Discrete differential geometry in homotopy type theory

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Motivation

Chern-Weil theory is an algorithm for manifold M, Lie group G

1. Form BG, study maps $I : BG \to B\mathbb{R}$ (invariant polynomials such as det, trace)
2. Fix a principal bundle $P : M \to BG$
3. Choose a connection A on P, compute curvature F_A
4. Compose $I \circ F_A$ (cohomology class)
5. I guarantees independence of class on connection

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- What is $BG \to B\mathbb{R}$ in HoTT?
- What are bundles, connections, and curvature in HoTT?
- What is cohomology; what are exact forms?

Questions

Where is the Bianchi identity? What is “the space of connections?”

Simplicial complexes

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \mathbb{M}_1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

Simplicial complexes

$$
\begin{tikzcd}
\partial P(2) \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & \partial P(2) \arrow[d, "*_{\partial\Delta^2}"] \\
P(2)_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] \arrow[ur, phantom, "\lrcorner", very near end] & \partial\Delta^2
\end{tikzcd}
$$

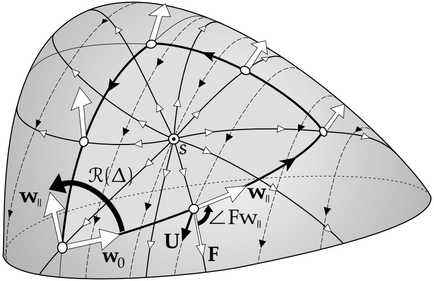
Simplicial complexes

$$
\begin{tikzcd}
 & M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "\overset{h_2}{\Nearrow}"] \arrow[dr, phantom, very near end, "\lrcorner"] & M_2 \arrow[d, "*_{\mathbb{M}_2}"] \\
\mathbb{M}_0 \arrow[r] \arrow[dr, phantom, "\underset{h_1}{\Searrow}"] \arrow[dr, phantom, very near end, "\ulcorner"] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times \partial\Delta^1 \arrow[u, "\mathbb{A}_0"] \arrow[r, "\mathrm{pr}_1"'] & M_1 \arrow[u, "*_{\mathbb{M}_1}"'] &
\end{tikzcd}
$$

Codomain

$$\mathrm{EM}(\mathbb{Z}, 1),\ \mathrm{K}(\mathbb{Z}, 2)$$

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi\mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F\mathbf{w}_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Tristan Needham, *Visual Differential Geometry and Forms*

Every aspect was informed by the classical proof, which is already discrete-flavored. The "$\angle F w_{||}$" looked a lot like a pathover.