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Combinatorial manifolds
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Vector fields
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Main theorem
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Discrete differential geometry in homotopy type theory

Greg Langmead

Carnegie Mellon University

April 2025

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Summary

Summary

This work brings to HoTT

- connections, curvature, and vector fields

- the index of a vector field

- a theorem in dimension 2 that total curvature = total index

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Classical → HoTT

Let M be a smooth, oriented 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes $x_1, \ldots, x_n$.

$$\frac{1}{2\pi} \int_M F_A = \sum_{i=1}^{n} \text{index}_X(x_i) = \chi(M)$$

$$\sum_{\text{faces } F} \flat_F \;=\!=\!=\; \sum_{\text{faces } F} L_F^X$$

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Classical index

Near an isolated zero there are only three possibilities: index 0, 1, −1.

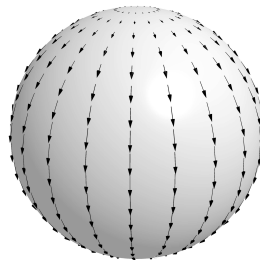Index is the winding number of the field as you move clockwise around the zero.

index 0   index +1   index +1   index +1   index +1   index −1

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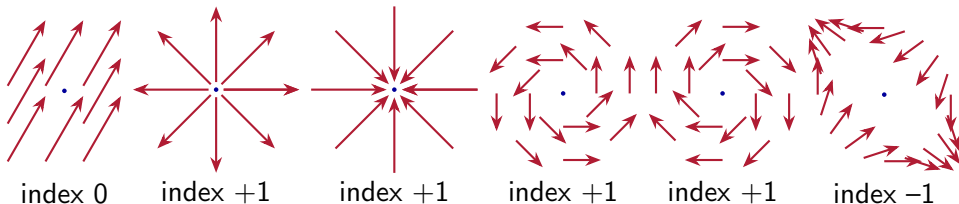
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.
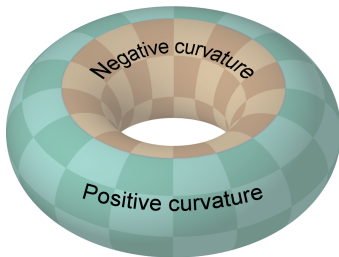
Examples:
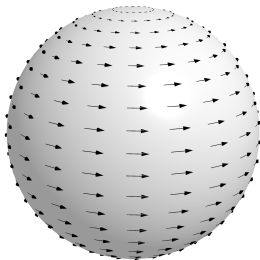
Rotation: index $+1$ at each pole $= \mathbf{2}$

Height: index $+1$ at each pole $= \mathbf{2}$

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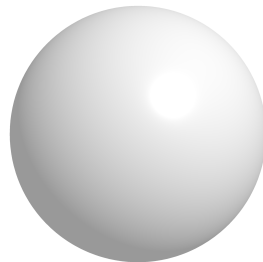○○○○○○○○○○○○○

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Gauss-Bonnet theorem

Total curvature divided by 2π is the Euler characteristic.

Curvature in 2D is a function $F_A : M \to \mathbb{R}$.

$\int_M F_A$ sums the values at every point.

Positive and negative curvature cancel: **0**

Constant curvature 1, area 4π: **2**

Plan

- Combinatorial manifolds

- Torsors and classifying maps

- Connections and curvature

- Vector fields

- Main theorem

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HoTT background

1 Symmetry,
Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
`https://github.com/UniMath/SymmetryBook`.

2 Central H-spaces and banded types,
Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
arXiv:2301.02636

3 Nilpotent types and fracture squares in homotopy type theory,
Scoccola, L. (2020)
MSCS 30(5). arXiv:1903.03245

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Combinatorial manifolds

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Manifolds in HoTT

- Recall the classical theory of simplicial complexes

- Define a realization procedure to construct types

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Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \overset{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of subsets of M_0 of cardinality $k + 1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j + 1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \overset{\text{def}}{=} [M_0, \ldots, M_k]$ the k-skeleton of M.

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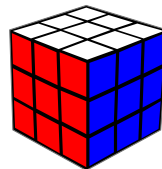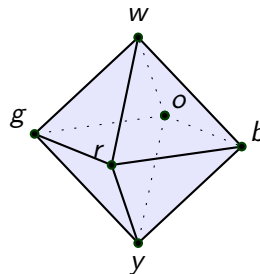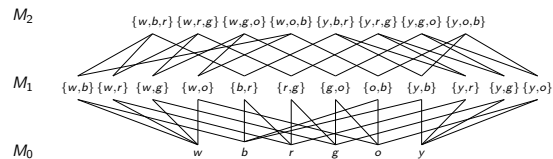000●00000000

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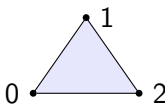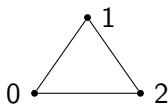Vector fields
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Simplicial complexes

Example

The complete simplex of dimension n, denoted $\Delta(n)$, is the set $\{0, \ldots, n\}$ and its power set. The $(n-1)$-skeleton $\Delta(n)_{\leq(n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$-sphere.

$\Delta(1)$ is visually $\quad 0 \bullet\!\!\!\longrightarrow\!\!\!\bullet 1$, $\partial\Delta(1)$ is visually $\quad 0 \bullet \qquad \bullet 1$,

$\Delta(2)$ is visually , $\partial\Delta(2)$ is visually

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Homotopy realization: dimension 0

We will realize simplicial complexes by means of a sequence of pushouts.

Base case: the realization $\mathbb{M}$ of a 0-dimensional complex M is M_0.

In particular the 0-sphere $\partial\Delta(1) \stackrel{\mathsf{def}}{=} \partial\Delta(1)_0$.

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Homotopy realization: dimension 1

For a 1-dim complex $M \overset{\mathsf{def}}{=} [M_0, M_1]$ the realization is given by

$$
\begin{array}{ccc}
M_1 \times \partial\Delta(1) & \xrightarrow{\ \mathsf{pr}_1\ } & M_1 \\
{\scriptstyle \mathbb{A}_0}\Big\downarrow & {\nearrow}_{h_1} & \Big\downarrow{\scriptstyle *_{\mathbb{M}_1}} \\
M_0 = \mathbb{M}_0 & \xrightarrow{\hspace{2cm}} & \mathbb{M}_1
\end{array}
$$

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Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\Delta(2) \overset{\text{def}}{=}$ is given by

$$
\begin{array}{ccc}
\partial\Delta(2)_1 \times \partial\Delta(1) & \longrightarrow & \partial\Delta(2)_1 \\
\downarrow & \nearrow_{h_1} & \downarrow \\
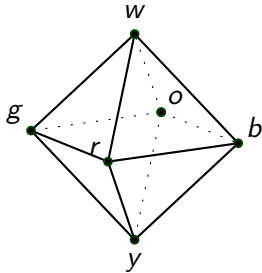\partial\Delta(2)_0 & \longrightarrow & \partial\Delta(2)
\end{array}
\qquad \text{i.e.} \qquad
\begin{array}{ccc}
\{\{0,1\},\{1,2\},\{2,0\}\}\times\{0,1\} & \longrightarrow & \{\{0,1\},\{1,2\},\{2,0\}\} \\
\downarrow & \nearrow_{h_1} & \downarrow \\
\{0,1,2\} & \longrightarrow & \partial\Delta(2)
\end{array}
$$

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Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron $\mathbb{O}$:

$$\begin{array}{ccc}
\{\{w, g\}, \ldots\} \times \{0, 1\} & \longrightarrow & \{\{w, g\}, \ldots\} \\
\downarrow & \overset{h_1}{\Longrightarrow} \nearrow & \downarrow \\
\{w, g, \ldots\} & \longrightarrow & \mathbb{O}_1
\end{array}$$

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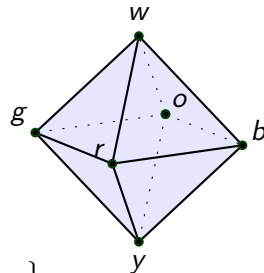Main theorem
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Homotopy realization: dimension 2

To realize $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\Delta(1), \partial\Delta(2)$:

$$
\begin{array}{ccc}
M_1 \times \partial\Delta(1) & \xrightarrow{\ \mathsf{pr}_1\ } & M_1 \\
{\scriptstyle \mathbb{A}_0}\downarrow & \nearrow{\scriptstyle h_1} & \downarrow{\scriptstyle *_{\mathbb{M}_1}} \\
M_0 = \mathbb{M}_0 \longrightarrow & \mathbb{M}_1 \longrightarrow & \mathbb{M}_2 \\
& {\scriptstyle \mathbb{A}_1}\uparrow \quad \searrow{\scriptstyle h_2} & \uparrow{\scriptstyle *_{\mathbb{M}_2}} \\
& M_2 \times \partial\Delta(2) & \xrightarrow[\ \mathsf{pr}_1\ ]{} M_2
\end{array}
$$

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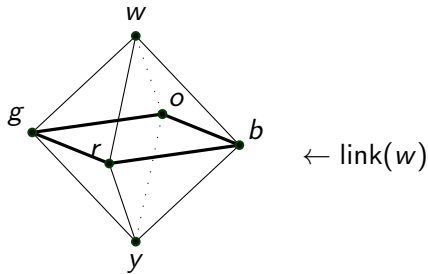0000000000

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Homotopy realization: dimension 2

The full octahedron $\mathbb{O}$:

$$
\begin{array}{ccc}
\{\{w, g\}, \ldots\} \times \{0, 1\} & \xrightarrow{\mathsf{pr}_1} & \{\{w, g\}, \ldots\} \\
\downarrow & & \downarrow \\
\{w, g, \ldots\} & \xrightarrow{} & \mathbb{O}_1 \longrightarrow \mathbb{O}_2 \\
& & \uparrow \\
& & \{\{w, g, r\}, \ldots\} \times \partial\Delta(2) \xrightarrow[\mathsf{pr}_1]{} \{\{w, g, r\}, \ldots\}
\end{array}
$$

with h_1 and h_2 homotopies.

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Homotopy realization: dimension 2

← link(w)

The link of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

A combinatorial manifold is a simplicial complex all of whose links are[*] simplicial spheres.

This will be our model of the tangent space.

[*]the (classical) geometric realization is homeomorphic to a sphere

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Combinatorial manifolds ↔ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$-sphere, i.e. its geometric realization is an $(n-1)$-sphere.

https://ncatlab.org/nlab/show/triangulation+theorem

Combinatorial manifolds $\leftrightarrow$ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$-sphere, i.e. its geometric realization is an $(n-1)$-sphere.
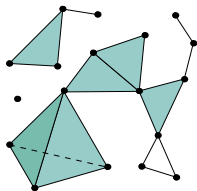
`https://ncatlab.org/nlab/show/triangulation+theorem`

Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:

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What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of torsors, also called principal bundles.

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Torsors

Let G be a (higher) group.

Torsors

Let G be a (higher) group.

Definition
- A right G-object is a type X equipped with a homomorphism $\phi : G^{\mathsf{op}} \to \mathsf{Aut}(X)$.

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Torsors

Let G be a (higher) group.

Definition

- A right G-object is a type X equipped with a homomorphism $\phi : G^{\mathrm{op}} \to \mathrm{Aut}(X)$.
- X is furthermore a G-torsor if it is inhabited and the map
  $(\mathrm{pr}_1, \phi) : X \times G \to X \times X$ is an equivalence.

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- The inverse is (pr_1, s) where $s : X \times X \to G$ is called subtraction (when G is commutative).

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Torsors

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- Let BG be the type of G-torsors.

Torsors

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Definition

- A right G-object is a type X equipped with a homomorphism $\phi : G^{\mathrm{op}} \to \mathrm{Aut}(X)$.
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- The inverse is (pr_1, s) where $s : X \times X \to G$ is called subtraction (when G is commutative).
- Let BG be the type of G-torsors.
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

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Facts

1. $\Omega(BG, G_{\text{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.

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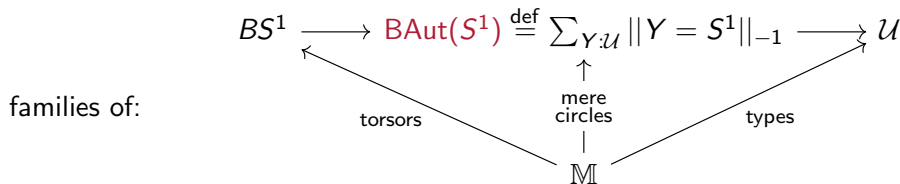## Facts

1. $\Omega(BG, G_{\text{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.

2. BG is connected.

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Facts

① $\Omega(BG, G_{\text{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.

② BG is connected.

③ 1 & 2 $\implies$ BG is a $K(G, 1)$.

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Facts

1. $\Omega(BG, G_{\mathrm{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.

2. BG is connected.

3. 1 & 2 $\implies$ BG is a $K(G, 1)$.

See the Buchholtz et. al. H-spaces paper for more.

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How to map into BS^1

To construct maps into BS^1 we lift a family of mere circles.

$$BS^1 \longrightarrow \mathrm{BAut}(S^1) \overset{\mathrm{def}}{=} \sum_{Y:\mathcal{U}} ||Y = S^1||_{-1} \longrightarrow \mathcal{U}$$

families of:

torsors

mere circles

types

$\mathbb{M}$

We will assume we have such a lift when we need it. (Remark: the lift is a choice of orientation.)

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torsors   mere circles   types

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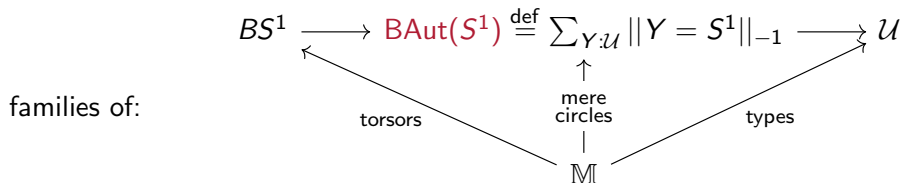
We will assume we have such a lift when we need it. (Remark: the lift is a choice of orientation.)

Other names:

- $\mathrm{BAut}(S^1) = BO(2) = \mathrm{EM}(\mathbb{Z}, 1)$ (where $\mathrm{EM}(G, n) \stackrel{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n))$)

- $BS^1 = BSO(2) = \mathrm{K}(\mathbb{Z}, 2)$

Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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Main theorem
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Connections and curvature

Summary
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Combinatorial manifolds
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Main theorem
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Connections

Connections are extensions of a bundle to higher skeleta.
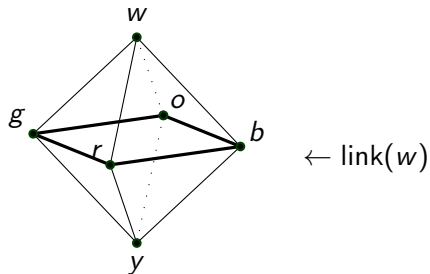
Summary
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Vector fields
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Main theorem
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Recall link

$\leftarrow \text{link}(w)$

The link of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

Define the tangent bundle on a combinatorial manifold to be
$$T_0 \overset{\text{def}}{=} \text{link} : \mathbb{M}_0 \to \text{BAut}(S^1).$$

Summary
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Vector fields
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Main theorem
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Connections on the tangent bundle

An extension T_1 of T_0 to $\mathbb{M}_1$ is called a connection on the tangent bundle.

$$\begin{array}{ccccc}
\mathbb{M}_0 & \longrightarrow & \mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
& \searrow & \downarrow {\scriptstyle T_1} & \swarrow & \\
T_0 \stackrel{\text{def}}{=} \text{link} & & \text{BAut}(S^1) & &
\end{array}$$

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Summary
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Vector fields
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Main theorem
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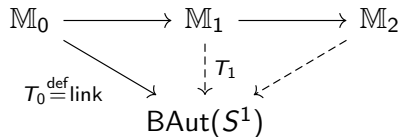
$T_1 : \mathbb{M}_1 \to \mathsf{BAut}(S^1)$ extending link

We will define T_1 on the edge wb, so we need a term
$T_1(wb) : \mathsf{link}(w) =_{\mathsf{BAut}(S^1)} \mathsf{link}(b)$.

We imagine tipping:
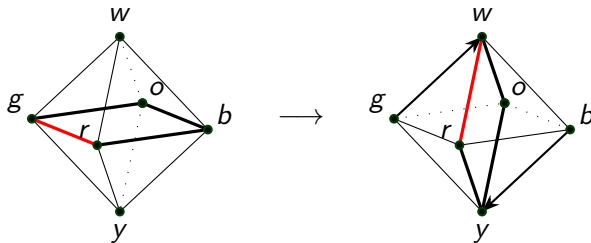
$T_1(g : \mathsf{link}(w)) \stackrel{\mathsf{def}}{=} w : \mathsf{link}(b), \ldots.$

Use this method to define T_1 on every edge.

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$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

Denote the path $wb \cdot br \cdot rw$ by $\partial(wbr)$. Consider $T_1(\partial(wbr))$:
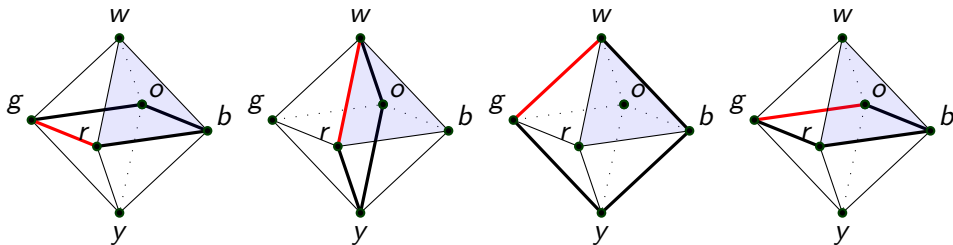
We come back rotated by 1/4 turn. Call this rotation $R : \mathrm{link}(w) =_{BAut(S^1)} \mathrm{link}(w)$.

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Combinatorial manifolds
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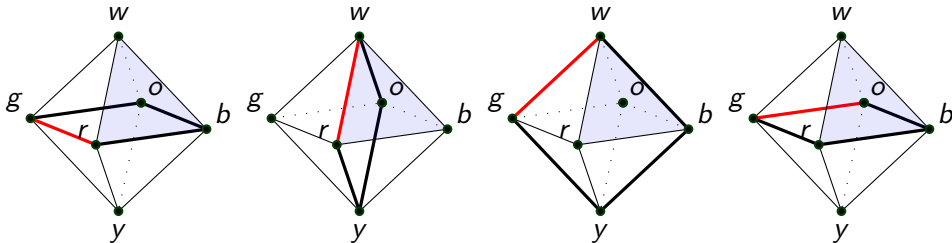# Extending T_1 to a face

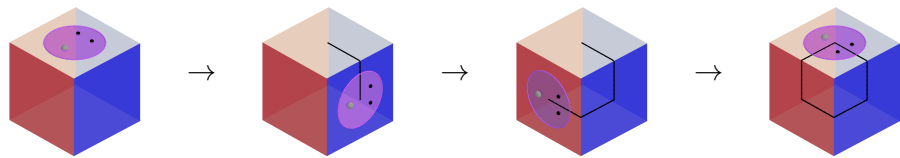Let $H_{wbr} : \text{refl}_w =_{w=_\mathbb{M} w} \partial(wbr)$ be the filler homotopy of the face.

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Vector fields
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Main theorem
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Extending T_1 to a face

Let $H_{wbr} : \mathrm{refl}_w =_{w =_{\mathbb{M}} w} \partial(wbr)$ be the filler homotopy of the face.

T_2 must live in $T_1(\mathrm{refl}_w) =_{(\mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(w))} T_1(\partial(wbr)) = R$

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Connections and curvature
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Vector fields
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Main theorem
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Extending T_1 to a face

Let $H_{wbr} : \mathrm{refl}_w =_{w=_\mathbb{M} w} \partial(wbr)$ be the filler homotopy of the face.

T_2 must live in $T_1(\mathrm{refl}_w) =_{(\mathrm{link}(w)=_{\mathrm{BAut}(S^1)}\mathrm{link}(w))} T_1(\partial(wbr)) = R$

T_2 must be a homotopy $H_R : \mathrm{id} = R$ between automorphisms of $\mathrm{link}(w)$.

Summary
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Combinatorial manifolds
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Connections and curvature
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Vector fields
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Main theorem
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Extending T_1 to a face

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T_2 must be a homotopy $H_R : \mathrm{id} = R$ between automorphisms of $\mathrm{link}(w)$.

For example, a path $H_R(g) : g = Rg = o$. Choose go.

Summary
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Connections and curvature
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Original inspiration

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Main theorem
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The definition of a connection

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is the realization of a combinatorial manifold and all the triangles commute in the diagram:

$$\mathbb{M}_0 \xrightarrow{\imath_0} \mathbb{M}_1 \xrightarrow{\imath_1} \mathbb{M}_2 \xrightarrow{\imath_2} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$$

with maps f_0, f_1, f_2, f_n to $\mathcal{U}$.
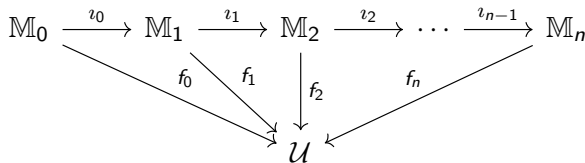
- The map f_k is a k-bundle on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-connection on the $(k-1)$-bundle f_{k-1}.

Summary
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Vector fields
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Main theorem
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The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:

$$
\begin{array}{ccc}
M_2 \times \partial\Delta(2) & \xrightarrow{\mathrm{pr}_1} & M_2 \\
{\scriptstyle \mathbb{A}_1}\downarrow & {\scriptstyle h_2}\nearrow\nearrow & \downarrow \\
\mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
& {\scriptstyle T_1}\searrow \nwarrow\nwarrow & \downarrow {\scriptstyle T_2} \\
& & \mathcal{U}
\end{array}
$$

Here's the outer square for a single face F:

$$
\begin{array}{ccc}
\{F\} \times \partial\Delta(2) & \xrightarrow{\mathrm{pr}_1} & \{F\} \\
{\scriptstyle \mathbb{A}_1}\downarrow & {\scriptstyle \flat_F}\swarrow\swarrow & \downarrow \\
\mathbb{M}_1 & \longrightarrow & \mathcal{U}
\end{array}
$$

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Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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Main theorem
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The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:

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\begin{array}{ccc}
M_2 \times \partial\Delta(2) & \xrightarrow{\mathrm{pr}_1} & M_2 \\
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\mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
& {\scriptstyle T_1}\searrow & \downarrow{\scriptstyle T_2} \\
& & \mathcal{U}
\end{array}
$$

Here's the outer square for a single face F:

$$
\begin{array}{ccc}
\{F\} \times \partial\Delta(2) & \xrightarrow{\mathrm{pr}_1} & \{F\} \\
{\scriptstyle \mathbb{A}_1}\downarrow & {\scriptstyle \flat_F}\swarrow & \downarrow \\
\mathbb{M}_1 & \longrightarrow & \mathcal{U}
\end{array}
$$

$T_1(\partial(F))$ is the curvature at the face F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a flatness structure for the face F.
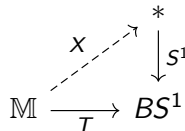
The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:

$$
\begin{array}{ccc}
M_2 \times \partial\Delta(2) & \xrightarrow{\mathrm{pr}_1} & M_2 \\
{\scriptstyle \mathbb{A}_1}\downarrow & {\scriptstyle h_2}\nearrow\!\!\!\nearrow & \downarrow \\
\mathbb{M}_1 & \longrightarrow & \mathbb{M}_2 \\
& {\scriptstyle T_1}\searrow\!\!\!\swarrow & \downarrow{\scriptstyle T_2} \\
& & \mathcal{U}
\end{array}
$$

Here's the outer square for a single face F:

$$
\begin{array}{ccc}
\{F\} \times \partial\Delta(2) & \xrightarrow{\mathrm{pr}_1} & \{F\} \\
{\scriptstyle \mathbb{A}_1}\downarrow & {\scriptstyle \flat_F}\swarrow\!\!\!\swarrow & \downarrow \\
\mathbb{M}_1 & \longrightarrow & \mathcal{U}
\end{array}
$$

$T_1(\partial(F))$ is the curvature at the face F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a flatness structure for the face F.

The distinction between the path $\flat_F$ and the endpoint $T_1(\partial(F))$ is small enough to be confusing.

Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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Main theorem
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Vector fields

Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
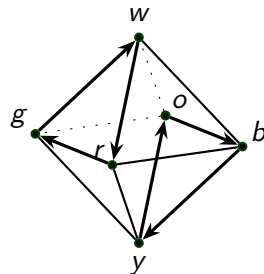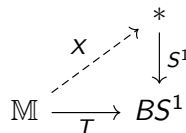○○○○○○○○○○

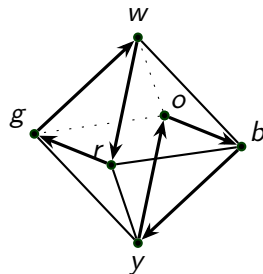Vector fields
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Main theorem
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Vector fields

Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model nonzero tangent vectors.
- A global section of this family would be a trivialization of T, so that's not a good definition.

$$
\begin{array}{ccc}
 & & * \\
 & \overset{x}{\nearrow} & \downarrow{\scriptstyle S^1} \\
\mathbb{M} & \underset{T}{\longrightarrow} & BS^1
\end{array}
$$

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Summary
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Combinatorial manifolds
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Vector fields
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Main theorem
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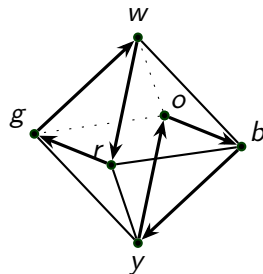## Vector fields

Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model nonzero tangent vectors.
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Our solution:

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Summary
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Combinatorial manifolds
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Vector fields
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Main theorem
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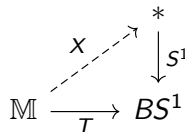# Vector fields

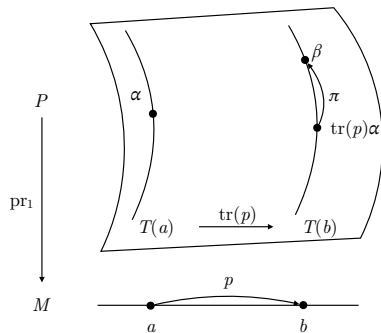Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model nonzero tangent vectors.
- A global section of this family would be a trivialization of T, so that's not a good definition.

Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.

$$\begin{array}{ccc} & & * \\ & \nearrow^{x} & \downarrow^{S^1} \\ \mathbb{M} & \xrightarrow{T} & BS^1 \end{array}$$

Summary
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Combinatorial manifolds
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Vector fields

Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model nonzero tangent vectors.
- A global section of this family would be a trivialization of T, so that's not a good definition.

Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical nonvanishing vector field on the 1-skeleton.

$$
\begin{array}{ccc}
 & & * \\
 & \overset{x}{\nearrow} & \downarrow{\scriptstyle s^1} \\
\mathbb{M} & \underset{T}{\longrightarrow} & BS^1
\end{array}
$$

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Vector fields

Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

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Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

Summary
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Vector fields
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Main theorem
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Reminder: pathovers

- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display π, the path over p.
- Dependent functions map paths to pathovers: $\mathrm{apd}(X)(p) : \mathrm{tr}_p(X(a)) = X(b)$ (simply denoted $X(p)$).

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Next goal: define the index of a vector field on a face by computing $X(\partial F)$ around a face.
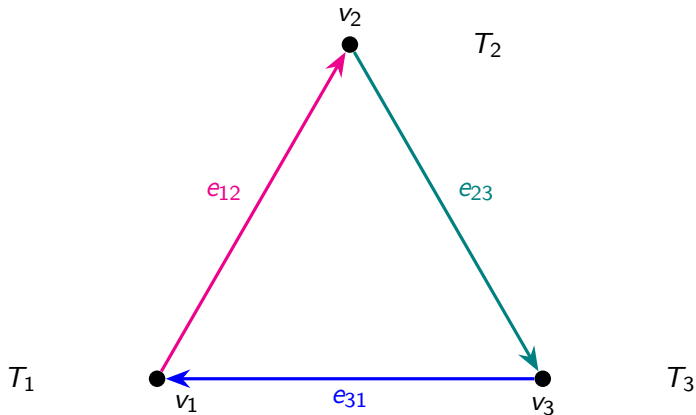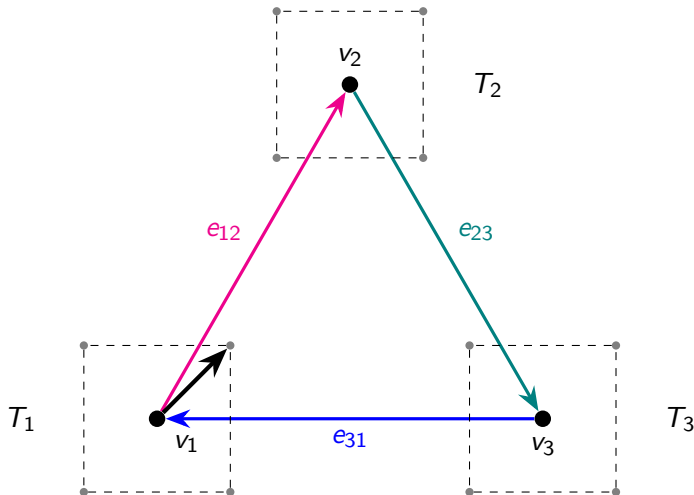
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Main theorem
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v_2

T_2

e_{12}

e_{23}

T_1

e_{31}

v_1

v_3

T_3

An example of swirling and index at this face.

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Main theorem
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An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.

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Summary
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Connections and curvature
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Vector fields
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Main theorem
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An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
-
-

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Vector fields
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Main theorem
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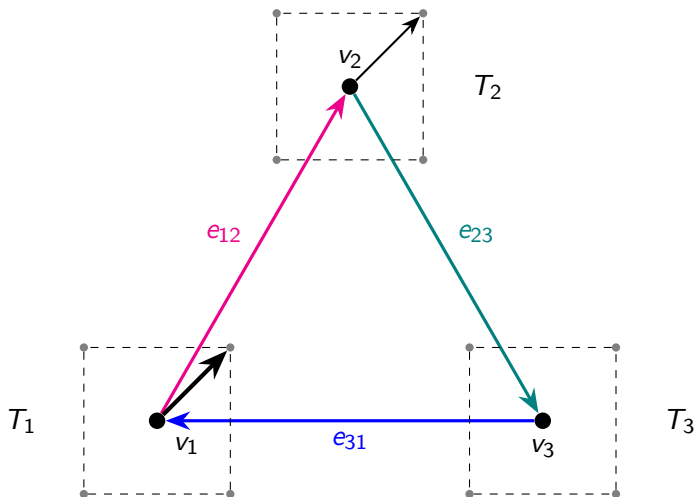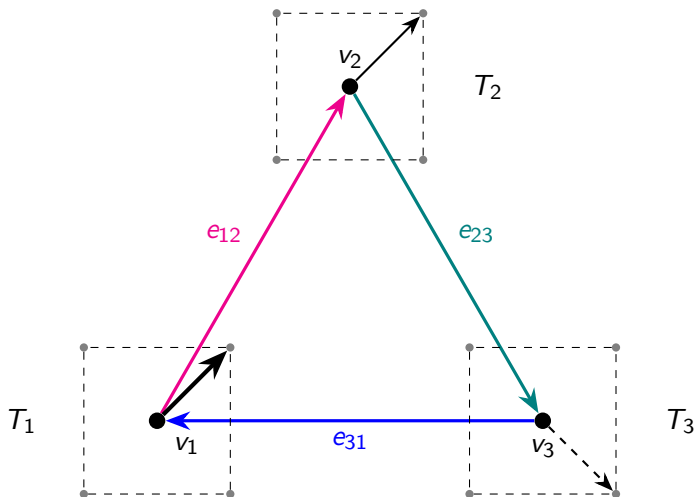
An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
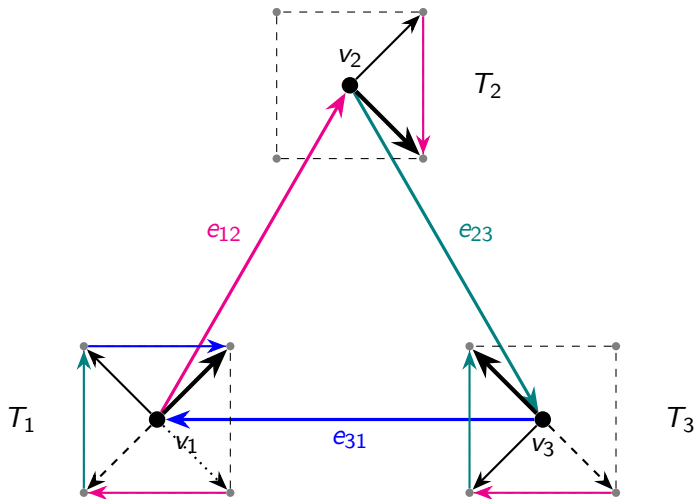- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
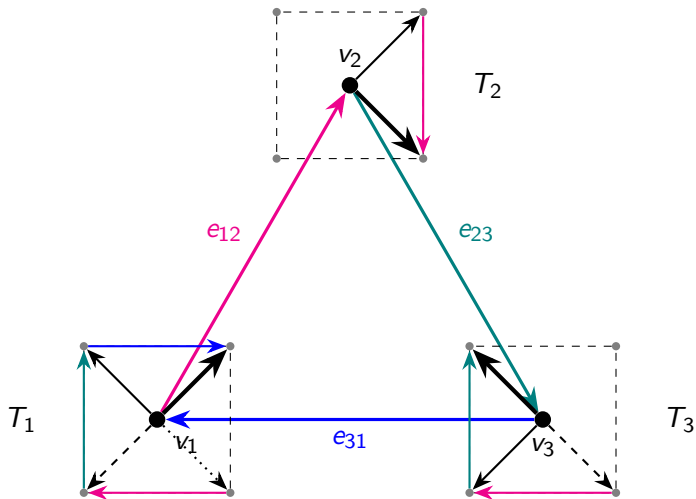-

An example of swirling and index at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
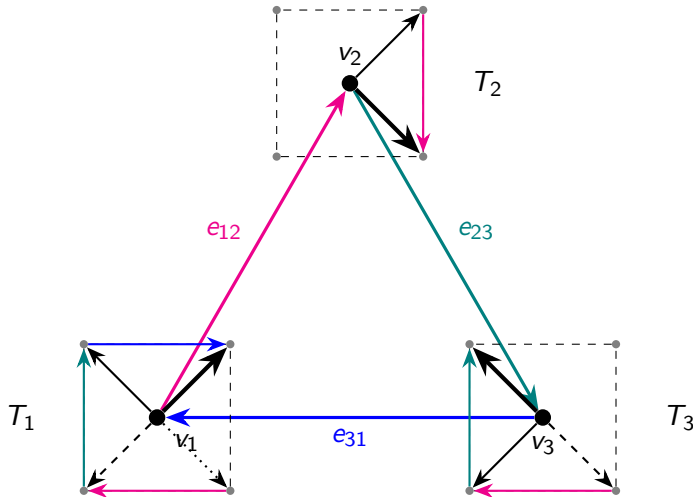- Say T_{13} is trivial. The thrice-transported vecor is dotted.
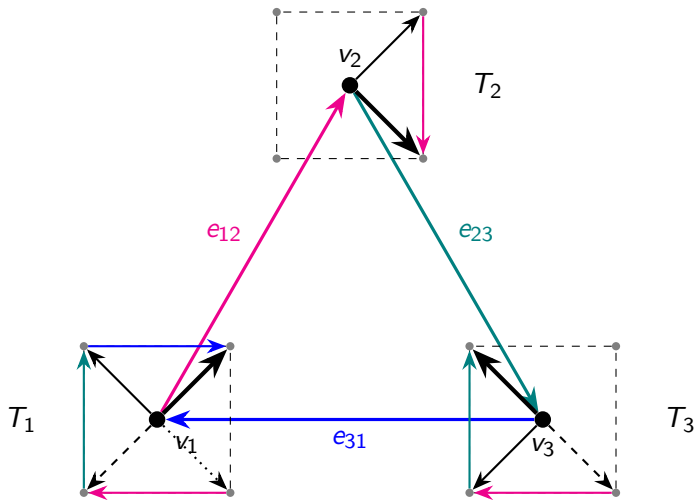
- X on e_{12} is red, etc.

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Vector fields
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Main theorem
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- X on e_{12} is red, etc.
- We translated all pathover data to the end of the loop.

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Vector fields
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Main theorem
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- X on e_{12} is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)

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Vector fields
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Main theorem
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- X on e_{12} is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)
- The total pathover $X(\partial F)$ is called the swirling X_F of X at the face F.

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Combinatorial manifolds
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Vector fields
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Main theorem
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Symbolic version

$$T_1 \xrightarrow{\;T_{21}\;} T_2 \xrightarrow{\;T_{32}\;} T_3 \xrightarrow{\;\;T_{13}\;\;} T_1$$

$$T_{13}\,T_{32}\,T_{21}X_1$$
$${\color{red}T_{13}\,T_{32}X_{21}:\|}$$

$$T_{32}\,T_{21}X_1 \qquad T_{13}\,T_{32}X_2$$
$${\color{red}T_{32}X_{21}:\|} \qquad {\color{red}T_{13}X_{32}:\|}$$

$$T_{21}X_1 \qquad T_{32}X_2 \qquad T_{13}X_3$$
$${\color{red}X_{21}:\|} \qquad {\color{red}X_{32}:\|} \qquad {\color{red}X_{13}:\|}$$

$$X_1 \qquad X_2 \qquad X_3 \qquad X_1$$

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Main theorem
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$$\mathrm{tr}_F \overset{\mathsf{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \overset{\mathsf{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$

$$X_F \overset{\mathsf{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

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$$\mathrm{tr}_F \overset{\text{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \quad \text{flatness}$$
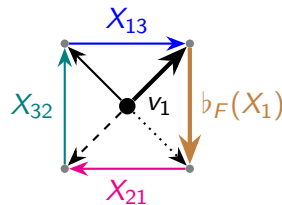
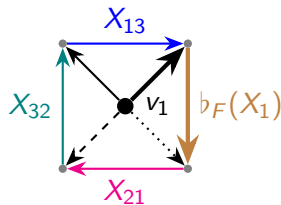$$X_F \overset{\text{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

Definition

The flattened swirling of the vector field X on the face F is the loop

$$L_F^X \overset{\text{def}}{=} \flat_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The index of the vector field X on the face F is the integer I_F^X such that $\mathrm{loop}^{I_F^X} =_{S^1} (L_F^X) - X_1$.

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$$\text{tr}_F \stackrel{\text{def}}{=} \text{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \stackrel{\text{def}}{=} \flat(\partial F) \quad : \text{id} =_{(T_1 =_{BS^1} T_1)} \text{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\text{def}}{=} X(\partial F) \quad : \text{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

(*Recall that T_1 being an S^1-torsor means we can use subtraction to obtain an equivalence $s(-, X_1) : T_1 \xrightarrow{x \mapsto x - X_1} S^1$.*)

Definition
The flattened swirling of the vector field X on the face F is the loop

$$L_F^X \stackrel{\text{def}}{=} \flat_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The index of the vector field X on the face F is the integer I_F^X such that $\text{loop}^{I_F^X} =_{S^1} (L_F^X) - X_1$.

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Main theorem
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Main theorem

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Simplifying swirling

Swirling involves concatenating dependent paths. Can we simplify that?

Pay off all our assumptions 1: torsor structure, vector field

$$T_1$$

$$T_{13}\, T_{32}\, T_{21} X_1$$
$T_{13}\, T_{32} X_{21}: \|$
$$T_{13}\, T_{32} X_2$$
$T_{13} X_{32}: \|$
$$T_{13} X_3$$
$X_{13}: \|$
$$X_1$$

Summary
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Combinatorial manifolds
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Vector fields
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Main theorem
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Pay off all our assumptions 1: torsor structure, vector field

T_1

- Def: $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \stackrel{\sim}{\to} S^1$ (trivialization on 0-skeleton).

$T_{13}\, T_{32}\, T_{21} X_1$

$T_{13}\, T_{32} X_{21}:\|$

$T_{13}\, T_{32} X_2$

$T_{13} X_{32}:\|$

$T_{13} X_3$

$X_{13}:\|$

X_1

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Main theorem
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Pay off all our assumptions 1: torsor structure, vector field

T_1

$T_{13} \, T_{32} \, T_{21} X_1$

$T_{13} \, T_{32} X_{21}: \big\|$

$T_{13} \, T_{32} X_2$

$T_{13} X_{32}: \big\|$

$T_{13} X_3$

$X_{13}: \big\|$

X_1

- Def: $\alpha_i \overset{\text{def}}{=} s(-, X_i) : T_i \overset{\sim}{\to} S^1$ (trivialization on 0-skeleton).
- Def: $\rho_{ji} \overset{\text{def}}{=} \alpha_j(T_{ji}(X_i))$ is the rotation of T_{ji}.

$$
\begin{array}{ccc}
T_i & \xrightarrow{\;T_{ji}\;} & T_j \\[4pt]
{\scriptstyle\text{base}\mapsto X_i}\,\big(\alpha_i \big\downarrow & & \big\downarrow \alpha_j\big)\,{\scriptstyle\text{base}\mapsto X_j} \\[4pt]
S^1 & \xrightarrow[(-)+\rho_{ji}]{} & S^1
\end{array}
$$

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Pay off all our assumptions 1: torsor structure, vector field

T_1

- Def: $\alpha_i \stackrel{\text{def}}{=} s(-, X_i) : T_i \stackrel{\sim}{\to} S^1$ (trivialization on 0-skeleton).
- Def: $\rho_{ji} \stackrel{\text{def}}{=} \alpha_j(T_{ji}(X_i))$ is the rotation of T_{ji}.

$T_{13} T_{32} T_{21} X_1$

$T_{13} T_{32} X_{21}: \|$

$T_{13} T_{32} X_2$

$T_{13} X_{32}: \|$

$T_{13} X_3$

$X_{13}: \|$

X_1

$$
\begin{array}{ccc}
T_i & \xrightarrow{T_{ji}} & T_j \\
\text{base} \mapsto X_i \left(\alpha_i \downarrow \right. & & \left. \downarrow \alpha_j \right) \text{base} \mapsto X_j \\
S^1 & \xrightarrow[(-)+\rho_{ji}]{} & S^1
\end{array}
$$

- Lemma: $\rho_{ij} = -\rho_{ji}$ because in T_j:
  $\rho_{ij} + \rho_{ji} + X_j = \rho_{ij} + T_{ji} X_i = T_{ji}(\rho_{ij} + X_i) = T_{ji} T_{ij} X_j = X_j$.

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Main theorem
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Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

$T_{13}\,T_{32}\,T_{21}X_1$

$_{T_{13}\,T_{32}\,X_{21}}:\|$

$T_{13}\,T_{32}X_2$

$_{T_{13}\,X_{32}}:\|$

$T_{13}X_3$

$_{X_{13}}:\|$

X_1

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Combinatorial manifolds
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Vector fields
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Main theorem
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Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1}$ base,.

$T_{13} \, T_{32} \, T_{21} X_1$

$T_{13} \, T_{32} X_{21} : \big\|$

$T_{13} \, T_{32} X_2$

$T_{13} X_{32} : \big\|$

$T_{13} X_3$

$X_{13} : \big\|$

X_1

Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

$T_{13}\,T_{32}\,T_{21}X_1$

$T_{13}\,T_{32}X_{21}:\Big\|$

$T_{13}\,T_{32}X_2$

$T_{13}X_{32}:\Big\|$

$T_{13}X_3$

$X_{13}:\Big\|$

X_1

- Define $\sigma_{ji} \overset{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1}$ base,.
- Paths of the form $(a =_{S^1}$ base$)$ can be added:
  - $+ : (a = \text{base}) \times (b = \text{base}) \to (a + b = \text{base})$.
  - $p + q = (p + b) \cdot q$.

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Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

$T_{13} \, T_{32} \, T_{21} X_1$

$T_{13} \, T_{32} X_{21} : \Big\|$

$T_{13} \, T_{32} X_2$

$T_{13} X_{32} : \Big\|$

$T_{13} X_3$

$X_{13} : \Big\|$

X_1

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base},$.
- Paths of the form $(a =_{S^1} \text{base})$ can be added:
  - $+ : (a = \text{base}) \times (b = \text{base}) \to (a + b = \text{base})$.
  - $p + q = (p + b) \cdot q$.
- Lemma: $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}$.

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Main theorem
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Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

$T_{13} \, T_{32} \, T_{21} X_1$

$T_{13} \, T_{32} X_{21}$:‖

$T_{13} \, T_{32} X_2$

$T_{13} X_{32}$:‖

$T_{13} X_3$

X_{13}:‖

X_1

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1}$ base,.
- Paths of the form $(a =_{S^1}$ base$)$ can be added:
    - $+ : (a = \text{base}) \times (b = \text{base}) \to (a + b = \text{base})$.
    - $p + q = (p + b) \cdot q$.
- Lemma: $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}$.
- Proof: $\text{apd}(X)(\text{refl}) = \text{refl}$
  $\implies X_{ij} \cdot T_{ij} X_{ji} = \text{refl}_{X_i}$
  $\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}$ (T_{ij} just translates X_{ji} to cat with X_{ij}).

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Combinatorial manifolds
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Pay off all our assumptions 1: torsor structure, vector field (cont.)

T_1

$T_{13}\, T_{32}\, T_{21} X_1$

$T_{13}\, T_{32} X_{21}$:$\|$

$T_{13}\, T_{32} X_2$

$T_{13} X_{32}$:$\|$

$T_{13} X_3$

X_{13}:$\|$

X_1

- Define $\sigma_{ji} \overset{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1}$ base,.
- Paths of the form ($a =_{S^1}$ base) can be added:
  - $+ : (a = \text{base}) \times (b = \text{base}) \to (a + b = \text{base})$.
  - $p + q = (p + b) \cdot q$.
- Lemma: $\sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}$.
- Proof: $\text{apd}(X)(\text{refl}) = \text{refl}$
  $\implies X_{ij} \cdot T_{ij} X_{ji} = \text{refl}_{X_i}$
  $\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}$ (T_{ij} just translates X_{ji} to cat with X_{ij}).

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Combinatorial manifolds
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Connections and curvature
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Vector fields
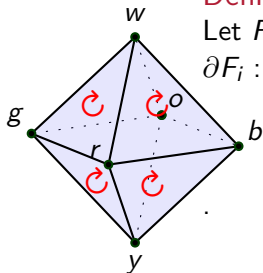○○○○○○○○○○○

Main theorem
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Pay off all our assumptions 2: no boundary, commutativity

Definition

Let $F_1, \ldots, F_n$ be the faces of $\mathbb{M}$, $v_i : F_i$ be designated vertices, and $\partial F_i : v_i = v_i$ be the triangular boundaries. The total swirling is

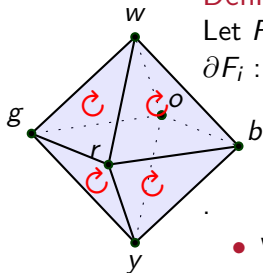$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \cdots + \sigma_{\partial F_n}$$

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Combinatorial manifolds
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Pay off all our assumptions 2: no boundary, commutativity

Definition

Let $F_1, \ldots, F_n$ be the faces of $\mathbb{M}$, $v_i : F_i$ be designated vertices, and $\partial F_i : v_i = v_i$ be the triangular boundaries. The total swirling is

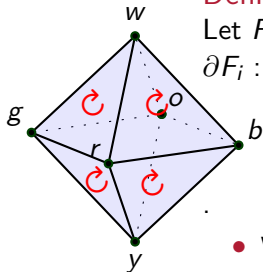$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \cdots + \sigma_{\partial F_n}$$

- We assume that this expression involves every edge once in each direction.

Summary
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Combinatorial manifolds
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Pay off all our assumptions 2: no boundary, commutativity

Definition

Let $F_1, \ldots, F_n$ be the faces of $\mathbb{M}$, $v_i : F_i$ be designated vertices, and $\partial F_i : v_i = v_i$ be the triangular boundaries. The total swirling is

$$X_{\text{tot}} \overset{\text{def}}{=} \sigma_{\partial F_1} + \cdots + \sigma_{\partial F_n}$$

- We assume that this expression involves every edge once in each direction.
- S^1 is commutative, hence complete cancellation.

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Main theorem
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Consequence

$$\mathsf{tr}_F \overset{\text{def}}{=} \mathsf{tr}(\partial F) \qquad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \qquad : \mathsf{id} =_{(T_1 =_{BS^1} T_1)} \mathsf{tr}_F \quad \text{flatness}$$

$$X_F \overset{\text{def}}{=} X(\partial F) \qquad : \mathsf{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

$$L_F^X \overset{\text{def}}{=} \flat_F(X_1) \cdot X_F \quad : (X_1 =_{T_1} X_1) \qquad \text{flattened swirling}$$

Consequence

$$\mathrm{tr}_F \overset{\text{def}}{=} \mathrm{tr}(\partial F) \qquad : T_1 =_{BS^1} T_1 \qquad\qquad \text{curvature}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \qquad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \qquad \text{flatness}$$

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These can all be totaled in S^1 to give

$$\mathrm{tr}_{\mathrm{tot}} \overset{\text{def}}{=} \sum_i \rho_{\partial F} = \mathrm{base} \qquad\qquad X_{\mathrm{tot}} \overset{\text{def}}{=} \sum_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

$$\flat_{\mathrm{tot}} \overset{\text{def}}{=} \sum_i \flat_{\partial F} \qquad\qquad L_{\mathrm{tot}}^X \overset{\text{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

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Consequence

$$\text{tr}_F \stackrel{\text{def}}{=} \text{tr}(\partial F) \qquad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \stackrel{\text{def}}{=} \flat(\partial F) \qquad : \text{id} =_{(T_1 =_{BS^1} T_1)} \text{tr}_F \quad \text{flatness}$$

$$X_F \stackrel{\text{def}}{=} X(\partial F) \qquad : \text{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

$$L_F^X \stackrel{\text{def}}{=} \flat_F(X_1) \cdot X_F \quad : (X_1 =_{T_1} X_1) \qquad \text{flattened swirling}$$

These can all be totaled in S^1 to give

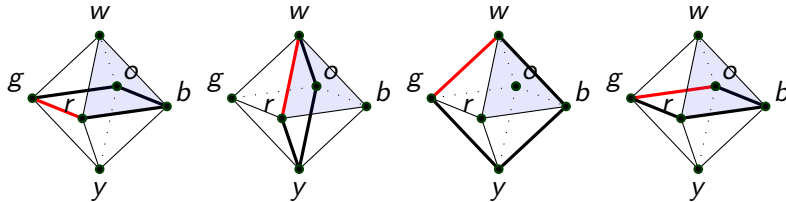$$\text{tr}_{\text{tot}} \stackrel{\text{def}}{=} \sum_i \rho_{\partial F} = \text{base} \qquad\qquad X_{\text{tot}} \stackrel{\text{def}}{=} \sum_i \sigma_{\partial F} = \text{refl}_{\text{base}}$$

$$\flat_{\text{tot}} \stackrel{\text{def}}{=} \sum_i \flat_{\partial F} \qquad\qquad L_{\text{tot}}^X \stackrel{\text{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$
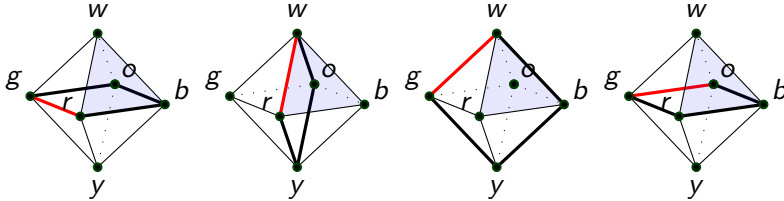
So in our lingo: the total flatness equals the total flattened swirling. □

Summary
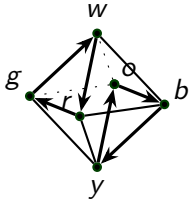○○○○○○○○

Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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Main theorem
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Examples

Each face contributes $\flat_F = H_R$, a 1/4-rotation. Total: 2.

Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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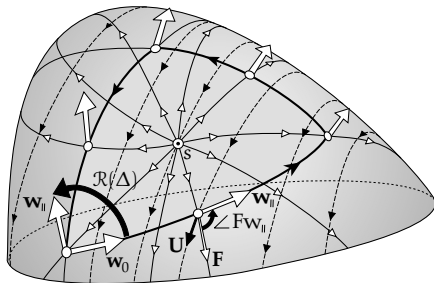Main theorem
○○○○○○○●○○

Examples

Each face contributes $\flat_F = H_R$, a 1/4-rotation. Total: 2.

For total index one obtains $+1$ from F_{wrg}, $+1$ from F_{ybo}, $+0$ from others. Total: 2.

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Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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Main theorem
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Classical proof

[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle Fw_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

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Summary
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Combinatorial manifolds
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Torsors
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Connections and curvature
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Vector fields
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Main theorem
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Thank you!

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