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DRAFT: Discrete differential geometry in homotopy type theory

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Motivation

To use HoTT to study connections and explain their applicability to algebraic topology, via

- the Gauss-Bonnet theorem

- its vast generalization, Chern-Weil theory

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Theorem (Gauss-Bonnet)

Let M be a compact 2-manifold without boundary, equipped with a Riemannian metric. Let K be the Gaussian curvature of M and let $\chi(M)$ be the Euler characteristic. Then

$$\frac{1}{2\pi} \int_M K \, dA = \chi(M).$$

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Theorem (Poincaré-Hopf)

Let M be a compact smooth manifold without boundary. Let X be a vector field on M with isolated zeroes $x_1, \ldots, x_n$. Then

$$\sum_{i=1}^{n} \text{index}_{x_i} = \chi(M).$$

index 0   index $+1$ index $+1$ index $+1$ index $+1$ index -1
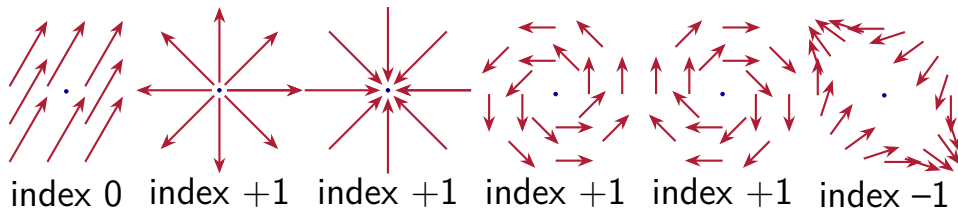
Plan

- Manifolds

- Classifying maps

- Connections and curvature

- Theorems

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HoTT background

1. Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-) Symmetry. `https://github.com/UniMath/SymmetryBook`.

2. Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023) Central H-spaces and banded types. arXiv:2301.02636

3. Scoccola, L. (2020) Nilpotent types and fracture squares in homotopy type theory, MSCS 30(5). arXiv:1903.03245

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Discrete manifolds in HoTT

- Recall the classical theory of simplicial complexes

- Define a realization procedure to turn them into homotopy pushouts

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Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \overset{\text{def}}{=} [M_0, \dots, M_n]$ consisting of

- a set M_0 of $(n+1)$ vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \overset{\text{def}}{=} [M_0, \dots, M_k]$ the k-skeleton of M.
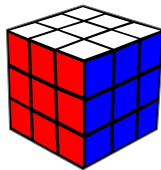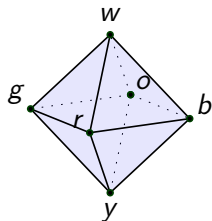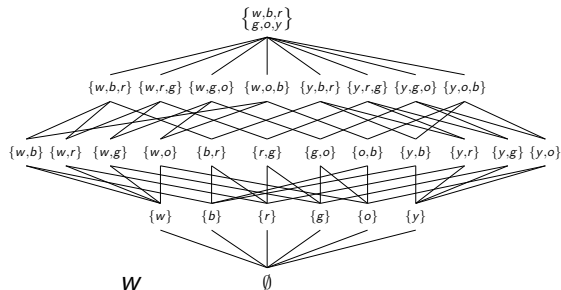
Simplicial complexes

Example

The complete simplex of dimension n, denoted $P(n)$, is the set $\{1, \ldots, n+1\}$ and its power set. The $(n-1)$-skeleton $P(n)_{\leq(n-1)}$ is denoted $\partial P(n)$ and will serve as a combinatorial $(n-1)$-sphere.

e.g., $\partial P(2)$ is an abstract triangle with 3 vertices and 3 edges (lacks its face).

Simplicial complexes

Here is a Hasse diagram of an abstract octahedron (vertices named for the colors on a Hungarian Cube)

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Homotopy realization

We will realize simplicial complexes as pushouts.

The realization of a 0-dimensional complex M_0 is the set M_0.

In particular the 0-sphere $\partial \Delta^1 \overset{\mathrm{def}}{=} \partial P(1)$.

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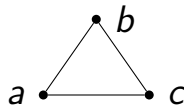ooooo

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Homotopy realization

For a 1-dim complex $M \stackrel{\text{def}}{=} [M_0, M_1]$ form

$$
\begin{array}{ccc}
M_1 \times \partial \Delta^1 & \xrightarrow{\;\mathrm{pr}_1\;} & M_1 \\
{\scriptstyle \mathbb{A}_0}\Big\downarrow & {\nearrow}_{h_1} & \Big\downarrow{\scriptstyle *_{\mathbb{M}_1}} \\
M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1
\end{array}
$$

$$\{\{w\}, \{g\}\} \leftarrow \{\{w, g\}\} \times \{0, 1\} \rightarrow \{\{w, g\}\}$$

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Next construct a 1-sphere $\partial\Delta^2 \stackrel{\text{def}}{=}$

as the realization

of $\partial P(2)$:

$$\begin{array}{ccc}
\partial P(2)_1 \times \partial\Delta^0 & \longrightarrow & \partial P(2)_1 \\
\downarrow & & \downarrow \\
\partial P(2)_0 & \longrightarrow & \partial\Delta^2
\end{array}$$

$$\begin{array}{ccc}
\{\{a,b\},\{b,c\},\{c,a\}\} \times \{0,1\} & \longrightarrow & \{\{a,b\},\{b,c\},\{c,a\}\} \\
\downarrow & & \downarrow \\
\{\{a\},\{b\},\{c\}\} & \longrightarrow & \partial\Delta^2
\end{array}$$

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Homotopy realization

Then use $\partial\Delta^1, \partial\Delta^2$ to realize $M \overset{\mathrm{def}}{=} [M_0, M_1, M_2]$.

$$
\begin{array}{ccccc}
M_1 \times \partial\Delta^1 & \xrightarrow{\mathrm{pr}_1} & M_1 & & \\
\Big\downarrow{\scriptstyle\mathbb{A}_0} & \overset{\nearrow}{\scriptstyle h_1} & \Big\downarrow{\scriptstyle *\mathbb{M}_1} & & \\
M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
& & \Big\uparrow{\scriptstyle\mathbb{A}_1} & \overset{h_2}{\searrow} & \Big\uparrow{\scriptstyle *\mathbb{M}_2} \\
& & M_2 \times \partial\Delta^2 & \xrightarrow{\mathrm{pr}_1} & M_2
\end{array}
$$

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Homotopy realization

$*_{\mathbb{M}_1}, *_{\mathbb{M}_2}$ provide hubs. h_1, h_2 provide spokes.

$$
\begin{array}{ccccc}
M_1 \times \partial\Delta^1 & \xrightarrow{\mathrm{pr}_1} & M_1 & & \\
\Big\downarrow{\mathbb{A}_0} & \underset{h_1}{\nearrow\nearrow} & \Big\downarrow{*_{\mathbb{M}_1}} & & \\
M_0 = \mathbb{M}_0 & \longrightarrow & \mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
& & \Big\uparrow{\mathbb{A}_1} & \underset{h_2}{\searrow\searrow} & \Big\uparrow{*_{\mathbb{M}_2}} \\
& & M_2 \times \partial\Delta^2 & \xrightarrow{\mathrm{pr}_1} & M_2
\end{array}
$$

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Homotopy realization
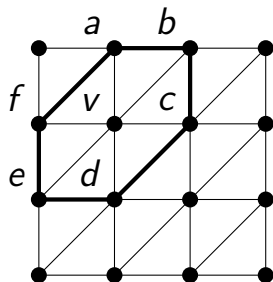
The link of a vertex v in a 2-complex is the polygon of edges not containing v but whose union with v is a face.

This will be our model of the tangent space.

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Theorem (Whitehead (1940))

Every smooth manifold has a compatible structure of a
combinatorial manifold: a simplicial complex of dimension n such
that the link is the geometric realization of an $(n-1)$-sphere.

`https://ncatlab.org/nlab/show/triangulation+theorem`

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Definition

Let G be a group with identity element e. A G-set is a set X equipped with a homomorphism $\phi : (G, e) \to \text{Aut}(X)$. If we have

$$\text{is\_torsor}(X, \phi) \overset{\text{def}}{=} ||X||_{-1} \times \prod_{x:X} \text{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

we say (X, ϕ) is a G-torsor. Denote the type of G-torsors by BG.

Lemma

Point BG at G_{reg}, the G-torsor G acting on itself on the right. Then $\Omega_{G_{\text{reg}}} BG \simeq G$, so BG is a $\text{K}(G, 1)$.

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- $S^1 : \mathcal{U}$ is not an Aut S^1-torsor.

- It's a torsor for $(S^1 = S^1)_{(\mathsf{id})}$, the identity component.

- This omits the flip, the reversal of orientation.
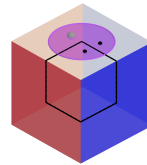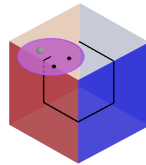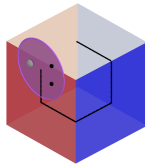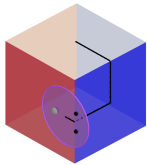
- See the Buchholtz et. al. H-spaces paper for more.

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Definition

$$\mathrm{EM}(G, n) \overset{\mathrm{def}}{=} \mathsf{BAut}(\mathrm{K}(G, n)) \overset{\mathrm{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$$

Definition

A $\mathrm{K}(G, n)$-bundle on a type M is a map $f : M \to \mathrm{EM}(G, n)$.

We further assume f factors through $\mathrm{K}(G, n + 1)$ and so is principal.

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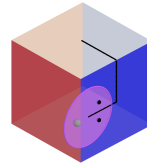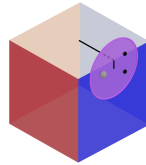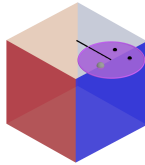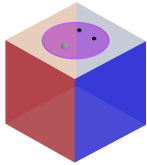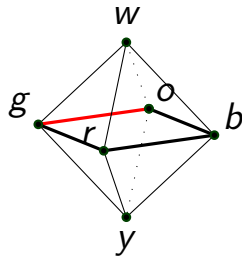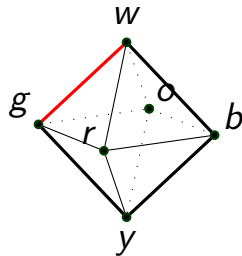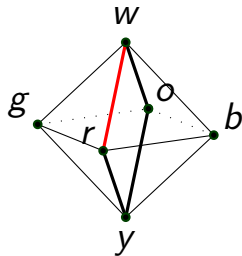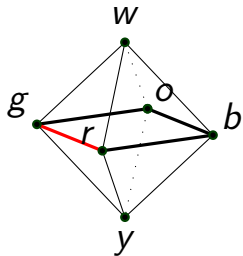
Rotation

Let $R : [abcd] \to [abcd]$ send $a \mapsto b, b \mapsto c, c \mapsto d, d \mapsto a$.

Extend R to edges.

Lemma
$[\![R]\!] : [\![abcd]\!] \to [\![abcd]\!]$ *is homotopic to the identity, i.e. we have*
$\prod_{x:[\![abcd]\!]} x = [\![R]\!](x).$

Proof.
Use edges. □

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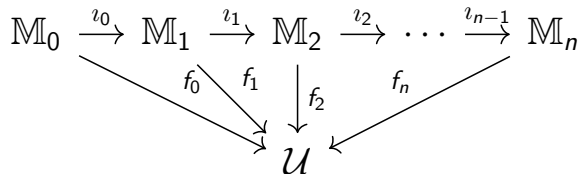○○○

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Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a cellular type and all the triangles commute in the diagram:

$$\mathbb{M}_0 \xrightarrow{\imath_0} \mathbb{M}_1 \xrightarrow{\imath_1} \mathbb{M}_2 \xrightarrow{\imath_2} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$$

with maps f_0, f_1, f_2, f_n converging to $\mathcal{U}$.

- The map f_k is a *k-bundle* on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a *k-connection* on the $(k-1)$-bundle f_{k-1}.
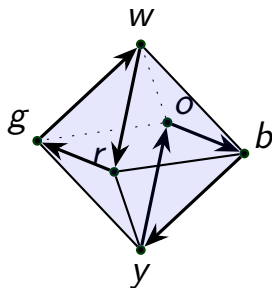
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Definition

$$
\begin{array}{ccc}
M_k \times \partial\Delta^k & \xrightarrow{\mathrm{pr}_1} & M_k \\
{\scriptstyle \mathbb{A}_{k-1}}\downarrow \quad {\nearrow\scriptstyle h_k} & & \downarrow {\scriptstyle *_{\mathbb{M}_k}} \\
\mathbb{M}_{k-1} & \xrightarrow{\imath_{k-1}} & \mathbb{M}_k \\
& {\scriptstyle f_{k-1}}\searrow \quad & \downarrow {\scriptstyle f_k} \\
& & \mathcal{U}
\end{array}
\qquad
\begin{array}{ccc}
\{m_k\} \times \partial\Delta^k & \xrightarrow{\;!\;} & \mathbf{1} \\
{\scriptstyle \mathbb{A}_{k-1}}\downarrow \quad {\swarrow\scriptstyle \flat_k} & & \downarrow {\scriptstyle *_{\mathbb{M}_k}} \\
\mathbb{M}_{k-1} & \xrightarrow{\hspace{2cm}} & \mathcal{U}
\end{array}
$$

the filler $\flat_k$ is called a flatness structure for the face m_k, and its ending path is called curvature at the face m_k.

Vector fields

Let $T : \mathbb{M}_2 \to \mathrm{K}(\mathbb{Z}, 2)$ be an oriented tangent bundle on an oriented 2-dim cellular type

- A vector field is a term $X : \prod_{m : \mathbb{M}_1} Tm$.
- It's a nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

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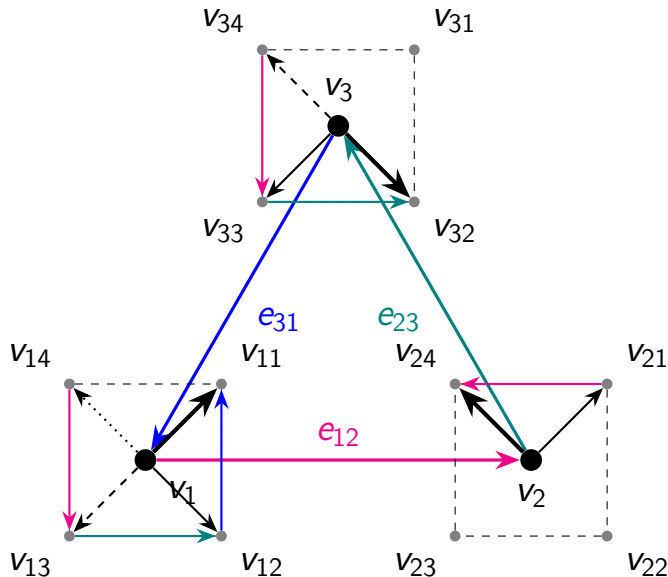
- $\partial F \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$
- We access pathovers asymmetrically: $X_{12} : T_{12}X_1 = X_2$
- $X(\partial F)$ is 3-sided inside a square
- To make a loop we cat with $\flat(\partial F)$

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$$\text{tr}_F \overset{\text{def}}{=} \text{tr}(\partial F) \quad : Tm = Tm \qquad\qquad \text{holonomy}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \quad : \text{id} =_{Tm=Tm} \text{tr}(\partial F) \qquad \text{flatness}$$

$$X_F \overset{\text{def}}{=} X(\partial F) \quad : \text{tr}(\partial F)(X(m)) =_{Tm} X(m) \quad \text{swirling}$$

Definition

The index of the vector field X on the face F is the integer

$$I_F^X \overset{\text{def}}{=} \Omega(\flat_F(X(m)) \cdot X_F) : \Omega(X(m) =_{Tm} X(m)).$$

$$
\begin{array}{ccc}
 & \flat_F(X_1) & \\
v_{14} \bullet & \longleftarrow & \bullet\, v_{11} \\
X_{12} \Big\downarrow & Tv_1 & \Big\uparrow X_{31} \\
v_{13} \bullet & \longrightarrow & \bullet\, v_{12} \\
 & X_{23} &
\end{array}
$$

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$$\mathsf{tr}_F \stackrel{\mathrm{def}}{=} \mathsf{tr}(\partial F) \quad : Tm = Tm \qquad\qquad \text{holonomy}$$

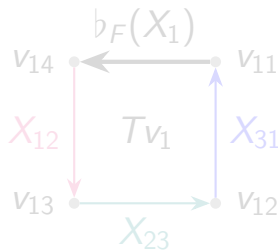$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \mathsf{id} =_{Tm=Tm} \mathsf{tr}(\partial F) \qquad \text{flatness}$$
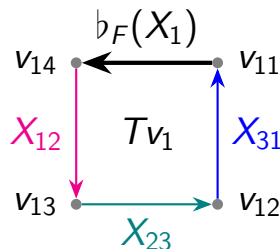
$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathsf{tr}(\partial F)(X(m)) =_{Tm} X(m) \quad \text{swirling}$$

Definition
The index of the vector field X on the face F is the integer
$$I_F^X \stackrel{\mathrm{def}}{=} \Omega(\flat_F(X(m)) \cdot X_F) : \Omega(X(m) =_{Tm} X(m)).$$

Classical proof

[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle Fw_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

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Thank you.