Discrete differential geometry in homotopy type theory

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April 2025

Summary

Summary

This work brings to HoTT

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature = total index

Classical $\to$ HoTT

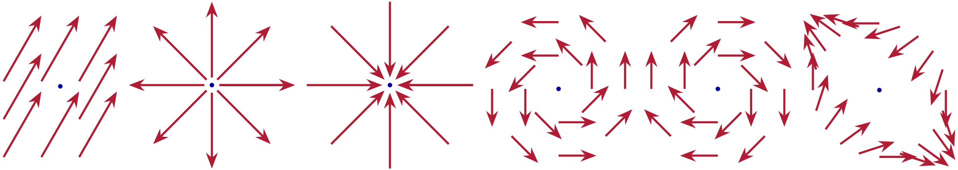
Let M be a smooth, oriented 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes $x_1, \ldots, x_n$.

$$
\begin{tikzcd}
\frac{1}{2\pi}\int_M F_A \arrow[d] & = \sum_{i=1}^{n} \mathrm{index}_X(x_i) \arrow[d] & = \chi(M) \\
\sum_{\text{faces } F} \flat_F \arrow[r, equal] & \sum_{\text{faces } F} L_F^X &
\end{tikzcd}
$$

Classical index

Near an isolated zero there are only three possibilities: index 0, 1, –1.

Index is the winding number of the field as you move clockwise around the zero.

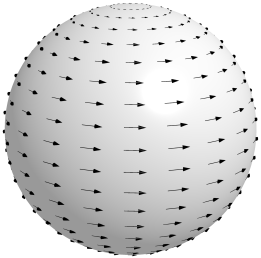


index 0 index +1 index +1 index +1 index +1 index –1

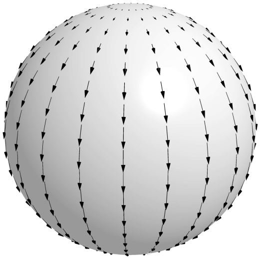
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index $+1$ at each pole $= \mathbf{2}$



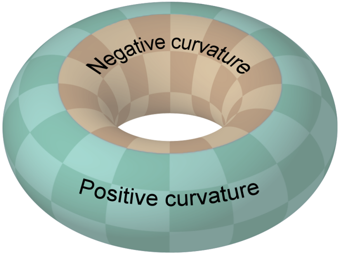
Height: index $+1$ at each pole $= \mathbf{2}$

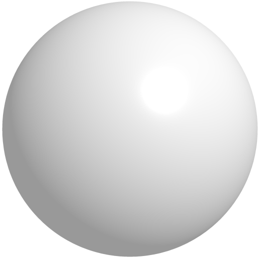
Gauss-Bonnet theorem

Total curvature divided by 2π is the Euler characteristic.

Curvature in 2D is a function $F_A : M \to \mathbb{R}$.

$\int_M F_A$ sums the values at every point.





Positive and negative curvature cancel: **0**

Constant curvature 1, area 4π: **2**

Plan

- Combinatorial manifolds
- Torsors and classifying maps
- Connections and curvature
- Vector fields
- Main theorem

HoTT background

1. Symmetry,
   Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
   `https://github.com/UniMath/SymmetryBook`.
2. Central H-spaces and banded types,
   Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
   arXiv:2301.02636
3. Nilpotent types and fracture squares in homotopy type theory,
   Scoccola, L. (2020)
   MSCS 30(5). arXiv:1903.03245

Combinatorial manifolds

Manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization procedure to construct types

Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \overset{\text{def}}{=} [M_0, \dots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \overset{\text{def}}{=} [M_0, \dots, M_k]$ the k-skeleton of M.

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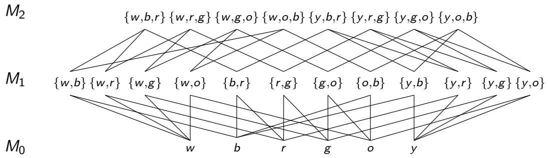
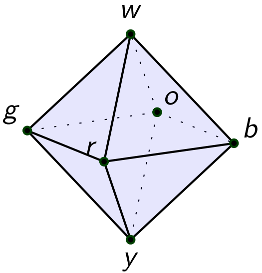
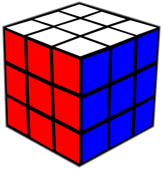


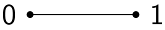




Simplicial complexes

Example

The complete simplex of dimension n, denoted $\Delta(n)$, is the set $\{0, \dots, n\}$ and its power set. The $(n-1)$-skeleton $\Delta(n)_{\leq(n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$-sphere.

$\Delta(1)$ is visually 

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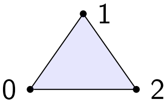
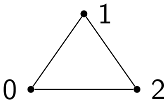
$\Delta(2)$ is visually 

, $\partial\Delta(2)$ is visually 

Homotopy realization: dimension 0

We will realize simplicial complexes by means of a sequence of pushouts.

Base case: the realization $\mathbb{M}$ of a 0-dimensional complex M is M_0.

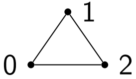
In particular the 0-sphere $\partial\mathbf{\Delta}(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$.

Homotopy realization: dimension 1

For a 1-dim complex $M \stackrel{\text{def}}{=} [M_0, M_1]$ the realization is given by

$$
\begin{tikzcd}
M_1 \times \partial\Delta(1) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] \arrow[ur, phantom, "\Nearrow h_1"] & \mathbb{M}_1 \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}
$$

Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\mathbb{\Delta}(2) \overset{\text{def}}{=}$ 

is given by

$$\begin{tikzcd} \partial\Delta(2)_1 \times \partial\mathbb{\Delta}(1) \arrow[r] \arrow[d] & \partial\Delta(2)_1 \arrow[d] \\ \partial\Delta(2)_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \partial\mathbb{\Delta}(2) \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

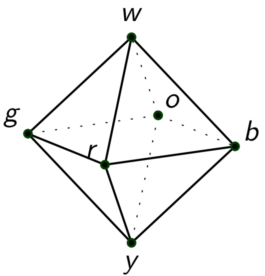
i.e.

$$\begin{tikzcd} \{\{0,1\},\{1,2\},\{2,0\}\}\times\{0,1\} \arrow[r] \arrow[d] & \{\{0,1\},\{1,2\},\{2,0\}\} \arrow[d] \\ \{0,1,2\} \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \partial\mathbb{\Delta}(2) \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron $\mathbb{O}$:

$$
\begin{tikzcd}
\{\{w,g\},\dots\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{w,g\},\dots\} \arrow[d] \\
\{w,g,\dots\} \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \mathbb{O}_1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$



Homotopy realization: dimension 2

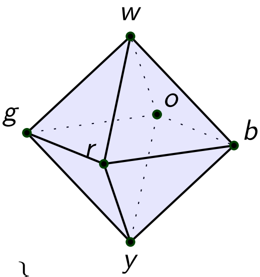
To realize $M \overset{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\mathbf{\Delta}(1), \partial\mathbf{\Delta}(2)$:

$$\begin{tikzcd}
M_1 \times \partial\mathbf{\Delta}(1) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[ul, phantom, very near start, "\lrcorner"] \arrow[r] \arrow[dr, phantom, "\Searrow h_2"] & \mathbb{M}_2 \arrow[dl, phantom, very near start, "\urcorner"] \\
& M_2 \times \partial\mathbf{\Delta}(2) \arrow[u, "\mathbb{A}_1"] \arrow[r, "\mathrm{pr}_1"'] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}$$

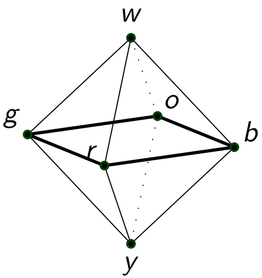
Homotopy realization: dimension 2

The full octahedron $\mathbb{O}$:

$$
\begin{tikzcd}
\{\{w,g\},\dots\} \times \{0,1\} \arrow[r, "\mathrm{pr}_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow h_1"] & \{\{w,g\},\dots\} \arrow[d] & \\
\{w,g,\dots\} \arrow[r] & \mathbb{O}_1 \arrow[r] \arrow[dr, phantom, "\Rightarrow h_2"] & \mathbb{O}_2 \\
& \{\{w,g,r\},\dots\} \times \partial\Delta(2) \arrow[u] \arrow[r, "\mathrm{pr}_1"'] & \{\{w,g,r\},\dots\} \arrow[u]
\end{tikzcd}
$$



Homotopy realization: dimension 2



$\leftarrow \text{link}(w)$

The link of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

A combinatorial manifold is a simplicial complex all of whose links are* simplicial spheres.

This will be our model of the tangent space.

*the (classical) geometric realization is homeomorphic to a sphere

Combinatorial manifolds $\leftrightarrow$ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$*-sphere, i.e. its geometric realization is an* $(n-1)$*-sphere.*

`https://ncatlab.org/nlab/show/triangulation+theorem`

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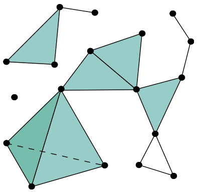
Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:



Torsors

What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of torsors, also called principal bundles.

Torsors

Let G be a (higher) group.

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- The inverse is (pr_1, s) where $s : X \times X \to G$ is called subtraction (when G is commutative).
- Let BG be the type of G-torsors.
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

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See the Buchholtz et. al. H-spaces paper for more.

How to map into BS^1

To construct maps into BS^1 we lift a family of mere circles.

families of:

$$
\begin{tikzcd}
BS^1 \arrow[r] & \mathrm{BAut}(S^1) \overset{\text{def}}{=} \sum_{Y:\mathcal{U}} \|Y = S^1\|_{-1} \arrow[r] & \mathcal{U} \\
& \mathbb{M} \arrow[ul, "\text{torsors}"] \arrow[u, "\text{mere circles}"] \arrow[ur, "\text{types}"'] &
\end{tikzcd}
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Other names:

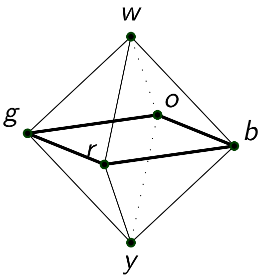
- $\mathrm{BAut}(S^1) = BO(2) = \mathrm{EM}(\mathbb{Z}, 1)$ (where $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n))$)
- $BS^1 = BSO(2) = \mathrm{K}(\mathbb{Z}, 2)$

Connections and curvature

Connections

Connections are extensions of the bundle to higher skeleta.

Recall link



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Define the tangent bundle on a combinatorial manifold to be $T_0 \stackrel{\mathrm{def}}{=} \mathrm{link} : \mathbb{M}_0 \to \mathrm{BAut}(S^1)$.

Connections on the tangent bundle

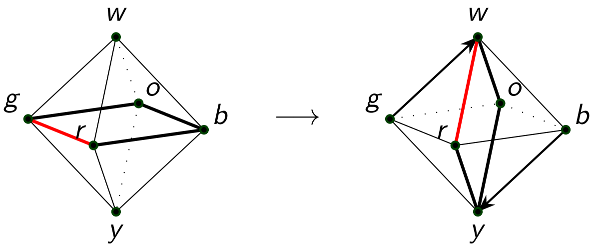
An extension T_1 of T_0 to $\mathbb{M}_1$ is called a connection on the tangent bundle.

$$
\begin{tikzcd}
\mathbb{M}_0 \arrow[r] \arrow[dr, "T_0 \overset{\text{def}}{=} \text{link}"'] & \mathbb{M}_1 \arrow[r] \arrow[d, dashed, "T_1"] & \mathbb{M}_2 \arrow[dl, dashed] \\
& \mathrm{BAut}(S^1) &
\end{tikzcd}
$$

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

We will define T_1 on the edge wb, so we need a term $T_1(wb) : \mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(b)$.

We imagine tipping:

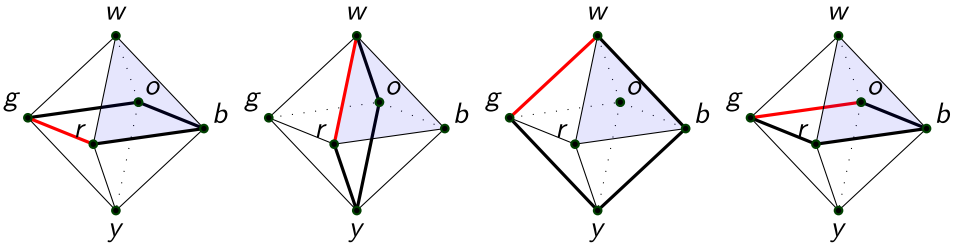


$T_1(g : \mathrm{link}(w)) \stackrel{\text{def}}{=} w : \mathrm{link}(b), \ldots$.

Use this method to define T_1 on every edge.

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

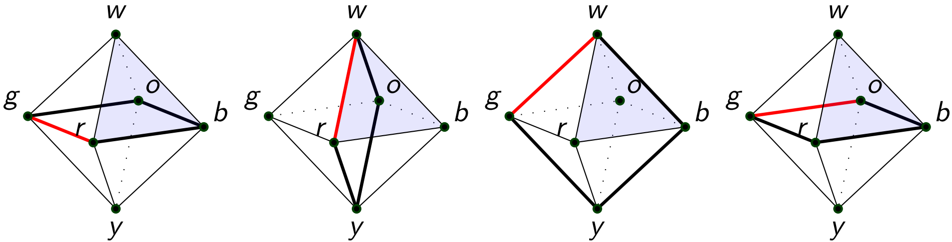
Denote the path $wb \cdot br \cdot rw$ by $\partial(wbr)$. Consider $T_1(\partial(wbr))$:



We come back rotated by $1/4$ turn. Call this rotation $R : \mathrm{link}(w) =_{BAut(S^1)} \mathrm{link}(w)$.

Extending T_1 to a face

Let $H_{wbr} : \mathrm{refl}_w =_{w =_{\mathbb{M}} w} \partial(wbr)$ be the filler homotopy of the face.



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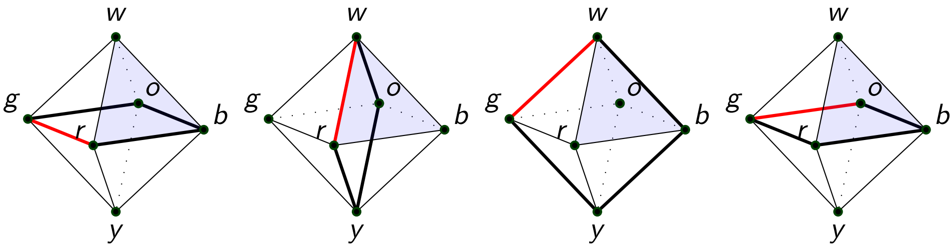

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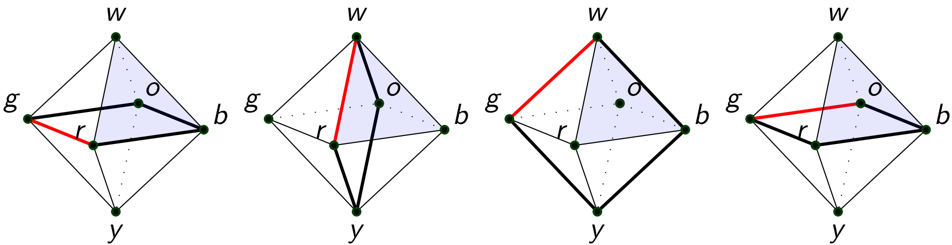

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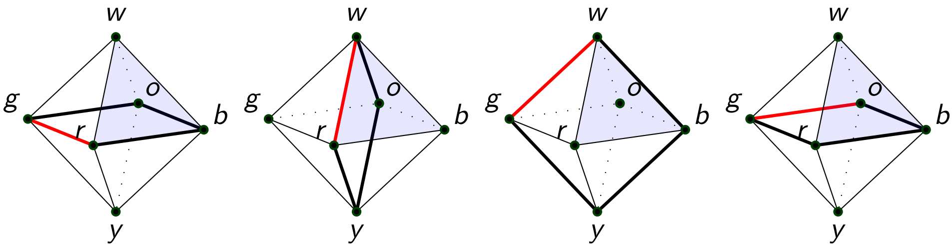
For example, a path $H_R(g) : g = Rg = o$. Choose go.



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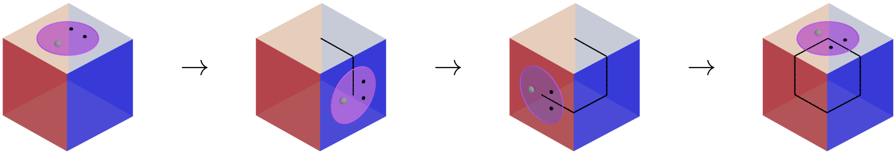
Original inspiration



The definition of a connection

Definition

If $\mathbb{M} \overset{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is the realization of a combinatorial manifold and all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"'] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"'] \\
 & & \mathcal{U} & &
\end{tikzcd}$$

- The map f_k is a k-bundle on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-connection on the $(k-1)$-bundle f_{k-1}.

The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:

$$
\begin{tikzcd}
M_2 \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Nearrow h_2"] & M_2 \arrow[d] \\
\mathbb{M}_1 \arrow[r] \arrow[dr, "T_1"'] \arrow[dr, phantom, shift left=3, "\Swarrow"] & \mathbb{M}_2 \arrow[d, "T_2"] \\
& \mathcal{U}
\end{tikzcd}
$$

Here's the outer square for a single face F:

$$
\begin{tikzcd}
\{F\} \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Swarrow \flat_F"] & \{F\} \arrow[d] \\
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\mathbb{M}_1 \arrow[r] \arrow[dr, "T_1"'] & \mathbb{M}_2 \arrow[d, "T_2"] \arrow[dl, phantom, "\Swarrow"] \\
& \mathcal{U}
\end{tikzcd}$$

Here's the outer square for a single face F:

$$\begin{tikzcd}
\{F\} \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] & \{F\} \arrow[d] \arrow[dl, phantom, "\overset{\flat_F}{\Swarrow}"] \\
\mathbb{M}_1 \arrow[r] & \mathcal{U}
\end{tikzcd}$$

$T_1(\partial(F))$ is the curvature at the face F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a flatness structure for the face F.

The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:

$$\begin{tikzcd}
M_2 \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathsf{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Nearrow h_2"] & M_2 \arrow[d] \\
\mathbb{M}_1 \arrow[r] \arrow[dr, "T_1"'] \arrow[r, phantom, shift right=4ex, "\Swarrow"] & \mathbb{M}_2 \arrow[d, "T_2"] \\
& \mathcal{U}
\end{tikzcd}$$

Here's the outer square for a single face F:

$$\begin{tikzcd}
\{F\} \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathsf{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Swarrow \flat_F"] & \{F\} \arrow[d] \\
\mathbb{M}_1 \arrow[r] & \mathcal{U}
\end{tikzcd}$$

$T_1(\partial(F))$ is the curvature at the face F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a flatness structure for the face F.

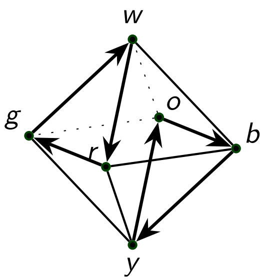
The distinction between the path $\flat_F$ and the endpoint $T_1(\partial(F))$ is small enough to be confusing.

Vector fields

Vector fields

Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim combinatorial manifold.

- Our bundles of mere circles can only model nonzero tangent vectors.
- A global section would be a trivialization of T, so there is an obstruction.



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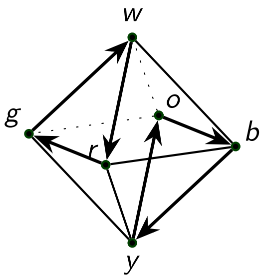

Our solution:

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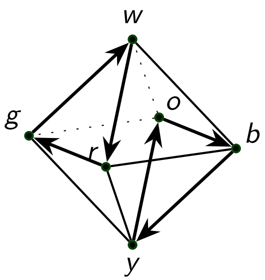
Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.



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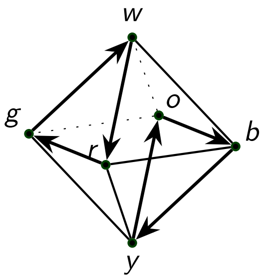
Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical nonvanishing vector field on the 1-skeleton.



Vector fields

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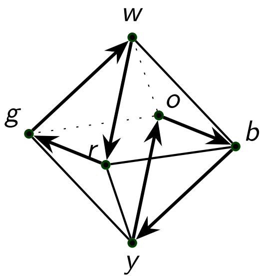
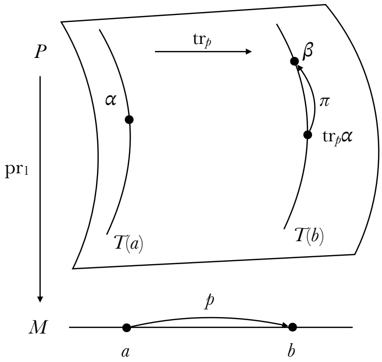
Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

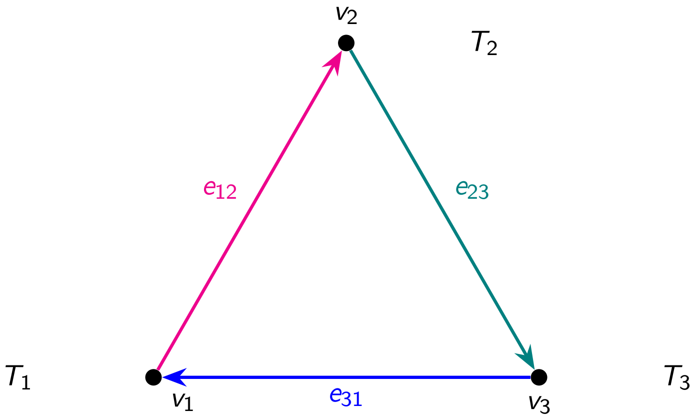


Reminder: pathovers

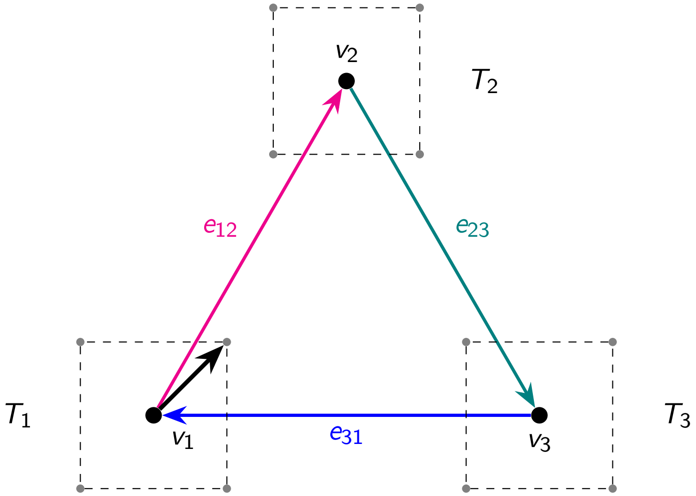


- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display π, the path over p.
- Dependent functions map paths to pathovers: $\mathrm{apd}(X)(p) : \mathrm{tr}_p(X(a)) = X(b)$ (simply denoted $X(p)$).

Next goal: define the index of a vector field on a face.

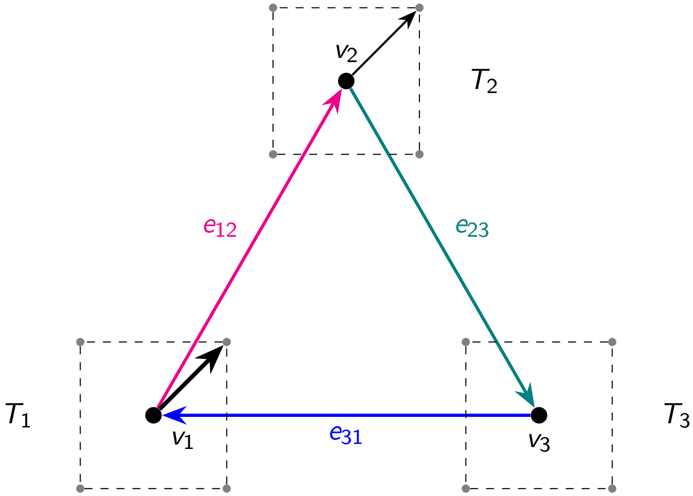


An example of swirling and index at this face.



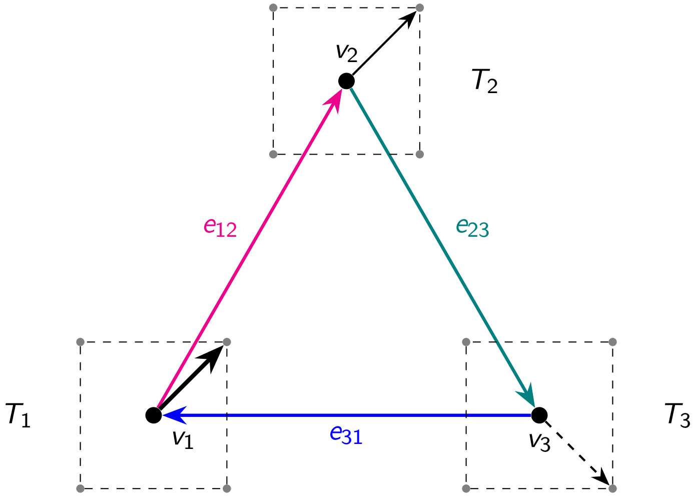
An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
-
-
-



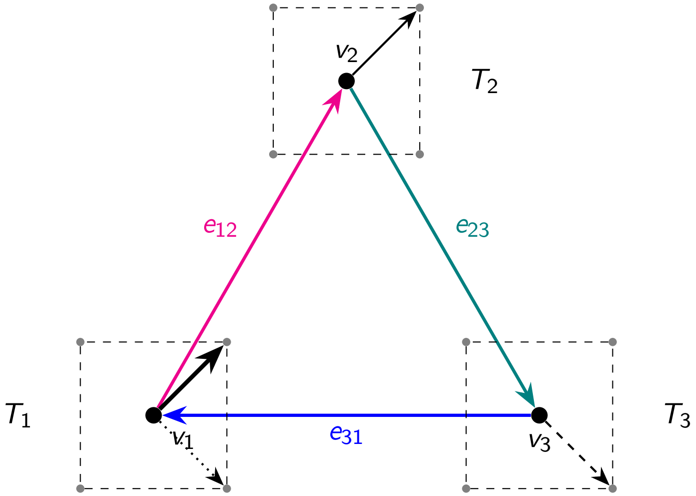
An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{12} is trivial. Denote the transported vector as thinner.
-
-



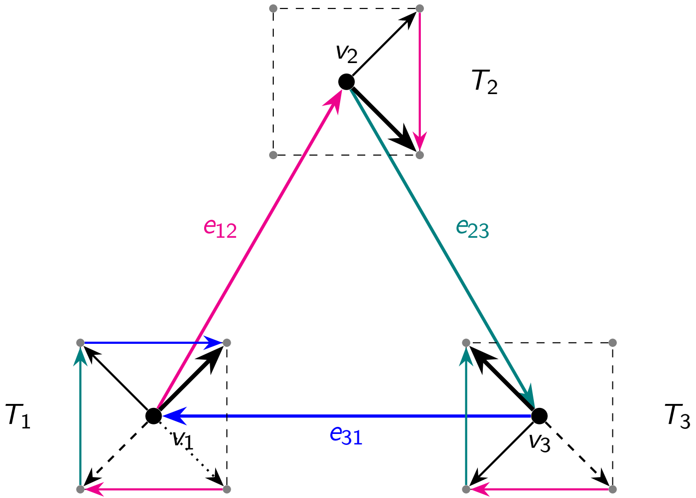
An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{12} is trivial. Denote the transported vector as thinner.
- Say T_{23} rotates clockwise. Denote the twice-transported vector as dashed.
-

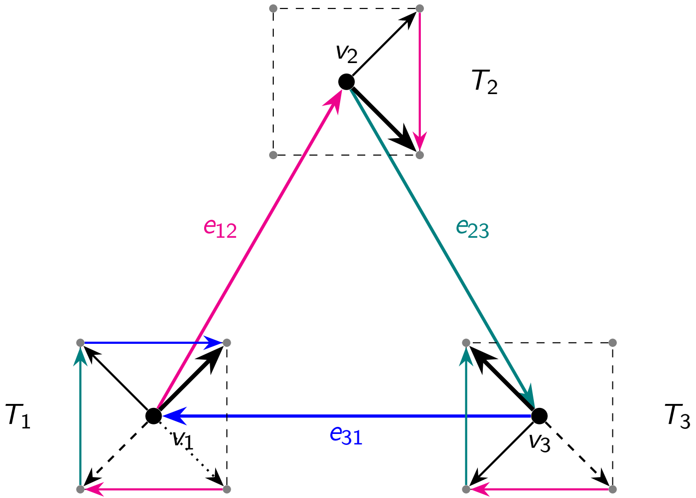


An example of swirling and index at this face.

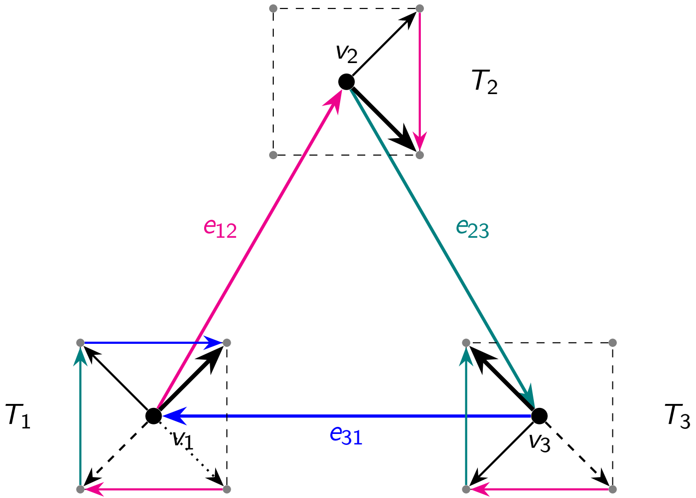
- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{12} is trivial. Denote the transported vector as thinner.
- Say T_{23} rotates clockwise. Denote the twice-transported vector as dashed.
- Say T_{31} is trivial. The thrice-transported vecor is dotted.



- X on e_{12} is red, etc.



- X on e_{12} is red, etc.
- We translated all results to the end of the loop.



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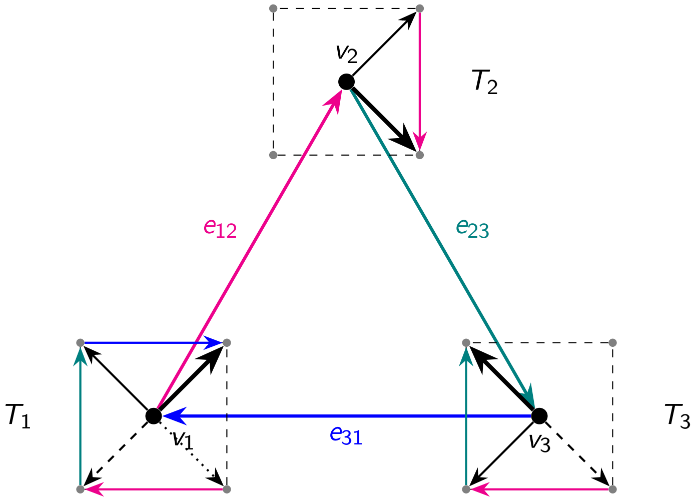


- X on e_{12} is red, etc.
- We translated all results to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)
- The total pathover is called the swirling X_F of X at the face F.

Symbolic version

$$
\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1 \\
 & & & T_{13}\,T_{32}\,T_{21}X_1 \arrow[d, equal, "T_{13}\,T_{32}X_{12}:"'] \\
 & & T_{32}\,T_{21}X_1 \arrow[d, equal, "T_{32}X_{12}:"'] & T_{13}\,T_{32}X_2 \arrow[d, equal, "T_{13}X_{23}:"'] \\
 & T_{21}X_1 \arrow[d, equal, "X_{12}:"'] & T_{32}X_2 \arrow[d, equal, "X_{23}:"'] & T_{13}X_3 \arrow[d, equal, "X_{31}:"'] \\
X_1 & X_2 & X_3 & X_1
\end{tikzcd}
$$

Index

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

Index

$$\mathrm{tr}_F \overset{\text{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \qquad \text{flatness}$$

$$X_F \overset{\text{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

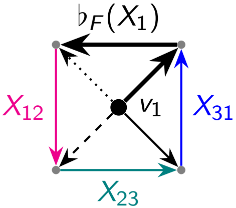
Recall that T_1 being an S^1-torsor means we can use subtraction to obtain an equivalence $s(-, X_1) : T_1 \xrightarrow{x \mapsto x - X_1} S^1$.

Definition

The flattened swirling of the vector field X on the face F is the loop

$$L_F^X \overset{\text{def}}{=} \flat_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The index of the vector field X on the face F is the integer I_F^X such that $\mathrm{loop}^{I_F^X} =_{S^1} (L_F^X) - X_1$.



Main theorem

Simplifying swirling

Swirling involves concatenating dependent paths. Can we simplify that?

Pay off all our assumptions 1: torsor structure, vector field

$$T_1$$

$$\begin{array}{rc}
 & T_{13}\,T_{32}\,T_{21}X_1 \\
T_{13}\,T_{32}X_{21}: & \Vert \\
 & T_{13}\,T_{32}X_2 \\
T_{13}X_{32}: & \Vert \\
 & T_{13}X_3 \\
X_{13}: & \Vert \\
 & X_1
\end{array}$$

Pay off all our assumptions 1: torsor structure, vector field

$$T_1$$

- Def: $\alpha_i \overset{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$ (trivialization on 0-skeleton).

$$\begin{array}{r}
T_{13}\,T_{32}\,T_{21}X_1 \\
T_{13}\,T_{32}X_{21} : \Big\| \\
T_{13}\,T_{32}X_2 \\
T_{13}X_{32} : \Big\| \\
T_{13}X_3 \\
X_{13} : \Big\| \\
X_1
\end{array}$$

Pay off all our assumptions 1: torsor structure, vector field

$$\begin{array}{c} T_1 \\ \\ T_{13}T_{32}T_{21}X_1 \\ {\color{red}T_{13}T_{32}X_{21}}: \Big\| \\ T_{13}T_{32}X_2 \\ {\color{red}T_{13}X_{32}}: \Big\| \\ T_{13}X_3 \\ {\color{red}X_{13}}: \Big\| \\ X_1 \end{array}$$

- Def: $\alpha_i \overset{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$ (trivialization on 0-skeleton).
- Def: $\rho_{ji} \overset{\text{def}}{=} \alpha_j(T_{ji}(X_i))$ is the rotation of T_{ji}.

$$\begin{tikzcd} T_i \arrow[r, "T_{ji}"] \arrow[d, "\alpha_i"'] & T_j \arrow[d, "\alpha_j"] \\ S^1 \arrow[u, bend left, "\text{base}\mapsto X_i"] \arrow[r, "(-)\odot\rho_{ji}"'] & S^1 \arrow[u, bend right, "\text{base}\mapsto X_j"'] \end{tikzcd}$$

Pay off all our assumptions 1: torsor structure, vector field

$$\begin{aligned} &T_1 \\ &T_{13}T_{32}T_{21}X_1 \\ T_{13}T_{32}X_{21}: &\| \\ &T_{13}T_{32}X_2 \\ T_{13}X_{32}: &\| \\ &T_{13}X_3 \\ X_{13}: &\| \\ &X_1 \end{aligned}$$

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$$\begin{tikzcd} T_i \arrow[r, "T_{ji}"] \arrow[d, "\alpha_i"', bend left] & T_j \arrow[d, "\alpha_j", bend right] \\ S^1 \arrow[u, bend left, "\text{base}\mapsto X_i"] \arrow[r, "(-)\odot\rho_{ji}"'] & S^1 \arrow[u, bend right, "\text{base}\mapsto X_j"'] \end{tikzcd}$$

- Lemma: $\rho_{ij} = \rho_{ji}^{-1}$ because in T_j:
$\rho_{ij} \odot \rho_{ji} \odot X_j = \rho_{ij} \odot T_{ji}X_i = T_{ji}(\rho_{ij} \odot X_i) = T_{ji}T_{ij}X_j = X_j$.

Pay off all our assumptions 1: torsor structure, vector field (cont.)

$$\begin{array}{c} T_1 \\ \\ T_{13}T_{32}T_{21}X_1 \\ {\scriptstyle T_{13}T_{32}X_{21}:}\Big\| \\ T_{13}T_{32}X_2 \\ {\scriptstyle T_{13}X_{32}:}\Big\| \\ T_{13}X_3 \\ {\scriptstyle X_{13}:}\Big\| \\ X_1 \end{array}$$

Pay off all our assumptions 1: torsor structure, vector field (cont.)

$$\begin{array}{c} T_1 \\ \\ \\ T_{13}\,T_{32}\,T_{21}X_1 \\ {\scriptstyle T_{13}T_{32}X_{21}:}\Big\| \\ T_{13}\,T_{32}X_2 \\ {\scriptstyle T_{13}X_{32}:}\Big\| \\ T_{13}X_3 \\ {\scriptstyle X_{13}:}\Big\| \\ X_1 \end{array}$$

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},$.

Pay off all our assumptions 1: torsor structure, vector field (cont.)

$$
\begin{array}{rc}
 & T_1 \\
 & \\
 & T_{13}\,T_{32}\,T_{21}X_1 \\
T_{13}\,T_{32}X_{21}: & \| \\
 & T_{13}\,T_{32}X_2 \\
T_{13}X_{32}: & \| \\
 & T_{13}X_3 \\
X_{13}: & \| \\
 & X_1
\end{array}
$$

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},.$
- Paths of the form $(a =_{S^1} \text{base})$ can be multiplied:
  - $\odot : (a = \text{base}) \times (b = base) \to (a \odot b = base)$.
  - $p \odot q = (p \odot b) \cdot q$.

Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

$$\begin{tikzcd} T_{13}T_{32}T_{21}X_1 \arrow[d, equal, "T_{13}T_{32}X_{21}:"'] \\ T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}X_{32}:"'] \\ T_{13}X_3 \arrow[d, equal, "X_{13}:"'] \\ X_1 \end{tikzcd}$$

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- Paths of the form $(a =_{S^1} \text{base})$ can be multiplied:
  - $\odot : (a = \text{base}) \times (b = base) \to (a \odot b = base)$.
  - $p \odot q = (p \odot b) \cdot q$.
- Lemma: $\text{apd}(X)(\text{refl}) = \text{refl} \implies X_{ij} \cdot T_{ij}X_{ji} = \text{refl}_{X_i}$
  $\implies \sigma_{ij} \odot \sigma_{ji} = \text{refl}_{\text{base}}$ (T_{ij} just translates X_{ji} to cat with X_{ji}).

Pay off all our assumptions 2: no boundary, commutativity

$$\begin{tikzcd}
T_1 \\
T_{13}T_{32}T_{21}X_1 \arrow[d, equal, "T_{13}T_{32}X_{21}:"'] \\
T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}X_{32}:"'] \\
T_{13}X_3 \arrow[d, equal, "X_{13}:"'] \\
X_1
\end{tikzcd}$$

Definition

Let $F_1, \ldots, F_n$ be the faces of $\mathbb{M}$, and $\partial F_1, \ldots, \partial F_n$ be the triangular boundaries. The total swirling is

$$X_{\mathrm{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} \odot \cdots \odot \sigma_{\partial F_n}$$

.

Pay off all our assumptions 2: no boundary, commutativity

$$\begin{array}{c}
T_1 \\
\\
T_{13}\,T_{32}\,T_{21}X_1 \\
{\color{red}T_{13}\,T_{32}X_{21}:}\Big\| \\
T_{13}\,T_{32}X_2 \\
{\color{red}T_{13}X_{32}:}\Big\| \\
T_{13}X_3 \\
{\color{red}X_{13}:}\Big\| \\
X_1
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- We assume that this expression involves every edge once in each direction.

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$$\begin{tikzcd}
T_{13}T_{32}T_{21}X_1 \arrow[d, equal, "T_{13}T_{32}X_{21}:"'] \\
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.

- We assume that this expression involves every edge once in each direction.
- S^1 is commutative, hence complete cancellation.

Consequence

$$
\begin{array}{llll}
\mathrm{tr}_F \stackrel{\text{def}}{=} \mathrm{tr}(\partial F) & : T_1 =_{BS^1} T_1 & \text{curvature} \\
\flat_F \stackrel{\text{def}}{=} \flat(\partial F) & : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F & \text{flatness} \\
X_F \stackrel{\text{def}}{=} X(\partial F) & : \mathrm{tr}_F(X_1) =_{T_1} X_1 & \text{swirling} \\
L_F^X \stackrel{\text{def}}{=} \flat_F(X_1) \cdot X_F & : (X_1 =_{T_1} X_1) & \text{flattened swirling}
\end{array}
$$

Consequence

$$\begin{array}{lll} \mathrm{tr}_F \overset{\mathrm{def}}{=} \mathrm{tr}(\partial F) & : T_1 =_{BS^1} T_1 & \text{curvature} \\ \flat_F \overset{\mathrm{def}}{=} \flat(\partial F) & : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F & \text{flatness} \\ X_F \overset{\mathrm{def}}{=} X(\partial F) & : \mathrm{tr}_F(X_1) =_{T_1} X_1 & \text{swirling} \\ L_F^X \overset{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F & : (X_1 =_{T_1} X_1) & \text{flattened swirling} \end{array}$$

These can all be totaled in S^1 to give

$$\mathrm{tr}_{\mathrm{tot}} \overset{\mathrm{def}}{=} \bigodot_i \rho_{\partial F} = \mathrm{base} \qquad X_{\mathrm{tot}} \overset{\mathrm{def}}{=} \bigodot_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

$$\flat_{\mathrm{tot}} \overset{\mathrm{def}}{=} \bigodot_i \flat_{\partial F} \qquad L^X_{\mathrm{tot}} \overset{\mathrm{def}}{=} \bigodot_i \flat_{\partial F} \odot \sigma_{\partial F} = \bigodot_i \flat_{\partial F}$$

Consequence

$$
\begin{array}{lll}
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\flat_F \stackrel{\text{def}}{=} \flat(\partial F) & : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F & \text{flatness} \\
X_F \stackrel{\text{def}}{=} X(\partial F) & : \mathrm{tr}_F(X_1) =_{T_1} X_1 & \text{swirling} \\
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\end{array}
$$

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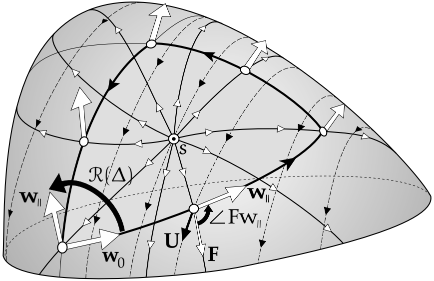
$$\mathrm{tr}_{\mathrm{tot}} \stackrel{\text{def}}{=} \bigodot_i \rho_{\partial F} = \mathrm{base} \qquad X_{\mathrm{tot}} \stackrel{\text{def}}{=} \bigodot_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

$$\flat_{\mathrm{tot}} \stackrel{\text{def}}{=} \bigodot_i \flat_{\partial F} \qquad L_{\mathrm{tot}}^X \stackrel{\text{def}}{=} \bigodot_i \flat_{\partial F} \odot \sigma_{\partial F} = \bigodot_i \flat_{\partial F}$$

So in our lingo: the total flatness equals the total flattened swirling.

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F w_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle F w_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Thank you!