DRAFT: Discrete differential geometry in homotopy type theory

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Summary

This work brings to HoTT

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature = total index

Classical → HoTT

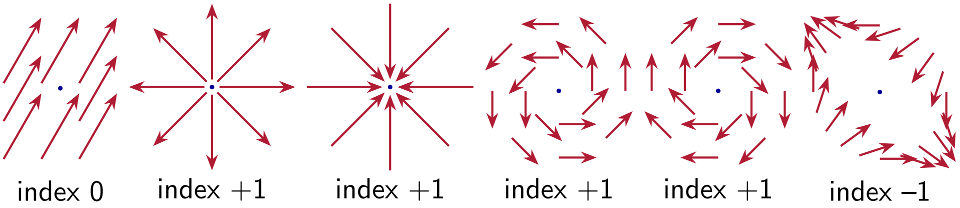
Let M be a smooth 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes $x_1, \ldots, x_n$.

$$
\begin{array}{ccccc}
\frac{1}{2\pi}\int_M F_A & = & \sum_{i=1}^{n} \mathrm{index}_X(x_i) & = & \chi(M) \\
\downarrow & & \downarrow & & \\
\Omega\left(\sum_{\text{faces } f} \flat_f\right) & = & \sum_{\text{faces } f} I_f^X & &
\end{array}
$$

Classical index

Near an isolated zero there are only three possibilities: index 0, 1, –1.

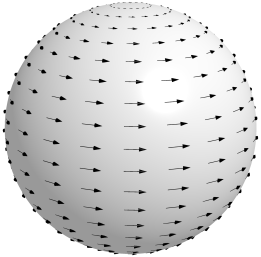
Index is the winding number of the field as you move clockwise around the zero.



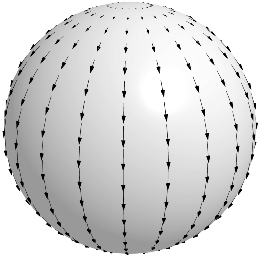
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index $+1$ at each pole $=$ **2**

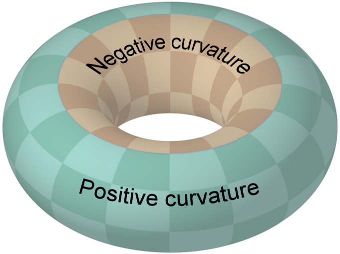


Height: index $+1$ at each pole $=$ **2**

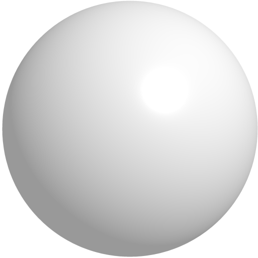
Gauss-Bonnet theorem

Curvature in 2D is a function $F_A : M \to \mathbb{R}$.

$\int_M F_A$ sums the values at every point.



Positive and negative curvature cancel: **0**



Constant curvature 1, area 4π: **2**

Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

HoTT background

1. Symmetry,
   Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
   `https://github.com/UniMath/SymmetryBook`.
2. Central H-spaces and banded types,
   Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
   arXiv:2301.02636
3. Nilpotent types and fracture squares in homotopy type theory,
   Scoccola, L. (2020)
   MSCS 30(5). arXiv:1903.03245

Manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization procedure to construct types

Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \ldots, M_k]$ the k-skeleton of M.

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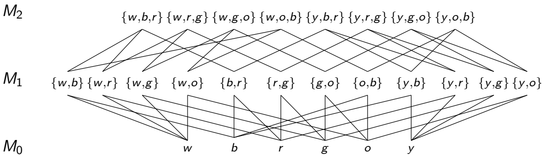
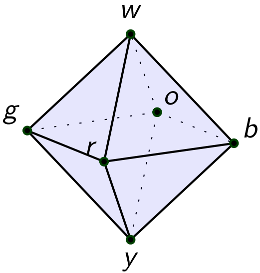
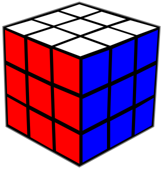


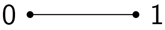




Simplicial complexes

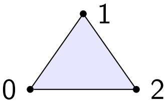
Example

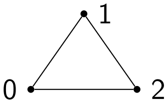
The complete simplex of dimension n, denoted $\Delta(n)$, is the set $\{0, \ldots, n\}$ and its power set. The $(n-1)$-skeleton $\Delta(n)_{\leq(n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$-sphere.

$\Delta(1)$ is visually 

, $\partial\Delta(1)$ is visually 

,

$\Delta(2)$ is visually 

, $\partial\Delta(2)$ is visually 

Homotopy realization: dimension 0

We will realize simplicial complexes by means of a sequence of pushouts.

Base case: the realization $\mathbb{M}$ of a 0-dimensional complex M is M_0.

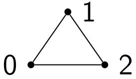
In particular the 0-sphere $\partial\mathbb{\Delta}(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$.

Homotopy realization: dimension 1

For a 1-dim complex $M \stackrel{\text{def}}{=} [M_0, M_1]$ the realization is given by

$$\begin{tikzcd} M_1 \times \partial\Delta(1) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \arrow[dl, phantom, "\Nearrow h_1"] \\ M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \end{tikzcd}$$

Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\boldsymbol{\Delta}(2) \overset{\text{def}}{=}$ 

is given by

$$\begin{tikzcd}
\partial\Delta(2)_1 \times \partial\boldsymbol{\Delta}(1) \arrow[r] \arrow[d] & \partial\Delta(2)_1 \arrow[d] \\
\partial\Delta(2)_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \partial\boldsymbol{\Delta}(2) \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

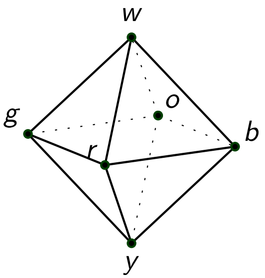
i.e.

$$\begin{tikzcd}
\{\{0,1\},\{1,2\},\{2,0\}\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{0,1\},\{1,2\},\{2,0\}\} \arrow[d] \\
\{0,1,2\} \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \partial\boldsymbol{\Delta}(2) \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron $\mathbb{O}$:

$$\begin{tikzcd}
\{\{w,g\},\ldots\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{w,g\},\ldots\} \arrow[d] \\
\{w,g,\ldots\} \arrow[r] \arrow[ur, phantom, "\Rightarrow h_1"] & \mathbb{O}_1 \arrow[ul, phantom, very near start, "\lrcorner"]
\end{tikzcd}$$



Homotopy realization: dimension 2

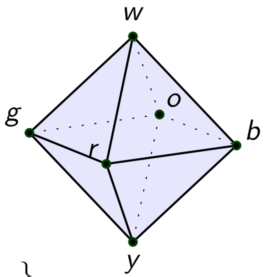
To realize $M \overset{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\mathbf{\Delta}(1), \partial\mathbf{\Delta}(2)$:

$$
\begin{tikzcd}
M_1 \times \partial\mathbf{\Delta}(1) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[dr, phantom, "\Searrow h_2"] & \mathbb{M}_2 \\
 & M_2 \times \partial\mathbf{\Delta}(2) \arrow[u, "\mathbb{A}_1"] \arrow[r, "\mathrm{pr}_1"'] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

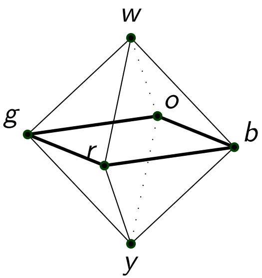
Homotopy realization: dimension 2

The full octahedron $\mathbb{O}$:

$$
\begin{tikzcd}
\{\{w,g\},\ldots\}\times\{0,1\} \arrow[r, "\mathrm{pr}_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow h_1"] & \{\{w,g\},\ldots\} \arrow[d] & \\
\{w,g,\ldots\} \arrow[r] & \mathbb{O}_1 \arrow[r] \arrow[dr, phantom, "\Rightarrow h_2"] & \mathbb{O}_2 \\
& \{\{w,g,r\},\ldots\}\times\partial\Delta(2) \arrow[u] \arrow[r, "\mathrm{pr}_1"'] & \{\{w,g,r\},\ldots\} \arrow[u]
\end{tikzcd}
$$



Homotopy realization: dimension 2



$\leftarrow \mathrm{link}(w)$

The link of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

A combinatorial manifold is a simplicial complex all of whose links are* simplicial spheres.

This will be our model of the tangent space.

*the (classical) geometric realization is homeomorphic to a sphere

Combinatorial manifolds $\leftrightarrow$ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$-sphere, i.e. its geometric realization is an $(n-1)$-sphere.

`https://ncatlab.org/nlab/show/triangulation+theorem`

Combinatorial manifolds $\leftrightarrow$ smooth manifolds

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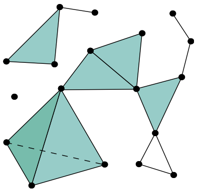
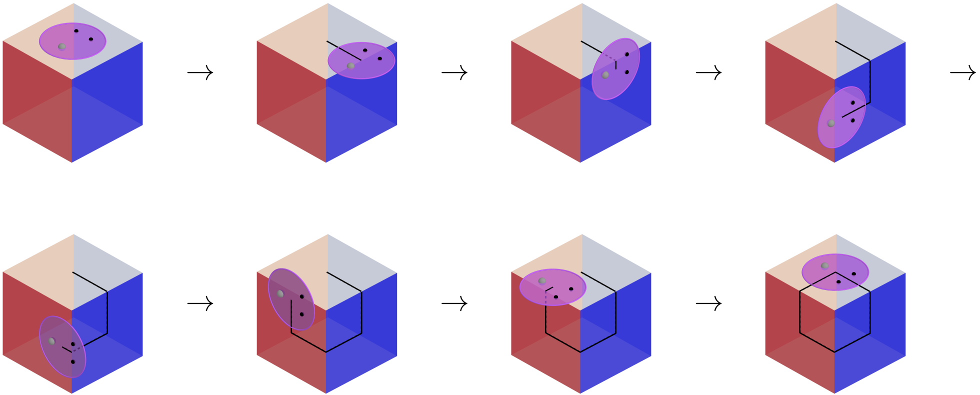
`https://ncatlab.org/nlab/show/triangulation+theorem`

Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:

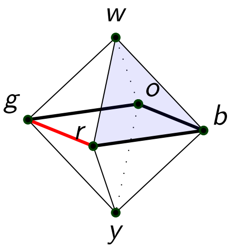


What we hope to capture and explain



$T : \mathbb{M} \to \mathsf{EM}(\mathbb{Z}, 1)$ extending link

We define T on edges by imaginging tipping:



$\mathrm{tr}(\partial(wbr)) : Tw = Tw$ is clockwise rotation by one notch.

We define T on the face wbr by the shortest homotopy $T(wbr) : \mathrm{id} = \mathrm{tr}(\partial(wbr))$.

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a realization and all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"'] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"] \\
& & \mathcal{U} & &
\end{tikzcd}$$

- The map f_k is a k-bundle on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-connection on the $(k-1)$-bundle f_{k-1}.

Definition (cont.)

$$
\begin{tikzcd}
M_k \times \partial\Delta^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\overset{h_k}{\Nearrow}"] & M_k \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r, "\imath_{k-1}"] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \\
 & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{F\} \times \partial\Delta^2 \arrow[r, "!"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\underset{\flat_F}{\Swarrow}"] & \mathbf{1} \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}
$$

the filler $\flat_F$ is called a flatness structure for the face F, and its ending path (the holonomy around the boundary) is called the k-curvature at the face F.

With these definitions we have now achieved one of our main goals.

Without a definition of Euler characterisitc we can't prove Gauss-Bonnet.

But once we add vector fields there is a lot more to say.

What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of torsors, called principal bundles.

Torsors

Let G be a (higher) group.

Definition

- A right G-object is a type X equipped with a homomorphism $\phi : G^{\mathrm{op}} \to \mathrm{Aut}(X)$.
- If we have a proof of

$$\mathrm{is\_torsor}(X, \phi) \stackrel{\mathrm{def}}{=} \|X\|_{-1} \times \prod_{x:X} \mathrm{is\_equiv}(\phi(-, x))$$

  we say (X, ϕ) is a G-torsor. Denote the type of G-torsors by BG.
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

Facts

- $\Omega(BG, G_{\mathrm{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.
- BG is connected.
- Previous 2 $\implies$ BG is a $\mathrm{K}(G, 1)$.
- $\mathrm{ev}(e) : (G_{\mathrm{reg}} =_{BG} X) \to X$ is an equivalence.
- See the Buchholtz et. al. H-spaces paper for more.

A connected component of $\mathcal{U}$?

Definition

The type of Eilenberg-Mac Lane spaces $\mathrm{EM}(G, n)$ is the connected component of $\mathrm{K}(G, n)$:

$$\mathrm{EM}(G, n) \stackrel{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \stackrel{\text{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$$

It is a property of a map $f : A \to \mathrm{EM}(G, n)$ to factor through $\mathrm{K}(G, n+1)$. See the Scoccola paper.

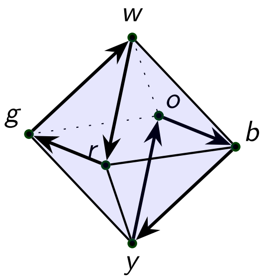
Coincidences of 2 dimensions

- S^1 is a $\mathsf{K}(\mathbb{Z}, 1)$ since $\Omega(S^1, \mathsf{base}) \simeq \mathbb{Z}$.
- So $\mathsf{EM}(\mathbb{Z}, 1)$ is a type of mere circles.
- But $S^1 =_{\mathsf{EM}(\mathbb{Z},1)} S^1$ contains an order 2 flip, so $\not\simeq S^1$.
- For a map $f : A \to \mathsf{EM}(\mathbb{Z}, 1)$ to factor through $\mathsf{K}(\mathbb{Z}, 2)$, it must somehow avoid flips.
- This deserves to be called orientability.
- $\mathsf{link} : \mathbb{M}_0 \to \mathsf{EM}(\mathbb{Z}, 1)$ is a great starting point.

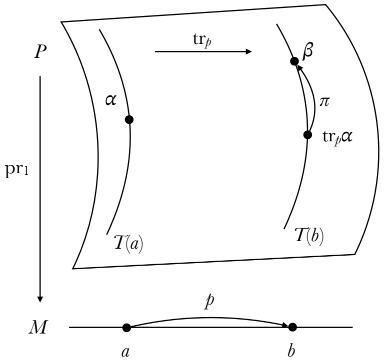
Vector fields

Let $T : \mathbb{M}_2 \to \mathrm{K}(\mathbb{Z}, 2)$ be an oriented tangent bundle on a 2-dim cellular type

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It's a nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.



Pathovers

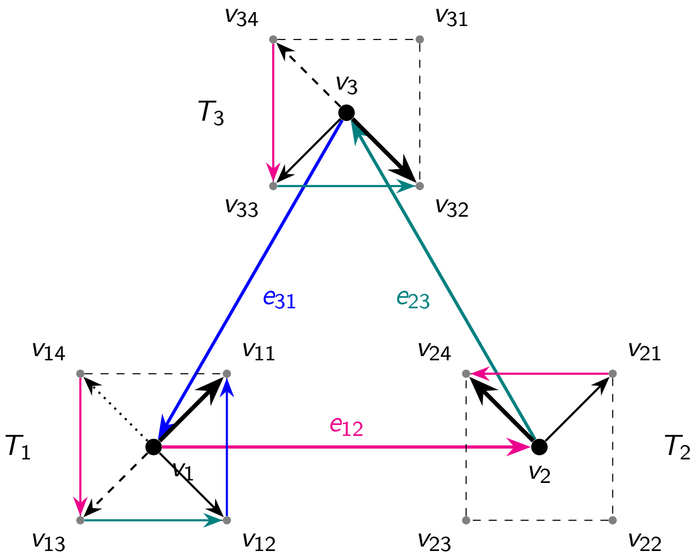


- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display it.
- Dependent functions map paths to pathovers (apd).

Building up a triangle-over

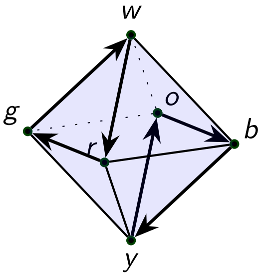
$$
\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & T_{13}T_{32}T_{21}X_1 \arrow[d, equal, "T_{13}T_{32}X_{12}:"'] \\
 & & T_{32}T_{21}X_1 \arrow[d, equal, "T_{32}X_{12}:"'] & T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}X_{23}:"'] \\
 & T_{21}X_1 \arrow[d, equal, "X_{12}:"'] & T_{32}X_2 \arrow[d, equal, "X_{23}:"'] & T_{13}X_3 \arrow[d, equal, "X_{31}:"'] \\
X_1 & X_2 & X_3 & X_1
\end{tikzcd}
$$



- $\partial F \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$.
- tr thins out arrows.
- X on a path is drawn in the path's color.
- $X(\partial F)$ traces 3 sides of a square.

Angle



- We want to extract from each X_{ij} just the angle, a non-dependent quantity.
- e.g. in this example: 3 copies of "+1 notch" and 3 of "–1 notch."
- The total swirled angle is 0.

Angle

Observation 1: Use the torsor structure. If we choose $m : \mathbb{M}$ then $T_m = T_m$ acts on all fibers. We can define subtraction $T_i \times T_i \to (T_m = T_m)$.

Angle

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Observation 2: Use the vector field. Given $X_i : T_i$ we can form subtraction $-X_i : T_i \to (T_m = T_m)$. $X_{ij} - X_j : T_{ji} X_i - X_j =_{T_m = T_m} 0$.

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Observation 3: Use ap of addition. We can add $\alpha : a =_{\mathbb{C}(4)} 0$ and $\beta : b =_{\mathbb{C}(4)} 0$ to form $\alpha + \beta : (a + b) =_{\mathbb{C}(4)} 0$.

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Observation 3: Use ap of addition. We can add $\alpha : a =_{\mathbb{C}(4)} 0$ and $\beta : b =_{\mathbb{C}(4)} 0$ to form $\alpha + \beta : (a + b) =_{\mathbb{C}(4)} 0$.

Together these remove the dependency. We can compute $\flat, I, X$ on each face independently and total them in $T_m = T_m$.

Lemma

If G is a higher group with multiplication $\mu : G \times G \to G$ and proof of commutativity is_comm : $\prod_{a,b:G} \mu(a,b) = \mu(b,a)$ *then μ induces a function* $\mu_= : (x =_G y) \times (x' =_G y') \to (\mu(x,x') =_G \mu(y,y'))$.

Proof.

If $p : x =_G y$ and $p' : x' =_G y'$, then we can define $\mu_=(p,p')$ by concatenating the three paths

$$\begin{aligned} \mu(x',p) &: \mu(x',x) =_G \mu(x',y) \\ \text{is\_comm}(x',y) &: \mu(x',y) =_G \mu(y,x') \\ \mu(y,p') &: \mu(y,x') =_G \mu(y,y'). \end{aligned}$$

$\square$

$$
\begin{array}{lll}
\mathrm{tr}_F \overset{\mathrm{def}}{=} \mathrm{tr}(\partial F) & : T_1 =_{\mathsf{K}(\mathbb{Z},2)} T_1 & \text{holonomy} \\
\flat_F \overset{\mathrm{def}}{=} \flat(\partial F) & : \mathrm{id} =_{T_1 = T_1} \mathrm{tr}_F & \text{flatness} \\
X_F \overset{\mathrm{def}}{=} X(\partial F) & : \mathrm{tr}_F(X_1) =_{T_1} X_1 & \text{swirling}
\end{array}
$$

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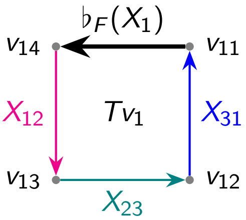
\end{array}
$$

Definition

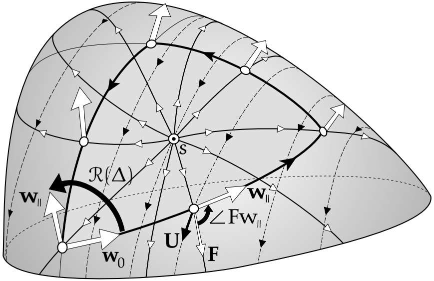
The index of the vector field X on the face F is the integer

$$I_F^X \overset{\text{def}}{=} \Omega(\flat_F(X_1) \cdot X_F) : \Omega(X_1 =_{T_1} X_1).$$



- On a single face we have $I_F^X = \Omega(\flat_F(X_1) \cdot X_F)$.
- Map $\flat_F(X_1)$ and X_F to angles in $T_m = T_m$.
- Sum over faces can be performed in $T_m = T_m$.
- Assume that each edge is traversed twice, once in each direction.
- Prove that the total angle $\sum_F X_F = 0$.
- Leaving us $I_{\text{tot}} = \Omega(\flat_{\text{tot}})$.

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi\,\mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F\mathrm{w}_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Thank you.