Discrete differential geometry in homotopy type theory

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Abstract

Homotopy type theory captures all the major concepts of differential geometry including forms, connections, curvature, and gauge theory. We show this by focusing on combinatorial manifolds, which are discrete in the sense of real cohesion[1], and drawing inspiration from the similarly young field of discrete differential geometry.

> "It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, *The Future Does Not Compute*[2]

1 Overview

We will define

- combinatorial 2-manifolds
- principal circle bundles of tangent bundles
- vector fields,

and then observe emerging from those definitions the presence of

- connections
- curvature
- the index of a vector field,

and prove

- the Gauss-Bonnet theorem
- the Poincaré-Hopf theorem
- and the Leibniz formula.

We will consider functions $M \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where M :

$\mathsf{Comb2Mfd}$ is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

Note that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. Therefore we can construct a map $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. This will serve as the circle bundle of the tangent bundle of the manifold, i.e. the principal bundle consisting of nonzero tangent vectors.

Now consider the type $\mathrm{EM}_\bullet(\mathbb{Z}, 1) \stackrel{\text{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ of pointed mere circles. We have the pullback

$$\begin{array}{ccc} P \stackrel{\text{def}}{=} \sum_{C:TM} C & \xrightarrow{\overline{T}} & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & {\scriptstyle \mathrm{pr}_1}\big\downarrow \\ M & \xrightarrow{T} & \mathrm{EM}(\mathbb{Z}, 1) \end{array}$$

from the univalent fibration on the right, forming the usual fiber of T as a sigma type. Such classifying maps are not always principal bundles; there is an extra condition on T that we will get into later. For now it's important only that we are mapping into a univalent fibration so that we can make use of type theory. Univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe.

We will investigate that the data in dimensions 1 and 2 of T can be thought of as a connection, notably one that is not necessarily flat. Moreover, lifting T to $\mathrm{EM}_\bullet(\mathbb{Z}, 1)$ can be thought of as a vector field. There will in general not be a total lift, just a partial function. The domain will have a boundary of circles, and the winding number on these can be thought of as the index of the vector field. We can then examine the total curvature and the total index and prove that they are equal, and equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds.

Taking the dimension 1 part of a function can be thought of as its derivative. If the codomain has an H-space structure then we can ask about how the action on paths interacts with pointwise multiplication. This will lead us to the Leibniz formula in this context, which emerges simply from horizontal composition in the codomain.

2 Torsors and principal bundles

The classical theory of principal bundles tells us to look for an appropriate classifying space of torsors to map into. Homotopy type theory tells us that classifying spaces are univalent fibrations. The type of torsors is not a priori such a fibration, so we'll do some work to make that happen. This will constitute the codomain of the investigation.

Definition 1. Let G be a group (a set with the usual classical structure and properties). A G**-set** is a set X equipped with a homomorphism $\phi : G \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : ||X||_{-1} \times \prod_{g:G} \mathsf{is\_equiv}(\phi(-, x) : G \to X)$$

then we call this data a G**-torsor**. Denote the type of G-torsors by TG.

If $(X, \phi), (Y, \psi) : TG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 1. There is a natural equivalence $(X =_{TG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for TG and we have a group isomorphism $\Omega TG \simeq G$.

Lemma 2. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 1. The pointed type $(TG, *)$ is a $\mathrm{K}(G, 1)$. □

2.1 Univalent replacement for torsors

The homotopy type theory of cohomology and bundles tells us that the type of G-bundles on a type M is the type $M \to \mathrm{K}(G, 1)$. So we will start there as well. But this is a type of structured types, a connected component of G-sets rather than a connected component of the universe. The paths *in the universe* between two G-sets is equivalent to the type of equivalences between the *underlying types*, not just the equivariant equivalences.

We'll resolve this problem with the following discussion, following Scoccola[3]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2. Let $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \overset{\text{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$. A $\mathrm{K}(G, n)-$*bundle* on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses the action on the universe of suspension and of forgetting a point to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{\Sigma} \mathrm{EM}_{\bullet\bullet}(G, n) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 3. Given $f : M \to \mathrm{EM}(G, n)$, the **associated action of** M **on** G, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ \Sigma \circ f$.

Theorem 1. (Scoccola[3] Proposition 2.39). A $\mathrm{K}(G, n)$ bundle $f : M \to \mathrm{EM}(G, n)$ is equivalent to a map in $M \to \mathrm{K}(G, n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Note that $\mathrm{EM}_\bullet(G, n) \simeq \mathrm{K}(\mathrm{Aut}G, 1)$. In the special case of $\mathrm{EM}(\mathbb{Z}, 1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\mathbb{Z}, 1)$ is contractible. This amounts to a choice of direction for all the circles, and so deserves to be renamed that M is *orientable*.

And so we can continue to work with the classifying space $\mathrm{EM}(G, 1)$ and study $\mathrm{K}(G, 1)$-fibrations, knowing that if the manifold is oriented then these are principal bundles.

2.2 Pathovers in principal bundles

Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$. Paths in a sigma type $\sum_{x:M} T(x)$ are given by pairs of paths: a path $p : x =_M y$ in the base, and a pathover $p' : \mathrm{tr}_p(x') =_{T(y)} y'$ between $x' : T(x)$ and $y' : T(y)$ in the fibers. We can't directly compare x' and y' since they are of different types, so we apply transport to one of them (which is asymmetrical, but equivalent to the alternatives). We say p' lies over p.

The individual fibers of T are polygons (the link of the vertex of which it is the fiber). Given a path $p : x =_M y$ in M, one of its pathovers consists of a path in $T(y)$. And given a face in M, a faceover is a homotopy from a pathover to refl.

3 Combinatorial manifolds

We will adapt to higher inductive types in a straightforward manner the classical construction of *combinatorial manifolds*. See for example the classic book by Kirby and Siebenmann[4]. These are a subclass of simplicial complexes.

Definition 4. An **abstract simplicial complex** M consists of a set M_0 of vertices, and for each $k > 0$ a set M_k of subsets of M_0 of cardinality $k + 1$, such that any $(j + 1)$-element subset of M_k is an element of M_j.

Note that we don't require all subsets of M_0 to be included – that would make M an individual simplex. A simplicial complex is a family of simplices that are identified along various faces.

Definition 5. In an abstract simplicial complex M, the **link** of a vertex v is the set containing every face $m \in M$ such that $v \notin m$ and $m \cup v$ is a face of M.

The link is all the neighboring vertices of v and the codimension 1 faces joining those to each other. See for example Figure 1.

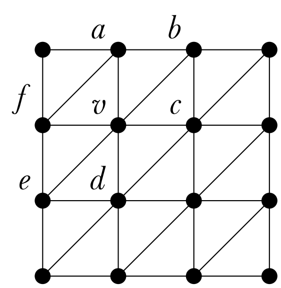

Figure 1: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$

.

Definition 6. A **combinatorial manfiold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a simplicial sphere of dimension $n - 1$ (i.e. its geometric realization is homeomorphic to an $n - 1$-sphere).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figures 2, 3, and 4 for some examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits a combinatorial triangulation[5]. So it appears reasonably well motivated to study this class of objects.

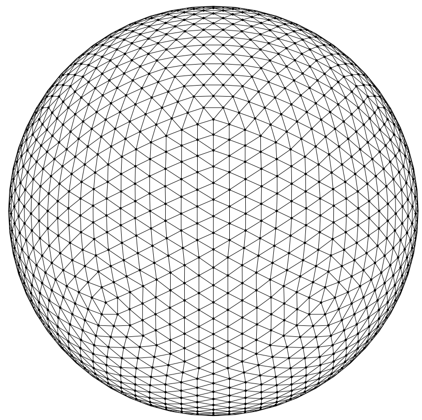

Figure 2: A combinatorial triangulation of a sphere, created with stripy.

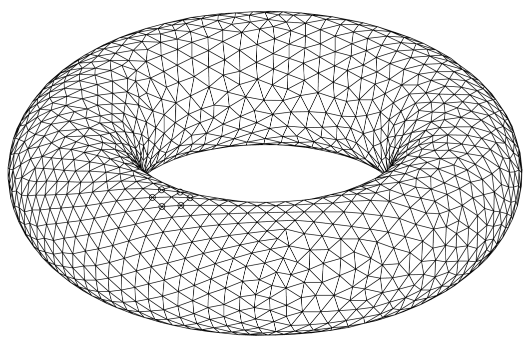

Figure 3: A torus with an interesting triangulation, from Wikipedia. The links have various vertex counts from 5-7. Clearly a constant value of 6 would also work. (By Ag2gaeh - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=30856793)

3.1 Higher inductive combinatorial manifolds

To convert a simplicial complex to a higher inductive type, we will convert the data in each classical dimension to a path constructor of the corresponding HoTT dimension. In order to specify the directionality of all the edges and faces, we need to first choose an ordering for each set in S:

Definition 7. The higher inductive type M' corresponding to the abstract simplicial complex M is given by

1. choosing an order for the elements of each M_k
2. vertices: a function $\mathsf{v}_0 : M_0 \to M'$ serving as the 0-dimensional constructors

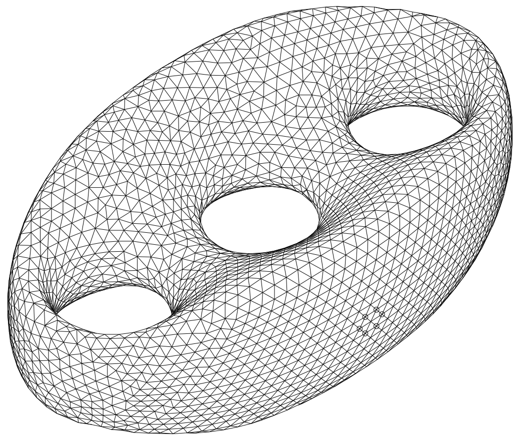

Figure 4: A 3-holes torus with triangulation, from Wikipedia. (By Ag2gaeh - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/wiki/File:Tri-brezel.svg)

3. edges: a function $\mathsf{v_1} : M_1 \to M'$ sending $\{a, b\} \mapsto \mathsf{v_0}(a) = \mathsf{v_0}(b)$
4. at dimension k, if $M_k = \{M_{k1}, \ldots, M_{kn}\}$, a path from $\mathsf{refl}_{M_{k1}}$ to the concatenation $\mathsf{v_{k-1}}(M_{k1}) \cdot \mathsf{v_{k-1}}(M_{k2}) \cdot \cdots \cdot \mathsf{v_{k-1}}(M_{kn})$).

We will drop the prime and work with higher inductive combinatorial manfiolds exclusively going forward.

3.2 Polygons

We will begin with a type that is important both for the domain and the codomain of mere circles: a square.

Definition 8. The higher inductive type C_4 (where C stands for "circle").

$$\begin{aligned} C_4 &: \mathsf{Type} \\ c_1, c_2, c_3, c_4 &: C_4 \\ c_1c_2 &: c_1 = c_2 \\ c_2c_3 &: c_2 = c_3 \\ c_3c_4 &: c_3 = c_4 \\ c_4c_1 &: c_4 = c_1 \end{aligned}$$

We may also think of C_4 as the join of the two-element sets $\{c_1, c_3\} * \{c_2, c_4\}$. The standard HoTT circle itself is a non-example of a combinatorial manifold since it lacks the second vertex of the edge:

Definition 9. The higher inductive type S^1:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

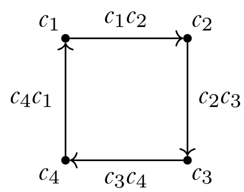

Figure 5: The HIT C_4.

Nonetheless, all polygons are equivalent to each other and to S^1.

Lemma 3. The function $\ell : C_4 \to S^1$ given by $\ell(c_i) = \mathsf{base}$ for all i, and $\ell(c_ic_j) = \mathsf{loop}$ for all i, j is an equivalence with inverse $\ell^{-1}(\mathsf{base}) = c_1$ and $\ell^{-1}(\mathsf{loop}) = c_1c_2 \cdot c_2c_3 \cdot c_3c_4 \cdot c_4c_1$. There are clearly other inverses for different choices of vertex.

Corollary 2. We have $(C_4, ||\ell||_{-1}) : \mathrm{EM}(\mathbb{Z}, 1)$.

Real-world triangulations of surfaces will often have links that are n-gons for a variable n. For example we can see hexagons and pentagons in Figure 2. This presumably introduces only a minor practical inconvenience and doesn't materially affect the discussion to come.

3.3 The higher inductive type $\mathbb{O}$

We will create our first combinatorial surface, a 2-sphere. We will adopt the convention that a subscript indicates the dimension of a subskeleton of a complex. For instance, we have $\mathsf{base} : S^1_0$.

Definition 10. The HIT $\mathbb{O}_0$ is just 6 points, intended as the 0-skeleton of an octahedron, with vertices named after the colors on the faces of a famous Central European puzzle cube.

$$w, y, b, r, g, o : \mathbb{O}_0$$

Definition 11. The HIT $\mathbb{O}_1$ is the 1-skeleton of an octahedron.

$$\begin{aligned}
w, y, b, r, g, o &: \mathbb{O}_1 \\
wb &: w = b \\
wr &: w = r \\
wg &: w = g \\
wo &: w = o \\
yb &: y = b \\
yr &: y = r \\
yg &: y = g \\
yo &: y = o \\
br &: b = r \\
rg &: r = g \\
go &: g = o \\
ob &: o = b
\end{aligned}$$

Definition 12. The HIT $\mathbb{O}$ is an octahedron:

$$
\begin{array}{llll}
w, y, b, r, g, o : \mathbb{O} & & & \\
& wb : w = b & br : b = r & wbr : wb \cdot br \cdot wr^{-1} = \mathsf{refl}_w \\
& wr : w = r & rg : r = g & wrg : wr \cdot rg \cdot wg^{-1} = \mathsf{refl}_w \\
& wg : w = g & go : g = o & wgo : wg \cdot go \cdot wo^{-1} = \mathsf{refl}_w \\
& wo : w = o & ob : o = b & wob : wo \cdot ob \cdot wb^{-1} = \mathsf{refl}_w \\
& yb : y = b & & yrb : yr \cdot rb \cdot yb^{-1} = \mathsf{refl}_y \\
& yr : y = r & & ygr : yg \cdot gr \cdot yr^{-1} = \mathsf{refl}_y \\
& yg : y = g & & yog : yo \cdot og \cdot yg^{-1} = \mathsf{refl}_y \\
& yo : y = o & & ybo : yb \cdot bo \cdot yo^{-1} = \mathsf{refl}_y
\end{array}
$$

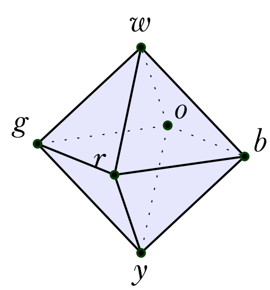

Figure 6: The HIT $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

We have obvious maps $\mathbb{O}_0 \xrightarrow{i_0} \mathbb{O}_1 \xrightarrow{i_1} \mathbb{O}$ that include each skeleton into the next-higher-dimensional skeleton.

3.4 The function link

Taking the link of a vertex gives us a map to polygons.

Definition 13. $\mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by induction:

$$
\begin{array}{ll}
\mathsf{link}(w) = brgo & \mathsf{link}(r) = wbyg \\
\mathsf{link}(y) = bogr & \mathsf{link}(g) = wryo \\
\mathsf{link}(b) = woyr & \mathsf{link}(o) = wgyb
\end{array}
$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

To extend link to the 1-skeleton, imagine how link changes as we slide from point to point. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 14. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending link as follows: Transport away from w:

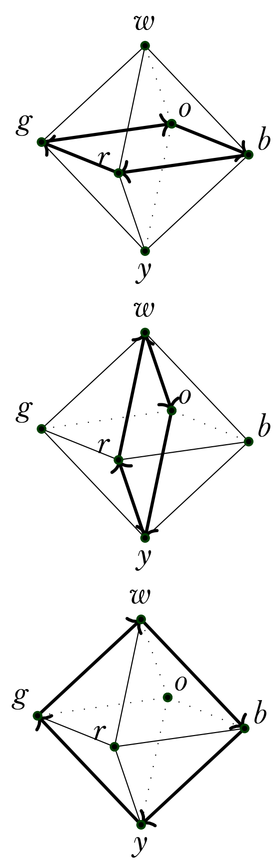

Figure 7: link for the verticies w, b and r.

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ (r, o fixed)
- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ (b, g fixed)
- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$
- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 7 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn.

Definition 15. The map $R : C_4 \to C_4$ rotates by one quarter turn, one "click":

- $R(c_1) = c_2$
- $R(c_2) = c_3$
- $R(c_3) = c_4$
- $R(c_4) = c_1$
- $R(c_1c_2) = c_2c_3$
- $R(c_2c_3) = c_3c_4$
- $R(c_3c_4) = c_4c_1$
- $R(c_4c_1) = c_1c_2$

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$.

Definition 16. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows:

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

where $H_R : R = \mathsf{refl}$ is the obvious homotopy.

All the faces do the same thing: they map to a homotopy between the identity and clockwise rotation by a quarter turn. Concatenating the eight faces in the 2-groupoid $\mathbb{O}$ would then map to a homotopy between the identity and two full rotations. This makes visible in HoTT the link between curvature and the Euler characteristic (which is 2 for the octahedron).

3.5 The torus

We can define a combinatorial torus as a similar HIT. This time each vertex will have six neighbors. So all the links will be merely equal to C_6 which is a hexagonal version of C_4. See Figure 8.

To help parse this figure, imagine instead Figure 9. We take this simple alternating-triangle pattern, then glue the left and right edges, then bend into Figure 8. The fact that each column in Figure 9 has four dots corresponds to the torus in Figure 8 having a square in front, diamonds in the middle, and a square in back.

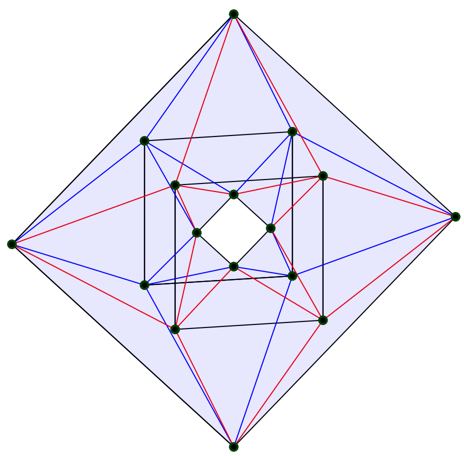

Figure 8: Torus embedded in 3-dimensional space. Black lines trace four square-shaped paths, red ones connect the front square to the middle diamonds, and blue ones connect the back path to the middle ones.

3.6 Vector fields

A vector field is a partial function $T_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ that lifts T. In other words, a pointing of some of the fibers. This aligns with the classical picture of a choice of nonzero vector at each point, except for some points where the vector field vanishes. We will have to omit such vanishing points from the domain because we are very opinionated that the codomain be mere circles, and so the function is a partial function.

Definition 17. If M is a combinatorial manifold and $Z \subset M_0$ is a set of vertices in M with members $Z = \{z_0, \ldots, z_n\}$, then denote by $M \setminus Z$ the type given by omitting the vertices in Z from the constructors in all dimensions where they appeared. Call the points of Z **isolated** if no two of them are neighbors, i.e. we have $\prod_{z:Z} \mathsf{link}(z) \cap Z = \emptyset$. In the isolated case $M \setminus Z$ has boundary circles where each vertex was removed.

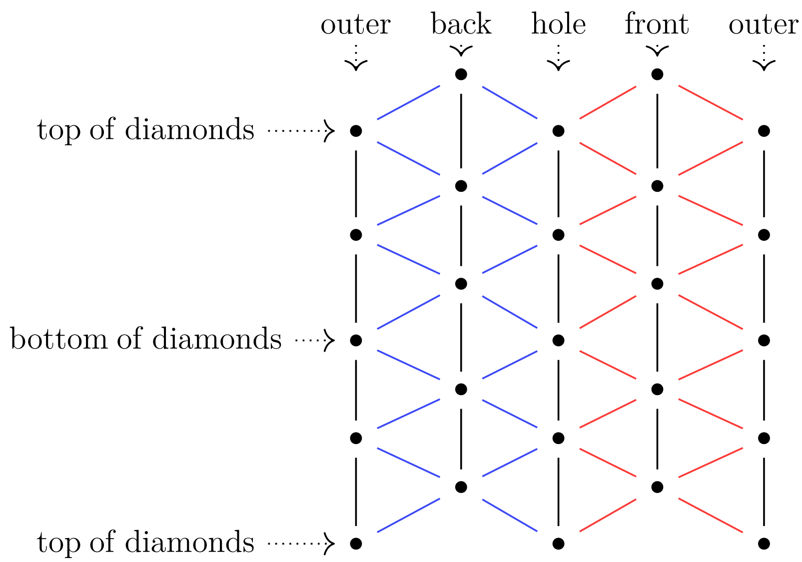

Figure 9: An inspiration for the torus. Identify the sides and then the top, definitionally, to get the actual torus.

Figure 10 illustrates what removing a point looks like. The resulting type is no longer a combinatorial manifold, since it fails the condition about every point having a circular link. We'll end up working around that when we need to.

Definition 18. Let M be a combinatorial manifold and Z an isolated set of vertices. A **vector field** X **on** M **with zero set** Z is a partial section of P, i.e. a term $X : \prod_{x:M\setminus Z} T(x)$. The **exponential map** $\exp : P \to M$ is the map sending points in a fiber to the corresponding point in the link of the base point: $\exp(x, y : \mathsf{link}(x)) = y$. In commutative diagram form we have:

$$\begin{tikzcd}
P \overset{\text{def}}{=} \sum_{C:TM} C \arrow[r, "\overline{T}"] \arrow[d, "\mathrm{pr}_1"] \arrow[d, bend left, "\exp"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"] \\
M \setminus Z \arrow[u, bend left, "X:\prod_{x:M\setminus Z} Tx"] \arrow[ur, "T_\bullet"] \arrow[r, "T"] & \mathrm{EM}(\mathbb{Z}, 1)
\end{tikzcd}$$

Where $T_\bullet = \overline{T} \circ X$. Note that exp is different from pr_1 since it spreads a fiber out onto the manifold. The composition $\exp \circ X$ is a map $M \setminus Z \to M$, and can be thought of as the flow of the vector field. It can be extended to a map $M \to M$ by taking the identity map on Z.

The vector field X is a map on all dimensions of $M \setminus Z$, not just the vertices. HoTT tells us that X also selects an "edgeover" for each edge, and "faceover" for each face. And these can be composed, so that we have an entire 2-groupoid $X(M \setminus Z)$ inside P. If $e_{12} : v_1 =_M v_2$ then $X(e_{12}) : \mathrm{tr}_{e_{12}}(X(v_1)) = X(v_2)$, which is a path in $T(v_2)$ that ends at the selected point $X(v_2) : T(v_2)$. The same goes for loops: given a loop $\ell : v_1 =_M v_1$ we have $X(\ell) : \mathrm{tr}_\ell(X(v_1)) = X(v_2)$, a path ending at $X(v_2) : T(v_2)$. It doesn't need to be a loop upstairs but it might be. Of course we always have $X(\mathsf{refl}_v) = \mathsf{refl}(X(v))$.

Faceovers are paths from $\mathsf{refl}_{X(v)}$ to a path $a = X(v)$ in some fiber $T(v)$. This amounts to contract-

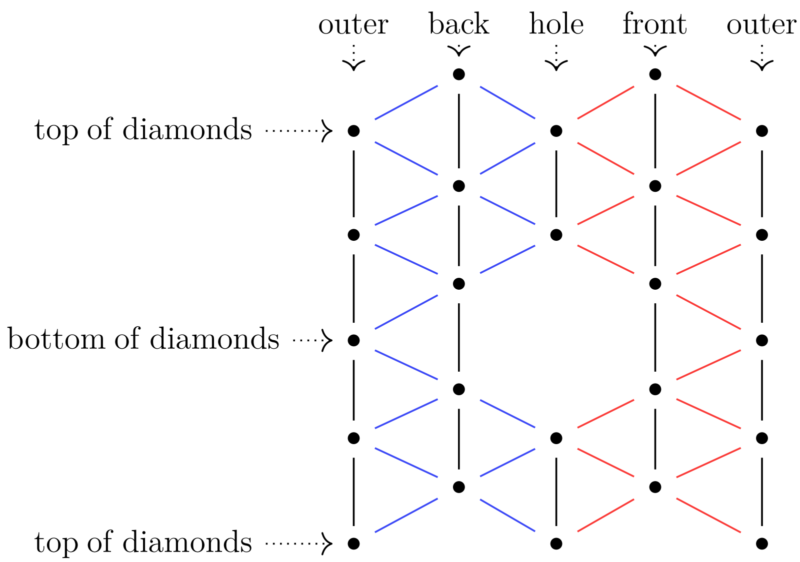

Figure 10: The flat torus with one vertex removed. This also removes the edges and faces containing that vertex.

ing the endpoint a back to $X(v)$ along the path, like a tape measure. In so doing we have squeezed a face of the manifold into the polygonal fiber $X(v)$.

Definition 19. The **index** of X at an isolated vertex $z : Z$ is the degree of the map $\exp \circ \mathrm{tr} : \mathsf{link}(z) \to \mathsf{link}(z)$. The map tr transports P and X along the spokes joining the link to the point z. Although z is not in the domain of X, the hypothesis of z being isolated implies that the link is in the domain.

Need pictures here. Show the classical examples of index +1 and -1.

4 Higher geometry

Here are the translations that are covered in the current paper:

Connections are infinitesimal splittings of a principal bundle.	Paths in a sigma type are equivalent to a pair of paths.
Differentials satisfy the Leibniz (product) rule.	Horizontal composition in an H-space is performed in two steps.
Connections with 0-truncated groups are covering spaces with unique flat connection.	Transport around contractible loops is refl when fibers are sets.
The group of gauge transformations (bundle automorphisms) acts on the space of connections.	Homotopies of classifying maps respect the splitting of paths in sigma types.

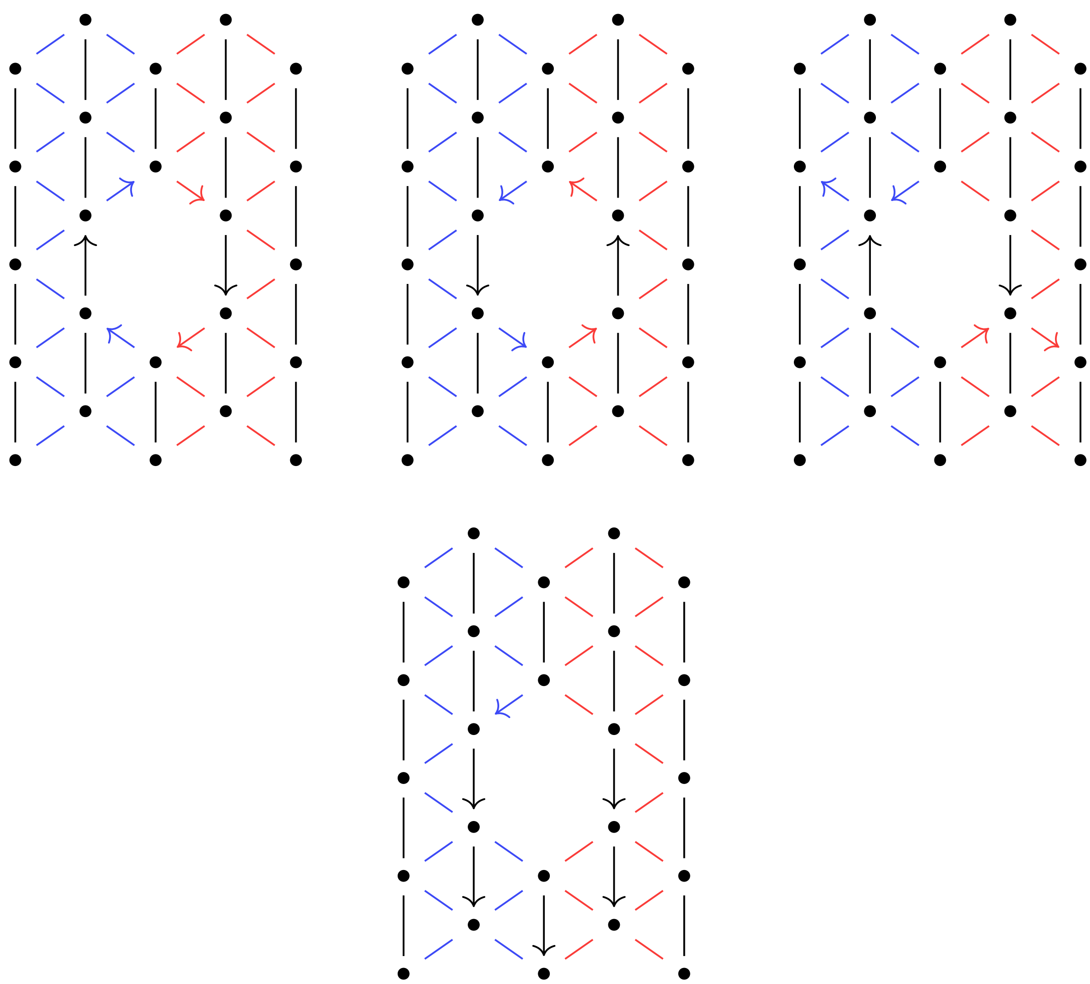

Figure 11: Four vector fields around a zero. The first has index 1 and circulates clockwise. The second also has index 1 despite circulating counterclockwise, because if you trace a clockwise path the vector also rotates clockwise. The third has index -1 since it circulates counterclockwise as you trace a clockwise path. The fourth has index 0.

4.1 Connections

The action on paths of the function $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ is what is called a connection. We'll spend some time drawing more complete parallels with the classical story, both to provide some evidence and to educate.

4.1.1 Connections as splittings

The classical story goes like this.

Definition 20. The **vertical bundle** VP of a principal bundle $\pi : P \to M$ with Lie group G is the kernel of the derivative $T\pi : TP \to TM$.

VP can be visualized as the collection of tangent vectors that point along the fibers. It should be clear that the group $\mathrm{Aut}P$ acts on VP: an automorphism $\phi : P \to P$ sends V_pP to $V_{\phi(p)}P$, where of course $\pi(p) = \pi(\phi(p))$.

Definition 21. An **Ehresmann connection** on a principal bundle $\pi : P \to M$ with Lie group G is a splitting $TP = VP \oplus HP$ at every point of P into vertical and "horizontal" subspaces, which is preserved by the action of $\mathrm{Aut}P$.

Being preserved by the action of $\mathrm{Aut}P$ means that the complementary horizontal subspaces in a given fiber of $\pi : P \to M$ are determined by the splitting at any single point in the fiber. The action of G on this fiber can then push the splitting around to all the other points.

The motivation for this definition is that we now have an isomorphism $T_p\pi : H_pP \to T_{\pi(p)}M$ between each horizontal space and the tangent space below it in M. This means that given a tangent vector at $x : M$ and a point p in $\pi^{-1}(x)$ we can uniquely lift the tangent vector to a horizontal vector at p. We can also lift vector fields and paths in this way. To lift a path $\gamma : [0, 1] \to M$ you must specify a lift for $\gamma(0)$ and then lift the tangent vectors of γ and prove that you can integrate the lift of that vector field upstairs in HP.

Then, armed with the lifting of paths one immediately obtains isomorphisms between the fibers of P. So the Ehresmann connection, the lifting of paths, and transport isomorphisms between fibers are all recapitulations of the structure that the connection adds to the bundle.

Moving now to HoTT, fix a type $M : \mathcal{U}$ and a type family $f : M \to \mathcal{U}$. Path induction gives us the transport isomorphism $\prod_{p:x=_My} \mathrm{tr}(p) : f(x) = f(y)$. We can use this to define a type of *dependent paths*, also called *pathovers* or *paths over* a given path.

Definition 22. With the above context and points $a : f(x), b : f(y)$ the type of **dependent paths over** p with endpoints a, b is denoted

$$a \xrightarrow[p]{=} b.$$

By induction we can assume p is refl_a in which case $a \xrightarrow[p]{=} b$ is $a =_{f(x)} a$.

See [6] for more discussion of dependent paths (where they use the term "path over"), including composition, and associativity thereof.

We recall now the identity type of sigma types:

Theorem 2. (HoTT book Theorem 2.7.2 [7]) If $f : M \to \mathcal{U}$ is a type family and $w, w' : \sum_{x:M} f(x)$ then there is an equivalence

$$\mathsf{split} : (w = w') \simeq \sum_{p:\mathrm{pr}_1(w) =_M \mathrm{pr}_1(w')} [\mathrm{tr}(p)(\mathrm{pr}_2(w))] = \mathrm{pr}_2(w').$$

In particular, given $p : x =_M y$ and $w : f(x)$ we have $w \xrightarrow[p]{=} tr(p)(w) \simeq tr(p)(w) =_{f(y)} tr(p)(w)$ which has the term refl which we can call "the horizontal lift of p starting at w." We can imitate the classical definition of a connection by defining $\omega \stackrel{\text{def}}{=} \mathrm{pr}_2 \circ \mathsf{split}$, the projection onto the vertical component. And thus in HoTT we can see the equivalence of transport and lifting of paths into horizontal and vertical components.

4.1.2 Covering spaces

If G is a 0-truncated group such as $\mathbb{Z}$ then the type of torsors (delooping) BG is 1-truncated. If $f : M \to BG$ is a type family then $\sum_{x:M} f(x)$ has fibers that are sets (G-torsors). So transport functions are set isomorphisms, and the transport of any contractible loop in M will be refl (the identity) of the fiber, which is what we mean by flat.

4.1.3 Gauge transformations

A *gauge transformation* is a term inherited from physics. It's an automorphism of a principal bundle $P \to M$, meaning a homeomorphism of P that commutes with the projection to M and so acts on each fiber. It is further required to be equivariant under the action of the group G, and so it's very similar to the act of multiplying each fiber by a continuously varying element of G.

In HoTT if the bundle is classified by $f : M \to \mathcal{U}$ then an automorphism is a homotopy $f \sim f$ and the group of gauge transformations is the loop space $\Omega_f(M \to \mathcal{U})$.

Recall that torsors have a physical interpretation as a quantity without a specified unit, such as mass, length, or time. When we choose a base point in a torsor it becomes the standard torsor G acting on itself (for example, the additive real numbers). A physicist is looking for properties or laws that are independent of such a choice. In the 20th century physicists further wondered about choices of units that vary from point to point, and began searching for laws that are invariant under this much larger space of transformations. And so they and we are led to explore quotienting by the action of the group of gauge transformations, and in particular the space of connections "mod gauge." In this scenario the base manifold M is spacetime, and a gauge transformation is a smoothly varying choice of gauge (units) at each point.

Gauge transformations act on connections. When we view connections as infinitesimal splittings of TP into vertical and horizontal sub-bundles, a gauge transformation that is constant in the neighborhood of a point will not change the splitting, it will just shift the fiber rigidly along itself, but one that is changing rapidly near a point will tilt the horizontal subspaces. So there are two effects: the effect of sliding along the fiber, and the effect of the rate of change of the gauge transformation. In classical geometry you'll see formulas like this:

Theorem 3. Let $P \to M$ be a principal bundle and $A \in \Omega^1(M, \mathfrak{g})$ a connection 1-form on P. Suppose that $H \in \mathscr{G}(P)$ is a bundle automorphism. Then H^*A is a connection 1-form and in a neighborhood U of a point $x \in M$ we can write H as a function $H_U : U \to G$ where $H_U(x) \in G$ is a group element multiplying the fiber at x, and then we have

$$H^*A = \mathrm{Ad}_{H_U(x)^{-1}} \circ A + H_U^*(\mu_G)$$

where $\mu_G : \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form on G.

This theorem is a combination of Theorems 5.2.2 and 5.4.4 in the excellent recent book on gauge theory for mathematicians interested in physics by Mark Hamilton[8].

It's not so important to fully understand this formula because we will re-explain it in HoTT terms in a moment. But notice that H^*A (the action of the gauge transformation on the connection 1-form) has contributions from two terms (both of which are vertical — connections always map onto the vertical bundle). The first is the adjoint action at the specific point x. This is always what we expect when we shift the base point in a torsor and look at the resulting group (or in this case, the Lie algebra). The second term involves the Maurer-Cartan form, which is the derivative of subtraction in the group. It takes tangent vectors at $g : G$ to a tangent vector at the identity (the Lie algebra, denoted $\mathfrak{g}$) by differentiating the action of multiplication by g^{-1}. If we think in terms of finite-length paths, then imagine a path $p : g = g'$ and the function $(g^{-1} \cdot -)$. The function will act on the path to give a path $g^{-1} \cdot p : e = (g' \cdot g^{-1})$ that starts at the identity. So the Maurer-Cartan form shifts paths to start at the identity by subtracting off the start point. Our Maurer-Cartan term is the pullback of the Maurer-Cartan form by H which records how H acts infinitesimally, i.e. the contribution from the gauge transformation H that comes from the rapidity of change from point to point. This term will be large when H_U has a large derivative.

In HoTT the connection's parallel transport is visible as the transport function, and the horizontal-vertical splitting is visible in the decomposition of paths in the sigma type (total space) into pairs of paths. What is the effect of applying a homotopy $H : f \sim f$ on transport, and on splitting?

H is a family of fiber automorphisms: $H : \prod_{a:M} f(a) = f(a)$ which we can assemble into an equivalence $H' : \sum_{a:M} f(a) = \sum_{a:M} f(a)$ that acts fiberwise. We want to compute the action of $\mathsf{ap}(H')$ on the horizontal-vertical decomposition of paths from Theorem 2 by computing $\omega \circ \mathsf{ap}(H') = \mathrm{pr}_2 \circ \mathsf{split} \circ \mathsf{ap}(H')$.

Denote $\sum_{a:M} f(a)$ by P. We'll adopt a convention of roman letters for structures in M and Greek for those upstairs in P. Let $p : a =_M b$ be a path in the base and let $\pi : (a, \alpha) =_P (b, \beta)$ be a path in P over p. Then $\omega(\pi) : \mathrm{tr}_p(\alpha) = \beta$.

Now let's apply H. We have $\mathsf{ap}(H')(\pi) : (a, H(a)(\alpha)) =_P (b, H(b)(\beta))$ which is still a path over p. Applying ω we get

$$\omega(\mathsf{ap}(H')(\pi)) : \mathrm{tr}_p(H(a)(\alpha)) = (H(b)(\beta))$$

. Using the lemma below we can if we wish rewrite this as

$$\omega(\mathsf{ap}(H')(\pi)) : H(b)\left[\mathrm{tr}_p(\alpha) = \beta\right]$$

which uses only $H(b)$.

Lemma 4. Given a function $f : M \to \mathcal{U}$ and homotopy $H : f \sim f$ the following square commutes and so in the type family we have $\mathrm{tr}(H(x) \cdot f(p)) = \mathrm{tr}(f(p) \cdot H(y))$.

$$\begin{tikzcd} f(a) \arrow[r, equal, "f(p)"] \arrow[d, equal, "H(a)"'] & f(b) \arrow[d, equal, "H(b)"] \\ f(a) \arrow[r, equal, "f(p)"'] & f(b) \end{tikzcd}$$

5 Leibniz, Gauss-Bonnet, Poincaré-Hopf

5.1 The Leibniz (product) rule

The Leibniz rule for exterior differentiation states that if $f, g : M \to \mathbb{R}$ are two smooth functions to the real numbers then $d(fg) = fdg + gdf$. Here fg is the function formed by taking the pointwise product of f and g. This is an interaction between multiplication in $\mathbb{R}$ and the action on vectors of smooth functions (the 1-forms df and dg).

To examine this situation in HoTT we need type-theoretic functions $f, g : M \to B$ from some type M to a central H-space B. Let $\mu : B \to B \to B$ be the H-space multiplication. How does μ act on paths? Suppose we have $a, a', b, b' : B$ and $p : a =_B a', q : b =_B b'$. Then we also have homotopies $\mu(p, -) : \mu(a, -) =_{B \to B} \mu(a', -)$ and $\mu(-, q) : \mu(-, b) =_{B \to B} \mu(-, b')$. Since $\mu(a, -) : B = B$ is an (unpointed) equivalence of B, and similarly for $\mu(b, -)$ and so on, this data assembles into the following diagram of higher groupoid morphisms:

$$\begin{tikzcd} B \arrow[r, bend left, "\mu(a,-)"{name=U}] \arrow[r, bend right, "\mu(a',-)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\mu(p,-)"'] & B \arrow[r, bend left, "\mu(-,b)"{name=U2}] \arrow[r, bend right, "\mu(-,b')"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\mu(-,q)"'] & B \end{tikzcd}$$

And so the two homotopies can be horizontally composed to give a path

$$\mu(p, -) \star \mu(-, q) : \mu(a, b) = \mu(a', b').$$

Horizontal composition is given by

$$\mu(p, -) \star \mu(-, q) \overset{\text{def}}{=} (\mu(p, -) \cdot_r \mu(-, b)) \cdot (\mu(a', -) \cdot_l \mu(-, q))$$

where

$$\mu(p, -) \cdot_r \mu(-, b) : \mu(a, b) = \mu(a', b)$$

and

$$\mu(a', -) \cdot_l \mu(-, q) : \mu(a', b) = \mu(a', b')$$

are defined by path induction. See the HoTT book Theorem 2.1.6 on the Eckmann-Hilton argument[7].

We can recognize the process of using whiskering to form horizontal composition in the Leibniz rule.

Quick aside: moving from infinitesimal calculus to finite groupoid algebra actually involves two changes. The first is the change from vectors to paths, forms to functions and so on. But it's also the case that tangent vectors have just the one basepoint, whereas paths have two endpoints. You can see this play out in this example, where a and a' were distinct points (and b and b').

The horizontal composition we build lives entirely in B and we didn't make use of M yet. The Leibniz rule will be a pointwise version of what's going on in B. Denote by $\mu \circ (f, g) : M \to B$ the map which sends $x \mapsto \mu(f(x), g(x))$.

Lemma 5. Given $f, g : M \to B$ and $p : x =_M y$ then

$$\begin{aligned} \mathsf{ap}(\mu \circ (f, g))(p) &= \mu(f(p), -) \star \mu(-, g(p)) \\ &= [\mu(f(p), -) \cdot_r \mu(-, g(x))] \cdot [\mu(f(y), -) \cdot_l \mu(-, g(p))] \\ &: \mu(f(x), g(x)) = \mu(f(y), g(y)) \end{aligned}$$

5.2 The total curvature

On the 2-face F bounding a loop $\ell : v =_M v$ the map $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ assigns a homotopy $T(F) : \mathsf{refl}_v = T(\ell)$, where $T(\ell)$ is an automorphism of $T(v)$. The various 2-faces can be composed as terms of the 2-groupoid M, and we will call the composition $T(M) : T(v) = T(v)$ the *total curvature*.

When the fibers are torsors we need loops to see what T is doing.

5.3 The total index

If we have $M \setminus Z$ for some isolated set of verticies Z, then for each $z : Z$ we can compose all the faces which contain z, forming a new face. In this way we produce an equivalent type $M_Z \simeq M$ but which is no longer combinatorial since we have erased some of the edges from some of the neighborhoods.

If X is a vector field with isolated zeroes on Z, then M_Z is a convenient replacement for M because X is a partial function defined on all of M_Z except for a collection of faces, each of which bounds one of the zeroes of Z.

If the combinatorial manifold was imported into type theory via some process of sampling or other, perhpaps more theoretical construction, then we should allow for zeroes to occur on verticies, edges, or faces. We can reduce the case of a zero on a vertex or edge to that of a face with the replacement method just described. The original combinatorial neighborhood structure continues to live in the fibers of T, it's only the base manifold that has been replaced.

The action of X around loops in M_Z respects the inverting of paths, i.e. traversing them backwards. Therefore in order to perform calculations we must also know the direction of the paths.

Definition 23. Given a connected loop $\ell : a =_{M_Z} a$ in M_Z that is running in the clockwise direction around a face of M_Z (i.e. the face is to the right of the path), we call the winding number of $X(\ell)$ the **index** of X around ℓ. If we drop the clockwise requirement, then the winding number is also known as the **degree** of the map X on ℓ. In other words index can be thought of as "clockwise degree."

5.4 Equality of total index and total curvature

If we compute $\mathrm{tr}_\ell X(a) = X(a)$ around every face boundary in M_Z then we can describe that in two ways. In homotopy type theory this is "the 1-skeleton-over," the pathover for the concatenation of every loop. It is also the total difference between the curvature around every loop, and the vector field. It is also zero.

Theorem 4. The total pathover of all face boundaries is refl_*, hence the total curvature is equal to the total index of X.

Proof. When we concatenate every loop we visit every edge twice in opposite directions. □

Corollary 3. The total index of a vector field with isolated zeroes is independent of the vector field.

Corollary 4. The total curvature is an integer.

The last step is to link this value to the Euler characteristic.

5.5 Identification with Euler characteristic

Combinatorial manifolds are intuitive objects that connect directly to the classical definition of Euler characteristic. We can argue using Morse theory, the study of smooth real-valued functions on smooth manifolds and their singularities. Classically the gradient of a Morse function is a vector field that can be used to decompose the manifold into its *handlebody decomposition*. This would be an excellent story to pursue in future work.

Imagine a combinatorial manifold of a genus g oriented surface standing upright with the holes forming a vertical sequence. Now install a vector field that points downward whenever possible. This vector field will have a zero at the top and bottom, and one at the top and bottom of each hole. The top and bottom will be index 1, and ones around the holes will be index -1.

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