X_1$ 

$$\begin{array}{ccc} T_i & \xrightarrow{T_{ji}} & T_j \\ \text{base} \mapsto X_i \left( \begin{array}{c} \alpha_i \downarrow \\ S^1 \end{array} \right. & & \left. \begin{array}{c} \downarrow \alpha_j \\ S^1 \end{array} \right) \text{base} \mapsto X_j \\ & \xrightarrow{(-) + \rho_{ji}} & \end{array}$$

- Lemma:  $\rho_{ij} = -\rho_{ji}$  because in  $T_j$ :  
 $\rho_{ij} + \rho_{ji} + X_j = \rho_{ij} + T_{ji}X_i = T_{ji}(\rho_{ij} + X_i) = T_{ji}T_{ij}X_j = X_j.$



# Pay off all our assumptions 1: torsor structure, vector field (cont.)

$$T_1$$

$$\begin{array}{c}
 T_{13} T_{32} T_{21} X_1 \\
 T_{13} T_{32} X_{21} : \parallel \\
 T_{13} T_{32} X_2 \\
 T_{13} X_{32} : \parallel \\
 T_{13} X_3 \\
 X_{13} : \parallel \\
 X_1
 \end{array}$$

# Pay off all our assumptions 1: torsor structure, vector field (cont.)

 $T_1$ 

- Define  $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},.$

 $T_{13} T_{32} T_{21} X_1$ 
 $T_{13} T_{32} X_{21} : \parallel$ 
 $T_{13} T_{32} X_2$ 
 $T_{13} X_{32} : \parallel$ 
 $T_{13} X_3$ 
 $X_{13} : \parallel$ 
 $X_1$

# Pay off all our assumptions 1: torsor structure, vector field (cont.)

 $T_1$ 

- Define  $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base}, .$
- Paths of the form  $(a =_{S^1} \text{base})$  can be added:

 $T_{13} T_{32} T_{21} X_1$   
 $T_{13} T_{32} X_{21} : \parallel$ 

- $+: (a = \text{base}) \times (b = \text{base}) \rightarrow (a + b = \text{base}).$

 $T_{13} T_{32} X_2$   
 $T_{13} X_{32} : \parallel$ 

- $p + q = (p + b) \cdot q.$

 $T_{13} X_3$   
 $X_{13} : \parallel$   
 $X_1$

# Pay off all our assumptions 1: torsor structure, vector field (cont.)

 $T_1$ 

- Define  $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base}, .$
- Paths of the form  $(a =_{S^1} \text{base})$  can be added:

 $T_{13} T_{32} T_{21} X_1$   
 $T_{13} T_{32} X_{21} : \parallel$ 

- $+: (a = \text{base}) \times (b = \text{base}) \rightarrow (a + b = \text{base}).$

 $T_{13} T_{32} X_2$   
 $T_{13} X_{32} : \parallel$ 

- $p + q = (p + b) \cdot q.$
- Lemma:  $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}.$

 $T_{13} X_3$   
 $X_{13} : \parallel$   
 $X_1$

# Pay off all our assumptions 1: torsor structure, vector field (cont.)

 $T_1$ 

- Define  $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},$
- Paths of the form  $(a =_{S^1} \text{base})$  can be added:

 $T_{13} T_{32} T_{21} X_1$   
 $T_{13} T_{32} X_{21} : \parallel$ 

- $+: (a = \text{base}) \times (b = \text{base}) \rightarrow (a + b = \text{base}).$

 $T_{13} T_{32} X_2$   
 $T_{13} X_{32} : \parallel$ 

- $p + q = (p + b) \cdot q.$
- Lemma:  $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}.$

 $T_{13} X_3$   
 $X_{13} : \parallel$   
 $X_1$ 

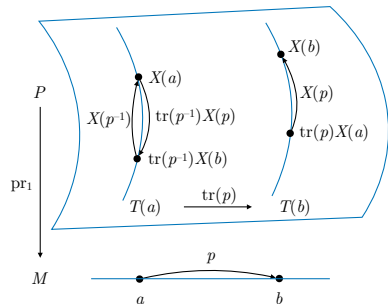
- Proof:  $\text{apd}(X)(\text{refl}) = \text{refl}$   
 $\implies X_{ij} \cdot T_{ij} X_{ji} = \text{refl}_{X_i}$   
 $\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}} \text{ (} T_{ij} \text{ just translates } X_{ji} \text{ to cat with } X_{ji} \text{).}$

# Pay off all our assumptions 1: torsor structure, vector field (cont.)

 $T_1$ 

$$\begin{array}{l}
 T_{13} T_{32} T_{21} X_1 \\
 T_{13} T_{32} X_{21} : \parallel \\
 T_{13} T_{32} X_2 \\
 T_{13} X_{32} : \parallel \\
 T_{13} X_3 \\
 X_{13} : \parallel \\
 X_1
 \end{array}$$

- Define  $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},$
- Paths of the form  $(a =_{S^1} \text{base})$  can be added:
  - $+: (a = \text{base}) \times (b = \text{base}) \rightarrow (a + b = \text{base}).$
  - $p + q = (p + b) \cdot q.$
- Lemma:  $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}.$
- Proof:  $\text{apd}(X)(\text{refl}) = \text{refl}$   
 $\implies X_{ij} \cdot T_{ij} X_{ji} = \text{refl}_{X_i}$   
 $\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}} \text{ (} T_{ij} \text{ just translates } X_{ji} \text{ to cat with } X_{ji} \text{).}$

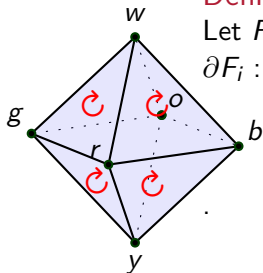


## Pay off all our assumptions 2: no boundary, commutativity

### Definition

Let  $F_1, \dots, F_n$  be the faces of  $\mathbb{M}$ ,  $v_i : F_i$  be designated vertices, and  $\partial F_i : v_i = v_i$  be the triangular boundaries. The **total swirling** is

$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \dots + \sigma_{\partial F_n}$$

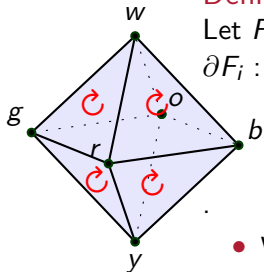


## Pay off all our assumptions 2: no boundary, commutativity

### Definition

Let  $F_1, \dots, F_n$  be the faces of  $\mathbb{M}$ ,  $v_i : F_i$  be designated vertices, and  $\partial F_i : v_i = v_i$  be the triangular boundaries. The **total swirling** is

$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \dots + \sigma_{\partial F_n}$$



- We assume that this expression involves **every edge once in each direction**.

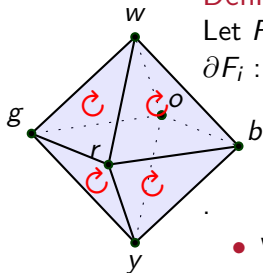


## Pay off all our assumptions 2: no boundary, commutativity

### Definition

Let  $F_1, \dots, F_n$  be the faces of  $\mathbb{M}$ ,  $v_i : F_i$  be designated vertices, and  $\partial F_i : v_i = v_i$  be the triangular boundaries. The **total swirling** is

$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \dots + \sigma_{\partial F_n}$$



- We assume that this expression involves **every edge once in each direction**.
- $S^1$  is commutative, hence **complete cancellation**.

# Consequence

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \quad \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \quad \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

$$L_F^X \stackrel{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F \quad : \quad (X_1 =_{T_1} X_1) \quad \text{flattened swirling}$$

## Consequence

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \quad \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \quad \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

$$L_F^X \stackrel{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F \quad : \quad (X_1 =_{T_1} X_1) \quad \text{flattened swirling}$$

These can all be totaled in  $S^1$  to give

$$\mathrm{tr}_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \rho_{\partial F} = \mathrm{base}$$

$$X_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

$$\flat_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F}$$

$$L_{\mathrm{tot}}^X \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

## Consequence

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \quad \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \quad \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

$$L_F^X \stackrel{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F \quad : \quad (X_1 =_{T_1} X_1) \quad \text{flattened swirling}$$

These can all be totaled in  $S^1$  to give

$$\mathrm{tr}_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \rho_{\partial F} = \mathrm{base}$$

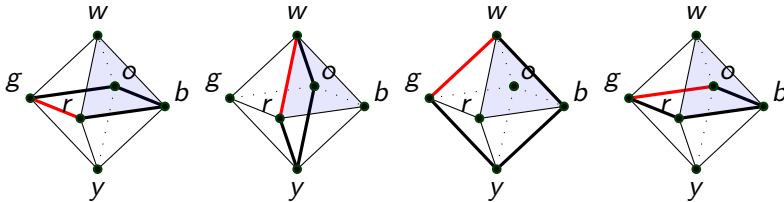
$$X_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

$$\flat_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F}$$

$$L_{\mathrm{tot}}^X \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

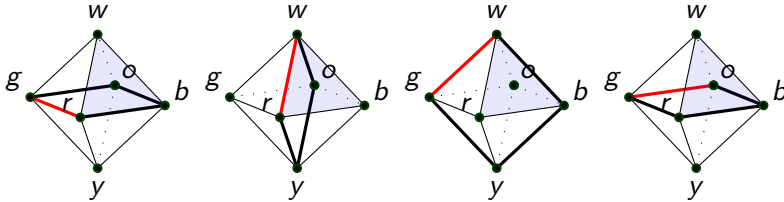
So in our lingo: the total flatness equals the total flattened swirling. □

## Examples

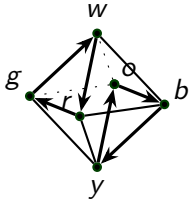


Each face contributes  $b_F = H_R$ , a  $1/4$ -rotation. Total: 2.

## Examples



Each face contributes  $b_F = H_R$ , a  $1/4$ -rotation. Total: 2.



For total index one obtains  $+1$  from  $F_{wrg}$ ,  $+1$  from  $F_{ybo}$ ,  $+0$  from others. Total: 2.

- The classical proof is discrete-flavored.
- “ $\angle Fw_{||}$ ” looked a lot like a pathover.
- Hopf’s  $\Phi$  is defined on edges, not loops. We imitated that too.

 $\angle Fw_{||}$ 

**Carnegie Mellon University**

Thank you!