$R: G\times E \longrightarrow E$

$TR: \mathfrak{g}\times E \longrightarrow TE$

$(u,X) \qquad \tilde{X}(u)$ fundamental vector field of X

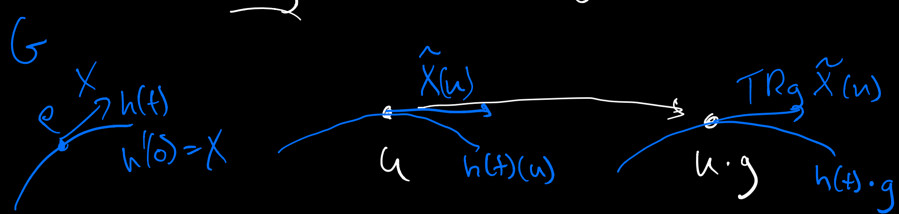
$(u,X) \overset{g}{\longmapsto} (ug, \mathrm{ad}_{g^{-1}}X)$

$$
\begin{tikzcd}
 & E\times\mathfrak{g} \arrow[r, "\cong"] \arrow[dl, "/G"'] \arrow[dd] & VE \arrow[r] & TE \arrow[r] \arrow[dl, "/G"'] \arrow[dd] & p^*TM \arrow[dl, "/G"'] \arrow[dd] \\
\mathrm{ad}E \arrow[rr] \arrow[dd] & & TE/G \arrow[r] \arrow[dd] & TM \arrow[dd] & \\
 & E \arrow[dl, "p"] & & E \arrow[dl] & E \arrow[dl] \\
M & & M & M &
\end{tikzcd}
$$

Lemma: $R: G\times E \longrightarrow E$

$\tilde{X}$ = f.v.f. of X

$$TR_g\, \tilde{X}(u) = \widetilde{\left(\mathrm{ad}\, g^{-1}(X)\right)}(u\cdot g)$$



$$
\begin{tikzcd}
 & \omega(v) & v & & \\
0 \arrow[r] & \mathrm{ad}E \arrow[r] \arrow[dr, bend right, "p_1"'] & TE/G \arrow[l, bend right, "\omega"'] \arrow[r] \arrow[d, shift left, "p_2"] & TM \arrow[r] \arrow[dl, "p_3"'] & 0 \\
 & & M \arrow[ul, bend left] \arrow[u, shift left] \arrow[ur, bend right] & &
\end{tikzcd}
$$

connection : $TE = VE \oplus HE$

$$\Gamma \mathrm{ad}E := \{ s: M \to \mathrm{ad}E \ \times \ p_1 \circ s = \mathrm{id}_M \}$$

$$\begin{tikzcd} 0 \arrow[r] & \underline{\Gamma \mathrm{ad}E} \arrow[r, hook] & \underline{\overbrace{\Gamma TE/G}^{G\text{-inv.}}} \arrow[r] \arrow[l, dashed, bend right, "?"'] & \underline{\Gamma TM} \arrow[r] \arrow[l, bend right, "\text{lift}"'] & 0 \end{tikzcd}$$

exact sequence of Lie algebras

$[X, Y]$ bracket of vector fields

$$F_\omega(X, Y) = \omega([\text{lift}\, X, \text{lift}\, Y])$$

$X, Y : \Gamma TM$ $\updownarrow$

$F_\omega(X, Y) = 0$ iff $[\text{lift}\, X, \text{lift}\, Y]$ is horizontal

A splitting of $\Gamma TE/G = \Gamma TM \oplus \Gamma \mathrm{ad}P$

would mean $Y, X : M \to TE/G$.

$X = (X_1, X_2)$ $\quad Y_1, X_1 : \Gamma \mathrm{ad}P$

$Y = (Y_1, Y_2)$ $\quad Y_2, X_2 : \Gamma TM$

$$[X, Y] \overset{?}{=} [[X_1, Y_1], \underline{[X_2, Y_2]}]$$

$\uparrow$

yes iff $F_\omega = 0$

$$\begin{tikzcd} E|_{U,V} \arrow[r, hook] \arrow[d] & E \arrow[d] \\ U,V \arrow[r, hook] & M \end{tikzcd}$$

U, V open in M

$U \cap V \neq \emptyset$

$\phi_U : E|_U \to U \times G$

$\phi_V : E|_V \to V \times G$

$\phi_U \circ \phi_V^{-1} : (U \cap V) \times G \to (U \cap V) \times G$

$\longrightarrow \phi_{uv} : u \cap v \longrightarrow G$

$\Big(u \cap v \longrightarrow GL(V)$ in a v.b. $\Big)$

$\rightarrow s : U \cup V \longrightarrow E$ section

$\phi_u \circ s : U \longrightarrow U \times G$

$\phi_v \circ s : V \longrightarrow V \times G$

$pr_2 \circ \phi_u \circ s : U \longrightarrow G$

$pr_2 \circ \phi_v \circ s : V \longrightarrow G$

$pr_2 \circ \phi_u \circ s = (pr_2 \circ \phi_v \circ s) \cdot \underline{\underline{\phi_{uv}}}$ on $U \cap V$

vertical vectors

$A : \Omega^1(U \cup V ; \mathfrak{g})$

$A|_u(m) : \mathfrak{g}$

$A|_v(m) : \mathfrak{g}$

$m : U \cap V$

pullback of forms

$A|_u = \phi_{uv}^{-1} A|_v \phi_{uv} + \phi_{uv}^{*}(\mu)$

$\mu : \Omega^1(G ; \mathfrak{g})$ Maurer-Cartan form

$TG \longrightarrow \mathfrak{g}$

$v_g \longmapsto T_{g^{-1}} v_g : T_e G$

$A|_u = g^{-1} A|_v g + g^{-1} dg$

Maurer-Cartan form on a matrix group

$g = e^A$

0)

$$\begin{tikzcd} E\times G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] & G \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ E \arrow[r] & M \arrow[r] & BG & * \arrow[l] \end{tikzcd}$$

Additional axioms

- open immersion classifier $\mathcal{O}$
- ring object R, R^n
- $R^n \to BG$ is constant

$$\begin{tikzcd} R^n \arrow[r] \arrow[dr] & BG \\ & * \arrow[u] \end{tikzcd}$$ commutes

BG^{R^n}

- Objects w/ atlases "good open covers"

$$\coprod_{\text{family}} R^n \to M$$

n:N fixed
D_i discrete

$$R^n \times D_1 \overset{s}{\underset{t}{\rightrightarrows}} R^n \times D_0 \to M$$

Groupoid, in fact equiv. rel. (quotient is 0-type)
s,t open immersions on $R^n \times (d:D_1)$
(c.f. quasi-compact, quasi-separated schemes)

$$
\begin{tikzcd}
R^n \times D_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & E \times G \arrow[r, "pr_2"] \arrow[d, "\text{action}"] \arrow[dr, phantom, "\lrcorner"] & G \arrow[d, shift left] \arrow[d, shift right] \\
R^n \times D_0 \arrow[r] \arrow[rr, bend left] \arrow[dr] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 1 \arrow[d] \\
\text{given} & M \arrow[r, "\phi"] & BG
\end{tikzcd}
$$

R^n, D_i projective 0-types w.r.t. $1 \longrightarrow BG$

weakly orthogonal

$$
R^n \to BG \quad \sim \quad
\begin{tikzcd}
P \cong R^n \times G \arrow[d] \\
R^n
\end{tikzcd}
$$

Theorem (Gleason) 2.1. *Extremally disconnected topological spaces are precisely the projective objects in the category of compact Hausdorff topological spaces.*

1) $Y \to X$ special class of maps s.t. locally triv.

"open covers"

Y "=" $\coprod U_i$

$$\begin{tikzcd} Y\times G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d] & * \arrow[d] \\ Y \arrow[r] & X \arrow[r] & BG \end{tikzcd}$$

2) open inclusions $U \hookrightarrow X$ s.t. (axioms about open, covering)

$$\begin{tikzcd} U\times G \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & * \arrow[d] \\ U \arrow[r] & X \arrow[r] & BG \end{tikzcd}$$