Discrete differential geometry in homotopy type theory

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Abstract

Type families on higher inductive types such as pushouts can capture the homotopical properties of differential geometric constructions including connections, curvature, gauge transformations, and vector fields. We define a class of combinatorial pushouts, then define principal bundles, connections, and curvature on these. We provide an example of a tangent bundle but do not prove when these must exist. We draw inspiration in part from the young field of discrete differential geometry, and in part from the original classical proofs, which often make use of triangulations and other discrete arguments. We prove an equality relating the Gauss-Bonnet theorem to the Poincaré-Hopf theorem. We also attempt to map out future directions.

This thesis is dedicated to John Baez, Sean M. Carroll, Sabine Hossenfelder, and other communicators who are carrying the torch of science forward in the spirit of my hero Carl Sagan. I have followed you all for many years, and you have inspired me to continue my studies alongside my career. Thank you.

"It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, The Future Does Not Compute*[1]*

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Contents

1 *Overview*

The outline is that we will define

- principal bundles in Section 2,
- simplicial complexes, and homotopical realizations of these in Section 3,
- tangent bundles and vector fields in Section 5,

and observe emerging from those definitions the presence of

- connections and curvature in Section 4,
- the index of a vector field in Section 6,

and then define in Section 6, for a 2-dimensional simplicial complex

- the total curvature, as in the Gauss-Bonnet theorem
- the total index of a vector field, as in the Poincaré-Hopf theorem,
- and prove the equality of these to each other when the complex is oriented (Theorem 6.8).

We will build up an example of all of these structures on an octahedron model of the sphere, and compute its Euler characteristic of 2. We will not, however, be supplying a separate definition of Euler characteristic so as to truly reproduce the Gauss-Bonnet and Poincaré-Hopf theorems.

Once we have defined homotopical realizations of simplicial complexes in Section 3, we will focus in most of this note on dimensions 1 and 2. In dimension 1 we obtain polygons, which we prove are equivalent to S^1, and so give terms in the type $\mathrm{EM}(\mathbb{Z},1) \stackrel{\text{def}}{=} \sum_{A:\mathcal{U}} ||A \simeq S^1||_1$. We can call this component of the universe "mere circles." In dimension 2 we will focus on a subset of complexes where the neighboring vertices and edges of each vertex (the vertex's "link") form a polygon. The homotopical realization $\mathbb{M}$ of such a complex then has a map link from each vertex to a homotopical polygon, i.e. a map to $\mathrm{EM}(\mathbb{Z},1)$. We do not know under what conditions this map necessarily extends to the higher cells of the realization.

Given a map $\mathbb{M} \to \mathrm{EM}(\mathbb{Z},1)$ we can form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{EM}_\bullet(\mathbb{Z},1) \arrow[d, "\mathrm{pr}_1"'] \\ \mathbb{M} \arrow[r, "\mathsf{link}"] & \mathrm{EM}(\mathbb{Z},1) \end{tikzcd}$$

to obtain a bundle of mere circles. We will discuss how, if $\mathrm{K}(\mathbb{Z},2)$ is an Eilenberg-Mac Lane space and if link factors through a map $\mathrm{K}(\mathbb{Z},2) \to \mathrm{EM}(\mathbb{Z},1)$ then the pullback is a principal fibration.

Then in Section 4 we will name various elements of the above construction, indicating their relationship to classical definitions.

In Section 5 we will define vector fields, which require a tangent bundle. We will introduce a method for computing vector fields along concatenations of paths.

Finally, in Section 6 we will define a method for visiting all the faces of a manifold in order to form "totals" of local objects. We will examine the total curvature and the total index and prove that they are equal. Our proof tracks very closely with the classical proof of Hopf[2], presented in detail in Needham[3]. In their case they can go on to prove that these values are both equal to the Euler characteristic, but we would need an independent definition to prove agreement with, which we do not currently have.

2 *Torsors and principal bundles*

Differential geometry is the study of principal bundles and their automorphisms. Principal bundles are bundles of torsors, so we start there.

2.1 Torsors

We will review some definitions and facts, drawing on the excellent resource [4].

Definition 2.1. Let G be a group with identity element e (with the usual classical structure and properties). A G-**set** is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If in addition we have a term of type

$$\mathsf{is\_torsor}(X, \phi) \stackrel{\text{def}}{=} ||X||_{-1} \times \prod_{x:X} \mathsf{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

then we say (X, ϕ) is a G-**torsor**. Denote the type of G-torsors by BG. Denote the G-torsor given by G itself under right-multiplication by G_{reg}.

A G-equivariant map is a function $f : X \to Y$ such that $f(\phi(g, x)) = \psi(g, f(x))$. Denote the type of G-equivariant maps by $X \to_G Y$.

Lemma 2.2. ([4] Lemma 5.2). If $(X, \phi), (Y, \psi) : BG$ then there is a natural equivalence $(X =_{BG} Y) \simeq (X \to_G Y)$. □

Lemma 2.3. ([4] Proposition 5.4). A G-set (X, ϕ) is a G-torsor if and only if there merely exists a G-equivariant equivalence $G_{\mathrm{reg}} \to_G X$. □

Corollary 2.4. ([4] Corollary 5.5). The pointed type $(BG, *)$ is a $\mathrm{K}(G, 1)$. □

In particular, to classify principal S^1-bundles we map into the space $\mathrm{K}(S^1, 1)$, a type of torsors of the circle. Since S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$, we have $\mathrm{K}(S^1, 1) \simeq \mathrm{K}(\mathbb{Z}, 2)$.

Suppose we have a basepoint $b : \mathrm{K}(\mathbb{Z}, 2)$. A logical one to choose would be $S^1 : \mathrm{K}(\mathbb{Z}, 2)$ itself. Following [4] Section 4.3, a term $X : \mathrm{K}(\mathbb{Z}, 2)$ is a torsor for the higher group $\Omega_b(\mathrm{K}(\mathbb{Z}, 2)) \stackrel{\text{def}}{=} (b =_{\mathrm{K}(\mathbb{Z},2)} b)$. In particular there is an action $\alpha : (b =_{\mathrm{K}(\mathbb{Z},2)} b) \times X \to X$, and an equivalence $(\alpha, \mathrm{pr}_2) : (b =_{\mathrm{K}(\mathbb{Z},2)} b) \times X \xrightarrow{\sim} (X \times X)$.

2.2 Bundles of Eilenberg-Mac Lane spaces

To construct maps into $\mathrm{K}(\mathbb{Z}, 2)$ we will follow Scoccola[5]. When can a map into a connected component of the universe be factored through an Eilenberg-Mac Lane space?

Definition 2.5. Let G be a group. Let $\mathrm{EM}(G, n) \stackrel{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \stackrel{\text{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$ be the connected component of $\mathcal{U}$ containing $\mathrm{K}(G, n)$. A $\mathrm{K}(G, n)$-**bundle** on a type M is a map $M \to \mathrm{EM}(G, n)$.

Scoccola uses

- suspension $\Sigma : \mathrm{EM}(G, n) \to \mathrm{EM}_{\bullet\bullet}(G, n)$ (see [6] §6.5), which maps into types with two points (denoted by the bullets),

- $(n+1)$-truncation (see [6] §7.3),
- forgetting a point $F_\bullet : \mathrm{EM}_{\bullet\bullet}(G,n) \to \mathrm{EM}_\bullet(G,n)$,

to form the composition

$$\mathrm{EM}(G,n) \xrightarrow{||\Sigma||_{n+1}} \mathrm{EM}_{\bullet\bullet}(G,n+1) \xrightarrow{F_\bullet} \mathrm{EM}_\bullet(G,n+1)$$

from types to types with two points (north and south), then to pointed types (by forgetting the south point).

Definition 2.6. Given $f : M \to \mathrm{EM}(G,n)$, the **associated action of** M **on** G, denoted by $f_\bullet$ is defined to be $f_\bullet = F_\bullet \circ ||\Sigma||_{n+1} \circ f$.

Theorem 2.7. (Scoccola[5] Proposition 2.39). A $\mathrm{K}(G,n)$ bundle $f : M \to \mathrm{EM}(G,n)$ factors through a map $M \to \mathrm{K}(G,n+1)$, and so is a principal fibration, if and only if the associated action $f_\bullet$ is merely homotopic to a constant map.

Remark 2.8. Iterating the loop map gives the isomorphism $\Omega^{(n+1)} : \mathrm{EM}_\bullet(G,n+1) \simeq \mathrm{K}(\mathrm{Aut}\, G,1)$ (see [5] Lemma 2.7). Theorem 2.7 therefore says that the map f factors through $\mathrm{K}(G,n+1)$ if and only if the map into $\mathrm{K}(\mathrm{Aut}\, G,1)$ is homotopic to a constant. In the case of $G = \mathbb{Z}$, the map $f_\bullet : M \to \mathrm{K}(\mathrm{Aut}\, \mathbb{Z},1)$ deserves to be called the first Stiefel-Whitney class of f and one can interpret its triviality as *orientability*. . This point of view is discussed in Schreiber[7] (starting with Example 1.2.138) and in Myers[8].

2.3 Pathovers

Here we will recall basic facts from homotopy type theory. See for example [6] or [9]. Suppose we have $T : M \to \mathcal{U}$ and $P \overset{\text{def}}{=} \sum_{x:M} Tx$. We adopt a convention of naming objects in M with Latin letters, and the corresponding structures in P with Greek letters. Recall that if $p : a =_M b$ then T acts on p with what's called the *action on paths*, denoted $\mathsf{ap}(T)(p) : Ta = Tb$ ([6] §2.2). This is a path in the codomain $\mathcal{U}$ of T. Type theory also provides a function called *transport*, denoted $\mathsf{tr}(p) : Ta \to Tb$ ([6] §2.3) which acts on the fibers of P. $\mathsf{tr}(p)$ is a function, acting on the terms of the types Ta and Tb, and univalence tells us this is the isomorphism corresponding to $\mathsf{ap}(T)(p)$.

Type theory also tells us that paths in P are given by pairs of paths: a path $p : a =_M b$ in the base, and a pathover $\pi : \mathsf{tr}(p)(\alpha) =_{Tb} \beta$ between $\alpha : Ta$ and $\beta : Tb$ in the fibers ([6] §2.7). We can't directly compare α and β since they are of different types, so we apply transport to one of them. We say π lies over p. See Figure 1.

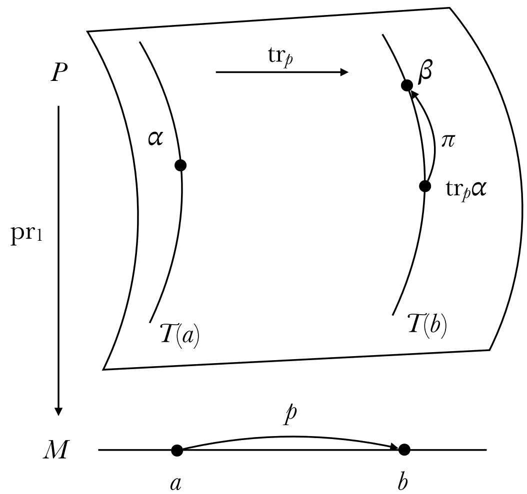

Figure 1: A path π over the path p in the base involves the transport function.

Given functions $\phi, \psi : A \to B$ between two arbitrary types we can form a type family of paths $\alpha : A \to \mathcal{U}$ by $\alpha(a) \overset{\text{def}}{=} (\phi(a) =_B \psi(a))$. Transport in this family is given by concatenation as follows (see Figure 2), where $p : a =_A a'$ and $q : \phi(a) = \psi(a)$ ([6] Theorem 2.11.3):

$$\mathsf{tr}(p)(q) = \phi(p)^{-1} \cdot q \cdot \psi(p)$$

which gives a path in $\phi(a') = \psi(a')$ by connecting dots between the terms $\phi(a'), \phi(a), \psi(a), \psi(a')$. This relates a would-be homotopy $\phi \sim \psi$ specified at a single point, to a point at the end of a path. We will use this to help construct such homotopies.

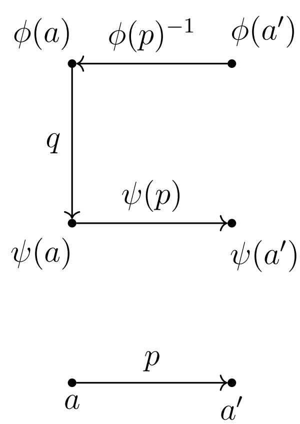

Figure 2: Transport along p in the fibers of a family of paths. The fiber over a is $\phi(a) = \psi(a)$ where $\phi, \psi : A \to B$.

Finally, recall that in the presence of a section $X : M \to P$ there is a dependent generalization of ap called apd: $\mathsf{apd}(X)(p) : \mathsf{tr}(p)(X(a)) = X(b)$ which is a pathover between the two values of the section over the basepoints of the path p ([6] Lemma 2.3.4).

3 *Discrete manifolds*

We will remind ourselves of the definition of a classical simplicial complex, in sets. Then we will create a type that realizes the data of a complex, using pushouts.

3.1 Abstract simplicial complexes

Definition 3.1. An **abstract simplicial complex** M **of dimension** n is an ordered list $M \overset{\text{def}}{=} [M_0, \dots, M_n]$ consisting of a set M_0 of vertices, and for each $0 < k \le n$ a set M_k of subsets of M_0 of cardinality $k+1$, such that for any $j < k$, any $(j+1)$-element subset of an element of M_k is an element of M_j. The elements of M_k are called k**-faces**. Denote by $\mathsf{SimCom_n}$ the type of abstract simplicial complexes of dimension n. Let $M_{\le k} \overset{\text{def}}{=} [M_0, \dots, M_k]$ and note that $M_{\le n} = M$. We call $M_{\le k}$ the k**-skeleton** of M, and it is a (k-)complex in its own right. A **morphism** f from $M = [M_0, \dots, M_m]$ to $N = [N_0, \dots, N_n]$ is a function on vertices $f : M_0 \to N_0$ such that for any face of M the image under f is a face of N. M is automatically equipped with a chain of inclusions of the skeleta $M_0 \hookrightarrow M_{\le 1} \hookrightarrow \cdots \hookrightarrow M_{\le n} = M$, which are simply inclusions of lists.

We can imagine constructing a simplicial complex dimension by dimension.

Definition 3.2. Denote by $P(n)$ the simplicial complex obtained by taking the set of all subsets of the standard $(n+1)$-element set $P(n)_0 \overset{\text{def}}{=} \{0, \dots, n\}$. We call $P(n)$ the **complete** n**-simplex**. We will refer to the $(n-1)$-skeleton $P(n)_{\le(n-1)}$ with the suggestive notation $\partial P(n)$.

Note that $P(n)_{(n-1)}$ has $n+1$ elements. For example, $P(2)_1$ consists of the three 2-element subsets of $\{0, 1, 2\}$.

Given $M = [M_0, \dots, M_n]$ and $k \le n$, a face $f_k : M_k$ is the union of $(k+1)$ faces in M_{k-1}, and so the k-skeleton is obtained from the $(k-1)$-skeleton by forming the following pushout of sets:

$$\begin{tikzcd} M_k \times \partial P(k) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "a_k"'] & M_k \arrow[d] \\ M_{\le(k-1)} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & M_{\le k} \end{tikzcd} \tag{3.1}$$

where the vertical "attach" map $a_k(f_k, -)$ picks out the $k+1$ subsets of f_k.

Definition 3.3. In an abstract simplicial complex M of dimension n, the **link** of a vertex v is the $(n-1)$-dimensional subcomplex containing every face $m \in M_{n-1}$ such that $v \notin m$ and $m \cup v$ is an n-face of M.

The link is easier to understand as all the neighboring vertices of v and the subcomplex containing these. See for example Figure 3.

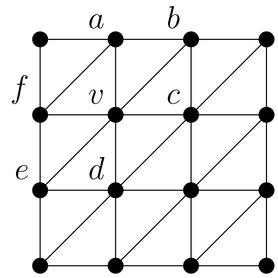

Figure 3: The link of v in this complex consists of the vertices $\{a, b, c, d, e, f\}$ and the edges $\{ab, bc, cd, de, ef, fa\}$, forming a hexagon.

> **Remark 3.4.** The *geometric realization* of a complex uses the combinatorial data to form pushouts of standard simplices inside the category of topological spaces. A *simplicial sphere* is a simplicial complex whose geometric realization is homeomorphic to a sphere. A classical 1940 result of Whitehead, building on Cairn, states that every smooth n-manifold is the geometric realization of a simplicial complex of dimension n such that the link is the geometric realization of an $(n-1)$-sphere[10]. For more of this theory see the classic book by Kirby and Siebenmann[11].

3.2 Higher inductive combinatorial manifolds

We will realize a simplicial complex as a higher inductive type by forming a sequence of pushouts. We will work upward by dimension so as to define the appropriate standard sphere that we need in the next dimension.

Definition 3.5. The **realization** $\mathbb{M}_0$ **of a 0-dimensional simplicial complex** M_0 is simply the set M_0.

Definition 3.6. The **simplicial 0-sphere** $\partial\Delta^1$ is the realization of $\partial P(1)$.

Definition 3.7. The **realization** $\mathbb{M}_1$ **of a 1-dimensional simplicial complex** $[M_0, M_1]$ is the pushout

$$\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "a_0"'] \arrow[dr, phantom, "\overset{h_1}{\Rightarrow}\ \lrcorner"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1
\end{tikzcd}$$

where a_k is the attachment data of the edges in M_1, as in diagram 3.1. The right vertical map $*_1$ provides a hub point for each edge, and the homotopy h_1 provides the spokes that connect the hub to the vertices.

Definition 3.8. The **simplicial 1-sphere** $\partial\Delta^2$ is the realization of $\partial P(2)$.

$$\begin{tikzcd}
\partial P(2) \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "a_0"'] \arrow[dr, phantom, "\overset{h_1}{\Rightarrow}\ \lrcorner"] & \partial P(2) \arrow[d, "*_{\partial\Delta^2}"] \\
\partial P(1) \arrow[r] & \partial\Delta^2
\end{tikzcd}$$

Definition 3.9. A **realization** $\mathbb{M}_2$ **of a 2-dimensional simplicial complex** $[M_0, M_1, M_2]$ is the pushout

$$\begin{tikzcd}
& M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\overset{h_2}{\Nearrow}"] & M_2 \arrow[d, "*_{\mathbb{M}_2}"] \\
\mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times \partial\Delta^1 \arrow[u, "a_0"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\overset{h_1}{\Searrow}"] & M_1 \arrow[u, "*_{\mathbb{M}_1}"']
\end{tikzcd}$$

where $\mathbb{A}_1$ is such that this additional diagram commutes

$$\begin{tikzcd}
M_2 \times \partial P(2) \arrow[r, "\mathrm{id}\times *_{\partial\Delta^2}"] \arrow[d, "a_1"'] & M_2 \times \partial\Delta^2 \arrow[d, "\mathbb{A}_1"] \\
M_{\leq 1} \arrow[r, "*_{\mathbb{M}_{\leq 1}}"'] & \mathbb{M}_1
\end{tikzcd}$$

In this diagram a_1 is the simplicial attaching map from 3.1, and $*_{\mathbb{M}_{\leq 1}}$ simply gathers the hub maps from M_0 and M_1 into $\mathbb{M}_1$. The commutativity then says that $\mathbb{A}_1$ reflects the attachment data.

Definition 3.10. Given a notion of realization in dimension $n-1$, a **realization** $\mathbb{M}_n$ **of an** n**-dimensional simplicial complex** $[M_0, \ldots, M_n]$ is the pushout

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, phantom, "\overset{h_{n-2}}{\Nearrow}"] & M_{n-2} \arrow[d, "*_{\mathbb{M}_{n-2}}"] & M_n \times \partial\Delta^n \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{n-1}"'] \arrow[dr, phantom, "\overset{h_n}{\Nearrow}"] & M_n \arrow[d, "*_{\mathbb{M}_n}"] \\
\cdots \arrow[r] & \mathbb{M}_{n-2} \arrow[r] & \mathbb{M}_{n-1} \arrow[r] & \mathbb{M}_n \\
& \cdots \arrow[r] \arrow[ur, phantom, "\overset{h_{n-1}}{\Searrow}"] & M_{n-1} \arrow[u, "*_{\mathbb{M}_{n-1}}"'] &
\end{tikzcd}$$

where $\mathbb{A}_n$ is such that the following commutes

$$\begin{tikzcd}
M_n \times \partial P(n) \arrow[r, "\mathrm{id}\times *_{\partial\Delta^n}"] \arrow[d, "a_1"'] & M_n \times \partial\Delta^n \arrow[d, "\mathbb{A}_n"] \\
M_{\leq n} \arrow[r, "*_{\mathbb{M}_{\leq n}}"'] & \mathbb{M}_n
\end{tikzcd}$$

Gathering some of the inductive process into one definition, we have

Definition 3.11. A **realization** $\mathbb{M}$ corresponding to an abstract simplicial complex $M : \mathsf{SimCom_n}$ consists of

1. $n+1$ types $\mathbb{M}_0, \ldots, \mathbb{M}_n$ where $\mathbb{M}_0 \overset{\text{def}}{=} M_0$,
2. n spans $\mathbb{M}_i \xleftarrow{\mathbb{A}_i} M_{i+1} \times \partial\Delta^{i+1} \xrightarrow{\mathrm{pr}_1} M_{i+1}$, $i = 0, \ldots, n-1$, where $\partial\Delta^{i+1}$ is the realization of the boundary of a standard simplex and $\mathbb{A}_i$ are called **attachment**

maps,

3. n pushout squares from each span to $\mathbb{M}_{i+1}$, with induced maps $\imath_i : \mathbb{M}_i \to \mathbb{M}_{i+1}$, $*_{\mathbb{M}_{i+1}} : M_{i+1} \to \mathbb{M}_{i+1}$ and proof of commutativity h_{i+1}.

A **cellular type** is a sequence of types $\mathbb{M}_0 \xrightarrow{\imath_0} \mathbb{M}_1 \xrightarrow{\imath_1} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$, together with a proof of existence of some $\mathbb{M}$ inducing this sequence.

3.3 Polygons

The 1-type $\partial\Delta^2$ has three vertices. In order to define the link of an arbitrary realization, we will need to have n-gons for $n \geq 3$. And since S^1 could be called a 1-gon, we will also define a 2-gon.

Definition 3.12. Define $C(n)$, $n \geq 3$ to be a simplicial complex with $C(n)_0 = \{v_1, \ldots, v_n\}$ and edges $e_1 = \{v_1, v_2\}$, …, $e_{n-1} = \{v_{n-1}, v_n\}$, and $e_n = \{v_n, v_0\}$. We call $C(n)$ a **polygon** or n**-gon**. The realization of $C(n)$ will be denoted $\mathbb{C}(n)$. When it is convenient we may refer to an n-gon by $[v_1 \cdots v_n]$ and to its realization by $[\![v_1 \cdots v_n]\!]$.

We also have two special polygons:

Definition 3.13. The higher inductive type S^1, also denoted $\mathbb{C}(1)$, has constructors:

$$\begin{aligned} S^1 &: \mathsf{Type} \\ \mathsf{base} &: S^1 \\ \mathsf{loop} &: \mathsf{base} = \mathsf{base} \end{aligned}$$

Definition 3.14. The higher inductive type $\mathbb{C}(2)$, has constructors:

$$\begin{aligned} \mathbb{C}(2) &: \mathsf{Type} \\ v_1, v_2 &: \mathbb{C}(2) \\ \ell_{12}, r_{21} &: v_1 = v_2 \end{aligned}$$

Lemma 3.15. $\mathbb{C}(2) \simeq \mathbb{C}(1)$ and in fact $\mathbb{C}(n) \simeq \mathbb{C}(n-1)$.

Proof. (Compare to [6] Lemma 6.5.1.) In the case of $\mathbb{C}(1)$ we will denote its constructors by base and loop. For $\mathbb{C}(2)$ we will denote the points by v_1, v_2 and the edges by ℓ_{12}, r_{21}. For $\mathbb{C}(3)$ and higher we will denote the points by $v_1, \ldots, v_n$ and the edges by $e_{i,j} : v_i = v_j$ where $j = i + 1$ except for $e_{n,1}$.

First we will define $f : \mathbb{C}(2) \to \mathbb{C}(1)$ and $g : \mathbb{C}(1) \to \mathbb{C}(2)$, then prove they are inverses.

$$\begin{aligned} f(v_1) = f(v_2) &= \mathsf{base} \\ f(\ell_{12}) &= \mathsf{loop} \\ f(r_{21}) &= \mathsf{refl}_{\mathsf{base}} \end{aligned} \qquad \begin{aligned} g(\mathsf{base}) &= v_1 \\ g(\mathsf{loop}) &= \ell_{12} \cdot r_{21} \end{aligned}$$

We need to show that $f \circ g \sim \mathrm{id}_{\mathbb{C}(1)}$ and $g \circ f \sim \mathrm{id}_{\mathbb{C}(2)}$. Think of f as sliding v_2 along r_{21} to coalesce with v_1, as in Figure 4. This may help understand why the somewhat intricate proof is working.

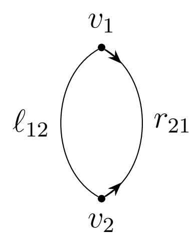

Figure 4: We imagine shrinking r_{21} down to become $\mathsf{refl}_{\mathsf{base}}$ in S^1.

We need terms $p : \prod_{a:\mathbb{C}(1)} f(g(a)) = a$ and $q : \prod_{a:\mathbb{C}(2)} g(f(a)) = a$. We will proceed by induction, defining appropriate paths on point constructors and then checking a condition on path constructors that confirms that the built-in transport of these type families respects the definition on points.

Looking first at $g \circ f$, which shrinks r_{21}, we have the following data to work with:

$$\begin{aligned} g(f(v_1)) = g(f(v_2)) &= v_1 \\ g(f(\ell_{12})) &= \ell_{12} \cdot r_{21} \\ g(f(r_{21})) &= \mathsf{refl}_{v_1}. \end{aligned}$$

We then need to supply a homotopy from this data to $\mathrm{id}_{\mathbb{C}(2)}$, which consists of a section and pathovers over $\mathbb{C}(2)$:

$$\begin{aligned} &p_1 : g(f(v_1)) = v_1 \\ &p_2 : g(f(v_1)) = v_2 \\ &H_\ell : \mathsf{tr}(\ell_{12})(p_1) = p_2 \\ &H_r : \mathsf{tr}(r_{21})(p_2) = p_1. \end{aligned}$$

which simplifies to

$$\begin{aligned} &p_1 : v_1 = v_1 \\ &p_2 : v_1 = v_2 \\ &H_\ell : g(f(\ell_{12}))^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\ &H_r := g(f(r_{21}))^{-1} \cdot p_2 \cdot r_{21} = p_1 \end{aligned}$$

and then to

$$\begin{aligned} &p_1 : v_1 = v_1 \\ &p_2 : v_1 = v_2 \\ &H_\ell : (\ell_{12} \cdot r_{21})^{-1} \cdot p_1 \cdot \ell_{12} = p_2 \\ &H_r : \mathsf{refl}_{v_1} \cdot p_2 \cdot r_{21} = p_1 \end{aligned}$$

To solve all of these constraints we can choose $p_1 \overset{\text{def}}{=} \mathsf{refl}_{v_1}$, which by consulting either H_ℓ or H_r requires that we take $p_2 \overset{\text{def}}{=} {r_{21}}^{-1}$.

Now examining $f \circ g$, we have

$$\begin{aligned} f(g(\mathsf{base})) &= \mathsf{base} \\ f(g(\mathsf{loop})) &= f(\ell_{12} \cdot r_{21}) = \mathsf{loop} \end{aligned}$$

and so we have an easy proof that this is the identity.

The proof of the more general case $\mathbb{C}(n) \simeq \mathbb{C}(n-1)$ is very similar. Take the maps $f : \mathbb{C}(n) \to \mathbb{C}(n-1)$, $g : \mathbb{C}(n-1) \to \mathbb{C}(n)$ to be

$$\begin{aligned} f(v_i) &= v_i \quad (i = 1, \ldots, n-1) & g(v_i) &= v_i & (i = 1, \ldots, n-1) \\ f(v_n) &= v_1 & g(e_{i,i+1}) &= e_{i,i+1} & (i = 1, \ldots, n-2) \\ f(e_{i,i+1}) &= e_{i,i+1} \quad (i = 1, \ldots, n-1) & g(e_{n-1,1}) &= e_{n-1,n} \cdot e_{n,1} \\ f(e_{n-1,n}) &= e_{n-1,1} \\ f(e_{n,1}) &= \mathsf{refl}_{v_1} \end{aligned}$$

where f should be thought of as shrinking $e_{n,1}$ so that v_n coalesces into v_1.

The proof that $g \circ f \sim \mathrm{id}_{\mathbb{C}(n)}$ proceeds as follows: the composition is definitionally the identity except

$$\begin{aligned} g(f(v_n)) &= v_1 \\ g(f(e_{n-1,n})) &= e_{n-1,n} \cdot e_{n,1} \\ g(f(e_{n,1})) &= \mathsf{refl}_{v_1}. \end{aligned}$$

Guided by our previous experience we choose ${e_{n,1}}^{-1} : g(f(v_n)) = v_n$, and define the pathovers by transport.

The proof that $f \circ g \sim \mathrm{id}_{\mathbb{C}(n-1)}$ requires only noting that $f(g(e_{n-1,1})) = f(e_{n-1,n} \cdot e_{n,1}) = e_{n-1,1} \cdot \mathsf{refl}_{v_1} = e_{n-1,1}$. □

Corollary 3.16. All polygons are equivalent to S^1, i.e. we have terms $e_n : \mathbb{C}(n) = S^1$, and hence we have constructed a map from the unit type $(\mathbb{C}(n), ||e_n||_{-1}) : \mathbf{1} \to \mathrm{EM}(\mathbb{Z}, 1)$.

Proof. The proofs in Lemma 3.15 can be concatenated to give $\mathbb{C}(n) \to \mathbb{C}(n-1) \to \cdots \to \mathbb{C}(2) \to S^1$. □

Definition 3.17. Let $R : [v_1 \cdots v_n] \to [v_1 \cdots v_n]$ (for "rotation") be the map sending $v_i \mapsto v_{i+1}$ and $v_n \mapsto v_0$. This map clearly preserves the edges, and so is a map of simplicial complexes, and extends to a map $[\![R]\!] : [\![v_1 \cdots v_n]\!] \to [\![v_1 \cdots v_n]\!]$.

A key point is that simplicial maps form a set, but the homotopical realization $[\![R]\!]$ has a path to the identity:

Lemma 3.18. The map $[\![R]\!] : [\![v_1 \cdots v_n]\!] \to [\![v_1 \cdots v_n]\!]$ is connected to $\mathsf{refl}_{[\![v_1 \cdots v_n]\!]}$ by a homotopy $H_R : \prod_{x : [\![v_1 \cdots v_n]\!]} [\![R]\!](x) = x$.

Proof. If x is a vertex, take $H_R(x)$ to be the obvious unique edge back to the starting vertex. This extends in the obvious functorial way to edges. □

3.4 The octahedron model of the sphere

We will create our first combinatorial surface, an octahedron. We will not prove that this type is equivalent to the sphere. In $\mathsf{SimCom_n}$ the combinatorial data of the faces can be represented with a *Hasse diagram*, which shows the poset of inclusions in a graded manner, with a special top and bottom element. We give an octahedron in Figure 5. The names of the vertices are short for white, yellow, blue, red, green, and orange, the colors of the faces of a Rubik's cube. The octahedron is the dual of the cube, with each vertex corresponding to a face.

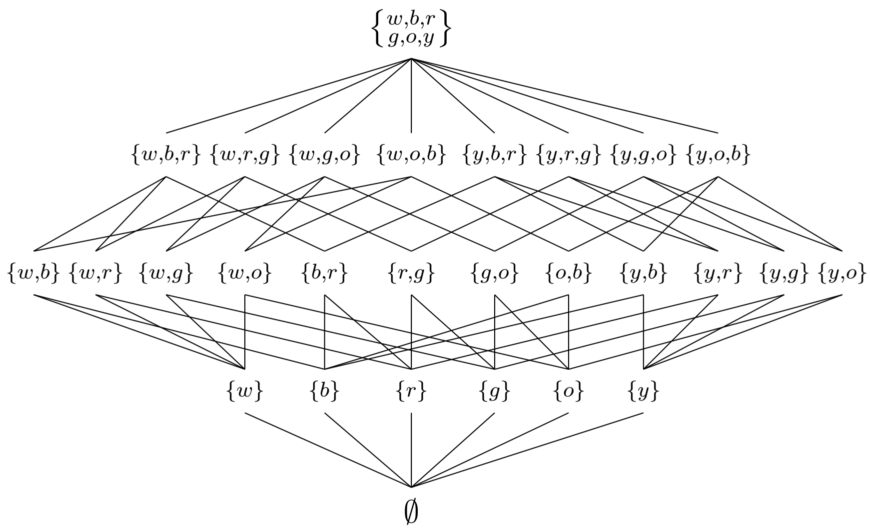

Figure 5: Hasse diagram of an octahedron O. The row of singletons is O_0 and above it are O_1 and O_2.

We can realize $O_0 \to O_1 \to O$ as a cellular type denoted $\mathbb{O}_0 \to \mathbb{O}_1 \to \mathbb{O}$.

Lemma 3.19. There is an equivalence $\mathbb{O} \simeq S^2$.

Proof. Omitted. □

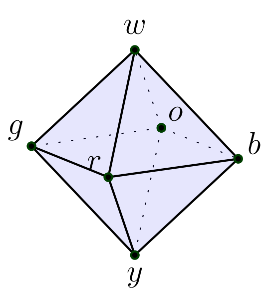

Figure 6: The type $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

4 *Bundles, connections, and curvature*

Bundles are simply maps into the universe. By using the extra cellular structure and the even more detailed combinatorial structure of higher realizations, we can identify inside of HoTT some additional classical definitions.

4.1 Definitions

Having the cellular structure allows us to define bundles and connections.

Definition 4.1. If $\mathbb{M} \overset{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a cellular type and $f_k : \mathbb{M}_k \to \mathcal{U}$ are type families on each skeleton such that all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"] \\
 & & \mathcal{U} & &
\end{tikzcd}$$

then we say

- The map f_k is a k**-bundle** on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$ that f_k extends f_{k-1} is called a k**-connection on the** $(k-1)$**-bundle** f_{k-1}.

Having the additional structure of a simplicial complex allows us to further define curvature, which is a local concept.

Definition 4.2. If $\mathbb{M}$ is the realization of a simplicial complex, such that for each pushout defining $\mathbb{M}_k$ we have the diagram

$$\begin{tikzcd}
M_k \times \partial\Delta^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\Nearrow h_k"] & M_k \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r, "\imath_{k-1}"] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \arrow[dl, phantom, "\Swarrow"] \\
 & \mathcal{U}
\end{tikzcd}$$

the outer square of which restricts on each face to the diagram

$$\begin{tikzcd}
\{m_k\} \times S^k \arrow[r, "!"] \arrow[d, "\partial_{k-1}"'] \arrow[dr, phantom, "\Swarrow \flat_k"] & \mathbf{1} \arrow[d, "f_k \circ *_k"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

then we say the filler $\flat_k$ is called a **flatness structure for the face** m_k, and its ending path is called **curvature at the face** m_k.

The definitions can be digested to give

Lemma 4.3. Given f_{k-1} as above, a k-connection exists if and only if there exists a flatness structure for each k-face.

On a 2-dimensional cellular type $\mathbb{M} \overset{\text{def}}{=} \mathbb{M}_0 \to \mathbb{M}_1 \to \mathbb{M}_2$ the terminology works out as follows: a 0-bundle on $\mathbb{M}$ is a map $T_0 : \mathbb{M}_0 \to \mathcal{U}$. A 1-connection on T_0 is an extension $T_1 : \mathbb{M}_1 \to \mathcal{U}$. A 2-connection on T_1 is an extension $T_2 : \mathbb{M}_2 \to \mathcal{U}$. Classically, a 1-extension that extends to a 2-connection is called *flat*.

4.2 Flat connections as local trivializations

This section can ve viewed as an extended remark. The observation we want to make is that the data of a 2-bundle on a realization of a 2-dimensional simplicial complex is related to the construction of local trivializations: the fiber at one vertex can be extended throughout a single face coherently, using the connection (the extension of the classifying map to the edges) to specify isomorphisms with the fibers at the other points, and the higher connections to establish commutativity between these.

We introduce a notation more suitable for the algebra of charts and overlaps: denote the fiber at v_i by T_i and denote transport along $e_{ij} : v_i = v_j$ by $T_{ji} : T_i \to T_j$. The indices are ordered from right to left, which is compatible with function composition notation. Denote the inverse function by swapping indices: $T_{ij} \overset{\text{def}}{=} T_{ji}^{-1}$. Assume we have some fixed isomporphism $T_i = S^1$, and to avoid composing everything with this function we will assume it is id. In the diagram below we see the data arranged so that our bundles fibers are on the left, and the fiber of a trivial bundle is on the right.

$$\begin{tikzcd}
T_i \arrow[r, equal, "\mathrm{id}"] \arrow[d, "T_{ji}"'] \arrow[dd, bend right=60, "T_{ki}"'] \arrow[dd, phantom, shift right=4ex, "\overset{\flat_{ijk}}{\Leftarrow}"] & S^1 \arrow[d, equal] \arrow[dd, bend left=60, equal] \\
T_j \arrow[r, "T_{ij}"'] \arrow[d, "T_{kj}"'] & S^1 \arrow[d, equal] \\
T_k \arrow[r, "T_{ij}T_{jk}"'] & S^1
\end{tikzcd}$$

The two middle squares commute definitionally. Call these two squares together the back face. The left triangle is filled by the flatness structure on the face, and the right triangular filler is trivial. There is also a filler needed for the front, i.e. the outer square. This requires proving that $T_{ki}T_{ij}T_{jk} = \mathrm{id}$, which is supplied by the flatness structure. There is also a 3-cell filling the interior of this prism, mapping the back face plus the left triangle filler to the front face plus the right triangle filler. These two faces both consist of one or two identities concatenated with the flatness structure, and so the 3-cell is definitional.

This relationship between flatness structure and local triviality of a chart can be compared to the classical result that on a paracompact, simply connected manifold (such as a single chart), a connection on a principal bundle is flat if and only if the bundle is trivial. See for example [12] Corollary 9.2.

4.3 The tangent bundle of the sphere

We will build up a map T out of $\mathbb{O}_0 \to \mathbb{O}_1 \to \mathbb{O}$ which is meant to be a model of the tangent bundle of the sphere. The link function will serve as our approximation to the tangent space. Taking the link of a vertex gives us a map from vertices to polygons, so the codomain is $\mathrm{EM}(\mathbb{Z}, 1)$.

If $\{b, r, g, o\}$ are four vertices in $\mathbb{O}$, the notation $[\![brgo]\!]$ refers to the 4-gon spanned by these four vertices and the edges between them.

Definition 4.4. $T_0 \stackrel{\text{def}}{=} \mathsf{link} : \mathbb{O}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is given by:

$$\begin{aligned}
\mathsf{link}(w) &= [\![brgo]\!] & \qquad \mathsf{link}(r) &= [\![wbyg]\!] \\
\mathsf{link}(y) &= [\![bogr]\!] & \qquad \mathsf{link}(g) &= [\![wryo]\!] \\
\mathsf{link}(b) &= [\![woyr]\!] & \qquad \mathsf{link}(o) &= [\![wgyb]\!]
\end{aligned}$$

We chose these orderings for the vertices in the link, by visualizing standing at the given vertex as if it were the north pole, then looking south and enumerating the link in clockwise order, starting from w if possible, else b.

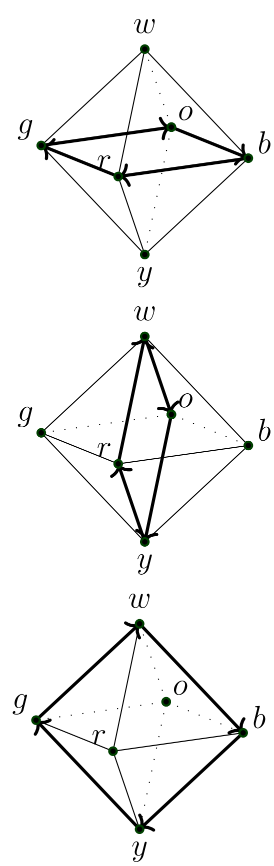

Figure 7: link for the vertices w, b and r.

To extend T_0 to a function T_1 on the 1-skeleton we have some freedom. We will do something motivated by the figures we have been drawing of an octahedron embedded in

3-dimensional space. We will imagine how T_1 changes as we slide from point to point in the embedding shown in the figures. Sliding from w to b and tipping the link as we go, we see $r \mapsto r$ and $o \mapsto o$ because those lie on the axis of rotation. Then $g \mapsto w$ and $b \mapsto y$.

Definition 4.5. Define $T_1 : \mathbb{O}_1 \to \mathrm{EM}(\mathbb{Z}, 1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:

- $T_1(wr) : [\![brgo]\!] \mapsto [\![bygw]\!]$ (b, g fixed)
- $T_1(wg) : [\![brgo]\!] \mapsto [\![wryo]\!]$
- $T_1(wb) : [\![brgo]\!] \mapsto [\![yrwo]\!]$ (r, o fixed)
- $T_1(wo) : [\![brgo]\!] \mapsto [\![bwgy]\!]$

Transport away from y:

- $T_1(yb) : [\![bogr]\!] \mapsto [\![woyr]\!]$
- $T_1(yr) : [\![bogr]\!] \mapsto [\![bygw]\!]$
- $T_1(yg) : [\![bogr]\!] \mapsto [\![yowr]\!]$
- $T_1(yo) : [\![bogr]\!] \mapsto [\![bwgy]\!]$

Transport along the equator:

- $T_1(br) : [\![woyr]\!] \mapsto [\![wbyg]\!]$
- $T_1(rg) : [\![wbyg]\!] \mapsto [\![wryo]\!]$
- $T_1(go) : [\![wryo]\!] \mapsto [\![wgyb]\!]$
- $T_1(ob) : [\![wgyb]\!] \mapsto [\![woyr]\!]$

It's very important to be able to visualize what T_1 does to triangular paths such as $wb \cdot br \cdot rw$ (which circulates around the boundary of face wbr). You can see it if you imagine Figure 7 as the frames of a short movie. Or you can place your palm over the top of a cube and note where your fingers are pointing, then slide your hand to an equatorial face, then along the equator, then back to the top. The answer is: you come back rotated clockwise by a quarter-turn, which we saw in Definition 3.17 where it is called $[\![R]\!]$.

Now let's extend T_1 to all of $\mathbb{O}$ by providing values for the eight faces. The face wbr is a path from refl_w to the concatenation $wb \cdot br \cdot rw$, and so the image of wbr under the extended version of T_1 must be a homotopy from $\mathsf{refl}_{T_1(w)}$ to $T_1(wb \cdot br \cdot rw)$. Here *there is no additional freedom.*

Definition 4.6. Define $T_2 : \mathbb{O} \to \mathrm{EM}(\mathbb{Z}, 1)$ by extending T_1 to the faces as follows (making use of H_R from Lemma 3.18):

- $T_2(wbr) = H_R$
- $T_2(wrg) = H_R$
- $T_2(wgo) = H_R$
- $T_2(ybo) = H_R$
- $T_2(yrb) = H_R$
- $T_2(ygr) = H_R$

- $T_2(yog) = H_R$
- $T_2(ybo) = H_R$

Defining these flatness structures suffices to define T_2 by Lemma 4.3.

4.4 Existence of connections

How confident can we be that we can always define a connection on an arbitrary combinatorial manifold? Two things make the octahedron example special: the link is a 4-gon at every vertex (as opposed to having a variable number of vertices), and every transport map extends to a rotation of the entire octahedron in 3-dimensional space. This imposed a coherence on the interactions of all the choices we made for the connection, which we can worry may not exist for more complex combinatorial data.

We know as a fact outside of HoTT that any combinatorial surface that has been realized as a triangulated surface embedded in 3-dimensional euclidean space can inherit the parallel transport entailed in the embedding. We could then approximate that data to arbitrary precision with enough subdivision of the fibers of T.

What would a proof inside of HoTT look like? We will leave this as an open question.

5 Vector fields

5.1 Definition

Vector fields are sections of the tangent bundle of a manifold. We do not have a general theory of tangent bundles, even for 2-dimensional cellular types, since we cannot yet prove that connections always exist on the 1-skeleton. But if $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_o \to \mathbb{M}_1 \to \mathbb{M}_1$ is a cellular type, and *given* an extension $T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ of the link function, we can consider the type of sections $\prod_{x:\mathbb{M}_1} T_1(x)$.

In this section and for the remainder of the note, we will assume that the bundle is oriented, that is that the bundle is in fact a map $T : \mathbb{M} \to \mathrm{K}(\mathbb{Z}, 2)$, the type of S^1-torsors.

Definition 5.1. Given a 2-dimensional cellular type $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_o \to \mathbb{M}_1 \to \mathbb{M}_1$ equipped with type family $T : \mathbb{M} \to \mathrm{K}(\mathbb{Z}, 2)$ extending link, a **vector field** on $\mathbb{M}$ is a term $X : \prod_{x:\mathbb{M}_1} T_1(x)$.

> **Remark 5.2.** The circle bundle extending link captures the *unit spheres* of the classical tangent bundle. A section of this bundle is therefore analogous to a classical *nonvanishing* vector field. To allow for classical vector fields *with zeroes*, we are limiting the section X to the 1-skeleton of $\mathbb{M}$.

On the 0-skeleton X picks a point in each link, i.e. a neighbor of each vertex. On a path $p : x =_{\mathbb{M}} y$, X assigns a dependent path over p, which as we know is a term $\pi : \mathsf{tr}(p)(X(x)) =_{T_y} X(y)$. We are very interested in working with the concatenation operation on dependent paths, which we call *swirling*.

5.2 Swirling of dependent paths

Consider the vertex $v_1 : \mathbb{M}$, a face F containing vertices v_1, v_2, v_3, and the boundary path $\ell \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$. As we did in Section 4.2, denote $T(v_i)$ by T_i and $T(e_{ij})$ by T_{ji}. The indices are swapped so that we can have expressions that respect function-composition order, such as $T_{32}T_{21}(X_1) : T_3$. We retain the opposite convention (which we can call path-concatenation order) for the vector field, e.g. X_{ij} which is a path in T_j, as these are paths. Figure 8 shows in tabular form how we concatenate the dependent paths over $e_{12} \cdot e_{23} \cdot e_{31}$. Figure 9 shows visually a possible example.

As we traverse an edge, say e_{12}, we get a path in T_2 which is the image of e_{12} under X, denoted X_{12}. As we traverse an additional edge, X_{12} is simply mapped to the next vertex by transport. The image is carried first to $T_{32}(X_{12})$ then to $T_{13} \circ T_{32}(X_{12})$.

We wish to simplify expressions such as $T_{13}T_{32}X_{12} \cdot T_{13}X_{23} \cdot X_{31}$, which take place in a particular fiber (T_1 in this case), and which depend on arranging for the endpoint of one segment to agree with the start of another. The simplification will empower us to easily perform calculations over the whole manifold, and to prove our main theorem 6.8.

First we will choose a specific group that acts on all the fibers of T. Recall from Section 2.1

$$
\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1 \\
X_1 & X_2 \arrow[d, equal, "X_{12}:"'] & X_3 \arrow[d, equal, "X_{23}:"'] & X_1 \arrow[d, equal, "X_{31}:"'] \\
 & T_{21}X_1 & T_{32}X_2 \arrow[d, equal, "T_{32}X_{12}:"'] & T_{13}X_3 \arrow[d, equal, "T_{13}X_{23}:"'] \\
 & & T_{32}T_{21}X_1 & T_{13}T_{32}X_2 \arrow[d, equal, "T_{13}T_{32}X_{12}:"'] \\
 & & & T_{13}T_{32}T_{21}X_1
\end{tikzcd}
$$

Figure 8: The data in each fiber as we move around a triangle with vertices indexed 1, 2, and 3. Double-lines indicate identity types between two types, and their labels are terms of this type. Items with one index are terms of some type at a vertex, and items with two indices are terms of a type on an edge.

that given some basepoint $b : \mathrm{K}(\mathbb{Z}, 2)$ and a type $T_i : \mathrm{K}(\mathbb{Z}, 2)$, the group $b =_{\mathrm{K}(\mathbb{Z},2)} b$ acts on T_i. Suppose we have a master basepoint $m : \mathbb{M}$, and choose $T_m : \mathrm{K}(\mathbb{Z}, 2)$ as the basepoint, so that all the fibers of T are now equipped with an action of $T_m =_{\mathrm{K}(\mathbb{Z},2)} T_m$.

We will give this group a nickname for brevity. Recall the bundle $\mathrm{Aut}\, T$ from Section A.1.

Definition 5.3. Define $\mathscr{G} \overset{\mathrm{def}}{=} \mathrm{Aut}\, T$, so that $\mathscr{G}(m) = (T_m =_{\mathrm{K}(\mathbb{Z},2)} T_m)$. Since this group is commutative, we'll denote the group operation by $+ : \mathscr{G}(m) \times \mathscr{G}(m) \to \mathscr{G}(m)$.

We need to derive two operations from the action $\mathscr{G}(m)$.

Definition 5.4. The map $\mathrm{pr}_1 \circ (\alpha, \mathrm{pr}_2)^{-1} : T_i \times T_i \to \mathscr{G}(m)$ is called **subtraction**. It maps (x, y) to the unique term $\delta : \mathscr{G}(m)$ such that $\alpha(\delta, x) = y$. For brevity we denote $\mathrm{pr}_1 \circ (\alpha, \mathrm{pr}_2)^{-1}(x, y)$ by $y - x$.

Lemma 5.5. If G is a higher group with multiplication $\mu : G \times G \to G$ and proof of commutativity $\mathsf{is\_comm} : \prod_{a,b:G} \mu(a, b) = \mu(b, a)$ then μ induces a function $\mu_= : (x =_G y) \times (x' =_G y') \to (\mu(x, x') =_G \mu(y, y'))$.

Proof. If $p : x =_G y$ and $p' : x' =_G y'$, then we can define $\mu_=(p, p')$ by concatenating the three paths

$$
\begin{aligned}
\mu(x', p) &: \mu(x', x) =_G \mu(x', y) \\
\mathsf{is\_comm}(x', y) &: \mu(x', y) =_G \mu(y, x') \\
\mu(y, p') &: \mu(y, x') =_G \mu(y, y').
\end{aligned}
$$

□

Each fiber T_i is pointed by X_i, so we can define the map $T_i \xrightarrow{-X_i} \mathscr{G}(m)$, and then give a

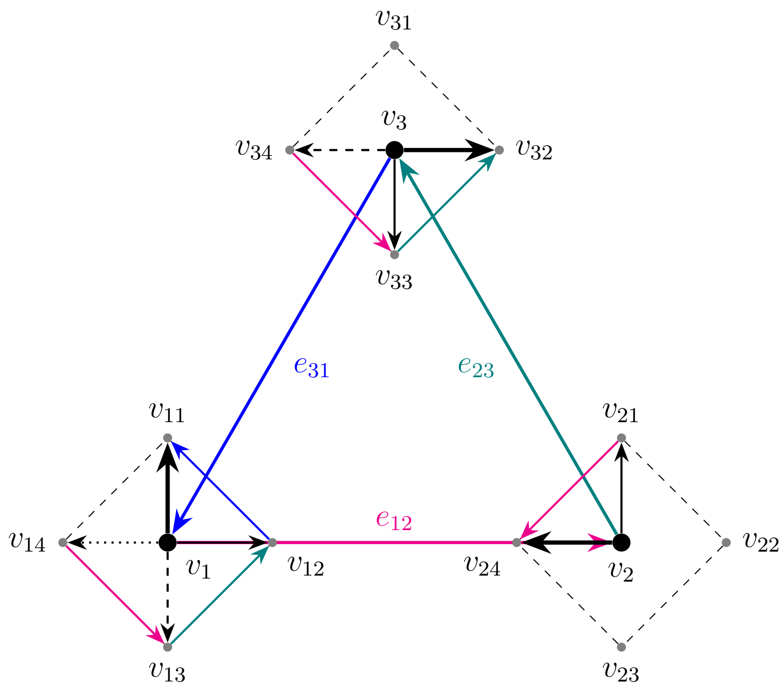

Figure 9: A vector field swirling counterclockwise around a face, in a bundle of squares. Imagine that transport along e_{12} does not rotate along the page, that transport along e_{23} rotates counterclockwise by 90 degrees, and that transport along e_{31} again does not rotate along the page. Thick black vectors are the vector field at a point. Thin vectors are transported once, dashed twice, and dotted three times. The vertices v_{ij} are in the tangent fibers. If you have a colorized version of the document, the colors of the arrows correspond: the red edge produced the red edge in the fibers.

name to the special term $\rho_{ij} \stackrel{\text{def}}{=} T_{ij}X_j - X_i$. It fits into a diagram

$$\begin{tikzcd} T_i \arrow[r, "T_{ji}"] \arrow[d, "(\text{-})-X_i"'] & T_j \arrow[d, "(\text{-})-X_j"] \\ \mathscr{G}(m) \arrow[r, "(\text{-})+\rho_{ji}"'] & \mathscr{G}(m) \end{tikzcd} \tag{5.1}$$

Lemma 5.6. $\rho_{ij} + \rho_{ji} =_{\mathscr{G}(m)} 0$.

Proof.

$$\begin{aligned} (X_i + \rho_{ij}) + \rho_{ji} &=_{T_i} T_{ij}X_j + \rho_{ji} && \text{by definition of } \rho_{ij} \\ &=_{T_i} T_{ij}(X_j + \rho_{ji}) && \text{by equivariance of transport} \\ &=_{T_i} T_{ij}(T_{ji}(X_i)) && \text{by definition of } \rho_{ji} \\ &=_{T_i} X_i && \text{by definition of } T \end{aligned}$$

□

We can obtain a path in $\mathscr{G}(m)$ from a dependent path by again subtracing the basepoint, e.g. $\sigma_{ij} \stackrel{\text{def}}{=} X_{ij} - X_j : T_{ji}X_i - X_j =_{\mathscr{G}(m)} 0$, which type also has the name $\rho_{ji} =_{\mathscr{G}(m)} 0$.

Notice the reversal in indices between σ_{ij} and ρ_{ji}, which reflects our opposite conventions for X_{ij} and T_{ji}.

The key technical lemma is the following. Recall that dependent functions such as our vector field X send refl_{v_i} to refl_{X_i} (by path induction, see for example [6] Lemma 2.3.4). As stated, this can only be used when we have a path that is refl, for example when traversing $e_{ij} \cdot e_{ji}$, i.e. an edge followed by its inverse. We will use subtraction together with the operation of Lemma 5.5 to lift this requirement, by obtaining two paths in $\mathscr{G}(m)$ that can be added without needing to be concatenated directly.

Lemma 5.7. With the notation $\sigma_{ij} \overset{\text{def}}{=} X_{ij} - X_j$, and with addition of paths as in Lemma 5.5, we have $\sigma_{ij} + \sigma_{ji} =_{0=_{\mathscr{G}(m)}0} \mathsf{refl}_0$.

Proof. First we need to show that the sum is a loop, then we can prove that it is refl_0. The terms have these types:

$$\begin{aligned}
&\sigma_{ij} : \rho_{ji} =_{\mathscr{G}(m)} 0 \\
&\sigma_{ji} : \rho_{ij} =_{\mathscr{G}(m)} 0
\end{aligned}$$

so when we add these paths with $+$ we obtain a path $\sigma_{ij} + \sigma_{ji} : \rho_{ji} + \rho_{ij} =_{\mathscr{G}(m)} 0$ which by Lemma 5.6 is $0 =_{\mathscr{G}(m)} 0$. We compute $+$ using the concatenations in Lemma 5.5, which gives

$$\sigma_{ij} + \sigma_{ji} : (0+0) \overset{\text{Lemma 5.6}}{=\!=} (\rho_{ji} + \rho_{ij}) \overset{\sigma_{ij}+\rho_{ij}}{=\!=} (0 + \rho_{ij}) \overset{\mathsf{is\_comm}}{=\!=} (\rho_{ij} + 0) \overset{\sigma_{ji}}{=\!=} (0+0)$$

$$\begin{aligned}
\sigma_{ij} + \sigma_{ji} &=_{0=_{\mathscr{G}(m)}0} (\sigma_{ij} + \rho_{ij}) \cdot \sigma_{ji} \\
&=_{0=_{\mathscr{G}(m)}0} ((X_{ij} - X_j) + \rho_{ij}) \cdot (X_{ji} - X_i) && \text{definition of } \sigma s \\
&=_{0=_{\mathscr{G}(m)}0} (T_{ij}(X_{ij}) - X_i) \cdot (X_{ji} - X_i) && \text{action of } \rho_{ij} \text{ (see (5.1))} \\
&=_{0=_{\mathscr{G}(m)}0} \mathsf{refl}_{X_i} - X_i && \text{path induction for } X \\
&=_{0=_{\mathscr{G}(m)}0} \mathsf{refl}_0
\end{aligned}$$

□

Remark 5.8. The classical argument of Hopf [2], which is presented in more detail in the more readily available [3], makes an implicit assumption that we can concatenate two terms of different type such as X_{ij} and X_{kl}. The authors name such terms “change in angle across an edge.” The extra work we are doing in this section amounts to a partial formalization of this idea.

5.3 An example vector field on the sphere

define the example I worked out, compute swirling

6 The total construction

We will place holonomy, flatness, and vector fields on the same footing, and combine them. We will prove the equivalance of the total curvature of a tangent bundle, and total index of a vector field. This is the key relationship in proving both the Gauss-Bonnet theorem and the Poincaré-Hopf theorem.

6.1 Index of the vector field on a face

The index of a vector field is derived from other data that we have. Consider a 2-dimensional simplicial complex M and its 2-dimensional realization $\mathbb{M} = \mathbb{M}_0 \to \mathbb{M}_1 \to \mathbb{M}_2$. Consider a face $F : M_2$, with $m : \mathbb{M}$ a vertex of F and $\partial F : \partial\Delta^2 \to \mathbb{M}$ the boundary loop of the face. We have accumulated the following constructions:

$$\begin{aligned}
\mathsf{tr}_F &\overset{\text{def}}{=} \mathsf{tr}(\partial F) &&: Tm = Tm && \text{holonomy/curvature}\\
\flat_F &\overset{\text{def}}{=} \flat(\partial F) &&: \mathrm{id} =_{Tm=Tm} \mathsf{tr}(\partial F) && \text{flatness}\\
X_F &\overset{\text{def}}{=} X(\partial F) &&: \mathsf{tr}(\partial F)(X(m)) =_{Tm} X(m) && \text{swirling.}
\end{aligned}$$

The flatness is a path in automorphisms of Tm whereas the swirling is a path in Tm, but we can view the latter it as a path in automorphisms as well:

Proposition 6.1. Given a polygon $\mathbb{C}(n) : \mathrm{K}(\mathbb{Z}, 2)$ and a point $b : \mathbb{C}(n)$ the evaluation map $\mathrm{ev}(b) : (\mathbb{C}(n) =_{\mathrm{K}(\mathbb{Z},2)} \mathbb{C}(n)) \to \mathbb{C}(n)$ is an equivalence.

Proof. See [4]. □

Now $\sigma_F^X \overset{\text{def}}{=} \mathrm{ev}(X(m))^{-1}(X_F)$ is the automorphism of Tm that corresponds to the swirling. We now have:

$$\begin{aligned}
\mathsf{tr}_F &: Tm = Tm && \text{curvature on } F\\
\flat_F &: \mathrm{id} =_{Tm=Tm} \mathsf{tr}_F && \text{flatness on } F\\
\sigma_F^X &: \mathsf{tr}_F =_{Tm=Tm} \mathrm{id} && \text{swirling on } F.
\end{aligned}$$

These last two can be concatenated to make a loop. We can then obtain an integer by using the fact that the automorphisms are a circle $\Omega_{Tm}(\mathrm{K}(\mathbb{Z}, 2)) \simeq S^1$, plus the well known formula that loops in the circle are integers: $\Omega(S^1, \mathsf{base}) \simeq \mathbb{Z}$ (e.g. [6] Corollary 8.1.10).

Definition 6.2. The **index of the vector field** X **on the face** F is the integer $I_F^X \overset{\text{def}}{=} \Omega(\flat_F \cdot \sigma_F^X) : \Omega(\mathrm{id} =_{Tm=Tm} \mathrm{id})$.

We now have the following final list of ingredients at a single face:

$$\begin{aligned}
\mathsf{tr}_F &: Tm = Tm && \text{curvature on } F\\
\flat_F &: \mathrm{id} =_{Tm=Tm} \mathsf{tr}_F && \text{flatness on } F\\
\sigma_F^X &: \mathsf{tr}_F =_{Tm=Tm} \mathrm{id} && \text{swirling on } F\\
I_F^X &: \mathbb{Z} && \text{index on } F.
\end{aligned} \tag{6.1}$$

6.2 Total curvature and index on the sphere

Now we wish to compute a sum over all faces of the data. To do this we need the following.

Definition 6.3. An **oriented set of pointed faces** of a realization $\mathbb{M}$ of an oriented 2-dimensional simplicial complex $M = [M_0, M_1, M_2]$, is a choice $v_F : F$ of a vertex in each face $F : M_2$, and a choice of boundary path $\ell_F : v_F = v_F$ on each face, satisfying the **cancellation property**: for each face the boundary path is a concatenation of three edge paths, and for each edge $e = \{x, y\}$ in M_1, whose image under realization is denoted $p_e : x = y$, the path p_e appears as a concatenation term in exactly one loop, say ℓ_F, and p_e^{-1} appears in exactly one loop $\ell_{F'}$.

We have seen in Lemma 5.5 how to perform the required sum.

Definition 6.4. The **total flatness**, **total swirling**, and **total index** on an oriented set of pointed faces is

$$
\begin{aligned}
\mathsf{tr}_{\text{tot}} &\stackrel{\text{def}}{=} \sum_F \mathsf{tr}_F &&: Tm = Tm \\
\flat_{\text{tot}} &\stackrel{\text{def}}{=} \sum_F \flat_F &&: \mathrm{id} =_{Tm=Tm} \mathsf{tr}_{\text{tot}} \\
\sigma^X_{\text{tot}} &\stackrel{\text{def}}{=} \sum_F \sigma^X_F &&: \mathsf{tr}_{\text{tot}} =_{Tm=Tm} \mathrm{id} \\
I^X_{\text{tot}} &\stackrel{\text{def}}{=} \sum_F I^X_F &&: \mathbb{Z}.
\end{aligned}
\tag{6.2}
$$

The assumption about edges appearing twice, once in each direction, together with Lemma 5.6 proves the following

Proposition 6.5. *The total transport is 0 in $Tm = Tm$.*

Proof. With a vector field X chosen, we can compute

$$\sum_F \mathsf{tr}_F = \sum_{(\text{edges } e_{ij})} \rho_{ij} + \rho_{ji} =_{Tm=Tm} 0$$

and note that the result does not depend on X. ☐

And making use of Lemma 5.7 we obtain

Proposition 6.6. *The total swirling is refl_0 in $Tm = Tm$.*

Proof. $\sum_F \sigma^X_F = \sum_{(\text{edges } e_{ij})} \sigma_{ij} + \sigma_{ji} =_{Tm=Tm} \mathsf{refl}_0$ ☐

Corollary 6.7. *Total flatness is a loop: $\flat_{\text{tot}} : \mathrm{id} =_{Tm=Tm} \mathrm{id}$.*

Proof. Follows from Corollary 6.5. ☐

And finally, if we give the loop map Ω the nickname "winding number":

Theorem 6.8. *The winding number of total flatness equals the total index.*

Proof. Follows directly from $I^X_{\text{tot}} \stackrel{\text{def}}{=} \Omega(\flat_{\text{tot}} \cdot \sigma^X_{\text{tot}})$ and Proposition 6.6. ☐

A Appendix: the program ahead

Euler characteristic. Chern-Weil theory. Atiyah bundle. Space of connections is contractible. Formalization.

The results of this note can be extended in many directions. There are higher-dimensional generalizations of Gauss-Bonnet, including the theory of characteristic classes and Chern-Weil theory (which links characteristic classes to connections and curvature). These would involve working with nonabelian groups like $SO(n)$ and sphere bundles. Results from gauge theory could be imported into HoTT, as well as results from surgery theory and other topological constructions that may be especially amenable to this discrete setting. Relationships with computer graphics and discrete differential geometry[13][14] could be explored. Finally, a theory that reintroduces smoothness could allow more formal versions of the analogies explored here.

A.1 The bundle of automorphisms

Definition A.1. Suppose we have $T : M \to \mathrm{EM}(\mathbb{Z}, 1)$ and $P \overset{\text{def}}{=} \sum_{x:M} Tx$. Then we can form the type family $\operatorname{Aut} T : M \to \mathcal{U}$ given by $\operatorname{Aut} T(x) \overset{\text{def}}{=} (Tx = Tx)$. The total space $\operatorname{Aut} P \overset{\text{def}}{=} \sum_{x:M}(Tx = Tx)$, which is a bundle of groups, is called the **automorphism bundle** or the **gauge bundle** and sections $\prod_{x:M}(Tx = Tx)$, which are homotopies $T \sim T$, are called **automorphisms of** P or **gauge transformations**.

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