(Smooth manifolds)
(k-forms)
Connections $\longrightarrow$
Transport $\longrightarrow$
Curvature $\longrightarrow$
Principal bundles $\longrightarrow$

Gauge transformations $\longrightarrow$

Connections

Fiber bundle (E, p, M, F)

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & M \end{tikzcd}$$

1) E, M, F smooth mfds.
2) p smooth, surjective
3) Locally trivial
   can cover M with $\{U_i\}$

$$\begin{tikzcd} E|_{U_i} \arrow[rr] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & U_i \times F \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

(p^*TM)

$$\begin{tikzcd} 0 \arrow[r] & VE \arrow[r] & TE \arrow[r, "Tp"] & TM \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd} \quad \text{over } E$$

a connection is
a splitting of this e.s.

$$\begin{tikzcd} & p^*TM \arrow[d] \\ TE \arrow[r] \arrow[d, "Tp"'] & E \arrow[d, "p"] \\ TM \arrow[r] & M \end{tikzcd}$$

1) $TE = VE \oplus HE$

$$0 \longrightarrow VE \longrightarrow TE \longrightarrow p^*TM \longrightarrow 0 \quad \text{over } E$$

2) projection $\omega : TE \to VE$

$HE = \ker \omega$

$\omega : \Omega^1(E;VE) = \Gamma\, T^*E \otimes VE$ — T^*E dual tangent bundle

3) $\text{lift} : (v : T_m M) \to (e : E_m) \to T_e E$

in fact $H_e E$

Connections exist

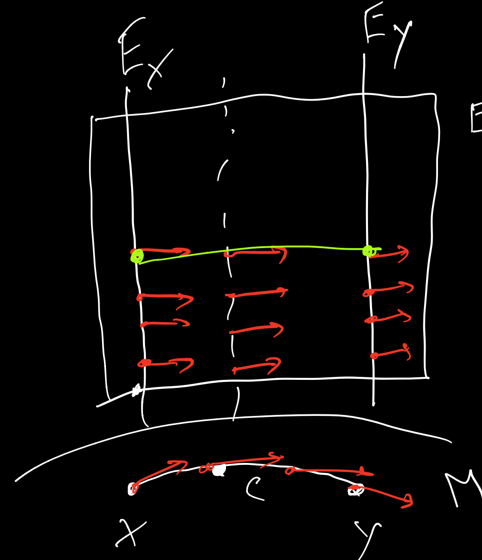
M paracompact (partition of unity, create a metric on M)

"distribution" sub-bundle

family of subspaces of a vector bundle

VE, HE

"infinitesimal splitting of E"



Transport

Parlay lift into tr

$c : [0,1] \to M$ smooth

c' vector field along c

$\text{lift}(c'(t), -)$

$* \, tr : (\omega : \Omega^1(E;VE)) \to \begin{pmatrix} c : [0,1] \to X \\ c(0) = x \\ c(1) = y \end{pmatrix} \to E_x \to E_y$

$x(1) = y$

Facts

1) $tr(\omega, c_{\leftarrow}) = tr(\omega, c_{\rightarrow})^{-1}$

2) $f : [0,1] \to [0,1]$ smooth $\Rightarrow tr(\omega, c \circ f) = tr(\omega, c)$
   $f(0) = 0$
   $f(1) = 1$

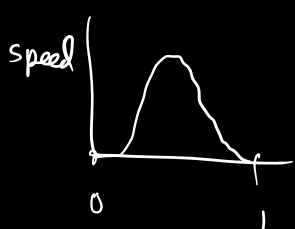
3) c, d form $c \cdot d$ $tr(\omega, c \cdot d) =$
   $c(1) = d(0)$ follow c then d $tr(\omega, d) \circ tr(\omega, c)$
   within $[0,1]$

Theory building

$\mathcal{P}_1(M)$ path groupoid

ob: pts of M

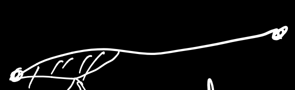
mor: smooth paths in M w/ sitting instants



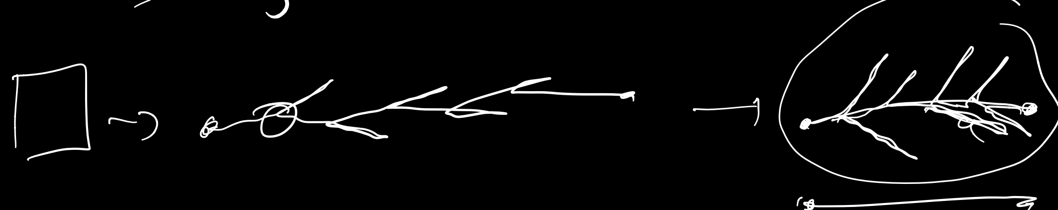
$\cdot$: concatenate + rescale

$\sim$: thin homotopy

$[0,1] \times [0,1] \to M$



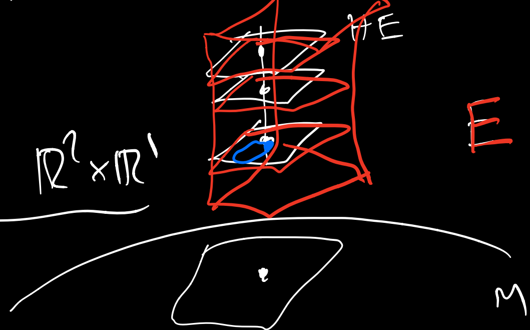
factors through a tree



Curvature

Distributions VE, HE

inf. $\rightsquigarrow$ local? "foliation"



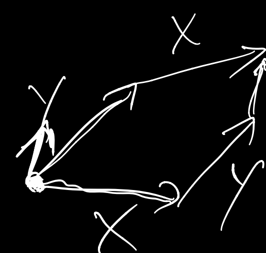
E is locally $\mathbb{R}^2 \times \mathbb{R}^1$

foliation

$X, Y : \Gamma HE$

$[X,Y] \overset{?}{:} \Gamma HE$

$[X,Y]f = X(Yf) - Y(Xf) \neq 0$



$[X,Y] \overset{?}{:} \Gamma HE \quad \underset{\text{Frobenius thm. 19th c.}}{\overset{\longleftarrow}{\longrightarrow}} \quad \exists$ foliation of HE

$\omega([X,Y])$ curvature

$X, Y : \Gamma HE \qquad \omega([X,Y]) = F(X,Y) : VE$

$F = 0 \iff$ foliation

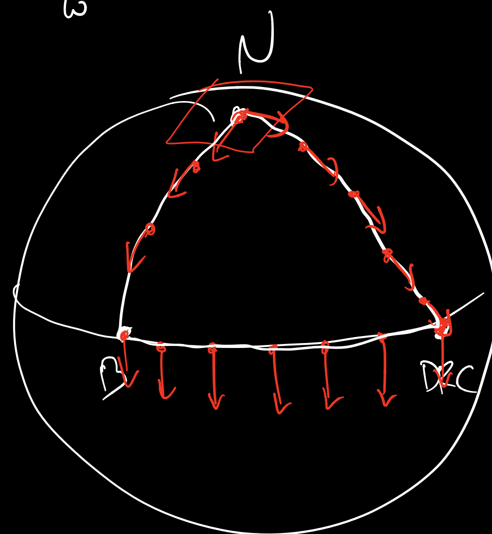
Flatness

trivial bundle $M \times F$

ω flat means ω is locally the obvious standard flat connection

<u>Theorem</u>

ω is flat iff $F_\omega = 0$.



$$E \downarrow S^2$$

$$\begin{array}{ccc} TTS^2 & & \\ \downarrow & & \\ TS^2 & \leftarrow & \mathbb{R}^2 = F \\ p \downarrow & & \\ S^2 & & \end{array}$$

$$\underbrace{TTS^2}_{\dim 4}$$

$\operatorname{Ker} Tp$

Principal bundle

Fiber bundle where F is a Lie group G and

PB1) $E \times G \to E$ free action of G on E on the right

PB2) $M = E/G$

PB3) $$\begin{tikzcd} E|_{U_i} \arrow[r, "\phi_i"] \arrow[dr] & U_i \times G \arrow[d] \\ & U_i \end{tikzcd}$$ $\phi_i(u) = (m, g)$

$\phi_i(u \cdot h) = (m, g \cdot h)$

Fibers are like affine versions of G.

E is trivial iff it has a section

e.g. $GL(V) \cong \underline{\{\text{bases of } V\}}$

$$\begin{array}{c} E' \\ \downarrow \\ M \end{array} \text{ vector bundle } n\text{-dim} \implies \begin{array}{c} E \leftarrow F \\ \downarrow \\ M \end{array} \begin{array}{l} \text{bases} \\ \text{of } E'_m \end{array}$$

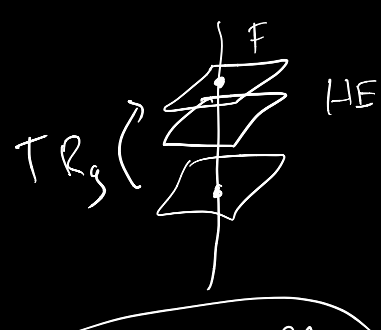
$$\begin{array}{c} TM \\ \downarrow \\ M \end{array} \quad \underline{GL(V)} \quad \begin{array}{c} E \\ \downarrow \\ M \end{array} \text{ frame bundle}$$

+ orientation $GL^+(V)$ $\quad \begin{array}{c} E' \\ \downarrow \\ M \end{array}$

+ metric $SO(V)$

Connections on (E, M, p, G)

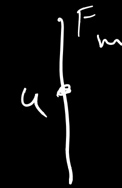
PB1) HE is G-invariant



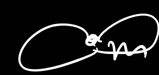
Facts

1) G acts on horizontal curves in E

2) $\Phi(\overset{E}{u}) = \mathrm{Image}\big(\mathrm{tr}(\omega, \{\text{loops}_m\})(u)\big)$



$\Phi_0(u) = \quad \ldots \quad \left\{\begin{matrix}\text{contractible}\\ \text{loops}_m\end{matrix}\right\}$



$\Phi(u) \subset G$ holonomy subgroup at u — Kobayashi–Nomizu I

$u \mapsto e$

$\Phi_0(u) \subset G$ reduced hol. subgrp.

2.5) Φ_0 is connected

3) Ambrose–Singer theorem

$\underline{\mathrm{Lie}}(\underline{\Phi_0}(u))$ = span of all $\mathfrak{g}$ of the form $F(X,Y)$ (X,Y horizontal)) — $VE = M \times \mathfrak{g}$

curvature is infinitesimal holonomy $\quad \omega : \Omega^1(E;\mathfrak{g})$

$X(u)$, $Y(u)$, extend locally $\quad$ ad $\mathfrak{g}$

$\Omega^1(E; VE)$ $\qquad$ Freed - Hopkins

$M \to \Omega^1$