DRAFT: Discrete differential geometry in homotopy type theory

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Summary

Summary

This work brings to HoTT

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature $=$ total index

Classical → HoTT

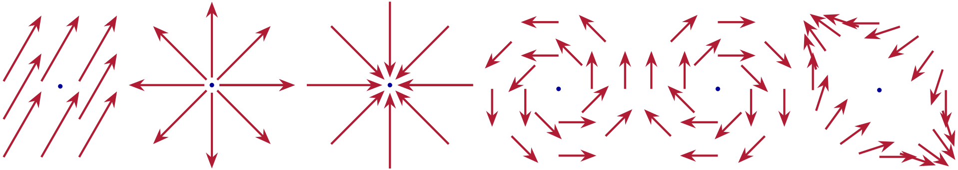
Let M be a smooth 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes $x_1, \ldots, x_n$.

$$
\begin{array}{ccc}
\displaystyle \frac{1}{2\pi}\int_M F_A & = & \displaystyle \sum_{i=1}^{n} \mathrm{index}_X(x_i) = \chi(M) \\
\downarrow & & \downarrow \\
\displaystyle \Omega\left(\sum_{\text{faces } f} \flat_f\right) & = & \displaystyle \sum_{\text{faces } f} I_f^X
\end{array}
$$

Classical index

Near an isolated zero there are only three possibilities: index 0, 1, –1.

Index is the winding number of the field as you move clockwise around the zero.

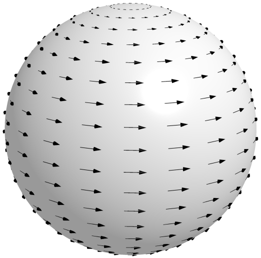


index 0 index +1 index +1 index +1 index +1 index –1

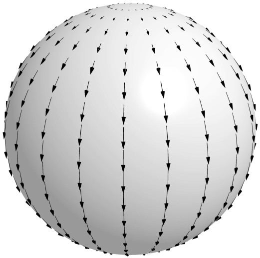
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index $+1$ at each pole $= \mathbf{2}$

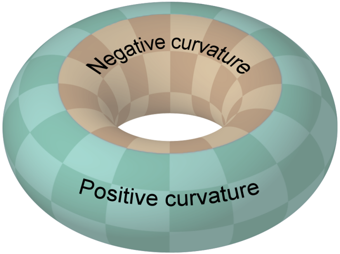


Height: index $+1$ at each pole $= \mathbf{2}$

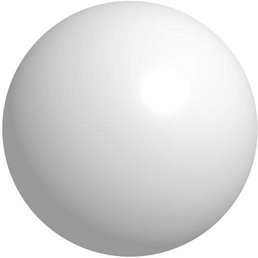
Gauss-Bonnet theorem

Curvature in 2D is a function $F_A : M \to \mathbb{R}$.

$\int_M F_A$ sums the values at every point.



Positive and negative curvature cancel: **0**



Constant curvature 1, area 4π: **2**

Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

HoTT background

1. Symmetry,
   Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
   `https://github.com/UniMath/SymmetryBook`.
2. Central H-spaces and banded types,
   Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
   arXiv:2301.02636
3. Nilpotent types and fracture squares in homotopy type theory,
   Scoccola, L. (2020)
   MSCS 30(5). arXiv:1903.03245

Combinatorial manifolds

Manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization procedure to construct types

Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \overset{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \overset{\text{def}}{=} [M_0, \ldots, M_k]$ the k-skeleton of M.

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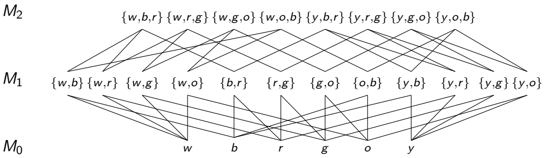
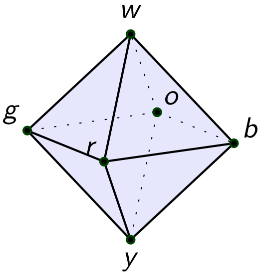
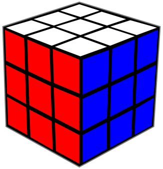


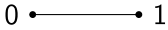




Simplicial complexes

Example

The complete simplex of dimension n, denoted $\Delta(n)$, is the set $\{0, \ldots, n\}$ and its power set. The $(n-1)$-skeleton $\Delta(n)_{\leq(n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$-sphere.

$\Delta(1)$ is visually 

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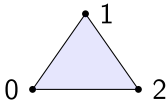
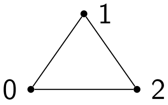
,

$\Delta(2)$ is visually 

, $\partial\Delta(2)$ is visually 

Homotopy realization: dimension 0

We will realize simplicial complexes by means of a sequence of pushouts.

Base case: the realization $\mathbb{M}$ of a 0-dimensional complex M is M_0.

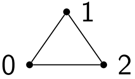
In particular the 0-sphere $\partial\mathbf{\Delta}(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$.

Homotopy realization: dimension 1

For a 1-dim complex $M \stackrel{\text{def}}{=} [M_0, M_1]$ the realization is given by

$$\begin{tikzcd}
M_1 \times \partial\Delta(1) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \arrow[dl, phantom, "\Uparrow h_1"] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\mathbb{\Delta}(2) \overset{\text{def}}{=}$ 

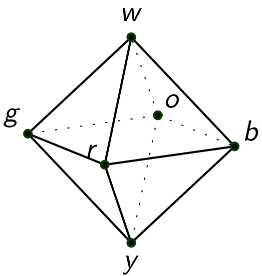
is given by

$$\begin{tikzcd}
\partial\Delta(2)_1 \times \partial\mathbb{\Delta}(1) \arrow[r] \arrow[d] & \partial\Delta(2)_1 \arrow[d] \\
\partial\Delta(2)_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \partial\mathbb{\Delta}(2)
\end{tikzcd}
\quad \text{i.e.} \quad
\begin{tikzcd}
\{\{0,1\},\{1,2\},\{2,0\}\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{0,1\},\{1,2\},\{2,0\}\} \arrow[d] \\
\{0,1,2\} \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \partial\mathbb{\Delta}(2)
\end{tikzcd}$$

Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron $\mathbb{O}$:

$$\begin{tikzcd}
\{\{w,g\},\ldots\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{w,g\},\ldots\} \arrow[d] \\
\{w,g,\ldots\} \arrow[r] \arrow[ur, Rightarrow, "h_1"'] \arrow[r, phantom, near end, "\lrcorner"] & \mathbb{O}_1
\end{tikzcd}$$



Homotopy realization: dimension 2

To realize $M \overset{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\mathbf{\Delta}(1), \partial\mathbf{\Delta}(2)$:

$$
\begin{tikzcd}
M_1 \times \partial\mathbf{\Delta}(1) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\overset{h_1}{\Nearrow}"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[dr, phantom, "\overset{h_2}{\Searrow}"] & \mathbb{M}_2 \\
& M_2 \times \partial\mathbf{\Delta}(2) \arrow[u, "\mathbb{A}_1"] \arrow[r, "\mathrm{pr}_1"'] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

Homotopy realization: dimension 2

The full octahedron $\mathbb{O}$:

$$
\begin{tikzcd}
\{\{w,g\},\dots\} \times \{0,1\} \arrow[r, "\mathrm{pr}_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow h_1"] & \{\{w,g\},\dots\} \arrow[d] & \\
\{w,g,\dots\} \arrow[r] & \mathbb{O}_1 \arrow[r] \arrow[dr, phantom, "\Rightarrow h_2"] & \mathbb{O}_2 \\
& \{\{w,g,r\},\dots\} \times \partial\Delta(2) \arrow[r, "\mathrm{pr}_1"'] \arrow[u] & \{\{w,g,r\},\dots\} \arrow[u]
\end{tikzcd}
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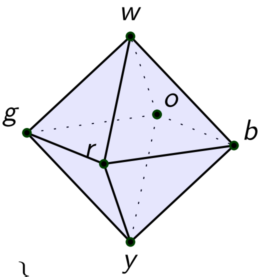
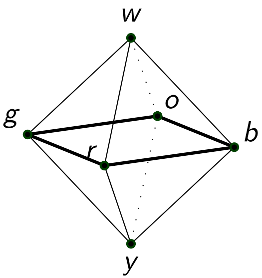


Homotopy realization: dimension 2



$\leftarrow$ link(w)

The link of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

A combinatorial manifold is a simplicial complex all of whose links are* simplicial spheres.

This will be our model of the tangent space.

*the (classical) geometric realization is homeomorphic to a sphere

Combinatorial manifolds $\leftrightarrow$ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$*-sphere, i.e. its geometric realization is an* $(n-1)$*-sphere.*

`https://ncatlab.org/nlab/show/triangulation+theorem`

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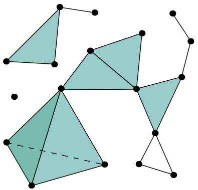
Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:



Torsors

What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of torsors, also called principal bundles.

Torsors

Let G be a (higher) group.

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- The inverse is (pr_1, s) where $s : X \times X \to G$ is called subtraction (when G is commutative).
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

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3. 1 & 2 $\implies$ BG is a $\mathrm{K}(G, 1)$.
4. $\mathrm{ev}(e) : (G_{\text{reg}} =_{BG} X) \to X$ is an equivalence (needed when we have vector fields).

See the Buchholtz et. al. H-spaces paper for more.

How to map into BS^1

To construct maps into BS^1 we lift a family of mere circles. (Remark: the lift is a choice of orientation.)

families of:

$$
\begin{tikzcd}
BS^1 \arrow[r] & \mathrm{BAut}(S^1) \arrow[r] & \mathcal{U} \\
& \mathbb{M} \arrow[ul, "\text{torsors}"] \arrow[u, "\text{mere circles}"] \arrow[ur, "\text{types}"'] &
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We will assume we have such a lift when we need it.

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Other names:

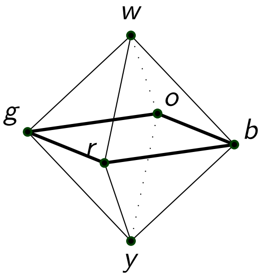
- $\mathrm{BAut}(S^1) = BO(2) = \mathrm{EM}(\mathbb{Z}, 1)$ (where $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n))$)
- $BS^1 = BSO(2) = \mathrm{K}(\mathbb{Z}, 2)$

Connections and curvature

Connections

Connections are extensions of the bundle to higher skeleta.

Recall link



$\leftarrow \text{link}(w)$

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Connections on the tangent bundle

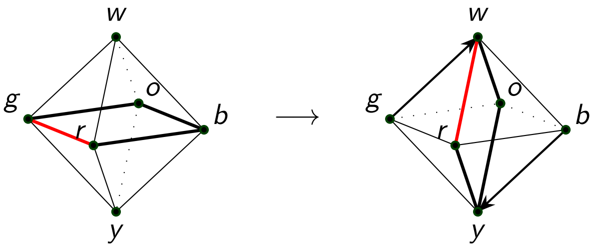
An extension T_1 of link to $\mathbb{M}_1$ is called a connection on the tangent bundle.

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r] \arrow[dr, "\mathrm{link}"'] & \mathbb{M}_1 \arrow[r] \arrow[d, dashed, "T_1"] & \mathbb{M}_2 \arrow[dl, dashed] \\
& \mathrm{BAut}(S^1) &
\end{tikzcd}$$

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

We will define T_1 on the edge wb, so we need a term $T_1(wb) : \mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(b)$.

We imagine tipping:

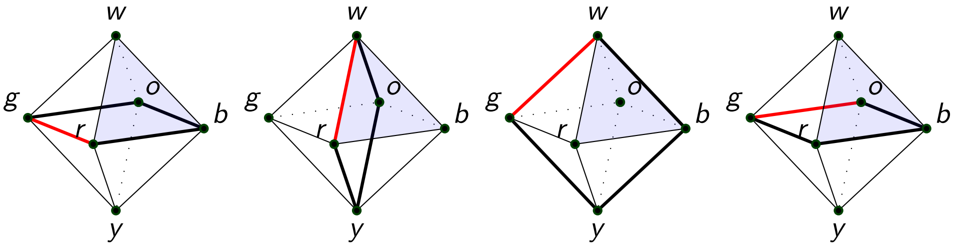


$$T_1(g : \mathrm{link}(w)) \stackrel{\mathrm{def}}{=} w : \mathrm{link}(b), \ldots.$$

Use this method to define T_1 on every edge.

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

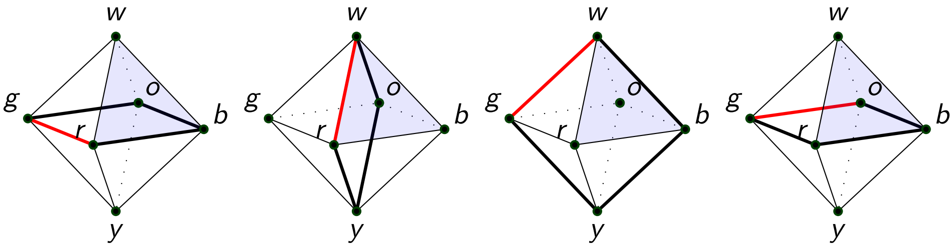
Denote the path $wb \cdot br \cdot rw$ by $\partial(wbr)$. Consider $T_1(\partial(wbr))$:



We come back rotated by 1/4 turn. Call this rotation $R : \mathrm{link}(w) =_{BAut(S^1)} \mathrm{link}(w)$.

Extending T_1 to a face

Let $H_{wbr} : \mathrm{refl}_w =_{w =_{\mathbb{M}} w} \partial(wbr)$ be the filler homotopy of the face.



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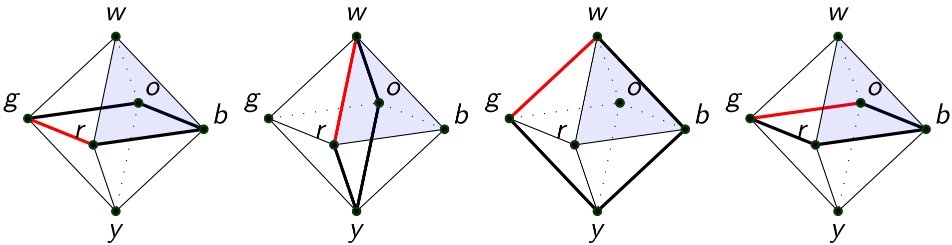
T_2 must live in $T_1(\mathrm{refl}_w) =_{(\mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(w))} T_1(\partial(wbr)) = R$



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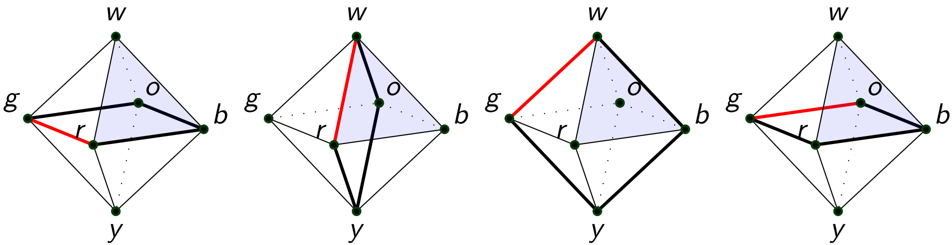
T_2 must be a homotopy $H_R : \mathsf{id} = R$ between automorphisms of $\mathsf{link}(w)$.



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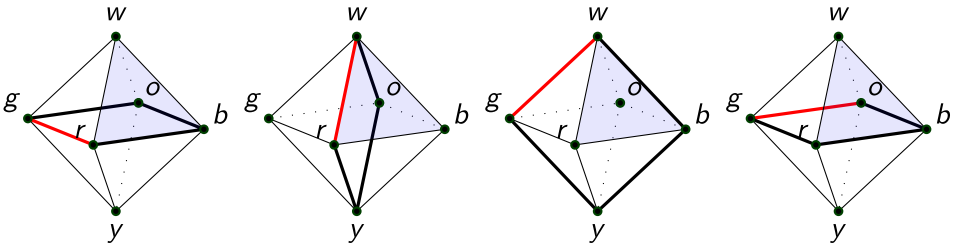
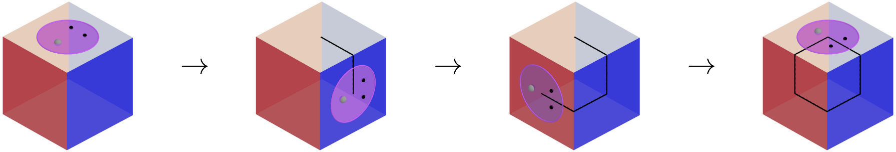
For example, a path $H_R(g) : g = Rg = o$. Choose go.



Original inspiration



The definition of a connection

Definition

If $\mathbb{M} \overset{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a combinatorial manifold and all the triangles commute in the diagram:

$$
\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"'] \\
& & \mathcal{U} & &
\end{tikzcd}
$$

- The map f_k is a k-bundle on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-connection on the $(k-1)$-bundle f_{k-1}.

The definition of curvature

Definition (cont.)

The pushout consists of M_2-many extensions:

$$\begin{tikzcd}
M_2 \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\overset{h_2}{\nearrow}"] & M_2 \arrow[d] \\
\mathbb{M}_1 \arrow[r] \arrow[dr, "T_1"'] \arrow[r, phantom, shift right=4ex, "\swarrow"] & \mathbb{M}_2 \arrow[d, "T_2"] \\
 & \mathcal{U}
\end{tikzcd}$$

Here's the outer square for a single face F:

$$\begin{tikzcd}
\{F\} \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\overset{\flat_F}{\swarrow}"] & \{F\} \arrow[d] \\
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\mathbb{M}_1 \arrow[r] & \mathcal{U}
\end{tikzcd}$$

$T_1(\partial(F))$ is the curvature at the face F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a flatness structure for the face F.

The definition of curvature

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The pushout consists of M_2-many extensions:

$$\begin{tikzcd}
M_2 \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Nearrow h_2"] & M_2 \arrow[d] \\
\mathbb{M}_1 \arrow[r] \arrow[dr, "T_1"'] \arrow[r, phantom, shift right=4, "\Swarrow"] & \mathbb{M}_2 \arrow[d, "T_2"] \arrow[ul, phantom, "\lrcorner", very near start] \\
 & \mathcal{U}
\end{tikzcd}$$

Here's the outer square for a single face F:

$$\begin{tikzcd}
\{F\} \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] & \{F\} \arrow[d] \arrow[dl, phantom, "\Swarrow \flat_F"] \\
\mathbb{M}_1 \arrow[r] & \mathcal{U}
\end{tikzcd}$$

$T_1(\partial(F))$ is the curvature at the face F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a flatness structure for the face F.

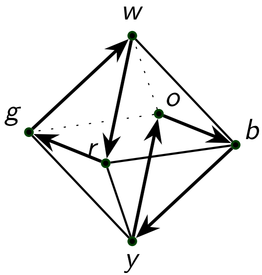
The distinction between the path $\flat_F$ and the endpoint $T_1(\partial(F))$ is small enough to be confusing.

Vector fields

Vector fields

Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim combinatorial manifold.

- Bundles of circles can only model nonzero tangent vectors.
- A global section would be a trivialization of T, so there is an obstruction.



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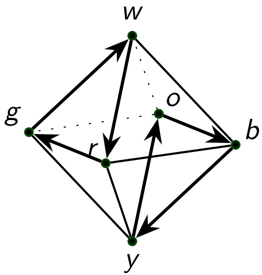

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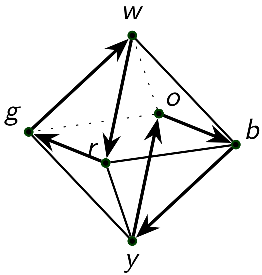
Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.



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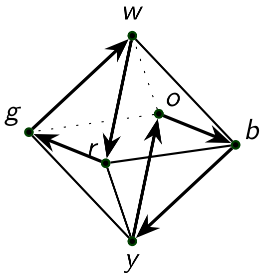

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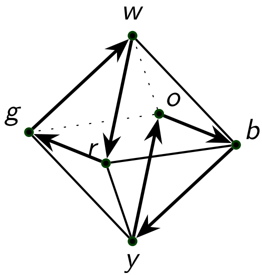
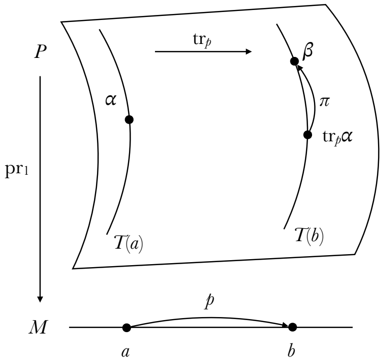
Our solution:

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It's a nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

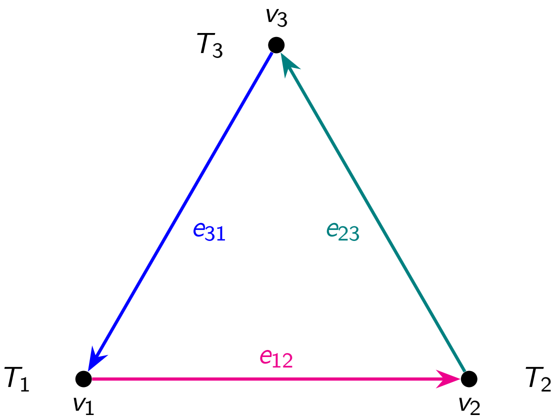


Reminder: pathovers



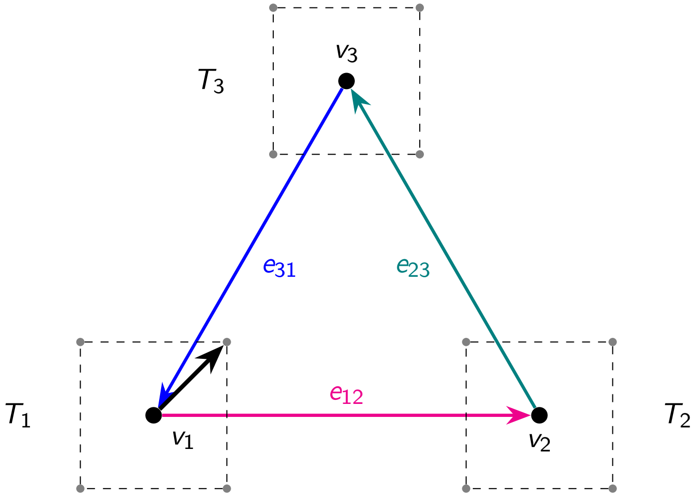
- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display π, the path over p.
- Dependent functions map paths to pathovers: $\mathsf{apd}(X)(p) : \mathsf{tr}_p(X(a)) = X(b)$ (simply denoted $X(p)$).

Next goal: define the index of a vector field on a face.

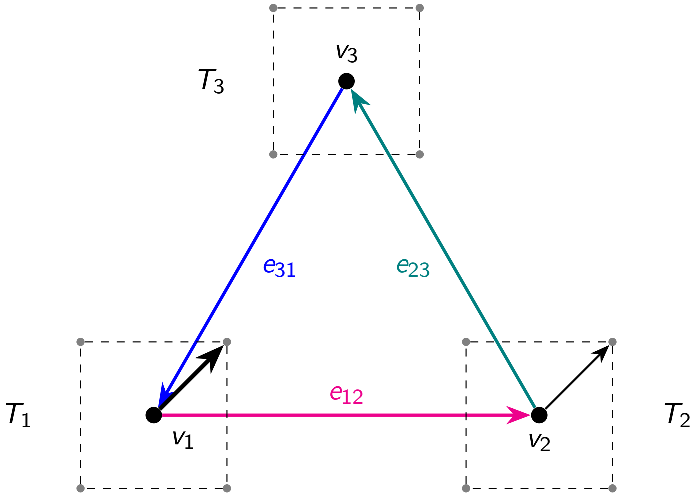


We will try to show the three ingredients of X on this face:

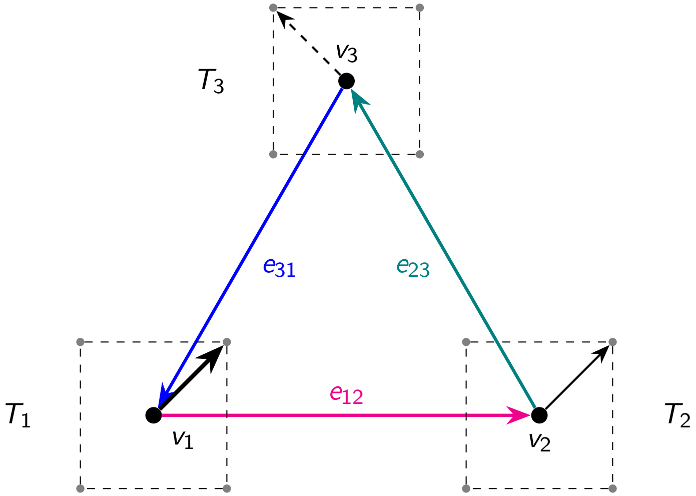
- The value of X on vertices.
- The value of X on edges.
- The transport between vertices, interacting with X.



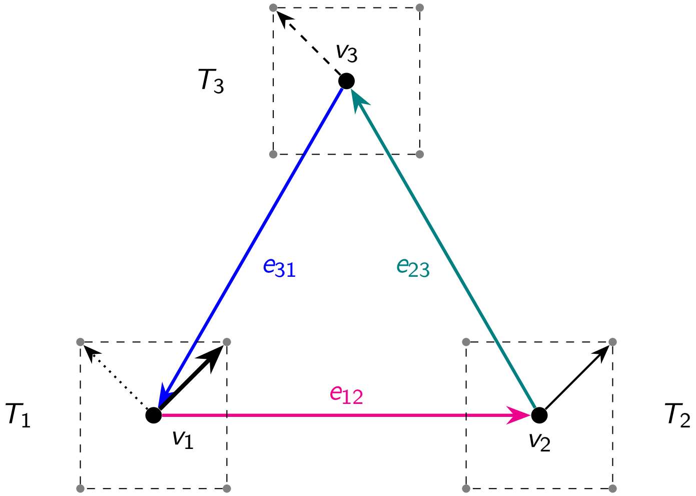
- Denote by X_1 this vector $X(v_1) : T_1$.
-
-
-



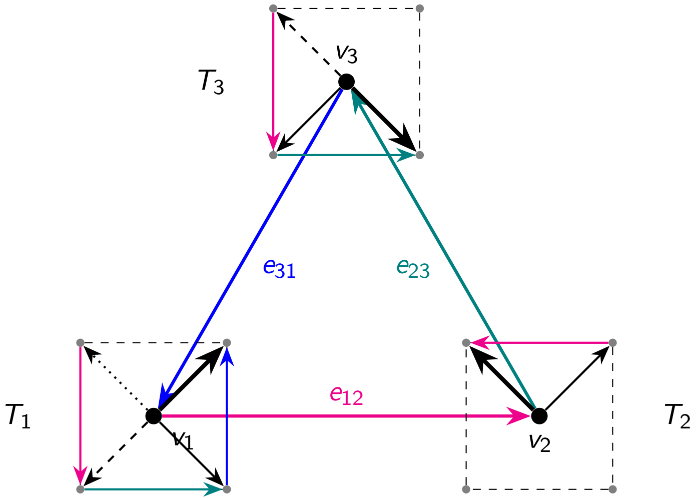
- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{12} is trivial. Denote the transported vector as thinner.
-
-



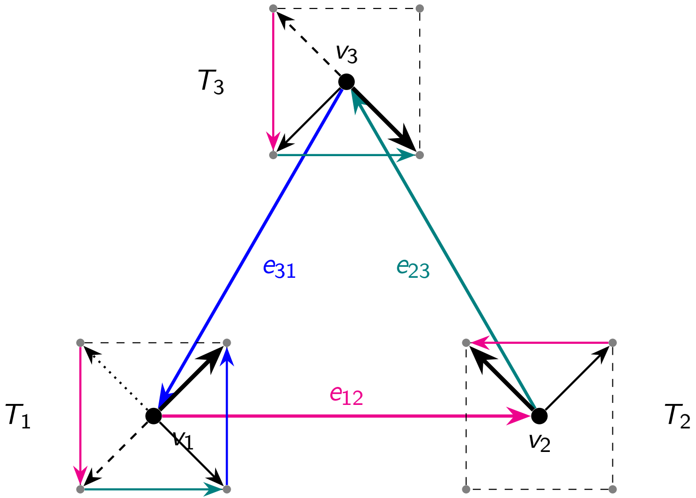
- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{12} is trivial. Denote the transported vector as thinner.
- Say T_{23} rotates counterclockwise. Denote the twice-transported vector as dashed.
-



- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{12} is trivial. Denote the transported vector as thinner.
- Say T_{23} rotates counterclockwise. Denote the twice-transported vector as dashed.
- Say T_{31} is trivial. The thrice-transported vecor is dotted.



- X on e_{12} is red, etc.



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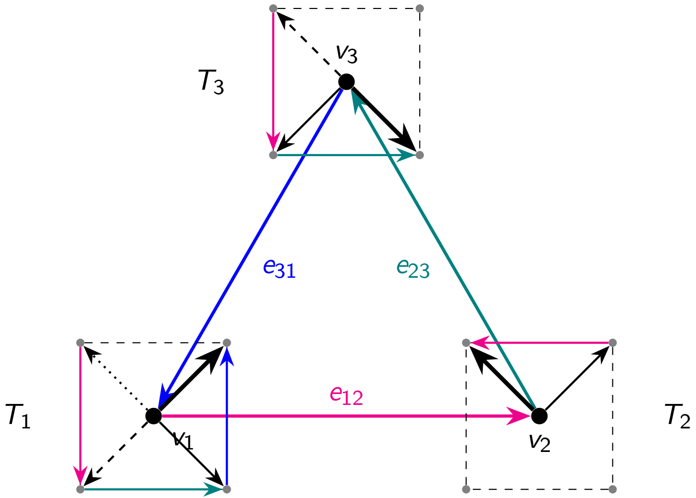

- X on e_{12} is red, etc.
- We translated all results to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)

Symbolic version

$$
\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1 \\
& & & T_{13}T_{32}T_{21}X_1 \arrow[d, equal, "T_{13}T_{32}X_{12}:"'] \\
& & T_{32}T_{21}X_1 \arrow[d, equal, "T_{32}X_{12}:"'] & T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}X_{23}:"'] \\
& T_{21}X_1 \arrow[d, equal, "X_{12}:"'] & T_{32}X_2 \arrow[d, equal, "X_{23}:"'] & T_{13}X_3 \arrow[d, equal, "X_{31}:"'] \\
X_1 & X_2 & X_3 & X_1
\end{tikzcd}
$$

Index

$$\mathrm{tr}_F \overset{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \qquad \text{curvature}$$

$$\flat_F \overset{\mathrm{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \qquad \text{flatness}$$

$$X_F \overset{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

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$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \qquad \text{swirling}$$

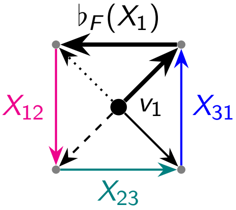
Recall that T_1 being an S^1-torsor means we can use subtraction to obtain an equivalence $s(-, X_1) : T_1 \xrightarrow{x \mapsto x - X_1} S^1$.

Definition

The flattened swirling of the vector field X on the face F is the loop

$$L_F^X \stackrel{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The index of the vector field X on the face F is the integer I_F^X such that $\mathrm{loop}^{I_F^X} =_{S^1} (L_F^X) - X_1$.



Main theorem

The main lemma we need is that the total swirling is 0. We will use four facts:

- The fibers T_i are torsors and they are pointed.
- A sum over faces is a sum over edges, and each edge is appears once in each direction.
- S^1 is commutative.
- Hence there is total cancellation.

Vector fields when the tangent bundle is principal

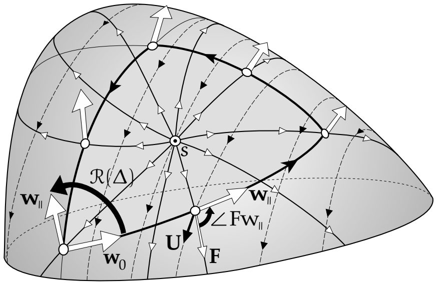
$$\begin{array}{r}
T_1 \\[2em]
T_{13}T_{32}T_{21}X_1 \\
{\color{red}T_{13}T_{32}X_{21}} : \Vert \\
T_{13}T_{32}X_2 \\
{\color{red}T_{13}X_{32}} : \Vert \\
T_{13}X_3 \\
{\color{red}X_{13}} : \Vert \\
X_1
\end{array}$$

- Def: $\alpha_i \overset{\text{def}}{=} s(-, X_i) : T_i \overset{\sim}{\to} S^1$ (trivialization on 0-skeleton).
- Def: $\rho_{ji} \overset{\text{def}}{=} \alpha(T_{ji}(X_i))$ is the rotation of T_{ji}.
- Lemma: $\rho_{ij} = \rho_{ji}^{-1}$ because
  $\rho_{ij} \cdot \rho_{ji} \cdot X_j =_{T_j} \rho_{ij} \cdot T_{ji}X_i =_{T_j} T_{ji}(\rho_{ij} \cdot X_i) =_{T_j} T_{ji}T_{ij}X_j =_{T_j} X_j$.
- Now to paths:
- Define $\sigma_{ji} \overset{\text{def}}{=} s(X_{ji}, X_j) : \rho_{ji} =_{S^1} \text{base}$, a path from a rotation to the identity.
- Paths of the form $a =_{S^1} \text{base}$ can be multiplied:
  $\mu : (a = \text{base}) \times (b = \mathit{base}) \to \mu(a, b) = \mathit{base}$,
  $\mu(p, q) = \mu(p, b) \cdot q$.
- So if $X_{\text{tot}} = X_{21} \cdot X_{32} \cdots$ then $\alpha(X_{\text{tot}}) = \sigma_{21} \cdot \sigma_{32} \cdots$.
- Lemma:
  $\text{apd}(\text{refl}) = \text{refl} \implies X_{ij} \cdot T_{ij}X_{ji} = \text{refl}_{X_i} \implies \sigma_{ij} \cdot \sigma_{ji} = \text{refl}_{\text{base}}$.
- We assume that summing over faces visits every edge once in each direction.
- S^1 is commutative, hence complete cancellation.

Consequence

Yeah.

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F w_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle F w_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Thank you!