DRAFT: Discrete differential geometry in homotopy type theory

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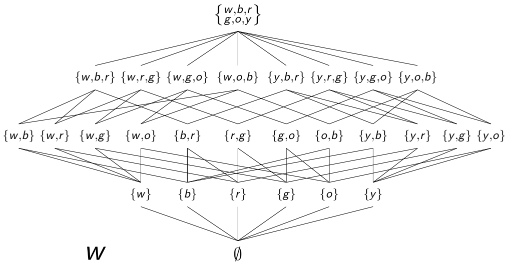
Motivation

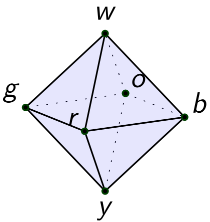
The motivation is to provide a deeper explanation for Chern-Weil theory by finding connections and curvature in HoTT principal bundles.

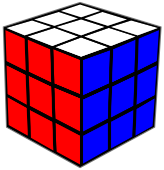
- Construct a type of manifolds
- Construct a classifying space of principal bundles
- Identify connections and curvature
- Put these to use in 2-d to prove that total curvature is an integer

- The classical theory of simplicial complexes
- A realization functor via higher inductive types (pushouts)

Simplicial complexes

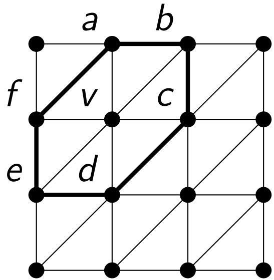






A Hasse diagram of a simplicial complex (vertices named for the colors on a Hungarian Cube)

Higher realization



The **link** of a vertex v in an n-complex is the $(n-1)$-subcomplex of faces not containing v but whose union with v is a face.

This will be our model of the tangent space.

Higher realization

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\overset{h_1}{\Nearrow}"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] \arrow[ur, phantom, very near end, "\lrcorner"] & \mathbb{M}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\overset{h_2}{\Nearrow}"] & M_2 \arrow[d, "*_{\mathbb{M}_2}"] \\
\mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"'] \arrow[u, "\mathbb{A}_0"] \arrow[ur, phantom, "\underset{h_1}{\Searrow}"] & M_1 \arrow[u, "*_{\mathbb{M}_1}"'] &
\end{tikzcd}
$$

Form a pushout of edges to create a 1-type.

Then push out maps from a 1-type triangle to from a 2-dim type.

$$
\begin{tikzcd}
 & M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Nearrow h_2"] & M_2 \arrow[d, "*_{\mathbb{M}_2}"] \\
\mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times \partial\Delta^1 \arrow[u, "\mathbb{A}_0"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\Searrow h_1"] & M_1 \arrow[u, "*_{\mathbb{M}_1}"'] &
\end{tikzcd}
$$

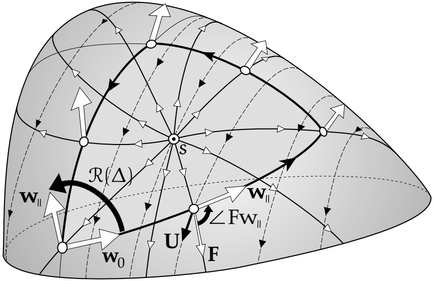
$*_{\mathbb{M}_1}, *_{\mathbb{M}_2}$ provide hubs.

h_1, h_2 provide spokes.

Torsors

$$\mathrm{EM}(\mathbb{Z}, 1),\ \mathrm{K}(\mathbb{Z}, 2)$$

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F\mathbf{w}_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: from Tristan Needham, *Visual Differential Geometry and Forms*

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.