

Discrete differential geometry in homotopy type theory

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Summary

Summary

This work brings to HoTT

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature = total index

Classical \rightarrow HoTT

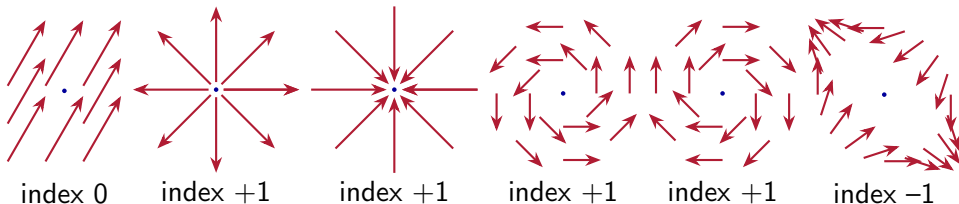
Let M be a smooth, oriented 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes x_1, \dots, x_n .

$$\begin{array}{ccc} \frac{1}{2\pi} \int_M F_A = \sum_{i=1}^n \text{index}_X(x_i) = \chi(M) & & \\ \downarrow & & \downarrow \\ \sum_{\text{faces } F} b_F = & \Longleftrightarrow & \sum_{\text{faces } F} L_F^X \end{array}$$

Classical index

Near an isolated zero there are only three possibilities: index 0, 1, -1 .

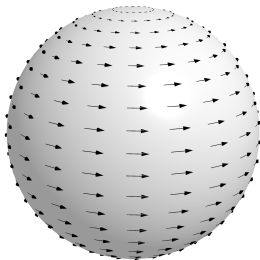
Index is the winding number of the field as you move clockwise around the zero.



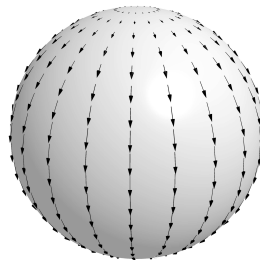
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index $+1$ at each pole $= 2$



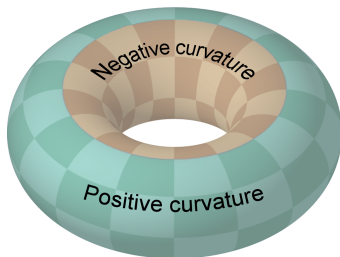
Height: index $+1$ at each pole $= 2$

Gauss-Bonnet theorem

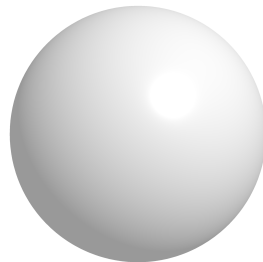
Total curvature divided by 2π is the Euler characteristic.

Curvature in 2D is a function $F_A : M \rightarrow \mathbb{R}$.

$\int_M F_A$ sums the values at every point.



Positive and negative curvature cancel: **0**



Constant curvature 1, area 4π : **2**

Plan

- Combinatorial manifolds
- Torsors and classifying maps
- Connections and curvature
- Vector fields
- Main theorem

HoTT background

- ① **Symmetry**,
Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
<https://github.com/UniMath/SymmetryBook>.
- ② **Central H-spaces and banded types**,
Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
arXiv:2301.02636
- ③ **Nilpotent types and fracture squares in homotopy type theory**,
Scoccola, L. (2020)
MSCS 30(5). arXiv:1903.03245

Combinatorial manifolds

Manifolds in HoTT

- Recall the classical theory of **simplicial complexes**
- Define a **realization** procedure to construct types

Simplicial complexes

Definition

An **abstract simplicial complex** M of **dimension** n is an ordered list of sets

$M \stackrel{\text{def}}{=} [M_0, \dots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of subsets of M_0 of cardinality $k + 1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j + 1$ then $G \in M_j$

We call the truncated list

$M_{\leq k} \stackrel{\text{def}}{=} [M_0, \dots, M_k]$ **the k -skeleton of M .**

Simplicial complexes

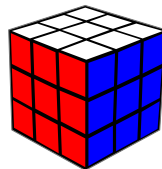
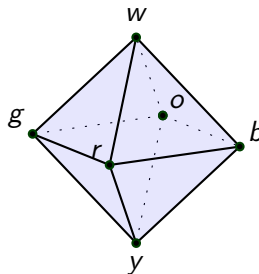
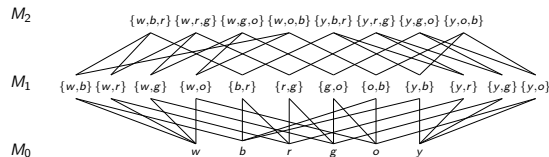
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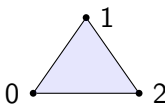
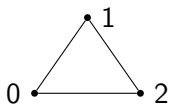


Simplicial complexes

Example

The **complete simplex of dimension n** , denoted $\Delta(n)$, is the set $\{0, \dots, n\}$ and its power set. The $(n-1)$ -skeleton $\Delta(n)_{\leq (n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$ -sphere.

$\Delta(1)$ is visually $0 \bullet \text{---} \bullet 1$, $\partial\Delta(1)$ is visually $0 \bullet \quad \bullet 1$,

$\Delta(2)$ is visually , $\partial\Delta(2)$ is visually 

Homotopy realization: dimension 0

We will **realize** simplicial complexes by means of **a sequence of pushouts**.

Base case: the realization \mathbb{M} of a 0-dimensional complex M is M_0 .

In particular the 0-sphere $\partial\Delta(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$.

Homotopy realization: dimension 1

For a 1-dim complex $M \stackrel{\text{def}}{=} [M_0, M_1]$ the realization is given by

$$\begin{array}{ccc}
 M_1 \times \partial\Delta(1) & \xrightarrow{\text{pr}_1} & M_1 \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *_{M_1} \\
 M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1
 \end{array}$$

Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\Delta(2) \stackrel{\text{def}}{=} \begin{array}{c} \bullet 1 \\ \diagup \quad \diagdown \\ 0 \bullet \quad \bullet 2 \end{array}$ is given by

$$\begin{array}{ccc} \partial\Delta(2)_1 \times \partial\Delta(1) & \longrightarrow & \partial\Delta(2)_1 \\ \downarrow & \nearrow h_1 & \downarrow \\ \partial\Delta(2)_0 & \longrightarrow & \partial\Delta(2) \end{array}$$

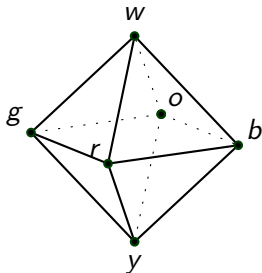
i.e.

$$\begin{array}{ccc} \{\{0,1\}, \{1,2\}, \{2,0\}\} \times \{0,1\} & \longrightarrow & \{\{0,1\}, \{1,2\}, \{2,0\}\} \\ \downarrow & \nearrow h_1 & \downarrow \\ \{0,1,2\} & \longrightarrow & \partial\Delta(2) \end{array}$$

Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron \mathbb{O} :

$$\begin{array}{ccc}
 \{\{w, g\}, \dots\} \times \{0, 1\} & \longrightarrow & \{\{w, g\}, \dots\} \\
 \downarrow & \nearrow h_1 & \downarrow \\
 \{w, g, \dots\} & \xrightarrow{\quad \perp \quad} & \mathbb{O}_1
 \end{array}$$



Homotopy realization: dimension 2

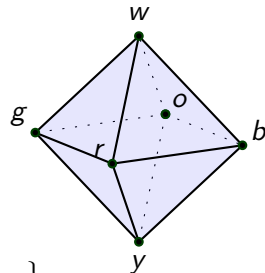
To realize $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\Delta(1), \partial\Delta(2)$:

$$\begin{array}{ccccc}
 M_1 \times \partial\Delta(1) & \xrightarrow{\text{pr}_1} & M_1 & & \\
 \mathbb{A}_0 \downarrow & \nearrow h_1 & \downarrow *M_1 & & \\
 M_0 = \mathbb{M}_0 & \xrightarrow{\quad} & \mathbb{M}_1 & \xrightarrow{\quad} & \mathbb{M}_2 \\
 & & \uparrow \mathbb{A}_1 & \nwarrow h_2 & \uparrow *M_2 \\
 & & M_2 \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & M_2
 \end{array}$$

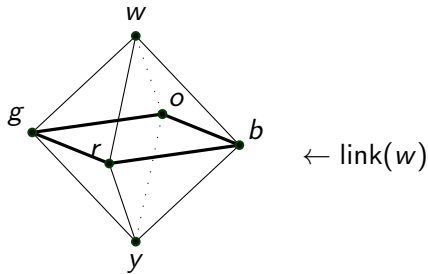
Homotopy realization: dimension 2

The full octahedron \mathbb{O} :

$$\begin{array}{ccccc}
 \{\{w, g\}, \dots\} \times \{0, 1\} & \xrightarrow{\text{pr}_1} & \{\{w, g\}, \dots\} & & \\
 \downarrow & \nearrow h_1 & \downarrow & & \\
 \{w, g, \dots\} & \xrightarrow{\quad} & \mathbb{O}_1 & \xrightarrow{\quad} & \mathbb{O}_2 \\
 & & \uparrow & \searrow h_2 & \uparrow \\
 & & \{\{w, g, r\}, \dots\} \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & \{\{w, g, r\}, \dots\}
 \end{array}$$



Homotopy realization: dimension 2



The **link** of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

A **combinatorial manifold** is a simplicial complex all of whose links are* simplicial spheres.

This will be our model of the **tangent space**.

*the (classical) geometric realization is homeomorphic to a sphere

Combinatorial manifolds \leftrightarrow smooth manifolds

Theorem (Whitehead (1940))

*Every smooth n -manifold has a compatible structure of a **combinatorial manifold**: a simplicial complex of dimension n such that the link is a combinatorial $(n - 1)$ -sphere, i.e. its geometric realization is an $(n - 1)$ -sphere.*

<https://ncatlab.org/nlab/show/triangulation+theorem>

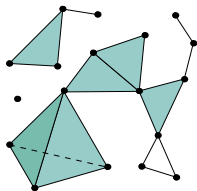
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Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:



Torsors

What type families $\mathbb{M} \rightarrow \mathcal{U}$ will we consider? Families of **torsors**, also called **principal bundles**.

Torsors

Let G be a (higher) group.

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- Let **BG** be the type of G -torsors.
- Let G_{reg} be the G -torsor consisting of G acting on itself on the right.

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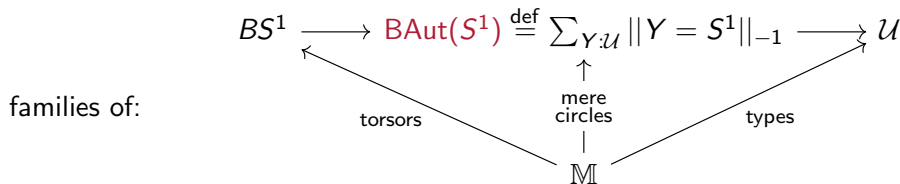
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- ② BG is connected.
- ③ $1 \text{ \& } 2 \implies BG$ is a $K(G, 1)$.

See the Buchholtz et. al. H-spaces paper for more.

How to map into BS^1

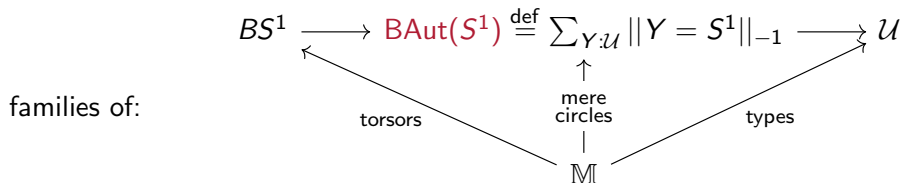
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Other names:

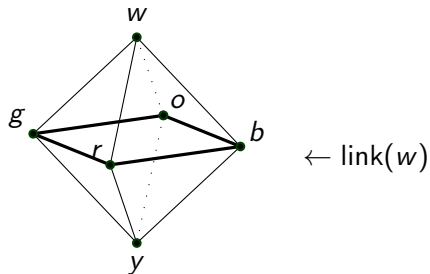
- $\mathbf{BAut}(S^1) = BO(2) = \mathbf{EM}(\mathbb{Z}, 1)$ (where $\mathbf{EM}(G, n) \stackrel{\text{def}}{=} \mathbf{BAut}(K(G, n))$)
- $BS^1 = BSO(2) = K(\mathbb{Z}, 2)$

Connections and curvature

Connections

Connections are extensions of a bundle to higher skeleta.

Recall link



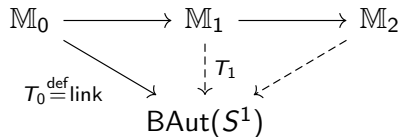
The **link** of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

Define the **tangent bundle** on a combinatorial manifold to be

$$T_0 \stackrel{\text{def}}{=} \text{link} : \mathbb{M}_0 \rightarrow \text{BAut}(S^1).$$

Connections on the tangent bundle

An extension T_1 of T_0 to \mathbb{M}_1 is called **a connection on the tangent bundle**.

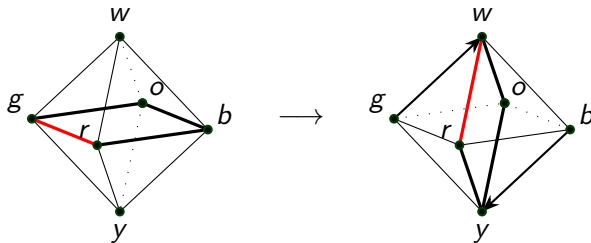


$T_1 : \mathbb{M}_1 \rightarrow \text{BAut}(S^1)$ extending link

We will define T_1 on the edge wb , so we need a term

$$T_1(wb) : \text{link}(w) =_{\text{BAut}(S^1)} \text{link}(b).$$

We imagine tipping:

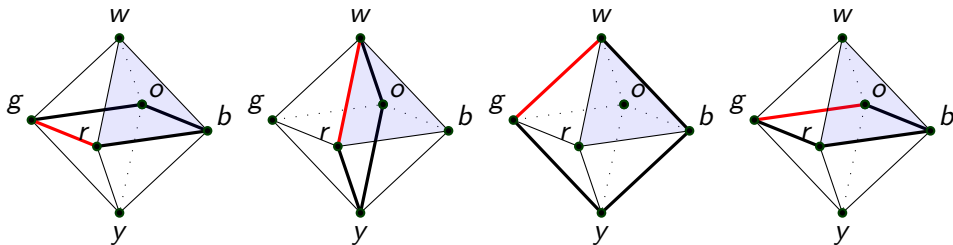


$$T_1(g : \text{link}(w)) \stackrel{\text{def}}{=} w : \text{link}(b), \dots$$

Use this method to define T_1 on every edge.

$T_1 : \mathbb{M}_1 \rightarrow \text{BAut}(S^1)$ extending link

Denote the path $wb \cdot br \cdot rw$ by $\partial(wbr)$. Consider $T_1(\partial(wbr))$:

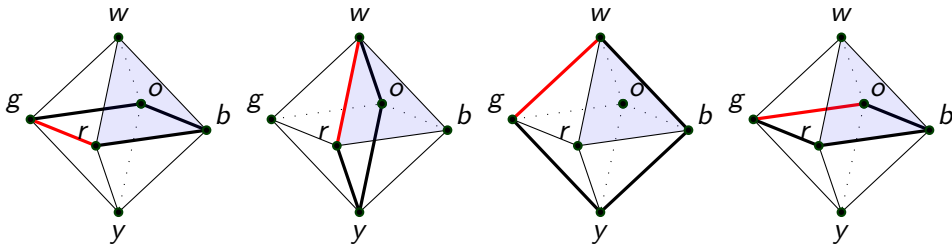


We come back rotated by $1/4$ turn. Call this rotation $R : \text{link}(w) =_{\text{BAut}(S^1)} \text{link}(w)$.

Extending T_1 to a face

Let $H_{wbr} : \text{refl}_w =_{w=\mathbb{M}w} \partial(wbr)$ be the filler homotopy of the face.

T_2 must live in $T_1(\text{refl}_w) =_{(\text{link}(w)=_{\text{BAut}(S^1)}\text{link}(w))} T_1(\partial(wbr)) = R$

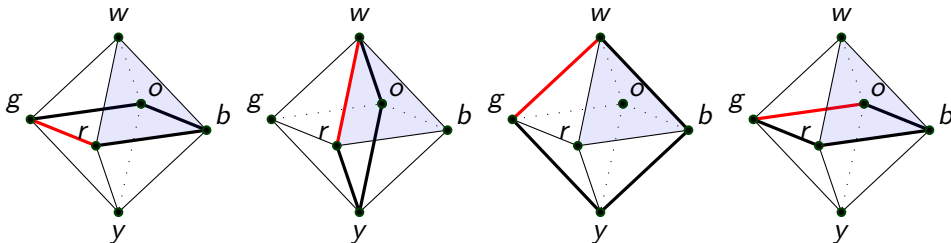


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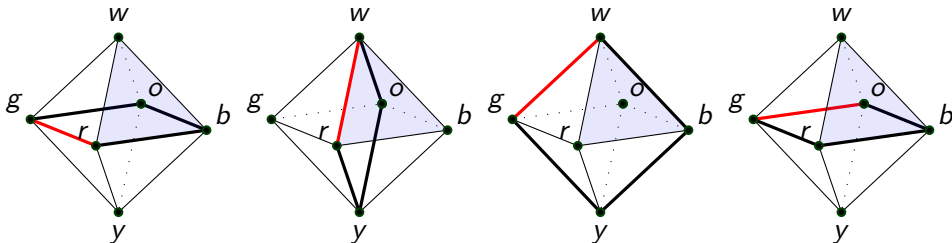
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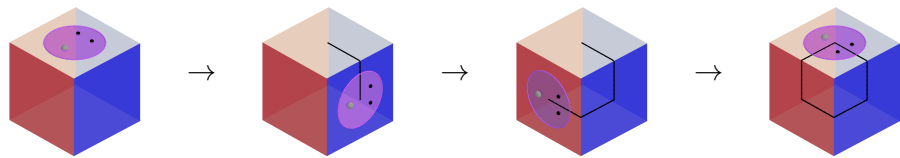
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For example, a path $H_R(g) : g = Rg = o$. Choose go .



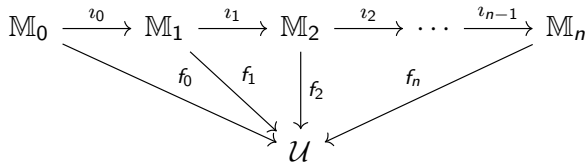
Original inspiration



The definition of a connection

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\iota_0} \dots \xrightarrow{\iota_{n-1}} \mathbb{M}_n$ is the realization of a combinatorial manifold and all the triangles commute in the diagram:



- The map f_k is a **k -bundle** on \mathbb{M} .
- The pair given by the map f_k and the proof $f_k \circ \iota_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a **k -connection on the $(k-1)$ -bundle f_{k-1}** .

The definition of curvature

Definition (cont.)

An extension consists of M_2 -many extensions to faces:

$$\begin{array}{ccc}
 M_2 \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & M_2 \\
 \mathbb{A}_1 \downarrow & \nearrow h_2 & \downarrow \\
 \mathbb{M}_1 & \xrightarrow{\quad} & \mathbb{M}_2 \\
 & \searrow T_1 & \downarrow T_2 \\
 & & \mathcal{U}
 \end{array}$$

Here's the outer square for a single face F :

$$\begin{array}{ccc}
 \{F\} \times \partial\Delta(2) & \xrightarrow{\text{pr}_1} & \{F\} \\
 \mathbb{A}_1 \downarrow & \nwarrow b_F & \downarrow \\
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$T_1(\partial(F))$ is the curvature at the face F and the filler $b_F : \text{id} = T_1(\partial F)$ is called a flatness structure for the face F .

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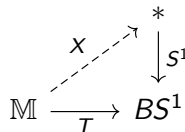
The distinction between the path b_F and the endpoint $T_1(\partial(F))$ is small enough to be confusing.

Vector fields

Vector fields

Let $T : \mathbb{M} \rightarrow BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of T , so that's not a good definition.

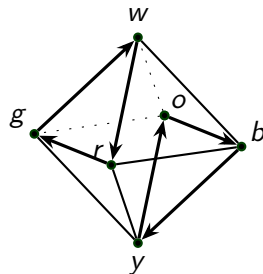
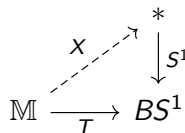


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Our solution:



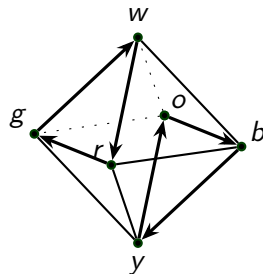
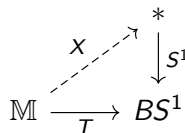
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- A global section of this family would be a trivialization of T , so that's not a good definition.

Our solution:

- A **vector field** is a term $X : \prod_{m:\mathbb{M}_1} Tm$.



Vector fields

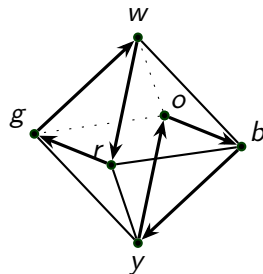
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- A **vector field** is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical **nonvanishing** vector field on the 1-skeleton.

$$\begin{array}{ccc} & & * \\ & \nearrow X & \downarrow s^1 \\ \mathbb{M} & \xrightarrow{T} & BS^1 \end{array}$$



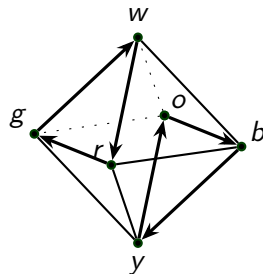
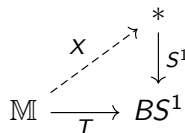
Vector fields

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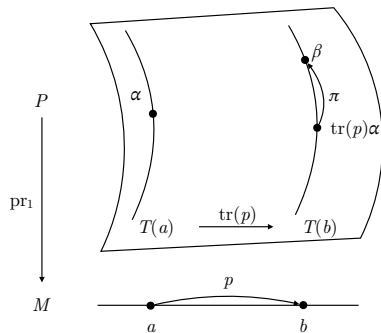
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Our solution:

- A **vector field** is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical **nonvanishing** vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

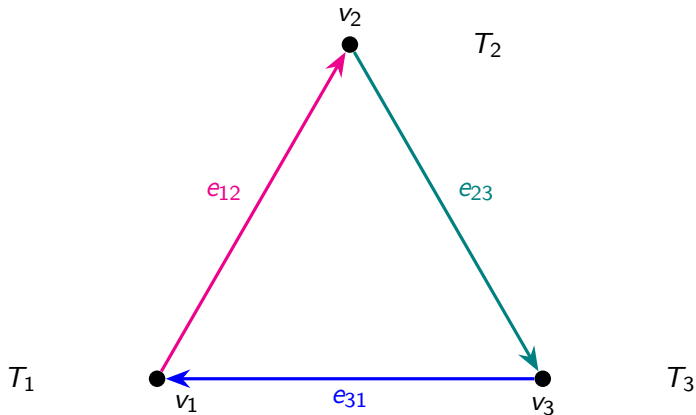


Reminder: pathovers

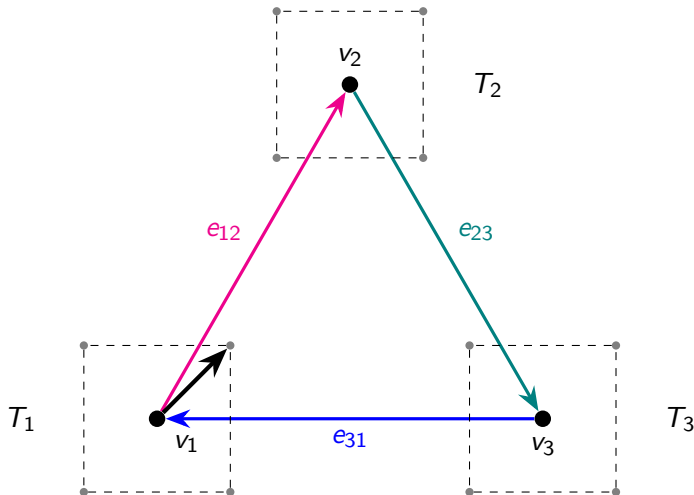


- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display π , the path over p .
- Dependent functions map paths to pathovers:
 $\text{apd}(X)(p) : \text{tr}_p(X(a)) = X(b)$ (simply denoted $X(p)$).

Next goal: define the index of a vector field on a face by computing $X(\partial F)$ around a face.



An example of **swirling** and **index** at this face.



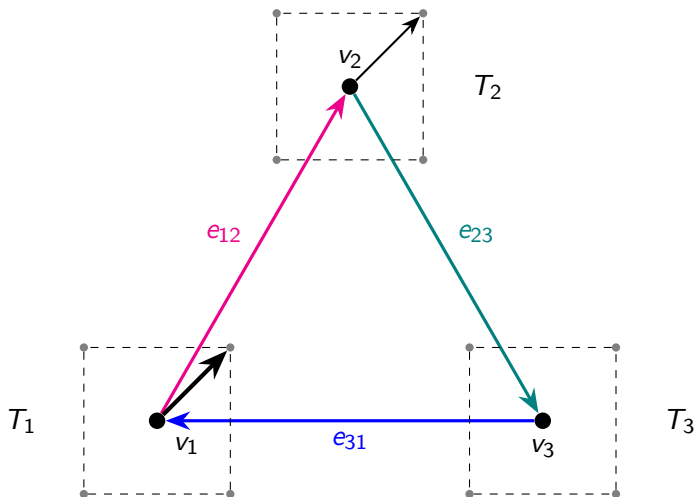
An example of **swirling** and **index** at this face.

- Denote by X_1 this vector
 $X(v_1) : T_1$.

•

•

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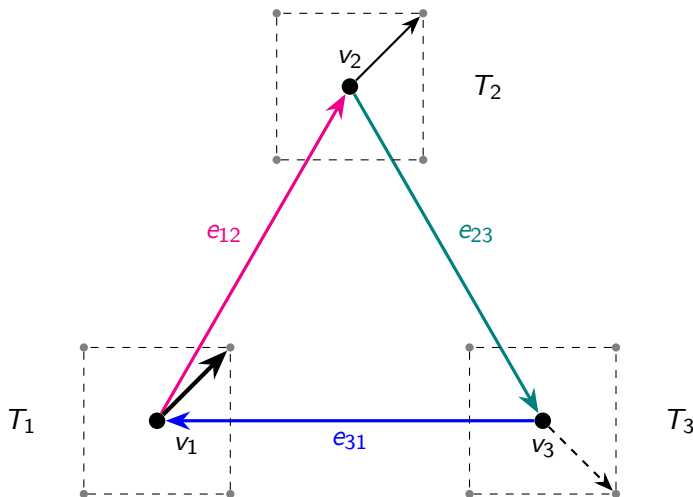


An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.

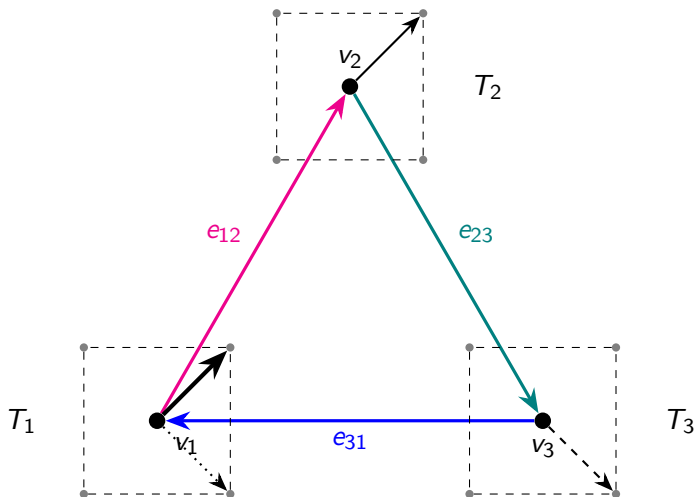
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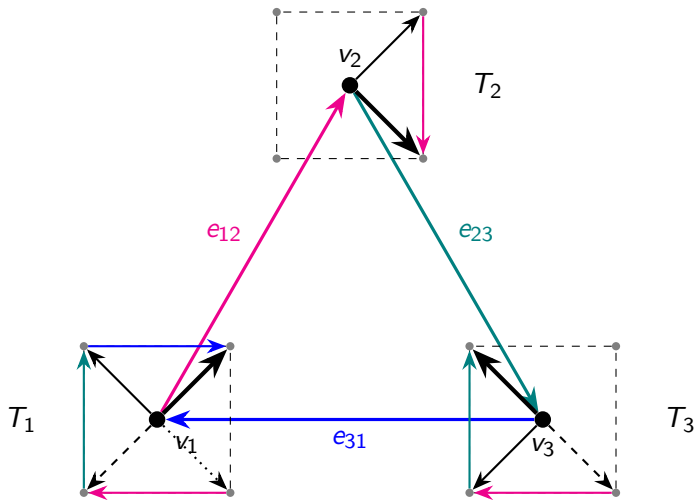
An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
-

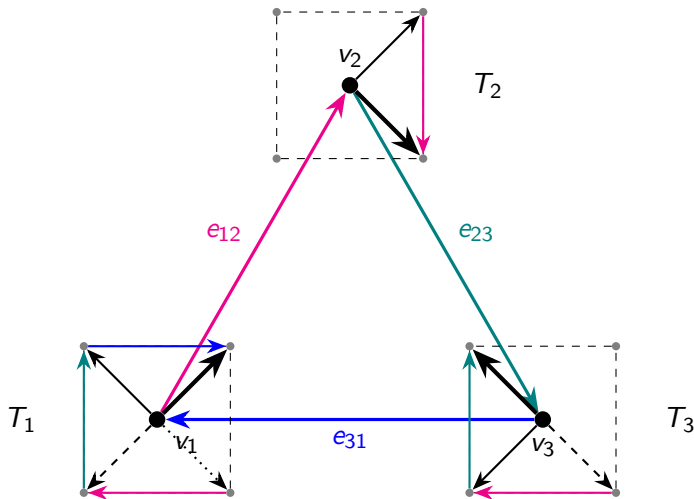


An example of **swirling** and **index** at this face.

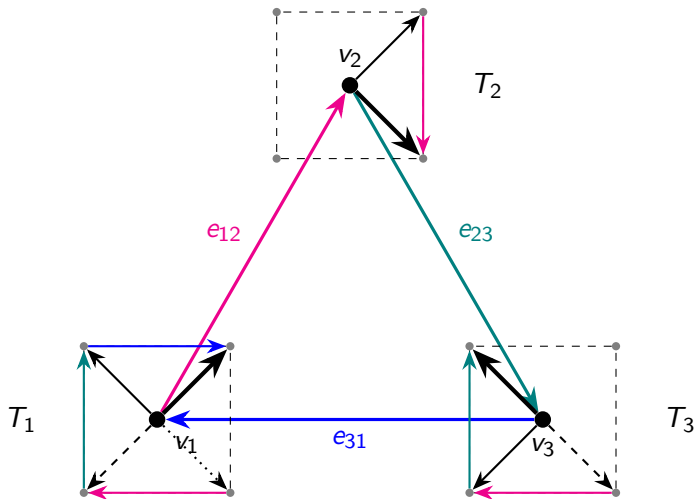
- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
- Say T_{13} is trivial. The thrice-transported vector is dotted.



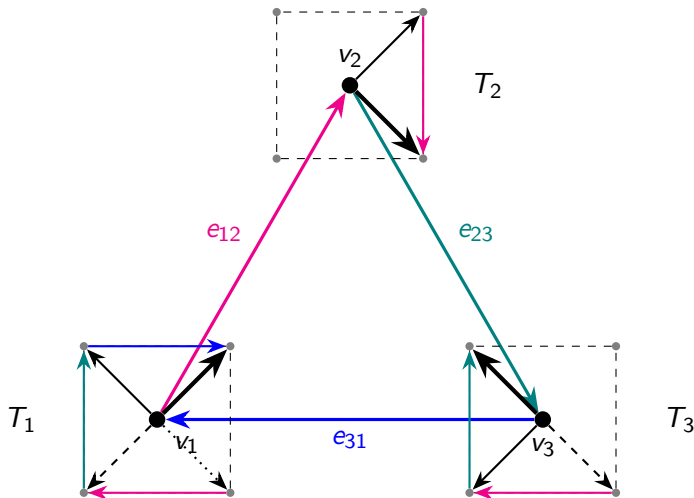
- X on e_{12} is red, etc.



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- We translated all pathover data to the end of the loop.



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- (Reminds me of scooping ice cream towards the last fiber.)



- X on e_{12} is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)
- The total pathover $X(\partial F)$ is called **the swirling X_F** of X at the face F .

Symbolic version

$$T_1 \xrightarrow{T_{21}} T_2 \xrightarrow{T_{32}} T_3 \xrightarrow{T_{13}} T_1$$

$$\begin{array}{ccccccc}
 & & & & T_{13} T_{32} T_{21} X_1 & & \\
 & & & & T_{13} T_{32} X_{21} : \parallel & & \\
 & & T_{32} T_{21} X_1 & & T_{13} T_{32} X_2 & & \\
 & & T_{32} X_{21} : \parallel & & T_{13} X_{32} : \parallel & & \\
 & & T_{32} X_2 & & T_{13} X_3 & & \\
 & & X_{32} : \parallel & & X_{13} : \parallel & & \\
 & & X_3 & & X_1 & & \\
 & T_{21} X_1 & & & & & \\
 X_{21} : \parallel & & & & & & \\
 X_1 & X_2 & & & & &
 \end{array}$$

Index

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$\chi_F \stackrel{\mathrm{def}}{=} \chi(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

Index

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$b_F \stackrel{\mathrm{def}}{=} b(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

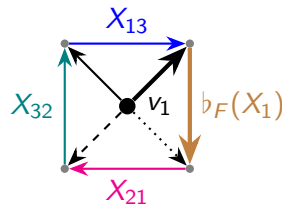
$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

Definition

The **flattened swirling** of the vector field X on the face F is the loop

$$L_F^X \stackrel{\mathrm{def}}{=} b_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The **index** of the vector field X on the face F is the integer I_F^X such that $\mathrm{loop}_{F^X}^{I_F^X} =_{S^1} (L_F^X) - X_1$.



Index

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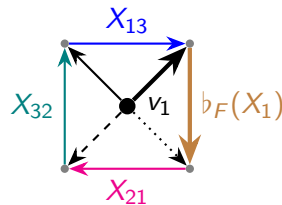
(Recall that T_1 being an S^1 -torsor means we can use subtraction to obtain an equivalence $s(-, X_1) : T_1 \xrightarrow{x \mapsto x - X_1} S^1$.)

Definition

The **flattened swirling** of the vector field X on the face F is the loop

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Main theorem

Simplifying swirling

Swirling involves concatenating dependent paths. Can we simplify that?

Pay off all our assumptions 1: torsor structure, vector field

$$T_1$$

$$\begin{array}{c}
 T_{13} T_{32} T_{21} X_1 \\
 T_{13} T_{32} X_{21}: \parallel \\
 T_{13} T_{32} X_2 \\
 T_{13} X_{32}: \parallel \\
 T_{13} X_3 \\
 X_{13}: \parallel \\
 X_1
 \end{array}$$

Pay off all our assumptions 1: torsor structure, vector field

 T_1

- Def: $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$ (trivialization on 0-skeleton).

 $T_{13} T_{32} T_{21} X_1$
 $T_{13} T_{32} X_{21} : \parallel$
 $T_{13} T_{32} X_2$
 $T_{13} X_{32} : \parallel$
 $T_{13} X_3$
 $X_{13} : \parallel$
 X_1

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- Def: $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$ (trivialization on 0-skeleton).
- Def: $\rho_{ji} \stackrel{\text{def}}{=} \alpha_j(T_{ji}(X_i))$ is the rotation of T_{ji} .

 $T_{13} T_{32} T_{21} X_1$
 $T_{13} T_{32} X_{21} : \parallel$
 $T_{13} T_{32} X_2$
 $T_{13} X_{32} : \parallel$
 $T_{13} X_3$
 $X_{13} : \parallel$
 X_1

$$\begin{array}{ccc}
 T_i & \xrightarrow{T_{ji}} & T_j \\
 \text{base} \mapsto X_i \left(\begin{array}{c} \nearrow \alpha_i \downarrow \\ \searrow \end{array} \right. & & \left. \downarrow \alpha_j \nearrow \right) \text{base} \mapsto X_j \\
 S^1 & \xrightarrow{(-) + \rho_{ji}} & S^1
 \end{array}$$

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 X_1

$$\begin{array}{ccc} T_i & \xrightarrow{T_{ji}} & T_j \\ \text{base} \mapsto X_i \left(\begin{array}{c} \alpha_i \downarrow \\ S^1 \end{array} \right. & & \left. \downarrow \alpha_j \right) \text{base} \mapsto X_j \\ & \xrightarrow{(-) + \rho_{ji}} & S^1 \end{array}$$

- Lemma: $\rho_{ij} = -\rho_{ji}$ because in T_j :
 $\rho_{ij} + \rho_{ji} + X_j = \rho_{ij} + T_{ji}X_i = T_{ji}(\rho_{ij} + X_i) = T_{ji}T_{ij}X_j = X_j.$

Pay off all our assumptions 1: torsor structure, vector field (cont.)

$$T_1$$

$$\begin{array}{c}
 T_{13} T_{32} T_{21} X_1 \\
 T_{13} T_{32} X_{21} : \parallel \\
 T_{13} T_{32} X_2 \\
 T_{13} X_{32} : \parallel \\
 T_{13} X_3 \\
 X_{13} : \parallel \\
 X_1
 \end{array}$$

Pay off all our assumptions 1: torsor structure, vector field (cont.)

 T_1

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},.$

 $T_{13} T_{32} T_{21} X_1$
 $T_{13} T_{32} X_{21} : \parallel$
 $T_{13} T_{32} X_2$
 $T_{13} X_{32} : \parallel$
 $T_{13} X_3$
 $X_{13} : \parallel$
 X_1

Pay off all our assumptions 1: torsor structure, vector field (cont.)

 T_1

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base}, .$
- Paths of the form $(a =_{S^1} \text{base})$ can be added:

 $T_{13} T_{32} T_{21} X_1$
 $T_{13} T_{32} X_{21} : \parallel$

- $+: (a = \text{base}) \times (b = \text{base}) \rightarrow (a + b = \text{base}).$

 $T_{13} T_{32} X_2$
 $T_{13} X_{32} : \parallel$

- $p + q = (p + b) \cdot q.$

 $T_{13} X_3$
 $X_{13} : \parallel$
 X_1

Pay off all our assumptions 1: torsor structure, vector field (cont.)

 T_1

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 $T_{13} T_{32} X_2$
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- $p + q = (p + b) \cdot q.$
- Lemma: $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}.$

 $T_{13} X_3$
 $X_{13} : \parallel$
 X_1

Pay off all our assumptions 1: torsor structure, vector field (cont.)

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 $T_{13} X_{32} :$

- Lemma: $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}.$

- Proof: $\text{apd}(X)(\text{refl}) = \text{refl}$

 $T_{13} X_3$

$$\implies X_{ij} \cdot T_{ij} X_{ji} = \text{refl}_{X_i}$$

 $X_{13} :$

$$\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}} \text{ (} T_{ij} \text{ just translates } X_{ji} \text{ to cat with } X_{ji} \text{).}$$

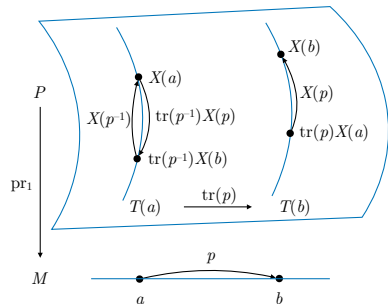
 X_1

Pay off all our assumptions 1: torsor structure, vector field (cont.)

 T_1

$$\begin{array}{l}
 T_{13} T_{32} T_{21} X_1 \\
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 \end{array}$$

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- Paths of the form $(a =_{S^1} \text{base})$ can be added:
 - $+: (a = \text{base}) \times (b = \text{base}) \rightarrow (a + b = \text{base}).$
 - $p + q = (p + b) \cdot q.$
- Lemma: $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}.$
- Proof: $\text{apd}(X)(\text{refl}) = \text{refl}$
 $\implies X_{ij} \cdot T_{ij} X_{ji} = \text{refl}_{X_i}$
 $\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}} \text{ (} T_{ij} \text{ just translates } X_{ji} \text{ to cat with } X_{ji} \text{).}$

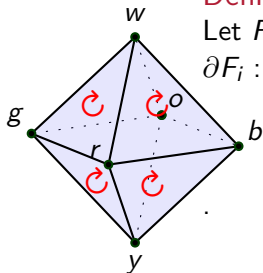


Pay off all our assumptions 2: no boundary, commutativity

Definition

Let F_1, \dots, F_n be the faces of \mathbb{M} , $v_i : F_i$ be designated vertices, and $\partial F_i : v_i = v_i$ be the triangular boundaries. The **total swirling** is

$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \dots + \sigma_{\partial F_n}$$

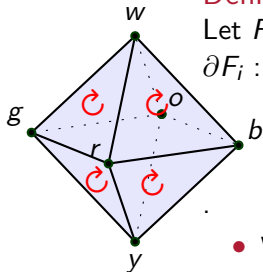


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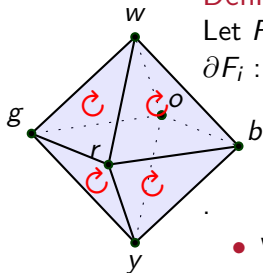
- We assume that this expression involves **every edge once in each direction**.

Pay off all our assumptions 2: no boundary, commutativity

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$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \dots + \sigma_{\partial F_n}$$



- We assume that this expression involves **every edge once in each direction**.
- S^1 is commutative, hence **complete cancellation**.

Consequence

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad T_1 =_{BS^1} T_1 \quad \text{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \quad \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \quad \mathrm{tr}_F(X_1) =_{T_1} X_1 \quad \text{swirling}$$

$$L_F^X \stackrel{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F \quad : \quad (X_1 =_{T_1} X_1) \quad \text{flattened swirling}$$

Consequence

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These can all be totaled in S^1 to give

$$\mathrm{tr}_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \rho_{\partial F} = \mathrm{base}$$

$$X_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

$$\flat_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F}$$

$$L_{\mathrm{tot}}^X \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

Consequence

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : \quad T_1 =_{BS^1} T_1 \quad \text{curvature}$$

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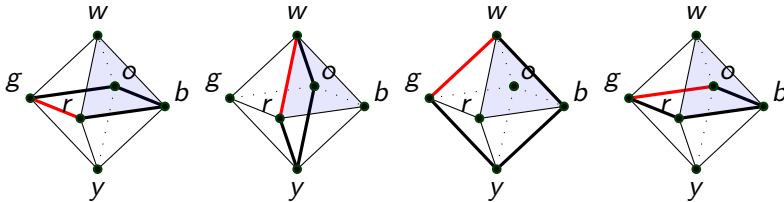
$$X_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \sigma_{\partial F} = \text{refl}_{\text{base}}$$

$$\flat_{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F}$$

$$L_{\mathrm{tot}}^X \stackrel{\mathrm{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

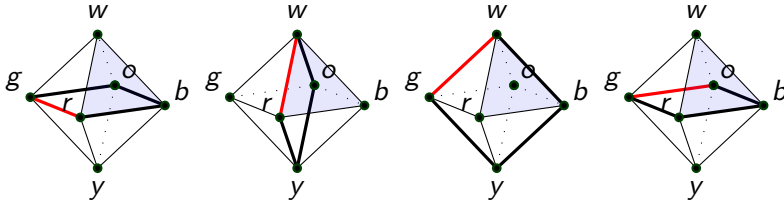
So in our lingo: the total flatness equals the total flattened swirling. □

Examples

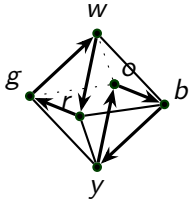


Each face contributes $b_F = H_R$, a $1/4$ -rotation. Total: 2.

Examples



Each face contributes $b_F = H_R$, a $1/4$ -rotation. Total: 2.



For total index one obtains $+1$ from F_{wrg} , $+1$ from F_{ybo} , $+0$ from others. Total: 2.

- The classical proof is discrete-flavored.
- “ $\angle Fw_{||}$ ” looked a lot like a pathover.
- Hopf’s Φ is defined on edges, not loops. We imitated that too.

 $\angle Fw_{||}$

Carnegie Mellon University

Thank you!