Discrete differential geometry in homotopy type theory

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Abstract

Higher inductive types can capture some concepts of differential geometry in two dimensions including connections, curvature, and vector fields. We define connections on higher inductive types. We then define tangent bundles and vector fields by looking at the special subclass of combinatorial manifolds, which are discrete in the sense of real cohesion[1], drawing inspiration from the field of discrete differential geometry. We prove the Gauss-Bonnet theorem and Poincaré-Hopf theorem for combinatorial manifolds.

This thesis is dedicated to John Baez, Sean M. Carroll, Sabine Hossenfelder, and other communicators who are carrying the torch of science forward in the spirit of my hero Carl Sagan. I have followed you all for many years, and you have inspired me to continue my studies alongside my unrelated career. Thank you.

"It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, The Future Does Not Compute*[2]*

Changelist

1 Overview

We will define

- combinatorial 2-manifolds
- circle bundles, and principal circle bundles of tangent bundles
- vector fields,

and then observe emerging from those definitions the presence of

- connections
- curvature
- the index of a vector field,

and attempt to[Greg] prove

- the Gauss-Bonnet theorem
- and the Poincaré-Hopf theorem.

We will consider functions $M \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where M is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

We will see in Section 3.3 that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. We will construct a map $\mathsf{link} : M \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. Then we can consider the type of pointed mere circles $\mathrm{EM}_\bullet(\mathbb{Z}, 1) \overset{\text{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ as well as the first projection that forgets the point. This is a univalent fibration (univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe[3]). If we form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"'] \\ M \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z}, 1) \end{tikzcd}$$

then we have a bundle of mere circles, with total space given by the $\sum$-type construction. We will show that this is not a principal bundle, i.e. a bundle of torsors. Torsors are types with the additional structure of a group action. But if link satisfies an additional property (amounting to an orientation) then the pullback is a principal fibration, i.e. link factors through a map $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$, where $\mathrm{K}(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space.

We will argue that extending link to a function T on paths can be thought of as constructing a connection, notably one that is not necessarily flat (trivial). Moreover, lifting T to $T_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ can be thought of as a nonvanishing vector field. There will in general not be a total lift, just a partial function. The domain of $T_\bullet$ will have a boundary of circles, and the degree (winding number) on the disjoint union of these can be thought of as the index of $T_\bullet$. We can then examine the total curvature and the total index and prove that they are equal, and argue that they are equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds. This is similar to the classical proof of Hopf[4], presented in detail in Needham[5].

1.1 Future work

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[6][7] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

2 Torsors and principal bundles

The classical theory of principal bundles tells us to look for an appropriate classifying space of torsors to map into.

Definition 2.1. Let G be a group with identity element e (with the usual classical structure and properties). A G**-set** is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : ||X||_{-1} \times \prod_{x:X} \mathsf{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

then we call this data a G**-torsor**. Denote the type of G-torsors by BG.

If $(X, \phi), (Y, \psi) : BG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 2.1. There is a natural equivalence $(X =_{BG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for BG and we have a group isomorphism $\Omega BG \simeq G$.

Lemma 2.2. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 2.1. The pointed type $(BG, *)$ is a $\mathrm{K}(G, 1)$. □

In particular, to classify principal S^1-bundles we map into the space $\mathrm{K}(S^1, 1)$, a type of torsors of the circle. Since S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$, we have $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$.

2.1 Bundles of mere circles

We find it illuminating to look also at the slightly more general classifying space of $\mathrm{K}(\mathbb{Z}, 1)$-bundles, that is bundles whose fiber are equivalent to $\mathrm{K}(\mathbb{Z}, 1)$. We can understand very well when these are in fact bundles of circle torsors, which will in turn shed light on orientation in this setting.

We will follow Scoccola[8]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2.2. Let $\mathrm{EM}(G, n) \overset{\mathrm{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \overset{\mathrm{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$. A $\mathrm{K}(G, n)$**-bundle** on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses two self-maps on the universe: suspension followed by $(n + 1)$-truncation $||\Sigma||_{n+1}$ and forgetting a point $F_\bullet$ to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{||\Sigma||_{n+1}} \mathrm{EM}_{\bullet\bullet}(G, n + 1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n + 1)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 2.3. Given $f : M \to \mathrm{EM}(G, n)$, the **associated action of** M **on** G, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ ||\Sigma||_{n+1} \circ f$.

Theorem 2.1. (Scoccola[8] Proposition 2.39). A $\mathrm{K}(G, n)$ bundle $f : M \to \mathrm{EM}(G, n)$ is equivalent to a map in $M \to \mathrm{K}(G, n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Let's relate this to *orientation*. Note that the obstruction in the theorem is about a map into $\mathrm{EM}_\bullet(G, n+1)$ and further note that $\mathrm{EM}_\bullet(G, n) \simeq \mathrm{K}(\mathrm{Aut}\, G, 1)$ (independent of n). The theorem says that the data of a map into $\mathrm{EM}(G, n)$ factors into data about a map into $\mathrm{K}(G, n+1)$ and one into $\mathrm{K}(\mathrm{Aut}\, G, 1)$. Informally, $\mathrm{EM}(G, n)$ is a little too large to be a $K(G, n+1)$, as it includes data about automorphisms of G.

In the special case of $\mathrm{EM}(\mathbb{Z}, 1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ is contractible. $\mathrm{Aut}\, \mathbb{Z}$ consists of the $\mathbb{Z}/2\mathbb{Z}$ worth of outer automorphisms given by multiplication by ± 1. If we look at the fiber sequence

$$\mathrm{K}(S^1, 1) \to \mathrm{BAut}\, S^1 \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$$

we see the automorphisms of the circle as an extension of the group of automorphisms that are homotopic to the identity (which are the torsorial actions) by the group that sends the loop in S^1 to its inverse. This is another way to see that a map $f : M \to \mathrm{BAut}\, S^1 \simeq \mathrm{EM}(\mathbb{Z}, 1)$ factors through $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$ if and only if the composition to $\mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ is trivial. This amounts to a choice of loop-direction for all the circles, and so deserves the name "f is *oriented*." In addition the map $\mathrm{BAut}\, S^1 \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z}, 1)$ deserves to be called the first Stiefel-Whitney class of f, and the requirement here is that it vanishes.

[Greg 1]
remark

***Remark* 1.** Bundles of oriented mere circles are principal, but this fact does not hold for bundles of higher-dimensional spheres. Since this note will focus on 2-dimensional oriented manifolds we will be making use of this coincidence.

2.2 Pathovers in circle bundles

Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ and $P \stackrel{\text{def}}{=} \sum_{x:M} T(x)$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : T(a) = T(b)$. This is a path in the codomain, which in this case is a type of types. Type theory also provides a function called *transport*, denoted $\mathsf{tr}(p) : T(a) \to T(b)$ which acts on the fibers of P. $\mathsf{tr}(p)$ is a function, acting on the terms of the types $T(a)$ and $T(b)$, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathsf{tr}(p)(\alpha) =_{T(b)} \beta$ between $\alpha : T(a)$ and $\beta : T(b)$ in the fibers. We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

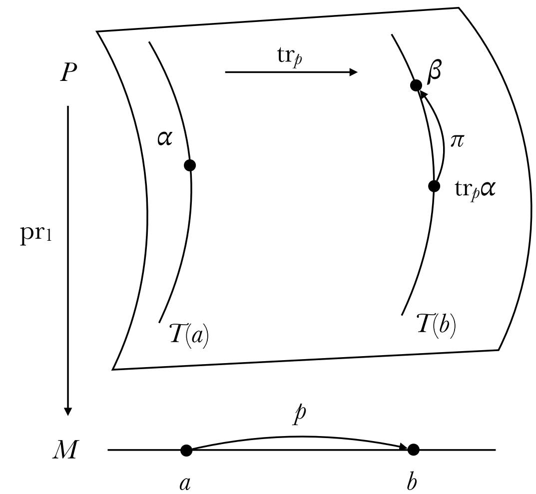

Figure 1: A path π over the path p in the base involves the transport function.

Lastly we want to recall that in the presence of a section $X : M \to P$ there is a dependent generalization of ap called apd: $\mathsf{apd}(X)(p) : \mathsf{tr}(p)(X(a)) = X(b)$ which is a pathover between the two values of the section over the basepoints of the path p.

3 Discrete manifolds

We will remind ourselves of the definition of a classical simplicial complex, in sets. Then we will create a higher type from the data of a complex, using pushouts. The type will both imitate the structure of the set-based complex and will contain explicit maps into it from the sets of the complex.

3.1 Abstract simplicial complexes

Definition 3.1. An **abstract simplicial complex** $M \overset{\text{def}}{=} [M_0, \ldots, M_n]$ **of dimension** n is an ordered list consisting of a set M_0 of vertices, and for each $0 < k \leq n$ a set M_k of subsets of M_0 of cardinality $k+1$, such that any $(j+1)$-element subset of M_k is an element of M_j. The elements of M_k are called **k-faces**. Denote by $\mathsf{SimpCompSet_n}$ the type of abstract simplicial complexes of dimension n (where the suffix Set reminds us that this is a type of sets). Let $M_{\leq k} = [M_0, \ldots, M_k]$ and note that $M_{\leq n} = M$. We call $M_{\leq k}$ the **k-skeleton** of M, and it is a (k-)complex in its own right. M is automatically equipped with a chain of inclusions of the skeleta $M_0 \hookrightarrow M_{\leq 1} \hookrightarrow \cdots \hookrightarrow M_{\leq n} = M$.

[Greg 2] Postponed enumerations/orientations of complexes to section 6 (new final section)

Definition 3.2. Let Δ^n be the standard n-simplex in $\mathbb{R}^n$ given by $\{x_1, \ldots, x_n | \sum_i x_i \leq 1\}$. Let $M : \mathsf{SimpCompSet_n}$. The **geometric realization** $|M| : \mathsf{Top}$ of M in the category of topological spaces is given inductively as follows: $|M_0| = M_0$, and given $|M_{\leq k-1}|$ we form $|M_{\leq k}|$ by the pushout in sets

$$\begin{tikzcd} M_k \times \partial\Delta^k \arrow[r, "\text{attach}"] \arrow[d, hook, "\text{id}\times\text{incl}"'] & {|M_{\leq k-1}|} \arrow[d, "i_k"] \\ M_k \times \Delta^k \arrow[r] & {|M_{\leq k}|} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

which attaches each k-simplex by taking the convex hull of the appropriate $k+1$ points in $|M_{\leq k-1}|$. The collection of vertical maps on the right gives a sequence of inclusion maps of skeleta $|M_0| \xrightarrow{i_1} |M_{\leq 1}| \xrightarrow{i_2} \cdots \xrightarrow{i_n} |M_{\leq n}| = |M|$.

Definition 3.3. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $n-1$-dimensional subcomplex containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

[Greg 3] removed enumeration of the link

The link is easier to understand as all the neighboring vertices of v and the subcomplex containing these. See for example Figure 2.

Definition 3.4. A **combinatorial manifold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a **simplicial sphere** of dimension $n-1$ (meaning its geometric realization is homeomorphic to an $n-1$-sphere). Denote by $\mathsf{CombMfdSet_n}$ the type of combinatorial manifolds of dimension n (which the notation again reminds us are sets).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figure 3 for some examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits

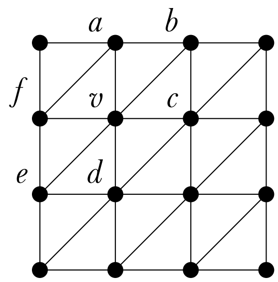

Figure 2: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

a combinatorial triangulation[9]. So it appears reasonably well motivated to study this class of objects. See for example the classic book by Kirby and Siebenmann[10]. There are important examples in four dimensions of topological manifolds that do not have any smoothness structure or triangulation. These will be out of reach of the theory we are building.

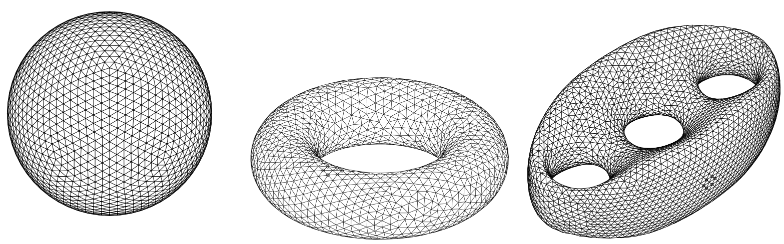

Figure 3: Combinatorial triangulations of a sphere, torus, and 3-holed torus. Sphere created with the tool `stripy`; torus from Wikipedia (By Ag2gaeh - Own work, CC BY-SA 3.0.); 3-holed torus from Wikipedia (By Ag2gaeh - Own work, CC BY-SA 3.0.)

3.2 Higher inductive combinatorial manifolds

Instead of $|M| : \mathsf{Top}$ we can use the simplicial complex to obtain $\mathbb{M} : \mathsf{Type}$ by forming a sequence of *homotopy* pushouts. For example in dimension 1 we could take the triangle with vertices $\{v_1, v_2, v_3\}$ and edges $\{e_{12}, e_{23}, e_{31}\}$ and form a polygon C_3:

$$
\begin{tikzcd}
C_{3,1} \times S^0 = \{e_{12}, e_{23}, e_{31}\} \times \{N, S\} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\substack{e_{12}\times\{N,S\}\mapsto\{v_1,v_2\} \\ e_{23}\times\{N,S\}\mapsto\{v_2,v_3\} \\ e_{31}\times\{N,S\}\mapsto\{v_3,v_1\}}"'] \arrow[dr, phantom, "\Nearrow h_1"] & C_{3,1} \arrow[d, "*_1"] \\
C_{3,0} = \{v_1, v_2, v_3\} \arrow[r] & C_3 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

The left vertical map expresses the connectivity between the edges and vertices in the set-based complex. The right vertical map $*_1$ provides a hub point for each edge, and the homotopy h_1 provides the spokes that connect the hub to the vertices. So in contrast to a geometric realization, the 1-dimensional cells in the HoTT sense are generated by the filler homotopy.

In dimension 2 we could fill in maps from C_3 into our higher type by adding faces (hence reusing the object we just built above):

$$
\begin{tikzcd}
 & M_2 \times C_3 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\Nearrow h_2"] & M_2 \arrow[d, "*_2"] \\
M_0 = \mathbb{M}_0 \arrow[r] \arrow[dr, phantom, "\Searrow h_1"] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times S^0 \arrow[u, "\partial_0"] \arrow[r, "\mathrm{pr}_1"'] & M_1 \arrow[u, "*_1"']
\end{tikzcd}
$$

The types C_3 and $\mathbb{M}_1$ are 1-types, $\mathbb{M}_2$ is a 2-type, and the rest are sets. The map ∂_0 maps each pair (e, S^0) to the pair of points this edge connects.

[Greg 4] removed a paragraph about how ∂_1 works

The h_i are the proofs of commutativity, and the two squares are also both homotopy pushouts. Note that the pushouts could be re-expressed as HIT constructors.

Whereas in geometric realization we use both Δ^i and $\partial\Delta^i$ to attach simplices, in this homotopy picture we need an explicit construction of "$\partial\Delta^i$" as a type equivalent to an $i-1$-dimensional sphere and the filling of this sphere with i-dimensional stuff has moved into the proof of commutativity. So in dimension n the picture would be

$$
\begin{tikzcd}
\cdots \arrow[r] & M_{n-2} \arrow[d, "*_{n-2}"] & M_n \times S^{n-1} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_{n-1}"'] \arrow[dr, phantom, "\Nearrow h_n"] & M_n \arrow[d, "*_n"] \\
\cdots \arrow[r] \arrow[ur, phantom, "\Nearrow h_{n-2}"] & \mathbb{M}_{n-2} \arrow[r] \arrow[dr, phantom, "\Searrow h_{n-1}"] & \mathbb{M}_{n-1} \arrow[r] & \mathbb{M}_n \\
 & M_{n-1} \times S^{n-2} \arrow[u, "\partial_{n-2}"] \arrow[r, "\mathrm{pr}_1"'] & M_{n-1} \arrow[u, "*_{n-1}"']
\end{tikzcd}
$$

Definition 3.5. A **higher realization** of an abstract simplicial complex M : $\mathsf{SimpCompSet_n}$ consists of

[Greg 5] new compact definition of realization (but I didn't go with "CW complex")

1. $n+1$ types $\mathbb{M}_0, \ldots, \mathbb{M}_n$ where $\mathbb{M}_0 \overset{\text{def}}{=} M_0$,
2. n spans $\mathbb{M}_i \xleftarrow{\partial_i} M_{i+1} \times S^i \xrightarrow{\mathrm{pr}_1} M_{i+1}$, $i = 1, \ldots, n$, where S^i is a HoTT i-sphere and ∂_i are called **attachment maps**,
3. n pushout squares from each span to $\mathbb{M}_{i+1}$, with induced maps $\iota_i : \mathbb{M}_i \to \mathbb{M}_{i+1}$, $*_{i+1} : M_{i+1} \to \mathbb{M}_{i+1}$ and proof of commutativity h_{i+1}.

We can call such a higher realization a **higher simplicial complex**, of in the case where the underlying complex M is a combinatorial manifold, a $highercombinatorialmanifold$.

We intend for the ∂_i to be isomorphisms between a boundary of a simplex in $\mathbb{M}_i$ and the i-sphere. To avoid spending too much time above dimension 2, we will leave the ∂_i underspecified in general. For 2-dimensional complexes we will see how the gluing can be done for triangles.

3.3 Polygons

The 1-type C_3 we created earlier by pushing out the combinatorial data of a set-based simplicial triangle is clearly an example of a type of marked presented polygons C_n, $n : \mathbb{N}$. The standard

HoTT circle itself is usually given as a HIT and is a non-example of a combinatorial manifold since it lacks the second vertex of the edge:

Definition 3.6. The higher inductive type S^1 which we can also call C_1:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

Denote by Gon the set of marked presented n-gons for some natural number n. We'll see below that the realization of an n-gon is a mere circle, i.e. we have a forgetful map $\mathsf{Gon} \to \mathrm{EM}(\mathbb{Z}, 1)$.

3.4 Adding and removing points from polygons

Recall that given functions $\phi, \psi : A \to B$ between two arbitrary types we can form a type family of paths $\alpha : A \to \mathcal{U}$ by $\alpha(a) \overset{\text{def}}{=} (\phi(a) =_B \psi(a))$. Transport in this family is given by concatenation as follows, where $p : a =_A a'$ and $q : \phi(a) = \psi(a)$ (see Figure 4):

$$\mathsf{tr}(p)(q) = \phi(p)^{-1} \cdot q \cdot \psi(p)$$

which gives a path in $\phi(a') = \psi(a')$ by connecting dots between the terms $\phi(a'), \phi(a), \psi(a), \psi(a')$. This relates a would-be homotopy $\phi \sim \psi$ specified at a single point, to a point at the end of a path. We will use this to help construct such homotopies.

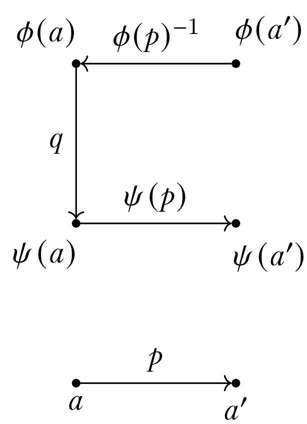

Figure 4: Transport along p in the fibers of a family of paths. The fiber over a is $\phi(a) = \psi(a)$ where $\phi, \psi : A \to B$.

Lemma 3.1. Let C_n be the marked presented polygon 1-type with n vertices. Then $C_2 \simeq C_1$ and in fact $C_n \simeq C_{n-1}$.

Proof. (Compare to [11] Lemma 6.5.1.) In the case of C_1 we will denote its constructors by base and loop. For C_2 we will denote the points by v_1, v_2 and the edges by ℓ_{12}, r_{21}. For C_3 and higher we will denote the points by $v_1, \ldots, v_n$ and the edges by $e_{i,j} : v_i = v_j$ where $j = i + 1$ except for $e_{n,1}$.

First we will define $f : C_2 \to C_1$ and $g : C_1 \to C_2$, then prove they are inverses.

$$\begin{aligned} f(v_1) = f(v_2) &= \mathsf{base} & g(\mathsf{base}) &= v_1 \\ f(\ell_{12}) &= \mathsf{loop} & g(\mathsf{loop}) &= \ell_{12} \cdot r_{21} \\ f(r_{21}) &= \mathsf{refl}_{\mathsf{base}} \end{aligned}$$

We need to show that $f \circ g \sim \mathrm{id}_{C_1}$ and $g \circ f \sim \mathrm{id}_{C_2}$. Think of f as sliding v_2 along r_{21} to coalesce with v_1. This may help understand why the unfortunately intricate proof is working.

We need terms $p : \prod_{a:C_1} f(g(a)) = a$ and $q : \prod_{a:C_2} g(f(a)) = a$. We will proceed by induction, defining appropriate paths on point constructors and then checking a condition on path constructors that confirms that the built-in transport of these type families respects the definition on points.

Looking first at $g \circ f$, which shrinks r_{21}, we have the following data to work with:

$$\begin{aligned} g(f(v_1)) = g(f(v_2)) &= v_1 \\ g(f(\ell_{12})) &= \ell_{12} \cdot r_{21} \\ g(f(r_{21})) &= \mathsf{refl}_{v_1}. \end{aligned}$$

We then need to supply a homotopy from this data to id_{C_2}, which consists of a section and pathovers over C_2:

$$\begin{aligned} p_1 &: g(f(v_1)) = v_1 \\ p_2 &: g(f(v_1)) = v_2 \\ H_\ell &: \mathsf{tr}(\ell_{12})(p_1) = p_2 \\ H_r &: \mathsf{tr}(r_{21})(p_2) = p_1. \end{aligned}$$

which simplifies to

$$\begin{aligned} p_1 &: v_1 = v_1 \\ p_2 &: v_1 = v_2 \\ H_\ell &: g(f(\ell_{12}))^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\ H_r &:= g(f(r_{21}))^{-1} \cdot p_2 \cdot r_{21} = p_1 \end{aligned}$$

and then to

$$\begin{aligned} p_1 &: v_1 = v_1 \\ p_2 &: v_1 = v_2 \\ H_\ell &: (\ell_{12} \cdot r_{21})^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\ H_r &: \mathsf{refl}_{v_1} \cdot p_2 \cdot r_{21} = p_1 \end{aligned}$$

To solve all of these constraints we can choose $p_1 \stackrel{\text{def}}{=} \mathsf{refl}_{v_1}$, which by consulting either H_ℓ or H_r requires that we take $p_2 \stackrel{\text{def}}{=} {r_{21}}^{-1}$.

Now examining $f \circ g$, we have

$$\begin{aligned} f(g(\mathsf{base})) &= \mathsf{base} \\ f(g(\mathsf{loop})) &= f(\ell_{12} \cdot r_{21}) = \mathsf{loop} \end{aligned}$$

and so we have an easy proof that this is the identity.

The proof of the more general case $C_n \simeq C_{n-1}$ is very similar. Take the maps $f : C_n \to C_{n-1}$, $g : C_{n-1} \to C_n$ to be

$$\begin{aligned} f(v_i) &= v_i \quad (i = 1, \ldots, n-1) & g(v_i) &= v_i & (i = 1, \ldots, n-1) \\ f(v_n) &= v_1 & g(e_{i,i+1}) &= e_{i,i+1} & (i = 1, \ldots, n-2) \\ f(e_{i,i+1}) &= e_{i,i+1} \quad (i = 1, \ldots, n-1) & g(e_{n-1,1}) &= e_{n-1,n} \cdot e_{n,1} & \\ f(e_{n-1,n}) &= e_{n-1,1} & & & \\ f(e_{n,1}) &= \mathsf{refl}_{v_1} & & & \end{aligned}$$

where f should be thought of as shrinking $e_{n,1}$ so that v_n coalesces into v_1.

The proof that $g \circ f \sim \mathrm{id}_{C_n}$ proceeds as follows: the composition is definitionally the identity except

$$\begin{aligned} g(f(v_n)) &= v_1 \\ g(f(e_{n-1,n})) &= e_{n-1,n} \cdot e_{n,1} \\ g(f(e_{n,1})) &= \mathsf{refl}_{v_1}. \end{aligned}$$

Guided by our previous experience we choose ${e_{n,1}}^{-1} : g(f(v_n)) = v_n$, and define the pathovers by transport.

The proof that $f \circ g \sim \mathrm{id}_{C_{n-1}}$ requires only noting that $f(g(e_{n-1,1})) = f(e_{n-1,n} \cdot e_{n,1}) = e_{n-1,1} \cdot \mathsf{refl}_{v_1} = e_{n-1,1}$. □

Corollary 3.1. All polygons are equivalent to S^1, i.e. we have a term in $\prod_{n:\mathbb{N}} \|C_n = S^1\|$, and hence Gon is a subtype of $\mathrm{EM}(\mathbb{Z}, 1)$.

Proof. We can add $n-1$ points to S^1 and use Lemma 3.1. □

Definition 3.7. For $k : \mathbb{N}$ define $m_k : \mathsf{Gon} \to \mathsf{Gon}$ where $m_k : C_n \to C_{kn}$ adds k vertices between each of the original verticies of C_n.

With m_k we can start with a collection of pentagons and hexagons and make the collection homogeneous: by applying m_6 to the pentagons and m_5 to the hexagons we obtain a collection of 30-gons. This will be useful when we work more with the link function.

3.5 The higher inductive type $\mathbb{O}$

We will create our first combinatorial surface, an octahedron. In $\mathsf{SimpCompSet_n}$ the combinatorial data of the faces can be represented with a *Hasse diagram*, which shows the poset of inclusions in a graded manner, with a special top and bottom element. We give an octahedron in Figure 5. The names of the vertices are short for white, yellow, blue, red, green, and orange, the

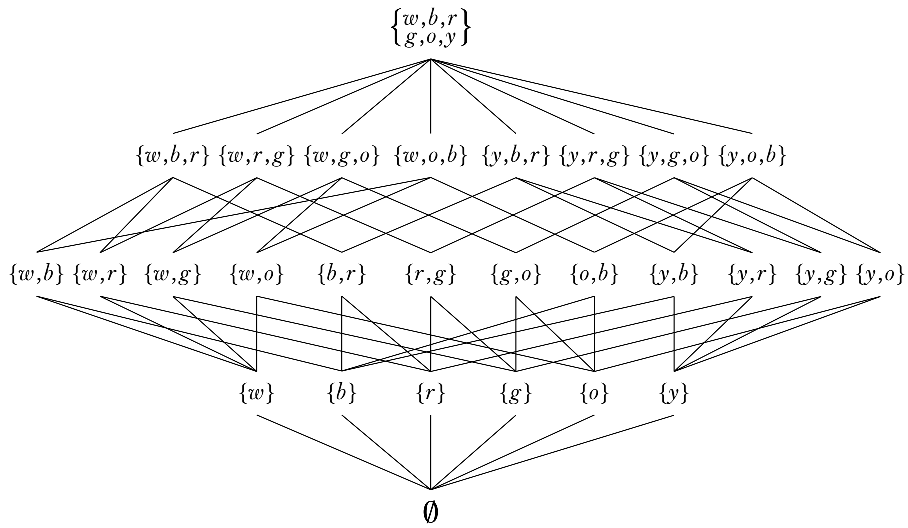

Figure 5: Hasse diagram of an octahedron O. The row of singletons is O_0 and above it are O_1 and O_2.

colors of the faces of a Rubik's cube. The octahedron is the dual of the cube, with each vertex corresponding to a face.

Applying the map $\mathcal{R}$ to $O_0 \to O_1 \to O$ gives the presented type $\mathbb{O}_0 \to \mathbb{O}_1 \to \mathbb{O}$.

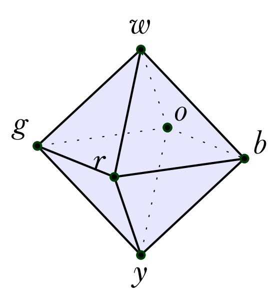

Figure 6: The marked presented type $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

4 Bundles, connections, and curvature

[Greg 6] Rewritten subsection on connections

A map out of a higher combinatorial manifold has various components. These line up with classical definitions, and identifying those is a primary purpose of this note.

Definition 4.1. If $\mathbb{M}$ is a higher combinatorial manifold and $f_k : \mathbb{M}_k \to \mathcal{U}$ are type families on each skeleton such that all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r] \arrow[drr, "f_0"] & \mathbb{M}_1 \arrow[r] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r] \arrow[d, "f_2"] & \cdots \arrow[r] & \mathbb{M} \arrow[dll, "f"'] \\
& & \mathcal{U} & &
\end{tikzcd}$$

and such that for each pushout defining $\mathbb{M}_k$ we have the diagram

$$\begin{tikzcd}
M_k \times S^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_{k-1}"'] \arrow[dr, phantom, "\Nearrow h_k"] & M_k \arrow[d, "*_k"] \\
\mathbb{M}_{k-1} \arrow[r] \arrow[dr, "f_{k-1}"'] \arrow[r, phantom, shift right=4, "\Swarrow"] & \mathbb{M}_k \arrow[d, "f_k"] \\
& \mathcal{U}
\end{tikzcd}$$

then we say

- The map f_k is a k**-bundle** on $\mathbb{M}_k$.
- Given f_{k-1}, the pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$ that f_k extends f_{k-1} is called a k**-connection on the bundle** f_{k-1}.
- For each k-face $m_k : M_k$ the proof of commutativity of the outer square of the second diagram is called a **flatness structure for the face** m_k.

***Remark* 2.** Another classical object we can identify here is a *gauge transformation*, which is a bundle automorphism $H : f \sim f \overset{\text{def}}{=} \prod_{m:\mathbb{M}} f(m) = f(m)$. If we have $||H = \mathrm{id}_f||_{-1}$ then the gauge transformation is called *small*, else *large*.

4.1 The function T on $\mathbb{O}$

We will build up a map T out of $\mathbb{O}$ which is meant to be like the circle bundle of a tangent bundle. And so we will begin with the intrinsic data of the link at each point: taking the link of a vertex gives us a map from vertices to polygons.

Definition 4.2. $T_0 \overset{\text{def}}{=} \mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned}
\mathsf{link}(w) &= brgo & \qquad \mathsf{link}(r) &= wbyg \\
\mathsf{link}(y) &= bogr & \qquad \mathsf{link}(g) &= wryo \\
\mathsf{link}(b) &= woyr & \qquad \mathsf{link}(o) &= wgyb
\end{aligned}$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

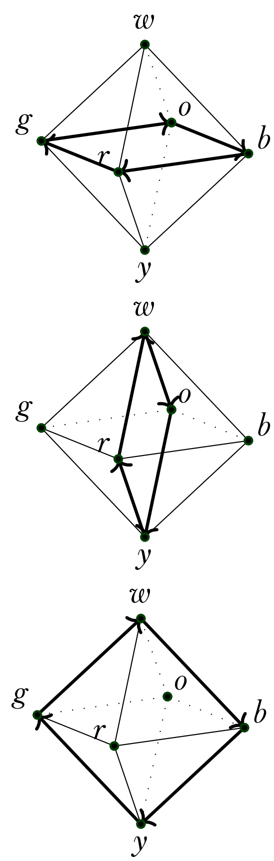

Figure 7: link for the vertices w, b and r.

To extend T_0 to a function T_1 on the 1-skeleton we have complete freedom. Defining a map by induction makes clear that the action on paths is its own structure. Two functions on the octahedron could agree on points but differ on edges. We are going to identify this 1-dimensional freedom with a connection:

Continuing the example, we will do something "tangent bundley", imagining how T_1 changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 4.3. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ (r, o fixed)
- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ (b, g fixed)
- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$
- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 7 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn.

Definition 4.4. The map $R : C_4 \to C_4$ rotates by one quarter turn, one "click":

- $R(c_1) = c_2$
- $R(c_2) = c_3$
- $R(c_3) = c_4$
- $R(c_4) = c_1$
- $R(c_1c_2) = c_2c_3$
- $R(c_2c_3) = c_3c_4$
- $R(c_3c_4) = c_4c_1$
- $R(c_4c_1) = c_1c_2$

Note by univalence the equivalence R gives a loop in the universe, a term of $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here *there is no additional freedom.*

Definition 4.5. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows:

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

where $H_R : R = \mathsf{refl}_{C_4}$ is the obvious homotopy given by composition with R^{-1}. Passing through univalence we get a 2-path between R and refl in the loop space $C_4 =_{\mathrm{EM}(\mathbb{Z},1)} C_4$.

4.2 Existence of connections

How confident can we be that we can always define a connection on an arbitrary combinatorial manifold? Two things make the octahedron example special: the link is a 4-gon at every vertex, and every edge extends to a symmetry of the entire octahedron, embedded in 3-dimensional space. This imposed a coherence on the interactions of all the choices we made for the connection, which we can worry may not exist for more complex combinatorial data.

We know as a fact outside of HoTT that any combinatorial surface that has been realized as a triangulated surface embedded in 3-dimensional euclidean space can inherit the parallel transport entailed in the embedding. We could then approximate that data to arbitrary precision with enough subdivision of the fibers of T.

What would a proof inside of HoTT look like? We will leave this as an open question.

5 Vector fields

[Greg 7] Rewritten vector field section

Vector fields are sections of the tangent bundle of a manifold. We do not have a general theory of tangent bundles, even for 2-dimensional higher combinatorial manifolds, since we cannot yet prove that connections always exist on the 1-skeleton. But *given* an extension T_1 of the link function to the 1-skeleton, we can consider the type of sections $\prod_{x:\mathbb{M}_1} T_1(x)$.

Definition 5.1. A **nonvanishing vector field** on a higher combinatorial 2-manifold $\mathbb{M}$ equipped with type family $T_1 : \mathbb{M}_1 \to \mathrm{K}(\mathbb{Z}, 2)$ that extends link is a term $X : \prod_{x:\mathbb{M}_1} T_1(x)$.

***Remark* 3.** We do not lose any classical generality by assuming we have such a nonvanishing field on the entire 1-skeleton. This corresponds to the fact that the zeros of a vector field can be perturbed to be isolated, and to live inside the faces of a triangulation.

***Remark* 4.** A section $\prod_{x:\mathbb{M}} f(x)$ for $f : \mathbb{M} \to \mathrm{K}(\mathbb{Z}, 2)$ is a trivialization of the bundle. The fact that an orientation suffices to factor the tangent bundle of a 2-manifold (which *a priori* maps to $\mathrm{EM}(\mathbb{Z}, 1)$) through a principal bundle classifier is special to dimension 2. For higher dimensional manifolds the tangent bundle is a bundle of spheres, and even if the bundle is oriented it is not necessarily a principal bundle. On the other hand, the n-truncation modal operator maps the type of n-spheres to the classifying space $\mathrm{K}(\mathbb{Z}, n)$, and so another way to phrase this remark is that S^1 is the only n-sphere which is n-truncated.

On the 0-skeleton X picks a point in each link, i.e. a neighbor of each vertex. On a path $p : x =_{\mathbb{M}} y$, X assigns a dependent path over p, which as we know is a term $\pi : \mathsf{tr}(p)(X(x)) =_{T_1 y} X(y)$.

***Remark* 5.** Concrete pathover terms such as π couple together the connection (the transport) and X. The transport is needed to provide a single type in which to compare the two values of X. The path π therefore reflects both the movement of X along the path p, and the moving lens that allows us to examine in a single fiber the two points $X(x)$ and $X(y)$ as well as $X(p)$.

5.1 Swirling

Classically, vector fields are seen to swirl around between points. We can see this in HoTT as well, though in contrast to the classical explanations we always have a connection that interacts with the vector field.

Consider a vertex $x : \mathbb{M}$, a face F containing vertices x, y, z, and the boundary path $\ell \stackrel{\text{def}}{=} e_{xy} \cdot e_{yz} \cdot e_{zx}$. We can calculate $X(\ell)$ by concatenating the data along the path. As we visit each point we accumulate more information:

1. In $T_1(x)$ we have
   (a) the point $X(x)$.
2. In $T_1(y)$ we have
   (a) point $X(y)$
   (b) point $\mathsf{tr}(e_{xy})(X(x))$

    (c) the path $X(e_{xy}) : \mathsf{tr}(e_{xy})(X(x)) = X(y)$ from (b) to (a).

3. In $T_1(z)$ we have
    (a) point $X(z)$
    (b) point $\mathsf{tr}(e_{yz})(X(y))$
    (c) point $\mathsf{tr}(e_{yz}) \circ \mathsf{tr}(e_{xy})(X(x))$
    (d) path $X(e_{yz}) : \mathsf{tr}(e_{yz})(X(y)) = X(z)$ from (b) to (a)
    (e) path $\mathsf{tr}(e_{yz})(X(e_{xy}))$ from (c) to (b)
4. Then back at $T_1(x)$ we have
    (a) point $X(x)$
    (b) point $\mathsf{tr}(e_{zx})(X(z))$
    (c) point $\mathsf{tr}(e_{zx}) \circ \mathsf{tr}(e_{yz})(X(y))$
    (d) point $\mathsf{tr}(e_{zx}) \circ \mathsf{tr}(e_{yz}) \circ \mathsf{tr}(e_{xy})(X(x))$, i.e. $\mathsf{tr}(\ell)(X(x))$
    (e) path $X(e_{zx}) : \mathsf{tr}(e_{zx})(X(z)) = X(x)$ from (b) to (a)
    (f) path $\mathsf{tr}(e_{zx})(X(e_{yz}))$ from (c) to (b)
    (g) path $\mathsf{tr}(e_{zx}) \circ \mathsf{tr}(e_{yz})(X(e_{xy}))$ from (d) to (c).

As we traverse an edge, say e_{xy}, we get a path in $T_1(y)$ which is the image of e_{xy} under X. Although this path is then transported along further edges of the triangle, it will help us to think of it as maintaining its relationship to e_{xy}:

Definition 5.2. If $e_{xy} : x =_{\mathbb{M}_1} y$, $e_{yz} : y =_{\mathbb{M}_1} z$ are paths in $\mathbb{M}_1$ and X is a nonvanishing vector field on $\mathbb{M}_1$, then we call $\mathsf{tr}(e_{yz})(X(e_{xy}))$ **the contribution on** e_{xy} **of** X, even though this is a path in $T_1(z)$.

As we traverse an additional edge this image is simply mapped to the next vertex through the lens of transport, which acts similarly on all the points in a fiber. The image is carried first to $\mathsf{tr}(e_{yz})(X(e_{xy}))$ then to $\mathsf{tr}(e_{zx}) \circ \mathsf{tr}(e_{yz})(X(x))$.

At stage 4 we have three paths, each consisting of X acting on a single edge of the triangle, mapped with some transport to $T_1(x)$. We see that these three can be concatenated to form a path $X(\ell) : \mathsf{tr}(\ell)(X(x)) = X(x)$.

***Remark* 6.** In Hopf [4] and in Needham [5], the value $X(e_{xy})$ is called "the change in angle between X and $X(x)_{||}$ along the edge e_{xy}," where by $X(x)_{||}$ we mean the transport of the fixed single vector $X(x)$ to the point $X(y)$, i.e. $\mathsf{tr}(e_{xy})(X(x))$. The concatenation $X(\ell)$ is called "the sum of the changes in angle along each edge." This remark is meant to help anyone intending to make further comparisons with classical results.

When T_1 has an extension to F we can obtain a second path in $\mathsf{tr}(\ell)(X(x)) = X(x)$! Namely given a filler path $\ell =_{v=_{\mathbb{M}}v} \mathsf{refl}(v)$ that fills the loop, an extension of T_1 to F provides by functoriality a corresponding path $T_1(\ell) =_{T_1(v)=T_1(v)} \mathrm{id}(T_1(v))$, which is a homotopy between these two automorphisms of $T_1(v)$ that we called the flatness structure on F. Evaluating this homotopy at $X(x)$ provides the term $\flat(\ell)(X(x)) : X(x) = \mathsf{tr}(\ell)(X(x))$ (and its inverse is hence the promised second path).

Concatenating $\flat(\ell)(X(x)) \cdot X(\ell)(X(x))$ gives a loop in the pointed type $(T_1(x), X(x))$. This is a term in $\mathbb{Z}$:

Definition 5.3. The **index** of X at x on the face F is $\flat(\ell)(X(x)) \cdot X(\ell)(X(x)) : \mathbb{Z}$.

***Remark* 7.** Intuitively, $X(\ell)$ combines the swirling of X with the twisting of the connection, and prepending with the flatness structure erases the effect of the connection, leaving just the swirling of X. This should line up with the classical definition of index, which is also an integer.

6 Total constructions

[Greg 8] Moved all "totaling" stuff into a section, all new material

We wish to make computations that combine contributions from every face of a combinatorial 2-manifold. To do this we will add some structure and then prove that the definitions are independent of that structure.

Definition 6.1. A **total enumeration of faces** for a combinatorial 2-manifold $\mathbb{M}$ with underlying simplicial complex $M = [M_0, M_1, M_2]$ consists of

- A "master basepoint" $m : M_0$.
- For each face $F : M_2$ with vertices $\{v_1, v_2, v_3\}$ an enumeration of its vertices $[v_1, v_2, v_3]$, including the choice of the first vertex in the enumeration as the basepoint of F, that is **globally compatible** with the choices for the other faces, meaning that when two faces F_1, F_2 share an edge $\{v, w\}$, then one of the faces includes the sublist $[v, w]$ and the other includes $[w, v]$.
- An ordering of the faces $[F_1, \ldots, F_n]$.
- For each face F a path $p_F : m =_{\mathbb{M}} F_v$ and a loop $\ell_F \overset{\text{def}}{=} p_F \cdot e_{12} \cdot e_{23} \cdot e_{31} \cdot p_F^{-1}$ where $e_{ij} : v_i = v_j$ are the paths corresponding to the edges.

Note that the last bullet of the definition is the only one that uses the realization $\mathbb{M}$.

***Remark* 8.** For such an enumeration to exist the underlying simplicial complex must be *orientable* in a classical sense. We are not going to explore this requirement internally in HoTT, nor prove any relationship between orientability of the set-based complex and orientation in the sense of factoring classifying maps through $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$.

Example 6.1. The octahedron: for $\mathbb{O}$ we might choose b as the master basepoint, as well as the basepoint for four of the faces. For the other four faces we could choose g as the basepoint. We could choose $br \cdot rg$ as the path between the basepoints, and we could order the faces like this: $[bwo, brw, boy, byr, gow, gwr, gry, gyo]$.

Definition 6.2. Given a total enumeration of faces, the concatenation $\ell_{\text{tot}} \overset{\text{def}}{=} \ell_{F_1} \cdot \cdots \cdot \ell_{F_n}$ is called **the total loop** of the enumeration.

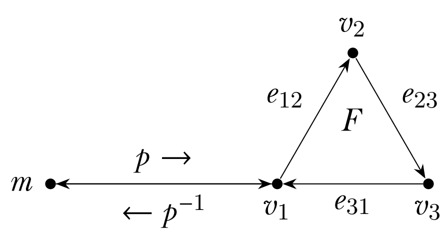

Figure 8: A loop $\ell_F : m =_{\mathbb{M}} m$ around the face F.

6.1 Total holonomy

Given a total enumeration of faces we can form the compositions $T \circ \ell_{F_i}$ with the associated loop to obtain holonomy isomorphisms $T \circ \ell_{F_i} : Tm = Tm$. We can concatenate all the loops

$\ell_{F_1} \cdot \cdots \cdot \ell_{F_n}$ and obtain a *total holonomy* $T \circ (\ell_{F_1} \cdot \cdots \cdot \ell_{F_n}) : Tm = Tm$.

Definition 6.3. The **total holonomy** of a total enumeration of faces is the map $T \circ (\ell_{F_1} \cdot \cdots \cdot \ell_{F_n}) : Tm = Tm$.

Proposition 6.1. The holonomy ℓ_F around a face F is conjugated if we change the path p_F or the basepoint m. The holonomy becomes the inverse automorphism if we change the enumeration of F by an odd permutation of its vertices (but keep the basepoint fixed).

Proof. If we change the path p_F to a homotopic path p'_F, then the change is made up of a finite sequence of moves like the one in Figure 9, where the triangle xyz bounds a face. Then p_F is given by

$$(p \cdot e_{xy}) \cdot (q \cdot e_{12} \cdot e_{23} \cdot e_{31} \cdot q^{-1}) \cdot (p \cdot e_{xy})^{-1}$$

where we grouped the parts that involve the change and do not involve it. So replacing e_{xy} by $e_{xz} \cdot e_{zy}$ is definitionally equal to

$$A \cdot (p \cdot e_{xy}) \cdot (q \cdot e_{12} \cdot e_{23} \cdot e_{31} \cdot q^{-1}) \cdot (p \cdot e_{xy})^{-1} \cdot A^{-1}$$

where $A \overset{\text{def}}{=} p \cdot e_{xz} \cdot e_{zy} \cdot e_{xy}^{-1} \cdot p^{-1}$ is the loop that circulates the difference between the paths. Said another way, we are making use of the observation that p_F is related to p'_F by conjugating with the loop, based at m, that goes around the discrepancy. If we add more such basic changes then we add additional conjugations.

If we change the basepoint m to m' then we need to make a new choice, which is a path $\pi : m' = m$. This lengthens the path p_F to the path $\pi \cdot p_F \cdot \pi^{-1}$. This conjugation relates two different groups, the automorphism groups $Tm = Tm$ and $Tm' = Tm'$. The two groups are not canonically identified because the identification relies on the choice of π. If we change π by a sequence of moves like we did with p_F then again the holonomy at m' will change by a conjugation.

Finally we observe that for the triangle F, an odd permutation that keeps the basepoint fixed is simply a traversal of the loop in the opposite direction, sending ℓ_F to $\ell_{F^{-1}}$. □

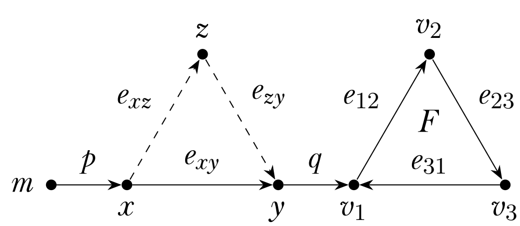

Figure 9: Two paths to the face, from x to y directly, or through z.

Corollary 6.1. In a bundle of mere circles $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ the total holonomy is independent of the choice of paths p_{F_i} and the basepoint m. It is inverted if we switch orientations.

Example 6.2. The total holonomy of T_1 on $\mathbb{O}$ (Definition 4.4) is R^8 = which is two full rotations of C_4.

6.2 Total index of a vector field

We defined the index on a face in Definition 5.3. We will extend this to a total index defined with a total enumeration of faces.

6.3 The main theorem

Theorem 6.1. The total index is constant on $\prod_{x:\mathbb{M}_1} T_1(x)$.

This appears to be a little less than what we have classically, which is:

1. A definition of total curvature of a connection on the tangent bundle.
2. A definition of total index, without using the connection.
3. A definition of Euler characteristic.
4. A proof that total index is independent of the vector field.
5. A proof that total curvature is equal to total index.

This collection of results provides enough redundancy to separately produce corollaries such as: total curvature is an integer; total curvature does not depend on the connection; a second proof that total index is independent of the vector field (since it is equal to total curvature, which does not so depend).

How much of this do we have?

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