

Towards Differential Geometry in Homotopy Type Theory

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Abstract

This thesis will show how to formalize parts of differential geometry in homotopy type theory.[1] [2] [3] [4] [5] [6]

1 Introduction

2 Notes from 6/23/2021 meeting

Effective epis. An effective epi is intuitively a map that preserves all the information. Every point goes somewhere, every morphism goes somewhere, every 2-morphism etc. This is apparently what a homotopy colimit is too, and so that's why we say the map is the colimit of its simplicial diagram. When a groupoid acts it can be non-effective if it discards some of the equivalences and maps points to the same point without also preserving the arrow as a self-arrow. Conservation of mass. In HoTT, all the higher structure is always preserved, and so the only reason a map can fail to be an effective epi is if it fails to be surjective, i.e. fails to hit some point, i.e. the fiber over some point is empty. Since a surjection is defined as a map with all nonempty fibers, all surjectives are effective epi in HoTT.

Local trivializations: One definition of locally trivial map we are proposing is: $E \rightarrow X$ is locally trivial there exists an effective epi $Y \rightarrow X$ such that the pullback of the bundle is trivial, i.e. isomorphic to a product.

$$\begin{array}{ccc} Y \times F & \xrightarrow{\cong} & E \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

Every HoTT principal bundle is locally trivial in this sense via the map... p itself??

Given an effective epi then we want to eventually interpret this in 0-types that behave like smooth manifolds and show the link with classical local triviality. We will use

Lemma 1. *If X has an open cover (in a sense TBD) and $p : X' \rightarrow X$ is effective epi, then the cover lifts:*

$$\begin{array}{ccc} & X' & \\ \nearrow \exists \sigma & \downarrow \forall \text{eff epi} & \\ \coprod U_{i \text{ open cover}} & \rightarrow & X \end{array}$$

Candidates for the definition of “open cover”.

1. A subset of effective epis that are a) stable under composition (a cover of a cover is a cover, which is apparently called a Σ -condition) because

$$\begin{array}{ccc} V & \xrightarrow{\simeq} & \sum_U V_U \\ \text{cover} \downarrow & & \downarrow \underline{\text{cover}} \\ U & \xrightarrow{\text{cover}} & X \end{array}$$

and b) stable under base change (which we will get for free in HoTT?).

2. A subset of opens inside the object classifier Ω . Also a definition of an open map. We might call the special collection of opens “blocks”. We would interpret them onto the \mathbb{R}^n s. We call a type X *geometric* with respect to these opens if there is an *open* map $\coprod_i U_i \rightarrow X$. These are the spaces that can be built by gluing together the blocks. In the interpretation it will sit in a chain of inclusions

$$\text{Manifolds} \subset \text{Geometric} \subset \text{Diffeological} \subset \text{Sh}(\mathbb{R}^n)$$

One of the non-manifolds that are geometric would be two copies of \mathbb{R} glued everywhere but at the origin, so a real line with two origins. This is geometric. But what is not geometric is the union of the x and y axes in \mathbb{R}^2 . Here the maps from \mathbb{R} are not open maps.

Submersions: These are smooth maps that locally look like projections from $\mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$. A Morse function is not a submersion. A bundle map is a submersion. The map $x^3 - y^2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is not a submersion because the fibers change topology due to singularities.

Principal bundles in HoTT, a review. A dependent type $p : E \rightarrow B$ is given by a classifying map $E : B \rightarrow \text{Type}$, and a principal bundle is one where E factors through BG , so $E : B \rightarrow BG$. One way to define BG is by

$$\sum_{E : \text{Type}^G} ||E \simeq_{\text{Type}^G} G||_{-1}$$

, where Type^G is the universe of types equipped with a G -action. The action of G on itself is via translation, i.e. left or right multiplication. I’ve been saying to myself that “ BG is everything equal to G ” but this is more precise.

We have a surjection followed by a mono

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \twoheadrightarrow & BG & \hookrightarrow & \text{Type}^G & \longrightarrow \text{Type} \\
 & & & \parallel & & & \\
 & & & \sum_{E:\text{Type}^G} ||E \simeq_{\text{Type}^G} G||_{-1} & & &
 \end{array}$$

Trivializing principal bundles. What are the fibers of p ?

$$\begin{array}{ccccccc}
 \sum_b \text{Id}(G, E_b) & \equiv & P & \equiv & \sum_b \text{Id}(F, E_b) & \xrightarrow{\quad} & B \xrightarrow{\quad} 1 \\
 & & & & \downarrow & \lrcorner & \downarrow \\
 & & & & 1 & \xrightarrow{F} & BG \subset \text{Type}
 \end{array}$$

(A dotted curved arrow points from B back to $\sum_b \text{Id}(F, E_b)$)

where the dotted arrow is a putative section, i.e. a trivialization of the bundle.

Adopt the point of view: are the fibers of the bundle all equivalent? What type is the type of the fibers? If we know the typical fiber is F , then we can look at the total space of identifications between F and the fiber. This is a dependent type over the base B . A section of this is a global choice of such an identification. I was surprised to learn that this is the same as a section of the bundle, a trivialization! But of course that tracks with the classical story: the fibers are only identifiable with G once you choose a basepoint. To do that globally is exactly a section. So this is just recapitulating that idea, plus the fact that in HoTT sections are continuous.

Consider that $\sum_b \text{Id}(G, E_b)$ is the type of all points $b : B$ in the base and identifications between G and the fiber E_b . Identifications of the fiber with G , in the case where G is the correct group, i.e. this type is inhabited, is exactly a copy of E_b because the identification is the same as a choice of basepoint.

References

- [1] D. S. Freed and M. J. Hopkins, “Chern-weil forms and abstract homotopy theory,” 2013. [Online]. Available: <https://arxiv.org/abs/1301.5959>
- [2] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, ser. Foundations of Differential Geometry. Interscience Publishers, 1963, no. v. 1. [Online]. Available: <https://books.google.com/books?id=wn4pAQAAMAAJ>
- [3] M. Hamilton, *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*, ser. Universitext. Springer International Publishing, 2017. [Online]. Available: <https://books.google.com/books?id=8zVCDwAAQBAJ>
- [4] A. Scorpan, *The Wild World of 4-Manifolds*. Providence, R.I: American Mathematical Society, May 2005.

- [5] J. Baez and J. Muniain, *Gauge Fields, Knots And Gravity*, ser. Series On Knots And Everything. World Scientific Publishing Company, 1994. [Online]. Available: <https://books.google.com/books?id=qvw7DQAAQBAJ>
- [6] M. F. Atiyah and R. Bott, “The yang-mills equations over riemann surfaces,” *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 308, no. 1505, pp. 523–615, 1983.