Discrete differential geometry in homotopy type theory

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Abstract

Type families on higher inductive types such as pushouts can capture the homotopical properties of differential geometric constructions including connections, curvature, gauge transformations, and vector fields. We define a class of combinatorial pushouts, then define principal bundles, tangent bundles, connections, curvature, and vector fields on these. We draw inspiration in part from the young field of discrete differential geometry, and in part from the original classical proofs, which often make use of triangulations and other discrete arguments. We prove an equality relating the Gauss-Bonnet theorem to the Poincaré-Hopf theorem. We also attempt to map out future directions.

This thesis is dedicated to John Baez, Sean M. Carroll, Sabine Hossenfelder, and other communicators who are carrying the torch of science forward in the spirit of my hero Carl Sagan. I have followed you all for many years, and you have inspired me to continue my studies alongside my career. Thank you.

"It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, The Future Does Not Compute*[1]*

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Todo list

Contents

1 *Overview*

The outline is that we will define

- principal bundles in Section 2,
- simplicial complexes, and homotopical realizations of these in Section 3,
- vector fields in Section 5,

and observe emerging from those definitions the presence of

- connections and curvature in Section 4,
- the index of a vector field in Section 6,

and then define in Section 6

- the total curvature, as in the Gauss-Bonnet theorem
- the total index of a vector field, as in Poincaré-Hopf theorem,
- and prove the equality of these to each other.

We will build up an example of all of these structures on an octahedron model of the sphere, and compute its Euler characteristic of 2. We will not, however, be supplying a separate definition of Euler characteristic so as to truly reproduce the Gauss-Bonnet and Poincaré-Hopf theorems.

We will consider functions $\mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ where $\mathrm{EM}(\mathbb{Z}, 1)$ is the connected component in the universe of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 1)$ which we will take to be S^1, and where $\mathbb{M}$ is a combinatorial manifold of dimension 2, which is a simplicial complex encoded in a higher inductive type, such that each vertex has a neighborhood that looks like a disk with a discrete circle boundary (i.e. a polygon). We can call terms $C : \mathrm{EM}(\mathbb{Z}, 1)$ "mere circles."

We will see in Section 3.3 that $\mathrm{EM}(\mathbb{Z}, 1)$ contains all the polygons. We will construct a map $\mathsf{link} : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ that maps each vertex to the polygon consisting of its neighbors. Then we can consider the type of pointed mere circles $\mathrm{EM}_\bullet(\mathbb{Z}, 1) \overset{\text{def}}{=} \sum_{Y:\mathrm{EM}(\mathbb{Z},1)} Y$ as well as the first projection that forgets the point. This is a univalent fibration (univalent fibrations are always equivalent to a projection of a type of pointed types to some connected component of the universe[2]). If we form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z}, 1) \arrow[d, "\mathrm{pr}_1"'] \\ \mathbb{M} \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z}, 1) \end{tikzcd}$$

then we have a bundle of mere circles, with total space given by the $\sum$-type construction. We will show that this is not quite a principal bundle, i.e. a bundle of torsors. Torsors are types with the additional structure of a free and transitive group action. But if link satisfies an additional property (amounting to an orientation) then the pullback is a principal fibration, i.e. link factors through a map $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$, where $\mathrm{K}(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space.

We will argue that extending link to a function T on paths can be thought of as constructing a connection, together with a "flatness structure," i.e. a proof of flatness. Moreover, lifting T to $X : \mathbb{M} \to \mathrm{EM}_{\bullet}(\mathbb{Z}, 1)$ can be thought of as a nonvanishing vector field. There will in general not be a total lift, just a lift on the 1-skeleton of $\mathbb{M}$. We will define the "index" of X at a face.

We will then define a method for visiting all the faces of a manifold in order to form "totals" of local objects. We will examine the total curvature and the total index and prove that they are equal, and argue that they are equal to the usual Euler characteristic. This will simultaneously prove the Poincaré-Hopf theorem and Gauss-Bonnet theorem in 2 dimensions, for combinatorial manifolds. This is similar to the classical proof of Hopf[3], presented in detail in Needham[4].

2 *Torsors and principal bundles*

Differential geometry is the study of principal bundles and their automorphisms. Principal bundles are bundles of torsors, so we start there.

2.1 Torsors

Definition 2.1. Let G be a group with identity element e (with the usual classical structure and properties). A G**-set** is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If in addition we have a term

$$\mathsf{is\_torsor} : ||X||_{-1} \times \prod_{x:X} \mathsf{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

then we call this data a G**-torsor**. Denote the type of G-torsors by BG.

If $(X, \phi), (Y, \psi) : BG$ then a G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 2.2. There is a natural equivalence $(X =_{BG} Y) \simeq (X \to_G Y)$. □

Denote by $*$ the torsor given by G actions on its underlying set by left-translation. This serves as a basepoint for BG and we have a group isomorphism $\Omega BG \simeq G$.

Lemma 2.3. A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $* \to_G X$. □

Corollary 2.4. The pointed type $(BG, *)$ is a $\mathrm{K}(G, 1)$. □

In particular, to classify principal S^1-bundles we map into the space $\mathrm{K}(S^1, 1)$, a type of torsors of the circle. Since S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$, we have $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$.

2.2 Bundles of Eilenberg-Mac Lane spaces

We find it illuminating to look also at the slightly more general classifying space of $\mathrm{K}(\mathbb{Z}, 1)$-bundles, that is bundles whose fiber are equivalent to $\mathrm{K}(\mathbb{Z}, 1)$. We can understand very well when these are in fact bundles of circle torsors, which will in turn shed light on orientation in this setting.

We will follow Scoccola[5]. We will state the definitions and theorems for a general $\mathrm{K}(G, n)$ but we will be focusing on $n = 1$ in this note.

Definition 2.5. Let $\mathrm{EM}(G, n) \stackrel{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \stackrel{\text{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$. A $\mathrm{K}(G, n)$**-bundle** on a type M is the fiber of a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses two self-maps on the universe: suspension followed by $(n + 1)$-truncation $||\Sigma||_{n+1}$ and forgetting a point $F_\bullet$ to form the composition

$$\mathrm{EM}(G, n) \xrightarrow{||\Sigma||_{n+1}} \mathrm{EM}_{\bullet\bullet}(G, n + 1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G, n + 1)$$

from types to types with two points (north and south), to pointed types (by forgetting the south point).

Definition 2.6. Given $f : M \to \mathrm{EM}(G,n)$, the **associated action of M on G**, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ ||\Sigma||_{n+1} \circ f$.

Theorem 2.7. (Scoccola[5] Proposition 2.39). A $\mathrm{K}(G,n)$ bundle $f : M \to \mathrm{EM}(G,n)$ is equivalent to a map in $M \to \mathrm{K}(G,n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is contractible.

Let's relate this to *orientation*. Note that the obstruction in the theorem is about a map into $\mathrm{EM}_\bullet(G,n+1)$ and further note that $\mathrm{EM}_\bullet(G,n) \simeq \mathrm{K}(\mathrm{Aut}\,G,1)$ (independent of n). The theorem says that the data of a map into $\mathrm{EM}(G,n)$ factors into data about a map into $\mathrm{K}(G,n+1)$ and one into $\mathrm{K}(\mathrm{Aut}\,G,1)$. Informally, $\mathrm{EM}(G,n)$ is a little too large to be a $K(G,n+1)$, as it includes data about automorphisms of G.

In the special case of $\mathrm{EM}(\mathbb{Z},1)$ the conditions of the theorem are met when $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z},1)$ is contractible. $\mathrm{Aut}\,\mathbb{Z}$ consists of the $\mathbb{Z}/2\mathbb{Z}$ worth of outer automorphisms given by multiplication by ± 1. If we look at the fiber sequence

$$\mathrm{K}(S^1,1) \to \mathrm{BAut}\,S^1 \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z},1)$$

we see the automorphisms of the circle as an extension of the group of automorphisms that are homotopic to the identity (which are the torsorial actions) by the group that sends the loop in S^1 to its inverse. This is another way to see that a map $f : M \to \mathrm{BAut}\,S^1 \simeq \mathrm{EM}(\mathbb{Z},1)$ factors through $\mathrm{K}(S^1,1) \simeq \mathrm{K}(\mathbb{Z},2)$ if and only if the composition to $\mathrm{K}(\mathrm{Aut}\,\mathbb{Z},1)$ is trivial. This amounts to a choice of loop-direction for all the circles, and so deserves the name "f is *oriented*." In addition the map $\mathrm{BAut}\,S^1 \to \mathrm{K}(\mathrm{Aut}\,\mathbb{Z},1)$ deserves to be called the first Stiefel-Whitney class of f, and the requirement here is that it vanishes. This point of view is discussed in Schreiber[6] (starting with Example 1.2.138) and in Myers[7].

> **Remark 2.8.** Bundles of oriented mere circles are principal, but this fact does not hold for bundles of higher-dimensional spheres. Since this note will focus on 2-dimensional oriented manifolds we will be making use of this coincidence.

2.3 Pathovers

Suppose we have $T : M \to \mathcal{U}$ and $P \stackrel{\text{def}}{=} \sum_{x:M} Tx$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : Ta = Tb$. This is a path in the codomain, which in this case is a type of types. Type theory also provides a function called *transport*, denoted $\mathsf{tr}(p) : Ta \to Tb$ which acts on the fibers of P. $\mathsf{tr}(p)$ is a function, acting on the terms of the types Ta and Tb, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathsf{tr}(p)(\alpha) =_{Tb} \beta$ between $\alpha : Ta$ and $\beta : Tb$ in the fibers. We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

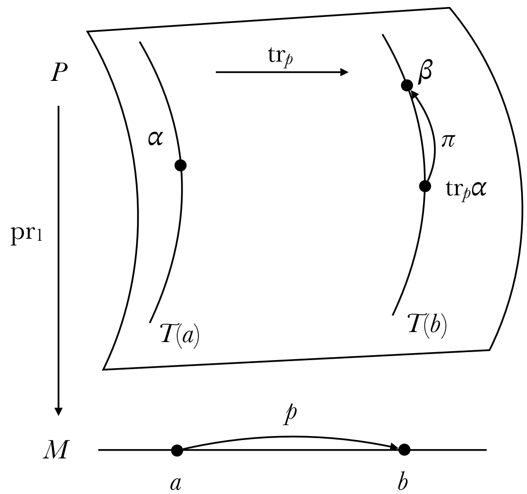

Figure 1: A path π over the path p in the base involves the transport function.

Lastly we want to recall that in the presence of a section $X : M \to P$ there is a dependent generalization of ap called apd: $\mathsf{apd}(X)(p) : \mathsf{tr}(p)(X(a)) = X(b)$ which is a pathover between the two values of the section over the basepoints of the path p.

2.4 The bundle of automorphisms

Definition 2.9. Again, suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ and $P \overset{\text{def}}{=} \sum_{x:M} Tx$. Then we can form the type family $\operatorname{Aut} T : M \to \mathcal{U}$ given by $\operatorname{Aut} T \overset{\text{def}}{=} \lambda(x : M) \to (Tx = Tx)$. The total space $\operatorname{Aut} P \overset{\text{def}}{=} \sum_{x:M}(Tx = Tx)$, which is a bundle of groups, is called the **automorphism bundle** or the **gauge bundle** and sections $\prod_{x:M}(Tx = Tx)$, which are homotopies $T \sim T$, are called **automorphisms of** P or **gauge transformations**.

If the automorphism group of the fibers is commutative then the automorphism bundle is trivial.

Lemma 2.10. Suppose M is connected and we have a point $m : M$. Then we have a trivialization $\operatorname{Aut} P \simeq M \times (Tm = Tm)$. This is equivalent to saying the map $\operatorname{Aut} T$ is equivalent to a constant map.

Proof. We will produce a term $\tau : \prod_{x:M}((Tx = Tx) \xrightarrow{\simeq} (Tm = Tm))$. Suppose we have a path $p : x =_M m$, and define $\tau x = \mathsf{tr}(p)$. Transport in the family $\operatorname{Aut} T$ is given by concatenation (see [8] Theorem 2.11.3), i.e. $\mathsf{tr}(p)(f) = T(p)^{-1} \cdot f \cdot T(p)$. To show that τ is independent of the choice of p, suppose we have $p, q : x =_M m$. Then $\mathsf{tr}(p \cdot q^{-1}) = T(p \cdot q^{-1})^{-1} \cdot - \cdot T(p \cdot q^{-1})$ which is conjugation in the commutative group $Tx = Tx$, hence is the identity. This proves that $\mathsf{tr}(p) = \mathsf{tr}(q)$. $\square$

3 Discrete manifolds

We will remind ourselves of the definition of a classical simplicial complex, in sets. Then we will create a higher type from the data of a complex, using pushouts. The type will both imitate the structure of the set-based complex and will contain explicit maps into it from the sets of the complex.

3.1 Abstract simplicial complexes

Definition 3.1. An **abstract simplicial complex** $M \overset{\text{def}}{=} [M_0, \ldots, M_n]$ **of dimension** n is an ordered list consisting of a set M_0 of vertices, and for each $0 < k \leq n$ a set M_k of subsets of M_0 of cardinality $k+1$, such that any $(j+1)$-element subset of M_k is an element of M_j. The elements of M_k are called k**-faces**. Denote by $\mathsf{SimpCompSet_n}$ the type of abstract simplicial complexes of dimension n (where the suffix Set reminds us that this is a type of sets). Let $M_{\leq k} = [M_0, \ldots, M_k]$ and note that $M_{\leq n} = M$. We call $M_{\leq k}$ the k**-skeleton** of M, and it is a (k-)complex in its own right. M is automatically equipped with a chain of inclusions of the skeleta $M_0 \hookrightarrow M_{\leq 1} \hookrightarrow \cdots \hookrightarrow M_{\leq n} = M$. A **morphism** f from $M = [M_0, \ldots, M_m]$ to $N = [N_0, \ldots, N_n]$ is a function on vertices $f : M_0 \to N_0$ such that for any face of M the image under f is a face of N.

Definition 3.2. Let Δ^n be the standard n-simplex in $\mathbb{R}^n$ given by $\{x_1, \ldots, x_n | \sum_i x_i \leq 1\}$. Let $M : \mathsf{SimpCompSet_n}$. The **geometric realization** $[M] : \mathsf{Top}$ of M in the category of topological spaces is given inductively as follows: $[M_0] = M_0$, and given $[M_{\leq k-1}]$ we form $[M_{\leq k}]$ by the pushout in sets

$$\begin{tikzcd} M_k \times \partial\Delta^k \arrow[r, "\text{attach}"] \arrow[d, hook, "\text{id}\times\text{incl}"'] & {[M_{\leq k-1}]} \arrow[d, "i_k"] \\ M_k \times \Delta^k \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & {[M_{\leq k}]} \end{tikzcd}$$

which attaches each k-simplex by taking the convex hull of the appropriate $k+1$ points in $[M_{\leq k-1}]$. The collection of vertical maps on the right gives a sequence of inclusion maps of skeleta $[M_0] \xrightarrow{i_1} [M_{\leq 1}] \xrightarrow{i_2} \cdots \xrightarrow{i_n} [M_{\leq n}] = [M]$.

Definition 3.3. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $n-1$-dimensional subcomplex containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

The link is easier to understand as all the neighboring vertices of v and the subcomplex containing these. See for example Figure 2.

Definition 3.4. A **combinatorial manifold** (or **combinatorial triangulation**) of dimension n is a simplicial complex of dimension n such that the link of every vertex is a **simplicial sphere** of dimension $n-1$ (meaning its geometric realization is homeomorphic to an $n-1$-sphere). Denote by $\mathsf{CombMfdSet_n}$ the type of combinatorial manifolds of dimension n (which the notation again reminds us are sets).

In a 2-dimensional combinatorial manifold the link is a polygon. See Figure 3 for some

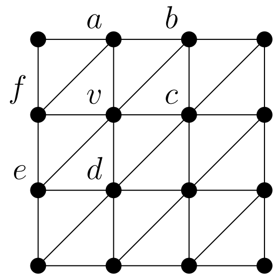

Figure 2: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

examples of 2-dimensional combinatorial manifolds of genus 0, 1, and 3.

A classical 1940 result of Whitehead, building on Cairn, states that every smooth manifold admits a combinatorial triangulation[9]. So it appears reasonably well motivated to study this class of objects. See for example the classic book by Kirby and Siebenmann[10]. There are important examples in four dimensions of topological manifolds that do not have any smoothness structure or triangulation. These will be out of reach of the theory we are building.

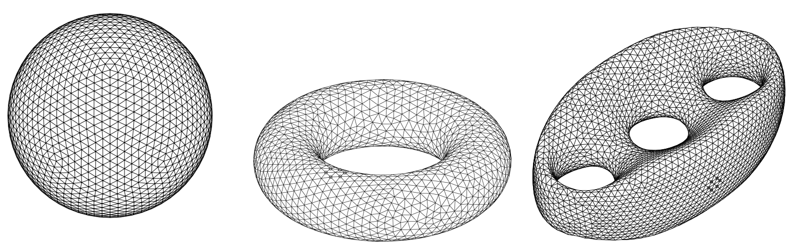

Figure 3: Combinatorial triangulations of a sphere, torus, and 3-holed torus. Sphere created with the tool `stripy`; torus from Wikipedia (By Ag2gaeh - Own work, CC BY-SA 3.0.); 3-holed torus from Wikipedia (By Ag2gaeh - Own work, CC BY-SA 3.0.)

3.2 Higher inductive combinatorial manifolds

Instead of building a set $[M] : \mathsf{Top}$, we can realize the simplicial complex as a (higher) type by forming a sequence of *homotopy* pushouts. For example in dimension 1 we could take the triangle with vertices $\{v_1, v_2, v_3\}$ and edges $\{e_{12}, e_{23}, e_{31}\}$ and form a polygon C_3:

$$\begin{tikzcd}
C_{3,1} \times S^0 = \{e_{12}, e_{23}, e_{31}\} \times \{N, S\} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\substack{e_{12}\times\{N,S\}\mapsto\{v_1,v_2\} \\ e_{23}\times\{N,S\}\mapsto\{v_2,v_3\} \\ e_{31}\times\{N,S\}\mapsto\{v_3,v_1\}}"'] \arrow[dr, phantom, "\Nearrow h_1"] & C_{3,1} \arrow[d, "*_1"] \\
C_{3,0} = \{v_1, v_2, v_3\} \arrow[r] & C_3 \arrow[ul, phantom, very near start, "\lrcorner"]
\end{tikzcd}$$

The left vertical map expresses the connectivity between the edges and vertices in the set-based complex. The right vertical map $*_1$ provides a hub point for each edge, and the homotopy h_1 provides the spokes that connect the hub to the vertices. So in contrast to a geometric realization, the 1-dimensional cells in the HoTT sense are generated by the filler homotopy.

In dimension 2 we could fill in maps from C_3 into our higher type by adding faces (hence reusing the object we just built above):

$$
\begin{tikzcd}
 & M_2 \times C_3 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\Nearrow h_2"] & M_2 \arrow[d, "*_2"] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times S^0 \arrow[u, "\partial_0"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\Searrow h_1"] & M_1 \arrow[u, "*_1"'] &
\end{tikzcd}
$$

The types C_3 and $\mathbb{M}_1$ are 1-types, $\mathbb{M}_2$ is a 2-type, and the rest are sets. The map ∂_0 maps each pair (e, S^0) to the pair of points this edge connects.

The h_i are the proofs of commutativity, and the two squares are also both homotopy pushouts. Note that the pushouts could be re-expressed as HIT constructors.

Whereas in geometric realization we use both Δ^i and $\partial\Delta^i$ to attach simplices, in this homotopy picture we need an explicit construction of "$\partial\Delta^i$" as a type equivalent to an $i-1$-dimensional sphere and the filling of this sphere with i-dimensional stuff has moved into the proof of commutativity. So in dimension n the picture would be

$$
\begin{tikzcd}
\cdots \arrow[r] & M_{n-2} \arrow[d, "*_{n-2}"] & M_n \times S^{n-1} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_{n-1}"'] \arrow[dr, phantom, "\Nearrow h_n"] & M_n \arrow[d, "*_n"] \\
\cdots \arrow[r] \arrow[ur, phantom, "\Nearrow h_{n-2}"] & \mathbb{M}_{n-2} \arrow[r] & \mathbb{M}_{n-1} \arrow[r] & \mathbb{M}_n \\
 & M_{n-1} \times S^{n-2} \arrow[u, "\partial_{n-2}"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\Searrow h_{n-1}"] & M_{n-1} \arrow[u, "*_{n-1}"'] &
\end{tikzcd}
$$

Definition 3.5. A **higher realization** $[\![M]\!]$ corresponding to an abstract simplicial complex $M : \mathsf{SimpCompSet_n}$ consists of

1. $n+1$ types $\mathbb{M}_0, \ldots, \mathbb{M}_n$ where $\mathbb{M}_0 \stackrel{\text{def}}{=} M_0$,
2. n spans $\mathbb{M}_i \xleftarrow{\partial_i} M_{i+1} \times S^i \xrightarrow{\mathrm{pr}_1} M_{i+1}$, $i = 1, \ldots, n$, where S^i is a HoTT i-sphere and ∂_i are called **attachment maps**,
3. n pushout squares from each span to $\mathbb{M}_{i+1}$, with induced maps $\imath_i : \mathbb{M}_i \to \mathbb{M}_{i+1}$, $*_{i+1} : M_{i+1} \to \mathbb{M}_{i+1}$ and proof of commutativity h_{i+1}.

A **cellular type** is a sequence $\mathbb{M}_0 \xrightarrow{\imath_0} \mathbb{M}_1 \xrightarrow{\imath_1} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$, together with a proof of existence of some $[\![M]\!]$ inducing this sequence. We define forgetful maps to the bare types with $[\![M]\!]_i \stackrel{\text{def}}{=} \mathbb{M}_i$.

We intend for the ∂_i to be isomorphisms between a boundary of a simplex in $\mathbb{M}_i$ and the i-sphere. To avoid spending too much time above dimension 2, we will leave the ∂_i underspecified in general. For 2-dimensional complexes we will see how the gluing can be done for triangles.

3.3 Polygons

Recall the 1-type C_3 that we just constructed from the triangle with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{e_{12}, e_{23}, e_{31}\}$:

$$\begin{tikzcd}
C_{3,1} \times S^0 = \{e_{12}, e_{23}, e_{31}\} \times \{N, S\} \arrow[r, "\mathrm{pr}_1"] \arrow[d, swap, "\substack{e_{12}\times\{N,S\}\mapsto\{v_1,v_2\} \\ e_{23}\times\{N,S\}\mapsto\{v_2,v_3\} \\ e_{31}\times\{N,S\}\mapsto\{v_3,v_1\}}"] \arrow[dr, phantom, "\Nearrow h_1"] & C_{3,1} \arrow[d, "*_1"] \\
C_{3,0} = \{v_1, v_2, v_3\} \arrow[r] & C_3 \arrow[ul, phantom, very near start, "\lrcorner"]
\end{tikzcd}$$

We will use a notational shorthand for polygons:

Definition 3.6. If C_n is a simplicial complex with $C_{n,0} = \{v_1, \ldots, v_n\}$ and edges $e_1 = \{v_1, v_2\}$, …, $e_{n-1} = \{v_{n-1}, v_n\}$, and $e_n = \{v_n, v_0\}$, we call C_n a **polygon** or **n-gon**. We can refer to C_n by $v_1v_2\cdots v_n$, to $[C_n]$ by $[v_1v_2\cdots v_n]$ and to $[\![C_n]\!]$ by $[\![v_1v_2\cdots v_n]\!]$. Forgetting the pushouts and vertices, we denote the bare 1-type by $[\![v_1v_2\cdots v_n]\!]_1$.

We also have the special circle type that is not the realization of a simplicial complex, but is nevertheless the natural basepoint in the type of polygons, as we shall see:

Definition 3.7. The higher inductive type S^1, also denoted $[\![C_1]\!]_1$ for convenience, despite C_1 not being a simplicial complex, has constructors:

$$\begin{aligned}
S^1 &: \mathsf{Type} \\
\mathsf{base} &: S^1 \\
\mathsf{loop} &: \mathsf{base} = \mathsf{base}
\end{aligned}$$

Lemma 3.8. $[\![C_2]\!]_1 \simeq [\![C_1]\!]_1$ and in fact $[\![C_n]\!]_1 \simeq [\![C_{n-1}]\!]_1$.

Proof. (Compare to [8] Lemma 6.5.1.) In the case of $[\![C_1]\!]_1$ we will denote its constructors by base and loop. For $[\![C_2]\!]_1$ we will denote the points by v_1, v_2 and the edges by ℓ_{12}, r_{21}. For $[\![C_3]\!]_1$ and higher we will denote the points by $v_1, \ldots, v_n$ and the edges by $e_{i,j} : v_i = v_j$ where $j = i + 1$ except for $e_{n,1}$.

First we will define $f : [\![C_2]\!]_1 \to [\![C_1]\!]_1$ and $g : [\![C_1]\!]_1 \to [\![C_2]\!]_1$, then prove they are inverses.

$$\begin{aligned}
f(v_1) = f(v_2) &= \mathsf{base} & \qquad g(\mathsf{base}) &= v_1 \\
f(\ell_{12}) &= \mathsf{loop} & g(\mathsf{loop}) &= \ell_{12} \cdot r_{21} \\
f(r_{21}) &= \mathsf{refl}_{\mathsf{base}}
\end{aligned}$$

We need to show that $f \circ g \sim \mathrm{id}_{[\![C_1]\!]_1}$ and $g \circ f \sim \mathrm{id}_{[\![C_2]\!]_1}$. Think of f as sliding v_2 along r_{21} to coalesce with v_1. This may help understand why the unfortunately intricate proof is working.

Recall that given functions $\phi, \psi : A \to B$ between two arbitrary types we can form a type family of paths $\alpha : A \to \mathcal{U}$ by $\alpha(a) \stackrel{\text{def}}{=} (\phi(a) =_B \psi(a))$. Transport in this family is given by concatenation as follows (compare Lemma 2.10), where $p : a =_A a'$ and $q : \phi(a) = \psi(a)$ (see Figure 4):

$$\mathsf{tr}(p)(q) = \phi(p)^{-1} \cdot q \cdot \psi(p)$$

which gives a path in $\phi(a') = \psi(a')$ by connecting dots between the terms $\phi(a'), \phi(a), \psi(a), \psi(a')$. This relates a would-be homotopy $\phi \sim \psi$ specified at a single point, to a point at the end of a path. We will use this to help construct such homotopies.

$$\begin{tikzcd}
\phi(a) \arrow[d, "q"'] & \phi(a') \arrow[l, "\phi(p)^{-1}"'] \\
\psi(a) \arrow[r, "\psi(p)"] & \psi(a') \\
a \arrow[r, "p"] & a'
\end{tikzcd}$$

Figure 4: Transport along p in the fibers of a family of paths. The fiber over a is $\phi(a) = \psi(a)$ where $\phi, \psi : A \to B$.

We need terms $p : \prod_{a:[\![C_1]\!]_1} f(g(a)) = a$ and $q : \prod_{a:[\![C_2]\!]_1} g(f(a)) = a$. We will proceed by induction, defining appropriate paths on point constructors and then checking a condition on path constructors that confirms that the built-in transport of these type families respects the definition on points.

Looking first at $g \circ f$, which shrinks r_{21}, we have the following data to work with:

$$\begin{aligned}
g(f(v_1)) = g(f(v_2)) &= v_1 \\
g(f(\ell_{12})) &= \ell_{12} \cdot r_{21} \\
g(f(r_{21})) &= \mathsf{refl}_{v_1}.
\end{aligned}$$

We then need to supply a homotopy from this data to $\mathrm{id}_{[\![C_2]\!]_1}$, which consists of a section and pathovers over $[\![C_2]\!]_1$:

$$\begin{aligned}
p_1 &: g(f(v_1)) = v_1 \\
p_2 &: g(f(v_1)) = v_2 \\
H_\ell &: \mathsf{tr}(\ell_{12})(p_1) = p_2 \\
H_r &: \mathsf{tr}(r_{21})(p_2) = p_1.
\end{aligned}$$

which simplifies to

$$\begin{aligned}
p_1 &: v_1 = v_1 \\
p_2 &: v_1 = v_2 \\
H_\ell &: g(f(\ell_{12}))^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\
H_r &:= g(f(r_{21}))^{-1} \cdot p_2 \cdot r_{21} = p_1
\end{aligned}$$

and then to

$$\begin{aligned}
p_1 &: v_1 = v_1 \\
p_2 &: v_1 = v_2 \\
H_\ell &: (\ell_{12} \cdot r_{21})^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\
H_r &: \mathsf{refl}_{v_1} \cdot p_2 \cdot r_{21} = p_1
\end{aligned}$$

To solve all of these constraints we can choose $p_1 \stackrel{\text{def}}{=} \mathsf{refl}_{v_1}$, which by consulting either H_ℓ or H_r requires that we take $p_2 \stackrel{\text{def}}{=} {r_{21}}^{-1}$.

Now examining $f \circ g$, we have

$$\begin{aligned}
f(g(\mathsf{base})) &= \mathsf{base} \\
f(g(\mathsf{loop})) &= f(\ell_{12} \cdot r_{21}) = \mathsf{loop}
\end{aligned}$$

and so we have an easy proof that this is the identity.

The proof of the more general case $[\![C_n]\!]_1 \simeq [\![C_{n-1}]\!]_1$ is very similar. Take the maps $f : [\![C_n]\!]_1 \to [\![C_{n-1}]\!]_1$, $g : [\![C_{n-1}]\!]_1 \to [\![C_n]\!]_1$ to be

$$\begin{aligned}
f(v_i) &= v_i \quad (i = 1, \ldots, n-1) & g(v_i) &= v_i & (i = 1, \ldots, n-1) \\
f(v_n) &= v_1 & g(e_{i,i+1}) &= e_{i,i+1} & (i = 1, \ldots, n-2) \\
f(e_{i,i+1}) &= e_{i,i+1} \quad (i = 1, \ldots, n-1) & g(e_{n-1,1}) &= e_{n-1,n} \cdot e_{n,1} \\
f(e_{n-1,n}) &= e_{n-1,1} \\
f(e_{n,1}) &= \mathsf{refl}_{v_1}
\end{aligned}$$

where f should be thought of as shrinking $e_{n,1}$ so that v_n coalesces into v_1.

The proof that $g \circ f \sim \mathrm{id}_{[\![C_n]\!]_1}$ proceeds as follows: the composition is definitionally the identity except

$$\begin{aligned}
g(f(v_n)) &= v_1 \\
g(f(e_{n-1,n})) &= e_{n-1,n} \cdot e_{n,1} \\
g(f(e_{n,1})) &= \mathsf{refl}_{v_1}.
\end{aligned}$$

Guided by our previous experience we choose ${e_{n,1}}^{-1} : g(f(v_n)) = v_n$, and define the pathovers by transport.

The proof that $f \circ g \sim \mathrm{id}_{[\![C_{n-1}]\!]_1}$ requires only noting that $f(g(e_{n-1,1})) = f(e_{n-1,n} \cdot e_{n,1}) = e_{n-1,1} \cdot \mathsf{refl}_{v_1} = e_{n-1,1}$. $\square$

Corollary 3.9. All polygons are equivalent to S^1, i.e. we have terms $e_n : ||[\![C_n]\!]_1 = S^1||_{-1}$, and hence we have constructed a map from the unit type $([\![C_n]\!]_1, e_n) : \mathbb{1} \to \mathrm{EM}(\mathbb{Z}, 1)$.

Proof. We can add $n-1$ points to S^1 and use Lemma 3.8. $\square$

If we cyclically permute the vertices of a simplicial polygon, what map is induced on the realization and the bare types within?

Definition 3.10. Let $R : [v_1 \cdots v_n] \to [v_1 \cdots v_n]$ (for "rotation") be the map sending $v_i \mapsto v_{i+1}$ and $v_n \mapsto v_0$. This map clearly preserves the edges, and so is a map of simplicial complexes, and extends to a map $[\![R]\!] : [\![v_1 \cdots v_n]\!] \to [\![v_1 \cdots v_n]\!]$.

A key point is that simplicial maps form a set, but the homotopical realization $[\![R]\!]$ has a path to the identity:

Lemma 3.11. The map $[\![R]\!]_1 : [\![v_1 \cdots v_n]\!]_1 \to [\![v_1 \cdots v_n]\!]_1$ is connected to $\mathsf{refl}_{[\![v_1 \cdots v_n]\!]_1}$ by a homotopy $H_R : \prod_{x:[\![v_1 \cdots v_n]\!]_1} [\![R]\!]_1(x) = x$.

Proof. If x is a vertex, take $H_R(x)$ to be the obvious unique edge back to the starting vertex. This extends in the obvious functorial way to edges. $\square$

3.4 The octahedron model of the sphere

We will create our first combinatorial surface, an octahedron. We will not prove that this type is equivalent to the sphere. In $\mathsf{SimpCompSet_n}$ the combinatorial data of the faces can be represented with a *Hasse diagram*, which shows the poset of inclusions in a graded manner, with a special top and bottom element. We give an octahedron in Figure 5. The names of the vertices are short for white, yellow, blue, red, green, and orange, the colors of the faces of a Rubik's cube. The octahedron is the dual of the cube, with each vertex corresponding to a face.

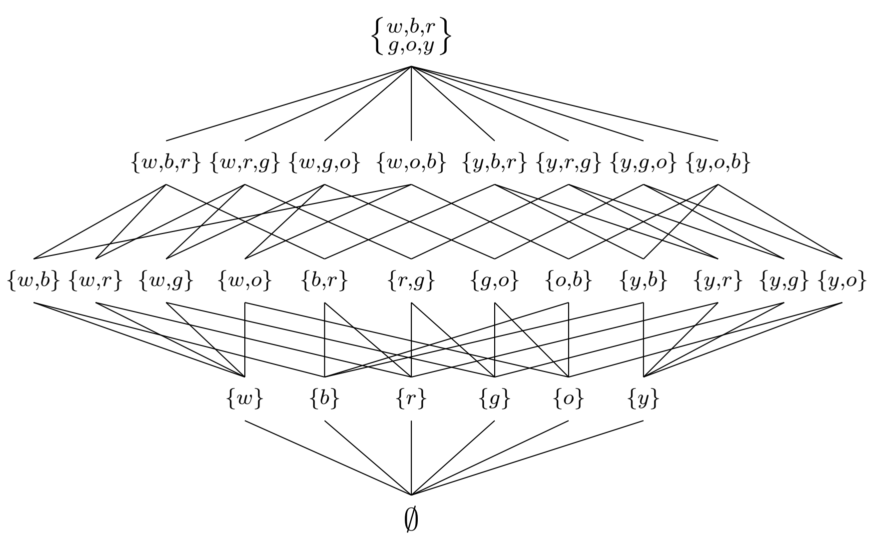

Figure 5: Hasse diagram of an octahedron O. The row of singletons is O_0 and above it are O_1 and O_2.

We can realize $O_0 \to O_1 \to O$ as a cellular type denoted $\mathbb{O}_0 \to \mathbb{O}_1 \to \mathbb{O}$.

Lemma 3.12. There is an equivalence $\mathbb{O} \simeq S^2$.

Proof. Omitted. $\square$

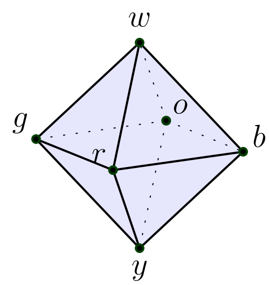

Figure 6: The type $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

4 Bundles, connections, and curvature

Bundles are simply maps into the universe. By using the extra cellular structure and the even more detailed combinatorial structure of higher realizations, we can identify inside of HoTT some additional classical definitions.

4.1 Definitions

Having the cellular structure allows us to define bundles and connections.

Definition 4.1. If $\mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a cellular type and $f_k : \mathbb{M}_k \to \mathcal{U}$ are type families on each skeleton such that all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drrr, "f_0"'] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[drr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[dr, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M} \arrow[dl, "f"'] \\
 & & & \mathcal{U} &
\end{tikzcd}$$

then we say

- The map f_k is a k-**bundle** on $\mathbb{M}_k$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$ that f_k extends f_{k-1} is called a k-**connection on the bundle** f_{k-1}.

Having the additional structure of a simplicial complex allows us to further define curvature, which is a local concept.

Definition 4.2. If $[\![M]\!]$ is a higher combinatorial n-manifold as above, and such that for each pushout defining $\mathbb{M}_k$ we have the diagram

$$\begin{tikzcd}
M_k \times S^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\partial_{k-1}"'] \arrow[dr, phantom, "\Nearrow h_k"] & M_k \arrow[d, "*_k"] \\
\mathbb{M}_{k-1} \arrow[r] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \arrow[dl, phantom, "\Swarrow"] \\
 & \mathcal{U}
\end{tikzcd}$$

the outer square of which restricts on each face to the diagram

$$\begin{tikzcd}
\{m_k\} \times S^k \arrow[r, "!"] \arrow[d, "\partial_{k-1}"'] \arrow[dr, phantom, "\Swarrow \flat_k"] & m_k \arrow[d, "f_k \circ *_k"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

then we say the filler $\flat_k$ is called a **flatness structure for the face** m_k, or, to align with classical terms, the **curvature at the face** m_k, and its ending path is called the **holonomy around the face**.

The definitions can be digested to give

Lemma 4.3. Given f_{k-1} as above, a k-connection exists if and only if there exists a flatness structure for each k-face.

4.2 Connections as local trivializations

This section can ve viewed as an extended remark. The observation we want to make is that the data of a 1-bundle is related to the construction of local trivializations: the fiber at one vertex can be extended throughout a single face coherently, using the connection (the extension of the classifying map to the edges) to specify isomorphisms with the fibers at the other points, and the higher connections to establish commutativity between these.

We introduce a notation more suitable for the algebra of charts and overlaps: denote by $g_{ji} : Ti \to Tj$ the isomorphism of transport along an edge $e_{ij} : i =_{\mathbb{M}} j$. The indices are ordered from right to left, which is compatible with function composition notation. Denote the inverse function by swapping indices: $g_{ij} \overset{\text{def}}{=} g_{ji}^{-1}$. Assume we have some fixed isomporphism $T_i = S^1$, and to avoid composing everything with this function we will assume it is id. In the diagram below we see the data arranged so that our bundles fibers are on the left, and the fiber of a trivial bundle is on the right.

$$
\begin{tikzcd}
& Ti \arrow[r, equal, "\mathrm{id}"] \arrow[d, "g_{ji}"] \arrow[dd, bend right=60, "g_{ki}"'] \arrow[dr, phantom, "/\!/"] & S^1 \arrow[d, equal] \arrow[dd, equal, bend left=60] & \\
{} & Tj \arrow[l, phantom, "\overset{\flat_{ijk}}{\Longleftarrow}"] \arrow[r, "g_{ij}"] \arrow[d, "g_{kj}"] \arrow[dr, phantom, "/\!/"] & S^1 \arrow[d, equal] \arrow[r, phantom, "="] & {} \\
& Tk \arrow[r, "g_{ij}g_{jk}"] & S^1 &
\end{tikzcd}
$$

The two middle squares commute definitionally. Call these two squares together the back face. The left triangular filler is filled by the flatness structure on the face, and the right triangular filler is trivial. There is also a filler needed for the front, i.e. the outer square. This requires proving that $g_{ki}g_{ij}g_{jk} = \mathrm{id}$, which is supplied by the flatness structure. There is also a 3-cell filling the interior of this prism, mapping the back face plus the left triangle filler to the front face plus the right triangle filler. These two faces both consist of one or two identities concatenated with the flatness structure, and so the 3-cell is definitional.

This relationship between flatness structure and local triviality of a chart can be compared to the classical result that on a paracompact, simply connected manifold (such as a single chart), a connection on a principal bundle is flat if and only if the bundle is trivial. See for example [11] Corollary 9.2.

4.3 The tangent bundle of the sphere

We will build up a map T out of $\mathbb{O}_0 \to \mathbb{O}_1 \to \mathbb{O}$ which is meant to be a model of the tangent bundle of the sphere. The link function will serve as our approximation to the tangent space. Taking the link of a vertex gives us a map from vertices to polygons, so the codomain is $\mathrm{EM}(\mathbb{Z}, 1)$.

Definition 4.4. $T_0 \stackrel{\text{def}}{=} \mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned} \mathsf{link}(w) &= [\![brgo]\!]_1 & \qquad \mathsf{link}(r) &= [\![wbyg]\!]_1 \\ \mathsf{link}(y) &= [\![bogr]\!]_1 & \qquad \mathsf{link}(g) &= [\![wryo]\!]_1 \\ \mathsf{link}(b) &= [\![woyr]\!]_1 & \qquad \mathsf{link}(o) &= [\![wgyb]\!]_1 \end{aligned}$$

Recall that the double-brackets denote the realization via pushouts of the combinatorial data, and the subscript takes the bare type of maximal dimension, forgetting the cellular structure. So T_0 is a composition of a map into $\mathsf{SimpCompSet}_2$ follows by the realization map to $\mathrm{EM}(\mathbb{Z}, 1)$.

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

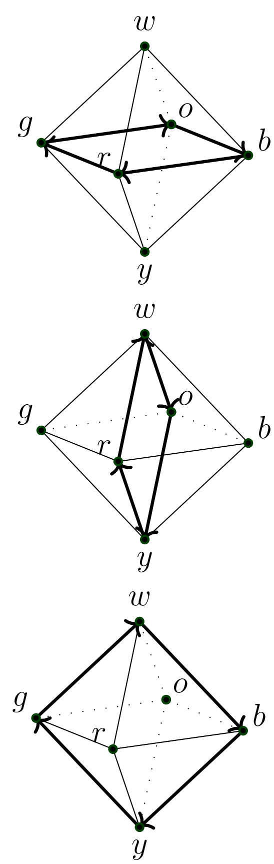

Figure 7: link for the vertices w, b and r.

To extend T_0 to a function T_1 on the 1-skeleton we have some freedom. We will do something motivated by the figures we have been drawing of an octahedron embedded in 3-dimensional space. We will imagine how T_1 changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 4.5. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:

- $T_1(wr) : [\![brgo]\!]_1 \mapsto [\![bygw]\!]_1$ (b, g fixed)
- $T_1(wg) : [\![brgo]\!]_1 \mapsto [\![wryo]\!]_1$
- $T_1(wb) : [\![brgo]\!]_1 \mapsto [\![yrwo]\!]_1$ (r, o fixed)
- $T_1(wo) : [\![brgo]\!]_1 \mapsto [\![bwgy]\!]_1$

Transport away from y:

- $T_1(yb) : [\![bogr]\!]_1 \mapsto [\![woyr]\!]_1$
- $T_1(yr) : [\![bogr]\!]_1 \mapsto [\![bygw]\!]_1$
- $T_1(yg) : [\![bogr]\!]_1 \mapsto [\![yowr]\!]_1$
- $T_1(yo) : [\![bogr]\!]_1 \mapsto [\![bwgy]\!]_1$

Transport along the equator:

- $T_1(br) : [\![woyr]\!]_1 \mapsto [\![wbyg]\!]_1$
- $T_1(rg) : [\![wbyg]\!]_1 \mapsto [\![wryo]\!]_1$
- $T_1(go) : [\![wryo]\!]_1 \mapsto [\![wgyb]\!]_1$
- $T_1(ob) : [\![wgyb]\!]_1 \mapsto [\![woyr]\!]_1$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 7 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn, which we saw in Definition 3.10 where it is called $[\![R]\!]$.

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here *there is no additional freedom.*

Definition 4.6. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows (making use of H_R from Lemma 3.11):

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$
- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

Defining these flatness structures suffices to define T_2 by Lemma 4.3.

4.4 Existence of connections

How confident can we be that we can always define a connection on an arbitrary combinatorial manifold? Two things make the octahedron example special: the link is a 4-gon at every vertex (as opposed to having a variable number of vertices), and every transport map extends to a rotation of the entire octahedron in 3-dimensional space. This imposed a coherence on the interactions of all the choices we made for the connection, which we can worry may not exist for more complex combinatorial data.

We know as a fact outside of HoTT that any combinatorial surface that has been realized as a triangulated surface embedded in 3-dimensional euclidean space can inherit the parallel transport entailed in the embedding. We could then approximate that data to arbitrary precision with enough subdivision of the fibers of T.

What would a proof inside of HoTT look like? We will leave this as an open question.

5 Vector fields

5.1 Definition

Vector fields are sections of the tangent bundle of a manifold. We do not have a general theory of tangent bundles, even for 2-dimensional cellular types, since we cannot yet prove that connections always exist on the 1-skeleton. But *given* an extension T of the link function, we can consider the type of sections $\prod_{x:\mathbb{M}_1} T_1(x)$.

Definition 5.1. A **vector field** on a higher combinatorial 2-manifold $\mathbb{M}$ equipped with type family $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ that extends link is a term $X : \prod_{x:\mathbb{M}_1} T_1(x)$. It may be possible to extend X to one or more faces of $\mathbb{M}$, but we call faces for which X cannot be extended **zeros of the vector field**.

prove that the index is an obstruction

Remark 5.2. A section $\prod_{x:\mathbb{M}} T(x)$ for $T : \mathbb{M} \to \mathrm{K}(\mathbb{Z}, 2)$ is a trivialization of the bundle. The fact that an orientation suffices to factor the tangent bundle of a 2-manifold (which *a priori* maps to $\mathrm{EM}(\mathbb{Z}, 1)$) through a principal bundle classifier is special to dimension 2. For higher dimensional manifolds the tangent bundle is a bundle of spheres, and even if the bundle is oriented it is not necessarily a principal bundle. On the other hand, the n-truncation modal operator maps the type of n-spheres to the classifying space $\mathrm{K}(\mathbb{Z}, n)$, and so another way to phrase this remark is that S^1 is the only n-sphere which is n-truncated.

On the 0-skeleton X picks a point in each link, i.e. a neighbor of each vertex. On a path $p : x =_{\mathbb{M}} y$, X assigns a dependent path over p, which as we know is a term $\pi : \mathsf{tr}(p)(X(x)) =_{Ty} X(y)$. We are very interested in working with the concatenation operation on dependent paths, which we call *swirling*.

5.2 Swirling

Consider the vertex $v_1 : \mathbb{M}$, a face F containing vertices v_1, v_2, v_3, and the boundary path $\ell \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$. For brevity, denote $T(v_i)$ by T_i and $T(e_{ij})$ by T_{ji}. The indices are swapped so that we can have expressions that respect function-composition order, such as $T_{32}T_{21}(X_1) : T_3$. Figure 8 shows in tabular form how we concatenate the dependent paths over $e_{12} \cdot e_{23} \cdot e_{31}$. Figure 9 shows visually a possible example.

As we traverse an edge, say e_{12}, we get a path in T_2 which is the image of e_{12} under X, denoted X_{12}. As we traverse an additional edge, X_{12} is simply mapped to the next vertex by transport. The image is carried first to $T_{32}(X_{12})$ then to $T_{13} \circ T_{32}(X_{12})$.

We wish to simplify expressions such as $T_{13}T_{32}X_{12} \cdot T_{13}X_{23} \cdot X_{31}$, which take place in a particular fiber (T_1 in this case), and which depend on arranging for the endpoint of one segment to agree with the start of another. The simplification will empower us to easily perform calculations over the whole manifold, and to prove our main theorem 6.15.

First we will choose a specific group that acts on all the fibers of T. Fix a point $m : \mathbb{M}$, and recall the map $\tau : \prod_{x:M}((Tx = Tx) \xrightarrow{\simeq} (Tm = Tm))$ from Lemma 2.10. This map is a trivialization of the automorphism bundle, so in particular provides an action

$$\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1 \\
X_1 & X_2 \arrow[d, equal, "X_{12}:"'] & X_3 \arrow[d, equal, "X_{23}:"'] & X_1 \arrow[d, equal, "X_{31}:"'] \\
& T_{21}X_1 & T_{32}X_2 \arrow[d, equal, "T_{32}X_{12}:"'] & T_{13}X_3 \arrow[d, equal, "T_{13}X_{23}:"'] \\
& & T_{32}T_{21}X_1 & T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}T_{32}X_{12}:"'] \\
& & & T_{13}T_{32}T_{21}X_1
\end{tikzcd}$$

Figure 8: The data in each fiber as we move around a triangle with vertices indexed 1, 2, and 3. Double-lines indicate identity types between two types, and their labels are terms of this type. Items with one index are terms of some type at a vertex, and items with two indices are terms of a type on an edge.

of $Tm = Tm$ on each Tx (or in our case T_i). Denote the action of $Tm = Tm$ on the torsor T_i by $\alpha : (Tm = Tm) \times T_i \to T_i$. This induces an equivalence $(\alpha, \mathrm{pr}_2) : (Tm = Tm) \times T_i \xrightarrow{\sim} T_i \times T_i$.

Definition 5.3. The map $\mathrm{pr}_1 \circ (\alpha, \mathrm{pr}_2)^{-1} : T_i \times T_i \to (Tm = Tm)$ is called **subtraction**. It maps (x, y) to the unique term $\delta : Tm = Tm$ such that $\alpha(\delta, x) = y$. For brevity we denote $\mathrm{pr}_1 \circ (\alpha, \mathrm{pr}_2)^{-1}(x, y)$ by $y - x$.

Each fiber T_i is pointed by X_i, so we can define the map $T_i \xrightarrow{-X_i} (Tm = Tm)$, and then give a name to the special term $\rho_{ji} \overset{\text{def}}{=} T_{ji}X_j - X_i$ which is the image in $Tm = Tm$ of the vector X_j in a neighboring vertex after transporting it to T_i.

Lemma 5.4. The following diagram is well-typed and commutes, and therefore $\rho_{ij} = -\rho_{ji}$.

$$\begin{tikzcd}
(X_j =_{T_j} T_{ji}X_i) \times (X_i =_{T_i} T_{ij}X_j) \arrow[r, "(\text{-})\cdot T_{ji}(\text{-})"] \arrow[d, "-X_j \times -X_i"'] & X_j =_{T_j} X_j \arrow[d, "-X_j"] \\
(\mathrm{id} =_{Tm=Tm} \rho_{ji}) \times (\mathrm{id} =_{Tm=Tm} \rho_{ij}) \arrow[r, "(\text{-})+(\rho_{ji}+(\text{-}))"'] & \mathrm{id} =_{Tm=Tm} \rho_{ij} + \rho_{ji}
\end{tikzcd}$$

Proof. The top map has the given codomain because $T_{ji}T_{ij} = \mathrm{id}$. The upper-right composition is $((\text{-}) \cdot (T_{ji}(\text{-})) - X_j$, and the lower-left composition is $((\text{-}) - X_j) + (\rho_{ji} + ((\text{-}) - X_i))$, so commutativity follows from the definition of ρ_{ji}. The final statement follows from comparing the types on the right. $\square$

We now know that the type in the bottom-right is $\mathrm{id} =_{Tm=Tm} \mathrm{id}$, so the vertical map $-X_j$ maps the path $X_{ij} \cdot X_{ji}$ to a path in $\mathrm{id} =_{Tm=Tm} \mathrm{id}$ and we can ask if the image is refl.

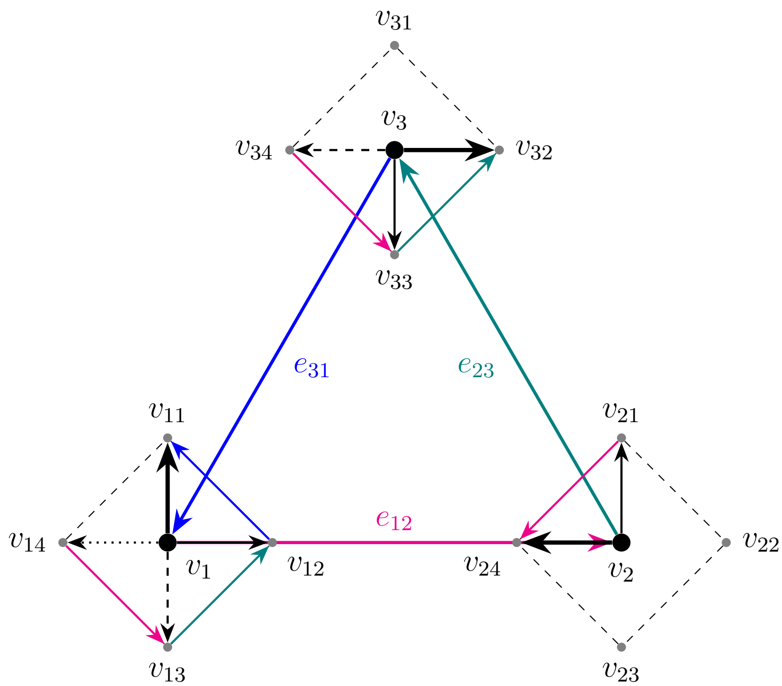

Figure 9: A vector field swirling around a face, in a bundle of squares. Imagine that transport along e_{12} does not rotate along the page, that transport along e_{23} rotates counterclockwise by 90 degrees, and that transport along e_{31} again does not rotate along the page. Thick black vectors are the vector field at a point. Thin vectors are transported once, dashed twice, and dotted three times. The vertices v_{ij} are in the tangent fibers. If you see color, the colors of the arrows correspond: the red edge produced the red edge in the fibers.

Lemma 5.5. $(X_{ij} \cdot X_{ji}) - X_j = \mathsf{refl}$.

Proof. We have

$$\begin{aligned} -X_j(X_{ij} \cdot X_{ji}) &= -X_j(X(e_{ji} \cdot e_{ij})) && \text{(by definition of dependent paths)} \\ &= -X_j(X(\mathsf{refl})) \\ &= -X_j(\mathsf{refl}) && \text{(by path induction)} \\ &= \mathsf{refl} \end{aligned}$$

□

Lemma 5.6. The group operation along the bottom of Lemma 5.4 is abelian, and the vertical maps form a homomorphism from the groupoid of dependent paths to the commutative group of paths in $Tm = Tm$.

Proof. Not entirely sure what remains here. □

By induction from 5.4, building up a long concat of dep. paths.

5.3 An example vector field on the sphere

define the example I

6 *The total construction*

We will place holonomy, flatness, and vector fields on the same footing, and combine them. We will prove the equivalance of the total curvature of a tangent bundle, and total index of a vector field. This is the key relationship in proving both the Gauss-Bonnet theorem and the Poincaré-Hopf theorem.

6.1 Index of the vector field

Given a polygon cellular type $C_{n,0} \to C_{n,1} = C_n$, and map $*_{C_n} : \nVdash \to \mathrm{K}(\mathbb{Z}, 2)$ giving it the structure of an S^1-torsor, we have the following general fact:

Proposition 6.1. Given a pointing $b : C_n$ the evaluation map $\mathrm{ev}(b) : (C_n = C_n) \to C_n$ is an equivalence.

Proof. See [12]. $\square$

Consider the cellular type $\mathbb{M}_0 \to \mathbb{M}_1 \to \mathbb{M}$ with point $m : \mathbb{M}$. Consider a loop $\ell : m =_{\mathbb{M}} m$ with proof of contractibility $f : \ell = \mathsf{refl}_m$. For example, we may have a face F in a combinatorial realization, with m a vertex of F and $\ell = \partial F$ the boundary loop. We have accumulated the following constructions:

$$\begin{aligned} \mathsf{tr}(\ell) &: Tm = Tm \\ \flat(\ell) &: \mathsf{tr}(\ell) =_{Tm=Tm} \mathrm{id} \\ X(\ell) &: \mathsf{tr}(\ell)(X(m)) =_{Tm=Tm} X(m) \end{aligned}$$

which invites us to make use of the equivalence 6.1 to define $X_E(\ell) \stackrel{\text{def}}{=} \mathrm{ev}(X(m))^{-1} \circ X(\ell) : \mathsf{tr}(\ell) = \mathrm{id}$ (where the subscript E stands for the extension to all of Tm) to obtain

$$\begin{aligned} \mathsf{tr}(\ell) &: Tm = Tm \\ \flat(\ell) &: \mathsf{tr}(\ell) =_{Tm=Tm} \mathrm{id} \\ X_E(\ell) &: \mathsf{tr}(\ell) =_{Tm=Tm} \mathrm{id} \end{aligned}$$

These last two can be concatenated to make a loop.

Definition 6.2. The **index of the vector field** X **around the loop** ℓ is the integer $I_X(\ell) \stackrel{\text{def}}{=} \flat(\ell)^{-1} \cdot X_E : \mathrm{id} =_{Tm=Tm} \mathrm{id}$.

> **Remark 6.3.** Classically the index around a loop makes use of a trivialization of the tangent bundle, and does not take place in the presence of a connection. We know from section 4.2 that the connection can serve as such a trivialization. Taking that point of view, the formula constitutes the difference between the swirling of the vector field inside the chart, and the twisting of the chart itself.

We now have the following list of ingredients given the loop ℓ:

$$\begin{aligned}
\mathsf{tr}(\ell) &: Tm = Tm \\
\flat(\ell) &: \mathsf{tr}(\ell) =_{Tm=Tm} \mathrm{id} \\
X_E(\ell) &: \mathsf{tr}(\ell) =_{Tm=Tm} \mathrm{id} \\
I_X(\ell) &: \mathrm{id} =_{Tm=Tm} \mathrm{id}
\end{aligned} \tag{1}$$

6.2 Cancellations

The next observation is that we can relate the quantities in Equation 1 at different points around a cellular surface by making use of the automorphism bundle $\sum_{x:\mathbb{M}}(Tx = Tx)$ and its trivialization $\tau : \prod_{x:M}(Tx = Tx) \xrightarrow{\simeq} (Tm = Tm)$ as in Lemma 2.10. We will use the action on paths of τ to better understand $\flat$ and X_E as *angles*.

Definition 6.4. Consider the contractible type $\sum_{\alpha:Tm=Tm} \alpha = \mathrm{id}$ with the addition law $(\alpha, p) + (\beta, q) = (\alpha \cdot \beta, p \cdot \alpha(q))$ where $\alpha(q) : \alpha = \alpha \cdot \beta$ (using concatenation notation instead of function composition). This is a commutative group with identity $\mathsf{refl}_{\mathrm{id}}$ which we call the **type of angles of** Tm.

Lemma 6.5. $\tau \circ X$ maps concatenation of paths to the commutative operation $+$ in the type of angles $\sum_{\alpha:Tm=Tm} \alpha = \mathrm{id}$.

Proof. Suppose we have $p : x =_{\mathbb{M}} y$ and $q : y =_{\mathbb{M}} z$. Then $X(p \cdot q) = \mathsf{tr}(q)(X(p)) \cdot X(q)$ which is exactly the operation $+$ but in the group $\sum_{\alpha:Tz=Tz} \alpha = \mathrm{id}$. We can then transport the result to m by an arbitrary path. □

Corollary 6.6. With e, p, ℓ as in Figure 10, we have $\tau(X_E(e \cdot \ell \cdot e^{-1})) = \tau(X_E(\ell))$.

So under τ we can cancel the contributions to X from the edge e after traversing it once in each direction. Meanwhile we have something similar for the transport automorphisms:

Lemma 6.7. With e, p, ℓ as in Figure 10, we have $\tau(\mathsf{tr}(e \cdot \ell \cdot e^{-1})) = \tau(\mathsf{tr}(\ell))$.

Proof. Recall that τ conjugates automorphisms by transport along some arbitrary path to m. To transport automorphisms from x to p, choose a path $p : m =_{\mathbb{M}} x$. Then make the specific choice $p \cdot e : m =_{\mathbb{M}} y$ to transport automorphisms from y to p. Then $\tau(\mathsf{tr}(e \cdot \ell \cdot e^{-1}))$ and $\tau(\mathsf{tr}(\ell))$ both compute to $\mathsf{tr}((p \cdot e) \cdot \ell \cdot (p \cdot e)^{-1})$. □

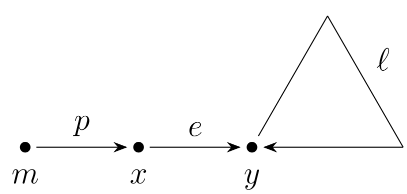

Figure 10: Comparing maps on the edge e at the master point m with and without traversing ℓ.

6.3 Total enumeration of faces

Definition 6.8. A **total enumeration of faces** for a combinatorial 2-manifold $\mathbb{M}$ with underlying simplicial complex $M = [M_0, M_1, M_2]$ consists of

1. A "master basepoint" $m : M_0$.
2. For each face $F : M_2$ with vertices $\{v_{F,1}, v_{F,2}, v_{F,3}\}$ an enumeration of its vertices $[v_{F,1}, v_{F,2}, v_{F,3}]$, including the choice of the first vertex in the enumeration as the basepoint of F, which is **globally compatible** with the choices for the other faces, meaning that when two faces F_1, F_2 share an edge $\{v, w\}$, then one of the faces includes the sublist $[v, w]$ and the other includes $[w, v]$.
3. An ordering of the faces $[F_1, \ldots, F_n]$.

Two enumerations that differ only in the ordering of vertices (2.) are said to have the **same orientation** if it is true for every face that the two orderings of vertices differ by an even orientation.

Remark 6.9. For such an enumeration to exist the underlying simplicial complex must be *orientable* in a classical sense. We are not going to explore this requirement internally in HoTT, nor prove any relationship between orientability of the set-based complex and orientation in the sense of factoring classifying maps through $\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{EM}(\mathbb{Z}, 1)$.

Example 6.10. The octahedron: for $\mathbb{O}$ we might choose b as the master basepoint, as well as the basepoint for four of the faces. For the other four faces we could choose g as the basepoint. We could choose $br \cdot rg$ as the path between the basepoints, and we could order the faces like this: $[bwo, brw, boy, byr, gow, gwr, gry, gyo]$.

Definition 6.11. The **total path** of an enumeration is the list $\ell_{F_1}, \ldots, \ell_{F_n}$ of loops where $\ell_{F_i} : v_{F_i,1} = v_{F_i,1}$ is the loop around F_i connecting vertices $v_{F,1}, v_{F,2}, v_{F,3}$.

Definition 6.12. Using Equation 1 we can define the **total curvature**, **total swirling** and **total index** of a total path as follows:

$$\begin{aligned} \flat_{\mathrm{tot}} &= \tau(\flat(\ell_{F_1})) + \cdots + \tau(\flat(\ell_{F_n})) \\ X_{\mathrm{tot}} &= \tau(X(\ell_{F_1})) + \cdots + \tau(X(\ell_{F_n})) \\ I_{\mathrm{tot}} &= \flat_{\mathrm{tot}}^{-1} \cdot X_{\mathrm{tot}} \end{aligned} \tag{2}$$

where the sums are taken in the group of angles $\sum_{\alpha : Tm = Tm} \alpha = \mathrm{id}$.

Lemma 6.13. The total path of an enumeration visits each edge an even number of times, equally in each direction.

Theorem 6.14. $\tau(X(\ell_{F_1})) \cdots \tau(X(\ell_{F_n})) = \mathsf{refl}$ in $\sum_{\alpha : Tm = Tm} \alpha = \mathrm{id}$.

Proof. Immediate from Lemma 6.13 and Corollary 6.6. $\square$

Corollary 6.15. The total index is equal to the opposite of the total curvature. The total curvature is an integer.

6.4 Total curvature and index on the sphere

6.5 Future directions

Euler characteristic. Chern-Weil theory. Atiyah bundle. Space of connections is contractible. Formalization.

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[13][14] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

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