Discrete differential geometry in homotopy type theory

Greg Langmead

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Abstract

Homotopy type theory can capture some of the major concepts of differential geometry including connections, curvature, and vector fields. We show this by focusing on combinatorial manifolds, which are discrete in the sense of real cohesion[1], and drawing inspiration from the field of discrete differential geometry. We prove the Gauss-Bonnet theorem and Poincaré-Hopf theorem in two dimensions.

> "It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, *The Future Does Not Compute*[2]

1 Overview

We will define

- combinatorial 2-manifolds
- principal circle bundles of tangent bundles
- vector fields,

and then observe emerging from those definitions the presence of

- connections
- curvature
- the index of a vector field,

and prove

- the Leibniz formula
- the Gauss-Bonnet theorem
- and the Poincaré-Hopf theorem.

We will consider functions $M \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where M is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

We will see in Section 3.2 that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. We will construct a map $\mathsf{link} : M \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. Then we can consider the type of pointed mere circles $\mathrm{EM}_\bullet(\mathbb{Z}, 1) \overset{\mathrm{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ as well as the first projection that forgets the point. This is a univalent fibration (univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe[3]). If we form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"'] \\ M \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z}, 1) \end{tikzcd}$$

then we have a bundle of mere circles, with total space given by the $\sum$-type construction. We will show that this is not a principal bundle, i.e. a bundle of torsors. Torsors are types with the additional structure of a group action. But if link satisfies an additional property (amounting to an orientation) then the pullback is a principal fibration, i.e. link factors through a map $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$, where $\mathrm{K}(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space.

We will investigate that the data in dimensions 1 and 2 of link can be thought of as a connection, notably one that is not necessarily flat. Moreover, lifting link to $\mathsf{link}_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ can be thought of as a nonvanishing vector field. There will in general not be a total lift, just a partial function. The domain of $\mathsf{link}_\bullet$ will have a boundary of circles, and the winding number on these can be thought of as the index of $\mathsf{link}_\bullet$. We can then examine the total curvature and the total index and prove that they are equal, and equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds. This is similar to the classical proof of Hopf[4], presented in detail in Needham[5].

Because the codomain $\mathrm{K}(\mathbb{Z}, 2)$ has an H-space structure, we might ask about how the action on paths of link, or any function for that matter, interacts with pointwise multiplication. This will lead us to the Leibniz formula, which emerges simply from horizontal composition in the codomain.

1.1 Future work

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[6][7] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

2 Torsors and principal bundles

The classical theory of principal bundles tells us to look for an appropriate classifying space of torsors to map into. Homotopy type theory tells us that classifying spaces are univalent fibrations. The type of torsors is not a priori such a fibration, so we'll do some work to make that happen. This will constitute the codomain of the investigation.

Definition 1. Let G be a group (a set with the usual classical structure and properties). A **G-set** is a set X equipped with a homomorphism $\phi : G \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : ||X||_{-1} \times \prod_{g:G} \mathsf{is\_equiv}(\phi(-, x) : G \to X)$$

then we call this data a **G-torsor**. Denote the type of G-torsors by TG.

If $(X, \phi), (Y, \psi) : TG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 1. There is a natural equivalence $(X =_{TG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for TG and we have a group isomorphism $\Omega TG \simeq G$.

Lemma 2. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 1. The pointed type $(TG, *)$ is a $\mathrm{K}(G, 1)$. □

2.1 Univalent replacement for torsors

The homotopy type theory of cohomology and bundles tells us that the type of principal G-bundles on a type M is the type $M \to \mathrm{K}(G, 1)$. But this is a type of structured types, a connected component of G-sets rather than a connected component of the universe. The paths *in the universe* between two G-sets is equivalent to the type of equivalences between the *underlying types*, not just the equivariant equivalences. We wish to work with a connected component of the universe $\mathcal{U}$.

We'll resolve this problem with the following discussion, following Scoccola[8]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2. Let $\mathrm{EM}(G, n) \overset{\mathrm{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \overset{\mathrm{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$. A **$\mathrm{K}(G, n)$-bundle** on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses two self-maps on the universe: suspension followed by $(n+1)$-truncation $||\Sigma||_{n+1}$ and forgetting a point $F_\bullet$ to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{||\Sigma||_{n+1}} \mathrm{EM}_{\bullet\bullet}(G, n+1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n+1)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 3. Given $f : M \to \mathrm{EM}(G, n)$, the **associated action of M on G**, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ ||\Sigma||_{n+1} \circ f$.

Theorem 1. (Scoccola[8] Proposition 2.39). A $\mathrm{K}(G,n)$ bundle $f : M \to \mathrm{EM}(G,n)$ is equivalent to a map in $M \to \mathrm{K}(G,n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Let's relate this to *orientation*. Note that the obstruction in the theorem is about a map into $\mathrm{EM}_\bullet(G,n+1)$ and further note that $\mathrm{EM}_\bullet(G,n) \simeq \mathrm{K}(\mathrm{Aut}\,G,1)$ (independent of n). The theorem says that the data of a map into $\mathrm{EM}(G,n)$ factors into data about a map into $\mathrm{K}(G,n+1)$ and one into $\mathrm{K}(\mathrm{Aut}\,G,1)$. Informally, $\mathrm{EM}(G,n)$ is a little too large to be a $K(G,n+1)$, as it includes data about automorphisms of G.

In the special case of $\mathrm{EM}(\mathbb{Z},1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z},1)$ is contractible. $\mathrm{Aut}\,\mathbb{Z}$ consists of the $\mathbb{Z}/2\mathbb{Z}$ worth of outer automorphisms given by multiplication by ± 1. Symmetries of the circle are our discrete stand-in for the matrix group $O(2)$, which contains both rotations and (orientation-reversing) reflections of the plane. Requiring a contractible induced map to ± 1 amounts to a choice of direction for all the circles, and so deserves the name "f is *orientable*." In addition $f_\bullet$ deserves to be called the first Stiefel-Whitney class of f, and the requirement here is that it vanishes.

Note 1. Reinterpreting more of the theory of characteristic classes would be an enlightening future project. Defining a Chern class and Euler class in 2 dimensions is a goal of this note, but we will not prove all the various laws these classes satisfy (the Whitney sum formula and so on). Nonabelian matrix Lie groups such as $SO(n)$ and $SU(n)$ are not fully imported into homotopy type theory, but recall that some classical results from the theory of characteristic classes are obtained by replacing the group with a maximal torus, which should be a smaller leap from what is presented here[9].

In summary, we can continue to work with the univalent fibration $\mathrm{EM}(G,1)$ and still know that we are also studying principal $\mathrm{K}(G,1)$-fibrations, if the bundle is orientable.

2.2 Pathovers in principal bundles

Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z},1)$ and $P \overset{\text{def}}{=} \sum_{x:M} T(x)$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : T(a) = T(b)$. This is a path in the codomain, which in this case is a type of types. Type theory also provides a function called *transport*, denoted $\mathrm{tr}_p : T(a) \to T(b)$ which acts on the fibers of P. tr_p acts on the terms of the types $T(a)$ and $T(b)$, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathrm{tr}_p(\alpha) =_{T(b)} \beta$ between $\alpha : T(a)$ and $\beta : T(b)$ in the fibers. We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

Lastly we want to recall that in the presence of a section $s : M \to P$ there is a dependent generalization of ap called apd: $\mathrm{apd}_p(s) : \mathrm{tr}_p(s(a)) = s(b)$ which is a pathover between the two values of the section over the basepoints of the path p.

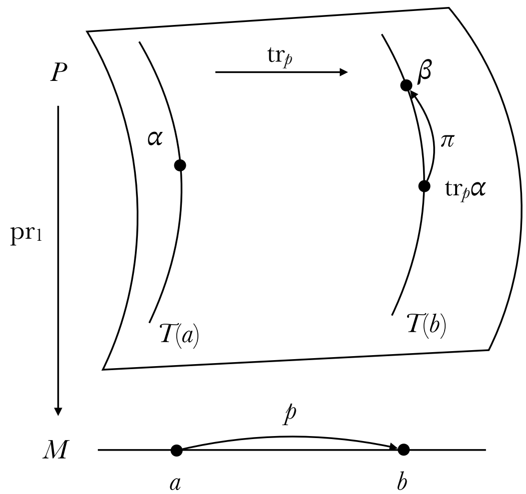

Figure 1: A path π over the path p in the base involves the transport function.

3 Combinatorial manifolds

We will adapt to higher inductive types in a straightforward manner the classical construction of *combinatorial manifolds*. See for example the classic book by Kirby and Siebenmann[10]. These are a subclass of simplicial complexes.

Definition 4. An **abstract simplicial complex** M **of dimension** n consists of a set M_0 of vertices, and for each $0 < k \leq n$ a set M_k of subsets of M_0 of cardinality $k+1$, such that any $(j+1)$-element subset of M_k is an element of M_j. The elements of M_k are called k**-faces**. Denote by $\mathsf{SimCompSet_n}$ the type of abstract simplicial complexes of dimension n (where the suffix Set reminds us that this is a type of sets).

Note that we don't require all subsets of M_0 to be included – that would make M an individual simplex. A simplicial complex is a family of simplices that are identified along various faces.

Definition 5. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $n-1$-face containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

The link is all the neighboring vertices of v and the codimension 1 faces joining those to each other. See for example Figure 2.

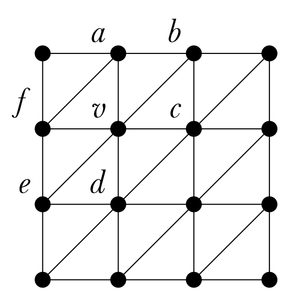

Figure 2: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

Definition 6. A **combinatorial manifold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a simplicial sphere of dimension $n-1$ (i.e. its geometric realization is homeomorphic to an $n-1$-sphere). Denote by $\mathsf{CombMfdSet_n}$ the type of combinatorial manifolds of dimension n (which the notation again reminds us are sets).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figures 3, 4, and 5 for some examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits a combinatorial triangulation[11]. So it appears reasonably well motivated to study this class of objects.

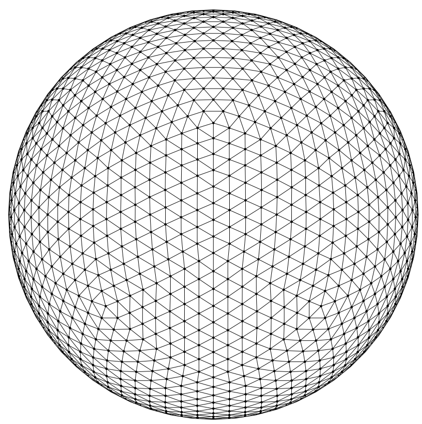

Figure 3: A combinatorial triangulation of a sphere, created with stripy.

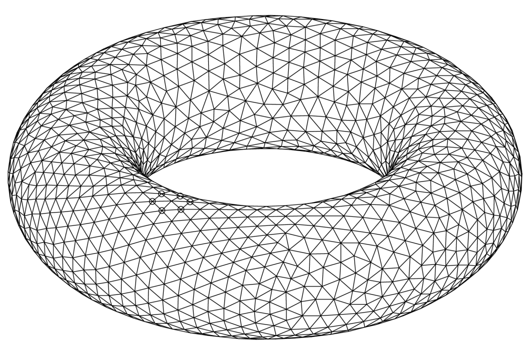

Figure 4: A torus with an interesting triangulation, from Wikipedia. The links have various vertex counts from 5-7. Clearly a constant value of 6 would also work. (By Ag2gaeh - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=30856793)

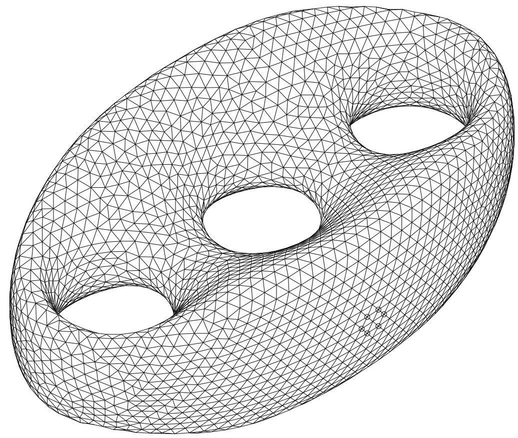

Figure 5: A 3-holes torus with triangulation, from Wikipedia. (By Ag2gaeh - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/wiki/File:Tri-brezel.svg)

3.1 Higher inductive combinatorial manifolds

To convert a simplicial complex M of dimension at most 2 to a higher inductive type, we will convert the data in each classical dimension to a path constructor of the corresponding HoTT dimension.

Definition 7. The function $\mathcal{R} : \mathsf{SimCompSet}_2 \to \mathsf{Type}$ is called **realization** and creates a higher inductive type from the data of a simplicial complex. Given $M : \mathsf{SimCompSet}_2$ then $\mathcal{R}(M)$ is given by

1. vertices: a function $\mathsf{v}_0 : M_0 \to \mathcal{R}(M)$ serving as the 0-dimensional constructors
2. edges: a function v_1 on 1-faces, sending $\{a, b\} \mapsto \mathsf{v}_0(a) = \mathsf{v}_0(b)$
3. 2-faces: a function v_2 on 2-faces, sending $\{a, b, c\} \mapsto \mathsf{refl}_a = \mathsf{v}_1(\{a, b\}) \cdot \mathsf{v}_1(\{b, c\}) \cdot \mathsf{v}_1(\{a, c\})^{-1}$.

3.2 Polygons

We will begin with a type that is important both for the domain and the codomain of mere circles: a square.

Definition 8. The higher inductive type C_4 (where C stands for "circle").

$$\begin{aligned} C_4 &: \mathsf{Type} \\ c_1, c_2, c_3, c_4 &: C_4 \\ c_1c_2 &: c_1 = c_2 \\ c_2c_3 &: c_2 = c_3 \\ c_3c_4 &: c_3 = c_4 \\ c_4c_1 &: c_4 = c_1 \end{aligned}$$

$$\begin{tikzcd}
c_1 \arrow[r, "c_1c_2"] & c_2 \arrow[d, "c_2c_3"] \\
c_4 \arrow[u, "c_4c_1"] & c_3 \arrow[l, "c_3c_4"]
\end{tikzcd}$$

Figure 6: The HIT C_4.

The standard HoTT circle itself is a non-example of a combinatorial manifold since it lacks the second vertex of the edge:

Definition 9. The higher inductive type S^1:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

Nonetheless, all polygons are equivalent to each other and to S^1.

Lemma 3. The function $\ell : C_4 \to S^1$ given by $\ell(c_i) = \mathsf{base}$ for all i, and $\ell(c_ic_j) = \mathsf{loop}$ for all i, j is an equivalence with inverse $\ell^{-1}(\mathsf{base}) = c_1$ and $\ell^{-1}(\mathsf{loop}) = c_1c_2 \cdot c_2c_3 \cdot c_3c_4 \cdot c_4c_1$. There are clearly other inverses for different choices of vertex.

Proof. We can adapt the proof in the HoTT Book[12] of Lemma 6.5.1 which proves that $\mathbf{\Sigma 2} \simeq S^1$. □

Recalling that terms of $\mathrm{EM}(\mathbb{Z}, 1)$ are pairs: a type, and a mere equivalence with S^1, we have:

Corollary 2. We have $(C_4, ||\ell||_{-1}) : \mathrm{EM}(\mathbb{Z}, 1)$.

Real-world triangulations of surfaces will often have links whose number of vertices varies across the surface. For example we can see hexagons and pentagons in Figure 3. This presumably introduces only a minor practical inconvenience and doesn't materially affect the discussion to come.

3.3 The higher inductive type $\mathbb{O}$

We will create our first combinatorial surface, a 2-sphere. We will adopt the convention that a subscript indicates the dimension of a subskeleton of a complex. For instance, we have $\mathsf{base} : S^1_0$.

Definition 10. The HIT $\mathbb{O}_0$ is just 6 points, intended as the 0-skeleton of an octahedron, with vertices named after the colors on the faces of a famous Central European puzzle cube.

$$w, y, b, r, g, o : \mathbb{O}_0$$

Definition 11. The HIT $\mathbb{O}_1$ is the 1-skeleton of an octahedron.

$$
\begin{aligned}
&w, y, b, r, g, o : \mathbb{O}_1 && yg : y = g \\
&\qquad wb : w = b && yo : y = o \\
&\qquad wr : w = r && br : b = r \\
&\qquad wg : w = g && rg : r = g \\
&\qquad wo : w = o && go : g = o \\
&\qquad yb : y = b && ob : o = b \\
&\qquad yr : y = r &&
\end{aligned}
$$

Definition 12. The HIT $\mathbb{O}$ is an octahedron:

$$
\begin{aligned}
&w, y, b, r, g, o : \mathbb{O} \\
&\qquad wb : w = b && br : b = r && wbr : wb \cdot br \cdot wr^{-1} = \mathsf{refl}_w \\
&\qquad wr : w = r && rg : r = g && wrg : wr \cdot rg \cdot wg^{-1} = \mathsf{refl}_w \\
&\qquad wg : w = g && go : g = o && wgo : wg \cdot go \cdot wo^{-1} = \mathsf{refl}_w \\
&\qquad wo : w = o && ob : o = b && wob : wo \cdot ob \cdot wb^{-1} = \mathsf{refl}_w \\
&\qquad yb : y = b && && yrb : yr \cdot rb \cdot yb^{-1} = \mathsf{refl}_y \\
&\qquad yr : y = r && && ygr : yg \cdot gr \cdot yr^{-1} = \mathsf{refl}_y \\
&\qquad yg : y = g && && yog : yo \cdot og \cdot yg^{-1} = \mathsf{refl}_y \\
&\qquad yo : y = o && && ybo : yb \cdot bo \cdot yo^{-1} = \mathsf{refl}_y
\end{aligned}
$$

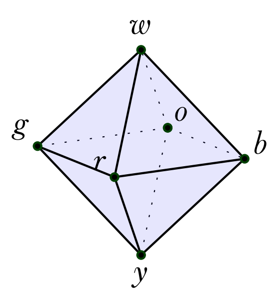

Figure 7: The HIT $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

We have obvious maps $\mathbb{O}_0 \xrightarrow{i_0} \mathbb{O}_1 \xrightarrow{i_1} \mathbb{O}$ that include each skeleton into the next-higher-dimensional skeleton.

3.4 Groupoid operations on higher inductive combinatorial manifolds

Let $M : \mathsf{SimCompSet}_2$ be a combinatorial 2-manifold and $\mathbb{M} \stackrel{\text{def}}{=} \mathcal{R}(M) : \mathsf{CombMfd}_2$ its realization as a higher inductive type. $\mathbb{M}$ has triangular 2-faces just as M does, except they are 2-paths in the HoTT sense. If two faces bca and bdc share the edge bc (see Figure 8), then we can form the type

$bca' : ba \cdot ac = bc$ by reorganizing the face as a path from the concatenation of two edges to the third edge. Similarly we can form $bdc' : bd \cdot dc = bc$. We can now concatenate these:

$$bca' \cdot bdc'^{-1} : ba \cdot ac = bd \cdot dc$$

which fills the 4-gon $abdc$, and so can be reorganized as a path $abdc : \mathsf{refl}_a = \mathsf{refl}_a$ or a path $bdca : \mathsf{refl}_b = \mathsf{refl}_b$ or other possibilities.

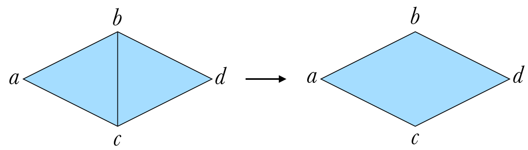

Figure 8: Concatenating the triangles abc and bdc gives the 4-gon $abdc$.

We will have two use cases for this operation. The first is to consider the concatenation of *all* the faces of $\mathbb{M}$, which if we choose a base point $x : \mathbb{M}$ is a path from refl_x to itself. This will play the role of the "fundamental homology class" from classical topology, which is an object on which 2-forms can be evaluated to compute their value on the whole manifold.

The second use case for concatenating faces is to create an *equivalent* type to $\mathbb{M}$ but without one of the point constructors. Figure 9 illustrates the equivalence.

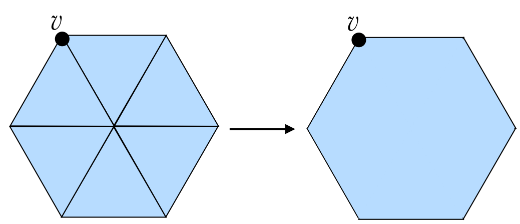

Figure 9: Concatenating the six triangles in the apprpropriate way produces a 2-path in $\mathsf{refl}_v = \mathsf{refl}_v$.

Definition 13. If $\mathbb{M}$ is a combinatorial manifold and $Z \subset \mathbb{M}_0$ is a set of vertices in $\mathbb{M}$ with members $Z = \{z_0, \ldots, z_n\}$, then denote by $\mathbb{M} \setminus Z$ the type given by omitting the vertices in Z from the constructors in all dimensions where they appeared. Call the points of Z **isolated** if no two of them are neighbors, i.e. we have $\prod_{z:Z} \mathsf{link}(z) \cap Z = \emptyset$. In the isolated case $\mathbb{M} \setminus Z$ has boundary circles where each vertex was removed.

Definition 14. If we have $\mathbb{M} \setminus Z$ for some isolated set of verticies Z, then for each $z : Z$ we can compose all the faces which contain z, forming a new face (see Figure 9). In this way we produce an equivalent type $\mathbb{M}_Z \simeq \mathbb{M}$ but which is no longer combinatorial since we have erased some of the edges from some of the neighborhoods. We call $\mathbb{M}_Z$ the **replacement of $\mathbb{M}$ without Z**.

3.5 The function link

Taking the link of a vertex gives us a map to polygons.

Definition 15. $\mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned} \mathsf{link}(w) &= brgo & \mathsf{link}(r) &= wbyg \\ \mathsf{link}(y) &= bogr & \mathsf{link}(g) &= wryo \\ \mathsf{link}(b) &= woyr & \mathsf{link}(o) &= wgyb \end{aligned}$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

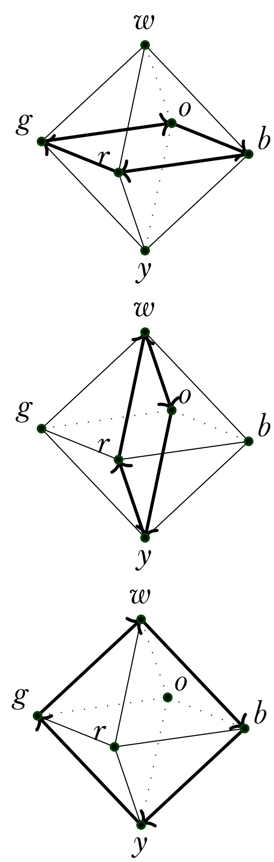

Figure 10: link for the verticies w, b and r.

To extend link to the 1-skeleton we have complete freedom. We will do something "tangent bundley", imagining how link changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 16. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending link as follows: Transport away from w:

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ (r, o fixed)
- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ (b, g fixed)
- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$

- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 10 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn.

Definition 17. The map $R : C_4 \to C_4$ rotates by one quarter turn, one "click":

- $R(c_1) = c_2$
- $R(c_2) = c_3$
- $R(c_3) = c_4$
- $R(c_4) = c_1$
- $R(c_1c_2) = c_2c_3$
- $R(c_2c_3) = c_3c_4$
- $R(c_3c_4) = c_4c_1$
- $R(c_4c_1) = c_1c_2$

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here there is no additional freedom.

Definition 18. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows:

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

where $H_R : R = \mathsf{refl}$ is the obvious homotopy.

All the faces do the same thing: they map to a homotopy between the identity and clockwise rotation by a quarter turn. Concatenating the eight faces in the 2-groupoid $\mathbb{O}$ would then map to a homotopy between the identity and two full rotations. This makes visible in HoTT the link between curvature and the Euler characteristic (which is 2 for the octahedron). More on that later.

3.6 The torus

We can define a combinatorial torus as a similar HIT. This time each vertex will have six neighbors. So all the links will be merely equal to C_6 which is a hexagonal version of C_4. See Figure 11.

To help parse this figure, imagine instead Figure 12. We take this simple alternating-triangle pattern, then glue the left and right edges, then bend into Figure 11. The fact that each column in Figure 12 has four dots corresponds to the torus in Figure 11 having a square in front, diamonds in the middle, and a square in back.

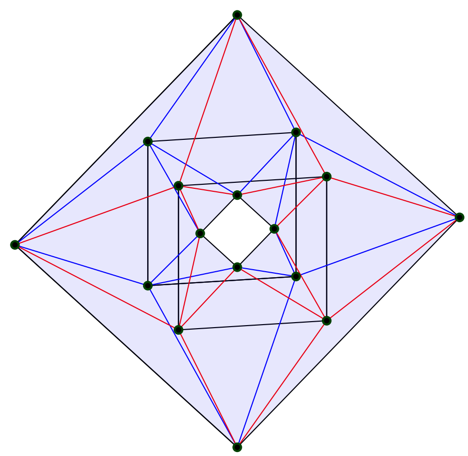

Figure 11: Torus embedded in 3-dimensional space. If you see color in your rendering then black lines trace four square-shaped paths, red ones connect the front square to the middle diamonds, and blue ones connect the back path to the middle ones.

3.7 Vector fields

A vector field is a partial function $T_\bullet : M \to \mathrm{EM}_\bullet(\mathbb{Z}, 1)$ that lifts T. In other words, a pointing of some of the fibers. This aligns with the classical picture of a choice of nonzero vector at each point, except for some points where the vector field vanishes. We will have to omit such vanishing points from the domain because we are very opinionated that the codomain be mere circles, and so the function is a partial function.

Figure 13 illustrates what removing a point looks like. The resulting type is no longer a combinatorial manifold, since it fails the condition about every point having a circular link.

Definition 19. Let M be a combinatorial manifold and Z an isolated set of vertices. A **vector field X on M with zero set Z** is a partial section of P, i.e. a term $X : \prod_{x:M\setminus Z} T(x)$. The **exponential map** $\exp : P \to M$ is the map sending points in a fiber to the corresponding point in the link of the base point: $\exp(x, y : \mathsf{link}(x)) = y$. In commutative diagram form we have:

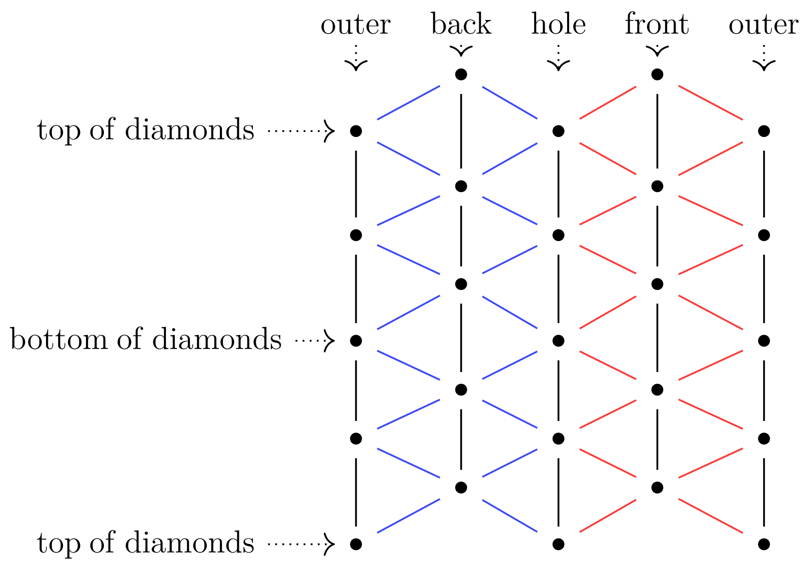

Figure 12: An inspiration for the torus. Identify the sides and then the top, definitionally, to get the actual torus.

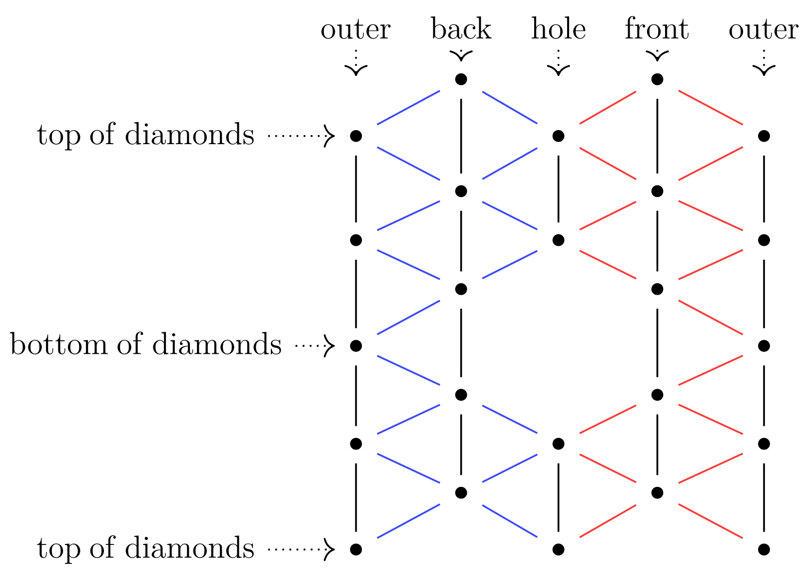

Figure 13: The flat torus with one vertex removed. This also removes the edges and faces containing that vertex.

$$
\begin{tikzcd}
P \overset{\text{def}}{=} \sum_{C:TM} C \arrow[r, "\overline{T}"] \arrow[d, "\mathrm{pr}_1"] \arrow[d, bend left, "\exp"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"] \\
M \setminus Z \arrow[r, "T"] \arrow[u, bend left, "X:\prod_{x:M\setminus Z} Tx"] \arrow[ur, "T_\bullet"] & \mathrm{EM}(\mathbb{Z}, 1)
\end{tikzcd}
$$

Where $T_\bullet = \overline{T} \circ X$. Note that exp is different from pr_1 since it spreads a fiber out onto the manifold. The composition $\exp \circ X$ is a map $M \setminus Z \to M$, and can be thought of as the flow of the vector field. It can be extended to a map $M \to M$ by taking the identity map on Z.

The vector field X is a map on all dimensions of $M \setminus Z$, not just the vertices. HoTT tells us that X also selects an "edgeover" for each edge, and "faceover" for each face. And these can be composed, so that we have an entire 2-groupoid $X(M \setminus Z)$ inside P. If $e_{12} : v_1 =_M v_2$ then $X(e_{12}) : \mathrm{tr}_{e_{12}}(X(v_1)) = X(v_2)$, which is a path in $T(v_2)$ that ends at the selected point $X(v_2) : T(v_2)$. The same goes for loops: given a loop $\ell : v_1 =_M v_1$ we have $X(\ell) : \mathrm{tr}_\ell(X(v_1)) = X(v_1)$. It doesn't need to be a loop upstairs but it might be. Of course we always have $X(\mathsf{refl}_v) = \mathsf{refl}_{X(v)}$.

Faceovers are homotopies from $\mathsf{refl}_{X(v)}$ to a path $a = X(v)$ in some fiber $T(v)$. The faceover implies that the loopover has winding number 0. Which brings us to index.

Definition 20. The **index** of X at an isolated vertex $z : Z$ is the degree of the map $\exp \circ \mathrm{tr} : \mathsf{link}(z) \to \mathsf{link}(z)$. The map tr transports P and X along the spokes joining the link to the point z. Although z is not in the domain of X, the hypothesis of z being isolated implies that the link is in the domain of X.

We will revisit the index later when we discuss the total index. Figure 14 shows some discrete versions of the classical examples of index $+1$, -1 and 0. One note: the figures are only illustrating the value of the vector field on vertices. The apd operation could in theory spin the vector many times around the link on its way from one vertex to the next. We will assume at all times that the smallest movement is taking place, and we leave it to the imagination whether that movement is clockwise or counterclockwise. A formalization would have to specify this fully of course.

4 The connection to connections

We don't have two definitions of connections in HoTT that we can prove are equivalent. Instead we'll compare some key features of the classical theory to the system we have sketched above. It is my hope that differential geometers could onboard into HoTT through this work, and that someone new to geometry could learn it here as well. In both cases a dictionary of sorts would be helpful.

4.1 Classical version

Definition 21. A **principal bundle** is a smooth map $\pi : P \to M$ between smooth manifolds such that

1. All the fibers of π are equivalent as a smooth manifold to a fixed Lie group G.
2. There is a smooth G-action $P \times G \to P$ on the right that acts on fibers, and does so freely and transitively.

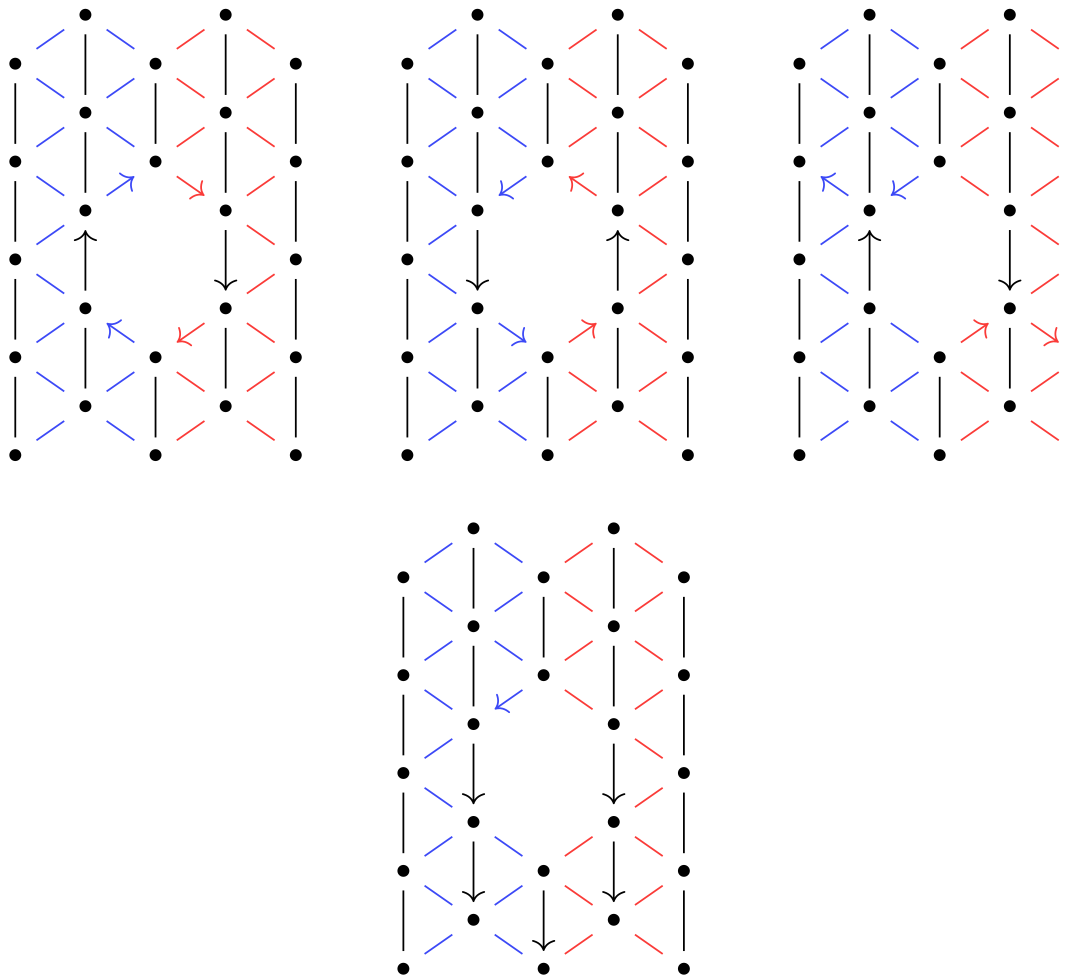

Figure 14: Four vector fields around a zero. The first has index 1 and circulates clockwise. The second also has index 1 despite circulating counterclockwise, because if you trace a clockwise path the vector also rotates clockwise. The third has index -1 since it circulates counterclockwise as you trace a clockwise path. The fourth has index 0 and could in fact be extended to the hole.

3. There exists an open cover $\{U_i\}$ of M and equivariant diffeomorphisms $\phi_i : P|_{U_i} \to U_i \times G$ (i.e. $\phi_i(p \cdot g) = \phi_i(p) \cdot g$).

Physicists call principal bundle automorphisms "gauge transformations":

Definition 22. A **gauge transformation** is a map $\Phi : P \to P$ commuting with the projection to M and which is G-equivariant, i.e. $\Phi(p \cdot g) = \Phi(p) \cdot g$. Denote the group of gauge transformations by $\operatorname{Aut} P$. In the literature it is sometimes denoted $\mathscr{G}(P)$.

Definition 23. The **vertical bundle** VP of a principal bundle $\pi : P \to M$ with Lie group G is the kernel of the derivative $T\pi : TP \to TM$.

VP can be visualized as the collection of tangent vectors that point along the fibers. It should be clear that at each point of M the group G acts on VP, sending vertical vectors to vertical vectors. In other words, $\operatorname{Aut} P$ acts on VP.

Definition 24. An **Ehresmann connection** on a principal bundle $\pi : P \to M$ with Lie group G is a splitting $TP = VP \oplus HP$ at every point of P into vertical and complementary "horizontal" subspaces, which is preserved by the action of $\operatorname{Aut} P$.

Being preserved by the action of $\operatorname{Aut} P$ implies that the complementary horizontal subspaces in a given fiber of $\pi : P \to M$ are determined by the splitting at any single point in the fiber. The action of G on this fiber can then push the splitting around to all the other points.

The utility and parsimony of this definition relates to the solvability of ordinary differential equations. We now have an isomorphism $T_p\pi : H_pP \simeq T_{\pi(p)}M$ between each horizontal space and the tangent space below it in M. This means that given a tangent vector at $x : M$ and a point p in $\pi^{-1}(x)$ we can uniquely lift the tangent vector to a horizontal vector at p. We can also lift vector fields and paths in this way. To lift a path $\gamma : [0, 1] \to M$ you must specify a lift for $\gamma(0)$ and then lift the tangent vectors of γ and prove that you can integrate the lift of that vector field upstairs in HP.

Armed with the lifting of paths one immediately obtains isomorphisms between the fibers of P: given a path in M we can map the starting point of a lift to the ending point. So the three constructions: the Ehresmann connection, the lifting of paths, and transport isomorphisms between fibers are all recapitulations of the structure that the connection adds to the bundle.

4.1.1 Gauge theory

Given a bundle $\pi : P \to M$ there is a space of connections $\mathscr{A}(P)$. The group $\operatorname{Aut} P$ acts on this space. For example, a gauge transformation that is constant in the neighborhood of a point will not change the splitting, it will just shift the fiber rigidly along itself. But at the other extreme, a gauge transformation that is changing rapidly near a point will tilt the horizontal subspaces rapidly. The field of **gauge theory** begins with a study of the quotient space $\mathscr{A}(P)/\operatorname{Aut} P$.

Note 2. Recall that torsors have a physical interpretation as a quantity without a specified unit, such as mass, length, or time. When we choose a base point in a torsor it becomes the standard torsor G acting on itself (for example, the additive real numbers). A physicist is looking for properties or laws that are independent of such a choice. In the 20th century physicists further wondered about choices of units that vary from point to point, and began searching for laws that are invariant under this much larger space of transformations. This led directly to the discovery of connections and curvature as useful fields that complement the matter fields, which are sections of associated vector bundles. They were then led to explore quotienting by the action of the group of gauge transformations, and in particular the space of connections "mod gauge." In this scenario the base manifold M is spacetime, and a gauge transformation is a smoothly varying choice of gauge (units) at each point.

We can characterize connections and curvature in terms of splittings of certain sequences. Atiyah and Bott ([13] equation 3.4) describe the space of vector fields on a total space P as a Lie algebra extension of ΓTM by $\Gamma \operatorname{ad} P$, respectively the Lie algebra of vector fields on the base and vertical vector fields on P. A non-flat connection will fail to split this sequence because horizontal vector fields may have a non-horizontal component when taking the Lie bracket. This extension is referred to as the *Atiyah sequence*.

In this century mathematicians in HoTT and HoTT-adjacent fields sought an *integrated Atiyah sequence*, including Urs Schreiber[14][15]. This would be a Lie groupoidal version of the Atiyah sequence on Lie algebras. If a groupoid extension could be examined, a link could be sought to Schreier theory. We'll return to these ideas in the next section.

4.2 Type theory version

Moving now to HoTT, fix a type $M : \mathcal{U}$ and a type family $f : M \to \mathcal{U}$. Path induction gives us the transport isomorphism $\prod_{p:a=_M b} \operatorname{tr}(p) : f(a) = f(b)$. We can use this to define a type of *dependent paths*, also called *pathovers* or *paths over* a given path.

Definition 25. With the above context and points $\alpha : f(a)$, $\beta : f(b)$ the type of **dependent paths over** p with endpoints α, β is denoted

$$\alpha \underset{p}{\overset{=}{\to}} \beta.$$

By induction we can assume p is refl_a in which case $\alpha \underset{p}{\overset{=}{\to}} \beta$ is $\alpha =_{f(a)} \alpha$.

See [16] for more discussion of dependent paths (where they use the term "path over"), including composition, and associativity thereof.

We recall now the identity type of sigma types:

Theorem 2. (HoTT book Theorem 2.7.2 [12]) If $f : M \to \mathcal{U}$ is a type family and $\alpha, \beta : \sum_{x:M} f(x)$ then there is an equivalence

$$\mathsf{split} : (\alpha = \beta) \simeq \sum_{p:\mathrm{pr}_1(\alpha)=_M \mathrm{pr}_1(\beta)} [\operatorname{tr}(p)(\mathrm{pr}_2(\alpha))] = \mathrm{pr}_2(\beta).$$

Definition 26. Given $p : a =_M b$ and $\alpha : f(a)$ we have

$$\left(\alpha \underset{p}{\overset{=}{\to}} \operatorname{tr}(p)(\alpha)\right) \simeq \left(\operatorname{tr}(p)(\alpha) =_{f(b)} \operatorname{tr}(p)(\alpha)\right)$$

which has the term refl which we can call **the horizontal lift of p starting at α**.

We can imitate the classical definition of a connection by defining $\omega \overset{\text{def}}{=} \mathrm{pr}_2 \circ \mathsf{split}$, the projection onto the vertical component.

In HoTT if the bundle is classified by $f : M \to \mathcal{U}$ then an automorphism is a homotopy $H : f \sim f$ and the group of gauge transformations is the loop space $\Omega_f(M \to \mathcal{U})$. What is the effect of applying a homotopy $H : f \sim f$ on transport, and on splitting?

H is a family of fiber automorphisms: $H : \prod_{a:M} f(a) = f(a)$ which we can assemble into an equivalence $H' : \sum_{a:M} f(a) = \sum_{a:M} f(a)$ that acts fiberwise. We want to compute the action of $\mathsf{ap}(H')$ on the horizontal-vertical decomposition of paths from Theorem 2 by computing $\omega \circ \mathsf{ap}(H') = \mathrm{pr}_2 \circ \mathsf{split} \circ \mathsf{ap}(H')$.

Denote $\sum_{a:M} f(a)$ by P. Let $p : a =_M b$ be a path in the base and let $\pi : (a, \alpha) =_P (b, \beta)$ be a path in P over p. Then $\omega(\pi) : \mathrm{tr}_p(\alpha) = \beta$.

Now let's apply H. We have $\mathsf{ap}(H')(\pi) : (a, H(a)(\alpha)) =_P (b, H(b)(\beta))$ which is still a path over p. Applying ω we get

$$\omega(\mathsf{ap}(H')(\pi)) : \mathrm{tr}_p(H(a)(\alpha)) = (H(b)(\beta)).$$

Using the lemma below we can if we wish rewrite this as

$$\omega(\mathsf{ap}(H')(\pi)) : H(b) \left[\mathrm{tr}_p(\alpha) = \beta\right]$$

which uses only $H(b)$. This is the action of gauge transformations on connections.

Lemma 4. Given a function $f : M \to \mathcal{U}$, path $p : a =_M b$, and homotopy $H : f \sim f$ the following square commutes and so in the type family we have $\mathrm{tr}(H(x) \cdot f(p)) = \mathrm{tr}(f(p) \cdot H(y))$.

$$\begin{tikzcd} f(a) \arrow[r, equal, "f(p)"] \arrow[d, equal, "H(a)"'] & f(b) \arrow[d, equal, "H(b)"] \\ f(a) \arrow[r, equal, "f(p)"'] & f(b) \end{tikzcd}$$

HoTT provides new ways to talk about trivializations of bundles, and flatness of connections.

Theorem 3. The following are equivalent for a map $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$:

1. The principal bundle $P \overset{\text{def}}{=} \sum_{x:M} Tx$ is a trivial bundle.
2. The dependent sum $\sum_{x:M} Tx$ is equivalent to a non-dependent sum.
3. There exists a total lifting $T_\bullet$ to pointed types.
4. T is contractible.

Proof. 1 and 2 are equivalent by definition. 1 implies 3 by choosing the basepoint of the second factor of $M \times S^1$. 3 implies 1 because the global choice of basepoint is a global isomorphism with S^1. 1 and 4 are equivalent because a trivialization is a contraction to $M \times S^1$. □

The classical definition of a flat connection is that contractible loops lift to horizontal loops, i.e. there is no holonomy around small loops. This implies that homotopic paths have the same transport. Here's how we'll describe this in HoTT:

Definition 27. We call a connection on $\sum_{x:M} Tx$ **flat** if T factors through the 1-truncation $||M||_1$.

Lemma 5. If $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ is flat and M is simply connected then T is trivial.

Proof. T factors through $||M||_1$ which is contractible. □

4.2.1 Gauge theory revisited

The sigma type $P \stackrel{\text{def}}{=} \sum_{x:M} Tx$ and projection map to M package the insights of the Atiyah sequence and observations about what does and doesn't split:

The fiber sequence $S^1 \to P \to M$ does not split unless P is trivial, by Theorem 3.

Paths $p : a =_M b$ do lift to P given a starting point $\alpha : Ta$. This is what we are calling the connection, and it is the finite version of the vertical/horizontal splitting $TP = VP \oplus HP$. Theorem 2 provides the factoring of pathovers into horizontal and vertical. So at the level of paths there *is* a splitting, a map from M to P.

But suppose we have two paths $p, q : a =_M b$ and a point α over a. If we concatenate the two horizontal lifts $(p, \mathsf{refl}_{\mathrm{tr}(p)(\alpha)})$ and $(q^{-1}, \mathsf{refl}_{\mathrm{tr}(q^{-1})\circ\mathrm{tr}(p)(\alpha)})$ into a loopover of $p \cdot q^{-1}$ then we get a term in $(p \cdot q^{-1}, \mathrm{tr}(q^{-1}) \circ \mathrm{tr}(p)(\alpha) = \alpha)$. As we have seen this can be a non-refl path in Ta! The concatenation of two horizontal lifts can be non-horizontal. This is the analogous statement to Atiyah's observation that the Lie bracket of two horizontal vectors can have a vertical component, and that this can be identified with curvature.

To work with connections mod gauge we don't really have to add anything to our existing picture, because we are always working with higher types such as $M \to \mathrm{EM}(\mathbb{Z}, 1)$ which have automorphisms (in this case gauge transformations, aka self-homotopies) baked in.

Thinking back to the desired link with Schreier theory, David Jaz Myers showed that in the case of higher groups we have an equivalence between the type of extensions of a group G by F and the actions of G on a delooping BF:

$$\mathrm{Ext}(G; F) \simeq (BG \cdot\!\!\to \mathrm{BAut}(BF))$$

(see [17] Theorem 2.5.7). Our type of classifying maps $M \to \mathrm{EM}(\mathbb{Z}, 1)$ can be seen as extensions of $\pi_1(M)$, or of M itself, by the group S^1. What a lovely reframing of principal bundles.

5 Leibniz, Gauss-Bonnet, Poincaré-Hopf

5.1 The Leibniz (product) rule

The intuition that motivates everything in this note is that derivatives and connections are visible in type theory through the action on paths, because a path is a finite version of an infinitesimal tangent vector. And if we think we have some new understanding of differentiation, then we should be able to see the Leibniz rule.

The Leibniz rule, or product rule, for differentiation states that if $f, g : M \to \mathbb{R}$ are two smooth functions to the real numbers then $d(fg) = fdg + gdf$. Here fg is the function formed by taking the pointwise product of f and g. This is an interaction between multiplication in $\mathbb{R}$ and the action on vectors of smooth functions (the 1-forms df and dg).

To examine this situation in HoTT we need type-theoretic functions $f, g : M \to B$ from some type M to a central H-space B. Let $\mu : B \to B \to B$ be the H-space multiplication. How does μ act on paths? Suppose we have $a, a', b, b' : B$ and $p : a =_B a', q : b =_B b'$. Then we also have homotopies $\mu(p, -) : \mu(a, -) =_{B \to B} \mu(a', -)$ and $\mu(-, q) : \mu(-, b) =_{B \to B} \mu(-, b')$. Since $\mu(a, -) : B = B$ is an (unpointed) equivalence of B, and similarly for $\mu(b, -)$ and so on, this data assembles into the following diagram of higher groupoid morphisms:

$$\begin{tikzcd} B \arrow[r, bend left, "\mu(a,-)"] \arrow[r, bend right, "\mu(a',-)"'] \arrow[r, phantom, "\mu(p,-)\Downarrow"] & B \arrow[r, bend left, "\mu(-,b)"] \arrow[r, bend right, "\mu(-,b')"'] \arrow[r, phantom, "\mu(-,q)\Downarrow"] & B \end{tikzcd}$$

And so the two homotopies can be horizontally composed to give a path

$$\mu(p, -) \star \mu(-, q) : \mu(a, b) = \mu(a', b').$$

Horizontal composition is given by

$$\mu(p, -) \star \mu(-, q) \overset{\text{def}}{=} (\mu(p, -) \cdot_r \mu(-, b)) \cdot (\mu(a', -) \cdot_l \mu(-, q))$$

where

$$\mu(p, -) \cdot_r \mu(-, b) : \mu(a, b) = \mu(a', b)$$

and

$$\mu(a', -) \cdot_l \mu(-, q) : \mu(a', b) = \mu(a', b')$$

are defined by path induction. See the HoTT book Theorem 2.1.6 on the Eckmann-Hilton argument[12].

We can recognize the process of using whiskering to form horizontal composition in the Leibniz rule.

Quick aside: moving from infinitesimal calculus to finite groupoid algebra actually involves two changes. The first is the change from vectors to paths, forms to functions and so on. But it's also the case that tangent vectors have just the one basepoint, whereas paths have two endpoints. You can see this play out in this example, where a and a' were distinct points (and b and b').

The horizontal composition we build lives entirely in B and we didn't make use of M yet. The Leibniz rule will be a pointwise version of what's going on in B. Denote by $\mu \circ (f, g) : M \to B$ the map which sends $x \mapsto \mu(f(x), g(x))$.

Lemma 6. Given $f, g : M \to B$ and $p : x =_M y$ then

$$\begin{aligned}\mathsf{ap}(\mu \circ (f, g))(p) &= \mu(f(p), -) \star \mu(-, g(p)) \\ &= [\mu(f(p), -) \cdot_r \mu(-, g(x))] \cdot [\mu(f(y), -) \cdot_l \mu(-, g(p))] \\ &: \mu(f(x), g(x)) = \mu(f(y), g(y))\end{aligned}$$

5.2 The total curvature

Definition 28. Let $F(M)$ denote the groupoid of faces of M generated by the faces in its constructor. Let $V : \prod_{f:F(M)} f$ be a selected vertex of every face, and let $\partial : \prod_{f:F(M)} Vf =_M Vf$ be the clockwise boundary path around a face. Let v be the basepoint of M. Then in particular we have $M : F(M)$ the concatenation of all faces (i.e. M itself), and we have $\partial(M) : v =_M v$ which is the concatenation of all boundaries.

Observe that each generating edge of M is visited once in each direction in the concatenation $\partial(M)$.

The bundle map T can be examined on each face. On a face f bounding a loop $\ell : v =_M v$ the map $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ assigns a homotopy $T(f) : \mathsf{refl}_v = T(\ell)$, where $T(\ell)$ is an automorphism of $T(v)$. On the composition of all faces we have $T(M) : T(v) = T(v)$, the **total curvature**.

It is crucial that $T(M)$ can be nontrivial (i.e. not refl_v) even though it's computed by transporting across every edge in $\partial(M)$ once in each direction!

5.3 The total index

See Figure 15 where the outlined hexagons are the composition of the six triangular faces around a zero.

If X is a vector field with isolated zeros on Z, then M_Z is a convenient replacement for M because X is a partial function defined on all of M_Z except for a collection of faces, each of which bounds one of the zeros of Z.

Note 3. If the combinatorial manifold was imported into type theory via some process of sampling or other, perhpaps more theoretical, construction then we should allow for zeros to occur on verticies, edges, or faces. We can reduce the case of a zero on a vertex or edge to that of a face with the replacement method just described. The original combinatorial neighborhood structure continues to live in the fibers of T, it's only the base manifold that has been replaced.

Definition 29. Given a face $f : F(M \setminus Z)$, define **the index of** X **on** f to be the winding number (also called the degree) of $X(\partial f)$. The **total index** of X on $M \setminus Z$ is the index of X on $\partial(M \setminus Z)$.

The index computation skips some faces of M and can most definitely be nonzero.

5.4 Equality of total index and total curvature

Here we closely follow the classical proof of Hopf[4], presented in detail in Needham[5].

Theorem 4. The total curvature's winding number is equal to the total index of X.

The fact that the index was already an integer, whereas the curvature is a path accounts for the factor of 2π in some classical statement of this theorem (e.g. [5] section 26.3).

Proof. Let C be a fixed pointed polygon (e.g. C_4 or C_6 which we met earlier). We can consider X to map each vertex of M_Z to C. On each face of M_Z (no faces are omitted) we traverse the boundary, which is in the domain of both T and of X. So we avoid the complication that X is not defined on some of the faces by remaining on the 1-skeleton. For a given edge $e_{12} : v_1 =_{M_Z} v_2$ the function $\mathrm{tr}_{e_{12}} X(v_1) = X(v_2)$ is a path in C. We will traverse every edge twice, once in each direction, which will cancel to give refl_v. In particular the boundaries around the zeros of X are themselves traversed twice in opposite directions, canceling the total index. □

Corollary 3. The total index of a vector field with isolated zeros is independent of the vector field.

Corollary 4. The total curvature is an integer.

The last step is to link this value to the Euler characteristic.

5.5 Identification with Euler characteristic

Combinatorial manifolds are intuitive objects that connect directly to the classical definition of Euler characteristic. We can argue using Morse theory, the study of smooth real-valued functions on smooth manifolds and their singularities. Classically the gradient of a Morse function is a vector field that can be used to decompose the manifold into its *handlebody decomposition*. This would be an excellent story to pursue in future work.

Imagine a combinatorial manifold of a genus g oriented surface standing upright with the holes forming a vertical sequence. Now install a vector field that points downward whenever possible. This vector field will have a zero at the top and bottom, and one at the top and bottom of each hole. The top and bottom will be index 1, and ones around the holes will be index -1. We include some sketches in the case of a torus. This illustrates how we obtain the formula for genus g: $\chi(M) = 2 - 2g$.

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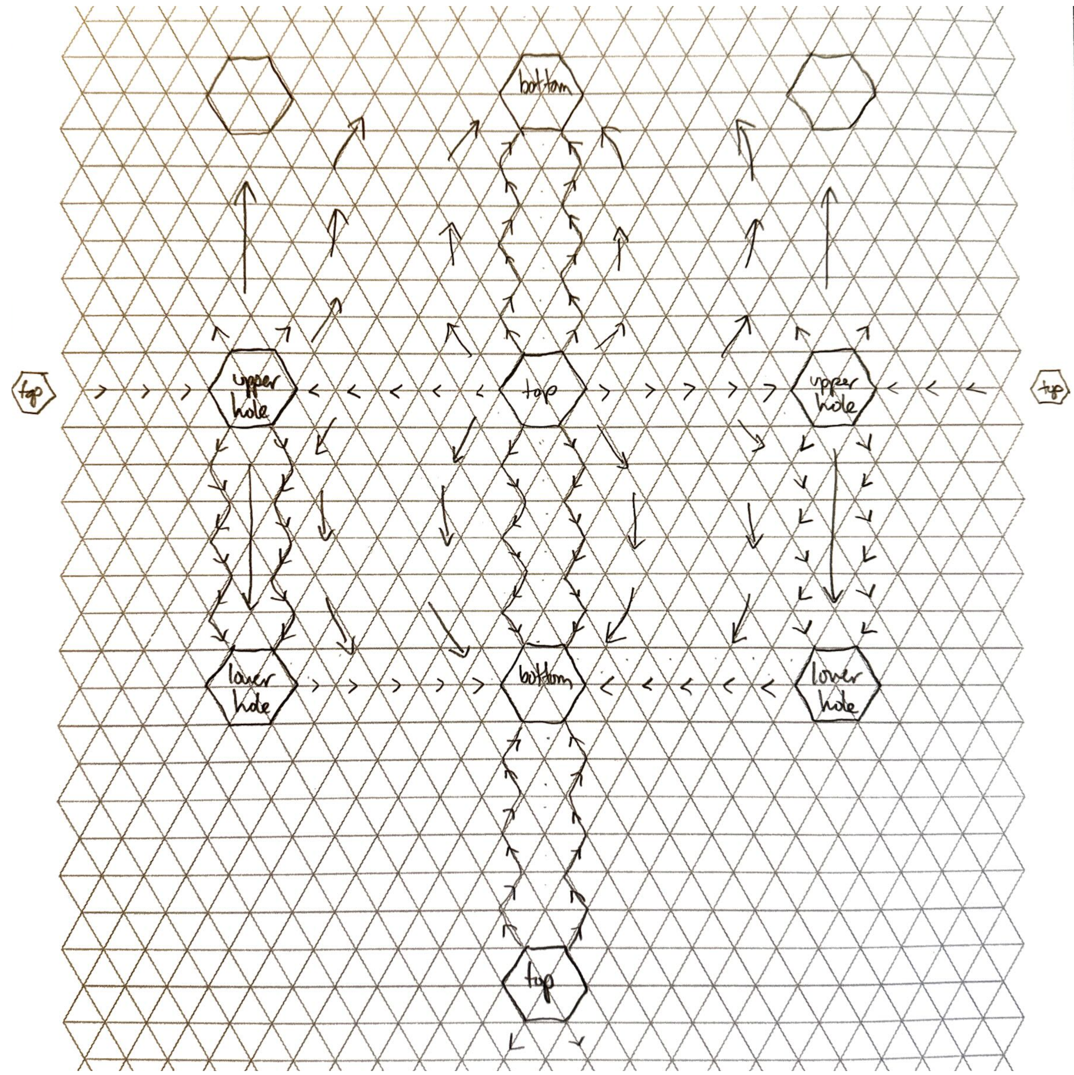

Figure 15: A sketch of downward flow on the torus.

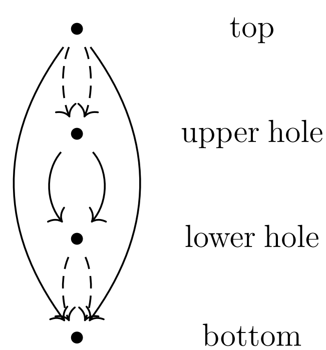

Figure 16: Schematic of the zeros of the downward flow of a torus.

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