Discrete differential geometry in homotopy type theory

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November 17, 2024

Abstract

Homotopy type theory can capture some of the major concepts of differential geometry including connections, curvature, and vector fields. We show this by focusing on combinatorial manifolds, which are discrete in the sense of real cohesion[1], and drawing inspiration from the field of discrete differential geometry. We prove the Gauss-Bonnet theorem and Poincaré-Hopf theorem in two dimensions.

> "It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, *The Future Does Not Compute*[2]

Math$_{\text{i}}$e$_{\text{u}}$matical todos

1. How to make a principal bundle that is a sigma type?
2. (done) Improve the introduction of HoTT terms like pathovers, `ap`, `tr`, and `apd`.
3. Prove we have terms and paths in the universe and not just maps between polygons.
4. (done) Introduce a "realization" operation from simplicial sets to HITs.
5. Spell out how the faces of a higher combinatorial manifold form a groupoid, with careful definitions.
6. Spell out what I mean by the replacement manifold $\mathbb{M}_Z$ that has points removed.
7. Emphasize that vector fields are nonvanishing and partial, because they cannot be total.
8. Commit to the definition of a connection, not relying on vibes.
9. Confirm definition of flatness.
10. Fix Leibniz rule or remove (it's independent of everything else, just fun).

Notes

curvature must be defined on any circle (polygon) in the manifold? but classical curvature is always local, doesn't need to be defined on non-bounding loops. holonomy is defined on arbitrary loops, curvature only on loops bounding a 2-cell.

total curvature: one approach: calculate total curvature ad-hoc in my examples, and point to what is needed in general compute total curv of octahedron twice and show they agree

define connections directly on the HIT constructor (dim 1) (higher connections are on the higher skeleta ctors)

ponder this falsehood: truer analogy to classical connections: define a tangent bundle on the HIT, and parallel the classical construction of lifting paths and so on

i don't need to bring in EMzn. EMz1 is the space of circles. keep it, due to link mapping there. i can even have nonorientable tangent bundles. can use scoccola to specify the condition on orientability and relate to kz2 S1 -> Aut S1 -> Z2, extension of Z2 by S1, using Z2 as flip take B of these (or K, 1) map to BAut S1 will lift to BS1 exactly when the map to BZ2 is trivial

1 Overview

We will define

- combinatorial 2-manifolds
- principal circle bundles of tangent bundles
- vector fields,

and then observe emerging from those definitions the presence of

- connections
- curvature
- the index of a vector field,

and prove

- the Leibniz formula
- the Gauss-Bonnet theorem
- and the Poincaré-Hopf theorem.

We will consider functions $M \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where M is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

We will see in Section 3.2 that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. We will construct a map $\mathsf{link} : M \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. Then we can consider the type of pointed mere circles $\mathrm{EM}_\bullet(\mathbb{Z},1) \stackrel{\text{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ as well as the first projection that forgets the point. This is a univalent fibration (univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe[3]). If we form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z},1) \arrow[d, "\mathrm{pr}_1"'] \\ M \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z},1) \end{tikzcd}$$

then we have a bundle of mere circles, with total space given by the $\sum$-type construction. We will show that this is not a principal bundle, i.e. a bundle of torsors. Torsors are types with the additional structure of a group action. But if link satisfies an additional property (amounting to an orientation) then the pullback is a principal fibration, i.e. link factors through a map $\mathrm{K}(\mathbb{Z},2) \to \mathrm{EM}(\mathbb{Z},1)$, where $\mathrm{K}(\mathbb{Z},2)$ is an Eilenberg-Mac Lane space.

We will investigate that the data in dimensions 1 and 2 of link can be thought of as a connection, notably one that is not necessarily flat. Moreover, lifting link to $\mathsf{link}_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z},1)$ can be thought of as a nonvanishing vector field. There will in general not be a total lift, just a partial function. The domain of $\mathsf{link}_\bullet$ will have a boundary of circles, and the winding number on these can be thought of as the index of $\mathsf{link}_\bullet$. We can then examine the total curvature and the total index and prove that they are equal, and equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds. This is similar to the classical proof of Hopf[4], presented in detail in Needham[5].

Because the codomain $\mathrm{K}(\mathbb{Z},2)$ has an H-space structure, we might ask about how the action on paths of link, or any function for that matter, interacts with pointwise multiplication. This will lead us to the Leibniz formula, which emerges simply from horizontal composition in the codomain.

1.1 Future work

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[6][7] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

2 Torsors and principal bundles

The classical theory of principal bundles tells us to look for an appropriate classifying space of torsors to map into. Homotopy type theory tells us that classifying spaces are univalent fibrations. The type of torsors is not a priori such a fibration, so we'll do some work to make that happen. This will constitute the codomain of the investigation.

Definition 2.1. Let G be a group (a set with the usual classical structure and properties). A **G-set** is a set X equipped with a homomorphism $\phi : G \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : ||X||_{-1} \times \prod_{g:G} \mathsf{is\_equiv}(\phi(-, x) : G \to X)$$

then we call this data a **G-torsor**. Denote the type of G-torsors by TG.

If $(X, \phi), (Y, \psi) : TG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 2.1. There is a natural equivalence $(X =_{TG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for TG and we have a group isomorphism $\Omega TG \simeq G$.

Lemma 2.2. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 2.1. The pointed type $(TG, *)$ is a $\mathrm{K}(G, 1)$. □

2.1 Univalent replacement for torsors

The homotopy type theory of cohomology and bundles tells us that the type of principal G-bundles on a type M is the type $M \to \mathrm{K}(G, 1)$. But this is a type of structured types, a connected component of G-sets rather than a connected component of the universe. The paths *in the universe* between two G-sets is equivalent to the type of equivalences between the *underlying types*, not just the equivariant equivalences. We wish to work with a connected component of the universe $\mathcal{U}$.

We'll resolve this problem with the following discussion, following Scoccola[8]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2.2. Let $\mathrm{EM}(G, n) \overset{\mathrm{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \overset{\mathrm{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$. A **$\mathrm{K}(G, n)$-bundle** on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses two self-maps on the universe: suspension followed by $(n+1)$-truncation $||\Sigma||_{n+1}$ and forgetting a point $F_\bullet$ to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{||\Sigma||_{n+1}} \mathrm{EM}_{\bullet\bullet}(G, n+1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n+1)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 2.3. Given $f : M \to \mathrm{EM}(G, n)$, the **associated action of M on G**, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ ||\Sigma||_{n+1} \circ f$.

Theorem 2.1. (Scoccola[8] Proposition 2.39). A $\mathrm{K}(G, n)$ bundle $f : M \to \mathrm{EM}(G, n)$ is equivalent to a map in $M \to \mathrm{K}(G, n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Let's relate this to *orientation*. Note that the obstruction in the theorem is about a map into $\mathrm{EM}_\bullet(G, n+1)$ and further note that $\mathrm{EM}_\bullet(G, n) \simeq \mathrm{K}(\mathrm{Aut}\, G, 1)$ (independent of n). The theorem says that the data of a map into $\mathrm{EM}(G, n)$ factors into data about a map into $\mathrm{K}(G, n+1)$ and one into

$\mathrm{K}(\mathrm{Aut}\, G, 1)$. Informally, $\mathrm{EM}(G, n)$ is a little too large to be a $K(G, n+1)$, as it includes data about automorphisms of G.

In the special case of $\mathrm{EM}(\mathbb{Z}, 1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ is contractible. $\mathrm{Aut}\, \mathbb{Z}$ consists of the $\mathbb{Z}/2\mathbb{Z}$ worth of outer automorphisms given by multiplication by ± 1. Symmetries of the circle are our discrete stand-in for the matrix group $O(2)$, which contains both rotations and (orientation-reversing) reflections of the plane. Requiring a contractible induced map to ± 1 amounts to a choice of direction for all the circles, and so deserves the name "f is *orientable*." In addition $f_\bullet$ deserves to be called the first Stiefel-Whitney class of f, and the requirement here is that it vanishes.

Note 2.1. Reinterpreting more of the theory of characteristic classes would be an enlightening future project. Defining a Chern class and Euler class in 2 dimensions is a goal of this note, but we will not prove all the various laws these classes satisfy (the Whitney sum formula and so on). Nonabelian matrix Lie groups such as $SO(n)$ and $SU(n)$ are not fully imported into homotopy type theory, but recall that some classical results from the theory of characteristic classes are obtained by replacing the group with a maximal torus, which should be a smaller leap from what is presented here[9].

In summary, we can continue to work with the univalent fibration $\mathrm{EM}(G, 1)$ and still know that we are also studying principal $\mathrm{K}(G, 1)$-fibrations, if the bundle is orientable.

2.2 Pathovers in principal bundles

Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ and $P \stackrel{\text{def}}{=} \sum_{x:M} T(x)$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : T(a) = T(b)$. This is a path in the codomain, which in this case is a type of types. Type theory also provides a function called *transport*, denoted $\mathsf{tr}_p : T(a) \to T(b)$ which acts on the fibers of P. tr_p acts on the terms of the types $T(a)$ and $T(b)$, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathsf{tr}_p(\alpha) =_{T(b)} \beta$ between $\alpha : T(a)$ and $\beta : T(b)$ in the fibers. We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

Lastly we want to recall that in the presence of a section $s : M \to P$ there is a dependent generalization of ap called apd: $\mathsf{apd}_p(s) : \mathsf{tr}_p(s(a)) = s(b)$ which is a pathover between the two values of the section over the basepoints of the path p.

3 Combinatorial manifolds

We will adapt to higher inductive types in a straightforward manner the classical construction of *combinatorial manifolds*. See for example the classic book by Kirby and Siebenmann[10]. These are a subclass of simplicial complexes.

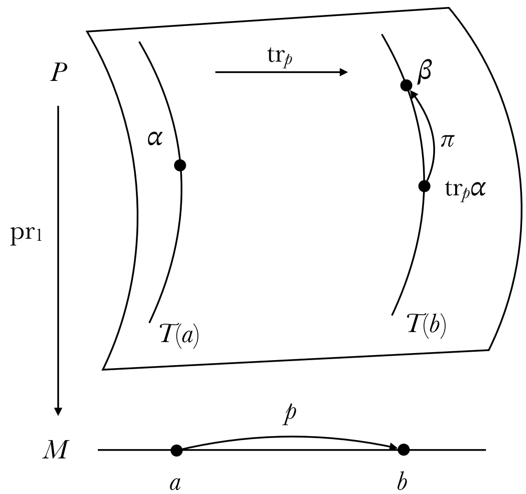

Figure 1: A path π over the path p in the base involves the transport function.

Definition 3.1. An **abstract simplicial complex** M **of dimension** n consists of a set M_0 of vertices, and for each $0 < k \le n$ a set M_k of subsets of M_0 of cardinality $k+1$, such that any $(j+1)$-element subset of M_k is an element of M_j. The elements of M_k are called k-**faces**. Denote by $\mathsf{SimpCompSet}_\mathsf{n}$ the type of abstract simplicial complexes of dimension n (where the suffix Set reminds us that this is a type of sets).

Note that we don't require all subsets of M_0 to be included – that would make M an individual simplex. A simplicial complex is a family of simplices that are identified along various faces.

Definition 3.2. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $n-1$-face containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

The link is all the neighboring vertices of v and the codimension 1 faces joining those to each other. See for example Figure 2.

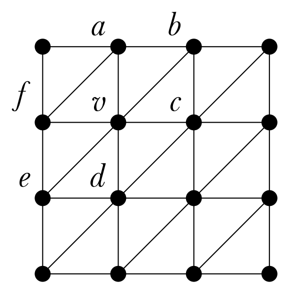

Figure 2: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

Definition 3.3. A **combinatorial manifold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a simplicial sphere of dimension $n-1$ (i.e. its geometric realization is homeomorphic to an $n-1$-sphere). Denote by $\mathsf{CombMfdSet}_\mathsf{n}$ the type of combinatorial manifolds of dimension n (which the notation again reminds us are sets).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figures 3, 4, and 5 for some examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits a combinatorial triangulation[11]. So it appears reasonably well motivated to study this class of objects.

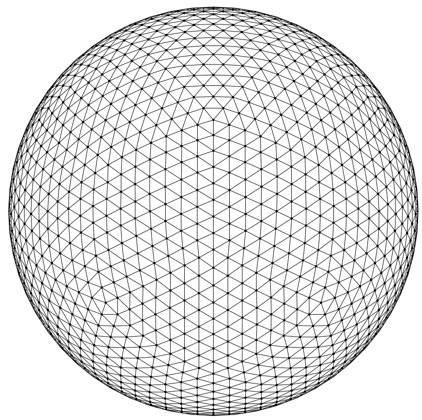

Figure 3: A combinatorial triangulation of a sphere, created with stripy.

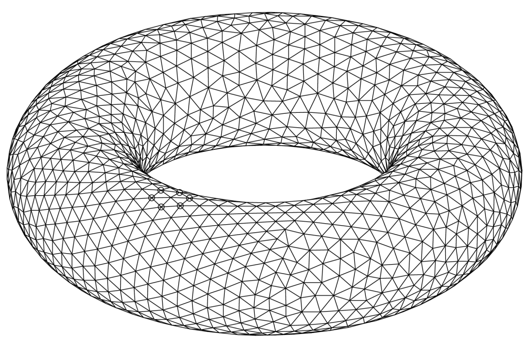

Figure 4: A torus with an interesting triangulation, from Wikipedia. The links have various vertex counts from 5-7. Clearly a constant value of 6 would also work. (By Ag2gaeh - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=30856793)

3.1 Higher inductive combinatorial manifolds

We will convert a simplicial complex M of dimension at most 2 to a higher inductive type, in two steps.

Definition 3.4. Define $\mathsf{CombMfd}_2$ to be the type of **higher inductive constructors of combinatorial manifolds of dimension at most 2** and let $\mathcal{H} : \mathsf{CombMfdSet}_2 \to \mathsf{CombMfd}_2$ be a map from a combinatorial manifold to such a HIT following this method:

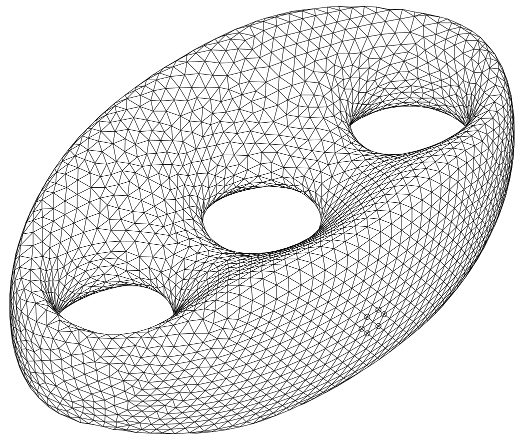

Figure 5: A 3-holes torus with triangulation, from Wikipedia. (By Ag2gaeh - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/wiki/File:Tri-brezel.svg)

1. vertices: a function $\mathsf{v}_0 : M_0 \to \mathcal{H}(M)$ serving as the 0-dimensional constructors
2. edges: a function v_1 on 1-faces, sending $\{a, b\} \mapsto \mathsf{v}_0(a) = \mathsf{v}_0(b)$
3. 2-faces: a function v_2 on 2-faces, sending $\{a, b, c\} \mapsto \mathsf{refl}_a = \mathsf{v}_1(\{a, b\}) \cdot \mathsf{v}_1(\{b, c\}) \cdot \mathsf{v}_1(\{a, c\})^{-1}$.

We will assume there is a rigorous theory of such HITs, and that at least up to dimension 2 there are no obstructions to simply copying over the combinatorial data to the HIT constructors. See for example David Wärn's recent work on pushouts[12].

Definition 3.5. Denote by $\mathcal{R} : \mathsf{CombMfd}_2 \to \mathsf{Type}$ the process of generating a type from the HIT data (which we refer to as **realization**). Note that $\mathcal{R}(\mathcal{H}(M))$ is not in general a set, and may not even be 2-truncated for an arbitrary 2-dimensional combinatorial manifold $M : \mathsf{CombMfdSet}_2$.

We're making the distinction between $\mathcal{H}$ and $\mathcal{R}$ because we will work so closely with the constructors when we map out of them, but of course type theory tells us that to define a map out of the type $\mathcal{R}(\mathcal{H}(M))$ we just have to define it on the constructors, so the distinction is not very sharp.

3.2 Polygons

We will now start looking at some examples, first by defining a type that is important both for the domain and the codomain of mere circles: a square.

Definition 3.6. The higher inductive type C_4 (where C stands for "circle").

$$\begin{aligned} C_4 &: \mathsf{Type} \\ c_1, c_2, c_3, c_4 &: C_4 \\ c_1c_2 &: c_1 = c_2 \\ c_2c_3 &: c_2 = c_3 \\ c_3c_4 &: c_3 = c_4 \\ c_4c_1 &: c_4 = c_1 \end{aligned}$$

$$\begin{tikzcd} c_1 \arrow[r, "c_1c_2"] & c_2 \arrow[d, "c_2c_3"] \\ c_4 \arrow[u, "c_4c_1"] & c_3 \arrow[l, "c_3c_4"] \end{tikzcd}$$

Figure 6: The HIT C_4.

The standard HoTT circle itself is a non-example of a combinatorial manifold since it lacks the second vertex of the edge:

Definition 3.7. The higher inductive type S^1:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

Nonetheless, all polygons are equivalent to each other and to S^1.

Lemma 3.1. Define the function $\ell : C_4 \to S^1$ by

$$\begin{aligned} \ell(c_i) &= \mathsf{base}, i = 1, 2, 3, 4 \\ \ell(c_1c_2) &= \mathsf{loop} \\ \ell(c_2c_3) = \ell(c_3c_4) = \ell(c_4c_1) &= \mathsf{refl}_{\mathsf{base}} \end{aligned}$$

and define the function $s_1 : S^1 \to C_4$ by $s_1(\mathsf{base}) = c_1$ and $s_1(\mathsf{loop}) = c_1c_2 \cdot c_2c_3 \cdot c_3c_4 \cdot c_4c_1$. (The subscript on s_1 reminds us that we chose vertex c_1 to map base to.) Then ℓ and s_1 constitute an equivalence $\mathsf{c4\_equiv} : \mathsf{is_equiv}(\ell, s_1)$.

Proof. We need $\prod_{x:C_4} s_1(\ell(x)) = x$ and $\prod_{y:S^1} \ell(s_1(y)) = y$. $\square$

Recalling that terms of $\mathrm{EM}(\mathbb{Z}, 1)$ are pairs: a type, and a mere equivalence with S^1, we have:

Corollary 3.1. We have $(C_4, ||\mathsf{c4\_equiv}||_{-1}) : \mathrm{EM}(\mathbb{Z}, 1)$.

More claims: given some other square $abcd$ and an equivalence with C_4, we get a path in $\mathrm{EM}(\mathbb{Z}, 1)$. This includes automorphisms of C_4. For example R. Furthermore there is a homotopy from R to the identity, which give a 2-path $\mathsf{refl}_{C_4} = R$ in the universe.

Real-world triangulations of surfaces will often have links whose number of vertices varies across the surface. For example we can see hexagons and pentagons in Figure 3. This presumably introduces only a minor practical inconvenience and doesn't materially affect the discussion to come.

3.3 The higher inductive type $\mathbb{O}$

We will create our first combinatorial surface, a 2-sphere. We will adopt the convention that a subscript indicates the dimension of a subskeleton of a complex. For instance, we have $\mathsf{base} : S^1_0$.

Definition 3.8. The HIT $\mathbb{O}_0$ is just 6 points, intended as the 0-skeleton of an octahedron, with vertices named after the colors on the faces of a famous Central European puzzle cube.

$$w, y, b, r, g, o : \mathbb{O}_0$$

Definition 3.9. The HIT $\mathbb{O}_1$ is the 1-skeleton of an octahedron.

$$\begin{array}{llll}
w, y, b, r, g, o : \mathbb{O}_1 & & & yg : y = g \\
& wb : w = b & & yo : y = o \\
& wr : w = r & & br : b = r \\
& wg : w = g & & rg : r = g \\
& wo : w = o & & go : g = o \\
& yb : y = b & & ob : o = b \\
& yr : y = r & &
\end{array}$$

Definition 3.10. The HIT $\mathbb{O}$ is an octahedron:

$$\begin{array}{llll}
w, y, b, r, g, o : \mathbb{O} & & & \\
& wb : w = b & br : b = r & wbr : wb \cdot br \cdot wr^{-1} = \mathsf{refl}_w \\
& wr : w = r & rg : r = g & wrg : wr \cdot rg \cdot wg^{-1} = \mathsf{refl}_w \\
& wg : w = g & go : g = o & wgo : wg \cdot go \cdot wo^{-1} = \mathsf{refl}_w \\
& wo : w = o & ob : o = b & wob : wo \cdot ob \cdot wb^{-1} = \mathsf{refl}_w \\
& yb : y = b & & yrb : yr \cdot rb \cdot yb^{-1} = \mathsf{refl}_y \\
& yr : y = r & & ygr : yg \cdot gr \cdot yr^{-1} = \mathsf{refl}_y \\
& yg : y = g & & yog : yo \cdot og \cdot yg^{-1} = \mathsf{refl}_y \\
& yo : y = o & & ybo : yb \cdot bo \cdot yo^{-1} = \mathsf{refl}_y
\end{array}$$

We have obvious maps $\mathbb{O}_0 \xrightarrow{i_0} \mathbb{O}_1 \xrightarrow{i_1} \mathbb{O}$ that include each skeleton into the next-higher-dimensional skeleton.

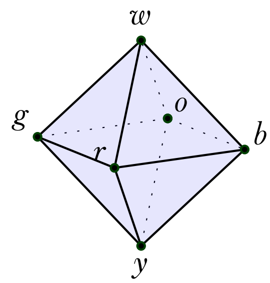

Figure 7: The HIT $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

3.4 Groupoid operations on higher inductive combinatorial manifolds

Let M : $\mathsf{SimpCompSet}_2$ be a combinatorial 2-manifold and $\mathbb{M} \stackrel{\text{def}}{=} \mathcal{H}(M) : \mathsf{CombMfd}_2$ the corresponding higher inductive type. $\mathbb{M}$ has triangular 2-faces just as M does, except they are 2-paths in the HoTT sense. If two faces bca and bdc share the edge bc (see Figure 8), then we can define an operation that combines the combinatorics of simplices with the higher groupoid operations generated by our HIT.

Consider Figure 8 and the 2-paths $bac : ba \cdot ac = bc$ and $bdc : bd \cdot dc = bc$. The 2-path concatenation $bac \cdot bdc^{-1}$ is a path in $ba \cdot ac = bd \cdot dc$. And from there we see we have a 4-gon $abdc : \mathsf{refl}_b = \mathsf{refl}_b$. In this way we can concatenate faces across common boundaries once we choose a common vertex (in this case b).

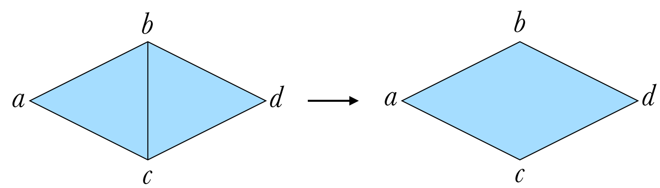

Figure 8: Concatenating the triangles bac and bdc gives the 4-gon $abdc$.

We will have two use cases for this operation. The first is to consider the concatenation of *all* the faces of $\mathbb{M}$, i.e. a term $f_{\mathbb{M}} : \mathsf{refl}_a = \mathsf{refl}_a$ corresponding to $\mathbb{M}$ itself. This will play the role of the "fundamental homology class" from classical topology, which is an object on which 2-forms can be evaluated to compute their value on the whole manifold.

Definition 3.11. If we have a combinatorial manifold $\mathbb{M} : \mathsf{CombMfd}_2$ (or a combinatorial manifold minus some isolated zeros $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{N} \setminus Z$) and $a : \mathbb{M}_0$ is a vertex, a **total face** of $\mathbb{M}$ is a term $f_{\mathbb{M}} : \mathsf{refl}_a = \mathsf{refl}_a$ given by any choice of ordering of the faces $\{f_i\}$, a vertex v_i in each face, and terms $a = v_i$ for each face.

Of course there are many choices in this definition of total face!

The second use case for concatenating faces is to create a HIT related to $\mathbb{M}$ but without one of the point constructors. Figure 9 illustrates the equivalence.

Definition 3.12. If $\mathbb{M} : \mathsf{CombMfd}_2$ is a combinatorial manifold and $Z \subset \mathbb{M}_0$ is a set of vertices in $\mathbb{M}$ with members $Z = \{z_0, \ldots, z_n\}$, then denote by $\mathbb{M} \setminus Z$ the type given by omitting the vertices

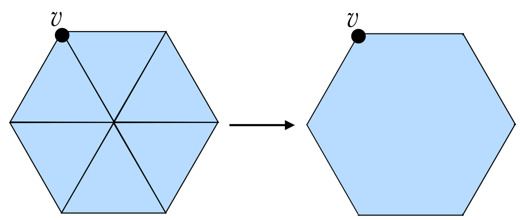

Figure 9: Concatenating the six triangles in the approrpiate way produces a 2-path in $\mathsf{refl}_v = \mathsf{refl}_v$.

in Z from the constructors in all dimensions where they appeared. Call the points of Z **isolated** if no two of them are neighbors, i.e. we have $\prod_{z:Z} \mathsf{link}(z) \cap Z = \emptyset$. In the isolated case $\mathbb{M} \setminus Z$ has boundary circles where each vertex was removed.

Definition 3.13. If we have $\mathbb{M} \setminus Z$ for some isolated set of verticies Z, then for each $z : Z$ we can compose all the faces which contain z, forming a new face (see Figure 9). In this way we produce a HIT called $\mathbb{M}_Z$, which is no longer combinatorial. We call $\mathbb{M}_Z$ the **replacement of** $\mathbb{M}$ **without** Z.

Lemma 3.2. If Z are isolated points of $\mathbb{M}$ then we have $\mathcal{R}(\mathbb{M}) =_{\mathsf{Type}} \mathcal{R}(\mathbb{M}_Z)$.

Proof. Concatenating all the faces around a given point $z : Z$ gives a map $\mathsf{concat}_z : \mathbb{M} \to \mathcal{R}(\mathbb{M}_Z)$. Compose these to form $\mathsf{concat} \stackrel{\text{def}}{=} \mathsf{concat}_{z_n} \circ \cdots \mathsf{concat}_{z_1} : \mathbb{M} \to \mathcal{R}(\mathbb{M}_Z)$. The rest of the data of $\mathbb{M}$ has an obvious inclusion into $\mathbb{M}_Z$ and hence $\mathcal{R}(\mathbb{M}_Z)$. In the other direction we map any face bouding a hexagon to the corresponding concatenation of faces in $\mathcal{R}(\mathbb{M})$ and map the rest of the data by the obvious inclusion. We omit the proof that these are inverses. □

4 Connections and vector fields

4.1 The function T

We will build up a map T out of $\mathbb{O}$ which is meant to be like a tangent bundle. And so we will begin with the intrinsic data of the link at each point: taking the link of a vertex gives us a map from vertices to polygons.

Definition 4.1. $T_0 \stackrel{\text{def}}{=} \mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned} \mathsf{link}(w) &= brgo & \qquad \mathsf{link}(r) &= wbyg \\ \mathsf{link}(y) &= bogr & \qquad \mathsf{link}(g) &= wryo \\ \mathsf{link}(b) &= woyr & \qquad \mathsf{link}(o) &= wgyb \end{aligned}$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

To extend T_0 to a function T_1 on the 1-skeleton we have complete freedom. In other words, defining a map one dimension at a time makes clear that on a HIT the action on paths is extra structure. Two functions on the octahedron could agree on points but differ on edges. We are going to identify this 1-dimensional freedom with a connection:

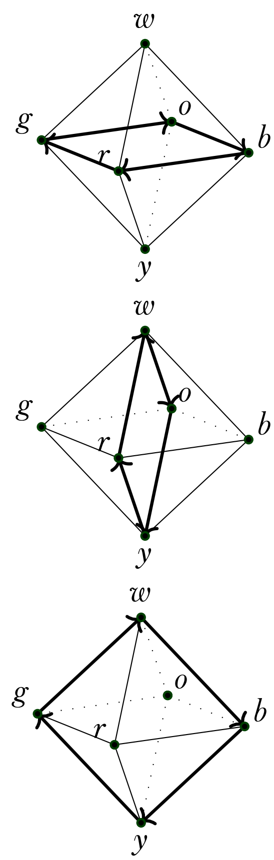

Figure 10: link for the vertices w, b and r.

Definition 4.2. A **connection** on a higher combinatorial manifold is an extension of a principal bundle from the 0-skeleton to the 1-skeleton.

Continuing the example, we will do something "tangent bundley", imagining how T_1 changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 4.3. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ (r, o fixed)
- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ (b, g fixed)
- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$
- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 10 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn.

Definition 4.4. The map $R : C_4 \to C_4$ rotates by one quarter turn, one "click":

- $R(c_1) = c_2$
- $R(c_2) = c_3$
- $R(c_3) = c_4$
- $R(c_4) = c_1$
- $R(c_1c_2) = c_2c_3$
- $R(c_2c_3) = c_3c_4$
- $R(c_3c_4) = c_4c_1$
- $R(c_4c_1) = c_1c_2$

Note that by composing R with the map c4_equiv from Lemma 3.1 we obtain a loop in the universe, a term of $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here *there is no additional freedom*.

Definition 4.5. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows:

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

where $H_R : R = \mathsf{refl}$ is the obvious homotopy.

Definition 4.6. The **curvature of a connection** on a type family $T : \mathbb{M} \to \mathcal{U}$ at a vertex v of a 2-face f with boundary path p_f of a higher combinatorial manifold $\mathbb{M}$ is the automorphism $\mathsf{tr}_{p_f}(Tv)$.

Note 4.1. We have defined a function on a cell by requiring it to correspond to the value on the boundary of that cell. This is familiar in classical differential topology, where it's called *the exterior derivative*. The duality of d and ∂ is recognizable in T_2, and we might say "curvature is the derivative of the connection."

4.2 T on concatenations of faces

Continuing with the classical analogies, we should seek a way to concatenate the curvature on two or more faces. This would correspond to integrating the curvature 2-form over a larger 2-cell, including integrating over a total face to compute *total curvature*. Look again at Figure 8 where we concatenated two faces that share an edge. HoTT maps respect groupoid operations, so we have $T_2(abc \cdot bdc) = T_2(abdc)$. We can double-check this by comparing transport around a 4-gon like $wbyr$.

Lemma 4.1.

$$T_1(wb \cdot by \cdot yr \cdot rw) = T_1(wb \cdot br \cdot rw \cdot wb \cdot by \cdot yr \cdot rb \cdot bw)$$

Proof. Both are equal to R^2 acting to permute $\{b, r, g, o\}$ to $\{g, o, b, r\}$. □

Similarly transport around the other vertical wedges $wryg$, $wgyo$ and $woyb$ are each R^2, and the four wedges can be composed to obtain $R^8 : \mathsf{link}(w) = \mathsf{link}(w)$. This implies that the total face given by concatenating all 8 faces (in this order) maps by T_2 to the homotopy that unwinds R^8 to the identity.

What if we chose another strategy for concatenating the faces of $\mathbb{O}$? Suppose we concatenate all four triangles in the upper hemisphere with w as the basepoint, then move to y and compose the four southern triangles, then move back up to w?

Lemma 4.2.

$$T_1(wb \cdot br \cdot rg \cdot go \cdot ow) = T_1(wb \cdot by \cdot (yr \cdot rb \cdot bo \cdot og \cdot gr \cdot ry) \cdot yb \cdot bw)$$

Proof. Both are equal to R^4. □

And therefore by concatenating the path on the left with the path on the right we see that this different ordering of the faces gives the same value of T_2 as before, namely R^8.

4.3 The torus

We can define a combinatorial torus as a similar HIT. This time each vertex will have six neighbors. So all the links will be merely equal to C_6 which is a hexagonal version of C_4. See Figure 11.

To help parse this figure, imagine instead Figure 12. We take this simple alternating-triangle pattern, then glue the left and right edges, then bend into Figure 11. The fact that each column in Figure 12 has four dots corresponds to the torus in Figure 11 having a square in front, diamonds in the middle, and a square in back.

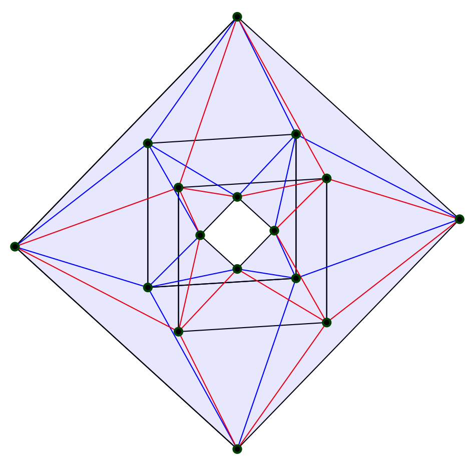

Figure 11: Torus embedded in 3-dimensional space. If you see color in your rendering then black lines trace four square-shaped paths, red ones connect the front square to the middle diamonds, and blue ones connect the back path to the middle ones.

This somewhat arbitrary and unfamiliar model of a torus has the helpful property that it is a combinatorial manifold that is somewhat minimal while still being representable by a donut shape. But the donut-shaped version suggests a very different tangent bundle than the flat model! Starting with the flat model, we can easily see how to define T_1 by sliding a link rigidly along the page to the link of some adjacent vertex. Then we can see that transport around any loop is the identity and so T_2 is always the homotopy $\mathsf{refl}_{\mathrm{id}}$ from the identity to itself.

The donut-shaped torus suggests a different tangent bundle, one determined by the embedding in 3-space that we have represented. But the easiest way to think about that bundle and its connection and curvature is to wait until we have a proof of the Poincaré-Hopf theorem, so that we can instead use a downward-flowing vector field inspired by Morse theory.

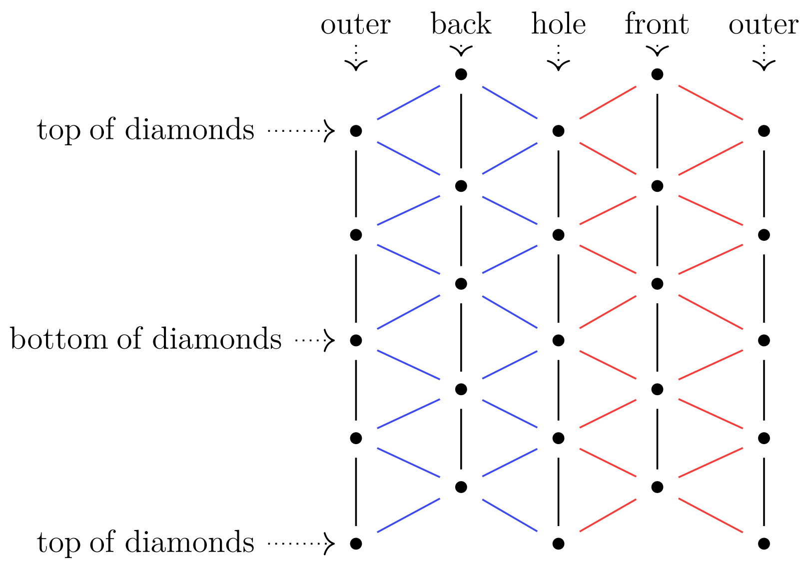

Figure 12: An inspiration for the torus. Identify the sides and then the top, definitionally, to get the actual torus.

4.4 Vector fields

Definition 4.7. A **partial function** $f : A \to B$ is a function $f : A \to B + \star$, the disjoint union of B with the 1-element type.

If $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ is a bundle of mere circles, then a vector field should be a partial function $T_\bullet : \mathbb{M} \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ that lifts T. In other words, a pointing of *some* of the fibers. This aligns with the classical picture of a choice of nonzero vector at each point, except for some points where the vector field vanishes. So instead of having some notion of the full tangent space (one candidate for which would be the disk at each point, i.e. link plus its spokes and filler triangles) we are mapping some vertices to their circular fibers, and others to $\star$. This lets us continue to work with $\mathrm{EM}(\mathbb{Z}, 1)$.

Figure 13 illustrates what removing a point looks like. The resulting type is no longer a combinatorial manifold, since it fails the condition about every point having a circular link.

Definition 4.8. Let $\mathbb{M} : \mathsf{CombMfd}_2$ be a combinatorial manifold and Z an isolated set of vertices. A **vector field** X **on** $\mathbb{M}$ **with zero set** Z is a partial section of P, i.e. a term $X : \prod_{x:\mathbb{M}\setminus Z} T(x)$ (and eliding the unique term of $Z \to \star$). The **exponential map** $\exp : P \to \mathbb{M}$ is the map sending points in a fiber to the corresponding point in the link of the base point: $\exp(x, y : \mathsf{link}(x)) = y$. In commutative diagram form we have:

$$\begin{tikzcd}
P \overset{\mathrm{def}}{=} \sum_{C:T(\mathbb{M}\setminus Z)} C \arrow[r, "\overline{T}"] \arrow[d, "\mathrm{pr}_1"] \arrow[d, bend left, "\exp"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"] \\
\mathbb{M} \setminus Z \arrow[u, bend left, "X:\prod_{x:\mathbb{M}\setminus Z} Tx"] \arrow[ur, "T_\bullet"] \arrow[r, "T"] & \mathrm{EM}(\mathbb{Z}, 1)
\end{tikzcd}$$

where $T_\bullet = \overline{T} \circ X$. Note that exp is different from pr_1 since it spreads a fiber out onto the manifold.

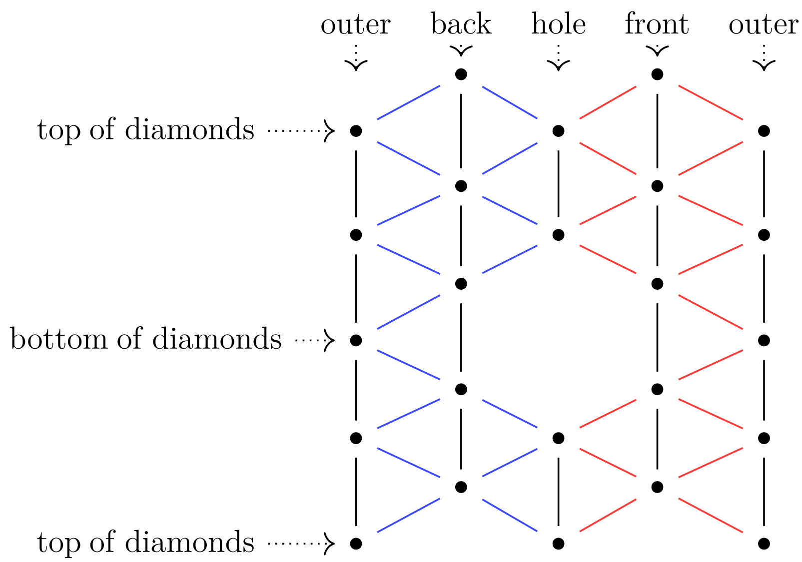

Figure 13: The flat torus with one vertex removed. This also removes the edges and faces containing that vertex.

The composition $\exp \circ X$ is a map $\mathbb{M} \setminus Z \to \mathbb{M}$, and can be thought of as the flow of the vector field.

Let's see a few examples.

Definition 4.9. The **spinning vector field** X_{spin} on $\mathbb{O} \setminus \{w, y\}$ is given by the following data. We compose with exp to keep the notation directly in $\mathbb{O}$. See Figure 14

$$\begin{aligned}
\exp \circ X_{\text{spin}}(b) &= r \\
\exp \circ X_{\text{spin}}(r) &= g \\
\exp \circ X_{\text{spin}}(g) &= o \\
\exp \circ X_{\text{spin}}(o) &= b
\end{aligned}$$

We must also define pathovers and faceovers. For example, $X_{\text{spin}}(b)$ is the point r in the link $woyr$. Transport along br takes the link of b to the link of r, mapping $r : Tb$ to $g : Tr$. This agrees with $X_{\text{spin}}(r)$ and so $X_{\text{spin}}(br) = \mathsf{refl}_g$ in Tr. We similarly obtain refl pathovers for the other equatorial edges. And since we have deleted all the faces when removing the zeros, there are no faceovers.

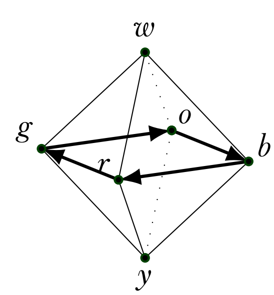

Figure 14: The vector field X_{spin} on $\mathbb{O}$, which circulates around the equator. w and y are zeros.

Definition 4.10. The **downward vector field** X_{down} on $\mathbb{O} \setminus \{w, y\}$ is given by the following data, where again we compose with exp to keep the notation directly in $\mathbb{O}$. See Figure 15

$$\begin{aligned}
\exp \circ X_{\text{spin}}(b) &= y \\
\exp \circ X_{\text{spin}}(r) &= y \\
\exp \circ X_{\text{spin}}(g) &= y \\
\exp \circ X_{\text{spin}}(o) &= y
\end{aligned}$$

We also need to select a pathover for each edge on the equator. Transport on all these edges takes y in one fiber to y in the next, so we choose the path refl_y in all four of these fibers. Again there are no faceovers to map.

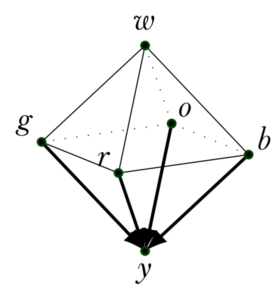

Figure 15: The vector field X_{down} on $\mathbb{O}$, which flows downward. w and y are zeros.

4.5 Index of a vector field

Index should be an integer that computes a winding number "of the vector field" around a zero. We can compute an integer from a map by taking its *degree*, which is a construction we will assume that we have, for example using [13].

Definition 4.11. Let $\mathbb{M} : \mathsf{CombMfd}_2$ and let $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ be the discrete tangent bundle given on $\mathbb{M}_0$ by link. Let $z : Z$ be a zero and let $\mathsf{link}\, z$ be its polygonal link in $\mathbb{M}$, with a clockwise orientation, say with ordered vertices $\{l_{z1}, \ldots, l_{zn}\}$. We call the degree of the map $\mathsf{tr}(\mathsf{link}\, z) : Tl_{z1} = Tl_{z1}$ the **index of** X **at** z. It does not depend on which vertex we use.

Lemma 4.3. The index of X_{spin} at both y and at w is 1, and the same for X_{down}.

Proof. $\mathsf{apd}(X_{\text{spin}})(br) = \mathsf{refl}_{X_{\text{spin}}(r)}$ and similarly for the other edges and for X_{down}. So apd on the loop around the equator is the identity, which has index 1. □

If these vector fields both have index +1, what does index −1 look like? We can see an example on the torus.

On the torus we can also consider both a spinning vector field and a downward vector field. Figure 16 shows one way to spin the torus, and in this case there are no zeros so the index is the degree of a map from the empty set, which is 0 (as it factors through a constant map).

Figure 17 shows a downward flow with four zeros. Although this is a picture of the flat torus, the vector field is derived from the shape of Figure 11 where we actually have a notion of up and down. We see at the position labeled (outer, top of diamonds), i.e. the top of the torus, an everywhere outward pointing vector field. At (outer, bottom of diamonds) we see an inward pointing vector field. But at (hole, top of diamonds), i.e. the top of the hole, we see something else. Looking first at the red arrows, then the downward pointing black arrow, then the blue arrows on the left, then the upward pointing black arrow (which is towards the bottom of the figure) we have moved clockwise on the page, but the arrows have rotated counterclockwise (leftward on the page, then downward, then rightward, then upward on the page). This is a zero of index -1, and so is the one at (hole, bottom of diamonds). Adding these four numbers we again get 0.

Summarizing what we've seen in our examples, vector fields with isolated zeros have an index, and this index tracks with the Euler characteristic.

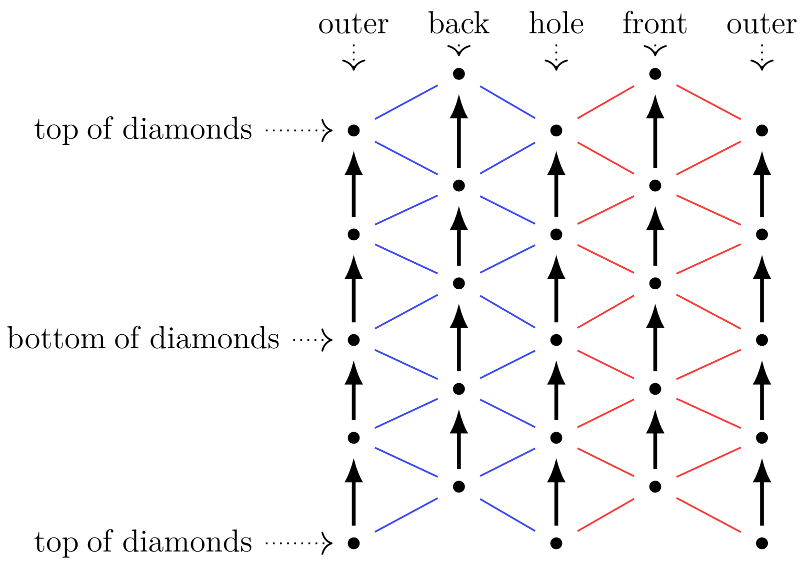

Figure 16: A vector field on the torus that spins and has no zeros.

4.6 Equality of total index and total curvature

Here we are inspired by the classical proof of Hopf[4], presented in detail in Needham[5].

Definition 4.12. An **enumeration** of a principal bundle with connection and vector field on a higher combinatorial manifold consists of the following data:

- a principal bundle $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ on some higher combinatorial manifold
- a nonvanishing vector field $X : \mathbb{M} \setminus Z \to P$ with isolated zeros Z
- a total face of the replacement $\mathbb{M}_Z$, that is
    - a basepoint $a : \mathbb{M}_Z$
    - a term $f_{\mathbb{M}_Z} : \mathsf{refl}_a = \mathsf{refl}_a$ given by

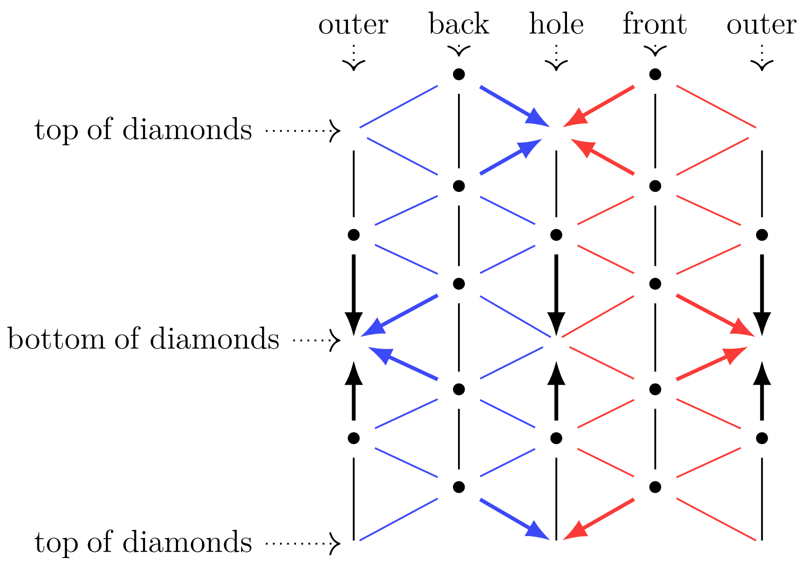

Figure 17: A vector field on the torus that flows downward, when viewed as a donut. The zeros are represented as missing dots. Every dot has one outgoing vector.

- an ordering of the face constructors $\{f_i\}$, with the sub-list of faces denoted $\{f_z k\}$ are the replacement faces at the zeros
- a vertex v_i in each face
- terms $a = v_i$ for each face

Note 4.2. An enumeration let us work both with $\mathbb{M} \setminus Z$ where the vector field is nonvanishing, while also having access to the disks that are missing from $\mathbb{M} \setminus Z$.

Lemma 4.4. The sub-list of faces $\{f_i\} - \{f_z k\}$ obtained by skipping the replacement faces at the zeros, is an ordering of face constructors for $\mathbb{M} \setminus Z$.

Proof. The algorithm that visits each face in order always starts and ends at a and so we can skip any faces we wish, to obtain an ordering of face constructors for the remaining union of faces. □

Lemma 4.5. The degree of Y is minus the total index of X.

Proof. First note that on $\mathbb{M} \setminus Z$ the vector field X trivializes the bundle by mapping into the contractible type of pointed types over $\mathrm{K}(\mathbb{Z}, 2)$. So $X \simeq Y : \mathbb{M} \setminus Z \to (Ta, a)$, the fixed pointed circle in the fiber of the basepoint a.

The ordering of faces $\{f_i\} - \{f_z k\}$ provides an ordering of all the edges, say $\{e_l\}$. Each edge appears an even number of times in this list, traversed in opposite directions, except those bounding a replacement face. These replacement-bounding edges are traversed an odd number of times and can be concatenated to traverse the boundary counterclockwise. Concatenation of paths in S^1 is abelian, so $Y(\{e_l\})$ cancels except on the boundary of the replacement faces, which gives a map from the disjoint union of boundary circles to S^1, with each boundary circle traversed in the counterclockwise direction. The orientation gives the minus sign. □

Consider now the total face $f_{\mathbb{M}_Z}$ of $\mathbb{M}_Z$ and its ordering of faces $\{f_i\}$. Y is only defined on some of these faces. We will define a new function on all the $\{f_i\}$.

Definition 4.13. The **coupling map** $C : f_i \to S^1$ is given by $C(f_i) \overset{\text{def}}{=} \mathsf{apd}(X)(\partial f_i)$ where $\partial f_i : v_i = v_i$ is the clockwise path around the face starting from the vertex supplied by the data of the total face.

Because apd uses both transport and the value of the vector field, it couples the connection with the vector field, hence the name. Of course in HoTT this coupling is built into the definition of apd, so that's another reminder that we aren't straying far from the foundations to find these geometrical concepts.

The fact that C is defined on all the faces, by using the value of the vector field only on the 1-skeleton of $\mathbb{M}_Z$ where it was always defined, lets us make the final part of the argument.

Lemma 4.6. $C : \mathbb{M}_Z \to S^1$ is constant.

Proof. The data of the total face provides an ordering of all the edges. Each edge appears an even number of times, traversed in opposite directions, including the edges in the replacement faces. Concatenation of paths in S^1 is abelian, so the paths all cancel. □

C is similar to X except that it is a total function. On any given edge it computes a path, that is, a homotopy from the function tr to the function X, which we can call "the difference between transport and the vector field on that edge." We have found a way to add all these homotopies together to get 0. We can call this total "the difference between the total index and the total curvature."

Recall now that we made some arbitrary choices in Definition 4.12 of an enumeration, and hence the function C. But since C is unconditionally constant, the space of extra data is contractible.

Corollary 4.1. The total index of a vector field with isolated zeros is independent of the vector field.

Corollary 4.2. The total curvature is an integer.

The last step is to link this value to the Euler characteristic.

4.7 Identification with Euler characteristic

Combinatorial manifolds are intuitive objects that connect directly to the classical definition of Euler characteristic. We can argue using Morse theory, the study of smooth real-valued functions on smooth manifolds and their singularities. Classically the gradient of a Morse function is a vector field that can be used to decompose the manifold into its *handlebody decomposition*. This would be an excellent story to pursue in future work.

Imagine a combinatorial manifold of a genus g oriented surface standing upright with the holes forming a vertical sequence. Now install a vector field that points downward whenever possible. This vector field will have a zero at the top and bottom, and one at the top and bottom of each hole. The top and bottom will be index 1, and ones around the holes will be index -1. We include some sketches in the case of a torus. This illustrates how we obtain the formula for genus g: $\chi(M) = 2 - 2g$.

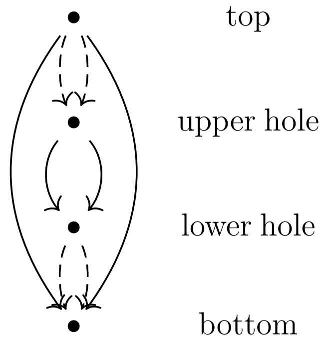

Figure 18: Schematic of the zeros of the downward flow of a torus.

5 Why this works

5.1 Classical connections

Definition 5.1. A **principal bundle** is a smooth map $\pi : P \to M$ between smooth manifolds such that

1. All the fibers of π are equivalent as a smooth manifold to a fixed Lie group G.
2. There is a smooth G-action $P \times G \to P$ on the right that acts on fibers, and does so freely and transitively.
3. There exists an open cover $\{U_i\}$ of M and equivariant diffeomorphisms $\phi_i : P|_{U_i} \to U_i \times G$ (i.e. $\phi_i(p \cdot g) = \phi_i(p) \cdot g$).

Physicists call principal bundle automorphisms "gauge transformations":

Definition 5.2. A **gauge transformation** is a map $\Phi : P \to P$ commuting with the projection to M and which is G-equivariant, i.e. $\Phi(p \cdot g) = \Phi(p) \cdot g$. Denote the group of gauge transformations by $\mathrm{Aut}\, P$. In the literature it is sometimes denoted $\mathscr{G}(P)$.

Definition 5.3. The **vertical bundle** VP of a principal bundle $\pi : P \to M$ with Lie group G is the kernel of the derivative $T\pi : TP \to TM$.

VP can be visualized as the collection of tangent vectors that point along the fibers. It should be clear that at each point of M the group G acts on VP, sending vertical vectors to vertical vectors. In other words, $\mathrm{Aut}\, P$ acts on VP.

Definition 5.4. An **Ehresmann connection** on a principal bundle $\pi : P \to M$ with Lie group G is a splitting $TP = VP \oplus HP$ at every point of P into vertical and complementary "horizontal" subspaces, which is preserved by the action of $\mathrm{Aut}\, P$.

Being preserved by the action of $\mathrm{Aut}\, P$ implies that the complementary horizontal subspaces in a given fiber of $\pi : P \to M$ are determined by the splitting at any single point in the fiber. The action of G on this fiber can then push the splitting around to all the other points.

The utility and parsimony of this definition relates to the solvability of ordinary differential equations. We now have an isomorphism $T_p\pi : H_pP \simeq T_{\pi(p)}M$ between each horizontal space and the tangent space below it in M. This means that given a tangent vector at $x : M$ and a point p in $\pi^{-1}(x)$ we can uniquely lift the tangent vector to a horizontal vector at p. We can also lift vector fields and paths in this way. To lift a path $\gamma : [0, 1] \to M$ you must specify a lift for $\gamma(0)$ and then lift the tangent vectors of γ and prove that you can integrate the lift of that vector field upstairs in HP.

Armed with the lifting of paths one immediately obtains isomorphisms between the fibers of P: given a path in M we can map the starting point of a lift to the ending point. So the three constructions: the Ehresmann connection, the lifting of paths, and transport isomorphisms between fibers are all recapitulations of the structure that the connection adds to the bundle.

5.1.1 Gauge theory

Given a bundle $\pi : P \to M$ there is a space of connections $\mathscr{A}(P)$. The group $\mathrm{Aut}\,P$ acts on this space. For example, a gauge transformation that is constant in the neighborhood of a point will not change the splitting, it will just shift the fiber rigidly along itself. But at the other extreme, a gauge transformation that is changing rapidly near a point will tilt the horizontal subspaces rapidly. The field of **gauge theory** begins with a study of the quotient space $\mathscr{A}(P)/\mathrm{Aut}\,P$.

Note 5.1. Recall that torsors have a physical interpretation as a quantity without a specified unit, such as mass, length, or time. When we choose a base point in a torsor it becomes the standard torsor G acting on itself (for example, the additive real numbers). A physicist is looking for properties or laws that are independent of such a choice. In the 20th century physicists further wondered about choices of units that vary from point to point, and began searching for laws that are invariant under this much larger space of transformations. This led directly to the discovery of connections and curvature as useful fields that complement the matter fields, which are sections of associated vector bundles. They were then led to explore quotienting by the action of the group of gauge transformations, and in particular the space of connections "mod gauge." In this scenario the base manifold M is spacetime, and a gauge transformation is a smoothly varying choice of gauge (units) at each point.

We can characterize connections and curvature in terms of splittings of certain sequences. Atiyah and Bott ([14] equation 3.4) describe the space of vector fields on a total space P as a Lie algebra extension of ΓTM by $\Gamma\,\mathrm{ad}\,P$, respectively the Lie algebra of vector fields on the base and vertical vector fields on P. A non-flat connection will fail to split this sequence because horizontal vector fields may have a non-horizontal component when taking the Lie bracket. This extension is referred to as the *Atiyah sequence*.

In this century mathematicians in HoTT and HoTT-adjacent fields sought an *integrated Atiyah sequence*, including Urs Schreiber[15][16]. This would be a Lie groupoidal version of the Atiyah sequence on Lie algebras. If a groupoid extension could be examined, a link could be sought to Schreier theory. We'll return to these ideas in the next section.

5.2 Type theory version

Moving now to HoTT, fix a type $M : \mathcal{U}$ and a type family $f : M \to \mathcal{U}$. Path induction gives us the transport isomorphism $\prod_{p:a=_M b} \mathsf{tr}(p) : f(a) = f(b)$. We can use this to define a type of *dependent paths*, also called *pathovers* or *paths over* a given path.

Definition 5.5. With the above context and points $\alpha : f(a)$, $\beta : f(b)$ the type of **dependent**

paths over p with endpoints α, β is denoted

$$\alpha \xrightarrow[p]{=} \beta.$$

By induction we can assume p is refl_a in which case $\alpha \xrightarrow[p]{=} \beta$ is $\alpha =_{f(a)} \alpha$.

See [17] for more discussion of dependent paths (where they use the term "path over"), including composition, and associativity thereof.

We recall now the identity type of sigma types:

Theorem 5.1. (HoTT book Theorem 2.7.2 [18]) If $f : M \to \mathcal{U}$ is a type family and $\alpha, \beta : \sum_{x:M} f(x)$ then there is an equivalence

$$\mathsf{split} : (\alpha = \beta) \simeq \sum_{p:\mathrm{pr}_1(\alpha) =_M \mathrm{pr}_1(\beta)} [\mathsf{tr}(p)(\mathrm{pr}_2(\alpha))] = \mathrm{pr}_2(\beta).$$

Definition 5.6. Given $p : a =_M b$ and $\alpha : f(a)$ we have

$$\left(\alpha \xrightarrow[p]{=} \mathsf{tr}(p)(\alpha)\right) \simeq \left(\mathsf{tr}(p)(\alpha) =_{f(b)} \mathsf{tr}(p)(\alpha)\right)$$

which has the term refl which we can call **the horizontal lift of** p **starting at** α.

We can imitate the classical definition of a connection by defining $\omega \overset{\text{def}}{=} \mathrm{pr}_2 \circ \mathsf{split}$, the projection onto the vertical component.

In HoTT if the bundle is classified by $f : M \to \mathcal{U}$ then an automorphism is a homotopy $H : f \sim f$ and the group of gauge transformations is the loop space $\Omega_f(M \to \mathcal{U})$. What is the effect of applying a homotopy $H : f \sim f$ on transport, and on splitting?

H is a family of fiber automorphisms: $H : \prod_{a:M} f(a) = f(a)$ which we can assemble into an equivalence $H' : \sum_{a:M} f(a) = \sum_{a:M} f(a)$ that acts fiberwise. We want to compute the action of $\mathsf{ap}(H')$ on the horizontal-vertical decomposition of paths from Theorem 5.1 by computing $\omega \circ \mathsf{ap}(H') = \mathrm{pr}_2 \circ \mathsf{split} \circ \mathsf{ap}(H')$.

Denote $\sum_{a:M} f(a)$ by P. Let $p : a =_M b$ be a path in the base and let $\pi : (a, \alpha) =_P (b, \beta)$ be a path in P over p. Then $\omega(\pi) : \mathsf{tr}_p(\alpha) = \beta$.

Now let's apply H. We have $\mathsf{ap}(H')(\pi) : (a, H(a)(\alpha)) =_P (b, H(b)(\beta))$ which is still a path over p. Applying ω we get

$$\omega(\mathsf{ap}(H')(\pi)) : \mathsf{tr}_p(H(a)(\alpha)) = (H(b)(\beta)).$$

Using the lemma below we can if we wish rewrite this as

$$\omega(\mathsf{ap}(H')(\pi)) : H(b)\left[\mathsf{tr}_p(\alpha) = \beta\right]$$

which uses only $H(b)$. This is the action of gauge transformations on connections.

Lemma 5.1. Given a function $f : M \to \mathcal{U}$, path $p : a =_M b$, and homotopy $H : f \sim f$ the following square commutes and so in the type family we have $\mathsf{tr}(H(x) \cdot f(p)) = \mathsf{tr}(f(p) \cdot H(y))$.

$$\begin{tikzcd}
f(a) \arrow[r, equal, "f(p)"] \arrow[d, equal, "H(a)"'] & f(b) \arrow[d, equal, "H(b)"] \\
f(a) \arrow[r, equal, "f(p)"'] & f(b)
\end{tikzcd}$$

HoTT provides new ways to talk about trivializations of bundles, and flatness of connections.

Theorem 5.2. The following are equivalent for a map $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$:

1. The principal bundle $P \stackrel{\text{def}}{=} \sum_{x:M} Tx$ is a trivial bundle.
2. The dependent sum $\sum_{x:M} Tx$ is equivalent to a non-dependent sum.
3. There exists a total lifting $T_\bullet$ to pointed types.
4. T is contractible.

Proof. 1 and 2 are equivalent by definition. 1 implies 3 by choosing the basepoint of the second factor of $M \times S^1$. 3 implies 1 because the global choice of basepoint is a global isomorphism with S^1. 1 and 4 are equivalent because a trivialization is a contraction to $M \times S^1$. □

The classical definition of a flat connection is that contractible loops lift to horizontal loops, i.e. there is no holonomy around small loops. This implies that homotopic paths have the same transport. Here's how we'll describe this in HoTT:

Definition 5.7. We call a connection on $\sum_{x:M} Tx$ **flat** if T factors through the 1-truncation $||M||_1$.

Lemma 5.2. If $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ is flat and M is simply connected then T is trivial.

Proof. T factors through $||M||_1$ which is contractible. □

5.2.1 Gauge theory revisited

The sigma type $P \stackrel{\text{def}}{=} \sum_{x:M} Tx$ and projection map to M package the insights of the Atiyah sequence and observations about what does and doesn't split:

The fiber sequence $S^1 \to P \to M$ does not split unless P is trivial, by Theorem 5.2.

Paths $p : a =_M b$ do lift to P given a starting point $\alpha : Ta$. This is what we are calling the connection, and it is the finite version of the vertical/horizontal splitting $TP = VP \oplus HP$. Theorem 5.1 provides the factoring of pathovers into horizontal and vertical. So at the level of paths there *is* a splitting, a map from M to P.

But suppose we have two paths $p, q : a =_M b$ and a point α over a. If we concatenate the two horizontal lifts $(p, \mathsf{refl}_{\mathsf{tr}(p)(\alpha)})$ and $(q^{-1}, \mathsf{refl}_{\mathsf{tr}(q^{-1})\circ\mathsf{tr}(p)(\alpha)})$ into a loopover of $p \cdot q^{-1}$ then we get a term in $(p \cdot q^{-1}, \mathsf{tr}(q^{-1}) \circ \mathsf{tr}(p)(\alpha) = \alpha)$. As we have seen this can be a non-refl path in Ta! The concatenation of two horizontal lifts can be non-horizontal. This is the analogous statement to Atiyah's observation that the Lie bracket of two horizontal vectors can have a vertical component, and that this can be identified with curvature.

To work with connections mod gauge we don't really have to add anything to our existing picture, because we are always working with higher types such as $M \to \mathrm{EM}(\mathbb{Z}, 1)$ which have automorphisms (in this case gauge transformations, aka self-homotopies) baked in.

Thinking back to the desired link with Schreier theory, David Jaz Myers showed that in the case of higher groups we have an equivalence between the type of extensions of a group G by F and the actions of G on a delooping BF:

$$\mathrm{Ext}(G; F) \simeq (BG \cdot\!\!\to \mathrm{BAut}(BF))$$

(see [19] Theorem 2.5.7). Our type of classifying maps $M \to \mathrm{EM}(\mathbb{Z}, 1)$ can be seen as extensions of $\pi_1(M)$, or of M itself, by the group S^1. What a lovely reframing of principal bundles.

5.3 The Leibniz (product) rule

The intuition that motivates everything in this note is that derivatives and connections are visible in type theory through the action on paths, because a path is a finite version of an infinitesimal tangent vector. And if we think we have some new understanding of differentiation, then we should be able to see the Leibniz rule.

The Leibniz rule, or product rule, for differentiation states that if $f, g : M \to \mathbb{R}$ are two smooth functions to the real numbers then $d(fg) = fdg + gdf$. Here fg is the function formed by taking the pointwise product of f and g. This is an interaction between multiplication in $\mathbb{R}$ and the action on vectors of smooth functions (the 1-forms df and dg).

To examine this situation in HoTT we need type-theoretic functions $f, g : M \to B$ from some type M to a central H-space B. Let $\mu : B \to B \to B$ be the H-space multiplication. How does μ act on paths? Suppose we have $a, a', b, b' : B$ and $p : a =_B a', q : b =_B b'$. Then we also have homotopies $\mu(p, -) : \mu(a, -) =_{B\to B} \mu(a', -)$ and $\mu(-, q) : \mu(-, b) =_{B\to B} \mu(-, b')$. Since $\mu(a, -) : B = B$ is an (unpointed) equivalence of B, and similarly for $\mu(b, -)$ and so on, this data assembles into the following diagram of higher groupoid morphisms:

$$\begin{tikzcd} B \arrow[r, bend left, "\mu(a,-)"] \arrow[r, bend right, "\mu(a',-)"'] \arrow[r, phantom, "\mu(p,-)\Downarrow"] & B \arrow[r, bend left, "\mu(-,b)"] \arrow[r, bend right, "\mu(-,b')"'] \arrow[r, phantom, "\mu(-,q)\Downarrow"] & B \end{tikzcd}$$

And so the two homotopies can be horizontally composed to give a path

$$\mu(p, -) \star \mu(-, q) : \mu(a, b) = \mu(a', b').$$

Horizontal composition is given by

$$\mu(p, -) \star \mu(-, q) \overset{\text{def}}{=} (\mu(p, -) \cdot_r \mu(-, b)) \cdot (\mu(a', -) \cdot_l \mu(-, q))$$

where

$$\mu(p, -) \cdot_r \mu(-, b) : \mu(a, b) = \mu(a', b)$$

and

$$\mu(a', -) \cdot_l \mu(-, q) : \mu(a', b) = \mu(a', b')$$

are defined by path induction. See the HoTT book Theorem 2.1.6 on the Eckmann-Hilton argument[18].

We can recognize the process of using whiskering to form horizontal composition in the Leibniz rule.

Quick aside: moving from infinitesimal calculus to finite groupoid algebra actually involves two changes. The first is the change from vectors to paths, forms to functions and so on. But it's also the case that tangent vectors have just the one basepoint, whereas paths have two endpoints. You can see this play out in this example, where a and a' were distinct points (and b and b').

The horizontal composition we build lives entirely in B and we didn't make use of M yet. The Leibniz rule will be a pointwise version of what's going on in B. Denote by $\mu \circ (f, g) : M \to B$ the map which sends $x \mapsto \mu(f(x), g(x))$.

Lemma 5.3. Given $f, g : M \to B$ and $p : x =_M y$ then

$$\begin{aligned}\mathsf{ap}(\mu \circ (f, g))(p) &= \mu(f(p), -) \star \mu(-, g(p)) \\ &= [\mu(f(p), -) \cdot_r \mu(-, g(x))] \cdot [\mu(f(y), -) \cdot_l \mu(-, g(p))] \\ &: \mu(f(x), g(x)) = \mu(f(y), g(y))\end{aligned}$$

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