Abstract

We identify connections, curvature, and gauge transformations within the structures of homotopy type theory. Whereas most classical treatments of these structures rely entirely on infinitesimal definitions, there is an equivalent discrete story of which the infinitesimal version is a limit, analogous to the relationship between smooth paths and tangent vectors, or between de Rham and Čech cohomology. We will show how to identify the elements of discrete gauge theory, provide some evidence that this is what we have found, and use it to prove some results from the 20th century mathematics of gauge theory that depend only on homotopy types.

1 Introduction

> ”It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world.” — Stephen Talbott, *The Future Does Not Compute*[1]

Homotopy type theory (HoTT) is a logical foundation for mathematics that adds to set theory several important mathematical patterns. Among these are *fibrations*, *(higher) groupoids*, and *classifying spaces* (called *universes*). It embodies the philosophy of constructivism by unifying mathematical statements with types and proofs with terms. It has an extensive syntax and a talented team of theory builders who have labored for a decade to equip the syntax with formal initiality proofs and categorical semantics that together generate living, breathing proofs simultaneously in every category with enough structure to support the semantics. It has implementations in several computer languages which have attracted multiple large formalization efforts. It benefits from an unusually strong interdisciplinary culture spanning computer science, mathematics, philosophy, and physics.

The hypothetical scope of HoTT is of course all of mathematics, past, present and future. It contains set theory, and the law of excluded middle and axiom of choice can be added whenever the intended semantics supports them. It therefore invites us to use the rest of its tools to enhance our understanding, and to seek better answers to *why* questions in the same way that category theory sometimes tells us *why* individual results across specific fields are true. Over the last decade many results have been verified in HoTT, and many have insightful new proofs. The topics that have received the most attention so far include homotopy theory, algebraic topology, group theory, category theory, and combinatorics. Our goal is to extend this list with topics from geometry: connections, curvature, gauge theory, characteristic classes, and Chern-Weil theory.

Gauge theory is a collection of tools inspired by, and in dialog with, the standard model of particle physics. The quantum field theories developed during the 20th century are mathematical models that supply spacetime with additional structures: principal bundles, vector bundles, sections of these bundles, and connections. The bundles provide internal dimensions to quantum particles, which are visible experimentally through phenomena such as charge, magnetism, mass, gravity, and quark color. Forces are mediated by massless particles that correspond to the connections. Particles are sections of associated vector bundles. Physical laws are expressed by real- or complex-valued functions called lagrangians, on the configuration space of all of this structure.

Mathematicians discovered that by adding their own gauge theory structures to smooth manifolds of interest they could define new invariants of the homotopy type, the homeomorphism class, or the diffeomorphism class of the manifold. HoTT is not equipped today to explore all of these scenarios since it is most developed in the world of homotopy types only. There is beautiful work to bring in notions of topology and smooth structure through the principal of *cohesion* (the study of spatial relationships by way of operators that *remove* such structure). We will touch on some of this and point the way to future work, but for now we will focus on homotopical results only.

For the HoTT-oriented audience my goal is to demonstrate that connections and gauge theory are present in the realm of discrete types. This can even serve as an introduction to classical differential geometry, and today's undergraduates who are drawn to HoTT can leverage their intuitions to bypass a lot of confusing material!

For the geometry audience my goal is to offer an entirely new perspective on the de Rham complex in general, and connections and curvature in particular. My grandiose hope is that in this framing we can find a natural and intuitive understanding of Chern-Weil theory, i.e. the link between curvature and characteristic classes.

$$
\begin{tikzcd}
b \arrow[r, "br"] & r \arrow[d, "rg"] \\
o \arrow[u, "ob"] & g \arrow[l, "go"]
\end{tikzcd}
$$

Figure 1: The HIT C_4 which is one of the types in $\mathrm{BAut}_1(S^1)$

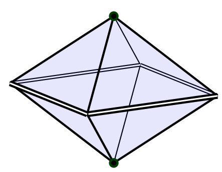

Figure 2: The HIT ΣC_4 which has 2 points, 4 1-paths, 4 2-paths.

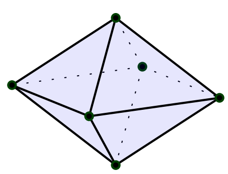

Figure 3: The HIT $\{w, y\} * C_4$ which has 6 points, 12 1-paths, 8 2-paths.

1.1 Connections are not necessarily infinitesimal

Given a principal bundle $\pi : P \to X$ a connection can equally well be defined *infinitesimally* as a map from tangent vectors in TX to horizontal lifts in TP, or via *holonomy* as a map from paths in X to horizontal lifts in P. Infinitesimal connections can be used to assign holonomy to finite length curves by integrating along them and using the exponential map $\mathfrak{g} \to G$ of a Lie algebra into its Lie group. Integration respects concatentation of paths and reversal of paths, and so respects the groupoidal nature of holonomy.

Likewise we may define holonomy directly on the space of smooth paths into a finite-dimensional smooth manifold $\mathcal{P}(M) \stackrel{\text{def}}{=} [0, 1] \to M$. This path space can be given a smooth structure and a groupoidal structure (via concatenation and reversal of paths). We can define a homomorphism $\mathcal{P}(M) \to G$ to assign holonomy to each path and then differentiate this assignment by taking a limit. For the rigorous details see [2] Appendix B and [3] II.3.

Working entirely with finite paths puts us within the use cases of real-cohesive modal homotopy type theory. This has been known for a decade, ever since we could write down expressions like $\int X \to BG$. But this is a space of maps out of a type that although it can have higher paths, it is spatially discrete. Discreteness in this sense means that the map const : $\int X \to (\mathbb{R} \to \int X)$ is an equivalence. Can connections be discrete as well?

Sure, the same way any function can be discrete. Take an n-manifold with a triangulation (a combinatorial cell structure where n-dimensional cells meet at $n - 1$-dimensional faces and so on down to 0-cells). We can define discrete functions that assign a single value in some codomain to each 0-cell, or to each 1-cell, or to each dual 1-cell (pair of adjoining n-cells) and so on. We can define discrete derivatives (i.e. differences), connections, curvature, and even Laplacian operators. See the literature from the field of *discrete differential geometry* (DDG) [4] [5] for a parallel train of thought that's focused on computer graphics.

Let's draw inspiration from DDG and define some higher inductive types (HITs) that stand in for the classical smooth manifolds of which they are the homotopy type. Assigning values to cells of various dimension is performed by functions, which already encode data at all dimensions in homotopy type theory. We can make some educated guesses as to what spaces to map our HITs into to mimic classical bundles with connection. We can then look more closely at standard HoTT constructions such as transport and the lifting of paths and see the very close agreement between the theories.

Why ought this to work? Why would we even expect connections to be so near to hand when classically they are an entirely independent and separate structure on a principal bundle? HoTT makes questions like this very simple to ask and to answer. First, HoTT makes it easy to switch between vertical and horizontal thinking: defining bundles as maps into the base X versus maps from X into a space of fibers. Second, HoTT packages functions and their "derivatives" (the action on paths, which are integrated versions of vector fields after all). Truncation to dimension 0 is the act of taking equivalence classes, and not truncating is the act of maintaining the groupoidal information about all the equivalences inside the domain, the codomain, and the maps. Classically we have the theorem that equivalence classes of principal bundles with group G over X are in bijection with homotopy classes of maps $X \to BG$ into a classifying space. That's the truncation of some groupoidal theorem. If we don't truncate then connections ought to appear at dimension 1: by specifying the lift of a path into the total space of a principal bundle, connections yield isomorphisms between fibers, which by univalence is a path in fiber space.

Of course examining untruncated maps into classifying spaces in HoTT doesn't automatically deserve to be called the study of bundles with connection. We'll need some justification. What criteria would make sense?

Claim 1. *We can calculate classical examples and derive classical formulas.*

Claim 2. *The classical theorems relating curvature to cohomology (Chern-Weil theory) are made transparent.*

The characteristic classes descend to the base. The closedness. Whitney sum formula. Independence of class from connection. ([3] XII)

Claim 3. *Gauge theory — the relating of bundles to the space of all connections modulo gauge transformations — is available.*

Claim 4. *We can prove theorems from gauge theory that depend only on the homotopy type of the manifold.*

The sections below will justify these claims. We will end with a survey of the relationships between topological manifolds, smooth manifolds, and various combinatorial manifolds such as CW complexes, PL manifolds and manifolds with triangulation, so we can consider how limited a theory of HIT manifolds may be.

1.2 Functions are multidimensional

Functions in homotopy type theory carry information in all dimensions at once. Given $f : X \to Y$ we can access the value of f on paths with terms like $\mathsf{ap}(f, -) : x =_X y \to f(x) =_Y f(y)$ and we can iterate this with more aps.

1.3 Pathovers

Given a type family $f : X \to \mathcal{U}$ and a path $p : x =_X x'$, and given terms $y : f(x), y' : f(x')$ in the two fibers, we can define the type $y \underset{p}{\overset{=}{\to}} y'$ by induction: if p is refl then $y = y'$ lives in a single fiber $f(x)$ where it reduces to the usual identity type.

There is an equivalence $y \underset{p}{\overset{=}{\to}} y' \simeq \mathrm{tr}(p, y) =_{f(x')} y'$. Furthermore if we also have a dependent function (section) $s : \prod_{x:X} f(x)$ then we can form $\mathrm{apd}(f, p) : s(x) \underset{p}{\overset{=}{\to}} s(x')$. In other words, there may be a whole type of paths between two points upstairs y, y' but if there is also a section then we have a special section-specific pathover between $s(x), s(x')$.

Lemma 1. *Given a type family $f : X \to \mathcal{U}$ and terms $x, x' : X, y : f(x), y' : f(x')$ then there is an equivalence*

$$\big((x, y) =_{\sum_{x:X} f(x)} (x', y')\big) \simeq (x = x') \times (y = y).$$

Proof. Both sides are identity types in a sigma type, so we must prove

$$\sum_{p:x=x'} y \underset{p}{\overset{=}{\to}} y' \simeq \sum_{p:x=x'} y = y.$$

This reduces to proving $y \underset{p}{\overset{=}{\to}} y' \simeq y = y$. By induction when $p = \mathsf{refl}$ both sides are $y = y$ which we prove with refl. (Compare to ([6] Lemma 4.14.1) which assumes a section.) □

So given a path p in the base and points upstairs over the endpoints, the dependent paths over p are equivalent to loops at one upstairs point, which are regular non-dependent paths. If p is a loop in $x = x$ then the type of dependent loops over that loop is equivalent to the non-dependent loops $y = y$.

1.4 Deloopings

A *delooping* of a group is a pointed type whose loop space is the group. The references we follow here are [6] and [7]. Especially:

Definition 1. *([7] Definition 4.1)* $\mathrm{BAut}_1(G) := \sum_{X:U} ||X = G||_0$.

Theorem 1. *([7] Theorem 4.26) The type of G-torsors is equivalent to* $\mathrm{BAut}_1(G)$.

The component $||X = G||_0$ is called a G-band. Its data consists of a choice of basepoint in X to map to the identity in G.

The loop space at the base point is the given group. The loop space at other points are equivalent groups (e.g. conjugations of the group). The path spaces between pairs of points are torsors of the given group.

1.5 Mapping into deloopings

A motivating example is $\mathbb{O} \stackrel{\text{def}}{=} \mathbb{2} * (\mathbb{2} * \mathbb{2})$. This is an octahedron with 6 points, 12 paths, and 8 2-paths. Think of it as representing a good open cover of a cube, where each vertex of $\mathbb{O}$ is one contractible open set covering a face of the cube (plus a little overlap to the four neighboring faces), each path in $\mathbb{O}$ is an overlap of two of these face open sets, and each 2-path is a 3-way overlap. We use a join because the join keeps all the 0-dimensional points of all of the $\mathbb{2}$s as 0-dimensional points in the final type, it doesn't send them up into a higher dimension like the suspension does.

Let's name the vertices after the colors of a Rubik's cube: (w)hite on top, (y)ellow on the bottom, (g)reen facing out of the page, (b)lue facing into the page, (r)ed to the right and (o)range to the left. So instead of using the same boolean type $\mathbb{2}$ repeatedly let's perform the join like this:

$$S = \{w, y\} * (\{b, g\} * \{r, o\})$$

so that we build up the cube with three pairs of opposite faces. The factor in parentheses $\{b, g\} * \{r, o\}$ is a square where b and g are at opposite corners and so are r and o. Then we join a "north and south pole" $\{w, y\}$ to this sqare, forming the octahedron. This differs from the suspension of the equatorial square $\{b, g\} * \{r, o\}$ because the square is not moved into dimensions 1 and 2, and because we have paths from the poles to the equator, not from pole to pole.

That's the domain of our map. Let's map it into $\mathrm{BAut}_1(S^1)$ with a certain basepoint. Consider a generalization of S^1, an *n-gon* C_n, generated by

- $v_1, \ldots, v_n : C_n$
- $e_1 : v_1 = v_2$
- $e_2 : v_2 = v_3$
- $\ldots$
- $e_{n-1} : v_{n-1} = v_n$
- $e_n : v_n = v_1$

There is a map $s_n : C_n \to C_n$ that sends $v_i \mapsto v_{i+1}$ and $v_n \mapsto v_1$, the generating cyclic permutation.

Lemma 2. *We have* $g : \prod_{n:\mathbb{N}} C_n \simeq S^1$

Proof. Generalize [8] Lemma 6.5.1. □

The reason we're interested in these is that they form arbitrarily fine-grained approximations to the smooth circle. We can consider an n-gon to be a circle that has a notion of "going around $1/n$th of the way".

Now let's re-point $\mathrm{BAut}_1(S^1)$ at C_4. We can use a + to denote a pointed type, and we can decorate it with the base point, like so: $\mathrm{BAut}_1(S^1)_{+C_4}$. We can then refer to the underlying type (first projection from the pointed type) with $\mathrm{BAut}_1(S^1)_{-}$. This is similar to the notation $\mathrm{BAut}_1(S^1)_{\div}$ you find in the Symmetry book [6].

On second thought let's use a suspension after all:

$$S' = \Sigma(\{b, g\} * \{r, o\}) = \Sigma C_4$$

so that we can more easily compare the tangent bundle with the Hopf bundle. Let's call the poles $\{w, y\}$, and now we have a copy of C_4 inside $w = y$.

Define the map $T : S' \to \mathrm{BAut}_1(S^1)$ as

- $T(w) = T(y) = C_4$
- $T(r) = T(o) = T(b) = T(g) = \mathrm{id}_{C_4}$
- $T(rg) = T(go) = T(ob) = T(br) =$ rotation by 180.

And therefore $T(rg \cdot go \cdot ob \cdot br) = 2$ rotations.

2 Higher covers

What makes this all work is that S^1 is a discrete 1-type, not a discrete set. A map from $\int X \to BG$ for a discrete set-level group G would indeed carry its unique *flat* connection, since homotopic paths must map to the same set element. But so long as we're willing to work with discrete spaces and discrete connections, we can see geometry inside homotopy type theory!

Following David Jaz Myers in [9], we define a cover as follows:

Definition 2. *A map* $\pi : E \to B$ *is a* cover *if it is* $\int_1$*-étale and its fibers are sets.*

Recall that π being $\int_1$-étale means that the naturality square

$$\begin{tikzcd} E \arrow[r, "(-)^{\int_1}"] \arrow[d, "\pi"'] & \int_1 E \arrow[d, "\int_1 \pi"] \\ B \arrow[r, "(-)^{\int_1}"'] & \int_1 B \end{tikzcd}$$

is a pullback, which means among other things that each corresponding fiber of the vertical maps is equivalent.

As David proves in [9], the type of $\int_1$-étale maps into B is equivalent to the function type $\int_1 B \to \mathsf{Type}_{\int_1}$. Since a cover has the further condition that the fibers are sets this implies

Lemma 3. *The type of covers over B is equivalent to $\int_1 B \to \mathsf{Type}_{\int_0}$.*

In homotopy type theory we pass easily between the vertical picture (maps into B) and the horizontal picture (classifying maps of B into some type of fibers). But when we unpack this a little bit we find important classical stories. If we equip B with a basepoint $*_B$ then a map $f : \int_1 B \cdot\!\to \mathsf{Type}_{\int_0}$ is an action of the group $\int_1 B$ on the set $f(*_B)$, which is the fiber over $*_B$.

Let us move up a dimension:

Definition 3. *A map $\pi : E \to B$ is a* 2-cover *if it is $\int_2$-étale and its fibers are groupoids.*

Lemma 4. *The type of 2-covers over B is equivalent to $\int_2 B \to \mathsf{Type}_{\int_1}$. Further, since $\mathsf{Type}_{\int_1}$ is 2-truncated, the type of 2-covers of B is equivalent to $\int B \to \mathsf{Type}_{\int_1}$.*

This is the type we will examine: maps from discrete n-types to the type of discrete 1-types.

For example, let S^1 be the higher inductive type generated by

- $\mathsf{base} : S^1$
- $\mathsf{loop} : \mathsf{base} = \mathsf{base}$

We can deloop S^1 by forming its type of torsors. This is equivalent to $\mathrm{BAut}_1(S^1) \overset{\text{def}}{=} \sum_{X:\mathsf{Type}} ||X = S^1||_0$, the type of pairs of a type together with an equivalence class of isomorphisms with S^1 (see [7]).

Lemma 5. *Terms of $\mathrm{BAut}_1(S^1)$ are discrete 1-types.*

Proof. TBD. ◻

3 Evidence that we found curved connections

3.1 The Hopf fibration

3.2 The hairy ball theorem

3.3 The Atiyah sequence

This is an exact sequence of bundles. Converting them to paths gives us the lemmas about pathovers.

3.4 Freed and Hopkins

Any claim to define connections in higher toposes must contend with Freed and Hopkins [10]. They introduce simplicial sets and simplicial sheaves in order to construct a classifying space for principal bundles with connection, which is a higher category. Actually they need higher categories even to classify principal bundles alone, since they wish to do so on the nose rather than up to homotopy, and so they need a groupoid-level classifying space.

In their language, the classifying space is the simplicial sheaf of trivializable principal G-bundles with connection. (G is a compact Lie group.) A connection is a classical, infinitesimal one, i.e. a 1-form with values in the Lie algebra $\mathfrak{g}$. And so their classifying space is the product (bundle classifier) $\times$ (connection). If we compose the 1-form with integration, we form a map from the paths in the classifying space to G. But the 1-dimensional structure of $\mathrm{BAut}_1(S^1)$ is already G since it's a delooping of G. So maybe they are just building the higher type $\mathrm{BAut}_1(S^1)$ as a product of two spaces.

If that's the case, then can we see in our picture the Weil algebra and so on that they end up proving is the cohomology of this classifying space?

3.5 Atiyah and Bott

Let $B_{+b} : \mathsf{Type}_{\int *}$ be a pointed discrete type and let $f : B_{+b} \cdot \to \mathrm{BAut}_1(S^1)_{+C_4}$. If we need to reference the proof of pointedness we'll call it $*_f : f(b) = C_4$.

Lemma 6. $\int(X \to Y) \simeq \int X \to \int Y$

.

Proof. TODO ◻

And so any results that depend only on the homotopy type of, say, a classifying map, can be studied in HoTT by replacing both the domain and the classifying type with their homotopy types.

Proposition 1. *([11] Proposition 2.4) Let BG be the classifying space for a compact Lie group G. Then in homotopy theory*

$$B\mathcal{G}(P) = \mathrm{Map}_P(M, BG)$$

where $\mathcal{G}(P)$ is the group of automorphisms of the principal bundle P, and where Map_P denotes the connected component of $M \to BG$ containing the classifying map of P.

Bibliography

[1] S. Talbott, *The Future Does Not Compute: Transcending the Machines in Our Midst*. O'Reilly & Associates, 1995. [Online]. Available: https://books.google.com/books?id=KcXaAAAAMAAJ

[2] D. S. Freed, "Classical chern-simons theory, part 1," 1992.

[3] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, ser. Foundations of Differential Geometry. Interscience Publishers, 1963. [Online]. Available: https://books.google.com/books?id=wn4pAQAAMAAJ

[4] K. Crane, "Discrete connections for geometry processing," Master's thesis, California Institute of Technology, 2010. [Online]. Available: http://resolver.caltech.edu/CaltechTHESIS:05282010-102307125

[5] K. Crane, Ed., *An Excursion Through Discrete Differential Geometry*. American Mathematical Society, 2020, vol. 76.

[6] M. Bezem, U. Buchholtz, P. Cagne, B. I. Dundas, and D. R. Grayson, "Symmetry," https://github.com/UniMath/SymmetryBook.

[7] U. Buchholtz, J. D. Christensen, J. G. T. Flaten, and E. Rijke, "Central h-spaces and banded types," 2023.

[8] Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[9] D. J. Myers, "Good fibrations through the modal prism," 2022.

[10] D. S. Freed and M. J. Hopkins, "Chern-weil forms and abstract homotopy theory," 2013. [Online]. Available: https://arxiv.org/abs/1301.5959

[11] M. F. Atiyah and R. Bott, "The yang-mills equations over riemann surfaces," *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 308, no. 1505, pp. 523–615, 1983.