Discrete differential geometry in homotopy type theory

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April 2025

Summary

Summary

This work brings to HoTT

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature = total index

Classical $\to$ HoTT

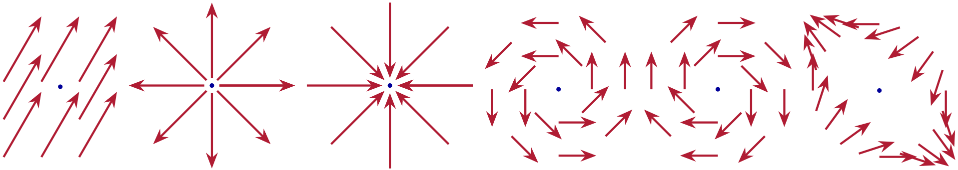
Let M be a smooth, oriented 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes $x_1, \ldots, x_n$.

$$\begin{array}{ccc}
\displaystyle\frac{1}{2\pi}\int_M F_A & = \displaystyle\sum_{i=1}^{n} \mathrm{index}_X(x_i) & = \chi(M) \\
\big\downarrow & \big\downarrow & \\
\displaystyle\sum_{\text{faces } F} \flat_F & = \displaystyle\sum_{\text{faces } F} L_F^X &
\end{array}$$

Classical index

Near an isolated zero there are only three possibilities: index 0, 1, –1.

Index is the winding number of the field as you move clockwise around the zero.

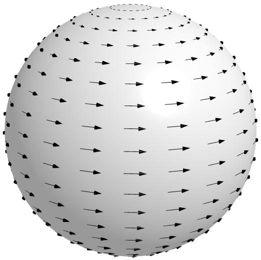


index 0 index $+1$ index $+1$ index $+1$ index $+1$ index –1

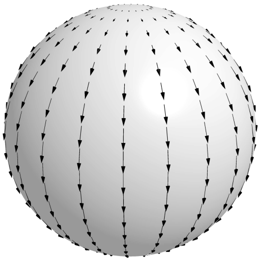
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index $+1$ at each pole $= \mathbf{2}$



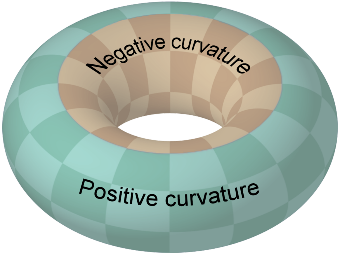
Height: index $+1$ at each pole $= \mathbf{2}$

Gauss-Bonnet theorem

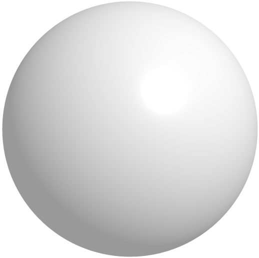
Total curvature divided by 2π is the Euler characteristic.

Curvature in 2D is a function $F_A : M \to \mathbb{R}$.

$\int_M F_A$ sums the values at every point.



Positive and negative curvature cancel: **0**



Constant curvature 1, area 4π: **2**

Plan

- Combinatorial manifolds
- Torsors and classifying maps
- Connections and curvature
- Vector fields
- Main theorem

HoTT background

1. **Symmetry**,
   Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
   `https://github.com/UniMath/SymmetryBook`.
2. **Central H-spaces and banded types**,
   Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
   arXiv:2301.02636
3. **Nilpotent types and fracture squares in homotopy type theory**,
   Scoccola, L. (2020)
   MSCS 30(5). arXiv:1903.03245

Combinatorial manifolds

Manifolds in HoTT

- Recall the classical theory of **simplicial complexes**
- Define a **realization** procedure to construct types

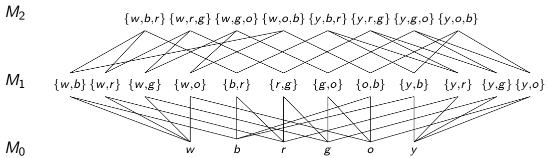
Simplicial complexes

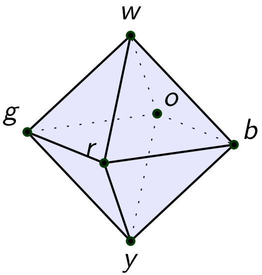
Definition

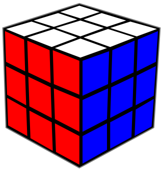
An **abstract simplicial complex** M **of dimension** n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \ldots, M_k]$ **the** k**-skeleton of** M.



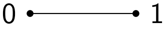




Simplicial complexes

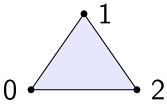
Example

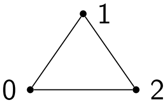
The **complete simplex of dimension** n, denoted $\Delta(n)$, is the set $\{0, \dots, n\}$ and its power set. The $(n-1)$-skeleton $\Delta(n)_{\leq(n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$-sphere.

$\Delta(1)$ is visually 

, $\partial\Delta(1)$ is visually 

,

$\Delta(2)$ is visually 

, $\partial\Delta(2)$ is visually 

Homotopy realization: dimension 0

We will **realize** simplicial complexes by means of **a sequence of pushouts**.

Base case: the realization $\mathbb{M}$ of a 0-dimensional complex M is M_0.

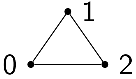
In particular the 0-sphere $\partial\mathbb{\Delta}(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$.

Homotopy realization: dimension 1

For a 1-dim complex $M \overset{\text{def}}{=} [M_0, M_1]$ the realization is given by

$$\begin{tikzcd}
M_1 \times \partial\boldsymbol{\Delta}(1) \arrow[r, "\mathsf{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\mathbb{\Delta}(2) \overset{\text{def}}{=}$ 

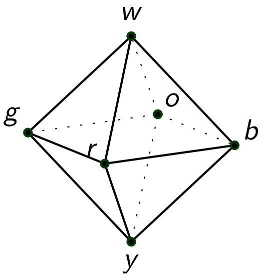
is given by

$$\begin{tikzcd} \partial\Delta(2)_1 \times \partial\mathbb{\Delta}(1) \arrow[r] \arrow[d] & \partial\Delta(2)_1 \arrow[d] \\ \partial\Delta(2)_0 \arrow[r] \arrow[ur, phantom, "\Rightarrow h_1"] & \partial\mathbb{\Delta}(2) \arrow[ul, phantom, very near start, "\lrcorner"] \end{tikzcd}
\quad \text{i.e.} \quad
\begin{tikzcd} \{\{0,1\},\{1,2\},\{2,0\}\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{0,1\},\{1,2\},\{2,0\}\} \arrow[d] \\ \{0,1,2\} \arrow[r] \arrow[ur, phantom, "\Rightarrow h_1"] & \partial\mathbb{\Delta}(2) \arrow[ul, phantom, very near start, "\lrcorner"] \end{tikzcd}$$

Homotopy realization: dimension 1

Or the 1-skeleton of the octahedron $\mathbb{O}$:

$$\begin{tikzcd}
\{\{w,g\},\ldots\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{w,g\},\ldots\} \arrow[d] \\
\{w,g,\ldots\} \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \mathbb{O}_1 \arrow[ul, phantom, "\lrcorner" very near start]
\end{tikzcd}$$



Homotopy realization: dimension 2

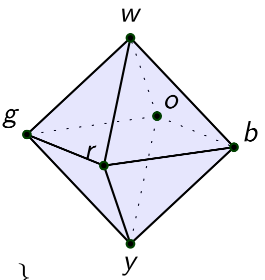
To realize $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\mathbf{\Delta}(1), \partial\mathbf{\Delta}(2)$:

$$
\begin{tikzcd}
M_1 \times \partial\mathbf{\Delta}(1) \arrow[r, "\mathsf{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Rightarrow h_1"] \arrow[dr, phantom, very near end, "\lrcorner"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[dr, phantom, "\Rightarrow h_2"] & \mathbb{M}_2 \\
& M_2 \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathsf{pr}_1"'] \arrow[u, "\mathbb{A}_1"] \arrow[ur, phantom, very near end, "\ulcorner"] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

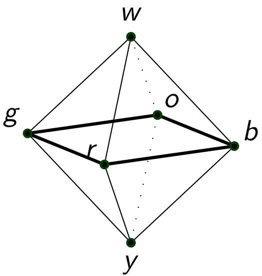
Homotopy realization: dimension 2

The full octahedron $\mathbb{O}$:

$$\begin{tikzcd}
\{\{w,g\},\dots\}\times\{0,1\} \arrow[r, "\mathrm{pr}_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow h_1"] & \{\{w,g\},\dots\} \arrow[d] & \\
\{w,g,\dots\} \arrow[r] & \mathbb{O}_1 \arrow[r] \arrow[dr, phantom, "\Rightarrow h_2"] & \mathbb{O}_2 \\
& \{\{w,g,r\},\dots\}\times\partial\mathbb{\Delta}(2) \arrow[u] \arrow[r, "\mathrm{pr}_1"'] & \{\{w,g,r\},\dots\} \arrow[u]
\end{tikzcd}$$



Homotopy realization: dimension 2



$\leftarrow \mathrm{link}(w)$

The **link** of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

A **combinatorial manifold** is a simplicial complex all of whose links are* simplicial spheres.

This will be our model of the **tangent space**.

*the (classical) geometric realization is homeomorphic to a sphere

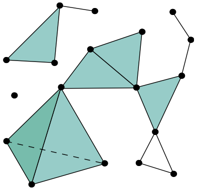
Combinatorial manifolds $\leftrightarrow$ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a **combinatorial manifold***: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$-sphere, i.e. its geometric realization is an $(n-1)$-sphere.*

https://ncatlab.org/nlab/show/triangulation+theorem

Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:



Torsors

What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of **torsors**, also called **principal bundles**.

Torsors

Let G be a (higher) group.

Definition

- A **right** G**-object** is a type X equipped with a homomorphism $\phi : G^{\mathrm{op}} \to \mathrm{Aut}(X)$.
- X is furthermore a G-**torsor** if it is inhabited and the map $(\mathrm{pr}_1, \phi) : X \times G \to X \times X$ is an equivalence.
- The inverse is (pr_1, s) where $s : X \times X \to G$ is called **subtraction** (when G is commutative).
- Let BG be the type of G-torsors.
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

Facts

1. $\Omega(BG, G_{\mathrm{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.
2. BG is connected.
3. $1 \ \&\ 2 \implies BG$ is a $\mathrm{K}(G, 1)$.

See the Buchholtz et. al. H-spaces paper for more.

How to map into BS^1

To construct maps into BS^1 we **lift** a family of **mere circles**.

$$\begin{tikzcd}[column sep=small]
 & BS^1 \arrow[r] & \mathrm{BAut}(S^1) \overset{\text{def}}{=} \sum_{Y:\mathcal{U}} \|Y = S^1\|_{-1} \arrow[r] & \mathcal{U} \\
\text{families of:} & & \mathbb{M} \arrow[ul, "\text{torsors}"] \arrow[u, "\text{mere circles}"] \arrow[ur, "\text{types}"'] &
\end{tikzcd}$$

We will assume we have such a lift when we need it. (Remark: the lift is a choice of **orientation**.)

Other names:

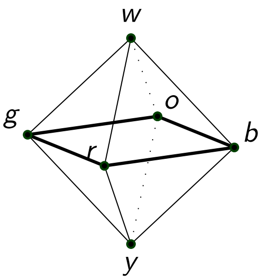
- $\mathrm{BAut}(S^1) = BO(2) = \mathrm{EM}(\mathbb{Z}, 1)$ (where $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n))$)
- $BS^1 = BSO(2) = \mathrm{K}(\mathbb{Z}, 2)$

Connections and curvature

Connections

Connections are extensions of a bundle to higher skeleta.

Recall link



$\leftarrow \mathrm{link}(w)$

The **link** of a vertex w in a 2-complex is: the sets not containing w but whose union with w is a face.

Define **the tangent bundle** on a combinatorial manifold to be $T_0 \stackrel{\text{def}}{=} \mathrm{link} : \mathbb{M}_0 \to \mathrm{BAut}(S^1)$.

Connections on the tangent bundle

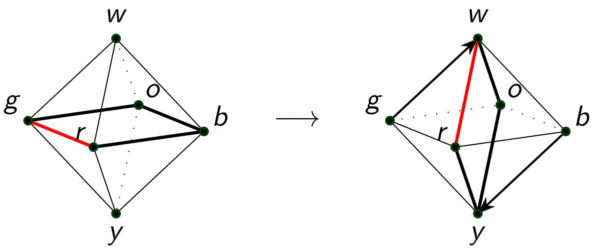
An extension T_1 of T_0 to $\mathbb{M}_1$ is called **a connection on the tangent bundle**.

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r] \arrow[dr, "T_0 \overset{\text{def}}{=} \text{link}"'] & \mathbb{M}_1 \arrow[r] \arrow[d, dashed, "T_1"] & \mathbb{M}_2 \arrow[dl, dashed] \\
& \mathrm{BAut}(S^1) &
\end{tikzcd}$$

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

We will define T_1 on the edge wb, so we need a term $T_1(wb) : \mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(b)$.

We imagine tipping:

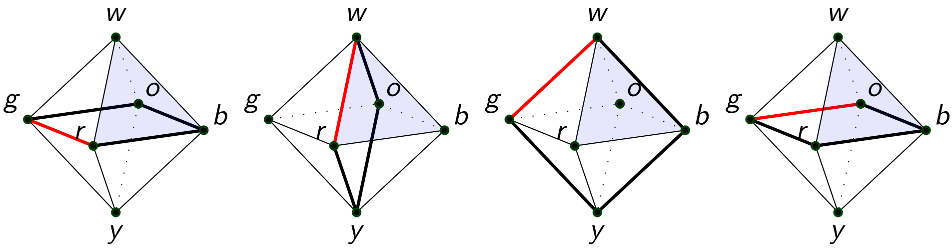


$T_1(g : \mathrm{link}(w)) \stackrel{\mathrm{def}}{=} w : \mathrm{link}(b), \ldots.$

Use this method to define T_1 on every edge.

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

Denote the path $wb \cdot br \cdot rw$ by $\partial(wbr)$. Consider $T_1(\partial(wbr))$:



We come back rotated by 1/4 turn. Call this rotation $R : \mathrm{link}(w) =_{BAut(S^1)} \mathrm{link}(w)$.

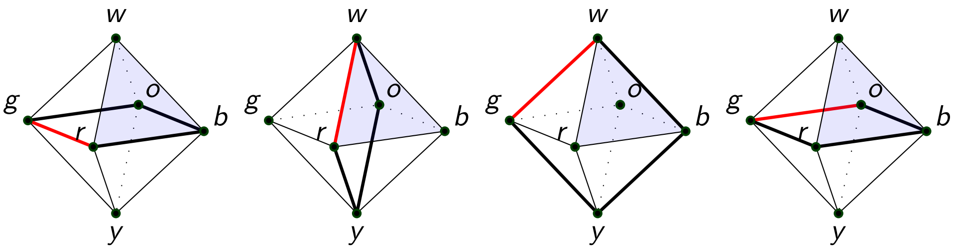
Extending T_1 to a face

Let $H_{wbr} : \mathrm{refl}_w =_{w =_{\mathbb{M}} w} \partial(wbr)$ be the filler homotopy of the face.

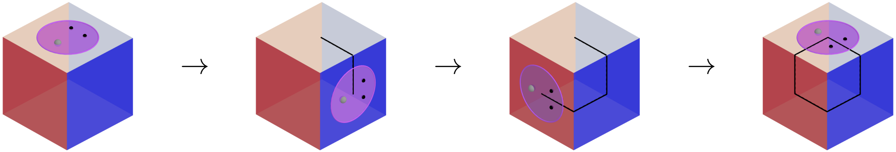
T_2 must live in $T_1(\mathrm{refl}_w) =_{(\mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(w))} T_1(\partial(wbr)) = R$

T_2 must be a homotopy $H_R : \mathrm{id} = R$ between automorphisms of $\mathrm{link}(w)$.

For example, a path $H_R(g) : g = Rg = o$. Choose go.



Original inspiration



The definition of a connection

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is the realization of a combinatorial manifold and all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"'] \\
& & \mathcal{U} & &
\end{tikzcd}$$

- The map f_k is a k-**bundle** on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-**connection on the** $(k-1)$-**bundle** f_{k-1}.

The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:

$$
\begin{tikzcd}
M_2 \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "h_2 \Nearrow"] & M_2 \arrow[d] \\
\mathbb{M}_1 \arrow[r] \arrow[dr, "T_1"'] & \mathbb{M}_2 \arrow[ul, phantom, very near start, "\lrcorner"] \arrow[d, "T_2"] \arrow[dl, phantom, "\Swarrow"] \\
& \mathcal{U}
\end{tikzcd}
$$

Here's the outer square for a single face F:

$$
\begin{tikzcd}
\{F\} \times \partial\mathbf{\Delta}(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\Swarrow \flat_F"] & \{F\} \arrow[d] \\
\mathbb{M}_1 \arrow[r] & \mathcal{U}
\end{tikzcd}
$$

$T_1(\partial(F))$ is **the curvature at the face** F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a **flatness structure for the face** F.

The distinction between the path $\flat_F$ and the endpoint $T_1(\partial(F))$ is small enough to be confusing.

Vector fields

Vector fields

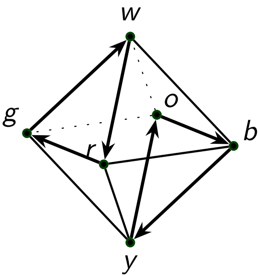
Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of T, so that's not a good definition.

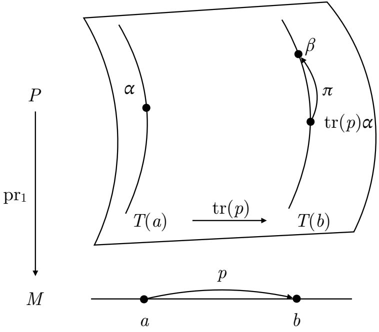
$$\begin{tikzcd} & * \arrow[d, "S^1"] \\ \mathbb{M} \arrow[ur, dashed, "X"] \arrow[r, "T"'] & BS^1 \end{tikzcd}$$

Our solution:

- A **vector field** is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical **nonvanishing** vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

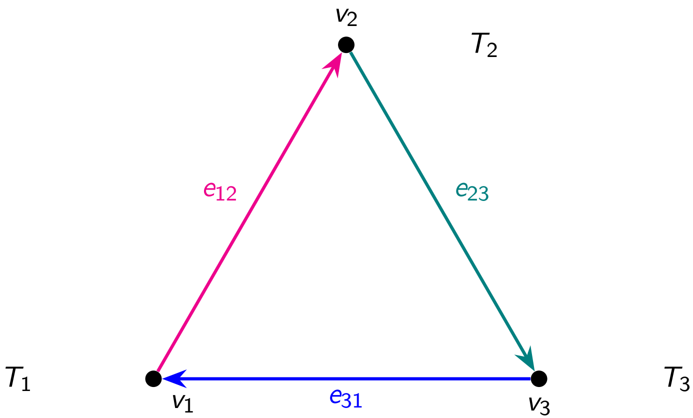


Reminder: pathovers

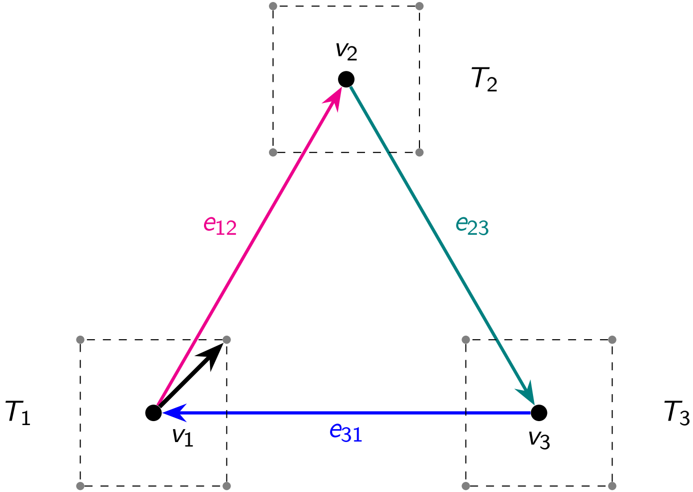


- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display π, the path over p.
- Dependent functions map paths to pathovers: $\mathsf{apd}(X)(p) : \mathsf{tr}_p(X(a)) = X(b)$ (simply denoted $X(p)$).

Next goal: define the index of a vector field on a face.

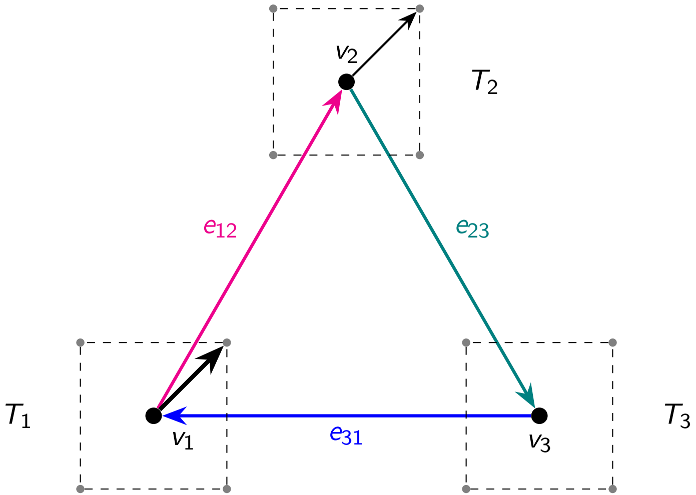


An example of **swirling** and **index** at this face.



An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
-
-
-



An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
-
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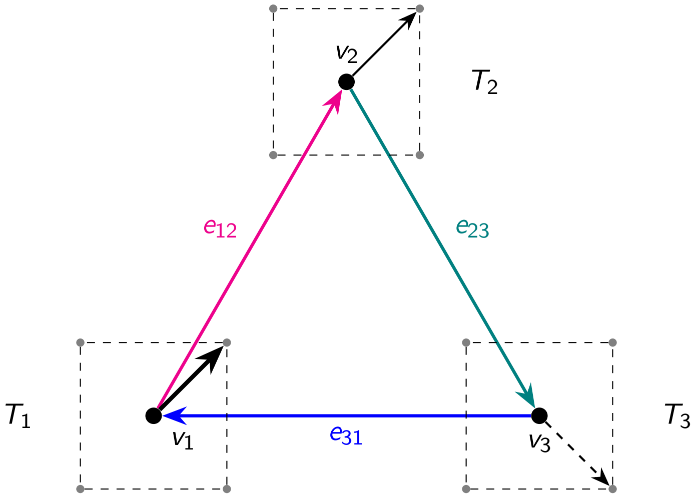
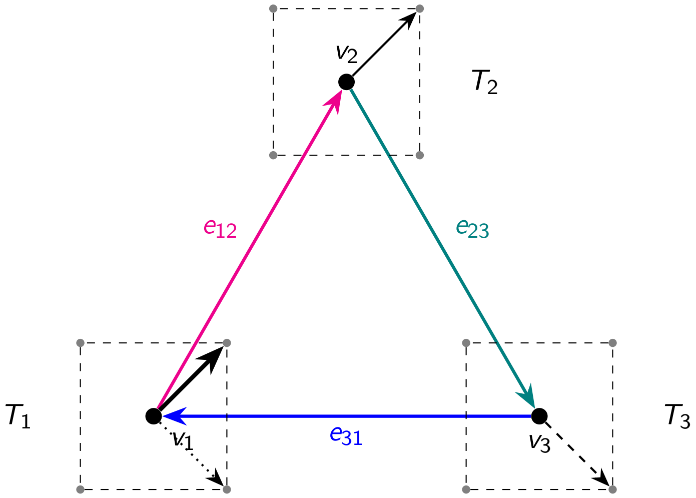


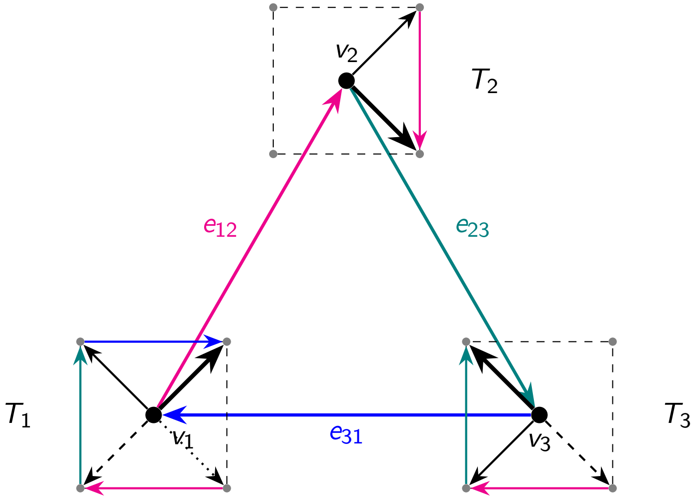
An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
-



An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
- Say T_{13} is trivial. The thrice-transported vecor is dotted.



- X on e_{12} is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)
- The total pathover $X(\partial F)$ is called **the swirling** X_F of X at the face F.

Symbolic version

$$
\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1 \\
& & & T_{13}\,T_{32}\,T_{21}X_1 \arrow[d, equal, "T_{13}\,T_{32}X_{21}:"'] \\
& & T_{32}\,T_{21}X_1 \arrow[d, equal, "T_{32}X_{21}:"'] & T_{13}\,T_{32}X_2 \arrow[d, equal, "T_{13}X_{32}:"'] \\
& T_{21}X_1 \arrow[d, equal, "X_{21}:"'] & T_{32}X_2 \arrow[d, equal, "X_{32}:"'] & T_{13}X_3 \arrow[d, equal, "X_{13}:"'] \\
X_1 & X_2 & X_3 & X_1
\end{tikzcd}
$$

Index

$$\begin{array}{llll} \mathrm{tr}_F \overset{\mathrm{def}}{=} \mathrm{tr}(\partial F) & : T_1 =_{BS^1} T_1 & \textbf{curvature} \\ \flat_F \overset{\mathrm{def}}{=} \flat(\partial F) & : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F & \textbf{flatness} \\ X_F \overset{\mathrm{def}}{=} X(\partial F) & : \mathrm{tr}_F(X_1) =_{T_1} X_1 & \textbf{swirling} \end{array}$$

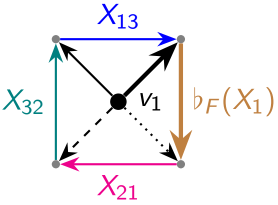
(Recall that T_1 being an S^1-torsor means we can use subtraction to obtain an equivalence $s(-, X_1) : T_1 \xrightarrow{x \mapsto x - X_1} S^1$.)

Definition

The **flattened swirling** of the vector field X on the face F is the loop

$$L_F^X \overset{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F : (X_1 =_{T_1} X_1).$$

The **index** of the vector field X on the face F is the integer I_F^X such that $\mathrm{loop}^{I_F^X} =_{S^1} (L_F^X) - X_1$.



Main theorem

Simplifying swirling

Swirling involves concatenating dependent paths. Can we simplify that?

Pay off all our assumptions 1: torsor structure, vector field

$$T_1$$

$$\begin{tikzcd}
T_{13}T_{32}T_{21}X_1 \arrow[d, equal, "T_{13}T_{32}X_{21}:"'] \\
T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}X_{32}:"'] \\
T_{13}X_3 \arrow[d, equal, "X_{13}:"'] \\
X_1
\end{tikzcd}$$

- Def: $\alpha_i \overset{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$ (**trivialization on 0-skeleton**).
- Def: $\rho_{ji} \overset{\text{def}}{=} \alpha_j(T_{ji}(X_i))$ is **the rotation of** T_{ji}.

$$\begin{tikzcd}
T_i \arrow[r, "T_{ji}"] \arrow[d, "\alpha_i"] & T_j \arrow[d, "\alpha_j"'] \\
S^1 \arrow[r, "(-)+\rho_{ji}"'] \arrow[u, bend left, "\mathrm{base}\mapsto X_i"] & S^1 \arrow[u, bend right, "\mathrm{base}\mapsto X_j"']
\end{tikzcd}$$

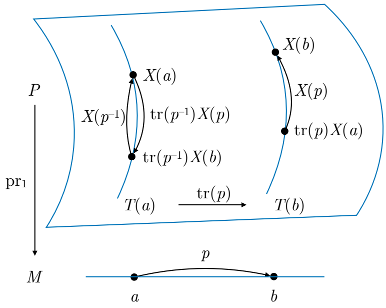
- Lemma: $\rho_{ij} = \rho_{ji}^{-1}$ because **in** T_j:
  $\rho_{ij} + \rho_{ji} + X_j = \rho_{ij} + T_{ji}X_i = T_{ji}(\rho_{ij} + X_i) = T_{ji}T_{ij}X_j = X_j$.

Pay off all our assumptions 1: torsor structure, vector field (cont.)

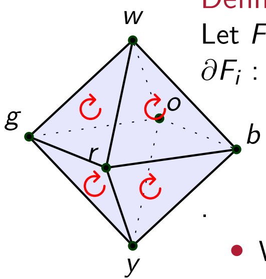
T_1

$$
\begin{array}{r c}
 & T_{13}T_{32}T_{21}X_1 \\
T_{13}T_{32}X_{21}: & \| \\
 & T_{13}T_{32}X_2 \\
T_{13}X_{32}: & \| \\
 & T_{13}X_3 \\
X_{13}: & \| \\
 & X_1
\end{array}
$$

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base}$,.
- Paths of the form $(a =_{S^1} \text{base})$ can be multiplied:
  - $+ : (a = \text{base}) \times (b = base) \to (a + b = base)$.
  - $p + q = (p + b) \cdot q$.
- Lemma: $\text{apd}(X)(\text{refl}) = \text{refl}$
  $\implies X_{ij} \cdot T_{ij}X_{ji} = \text{refl}_{X_i}$
  $\implies \sigma_{ij} + \sigma_{ji} = \text{refl}_{\text{base}}$ (T_{ij} just translates X_{ji} to cat with X_{ji}).



Pay off all our assumptions 2: no boundary, commutativity



Definition

Let $F_1, \ldots, F_n$ be the faces of $\mathbb{M}$, $v_i : F_i$ be designated vertices, and $\partial F_i : v_i = v_i$ be the triangular boundaries. The **total swirling** is

$$X_{\text{tot}} \stackrel{\text{def}}{=} \sigma_{\partial F_1} + \cdots + \sigma_{\partial F_n}$$

.

- We assume that this expression involves **every edge once in each direction**.
- S^1 is commutative, hence **complete cancellation**.

Consequence

$$\begin{aligned}
\mathrm{tr}_F &\stackrel{\text{def}}{=} \mathrm{tr}(\partial F) && : T_1 =_{BS^1} T_1 && \textbf{curvature}\\
\flat_F &\stackrel{\text{def}}{=} \flat(\partial F) && : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F && \textbf{flatness}\\
X_F &\stackrel{\text{def}}{=} X(\partial F) && : \mathrm{tr}_F(X_1) =_{T_1} X_1 && \textbf{swirling}\\
L_F^X &\stackrel{\text{def}}{=} \flat_F(X_1) \cdot X_F && : (X_1 =_{T_1} X_1) && \textbf{flattened swirling}
\end{aligned}$$

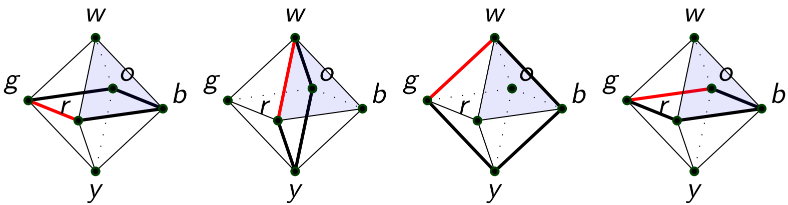
These can all be totaled in S^1 to give

$$\mathrm{tr}_{\mathrm{tot}} \stackrel{\text{def}}{=} \sum_i \rho_{\partial F} = \mathrm{base} \qquad X_{\mathrm{tot}} \stackrel{\text{def}}{=} \sum_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

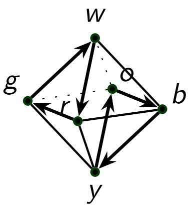
$$\flat_{\mathrm{tot}} \stackrel{\text{def}}{=} \sum_i \flat_{\partial F} \qquad L_{\mathrm{tot}}^X \stackrel{\text{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

So in our lingo: the total flatness equals the total flattened swirling. $\square$

Examples

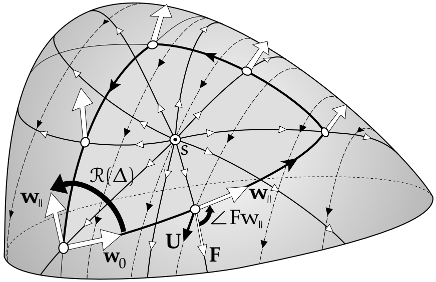


Each face contributes $\flat_F = H_R$, a $1/4$-rotation. Total: 2.



This is what it looks like: $+1$ from F_{wrg}, $+1$ from F_{ybo}, $+0$ from others.

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi\, \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F\mathbf{w}_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle F w_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Thank you!