A SUPERSYMMETRIC QUANTUM FIELD THEORY FORMULATION OF THE DONALDSON POLYNOMIAL INVARIANTS

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Abstract. We construct a mathematical framework for twisted $N = 2$ supersymmetric topological quantum field theory on a 4-manifold. Supersymmetry in flat space is defined and the twist homomorphism is constructed, giving us a supermanifold that is the total space of an odd vector bundle over the even 4-manifold. A special category of connections on this space is defined and a decomposition into so-called component fields is proved. The twisted supersymmetric action is computed, and the structure of the action, the decomposition, and the action of a special odd vector field are all shown to have a rich geometrical structure that was partially interpred by Atiyah and Jeffrey. [1] In short, the action is an infinite-dimensional analogue of the Euler class of the vector bundle of self-dual 2-forms over the space of connections mod gauge. This geometrical insight serves two purposes: first, it motivates the study of anti-self-dual connections, intersection theory, and the action of the group of gauge transformations, all of which appear by themselves after the twist. Secondly, it sets the stage for an eventual proof of Witten's Conjecture, relating the Donaldson and Seiberg-Witten invariants. What we build here amounts to a mathematical treatment of a physical treatment [2] of a mathematical construction of Donaldson. [3], [4].

Introduction

The primary goal of this paper is to present an alternative formulation of Donaldson theory [3], [4]. This will involve an exploration of supersymmetry and an important variation thereof. We will construct a very special eight-dimensional vector bundle SX over a compact, closed, simply connected riemannian four-manifold X that is the direct descendent of $N = 2$ supersymmetry in Euclidean space. We will examine the space of connections on a principal bundle over this space, building on results for connections over super Euclidean space. This space of connections comes equipped with a vector field, inherited from the supersymmetry algebra. The structure of the space together with the vector field is very rich and generalizes a beautiful finite dimensional geometrical picture. This geometry is further reflected in the action, a functional on superconnections that is the analog for SX of a similar construct in super Euclidean space. What we gain from this framework is a set of algebraic tools that are geared to one purpose: doing intersection theory on the moduli space of anti-self-dual (ASD) connections modulo the group of gauge transformations. Without having introduced ASD connections, or requiring that we divide by the gauge group, we see that these objects and operations are natural in this supersymmetric context.

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The secondary goal of this paper is discussed in the final section. Once we have seen that the Donaldson invariants fit into a supersymmetric quantum field theory framework, we can begin to address Witten's Conjecture [5]. This is the famous unproven relationship between the Donaldson and Seiberg-Witten ([6], [7]) invariants. Witten's "proof" of this result used the celebrated breakthroughs he obtained with Seiberg [6] on $N = 2$ supersymmetric gauge theories in Minkowski space, just like the gauge theory we consider here in Euclidean space. We will see a brief sketch of their proof, and attempt to point the way that leads from this paper to a mathematical proof of their result. We hope to convince the reader that the enlargement of Donaldson's picture presented here is the right place to begin to understand why Witten's conjecture is true, and maybe why and how it was discovered in the first place.

The following outline of this document should aid the reader. Super Euclidean and super Minkowski space are made from spin bundles in four dimensions, so we present these objects and the necessary volume forms and metrics. Super Euclidean space is the first important object we will encounter. This superspace has a special framing that reflects the spin structure, and which we will use to construct component fields of superfields. This framing is the second focal point of this paper. Next we define a very specific category of connections over superspace, called *semi-constrained*. The semi-constrained condition comes from physics, where objects are defined in terms of a dimensional reduction from six dimensions to four. In short, fully constrained connections are required to be flat in the odd directions, whereas semi-constrained ones can have two independent nonvanishing components of curvature in odd directions. Semi-constrained connections form the third focal center for this work. To study them, we describe the approach taken in physics. That is, we construct and prove an isomorphism between superconnections and a different space, a product space of objects defined on the underlying even principal bundle and even four-manifold. We rely heavily on the framing we constructed to define these "component fields." Finally, we put this all together and write down the $N = 2$ superspace version of the Yang-Mills action, which involves all of the component fields.

Section 2 repeats much of this discussion for the *twisted* picture. The twist is a representation theoretic operation on the two copies of spinors we have in $N = 2$ supersymmetry, that turns spinors into constants, 1-forms, and self-dual 2-forms. It is here that self-duality enters for the first time, and it is directly from this isomorphism of representations that we are eventually led to consider the ASD equations. We construct analogues of material from Section 1: the eight-dimensional bundle is now well-defined on any riemannian 4-manifold; the eight odd vector fields that formed our framing become three odd vector fields with bundle values; superconnections retain the same definition, though the nonvanishing odd-odd curvatures are described differently; the component fields have a more elegant and intrinsic definition, though we are careful to recognize that each new component field can be rewritten in a coordinate patch as its flat space counterpart. Then we write down the action of one of the three odd vector fields on the space of superconnections. This action forms part of an infinite-dimensional analogue of a beautiful finite-dimensional construction, which we take up in Section 3.

Section 3 is an introduction to two finite-dimensional geometrical constructions. The first is a special formula for the Thom class of a vector bundle, first constructed by Mathai and Quillen [8]. The second is a form on the total space of a principal bundle that allows integration of forms on the base to take place on the total space instead. Both constructions have an algebraic flavor that helped physicists connect with physics. We will see that the algebraic structure of both of these constructions is present on our 4-manifold in the form of: the space of superconnections, the action of the odd vector field, and the form of the twisted Yang-Mills action. This would be enough to prove the field theoretic formulas for the Donaldson polynomial invariants that we write down, but for the fact that there are no theorems along the geometrical lines that work in finite dimensions. However, we will prove the result another way, using formulas for linear and Gaussian path integrals that are formal but consistent with physical manipulations.

Finally, in the last section we introduce the reader to the issues that led to this work, namely Witten's Conjecture relating the Donaldson polynomial invariants to the Seiberg-Witten invariants. We will see that in the picture painted by modern physics, the quest for an easier formulation of the polynomial invariants is *completely natural*. The issue is that solving this problem is very hard. There is a physics proof, and any proper mathematical proof should address it, or at least parallel it. And so we are led to ask for a mathematical formulation of the physical formulation of the invariants, a need this paper is designed to address.

1. Introduction to supersymmetry

1.1. **A few super preliminaries.** A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$V = V_0 \oplus V_1.$$

The *parity* of an element $v \in V$, denoted $\pi(v)$, is 0 if $v \in V_0$, in which case v is called *even*, and is 1 if $v \in V_1$, in which case v is called *odd*. A morphism from V to W in this category is a grading-preserving linear transformation. The *parity reversal* of V, denoted ΠV is an isomorphism defined by

$$(\Pi V)_0 = V_1$$
$$(\Pi V)_1 = V_0.$$

Tensor products are defined using the tensor product of the underlying vector spaces, with grading given by

$$(V \otimes W)_k = \oplus_{i+j=k} V_i \otimes W_j.$$

The departure from simply defining a category of graded spaces comes with the definition of the commutativity isomorphism

$$V \otimes W \to W \otimes V$$

which we define to send

$$v \otimes w \to (-1)^{\pi(v)\pi(w)} w \otimes v. \tag{1}$$

If $t_1, \ldots, t_p$ is a basis for V_0 and $\theta_1, \ldots, \theta_q$ is a basis for V_1, the commutative $\mathbb{R}$-algebra $\mathbb{R}[t_1, \ldots, t_p, \theta_1, \ldots, \theta_q]$ is defined to be

$$S^*(t_1, \ldots, t_p) \otimes \wedge^*(\theta_1, \ldots \theta_q).$$

This should be thought of as a super version of the symmetric algebra on a vector space, where skew commutativity of the θ_i is part of the underlying properties of the odd generators. The space $\mathbb{R}^{p|q}$ is defined as the topological space $\mathbb{R}^p$ endowed with a sheaf $C^\infty(\mathbb{R}^p)(\theta^1, \dots \theta^q)$ of commutative super $\mathbb{R}$-algebras, freely generated over the sheaf $C^\infty(\mathbb{R}^p)$ by the odd quantities $\theta^1, \dots \theta^q$. A *super manifold* M is a topological space with a sheaf of super $\mathbb{R}$-algebras, that is locally isomorphic to $\mathbb{R}^{p|q}$. The ideal generated by all odd functions on a supermanifold M defines an even submanifold we will denote by M_{even}, where we use the usual algebro-geometric correspondence between ideals and varieties given by the set of common zeros of the ideal.

A morphism from a supermanifold S to $\mathbb{R}^{p|q}$ can be identified with a set of p even functions and q odd functions on S. This definition can be worked up into a definition of maps between supermanifolds, and in particular to vector bundles and principal bundles. If $\mathcal{P} \to M$ is a principal $SU(2)$ bundle over a supermanifold M, with fiber given by the even space $SU(2)$, then the restriction of this bundle to the even part M_{even} of M is a usual principal bundle we will consistently denote by P.

If $E \to M$ is a vector bundle over a supermanifold, with fiber isomorphic to $\mathbb{R}^{p|q}$, then there is another vector bundle we can form denoted $\Pi E \to M$, which is parity reversed on each fiber. In this case, the underlying even vector bundle has fiber isomorphic to $\mathbb{R}^q$, whereas the even vector bundle underlying E has fiber isomorphic to $\mathbb{R}^p$.

There is a concept of integration over odd variables called *Berezinian integration.* To compute

$$\int d\theta_1 \cdots d\theta_n f(x_1, \dots, x_k, \theta_1, \dots, \theta_l)$$

we expand f into a power series in the θ directions and take the coefficient of $\theta_1 \cdots \theta_n$,

$$\int d\theta_1 \cdots d\theta_n f(x_1, \dots, x_k, \theta_1, \dots, \theta_l) = f_{i_1 \cdots i_n}(x_1, \dots, x_k).$$

As an application of super geometry we prove the following trivial, but crucial, isomorphism.

Lemma 1. *Let X be an even manifold. Then $C^\infty(\Pi TX) \cong \Omega^*(X)$. Furthermore, there is a natural operator $Q = \sum_i \theta^i \partial_{x^i} \in C^\infty(\Pi TX)$ and under the isomorphism we have $Q \cong d$.*

Proof. Let $(x^1, \dots, x^n)$ be coordinates in a patch on X. Let $(\theta^1, \dots, \theta^n)$ be the induced coordinates in the odd $\partial/\partial x^1, \dots, \partial/\partial x^n$ directions of ΠTX. Then the isomorphism is simply given by

$$\theta^i \mapsto dx^i.$$

1.2. **Super Euclidean space.** Consider two 2-dimensional complex vector spaces S^+ and S^-. From now on, we shall make use of the notation $\pm$ to make pairs of statements or definitions at once. Let

$$\varepsilon^\pm : \wedge^2 S^\pm \to \mathbb{C}$$

be fixed isomorphisms. These maps have adjoints

$$\mathrm{ad}(\varepsilon^\pm) : S^\pm \to (S^\pm)^*$$

given by

$$\mathrm{ad}(\varepsilon^\pm)(s) = \varepsilon^\pm(s, \cdot).$$

The adjoint can be used to define the dual map

$$(\varepsilon^\pm)^* : \wedge^2 (S^\pm)^* \to \mathbb{C}$$

by mapping

$$(s_1^\pm, s_2^\pm) \mapsto (s_1^\pm, \mathrm{ad}(\varepsilon^\pm)^{-1}(s_2^\pm)) \mapsto s_1^\pm(\mathrm{ad}(\varepsilon^\pm)^{-1}(s_2^\pm)).$$

Now we choose a basis $\{e_+^1, e_+^2\}$ of S^+, such that $\varepsilon^+(e_+^1, e_+^2) = 1$. We also choose a basis $\{e_-^1, e_-^2\}$ of S^- such that $\varepsilon^-(e_-^1, e_-^2) = 1$. We denote the dual basis by e_i^+ and e_i^-. One computes that

$$\mathrm{ad}(\varepsilon^\pm)(e_\pm^1) = e_2^\pm \tag{2}$$

$$\mathrm{ad}(\varepsilon^\pm)(e_\pm^2) = -e_1^\pm \tag{3}$$

Also, for completeness we have

$$\mathrm{ad}(\varepsilon^\pm)^{-1}(e_1^\pm) = -e_\pm^2 \tag{4}$$

$$\mathrm{ad}(\varepsilon^\pm)^{-1}(e_2^\pm) = e_\pm^1 \tag{5}$$

A final easy computation shows that with the above definition, $(\varepsilon^\pm)^*(e_\pm^1, e_\pm^2) = 1$.

We will build a four-dimensional complex vector space $V_{\mathbb{C}}$ with special properties. First of all, this space has two possible real structures, which we can use to construct Minkowski space and Euclidean space. Second of all, the action of $V_{\mathbb{C}}$ on $S^\pm$ by Clifford multiplication is “included” in the structure of $V_{\mathbb{C}}$ itself. We'll see more of that shortly.

We define

$$V_{\mathbb{C}} = (S^+)^* \otimes (S^-)^*,$$

and now our backward convention of denoting basis vectors with upper indices and dual vectors with lower indices should seem justified: $V_{\mathbb{C}}$ is built from dual spinors. Thus, elements of $V_{\mathbb{C}}$ have lower indices as expected and only the spinor spaces themselves have reversed index conventions. We equip $V_{\mathbb{C}}$ with the metric

$$\langle, \rangle = \frac{1}{2}(\varepsilon^+)^* \otimes (\varepsilon^-)^*$$

or in other words

$$\left\langle s_1^+ \otimes s_1^-, s_2^+ \otimes s_2^- \right\rangle = (\varepsilon^+)^*(s_1^+, s_2^+) \cdot (\varepsilon^-)^*(s_1^-, s_2^-).$$

To move towards defining a real subspace of $V_{\mathbb{C}}$ we note that we can define maps on spaces with the opposite complex structure,

$$(\varepsilon^\pm)^{\mathrm{opp}} : \wedge^2 S^{\pm\,\mathrm{opp}} \to \mathbb{C},$$

by letting ε act normally on two elements, but then taking the complex conjugate of the result:

$$(\varepsilon^\pm)^{\mathrm{opp}}(a, b) = \overline{\varepsilon^\pm(a, b)}.$$

Now we introduce hermitian inner products $h^\pm$ on $S^\pm$, which give isomorphisms

$$h^\pm : S^\pm \to (S^\pm)^{*\mathrm{opp}}.$$

We require that $h^\pm$ preserve the ε tensors, and so we can choose the bases $e_\pm^i$ to be orthonormal.

We now use $h^\pm$ to define a real structure on $V_{\mathbb{C}}$. Consider the map

$$\mathrm{ad}(\varepsilon^+)^{-1} \otimes \mathrm{ad}(\varepsilon^-)^{-1} : (S^+)^* \otimes (S^-)^* \to S^+ \otimes S^-$$

and compose it with the map

$$h^+ \otimes h^- : S^+ \otimes S^- \to (S^+)^{*\mathrm{opp}} \otimes (S^-)^{*\mathrm{opp}}.$$

Call this composition τ. One easily sees that τ is anti-$\mathbb{C}$-linear and that τ^2 is the identity. We define $V \subset V_{\mathbb{C}}$ to be the set of fixed points of τ. Below we will see that V with the metric $\langle,\rangle$ is in fact Euclidean 4-space, E^4.

It is appropriate to mention the variation of the above construction that leads to Minkowski space. First, we begin by setting $S^- = (S^+)^{\mathrm{opp}}$. In other words, S^- and S^+ are opposite representations of $SL(2,\mathbb{C})$. We define $\varepsilon^- = (\varepsilon^+)^{\mathrm{opp}}$, i.e. $\varepsilon^-(a,b) = \overline{\varepsilon^+(a,b)}$. We define τ to be the anti-$\mathbb{C}$-linear map exchanging S^- and S^+, which is the identity on the underlying vector spaces, but which reverses the complex structure. The fixed set of τ is Minkowski space M^4, with the indefinite metric of signature $(3,1)$.

Let us discuss a permanent change of notation. Instead of denoting elements of S^- with minus subscripts, we will place bars over them and place dots over their indices. Therefore, in the new notation a basis for S^- is denoted $\{\bar{e}^{\dot{1}}, \bar{e}^{\dot{2}}\}$. The dual elements have lower indices. Note that the bars and dots do *not* indicate complex conjugate. This awkward-seeming notation is useful to make contact with the published physics literature, where Minkowski space is usually the context, and as we just saw the elements of S^- are the conjugates of corresponding elements from S^+.

Because we are translating parts of the physics literature into mathematics, we will have to make extensive use of index notation. So we need abbreviations for certain frequent notation. For example, we will denote the induced basis on the space $S^+ \otimes S^+$ by the four elements $e^{ab} = e^a_+ \otimes e^b_+$. A basis of the space $(S^+)^* \otimes S^+$ is given by elements $e^a_b = e^+_b \otimes e^a_+$. As a final example, a basis of the space $(S^+)^* \otimes (S^-)^* \cong V_{\mathbb{C}}$ is given by $e_{a\dot{b}} = e^+_a \otimes e^-_b$.

Lemma 2. *$\langle,\rangle : V \otimes V \to \mathbb{C}$ is real and positive definite.*

Proof. Define the following basis of $V_{\mathbb{C}} = (S^+)^* \otimes (S^-)^*$:

$$\begin{aligned} v_1 &= e_{1\dot{1}} + e_{2\dot{2}} \\ v_2 &= ie_{1\dot{1}} - ie_{2\dot{2}} \\ v_3 &= e_{1\dot{2}} - e_{2\dot{1}} \\ v_4 &= ie_{1\dot{2}} + ie_{2\dot{1}}. \end{aligned} \tag{6}$$

Direct computation shows that this basis is real, and that in this basis $\langle,\rangle$ is the identity matrix.

Definition 1. *$E^{4|4}$ is the subspace $V \times \Pi((S^+)^* \oplus (S^-)^*) \subset V_{\mathbb{C}} \times \Pi((S^+)^* \oplus (S^-)^*)$.*

Note that the base V is real, while the fibers do not have a real structure.

The automorphism group of $(S^{\pm}, \varepsilon^{\pm}, h^{\pm})$ is $SU(2)$, and so $SU(2) \times SU(2)$ acts by isometries on $V_{\mathbb{C}}$. This action leaves τ invariant and so preserves V, identifying $SU(2) \times SU(2)$ with the spin double cover of $SO(4)$.

1.2.1. *Clifford multiplication.* The special definition of $V_{\mathbb{C}}$ makes describing Clifford multiplication particularly easy. The action

$$V_{\mathbb{C}} \otimes S^+ \to S^-$$

is given by

$$(S^+)^* \otimes (S^-)^* \otimes S^+ \xrightarrow{\mathrm{ev}} (S^-)^* \xrightarrow{\mathrm{ad}(\varepsilon^-)^{-1}} S^-,$$

where the first map is evaluation on the S^+ factors. The action of $V_{\mathbb{C}}$ on S^- is given by a similar composition

$$(S^+)^* \otimes (S^-)^* \otimes S^- \to (S^+)^* \to S^+.$$

These actions induce an action of the whole Clifford algebra $Cl(V_{\mathbb{C}})$ on $S^+ \oplus S^-$, as one can easily check. This boils down to checking that acting with v twice gives multiplication by $-\|v\|^2$.

Lemma 3. *Clifford multiplication induces isomorphisms*

$$\begin{aligned} V_{\mathbb{C}} &\cong (S^-)^* \otimes S^+ \\ \mathbb{C} \oplus \wedge^2_+ V_{\mathbb{C}} &\cong (S^+)^* \otimes S^+. \end{aligned}$$

Proof. The first isomorphism is given explicitly by

$$\begin{aligned} v_1 &\mapsto -e^2_{\dot 1} + e^1_{\dot 2} \\ v_2 &\mapsto -ie^2_{\dot 1} - ie^1_{\dot 2} \\ v_3 &\mapsto -e^2_{\dot 2} - e^1_{\dot 1} \\ v_4 &\mapsto -ie^2_{\dot 2} + ie^1_{\dot 1} \end{aligned} \tag{7}$$

which uses the definition of the v_i together with (4) and (5).

To prove the second isomorphism we compute as follows. Compute multiplication by $v_1 \cdot v_2$ (meaning multiply by v_2 and then multiply the result by v_1) with

$$\begin{aligned} v_1 \cdot v_2 \cdot e^m_+ &= v_1 \cdot \mathrm{ad}(\varepsilon^-)^{-1}((ie_{1\dot 1} - ie_{2\dot 2})(e^m_+)) \\ &= v_1 \cdot \mathrm{ad}(\varepsilon^-)^{-1}(i\delta^m_1 e^-_{\dot 1} - i\delta^m_2 e^-_{\dot 2}) \\ &= v_1 \cdot (-i\delta^m_1 e^2_- - i\delta^m_2 e^1_-) \\ &= \mathrm{ad}(\varepsilon^+)^{-1}((e_{1\dot 1} + e_{2\dot 2})(-i\delta^m_1 e^2_- - i\delta^m_2 e^1_-)) \\ &= \mathrm{ad}(\varepsilon^+)^{-1}(-i\delta^m_1 e^+_2 - i\delta^m_2 e^+_1) \\ &= -i\delta^m_1 e^1_+ + i\delta^m_2 e^2_+ \\ &= i(-1)^m e^m_+. \end{aligned}$$

Doing the calculation for $v_3 \cdot v_4$ yields an identical result, and so using the relationship between wedge product and Clifford product, we have computed that

$$v_1 \wedge v_2 + v_3 \wedge v_4 \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

One similarly obtains the rest of the maps

$$\begin{aligned} (1,0) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (0, v_1 \wedge v_2 + v_3 \wedge v_4) &\mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ (0, v_1 \wedge v_3 - v_2 \wedge v_4) &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ (0, v_1 \wedge v_4 + v_2 \wedge v_3) &\mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \end{aligned}$$

Noting that $\mathrm{Hom}(S^+, S^+) \cong (S^+)^* \otimes S^+$, we represent this as

$$\begin{aligned} (1,0) &\mapsto \quad e_1^1 + e_2^2 \\ (0, v_1 \wedge v_2 + v_3 \wedge v_4) &\mapsto -ie_1^1 + ie_2^2 \\ (0, v_1 \wedge v_3 - v_2 \wedge v_4) &\mapsto \quad e_2^1 - e_1^2 \\ (0, v_1 \wedge v_4 + v_2 \wedge v_3) &\mapsto -ie_2^1 - ie_1^2. \end{aligned} \tag{8}$$

The second isomorphism is now clear. This completes the proof.

The spaces $V_{\mathbb{C}}$ and $\mathbb{C} \oplus \wedge_+^2 V_{\mathbb{C}}$ have obvious real subrepresentations, a fact that will be important when we twist.

For later use, we provide a version of (8) with lowered indices, using (2), (3) (the raised index becomes the first lower index since $(S^+)^*$ is the first factor).

$$\begin{aligned} (1,0) &\mapsto \quad e_{21} - e_{12} \\ (0, v_1 \wedge v_2 + v_3 \wedge v_4) &\mapsto -ie_{21} - ie_{12} \\ (0, v_1 \wedge v_3 - v_2 \wedge v_4) &\mapsto \quad e_{22} + e_{11} \\ (0, v_1 \wedge v_4 + v_2 \wedge v_3) &\mapsto -ie_{22} + ie_{11}. \end{aligned} \tag{9}$$

1.2.2. *Invariant vector fields.* The remainder of this section follows [9].

We define a coordinate system on $E^{4|4}$ as follows. Using the orthonormal basis elements of S^+, S^-, and V given above we define coordinate functions θ^a and $\overline{\theta}^{\dot{a}}$ on $(S^+)^*$ and $(S^-)^*$ respectively (a and $\dot{a}$ take on values 1 or 2). On V we use coordinates that we denote by $y^{a\dot{b}}$ (again, each index takes on the values 1 or 2). So we have explicitly

$$\theta^a(e_b) = \delta_b^a \tag{10}$$

$$\overline{\theta}^{\dot{a}}(\overline{e}_{\dot{b}}) = \delta_{\dot{b}}^{\dot{a}} \tag{11}$$

$$y^{a\dot{b}}(e_{c\dot{d}}) = \delta_c^a \delta_{\dot{d}}^{\dot{b}}. \tag{12}$$

We denote differentiation in the θ^a and $\overline{\theta}^{\dot{a}}$ direction by ∂_a and $\overline{\partial}_{\dot{a}}$ respectively. We denote differentiation in the $y^{a\dot{b}}$ direction by $\partial_{a\dot{b}}$. We define

$$\begin{aligned} D_a &= \partial_a - \overline{\theta}^{\dot{b}} \partial_{a\dot{b}} \\ \overline{D}_{\dot{a}} &= \overline{\partial}_{\dot{a}} - \theta^b \partial_{b\dot{a}} \end{aligned}$$

and

$$\begin{aligned} Q_a &= \partial_a + \overline{\theta}^{\dot{b}} \partial_{a\dot{b}} \\ \overline{Q}_{\dot{a}} &= \overline{\partial}_{\dot{a}} + \theta^b \partial_{b\dot{a}}. \end{aligned}$$

These vector fields satisfy the bracket relations (remembering that for two odd vector fields you add instead of subtract to form brackets)

$$\begin{aligned} [D_a, D_b] &= [\overline{D}_{\dot{a}}, \overline{D}_{\dot{b}}] = 0 \\ [D_a, \overline{D}_{\dot{b}}] &= -2\partial_{a\dot{b}} \end{aligned}$$

and

$$\begin{aligned} [Q_a, Q_b] &= [\overline{Q}_{\dot{a}}, \overline{Q}_{\dot{b}}] = 0 \\ [Q_a, \overline{Q}_{\dot{b}}] &= 2\partial_{a\dot{b}}. \end{aligned}$$

1.2.3. $E^{4|8}$. Simply put, $N=2$ super Euclidean space, also called $E^{4|8}$, has two copies of $\Pi((S^+)^*\oplus(S^-)^*)$ instead of one. We don't need to repeat the above discussion, but there are some complications. First of all, we need to provide the odd coordinate functions and vector fields with another index, that can take on the values 1 or 2, to represent which copy of $\Pi((S^+)^*\oplus(S^-)^*)$ they live on. So, we now have the odd coordinate functions

$$\theta^{1(1)},\theta^{2(1)},\overline{\theta}^{\dot{1}(1)},\overline{\theta}^{\dot{2}(1)},\theta^{1(2)},\theta^{2(2)},\overline{\theta}^{\dot{1}(2)},\overline{\theta}^{\dot{2}(2)}$$

as well as the left-invariant vector fields

$$D_1^{(1)},D_2^{(1)},\overline{D}_{\dot{1}}^{(1)},\overline{D}_{\dot{2}}^{(1)}D_1^{(2)},D_2^{(2)},\overline{D}_{\dot{1}}^{(2)},\overline{D}_{\dot{2}}^{(2)},$$
$$Q_1^{(1)},Q_2^{(1)},\overline{Q}_{\dot{1}}^{(1)},\overline{Q}_{\dot{2}}^{(1)}Q_1^{(2)},Q_2^{(2)},\overline{Q}_{\dot{1}}^{(2)},\overline{Q}_{\dot{2}}^{(2)}.$$

The commutation relations are the same as before, with brackets of vector fields of differing upper index vanishing:

$$[D_a^{(i)},D_b^{(j)}]=[\overline{D}_{\dot{a}}^{(i)},\overline{D}_{\dot{b}}^{(j)}]=0$$
$$[D_a^{(i)},\overline{D}_{\dot{b}}^{(j)}]=-2\delta^{ij}\partial_{a\dot{b}} \tag{13}$$

and

$$[Q_a^{(i)},Q_b^{(j)}]=[\overline{Q}_{\dot{a}}^{(i)},\overline{Q}_{\dot{b}}^{(j)}]=0$$
$$[Q_a^{(i)},\overline{Q}_{\dot{b}}^{(j)}]=2\delta^{ij}\partial_{a\dot{b}}. \tag{14}$$

To sum up the index structure of these vector fields we make the following remark.

Observation 1. *The Q's and D's are sections of the Spin$(4)\times SU(2)$-bundle $(S^+)^*\otimes\mathbb{C}^2$ and the $\overline{Q}$'s and $\overline{D}$'s are sections of the Spin$(4)\times SU(2)$-bundle $(S^-)^*\otimes\mathbb{C}^2$.*

1.3. **Gauge theory on $E^{4|8}$.** Much of this section can be considered "standard material" and can be found in the literature. One thorough account can be found in [10]. Another good accounting, and the one whose notation we adopt here, is [9].

Deciding what category of connections we should work with is a subtle business. The correct formulation in $N=1$ theories from the physical standpoint is to examine *constrained* connections.

Definition 2. *A superconnection $\mathcal{A}$ on a supermanifold with odd distribution τ is said to be* constrained *if the curvature $\mathcal{F}$ of $\mathcal{A}$ vanishes along τ. That is, if $\mathcal{F}(x,y)=0$ whenever x and y are odd vector fields.*

There are physical reasons for requiring this, but from the mathematical perspective it's just a subcategory we happen to be focusing on. Things are different in $N=2$ theories, though. Here there are eight odd directions to consider in four-dimensional theories. One proceeds by considering $N=1$ superconnections in *six* dimensions, where the spin bundle has eight dimensions. We will reduce this picture to four dimensions by requiring translation invariance along two dimensions, say the span of v and w for $v,w\in\mathbb{R}^6$ linearly independent. So we examine a principal $SU(2)$ bundle $\mathcal{P}$ over $E^{6|8}$ that is trivial in the v and w directions. Then we work with constrained connections that are constant along v and w.

We will see in a moment that such a dimensionally reduced object is no longer constrained. Instead, it can have two independent nonvanishing scalar curvatures on the odd distribution of $E^{4|8}$.

Definition 3. *A superconnection on a supermanifold whose curvature vanishes identically along the odd distribution except for two two-dimensional subdistributions along which the curvature is unconstrained is called* semi-constrained.

Theorem 1. *The space of dimensionally reduced connections from $E^{6|8}$ to $E^{4|8}$ is isomorphic to the space of semi-constrained connections on $E^{4|8}$.*

Proof. It is easiest to use proper coordinates and vector fields from $E^{6|8}$, so we give a brief run-down of this. Details can be found in [9]. We will use the name y^{ab} and θ^{ai} for the coordinate system on $E^{6|8}$, and ∂_{ab} and ∂_{ai} for the corresponding vector fields. Here, a, b take on the values 1 through 4, but with $a < b$. i can be 1 or 2. We denote the ε tensor in coordinates as ε_{ij} (i and j can take on values 1 or 2).

The eight left-invariant vector fields are given by

$$D_{ai} = \partial_{ai} - \varepsilon_{ij}\theta^{bj}\partial_{ab} \tag{15}$$

and the right-invariant ones by

$$Q_{ai} = \partial_{ai} + \varepsilon_{ij}\theta^{bj}\partial_{ab} \tag{16}$$

with the commutation relations

$$\begin{aligned} [D_{ai}, D_{bj}] &= -\varepsilon_{ij}\partial_{ab} \\ [Q_{ai}, Q_{bj}] &= +\varepsilon_{ij}\partial_{ab}. \end{aligned}$$

By dimensional reduction we mean the restriction to $E^{4|8}$, which is just the standard embedding of $\mathbb{R}^4$ into $\mathbb{R}^6$ by setting two coordinates on $\mathbb{R}^6$ to zero. The effect on the coordinate systems we've been using is

$$\begin{aligned} y^{12} &= 0 & y^{23} &= y^{2\dot{1}} \\ y^{13} &= y^{1\dot{1}} & y^{24} &= y^{2\dot{2}} \\ y^{14} &= y^{1\dot{2}} & y^{34} &= 0. \end{aligned} \tag{17}$$

We do not reduce the number of odd coordinates, though, and we can make a dictionary of left-invariant vector fields

$$\begin{aligned} D_{11} &= D_1^{(1)} & D_{12} &= D_1^{(2)} \\ D_{21} &= D_2^{(1)} & D_{22} &= D_2^{(2)} \\ D_{31} &= -\overline{D}_{\dot{1}}^{(2)} & D_{32} &= \overline{D}_{\dot{1}}^{(1)} \\ D_{41} &= -\overline{D}_{\dot{2}}^{(2)} & D_{42} &= \overline{D}_{\dot{2}}^{(1)}. \end{aligned} \tag{18}$$

We are reducing by two dimensions, from $E^{6|8}$ to $E^{4|8}$, by setting the coordinates y^{12} and y^{34} to zero. Whereas we have on $E^{6|8}$ the relation

$$[D_{31}, D_{42}] = -\partial_{34}, \tag{19}$$

under the reduction correspondence (18), $D_{31} = -\overline{D}_{\dot{1}}^{(2)}$, $D_{42} = \overline{D}_{\dot{2}}^{(1)}$, we instead have on $E^{4|8}$ the equation

$$-[\overline{D}_{\dot{1}}^{(2)}, \overline{D}_{\dot{2}}^{(1)}] = 0. \tag{20}$$

The covariant version of (19) is

$$[\mathcal{D}_{31}, \mathcal{D}_{42}] = -\nabla_{34}, \tag{21}$$

and we can wonder, What happens to this equation after we dimensionally reduce? The two odd vector fields become two of the vector fields on $E^{4|8}$, but ∂_{34} becomes zero, so what is the reduction of this covariant equation? The answer is that if the principal bundle $\mathcal{P}$ and the connection are invariant under translations in the y^{34} direction, then there is a trivial lift of ∂_{34} which we will call $\tilde{\partial}_{34}$; it is the lift of ∂_{34} using the product connection in this trivial direction. The difference $\nabla_{34} - \tilde{\partial}_{34}$ is a vertical vector field, or a section of the adjoint bundle. Dimensional reduction simply states that there can be no component of the lift of $[\overline{D}^{(2)}_{\dot{1}}, \overline{D}^{(1)}_{\dot{2}}]$ in the $\tilde{\partial}_{34}$ direction, and so this bracket must lift to the *vertical part* of $-\nabla_{34}$. We define

$$\Sigma = \nabla_{34} - \tilde{\partial}_{34}$$

and thus have the dimensionally reduced equation

$$[\overline{\mathcal{D}}^{(2)}_{\dot{1}}, \overline{\mathcal{D}}^{(1)}_{\dot{2}}] = \Sigma. \tag{22}$$

This equation tells us that this particular component of odd-odd curvature *need not vanish*.

Similarly, we have

$$[D_{41}, D_{32}] = \partial_{34}$$

so using the correspondence $D_{41} = -\overline{D}^{(2)}_{\dot{2}}$, $D_{32} = \overline{D}^{(1)}_{\dot{1}}$ we get

$$-[\overline{D}^{(2)}_{\dot{2}}, \overline{D}^{(1)}_{\dot{1}}] = \Sigma. \tag{23}$$

Now, we consider constancy in the y^{12} direction. This leads to a second section of the adjoint bundle that we'll call $\overline{\Sigma}$

$$\overline{\Sigma} = \nabla_{12} - \tilde{\partial}_{12}.$$

This in turn leads to the equations

$$-\left[\mathcal{D}^{(2)}_{2}, \mathcal{D}^{(1)}_{1}\right] = \overline{\Sigma} \tag{24}$$

$$\left[\mathcal{D}^{(2)}_{1}, \mathcal{D}^{(1)}_{2}\right] = \overline{\Sigma}. \tag{25}$$

This completes the proof.

We will see this theorem play out in the twisted context as well, where we will have two independent odd-odd curvatures that are not required to vanish.

1.3.1. *Component Fields.* The component fields of a superconnection in $E^{4|8}$ are denoted $A, \sigma, \overline{\sigma}, \lambda, \overline{\lambda}, \chi, \overline{\chi}, E, F$. These are

$$\begin{aligned} A &= \text{a connection on } E^4 \\ \sigma &= \text{a section of ad}\, P \\ \overline{\sigma} &= \text{a section of ad}\, P \\ \lambda &= \text{a section of ad}\, P \otimes (S^+)^* \\ \overline{\lambda} &= \text{a section of ad}\, P \otimes (S^-)^* \\ \chi &= \text{a section of ad}\, P \otimes (S^+)^* \\ \overline{\chi} &= \text{a section of ad}\, P \otimes (S^-)^* \\ E &= \text{a section of ad}\, P \\ F &= \text{a section of ad}\, P \otimes \mathbb{C} \end{aligned} \tag{26}$$

These are defined as follows. A is the induced connection on the induced even bundle $P \to E^4$ sitting inside $\mathcal{P} \to E^{4|8}$. The others are defined by

$$\begin{aligned} \sigma &= i^*\Sigma \\ \overline{\sigma} &= i^*\overline{\Sigma} \\ \lambda_a &= i^* W^1_a = i^* \frac{1}{4} \varepsilon^{\dot{c}\dot{d}} [\overline{\mathcal{D}}^{(1)}_{\dot{c}}, \nabla_{a\dot{d}}] \\ \chi_a &= i^* W^2_a = i^* \frac{1}{4} \varepsilon^{\dot{c}\dot{d}} [\overline{\mathcal{D}}^{(2)}_{\dot{c}}, \nabla_{a\dot{d}}] \\ \overline{\lambda}_{\dot{a}} &= i^* \overline{W}^1_{\dot{a}} = i^* \frac{1}{4} \varepsilon^{cd} [\mathcal{D}^{(1)}_c, \nabla_{d\dot{a}}] \\ \overline{\chi}_{\dot{a}} &= i^* \overline{W}^2_{\dot{a}} = i^* \frac{1}{4} \varepsilon^{cd} [\mathcal{D}^{(2)}_c, \nabla_{d\dot{a}}] \\ E &= -i^* (\overline{\mathcal{D}}^{(1)}_{\dot{2}} \overline{\mathcal{D}}^{(2)}_{\dot{1}} \Sigma - \overline{\mathcal{D}}^{(1)}_{\dot{1}} \overline{\mathcal{D}}^{(2)}_{\dot{2}} \Sigma) \\ F &= i^* \overline{\mathcal{D}}^{(2)}_{\dot{2}} \overline{\mathcal{D}}^{(2)}_{\dot{1}} \Sigma \\ \overline{F} &= i^* \overline{\mathcal{D}}^{(1)}_{\dot{1}} \overline{\mathcal{D}}^{(1)}_{\dot{2}} \Sigma. \end{aligned} \tag{27}$$

Here, i^* is the pullback functor using the inclusion $i : E^4 \hookrightarrow E^{4|8}$.

Theorem 2. *The space of semi-constrained superconnections on $E^{4|8}$ is isomorphic to the space of component fields.*

See [9] for a discussion of this.

1.4. **The super Yang-Mills action.** Let τ be the complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{28}$$

The action on super Minkowski space $M^{4|8}$ is given by

$$S = \int d^4x \,\mathrm{Im} \left(d^4\theta \, \frac{\tau}{32\pi} \langle \Sigma, \Sigma \rangle \right) \tag{29}$$

We will write this action in its component formulation. This is obtained from (29) by integrating out the four odd variables, or equivalently, hitting the integrand with an appropriate combination of four odd derivatives. In this case that is

$$D_2^{(1)} D_1^{(1)} D_2^{(2)} D_1^{(2)}, \tag{30}$$

though other choices are appropriate as well, so long as they differ from this one by an exact term. See [9] for more information about this computation. The Dirac pairing seen below is defined by

$$\langle \lambda \not{D}_A \overline{\lambda} \rangle = \varepsilon^{ac} \varepsilon^{\dot{b}\dot{d}} \lambda_c \nabla_{a\dot{b}} \overline{\lambda}_{\dot{d}}. \tag{31}$$

The action in components is therefore given by

$$\begin{aligned} S = \int d^4x \, \frac{1}{g^2} \Big\{ &- \frac{1}{2}|F_A|^2 + \langle d_A \overline{\sigma}, d_A \sigma \rangle + \langle \lambda \not{D}_A \overline{\lambda} \rangle + \langle \chi \not{D}_A \overline{\chi} \rangle \\ &- \varepsilon^{ab} \langle \overline{\sigma}, [\lambda_a, \chi_b] \rangle + \varepsilon^{\dot{a}\dot{b}} \big\langle [\overline{\lambda}_{\dot{a}}, \overline{\chi}_{\dot{b}}], \sigma \big\rangle - \frac{1}{2}|E|^2 \\ &+ \big\langle \overline{F}, F \big\rangle \Big\} + \frac{\theta}{16\pi^2} \langle F_A \wedge F_A \rangle. \end{aligned} \tag{32}$$

Note the presence of the usual Yang-Mills action (the first term), as well as the second Chern class (the last term). However, we are following the usual convention of having separate coupling coefficients for these two terms. This is because the topological Chern-Simons term has a different character in the physical theory, since it is locally constant on components of $\mathcal{A}$. We will find reason to revisit the value of the coefficient of the "theta term" when we twist the action in Section 2.4.

Next we write the result of Wick rotating this action to $E^{4|8}$. This is a procedure we will not carry out explicitly, but merely write the result. For more information, see [11]. It differs only in the signs and some coefficients of i.

$$\begin{aligned} S = \int d^4x \, \frac{1}{g^2} \Big\{ &- \frac{1}{2}|F_A|^2 - i \langle d_A \overline{\sigma}, d_A \sigma \rangle + i \langle \lambda \not{D}_A \overline{\lambda} \rangle + i \langle \chi \not{D}_A \overline{\chi} \rangle \\ &+ i\varepsilon^{ab} \langle \overline{\sigma}, [\lambda_a, \chi_b] \rangle + i\varepsilon^{\dot{a}\dot{b}} \big\langle \sigma, [\overline{\lambda}_{\dot{a}}, \overline{\chi}_{\dot{b}}] \big\rangle - i\frac{1}{2}|E|^2 \\ &+ i \big\langle \overline{F}, F \big\rangle \Big\} + \frac{\theta}{16\pi^2} \langle F_A \wedge F_A \rangle. \end{aligned} \tag{33}$$

The material in this section would greatly benefit from a treatment more within the philosophical scope of this paper. This remark applies equally to the computation of the twisted counterpart to the above action formula, which is a main result of the next section. The enterprising reader should perhaps focus on the operator (30), in search of a generalized interpretation and formula for an operator that integrates over the odd components of a superspace.

2. The superspace SX

2.1. **The twist.** The twisting operation is a modification of the global structure of $E^{4|8}$. It will alter the structure of any theory over this base, and so we will obtain a class of theories that is very different from those on honest supersymmetric space. However, this trade-off allows us to construct an extension to any 4-manifold X that is analogous to extending E^4 to $E^{4|8}$.

The presence of the δ-function on the right hand side of the bracket relations (13) and (14) is a clue that there is an automorphism group of $E^{4|8}$ that preserves the even subspace. We can see immediately that the group that acts on the $D_a^{(i)}$

and $\overline{D}^{(i)}_{\dot{a}}$ preserving the bracket is the group $U(2)$ preserving a symmetric hermitian bilinear pairing on $\mathbb{C}^2$. One calls this $U(2)$ the *R-symmetry group*. We will only be discussing the subgroup $SU(2) \subset U(2)$. The quotient group $U(2)/SU(2) \cong U(1)$ plays quite a different role in the physics, and presumably in the mathematics as well, related to anomalies, but we will not encounter it further. We denote the $SU(2)$ part of the R-symmetry by $SU(2)^R$.

The odd fiber of $E^{4|8}$ is a representation of $Spin(4)$ as well as of $SU(2)^R$. We denote the decomposition of $Spin(4)$ as $Spin(4) \cong SU(2)^+ \times SU(2)^-$. The R-symmetry allows us to construct an interesting and important map $Spin(4)$ into $H = SU(2)^+ \times SU(2)^- \times SU(2)^R$. We define the *twist homomorphism*

$$T : SU(2)^+ \times SU(2)^- \to H$$

by

$$T(a, b) = (a, b, a) \tag{34}$$

which is the diagonal embedding of $SU(2)$ into $SU(2)^+ \times SU(2)^R$ combined with the identity mapping of $SU(2)^-$. Clearly H acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, so we can now form a new $Spin(4)$ associated vector bundle over E^4 with fiber $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ by precomposing with the mapping T. Another way to look at this operation is that we have declared that the index for the trivial $\mathbb{C}^2$ fiber of $E^{4|8}$ now labels another copy of S^+ instead.

We can use this to do something very special. We can use the isomorphisms in Lemma 3 to prove immediately that these bundles factor through $SO(4)$, and so can be formed on any riemannian 4-manifold. Since the twisted vector fields take values in $(S^+)^* \otimes S^+$ and $(S^-)^* \otimes S^+$, then after the twist one vector field takes values in $\mathbb{R}$, one of them takes values in $\wedge^2_+ V$, and one of them takes values in V. Explicitly we define

$$D_0 = D^1_1 + D^2_2 \tag{35}$$

$$\begin{aligned} D_1 = &\left(\overline{D}^1_{\dot{2}} - \overline{D}^2_{\dot{1}}\right) dv^1 + \left(-i\overline{D}^1_{\dot{2}} - i\overline{D}^2_{\dot{1}}\right) dv^2 \\ &+ \left(-\overline{D}^1_{\dot{1}} - \overline{D}^2_{\dot{2}}\right) dv^3 + \left(i\overline{D}^1_{\dot{1}} - i\overline{D}^2_{\dot{2}}\right) dv^4 \end{aligned} \tag{36}$$

$$\begin{aligned} D_2 = &\left(iD^1_1 + iD^2_2\right)(dv^1 \wedge dv^2 + dv^3 \wedge dv^4) \\ &+ \left(D^1_2 - D^2_1\right)(dv^1 \wedge dv^3 - dv^2 \wedge dv^4) \\ &+ \left(-iD^1_2 - iD^2_1\right)(dv^1 \wedge dv^4 + dv^2 \wedge dv^3). \end{aligned} \tag{37}$$

These three vector fields should have a more intrinsic description, one that does not make reference to supersymmetry or the spin bundles we have tensored together. The results along these lines are as follows.

Let X be a riemannian four-manifold with local coordinate functions x^i. Form the odd vector bundle

$$SX = \Pi\left((X \times \mathbb{C}) \oplus TX \oplus \wedge^2_+ TX\right).$$

The x^i induce local bases ∂_{x^i} for vector fields, and dx^i for one-forms. We obtain induced coordinates in the ΠTX directions that we will denote by θ^i (so θ^i is an odd coordinate function along the odd ∂_{x^i} direction). Similarly the x^i induce coordinates in the $\wedge^2_+ TX$ directions. We will denote by θ^{1234} the coordinate in the odd $\partial_{x^1} \wedge \partial_{x^2} + \partial_{x^3} \wedge \partial_{x^4}$ direction, θ^{1324} in the odd $\partial_{x^1} \wedge \partial_{x^3} - \partial_{x^2} \wedge \partial_{x^4}$ direction, and θ^{1423} in the odd $\partial_{x^1} \wedge \partial_{x^4} + \partial_{x^2} \wedge \partial_{x^3}$ direction. Lastly, we will call the coordinate in the trivial odd direction θ.

Define

$$\begin{aligned} M = {} & \theta^{1234} \otimes (\partial_{x^1} \otimes dx^2 - \partial_{x^2} \otimes dx^1 + \partial_{x^3} \otimes dx^4 - \partial_{x^4} \otimes dx^3) \\ & + \theta^{1324} \otimes (\partial_{x^1} \otimes dx^3 - \partial_{x^3} \otimes dx^1 - \partial_{x^2} \otimes dx^4 + \partial_{x^4} \otimes dx^2) \\ & + \theta^{1423} \otimes (\partial_{x^1} \otimes dx^4 - \partial_{x^4} \otimes dx^1 + \partial_{x^2} \otimes dx^3 - \partial_{x^3} \otimes dx^2). \end{aligned}$$

We can now state intrinsic (although coordinate-dependent) formulas for the D_i.

Proposition 1. *$D_0 = \partial_\theta - \theta^i \partial_{x^i}$, $D_1 = \partial_{\theta^i} dx^i - \theta \partial_{x^i} dx^i - M$ and*

$$\begin{aligned} D_2 = {} & \partial_{\theta^{1234}}(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\ & + \partial_{\theta^{1324}}(dx^1 \wedge dx^3 - dx^2 \wedge dx^4) \\ & + \partial_{\theta^{1423}}(dx^1 \wedge dx^4 + dx^2 \wedge dx^3) \\ & - (\theta^1 \partial_{x^2} - \theta^2 \partial_{x^1} + \theta^3 \partial_{x^4} - \theta^4 \partial_{x^3})(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\ & - (\theta^1 \partial_{x^3} - \theta^3 \partial_{x^1} - \theta^2 \partial_{x^4} + \theta^4 \partial_{x^2})(dx^1 \wedge dx^3 - dx^2 \wedge dx^4) \\ & - (\theta^1 \partial_{x^4} - \theta^4 \partial_{x^1} + \theta^2 \partial_{x^3} - \theta^3 \partial_{x^2})(dx^1 \wedge dx^4 + dx^2 \wedge dx^3) \end{aligned}$$

Proof. First a bit of motivation. The vector fields D^a_b and $\overline{D}^a_b$ are made of two terms: a partial derivative in an odd direction and an odd coordinate function times an even partial derivative. Let us try to construct global objects with this form. Guessing at the formula for D_0, for example, is easy if you want to obtain this form. The others are more complex.

To construct the 1-form D_1, we can take advantage of the redundancy in having X and ΠTX both available, and use the isomorphism between the odd and even tangent spaces. This is what $\partial_{\theta^i} dx^i$ does. We can also construct de Rham d, the identity element in $Hom(TX, TX)$, and multiply it by θ. Lastly for D_1 we can try to find an element of

$$\Pi \wedge^2_+ TX \otimes TX \otimes T^*X$$

where the first factor are the θ^{abcd} coefficients, the second are the vector fields, and the third are the dx^i. Is there a canonical element of this bundle? Yes, look at

$$\mathrm{Id} \in \Pi \wedge^2_+ TX \otimes \wedge^2_+ T^*X$$

and map it through the inclusion

$$\Pi \wedge^2_+ TX \otimes \wedge^2_+ T^*X \hookrightarrow \Pi \wedge^2_+ TX \otimes T^*X \otimes T^*X$$

followed by taking the dual on the first T^*X using the metric. This is the element M.

For D_2 we can use the isomorphism between the even and the odd self-dual 2-vectors, which is what the first three terms of D_2 do. For the second set of three terms we take the element

$$\mathrm{Id} \in \wedge^2_+ TX \otimes \wedge^2_+ T^*X$$

and map it through the inclusion

$$\wedge^2_+ TX \otimes \wedge^2_+ T^*X \hookrightarrow TX \otimes TX \otimes \wedge^2_+ T^*X$$

followed by taking parity reversal on the first TX. Although these three constructions are very canonical and unique, they do not suffice to prove the relationship to the $E^{4|8}$ formulas. However, once we have proven this relationship rigorously, we should leave the Proposition with the sense that twisted supersymmetry has a very rich and deep relationship to intrinsic smooth objects.

We will prove the formula for D_0 as an example, and leave the rest as exercises. θ is the coordinate in the trivial line bundle direction, and we have from (8) that

$$\theta = \theta_1^1 + \theta_2^2.$$

To write out $\theta^i \partial_i$ we use (6) directly on the partials, and use (6) plus the usual change of variables formula for the $\theta^{a\dot{b}}$'s. The change of variables has the effect of taking the complex conjugate compared to the ∂ formulas, which is analogous to the relationship $z = x + iy, \partial_z = \partial_x - i\partial_y$.

$$\begin{aligned}
\theta^1\partial_1 &= (\theta^{1\dot{1}} + \theta^{2\dot{2}})(\partial_{1\dot{1}} + \partial_{2\dot{2}}) \\
\theta^2\partial_2 &= (-i\theta^{1\dot{1}} + i\theta^{2\dot{2}})(i\partial_{1\dot{1}} - i\partial_{2\dot{2}}) \\
\theta^3\partial_3 &= (\theta^{1\dot{2}} - \theta^{2\dot{1}})(\partial_{1\dot{2}} - \partial_{2\dot{1}}) \\
\theta^4\partial_4 &= (-i\theta^{1\dot{2}} - i\theta^{2\dot{1}})(i\partial_{1\dot{2}} + i\partial_{2\dot{1}}).
\end{aligned}$$

Adding this all up gives

$$\partial_1^1 + \theta^{1\dot{1}}\partial_{1\dot{1}} + \theta^{1\dot{2}}\partial_{1\dot{2}} + \partial_2^2 + \theta^{2\dot{1}}\partial_{2\dot{1}} + \theta^{2\dot{2}}\partial_{2\dot{2}} = Q_1^1 + Q_2^2,$$

as required.

We also define Q analogues of D_0, D_1 and D_2, but with plus signs instead of minus signs. Explicitly we have $Q_0 = \partial_\theta + \theta^i\partial_{x^i}$, $Q_1 = \partial_{\theta^i}dx^i + \theta\partial_{x^i}dx^i + M$ and

$$\begin{aligned}
Q_2 = {} & \partial_{\theta^{1234}}(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\
& + \partial_{\theta^{1324}}(dx^1 \wedge dx^3 - dx^2 \wedge dx^4) \\
& + \partial_{\theta^{1423}}(dx^1 \wedge dx^4 + dx^2 \wedge dx^3) \\
& + (\theta^1\partial_{x^2} - \theta^2\partial_{x^1} + \theta^3\partial_{x^4} - \theta^4\partial_{x^3})(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\
& + (\theta^1\partial_{x^3} - \theta^3\partial_{x^1} - \theta^2\partial_{x^4} + \theta^4\partial_{x^2})(dx^1 \wedge dx^3 - dx^2 \wedge dx^4) \\
& + (\theta^1\partial_{x^4} - \theta^4\partial_{x^1} + \theta^2\partial_{x^3} - \theta^3\partial_{x^2})(dx^1 \wedge dx^4 + dx^2 \wedge dx^3).
\end{aligned}$$

Using these formulas, it is trivial to check that all three Q's commute with all three D's. Also, other commutators that will interest us are

$$[D_0, D_1] = -d \tag{38}$$

$$[Q_0, Q_1] = d. \tag{39}$$

Here, and from now on, we will use the following definition of the bracket

$$[A, B] = \frac{1}{2}(AB - (-1)^{\pi(A)\pi(B)}BA). \tag{40}$$

2.2. **Superconnections on SX.** The general idea is that we will simply twist the picture presented in (26). For example, the two S^+-valued sections λ and χ of $\operatorname{ad} P$ will combine to form a single $(S^+)^* \otimes S^+$-valued section of $\operatorname{ad} P$, and so will decompose as a section of $\operatorname{ad} P$ and a section of $\operatorname{ad} P \otimes \wedge^2_+ TX$.

Let us begin with the spinors λ and χ. In flat space they are defined by

$$\chi_a = i^* W_a^1 = i^* \frac{1}{4} \varepsilon^{\dot{c}\dot{d}} [\overline{\mathcal{D}}_{\dot{c}}^{(1)}, \nabla_{a\dot{d}}] \tag{41}$$

$$\lambda_a = i^* W_a^2 = i^* \frac{1}{4} \varepsilon^{\dot{c}\dot{d}} [\overline{\mathcal{D}}_{\dot{c}}^{(2)}, \nabla_{a\dot{d}}]. \tag{42}$$

These can therefore equivalently be defined as the image of the expression

$$\frac{1}{4} [\overline{\mathcal{D}}_{\dot{c}}^{(i)}, \nabla_{a\dot{d}}]$$

under the mapping

$$(\varepsilon^-)^* : (S^-)^* \otimes S^+ \otimes (S^-)^* \otimes (S^+)^* \to S^+ \otimes (S^+)^*.$$

A computation shows that this operation can be interpreted quite simply in global language. Precisely, we have

Proposition 2. *The following diagram commutes*

$$\begin{array}{ccc} (S^-)^* \otimes S^+ \otimes (S^-)^* \otimes (S^+)^* & \xrightarrow{(\varepsilon^-)^*} & S^+ \otimes (S^+)^* \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ TX \otimes TX & \xrightarrow{\text{proj.}} & (X \times \mathbb{R}) \oplus \wedge_+^2 TX \end{array} \tag{43}$$

The barred spinors $\overline{\lambda}$ and $\overline{\chi}$ are defined by

$$\overline{\lambda}_{\dot{a}} = i^* \overline{W}_{\dot{a}}^1 = i^* \frac{1}{4} \varepsilon^{cd} [\mathcal{D}_c^{(1)}, \nabla_{d\dot{a}}] \tag{44}$$

$$\overline{\chi}_{\dot{a}} = i^* \overline{W}_{\dot{a}}^2 = i^* \frac{1}{4} \varepsilon^{cd} [\mathcal{D}_c^{(2)}, \nabla_{d\dot{a}}]. \tag{45}$$

These are the images of the expressions

$$\frac{1}{4} [\mathcal{D}_c^{(i)}, \nabla_{d\dot{a}}]$$

under the mapping

$$\varepsilon^+ : (S^+)^* \otimes S^+ \otimes (S^+)^* \otimes (S^-)^* \to S^+ \otimes (S^-)^*.$$

Here, we obtain another diagram that tells us how to interpret this mapping in global language.

Proposition 3. *The following diagram commutes.*

$$\begin{array}{ccc} (S^+)^* \otimes S^+ \otimes (S^+)^* \otimes (S^-)^* & \xrightarrow{\varepsilon^+} & S^+ \otimes (S^-)^* \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ ((X \times \mathbb{R}) \oplus \wedge_+^2 TX) \otimes TX & \xrightarrow{C} & TX \end{array} \tag{46}$$

where the mapping C is given on a fiber by

$$C((a, \omega) \otimes v) = av + i_{v^*}\omega. \tag{47}$$

Proof. To prove Proposition 2 we need to show that the bottom arrow in (43) is indeed given by the obvious projection. We use (6) and (9) to compute

$$\begin{aligned}
v_1 \otimes v_2 &= (e_{1\dot{1}} + e_{2\dot{2}}) \otimes (ie_{1\dot{1}} - ie_{2\dot{2}}) \\
&\overset{\varepsilon^+}{\mapsto} -ie_{12} - ie_{21} \\
&= v_1 \wedge v_2 + v_3 \wedge v_4 \\
v_1 \otimes v_3 &= (e_{1\dot{1}} + e_{2\dot{2}}) \otimes (e_{1\dot{2}} - e_{2\dot{1}}) \\
&\overset{\varepsilon^+}{\mapsto} e_{11} + e_{22} \\
&= v_1 \wedge v_3 - v_2 \wedge v_4 \\
v_1 \otimes v_4 &= (e_{1\dot{1}} + e_{2\dot{2}}) \otimes (ie_{2\dot{1}} + ie_{2\dot{1}}) \\
&\overset{\varepsilon^+}{\mapsto} ie_{11} - ie_{22} \\
&= v_1 \wedge v_4 + v_2 \wedge v_3
\end{aligned}$$

with similar formulas for $v_2 \otimes v_1$ etc. We also get

$$\begin{aligned}
v_1 \otimes v_1 &= (e_{1\dot{1}} + e_{2\dot{2}}) \otimes (e_{1\dot{1}} + e_{2\dot{2}}) \\
&\overset{\varepsilon^+}{\mapsto} e_{12} - e_{21} \\
&= 1 \in \mathbb{R}
\end{aligned}$$

with identical formulas for $v_i \otimes v_i$, $i = 2, 3, 4$. This completes the proof of Proposition 2. Proposition 3 is proved with a similar computation.

The $X \times \mathbb{R}$ component of the space on the bottom of (43) is just the trivial bundle spanned by the identity section of $T^*X \otimes TX$, followed by using the metric to map $T^*X \to TX$. So, what we have learned is that the unbarred spinors λ, χ, in twisted language, can be built as follows. The image of the $\overline{\mathcal{D}}^{(i)}_{\dot{c}}$ under the vertical map in the diagram is simply the horizontal lift of D_1, which we'll denote by $\mathcal{D}_1$. The image of the $\nabla_{d\dot{a}}$ is just the horizontal lift of de Rham d, which we usually denote by ∇. Let us decompose the projection on the bottom of the diagram with the maps

$$\varepsilon_0 : TX \otimes TX \to X \times \mathbb{R} \tag{48}$$

$$\varepsilon_2 : TX \otimes TX \to \wedge^2_+ TX. \tag{49}$$

Then if we define

$$W_0 = \varepsilon_0[\mathcal{D}_1, \nabla] \tag{50}$$

$$W_2 = \varepsilon_2[\mathcal{D}_1, \nabla] \tag{51}$$

$$\psi_0 = i^* W_0 \tag{52}$$

$$\psi_2 = i^* W_2 \tag{53}$$

we have met two goals. We have defined two component fields in global language on SX, but we have also proved with Proposition 2 that these two components can be rewritten in local coordinates as the λ and χ we saw before.

We see from (46) that the restriction of C to the subspace $(X \times \mathbb{R}) \otimes TX \cong TX$ has the same image as all of C. So, we can build the barred spinors in a global way by forming the bracket $[\mathcal{D}_0, \nabla]$.

$$W_1 = -\frac{1}{2}[\mathcal{D}_0, \nabla] \tag{54}$$

$$\psi_1 = i^* W_1. \tag{55}$$

Next define

$$\Phi = -[\mathcal{D}_0, \mathcal{D}_0]$$
$$\overline{\Phi} = -\frac{1}{4}\varepsilon_0[\mathcal{D}_1, \mathcal{D}_1]$$

and

$$\phi = i^*\Phi$$
$$\overline{\phi} = i^*\overline{\Phi}.$$

Proposition 4. *ϕ is the twisted version of $\overline{\sigma}$ and $\overline{\phi}$ is the twisted version of σ.*

Proof. We compute that

$$\begin{aligned}\Phi &= -[\mathcal{D}_0, \mathcal{D}_0] \\ &= -[\mathcal{D}_1^1 + \mathcal{D}_2^2, \mathcal{D}_1^1 + \mathcal{D}_2^2] \\ &= -2[\mathcal{D}_1^1, \mathcal{D}_2^2],\end{aligned}$$

which agrees with (24). Similarly,

$$\begin{aligned}-4\overline{\Phi} &= -\frac{1}{4}\varepsilon_0[\mathcal{D}_1, \mathcal{D}_1] \\ &= [\overline{\mathcal{D}}_{\dot 1}^{(2)} - \overline{\mathcal{D}}_{\dot 2}^{(1)}, \overline{\mathcal{D}}_{\dot 1}^{(2)} - \overline{\mathcal{D}}_{\dot 2}^{(1)}] + [i\overline{\mathcal{D}}_{\dot 1}^{(2)} + i\overline{\mathcal{D}}_{\dot 2}^{(1)}, i\overline{\mathcal{D}}_{\dot 1}^{(2)} + i\overline{\mathcal{D}}_{\dot 2}^{(1)}] \\ &+ [\overline{\mathcal{D}}_{\dot 2}^{(2)} + \overline{\mathcal{D}}_{\dot 1}^{(1)}, \overline{\mathcal{D}}_{\dot 2}^{(2)} + \overline{\mathcal{D}}_{\dot 1}^{(1)}] + [i\overline{\mathcal{D}}_{\dot 2}^{(2)} - i\overline{\mathcal{D}}_{\dot 1}^{(1)}, i\overline{\mathcal{D}}_{\dot 2}^{(2)} - i\overline{\mathcal{D}}_{\dot 1}^{(1)}] \\ &= 4[\overline{\mathcal{D}}_{\dot 2}^{(2)}, \overline{\mathcal{D}}_{\dot 1}^{(1)}] - 4[\overline{\mathcal{D}}_{\dot 1}^{(2)}, \overline{\mathcal{D}}_{\dot 2}^{(1)}],\end{aligned}$$

which is the sum of (22) and (23), completing the proof. However, if we compute

$$\begin{aligned}-[\mathcal{D}_2, \mathcal{D}_2] &= [-i\mathcal{D}_1^1 + i\mathcal{D}_2^2, -i\mathcal{D}_1^1 + i\mathcal{D}_2^2] + [\mathcal{D}_2^1 - \mathcal{D}_1^2, \mathcal{D}_2^1 - \mathcal{D}_1^2] \\ &+ [-i\mathcal{D}_2^1 - i\mathcal{D}_1^2, -i\mathcal{D}_2^1 - i\mathcal{D}_1^2] \\ &= -2[\mathcal{D}_1^1, \mathcal{D}_2^2] + 4[\mathcal{D}_2^1, \mathcal{D}_1^2]\end{aligned}$$

then we see that it is also very natural to form the object

$$-[\mathcal{D}_0, \mathcal{D}_0] - [\mathcal{D}_2, \mathcal{D}_2],$$

which agrees with the sum of (24) and (25). However, we do not adopt this alternate definition of Φ.

We now proceed to discuss the fact that in flat space there are three auxiliary fields. The natural guess is to find some self-dual two-vector field that can be written using the twist as these three fields. If we name this single twisted auxiliary field by the name E_2 then we claim

Proposition 5. *$E_2 = i^*\frac{1}{2}\varepsilon_2(\mathcal{D}_1[\mathcal{D}_0, \nabla])$ is the twisted version of the auxiliary fields E and F.*

Proof. We rewrite E_2 as

$$\begin{aligned} E_2 &= i^* \frac{1}{2} \varepsilon_2(\mathcal{D}_1([\mathcal{D}_0, \nabla])) \\ &= -i^* \varepsilon_2(\mathcal{D}_1 W_1) \\ &= -i^* \varepsilon_2(\mathcal{D}_1 \mathcal{D}_1 \Phi) \end{aligned}$$

and then consider what ε_2 does to $\mathcal{D}_1\mathcal{D}_1$. If we label the four components of $\mathcal{D}_1$ in a coordinate chart by $\mathcal{D}_1^i$ then we can compute that

$$\begin{aligned} \mathcal{D}_1^1\mathcal{D}_1^2 + \mathcal{D}_1^3\mathcal{D}_1^4 &= (\overline{\mathcal{D}}_1^1 + \overline{\mathcal{D}}_2^2)(\overline{\mathcal{D}}_2^1 - \overline{\mathcal{D}}_1^2) - (i\overline{\mathcal{D}}_1^1 - i\overline{\mathcal{D}}_2^2)(i\overline{\mathcal{D}}_2^1 + i\overline{\mathcal{D}}_1^2) \\ &= \overline{\mathcal{D}}_1^1\overline{\mathcal{D}}_2^1 - \overline{\mathcal{D}}_2^2\overline{\mathcal{D}}_1^2 \end{aligned}$$

and so checking with (27) we see that one of the three components of $\varepsilon_2\mathcal{D}_1\mathcal{D}_1\Phi$ is $\overline{F} - F$. Similar computations reveal that the second component, $D_1^1D_1^3 - D_1^2D_1^4$, is $F + \overline{F}$ and the third, $D_1^1D_1^4 + D_1^2D_1^3$, gives E, which completes the proof.

In summary we have proved the following key result.

Theorem 3. *Let $\mathcal{P}$ be a principal $SU(2)$ bundle over the supermanifold SX. Let $P \to X$ be the restriction of $\mathcal{P}$ to X. The space of semi-constrained superconnections on $\mathcal{P}$ is isomorphic to the space of fields $A, \phi, \overline{\phi}, \psi_0, \psi_1, \psi_2, E_2$, where A is an ordinary connection on the restriction $P \to X$ of $\mathcal{P} \to SX$, ϕ and $\overline{\phi}$ are sections of $\operatorname{ad} P$, $\psi_0 \in \Gamma(\Pi(X \times \mathbb{R}) \otimes \operatorname{ad} P))$, $\psi_1 \in \Gamma(\Pi TX \otimes \operatorname{ad} P)$, $\psi_2 \in \Gamma(\Pi \wedge_+^2 TX \otimes \operatorname{ad} P)$, and $E_2 \in \Gamma(\wedge_+^2 TX \otimes \operatorname{ad} P)$.*

Proof. It suffices to work in a coordinate patch, where by the preceding discussion the theorem reduces to its $N = 2$ flat space counterpart Theorem 1.

2.3. **The action of Q_0.** We will now go about computing the vector field on $S\mathcal{A}$ that is induced by Q_0. What we mean is that the vector field Q_0 on SX acts on functions and bundle sections by covariant differentiation, and so it acts on the points of the space $S\mathcal{A}$. The infinitesimal form of this action is again a vector field and we are going to try to express it in terms of components. Let η be an odd parameter. If $\mathcal{A}$ is a semi-constrained superconnection then denote by ξ the diffeomorphism generated by the even vector field $\eta\mathcal{Q}_0$. To get formulas in components, then, we are searching for the components of $\xi\mathcal{A}$. These components are defined using the $\mathcal{D}_i$, which all commute with ξ. Moreover, when restricted to the even submanifold $P \subset \mathcal{P}$ the vector fields $\mathcal{D}_0$ and $\mathcal{Q}_0$ agree with each other. This all implies that we can compute the action of $\mathcal{Q}_0$ by using $\mathcal{D}_0$ instead. And so, our approach will simply be to hit the component fields with $\mathcal{D}_0$ on the left and rewrite them in terms of each other after some dust settles. A key tool will be the super Bianchi identity, which is the Bianchi identity with parity taken into account.

Theorem 4 (Bianchi)**.** *Let $\mathcal{F}$ be the curvature of a connection $\mathcal{A}$ on a principal bundle $\mathcal{P} \to SX$. Let X, Y, Z be vector fields on SX. Let $\hat{X}, \hat{Y}, \hat{Z}$ denote the horizontal lifts to $\mathcal{P}$. If $\pi(X)$ denotes the parity of the vector field X then*

$$\begin{aligned} 0 = (\hat{X}\mathcal{F})(Y, Z) &+ (-1)^{\pi(X)\pi(Y)+\pi(X)\pi(Z)}(\hat{Y}\mathcal{F})(Z, X) \\ &+ (-1)^{\pi(Z)\pi(X)+\pi(Z)\pi(Y)}(\hat{Z}\mathcal{F})(X, Y), \end{aligned}$$

where the covariant derivative of a two-form is given by the super formula

$$(\hat{X}\mathcal{F})(Y, Z) = \hat{X}(\mathcal{F}(Y, Z)) - \mathcal{F}([X, Y], Z) + (-1)^{\pi(X)\pi(Y)}\mathcal{F}(Y, [X, Z]).$$

We apply this theorem as follows. The identity on the three vector fields D_0, D_0 and d yields

$$\begin{aligned} &\mathcal{D}_0(\mathcal{F}(D_0,d)) \; - \mathcal{F}([D_0,D_0],d) + \mathcal{F}(D_0,[D_0,d]) \\ - \; &\mathcal{D}_0(\mathcal{F}(d,D_0)) \; + \mathcal{F}([D_0,d],D_0) + \mathcal{F}(d,[D_0,D_0]) \\ + \; &\nabla(\mathcal{F}(D_0,D_0)) - \mathcal{F}([d,D_0],D_0) - \mathcal{F}(D_0,[d,D_0]) = 0 \end{aligned} \; .$$

In this expression, the second and sixth terms vanish due to $D_0^2 = 0$ and the third, fifth, eighth and ninth vanish due to the fact that the connection is semi-constrained. This leaves us with

$$\mathcal{D}_0(\mathcal{F}(D_0,d)) - \mathcal{D}_0(\mathcal{F}(d,D_0)) + \nabla(\mathcal{F}(D_0,D_0)) = 0,$$

which gives on restriction to X

$$\boxed{\mathcal{D}_0\psi_1 = -\nabla\phi.} \tag{56}$$

Next we examine the identity for three copies of D_0.

$$\begin{aligned} &\mathcal{D}_0(\mathcal{F}(D_0,D_0)) - \mathcal{F}([D_0,D_0],D_0) + \mathcal{F}(D_0,[D_0,D_0]) \\ + \; &\mathcal{D}_0(\mathcal{F}(D_0,D_0)) - \mathcal{F}([D_0,D_0],D_0) + \mathcal{F}(D_0,[D_0,D_0]) \\ + \; &\mathcal{D}_0(\mathcal{F}(D_0,D_0)) - \mathcal{F}([D_0,D_0],D_0) + \mathcal{F}(D_0,[D_0,D_0]) = 0 \end{aligned}$$

which immediately becomes

$$\boxed{\mathcal{D}_0\phi = 0.} \tag{57}$$

Next we work with D_0, D_1 and d to obtain

$$\begin{aligned} &\mathcal{D}_0(\mathcal{F}(D_1,d)) \; - \mathcal{F}([D_0,D_1],d) + \mathcal{F}(D_1,[D_0,d]) \\ - \; &\mathcal{D}_1(\mathcal{F}(d,D_0)) \; + \mathcal{F}([D_1,d],D_0) + \mathcal{F}(d,[D_1,D_0]) \\ + \; &\nabla(\mathcal{F}(D_0,D_1)) - \mathcal{F}([d,D_0],D_1) - \mathcal{F}(D_0,[d,D_1]) = 0 \end{aligned}$$

whose third, fifth, seventh, eighth and ninth terms vanish because the connection is semi-constrained. Using the fact that

$$[D_1,D_0] = [D_0,D_1] = d$$

gives

$$\mathcal{D}_0((\mathcal{F}(D_1,d)) - \mathcal{D}_1(\mathcal{F}(d,D_0)) = 0.$$

This yields the equation

$$\begin{aligned} \mathcal{D}_0\psi_2 &= i^*\varepsilon_2(\mathcal{D}_1(\mathcal{F}(d,D_0))) \\ &= E_2. \end{aligned} \tag{58}$$

so we have

$$\boxed{\mathcal{D}_0\psi_2 = E_2.} \tag{59}$$

Next we compute with D_0, D_1 and D_1.

$$\begin{aligned} &\mathcal{D}_0(\mathcal{F}(D_1,D_1)) - \mathcal{F}([D_0,D_1],D_1) + \mathcal{F}(D_1,[D_0,D_1]) \\ + \; &\mathcal{D}_1(\mathcal{F}(D_1,D_0)) - \mathcal{F}([D_1,D_1],D_0) + \mathcal{F}(D_1,[D_1,D_0]) \\ + \; &\mathcal{D}_1(\mathcal{F}(D_0,D_1)) - \mathcal{F}([D_1,D_0],D_1) + \mathcal{F}(D_0,[D_1,D_1]) = 0 \end{aligned} \; .$$

the fourth and seventh terms vanish by the semi-constrained condition. We are going to use this equation to compute $\mathcal{D}_0\overline{\phi}$ and so we need to take ε_0 of both sides. ε_0 takes the trace of this bracket, and because individual components of $\mathcal{D}_1$ square to zero we kill the terms with $[\mathcal{D}_1,\mathcal{D}_1]$. This leaves us with

$$\boxed{\mathcal{D}_0\overline{\phi} = \psi_0.} \tag{60}$$

Next let's work out $\mathcal{D}_0 A$. To compute a component of this one-form we'd examine the restriction to X of

$$i(\nabla_{x^\mu})L(\mathcal{D}_0)A.$$

To get the global version of this, we simply replace the partial with ∇.

$$\begin{aligned} i(\nabla)L(\mathcal{D}_0)A &= L(-\mathcal{D}_0)i(\nabla)A - i(2[\mathcal{D}_0,\nabla])A && (61)\\ &= 0 - 2[\mathcal{D}_0,\nabla] && (62)\\ &= W_1 && (63) \end{aligned}$$

so that we obtain

$$\boxed{\mathcal{D}_0 A = \psi_1.} \tag{64}$$

This leaves us with the computations for ψ_0 and E_2. These are in the image of $\mathcal{D}_0$ though, and we can argue as follows. If f is some component field, we can compute

$$\begin{aligned} i^*\mathcal{D}_0\mathcal{D}_0 f &= i^*[\mathcal{D}_0,\mathcal{D}_0]f && (65)\\ &= i^*\nabla_{F(D_0,D_0)}f && (66)\\ &= \nabla_{-\phi}f && (67)\\ &= -[\phi,f]. && (68) \end{aligned}$$

Using this we obtain

$$\boxed{\mathcal{D}_0\psi_0 = -[\phi,\overline{\phi}]} \tag{69}$$

$$\boxed{\mathcal{D}_0(F_A^+ - E_2) = -[\phi,\psi_2].} \tag{70}$$

The total result is then

$$\boxed{\mathcal{Q}_0\begin{pmatrix} A\\ \psi_1\\ \phi\\ \overline{\phi}\\ \psi_0\\ \psi_2\\ E_2 \end{pmatrix} = \begin{pmatrix} \psi_1\\ -\nabla\phi\\ 0\\ \psi_0\\ -\left[\phi,\overline{\phi}\right]\\ E_2\\ -\left[\phi,\psi_2\right] \end{pmatrix}_{(A,\psi_1,\phi,\overline{\phi},\psi_0,\psi_2,E_2)}} \tag{71}$$

This is a vector field on $S\mathcal{A}$, and so can be used as a differential operator on functions on $S\mathcal{A}$. We have decomposed the infinite dimensional space $S\mathcal{A}$ into seven subspaces,

$$S\mathcal{A} = \mathcal{A}\times\Pi\Omega^1\times\Omega^0\times\Omega^0\times\Pi\Omega^0\times\Pi\Omega^2_+\times\Omega^2_+ \tag{72}$$

(where we omit the $(X;\operatorname{ad}P)$'s from the notation for clarity). Suppose f is a function on $S\mathcal{A}$, and we would like to compute the derivative of f using $\mathcal{Q}_0$, i.e. $\mathcal{Q}_0 f$. Suppose further that we have an explicit expression for f as a combination of various component fields. In order to compute a similarly explicit expression for $\mathcal{Q}_0 f$, we use the Leibnitz rule and the chain rule, and then ask the question "what is $\mathcal{Q}_0$ evaluated on an individual component field?" One computes this derivative by taking the corresponding component of $\mathcal{Q}_0$.

An analogous situation is the following. Suppose we work on a finite-dimensional manifold X and use local coordinates $x^\mu, \mu = 1,\ldots,n$ to express a computation. Let $f(x^1,\ldots,x^n) = x^i$ for some fixed i in these coordinates. If $V =$

$\sum_k a^k(x^1,\dots,x^n)\frac{\partial}{\partial x^k}$ is a vector field in this patch, then $Vf(x^1,\dots,x^n) = a^i(x^1,\dots,x^n)$, the ith component of V.

2.4. **The action after the twist.** We will twist the formula (33). Many of the terms there do not respond to the twist, which affects only the fermions and auxiliary fields as we have seen. However, note that in our notation σ is $\overline{\phi}$ and $\overline{\sigma}$ is ϕ. With just this we find that the twisted action has the terms

$$\frac{1}{g^2}\left\{-|F_A|^2 + -i\left\langle\overline{\phi}, d_A^* d_A\phi\right\rangle - i|E_2|^2\right\} + \frac{\theta}{16\pi^2}\langle F_A, F_A\rangle.$$

It remains to twist the fermionic terms. First we examine

$$\langle\lambda \not{D}_A\overline{\lambda}\rangle + \langle\chi \not{D}_A\overline{\chi}\rangle.$$

Lemma 4. *In local coordinates on SX we have*

$$\langle\lambda \not{D}_A\overline{\lambda}\rangle + \langle\chi \not{D}_A\overline{\chi}\rangle = \langle\psi_0, d_A^*\psi_1\rangle + \left\langle\psi_2, d_A^+\psi_1\right\rangle \tag{73}$$

Proof. We express the right hand side in spinorial coordinates to prove the lemma. By way of motivation, examine (31). The element being hit with the two epsilon tensors is an element of

$$(S^+)^*\otimes S^+\otimes(S^+)^*\otimes(S^-)^*\otimes(S^-)^*\otimes S^+$$

and the ε's are contracting the two $(S^+)^*$ spaces and the two $(S^-)^*$ spaces. We formed a picture of these two contractions in Propositions 2 and 3. For instance, the ε^- contraction on the $(S^-)^*$ spaces will combine the $\nabla_{a\dot{b}}\lambda_{\dot{b}}$ (and its twin $\nabla_{a\dot{b}}\chi_{\dot{b}}$ which is not separate in this context) by mapping to $(X\times\mathbb{R})\oplus\wedge^2_+TX$. Clearly this will produce $d_A^*\psi_1$ and $d_A^+\psi_1$. Recalling (6), (7) and (8) we compute

$$2d_A^+\psi_1 = \begin{pmatrix} (\partial_{1\dot{1}}+\partial_{2\dot{2}})(-i\overline{\chi}_{\dot{1}} - i\overline{\lambda}_{\dot{2}}) - (i\partial_{1\dot{1}} - i\partial_{2\dot{2}})(-\overline{\chi}_{\dot{1}} + \overline{\lambda}_{\dot{2}}) \\ (\partial_{1\dot{1}}+\partial_{2\dot{2}})(-\overline{\chi}_{\dot{2}} - \overline{\lambda}_{\dot{1}}) - (\partial_{1\dot{2}} - \partial_{2\dot{1}})(-\overline{\chi}_{\dot{1}} + \overline{\lambda}_{\dot{2}}) \\ (\partial_{1\dot{1}}+\partial_{2\dot{2}})(-i\overline{\chi}_{\dot{2}} + i\overline{\lambda}_{\dot{1}}) - (i\partial_{1\dot{2}} + i\partial_{2\dot{1}})(-\overline{\chi}_{\dot{1}} + \overline{\lambda}_{\dot{2}}) \\ +(\partial_{1\dot{2}} - \partial_{2\dot{1}})(-i\overline{\chi}_{\dot{2}} + i\overline{\lambda}_{\dot{1}}) - (i\partial_{1\dot{2}} + i\partial_{2\dot{1}})(-\overline{\chi}_{\dot{2}} - \overline{\lambda}_{\dot{1}}) \\ -(i\partial_{1\dot{1}} - i\partial_{2\dot{2}})(-i\overline{\chi}_{\dot{2}} + i\overline{\lambda}_{\dot{1}}) + (i\partial_{1\dot{2}} + i\partial_{2\dot{1}})(-i\overline{\chi}_{\dot{1}} - i\overline{\lambda}_{\dot{2}}) \\ +(i\partial_{1\dot{1}} - i\partial_{2\dot{2}})(-\overline{\chi}_{\dot{2}} - \overline{\lambda}_{\dot{1}}) - (\partial_{1\dot{2}} - \partial_{2\dot{1}})(-i\overline{\chi}_{\dot{1}} - i\overline{\lambda}_{\dot{2}}) \end{pmatrix}$$

To express ψ_2 in spinor coordinates, we use (8) together with the fact that elements with upper index 1 are called χ and with upper index 2 are called λ (see (44) and (45)). We get

$$4\psi_2 = \begin{pmatrix} -i\chi_1 + i\lambda_2 \\ \chi_2 - \lambda_1 \\ -i\chi_2 - i\lambda_1 \end{pmatrix}.$$

Similarly,

$$4\psi_0 = \chi_1 + \lambda_2$$

and

$$\begin{aligned} 2d_A^*\psi_1 =& (\nabla_{1\dot{1}} + \nabla_{2\dot{2}})(-\overline{\chi}_{\dot{1}} + \overline{\lambda}_{\dot{2}}) + (i\nabla_{1\dot{1}} - i\nabla_{2\dot{2}})(-i\overline{\chi}_{\dot{1}} - i\overline{\lambda}_{\dot{2}}) \\ &+ (\nabla_{1\dot{2}} - \nabla_{2\dot{1}})(-\overline{\chi}_{\dot{2}} - \overline{\lambda}_{\dot{1}}) + (i\nabla_{1\dot{2}} + i\nabla_{2\dot{1}})(-i\overline{\chi}_{\dot{2}} + i\overline{\lambda}_{\dot{1}}) \end{aligned}$$

Computing $\langle\psi_0, d_A^*\psi_1\rangle + \left\langle\psi_2, d_A^+\psi_1\right\rangle$ is now a matter of combining these expressions and cancelling half of the terms, leaving us with the desired quantity.

Next we work with the terms involving brackets of spinors. Something surprising will result — a term that will not play a role in the geometrical picture that emerges in the next section.

Lemma 5. *In local coordinates on SX we have*

$$\varepsilon^{ab}[\lambda_a, \chi_b] = \frac{1}{4}[\psi_2, \psi_2] + \frac{1}{4}[\psi_0, \psi_0] \tag{74}$$

$$\varepsilon^{\dot{a}\dot{b}}[\overline{\lambda}_{\dot{a}}, \overline{\chi}_{\dot{b}}] = \varepsilon_0[\psi_1, \psi_1]. \tag{75}$$

Proof. We compute

$$\begin{aligned}\varepsilon_0[\psi_1, \psi_1] =& -\frac{1}{4}([-\overline{\chi}_{\dot{1}} + \overline{\lambda}_{\dot{2}}, -\overline{\chi}_{\dot{1}} + \overline{\lambda}_{\dot{2}}] + [-i\overline{\chi}_{\dot{1}} - i\overline{\lambda}_{\dot{2}}, -i\overline{\chi}_{\dot{1}} - i\overline{\lambda}_{\dot{2}}] \\ &+ [-\overline{\chi}_{\dot{2}} - \overline{\lambda}_{\dot{1}}, -\overline{\chi}_{\dot{2}} - \overline{\lambda}_{\dot{1}}] + [-i\overline{\chi}_{\dot{2}} + i\overline{\lambda}_{\dot{1}}, -i\overline{\chi}_{\dot{2}} + i\overline{\lambda}_{\dot{1}}]) \\ =& [\overline{\lambda}_{\dot{1}}, \overline{\chi}_{\dot{2}}] - [\overline{\lambda}_{\dot{2}}, \overline{\chi}_{\dot{1}}].\end{aligned}$$

And using the fact that the bracket uses the structure of the wedge product on forms, and that the components of ψ_2 wedge to zero except against themselves, we obtain

$$\begin{aligned}\frac{1}{4}[\psi_2, \psi_2] + \frac{1}{4}[\psi_0, \psi_0] =& \frac{1}{4}([-i\lambda_1 + i\chi_2, -i\lambda_1 + i\chi_2] + [\lambda_2 - \chi_1, \lambda_2 - \chi_1] \\ &+ [-i\lambda_2 - i\chi_1, -i\lambda_2 - i\chi_1] + [\lambda_1 + \chi_2, \lambda_1 + \chi_2]) \\ =& [\lambda_1, \chi_2] - [\lambda_2, \chi_1].\end{aligned}$$

This completes the proof and thus we have computed

$$\begin{aligned}&\int_{S\mathcal{A}} \exp\Big(\frac{1}{g^2}\Big(-\frac{1}{2}|F_A|^2 - i\left\langle\overline{\phi}, d_A^* d_A\phi\right\rangle + i\left\langle\psi_0, d_A^*\psi_1\right\rangle + i\left\langle\psi_2, (d_A\psi_1)^+\right\rangle \\ &\qquad + i\left\langle\psi_2, [\phi, \psi_2]\right\rangle + i\left\langle\phi, [\psi_0, \psi_0]\right\rangle + i\left\langle\overline{\phi}, [\psi_1, *\psi_1]\right\rangle \qquad (76) \\ &\qquad - i|E_2|^2\Big) + \frac{\theta}{16\pi^2}\left\langle F_A \wedge F_A\right\rangle\Big) \qquad (77)\end{aligned}$$

If we tweak the parameter θ, we can obtain the sum

$$-\frac{1}{2g^2}|F_A|^2 - \frac{1}{2g^2}\left\langle F_A \wedge F_A\right\rangle$$

which becomes

$$-\frac{1}{g^2}|F_A^+|^2$$

This particular value for θ will be fixed from now on, for it facilitates the geometric interpretation we will dwell on presently. Note that with this alteration the whole action has an overall coefficient of $\frac{1}{g^2}$. This is the coupling parameter for this physical theory, and when written outside the action it acts like Planck's constant h. Namely, we can see directly (if path integration makes sense) that if the coupling becomes vanishingly small then the minima of S are heavily weighed in a path integral computation, and we approach a classical limit. We will prefer a different interpretation for the coupling parameter and so we scale some of the fields as follows

$$\begin{aligned}\phi &\mapsto g^2\phi\\ \psi_0 &\mapsto g^2\psi_0\\ \psi_2 &\mapsto g^2\psi_2\\ E_2 &\mapsto gE_2,\end{aligned}$$

producing the formula we will use going forward:

$$\int_{S\mathcal{A}} \exp\Big(-\frac{1}{g^2}|F_A^+|^2 - i\left\langle\overline{\phi}, d_A^* d_A\phi\right\rangle + i\left\langle\psi_0, d_A^*\psi_1\right\rangle + i\left\langle\psi_2, (d_A\psi_1)^+\right\rangle \tag{78}$$

$$+ g^2 i\left\langle\psi_2, [\phi,\psi_2]\right\rangle + g^2 i\left\langle\phi, [\psi_0,\psi_0]\right\rangle + i\left\langle\overline{\phi}, [\psi_1, *\psi_1]\right\rangle - i|E_2|^2\Big) \tag{79}$$

We make one final remark about this computation. It should in principle be possible to compute the twisted action directly on SX, perhaps using a multiple of

$$\int_{\Pi TX} \operatorname{Tr}\Phi^2 - \int_{\Pi((X\times\mathbb{R})\oplus\wedge_2^+TX)} \operatorname{Tr}\overline{\Phi}^2.$$

which corresponds to (29). To compute this in components, one would hit each integrand with an appropriate differential operator. For example, the first odd integral could be carried out by hitting $\operatorname{Tr}\Phi^2$ with $(D_1)^4$, interpreted in an appropriate sense. Similarly, the second integral could be carried out with the help of $D_0\circ(D_2)^3$ where the cube is perhaps interpreted to mean the determinant on the third tensor power of the 3-dimensional bundle $\wedge_2^+TX$. This computation should be straightforward once the meaning of these operators is sorted out. Some insight into SX is sure to be gained by this exercise.

3. The polynomial invariants

The definitions of the component fields of semi-constrained superconnections give a decomposition of $S\mathcal{A}$. A central result of this paper is that this decomposition can be viewed as a rich algebraic structure living on the usual space of connections. Without having ever mentioned the ASD equations or the action of the group of gauge transformations, we will find that in a formal sense these are *automatically called for* by the structure of $S\mathcal{A}$.

Let P be a principal G-bundle over a base X and let V be a $2n$-dimensional representation of G. Form the associated vector bundle $E = P\times_G V$. On this vector bundle there is a Thom class $\mathcal{T}\in H_c^{2n}(E)$ in compactly supported cohomology. It has maximal degree along the fibers, and so is fully "vertical." One can pull back a representative of $\mathcal{T}$ to X by the zero section s and obtain a representative e of the Euler class of E. If one pulls $\mathcal{T}$ back by a nonzero section s, one can interpret the pullback $s^*(\mathcal{T})$ as the Poincaré dual to the zero set Z_s of s. And so, to integrate a differential form ω over Z_s one can integrate $\omega\wedge s^*(\mathcal{T})$ over all of X.

Mathai and Quillen [8] introduced a representative $\mathcal{T}_A$ for the Thom class that lives in the G-equivariant cohomology of $P\times V$. The A denotes a connection on P, which is used in the construction. In fact, they write an element of the Cartan algebra of V, which is an algebraic model of equivariant cohomology, and then use the connection to map it to an equivariant differential form on $P\times V$, using the Weil homomorphism. Mathai and Quillen showed that if s is an arbitrary section of E, then $s^*\mathcal{T}_A = e_{s,A}$ is a representative for the Euler class, and is independent of both A and s. To be totally explicit, they write

$$e_{s,A} = \frac{1}{(2\pi)^n} \int d\psi \, e^{-\|s\|^2 + \frac{1}{2}\langle \psi, \phi\psi \rangle + i\langle ds, \psi \rangle} \tag{80}$$

where ψ is an element of ΠV. The object ϕ is an element of the equivariant cohomology, and under the Weil homomorphism it maps to F_A, which we will discuss a little later. Note that this element has *rapid decrease* along the fiber V, but is not compactly supported. In fact, the inclusion of compactly supported forms into forms with rapid decrease induces an isomorphism of cohomology.

Taking $s = 0$ produces $\mathrm{Pfaff}(F_A)$, which restates the Gauss-Bonnet theorem. Taking s nonzero and multiplying by a constant γ to get γs and then taking $\gamma \to \infty$ *localizes* $e_{s,A}$ to the zero set of s. This can be proven by approximating (80) with the method of steepest descent.

Also relevant for us is a modification of this picture that lets us work upstairs. Let s be a section of E, and suppose we want to compute

$$\int_{Z_s} \omega$$

for some form ω. We know we can instead work with

$$\int_X \omega \wedge e_{s,A}.$$

However, we can further enlarge the space we integrate over to E if we can find an appropriate differential form that has maximal degree along the fibers of P and that integrates to 1 on a fiber. Such a form is called a *projection form*, and if we call it η_{proj} then we have

$$\int_{Z_s} \omega = \int_E \omega \wedge e_{s,A} \wedge \eta_{\mathrm{proj}}.$$

It is familiar in Donaldson theory that the ASD moduli space can be defined as the zero set of the section $F_A^+ : \mathcal{A}/\mathcal{G} \to \Omega^2_+(X; \mathrm{ad}\, P)$. If there were such an object as a Thom class in this infinite-dimensional context, we could hope that the pullback by F_A^+ would be in some sense Poincaré dual to the ASD moduli space. Surely such a geometrical construction could be carried out mathematically, but it has not yet been done. The problem is that the space $\mathcal{B}$ of connections modulo gauge transformations is infinite-dimensional and the fiber of the vector bundle, $\Omega^2_+(X; \mathrm{ad}\, P)$ is also infinite-dimensional. In addition, the group $\mathcal{G}$ has infinite dimension, so the concept of the projection form as a "top-dimensional" form along the fibers of $\mathcal{A} \to \mathcal{B}$ does not make sense. Nonetheless, if we ignore these issues we will see that a straightforward application of the above construction to F_A^+ produces the twisted action (78).

So the simple twisting operation has brought us from a physical supersymmetric theory all the way to the ASD moduli space, equipped with an Euler class to help us do intersection theory. All that is missing is Donaldson's μ-map, which has a beautiful manifestation in this context, as we will see below.

Much of this treatment of the Mathai-Quillen form and the projection form is based on [12]. The original insight into the geometry underlying the action is in Atiyah and Jeffrey's paper [1]. The following account differs from Atiyah and Jeffrey's, however, in two important respects. First of all, we build the geometrical constructions from the structure of $S\mathcal{A}$ itself, using the operator $\mathcal{Q}_0$ and the component fields to prove that the equivariant cohomological data we need is encoded in the twisted supersymmetry. This is a very important observation, as it uses the twist to show that the Mathai-Quillen and projection forms naturally arise from supersymmetry, and so *motivate* doing Donaldson theory rather than just *imitating* Donaldson theory. Thinking of Donaldson theory as an outgrowth of twisted supersymmetry may eventually prove to be useful for gaining additional insights about smooth structures on 4-manifolds. The second departure from Atiyah and Jeffrey's work is that we will try to de-emphasize the interpretation of the path integral as a representation of a nonexistent Euler class. Instead, we will discuss the physical approach to path integration and show how the localization to the ASD moduli space is obtained by examining the classical limit of the quantum theory. Strengthening the link with physics fits into our overarching strategy of initiating an investigation into Witten's Conjecture, but the reader should be clear on one point: understanding what an Euler class is in infinite dimensions will shed light on both the Donaldson invariants and on path integrals, and so we are not advocating that mathematicians should neglect to sort those ideas out.

3.1. **The algebraic structure of $S\mathcal{A}$.** We define two subspaces of $S\mathcal{A}$.

$$L(\mathcal{A}) = \Omega^0 \times \Omega^2_+ \times \Pi\Omega^2_+ \tag{81}$$

$$P(\mathcal{A}) = \mathcal{A} \times \Pi\Omega^1 \times \Omega^0 \times \Omega^0 \times \Pi\Omega^0 \tag{82}$$

where the shared copy of Ω^0 is the one given by elements we have been calling ϕ. (Sometimes we will want to use dual spaces of a few of these pieces but we will feel free to switch to the dual spaces as needed.) Keeping this structure in mind, we will digress temporarily to treat more carefully the two finite-dimensional geometrical ideas, the Mathai-Quillen form, and the projection form. Our presentation of these two forms relies on the algebraic structure of the Cartan model for equivariant cohomology. Then we will return to $S\mathcal{A}$ and see that we have the same algebraic picture present, in the guise of the vector field $\mathcal{Q}_0$ and in the twisted action (78).

3.1.1. *The Mathai-Quillen form.* In [8], Mathai and Quillen constructed a representative for the Euler class of a vector bundle that is built from a connection and a section. The proved that their form was closed and that its cohomology class depends neither on the section nor the connection. We will describe their construction now. Let G be a Lie group, let A be a principal G-bundle over a space B, and let $E = A \times_\rho V$ be an associated n-dimensional vector bundle, where $\rho : G \to GL(V)$ is a given representation. We are using finite dimensional A, B and G, but their names should suggest that we will eventually apply these ideas to the infinite dimensional spaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$.

Let $d\rho : \mathfrak{g} \to \mathrm{Hom}(V) \cong \mathrm{Vect}(V)$ be the Lie algebra map to vector fields on V. We will denote by i_ϕ the contraction operator in the direction of $d\rho(\phi)$. The Cartan algebra is the space $S^*(\mathfrak{g}^*) \otimes \Omega^*(V)$, equipped with a differential d_C given by

$$d_C(\phi \otimes 1) = 0 \tag{83}$$

$$d_C(\phi \otimes \omega) = 1 \otimes (d - i_\phi)\omega \tag{84}$$

where ϕ is a generator of $S^*(\mathfrak{g}^*)$, and one extends this formula to the full algebra by the Leibnitz rule. The cohomology of this complex computes the equivariant cohomology of V, $H^*_G(V)$. In case G acts freely, one has $H^*_G(V) \cong H^*(V/G)$, so this algebraic model is designed to help handle the cases where the action is not free.

To make closer contact with our work on superconnections, we can describe the Cartan algebra using a vector field on an odd space.

Proposition 6. *Let V be a vector space with inner product, together with an action of a Lie group G (not necessarily linear). Let a metric on $\mathfrak{g}$ be given. Let $\{v_i\}$ be a basis of V and let $\{v^i\}$ be the dual basis. Define $CV = \mathfrak{g} \times \Pi TV$. Let a vector field Q on CV be given by*

$$Q\begin{pmatrix}\phi_i\\ v_j\\ \lambda_k\end{pmatrix} = \begin{pmatrix}0\\ \lambda_j\\ -L(\phi_i)v_k\end{pmatrix}_{(\phi_i, w_j, \lambda_k)}$$

where ϕ_i is an element of a basis for $\mathfrak{g}$ and λ_j is the basis element of $\Pi V \cong \Pi T_{v_j}V$ corresponding to v_j. Then on the space $S^*(\mathfrak{g}) \otimes \Omega^*(V)$, Q induces the action of the Cartan differential.

Proof. We have already discussed in Lemma 1 how $C^\infty(\Pi TV) \cong \Omega^*(V)$ and so taking the space of polynomial functions on $\mathfrak{g}$, we have $S^*(\mathfrak{g}^*) \otimes \Omega^*(V) \subset C^\infty(\mathfrak{g} \times \Pi TV)$. Q induces an action on this space by differentiation, and so to complete the proof we compute this induced action. Let us denote the superspace analogue of a differential form ω by $\hat{\omega}$. So if

$$\omega = \phi^\alpha \cdot \sum_{i_1 < \cdots < i_k} a_{i_1\cdots i_k}(v) dv^{i_1} \wedge \cdots \wedge dv^{i_n}$$

is an element of the Cartan algebra, then the corresponding function on $\mathfrak{g} \times \Pi TV$ is

$$\hat{\omega} = \phi^\alpha \cdot \sum_{i_1 < \cdots < i_k} a_{i_1\cdots i_k}(v) \lambda_{i_1} \cdots \lambda_{i_n}.$$

Let us compute the action of Q on $\hat{\omega}$. We have

$$\begin{aligned} Q\hat{\omega} = & \sum a_{i_1\cdots i_k}(-1)^{\gamma+1}\lambda_{i_1}\cdots\lambda_{i_{\gamma-1}}(-L(\phi_\alpha)v^\gamma)\lambda_{i_{\gamma+1}}\cdots\lambda_{i_k} \\ & + \phi^\alpha \cdot \sum \frac{\partial a_{i_1\cdots i_k}}{\partial v_\gamma}\lambda_\gamma\lambda_{i_1}\cdots\lambda_{i_n}. \end{aligned} \tag{85}$$

Note that we computed only for a generator of $S^*(\mathfrak{g}^*)$ but this suffices as both Q and the Cartan differential are extended in the same way (the Leibnitz rule) to more general elements. Under the correspondence with differential forms, $\lambda_i \to dv^i$. Also, by the Cartan formula, $L = d \circ i + i \circ d$ and so

$$-L(\phi_\alpha)v^\gamma = -i(\phi_\alpha)dv^\gamma.$$

With these replacements, we can easily see from (85) that $Q\hat{\omega} \to (d - i(\phi_\alpha))\omega$, completing the proof.

We deliberately avoided using linearity of the G-action above, in order to be a little more general. However, in the case of a linear action (a representation) the action of ϕ on an element of V is just the vector field $\phi(v)_v$.

Let us examine the space $L'(A) = \mathfrak{g} \times \Pi TV \times \Pi V^* \times V^*$. (The number of components differs from the definition of $L(\mathcal{A})$ above, hence the primed notation; the ΠTV part of $L'(A)$ should be considered "extra" and we will see at the end of the section why its presence is not needed to discuss the Donaldson invariants.) We will install on this space the vector field

$$Q\begin{pmatrix}\phi\\ v_j\\ \lambda_k\\ \psi^l\\ E^m\end{pmatrix}=\begin{pmatrix}0\\ \lambda_j\\ -\phi(v_k)\\ E^l\\ -\phi(\psi^m)\end{pmatrix}_{(\phi_i,v_j,\lambda_k,\psi^l,E^m)}. \tag{86}$$

We will construct a special function Ψ_L on this space as follows. We will then integrate the exponential of the function $Q\Psi_L$ over the E and ψ variables, and we shall point out that the remaining function is Mathai and Quillen's element of the Cartan algebra. Let us proceed. We set

$$\Psi_L(\phi,v,\lambda,\psi,E)=-i\psi(v)-\langle\psi,E\rangle_{V^*} \tag{87}$$

and then obtain

$$\Phi_L=Q\Psi_L=-iE(v)+i\psi(\lambda)-\langle E,E\rangle_{V^*}-\langle\psi,\phi(\psi)\rangle_{V^*} \tag{88}$$

(remembering to pick up a minus sign when we move Q past the ψ in the second term). Now we compute

$$U(\phi,v,\lambda)=\frac{1}{(2\pi)^{2\dim V}}\int_{V^*\times\Pi V^*}d\mathrm{vol}(E)\,d\mathrm{vol}(\psi)e^{\Phi_L}.$$

We now use the fact that Gaussian integration gives

$$\int_V d\mathrm{vol}\,e^{-\langle v,Av\rangle+\langle B,v\rangle}=\int_V d\mathrm{vol}\,e^{-\frac{1}{4}\left\langle B,A^{-1}B\right\rangle}e^{-\left\langle v-A^{-1}v,A(v-A^{-1}v)\right\rangle} \tag{89}$$

$$=e^{-\frac{1}{4}\left\langle B,A^{-1}B\right\rangle}\left(\frac{\pi}{\det A}\right)^{\frac{\dim V}{2}} \tag{90}$$

and obtain

$$U(\phi,v,\lambda)=\frac{1}{(4\pi)^{\dim V}}\int_{\Pi V^*}d\mathrm{vol}(\psi)\,e^{-\frac{1}{4}\langle v,v\rangle_V+i\psi(\lambda)-\langle\psi,\phi(\psi)\rangle_{V^*}} \tag{91}$$

[[**Note: Am I off by a minus sign on that third term in the exponential?**]]

To obtain an element of the Cartan algebra, we use the fact that Φ_L is a linear function of λ and so can be identified with a 1-form on V. If we now choose a connection a on A then we can construct the map $w: S^*(\mathfrak{g}^*)\otimes\Omega^*(V)\to\Omega^*(P\times V)$ by sending $\phi\to F_a$. This is the *Weil homomorphism*. It is an equivariant map because F_a transforms in the adjoint representation, and so descends to a map on G-invariant forms

$$\overline{w}:\left(S^*(\mathfrak{g}^*)\otimes\Omega^*(V)\right)^G\to\Omega^*(E).$$

The form $w(U)$ is almost a representative of the Thom class $\mathcal{T}$. In fact, $w(U)$ fails to be fully horizontal, which is required for it to be the lift of a form from E. Use the connection to decompose $T_{(x,v)}(A\times V)$ into $T_{x,\mathrm{vert}}\oplus T_{x,\mathrm{horiz}}\oplus V$ and to form the projection p onto $T_{x,\mathrm{horiz}}\oplus V$. Then we denote

$$w(U)_{\mathrm{horiz}}(X_1,\ldots,X_n)=w(U)(p(X_1),\ldots,p(X_n)).$$

This horizontal element does in fact descend to $\Omega^*(E)$. The fact that U was already G-invariant but not horizontal perhaps indicates that the construction is better off living in the *Weil model* of equivariant cohomology, but we follow standard practice and take a horizontal projection. Note that we never had to use $\overline{w}$, since we projected horizontally after applying w.

Theorem 5 (Mathai-Quillen [8]). *$w(U)_{\text{horiz}}$ is a representative for the Thom class $\mathcal{T}$.*

Heuristically, we see that the Berezinian integration over ψ will give us the Pfaffian of F_a, just as in the Gauss-Bonnet formula. The Gaussian in v and the constants ensure the integral over a fiber is 1. We obtain a top-dimensional form in the V direction because of the $i\psi(\lambda)$ term and the correspondence between functions of λ and forms.

Now we get back to the point about v and λ. The Thom class can be pulled back by a section $s : B \to E$ to produce the Euler class of E. This is an n-form on B (recall that $\dim V = n$), unless $n > \dim B$ in which case the Euler class is defined to be zero. We can pull the Mathai-Quillen form back to B by s to obtain

$$s^* U(\phi, v, \lambda) = \frac{1}{(4\pi)^{\dim V}} \int_{\Pi V^*} d\mathrm{vol}(\psi)\, e^{-\frac{1}{4}\langle s,s\rangle_V + i\psi(ds) - \langle\psi,\phi(\psi)\rangle_{V^*}} \tag{92}$$

where we simply replaced v by s and λ by ds to effect the pullback. This form represents the Euler class of E.

If we replace the section s by ts for a real parameter t and rescale ψ by $\psi \to \frac{1}{t}\psi$ then this expression becomes

$$s^* U(\phi, v, \lambda) = \frac{1}{(4\pi)^{\dim V}} \int_{\Pi V^*} d\mathrm{vol}(\psi)\, e^{-\frac{1}{4t^2}\langle s,s\rangle_V + i\psi(ds) - t^2\langle\psi,\phi(\psi)\rangle_{V^*}}. \tag{93}$$

This version allows us to consider the two limits $t \to 0$ and $t \to \infty$ that link the Gauss-Bonnet formula with a formula that involves local data at the zero set of s.

We will now see that our action on $S\mathcal{A}$ has an Euler class part that is the pullback by $F_A^+ : \mathcal{B} \to \mathcal{A} \times_{\mathcal{G}} \Omega_+^2(X; \mathrm{ad}\, P)$. Under the identification of Ω_2^+ with V^* and $\mathrm{Lie}\, \mathcal{G}$ with $\mathfrak{g}$, we identify E_2 with E, ψ_2 with ψ and ϕ with $i\phi$. So $L(\mathcal{A})$ is an analogous space to the one we were just considering. Now let us compare (93) with the action (78). If we choose the section s to be the map

$$\frac{1}{2} F_A^+ : \mathcal{A} \to \mathcal{A} \times \Omega_2^+(X; \mathrm{ad}\, P)$$

then ds will take ψ_1 to $d_A^+\psi_1$. Also, the action of $\mathrm{Lie}\, \mathcal{G}$ on Ω_2^+ is $\psi_2 \mapsto [\phi, \psi_2]$ and so the analogy with (93) gives

$$(tF_A^+)^* U = \int_{\Pi(\Omega_2^+(X;\mathrm{ad}\, P))} d\psi_2\, e^{-\frac{t^2}{4}|F_A^+|^2 + i\left\langle \psi_2, d_A^+\psi_1\right\rangle - \frac{1}{t^2}\langle\psi_2,[\phi,\psi_2]\rangle} \tag{94}$$

which is part of (78) with t replaced by $1/g$. So not only does part of the twisted action represent the Poincaré dual of the ASD moduli space, but the physical coupling constant plays an analogous role to the scale of the section F_A^+! We will dwell on this after discussing the projection form.

3.1.2. *The projection form.* The construction of the projection form follows in the same vein as the Mathai-Quillen construction above. We will introduce a space together with a vector field. We will differentiate a function to obtain another function that we then exponentiate as before, and integrate over some of the variables. This form will interact with the Mathai-Quillen form, as they will share certain variables. In fact, the projection form will enforce both of the modifications we made to U above: it will kill off all but the horizontal part of U and it will produce a δ-function that is supported where ϕ is equal to the curvature of the chosen connection (the construction of the projection form involves a choice of connection).

Let $A \to B$ be a principal bundle with group G. Let $\langle , \rangle_{\mathfrak{g}}$ be an bi-invariant inner product on $\mathfrak{g}$. Suppose we are given a G-equivariant metric g on A. Then g induces a natural connection on B by using the metric to take the horizontal distribution to be the orthogonal complement of the vertical one. The action of the group G on A induces a map from $\mathfrak{g}$ to the vertical tangent spaces of A. We call this map C, so we have $C : \mathfrak{g} \to T_a A$. Using the metric, we can define the adjoint C^* of C, $C^* : T_a A \to \mathfrak{g}$. In other words, C^* is a Lie algebra-valued 1-form on A.

We will examine the space

$$P(A) = \mathfrak{g} \times A \times \Pi\Omega^1(X; \operatorname{ad} P) \times \mathfrak{g} \times \Pi\mathfrak{g}$$

with the vector field

$$Q' \begin{pmatrix} \phi \\ a \\ \psi' \\ \overline{\phi} \\ \overline{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ \psi' \\ -L(\phi)a \\ \overline{\psi} \\ -L(\phi)\overline{\phi} \end{pmatrix}_{(\phi,a,\psi',\overline{\phi},\overline{\psi})} . \tag{95}$$

We begin with the element

$$\Psi_P(\phi, a, \psi', \overline{\phi}, \overline{\psi}) = i \left\langle \overline{\phi}, C^* \right\rangle_{\mathfrak{g}} . \tag{96}$$

We won't be able to compute $Q'\Psi$ with just (95), though. This is because C^* is a genuine 1-form, not an odd object. However, by Proposition 6 we can work with the Cartan differential which operates by $(d - i(\phi))C^* = dC^* - C^*(C\phi)$. And so we have

$$\Phi_P = Q'\Psi_P = i \left\langle \overline{\psi}, C^* \right\rangle_{\mathfrak{g}} + i \left\langle \overline{\phi}, dC^* \right\rangle_{\mathfrak{g}} - i \left\langle \overline{\phi}, C^*(C\phi) \right\rangle_{\mathfrak{g}} . \tag{97}$$

The projection form is then

(98)

$$U'(\phi, a, \psi') = \frac{1}{(2\pi i)^{\dim G}} \int_{\mathfrak{g}\times\Pi\mathfrak{g}} \operatorname{dvol}(\overline{\phi}) \operatorname{dvol}(\overline{\psi})\, e^{i\left\langle \overline{\psi}, C^* \right\rangle_{\mathfrak{g}} + i\left\langle \overline{\phi}, dC^* \right\rangle_{\mathfrak{g}} - i\left\langle \overline{\phi}, C^*(C\phi) \right\rangle_{\mathfrak{g}}}$$

which lies in $S^*(\mathfrak{g}^*) \otimes \Omega^*(A)$.

Proposition 7. *Let ω be an element of $S^*(\mathfrak{g}^*) \otimes \Omega^*(V)$ for some vector space V with G-action. Then $\int_{\mathfrak{g}} d\phi\, \omega \wedge U' = w(\omega)_{\text{horiz}}$*

Proof. For a more detailed treatment of U', see Section 14.3.3 of [12]. The Berezinian integral over $\overline{\psi}$ picks off the piece of maximal degree in $\overline{\psi}$. If $\dim G = m$ then this yields an m-form built from the m-fold wedge product of C^*. Since C^* is a vertical 1-form (it vanishes on horizontal vectors in TA) this wedge product is an m-form along strictly vertical directions. In fact, it is an element of $\wedge^{\text{top}}(T^*_{\text{vert}}A)$. Any form on A with components along vertical directions is zero when wedged with such a fully vertical form, and so multiplying ω by U' picks off the horizontal part of U'.

Next, we note that the integral over $\overline{\phi}$ in U' gives the δ-function

$$\delta(dC^* - C^*(C\phi))$$

which is zero unless

$$\phi = (C^*C)^{-1} dC^*.$$

Lemma 6. $(C^*C)^{-1}dC^*$ *is the horizontal part of the curvature two-form* F *on* B *induced from the connection induced from the metric* g *on* A.

Proof. Let X, Y be two horizontal vector fields on A. The curvature two-form computes the opposite of the vertical part of the Lie bracket of X with Y. C^* vanishes on horizontal vectors and so

$$dC^*(X,Y) = XC^*(Y) - YC^*(X) - C^*([X,Y]) = -C^*([X,Y]).$$

Composing this with $(C^*C)^{-1}$ gives the curvature's vertical projection, which actually lives in $\mathfrak{g}$, but can be brought back to a vertical vector field on A by composing with one more C to give

$$C(C^*C)^{-1}C^*$$

which clarifies that this is a projection. This completes the proof of the Lemma which implies the Proposition.

Now let us examine the analogous objects over $\mathcal{A}$ and again compare with (78). We need to do a couple of computations first to get the right expressions. Here we follow [1]. First, the operator C^* is a $\operatorname{Lie}\mathcal{G}$-valued 1-form on $\mathcal{A}$, and a standard formula from gauge theory gives for a connection A

$$C^*(A)(\eta) = d_A^*(\eta)$$

where $\eta \in \Omega^1(X; \operatorname{ad} P)$. Similarly, the map C, which is a map from $\operatorname{Lie}\mathcal{G}$ into $T\mathcal{A}$ is given by

$$\phi \mapsto d_A\phi$$

for $\phi \in \Omega^0(X; \operatorname{ad} P) \cong \operatorname{Lie}\mathcal{G}$.

What about the map dC^*? This will be a 2-form on $\mathcal{A}$, and we can argue as follows. In finite dimensions, if a 1-form is given by $\sum f_i(x)dx^i$ then

$$d\left(\sum f_i(x)dx^i\right) = \sum \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i,$$

so what we are looking to do is differentiate d_A^* in the A direction. Let $\overline{\phi} \in \operatorname{Lie}\mathcal{G}$. Consider the expression $\left\langle \overline{\phi}, C^* \right\rangle$. On an element $Y_1 \in T\mathcal{A}$ this function gives

$$\left\langle \overline{\phi}, d_A^* Y_1 \right\rangle = \left\langle d_A\phi, Y_1 \right\rangle.$$

Differentiating this in the direction of the tangent vector Y_2 gives

$$\left\langle [Y_2, \overline{\phi}], Y_1 \right\rangle.$$

The invariance of the metric under the adjoint action of $\operatorname{Lie}\mathcal{G}$ implies

$$\left\langle [Y_2, \overline{\phi}], Y_1 \right\rangle = \left\langle \overline{\phi}, [Y_1, *Y_2] \right\rangle.$$

This is the 2-form $\left\langle \overline{\phi}, dc^* \right\rangle$ evaluated on Y_1 and Y_2, and now we wish to express this as a quadratic function on $\Pi T\mathcal{A}$. If $\psi_1 \in \Pi T\mathcal{A}$ then the corresponding function is just

$$\left\langle \overline{\phi}, [\psi_1, *\psi_1] \right\rangle. \tag{99}$$

And so the analog of the projection form is

$$U' = \int_{\operatorname{Lie}\mathcal{G} \times \Pi \operatorname{Lie}\mathcal{G}} D\overline{\phi}\, D\psi_0\, e^{i\langle \psi_0, d_A^*\psi_1\rangle + i\left\langle \overline{\phi}, [\psi_1, *\psi_1]\right\rangle - i\left\langle \overline{\phi}, d_A^* d_A \phi\right\rangle} \tag{100}$$

which forms another part of the action (78).

We have not discussed two of the terms in (78). Those are $-i|E_2|^2$ and $i\langle\phi, [\psi_0, \psi_0]\rangle$. These two terms are not used in the analogy with geometry that we have just constructed, but nor do they pose a problem. In fact, if one enforces the classical equations of motion for the auxiliary field E_2 one obtains

$$-i|E_2|^2 \to -i|[\phi, \overline{\phi}]|^2,$$

which we will not prove. The quantity

$$-i|[\phi, \overline{\phi}]|^2 + i\langle\phi, [\psi_0, \psi_0]\rangle$$

is in the image of a Cartan differential, just as the rest of the action was shown to be above. However, in this case passing to equivariant cohomology actually kills off these two terms, and so they are not important to our story. The $[\phi, \overline{\phi}]$ term is of crucial importance in studying the classical and quantum vacua of the physical theory on flat space.

3.2. **Path integrals.** Consider a path integral of the form

$$\int_{\text{fields}} e^{-S_q + \sum_i g_i S_{I,i}}$$

where the various $S_{I,i}$ are *interaction terms*, which just means they are each a cubic or higher order function on field space. S_q is a quadratic function on field space. Such an integral can be written as a formal power series in the g_i, the coupling coefficients. This series does not converge, and each term in the series diverges unless we renormalize. So what we're dealing with here is no better off mathematically than the Thom class idea is. However, the constant term (independent of all g_i) is computed by evaluating only the quadratic part S_q of the action, which is a Gaussian integral that can be rigorously defined using zeta function regularization of determinants and Pfaffians of infinite-dimensional operators. So, in the limit as the coupling parameters vanish, the quadratic part is the whole of the path integral. This is a *free theory*, which means it models a theory of particles that do not interact with each other.

Our theory has one coupling parameter, g and so to compute the free path integral we'd be taking the limit $g \to 0$. Furthermore, if an action has a moduli space of minima, one computes terms in the perturbation expansion by integrating over this moduli space and projecting the path integral onto the normal bundle of this space. That would be how we'd compute the Donaldson invariants as path integrals, too. The quadratic part of the action acting on the normal bundle to $\mathcal{M} \in \mathcal{B}$ has quadratic part consisting of the Laplacian $d_A^* d_A$ on even objects and d_A or d_A^* on fermions. When carefully computed, the resulting determinants and Pfaffians of these operators will cancel up to sign, the details of the sign depending on considerations involving the orientation of $\mathcal{M}$.

Something very special is happening here, though. The reciprocal of the coupling parameter plays the same role as the scale of the section F_A^+, as we saw when we compared (78) and (94). And so low coupling corresponds to taking the scale to infinity. The algorithm we use at low coupling to compute the path integral aligns exactly with the steepest descent computation one uses to show that the Mathai-Quillen form can be expressed in terms of local data on the zero set of the section.

Atiyah and Jeffrey made a related statement in [1]. They pointed out that the Mathai-Quillen construction could allow the definition of a *regularized* Euler class. Even if the base space and vector bundle are infinite-dimensional, if we choose a section that has a finite-dimensional zero set M, then we can define the regularized Euler class in terms of M. This is similar to saying that one can define a path integral at zero coupling. Perhaps results in either the path integral direction or the Thom class direction can inform the other.

3.3. **The quantum observables $\mathcal{O}^{(i)}$.** If we want to do intersection theory on the zero set of F_A^+ then we are all ready, because we have an Euler class to wedge forms against, which is equivalent to integrating the forms over the Poincaré dual of the Euler class, which is of course exactly the zero set in question. To do this within the field theory framework, we need superspace representatives of the forms used in Donaldson theory. These involve slant products with a Pontrjagin class, and so we will find a field theory representative for this construction.

The following is a standard construction from Donaldson theory. See [4] for more details. Let $P \to X$ be a principal $SU(2)$ bundle over an even riemannian four-manifold X. We denote by $\mathcal{A}$ the space of connections and by $\mathcal{G}$ the group of gauge transformations. Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be the respective complements of the set of reducible connections. Let $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$. The bundle

$$\mathbb{P} = \mathcal{A}^* \times_{\mathcal{G}} P \to \mathcal{B}^* \times X$$

is a principal $SO(3)$ bundle, and so has a first Pontrjagin class $p_1(\mathbb{P})$. We define a connection on this bundle by using a metric as follows. Give $\mathcal{A} \times \{p\}$ the usual metric on $\mathcal{A}$ and give $\{A\} \times P$ a metric by using the metric on X for horizontal vectors, and the connection A together with a metric on $\operatorname{Lie}\mathcal{G}$ for vertical vectors. There is an associated connection given by taking the orthogonal complements of the vertical subspaces of $T(\mathcal{A}^* \times P)$. The curvature of this connection, $\mathcal{F}$, is given by the following formulas. Let τ_1 and τ_2 be horizontal tangents to $\mathcal{A}$ at A, and let X_1 and X_2 be horizontal tangents to P at p. Then one computes

$$\mathcal{F}_{2,0}(A,p)(\tau_1,\tau_2) = -\frac{1}{d_A^* d_A} d_A^*([\tau_1,\tau_2]) \tag{101}$$

$$\mathcal{F}_{1,1}(A,p)(\tau_1,X_1) = \tau_1(X_1) \tag{102}$$

$$\mathcal{F}_{0,2}(A,p)(X_1,X_2) = F_A(X_1,X_2). \tag{103}$$

The subscripted indices denote the bigrading in $H^*(\mathcal{B}^* \times X)$. The bracket $[\tau_1,\tau_2]$ is a bracket of two vector fields on $\mathcal{A}$, not the bracket as sections of $\operatorname{ad} P$.

To see how to create a superspace representation $\mathbb{F}$ of $\mathcal{F}$, we just need to come up with an equivariant representative of each of these three 2-forms. The generator ϕ of $S^*(\operatorname{Lie}\mathcal{G}^*)$ maps to the curvature $\mathcal{F}_{2,0}$ under the Weil map, and so properly interpreted, ϕ is $\mathbb{F}_{2,0}$. What the generator ϕ means written alone is *the identity function* on $\operatorname{Lie}\mathcal{G}$. This is the element

$$\phi \in \operatorname{Lie}\mathcal{G}^* \times \operatorname{Lie}\mathcal{G} \cong \operatorname{Lie}\mathcal{G}^* \times \Omega^0(X; \operatorname{ad} P).$$

Similarly the expression "ψ_1," when written in isolation, is an identity function, this time on $\Pi T\mathcal{A}$. That makes ψ_1 a vector-valued function on $\Pi T\mathcal{A}$. Under the correspondence with forms, this becomes a vector-valued 1-form corresponding to the identity function on tangent vectors, otherwise known as de Rham d. So, $\psi_1 \leftrightarrow d$, which can be evaluated on a pair (τ_1, X_1) as above to give

$$d : (\tau_1, X_1) \to (\tau_1, X_1)$$

which is the identity, yielding a tangent vector to the space of connections and a tangent vector to the manifold. And so the final evaluation of ψ_1 is

$$\tau_1(X_1)$$

giving the identification between ψ_1 and $\mathcal{F}_{1,1}$. F_A is already a field in our theory and so we obtain

$$\mathbb{F}_{2,0} = \phi \tag{104}$$

$$\mathbb{F}_{1,1} = \psi_1 \tag{105}$$

$$\mathbb{F}_{0,2} = F_A. \tag{106}$$

It is worthwhile to note that if we compute the action of the vector field induced by Q_1 on ϕ we obtain

$$\begin{aligned} \mathcal{Q}_1\phi\ &= \psi_1 \\ \mathcal{Q}_1\psi_1 &= F_A + \mathcal{Q}_0(F(d, D_1)) \end{aligned} \tag{107}$$

so that if we compute modulo $\mathcal{Q}_0$ (i.e. we work on the level of equivariant cohomology and not equivariant forms) then $\mathcal{Q}_1$ permutes bigraded pieces of $\mathbb{F}$, keeping total grading invariant.

The Pontrjagin class $p_1(\mathbb{P})$ is given by

$$p_1(\mathbb{P}) = \frac{1}{2}\mathrm{Tr}(\mathcal{F}\wedge\mathcal{F}),$$

and so we will examine the equivariant representative

$$\frac{1}{2}\mathrm{Tr}(\mathbb{F}\wedge\mathbb{F}).$$

To compute the slant product with an i-dimensional homology class, we just integrate a bigraded piece over a smooth representative. So Donaldson's μ-map is

$$\mu([\Sigma_i]) = p_1(\mathbb{P})/[\Sigma_i] = \int_{\Sigma_i}\left(\frac{1}{2}\mathrm{Tr}(\mathbb{F}\wedge\mathbb{F})\right)_{4-i,i}.$$

For the record we list the bigraded pieces

$$p_1(\mathbb{P})_{4,0} = \frac{1}{2}\mathrm{Tr}(\phi^2) \tag{108}$$

$$p_1(\mathbb{P})_{3,1} = \mathrm{Tr}(\phi\psi_1) \tag{109}$$

$$p_1(\mathbb{P})_{2,2} = \mathrm{Tr}\left(\frac{1}{2}\psi_1\wedge\psi_1 + \phi F_A\right) \tag{110}$$

$$p_1(\mathbb{P})_{1,3} = \mathrm{Tr}(\psi_1\wedge F_A) \tag{111}$$

$$p_1(\mathbb{P})_{0,4} = \frac{1}{2}\mathrm{Tr}(F_A\wedge F_A) \tag{112}$$

3.4. **The path integral formulation of the polynomial invariants.** We already know that the projection form part of the path integral can be evaluated formally and gives the Weil homomorphism from equivariant cohomology to usual cohomology, by mapping ϕ to the curvature of a connection. And so we may use supersymmetric equivariant representatives for p_1 in the construction of a path integral, knowing that they will be mapped to the real thing. Thus we can construct a completely physical analogue of Donaldson theory, by integrating local operators over a space of fields, against the exponential of an action that is also built from local fields. Let $\mathcal{O}^{(i)} = p_1(\mathbb{P})_{4-i,i}$ be the equivariant representatives and let D be the Donaldson polynomial on second homology. We have hopefully motivated the following claim.

Claim 1. *The gaussian approximation to the path integral*

$$\int_{S\mathcal{A}} \left(\prod_{k=1}^{N} \int_{\Sigma_{i_k}} \mathcal{O}^{(i_k)} \right) \exp(-S)$$

agrees with the Donaldson polynomial

$$D(\Sigma_{i_1}, \ldots, \Sigma_{i_N}) = \int_{\mathcal{M}} \mu([\Sigma_{i_1}]) \wedge \cdots \wedge \mu([\Sigma_{i_N}]).$$

4. Outlook: Witten's conjecture

This work is the beginning of a program. The goal is to prove Witten's conjecture [5]. The physical insight that allowed Witten to make this remarkable conjecture comes from his celebrated paper with Seiberg [6]. My hope is that one day a mathematical version of that paper may be created, and this work marks the humble beginnings of that project.

It may be possible to prove Witten's conjecture using nonabelian monopoles ([13], [14], [15]). However, a proof that parallels the physical proof would have the advantage that it may reveal a broader picture of math/physics interaction, and help us to understand why so much recent mathematics has grown out of physics. In particular, it may shed light on either the mathematical relevance or the mathematical underpinnings of the renormalization group.

However, for that we need more mathematics. We have tried to cast Donaldson theory in physical language, along the lines that Witten hinted at in 1988 [2]. That was the starting point for Seiberg and Witten's breakthrough work of 1994 [6], and so it needs to be the starting point for the mathematical proof as well. Hopefully after reading this paper this picture of Donaldson theory seems somewhat natural from a mathematical standpoint. From a physical standpoint, it is simply a supersymmetric gauge theory, and the Minkowski space version (the "physical" theory, as it's known, as opposed to the twisted theory, which is called "topological" or "cohomological") can be treated with all the machinery of modern physics. As of 1994, there wasn't enough physics to understand this theory any better than we have already done in this paper. However, Seiberg and Witten created new physics that solved the theory. To understand what that means, we need to discuss energy scales.

4.1. **Energy scale and the renormalization group.**

We have discussed the coupling parameter that appears in both the physical and twisted gauge theory actions. We talked about how the constant term in the coupling expansion (the perturbation series) is the appropriate Donaldson invariant. We did not address what controls the value of the coupling constant or whether it is simply arbitrary. In fact

Claim 2. *Let $\gamma_t = e^{-t}\gamma$ be a one-parameter family of metrics on E^4. Then as t blows up, the coupling parameter g responds by shrinking to zero. In other words, g is a function of t and we have*

$$\lim_{t\to\infty} g(t) = 0.$$

This follows from the *asymptotic freedom* of nonabelian gauge theories. The significance of the metric scaling to zero is in the fact that this is the regime of *high energy*. Using the speed of light and Planck's constant, meters and energy units can be converted back and forth, just as meters and seconds can be converted using just the speed of light. Small distances correspond to large energy units, and so shrinking the metric to zero examines the theory at high energy. In this paper we have therefore examined correlation functions at the high energy limit, since as the energy approaches infinity, the coupling goes all the way to zero, leaving only the Donaldson invariants. On the other hand, the perturbation expansion, which is borderline meaningless anyway, only stands a chance of converging if the coupling is small, and so physicists believe that asymptotically free theories at high energy are quite under their control to compute with.

Physicists, however, would have us make the following analogy. Take some fundamental model of the basic forces of nature, like string theory or the standard model. These are hugely complicated asymptotically free theories that describe the behavior of our universe at the highest of energies and the smallest of distance scales. They are "fundamental" in that sense — they are the underlying physics of everything. However, more often than not we are interested in more mundane matters like fluid flows or planetary mechanics. In these cases, we are working on rather larger distance scales, and quite tiny energy regimes. The laws of physics are surprisingly simpler in this sort of context, becoming things like Kepler's laws or the Navier-Stokes equations. These are not gauge theories or string theories with infinitely many degrees of freedom. They are finite-dimensional PDEs or algebraic equations that, while perhaps difficult to solve completely, are provincial and comparatively easy to understand. The conceptual force at work here is the *renormalization group*, first understood in this way by Wilson [16], [17]. The renormalization group is simply the group of energy scaling. It is the t parameter in the above theorem. While the underlying group is simple, its action on a theory is not. Somehow, as it flows from high energy to low, the complexities of the fundamental theory are suppressed, and only a few parameters and degrees of freedom survive at low energy. We are not going to try to understand the renormalization group and its workings, but instead we are just trying to paint the picture that is in the back of all physicists' minds. There is no overarching description of the renormalization group, no actual flow that can be applied to a theory to get answers at different scales. It is something that is ill-understood at best, though perhaps the route to addressing it and constructing a workable theory is via the sort of topological application we are discussing here!

Witten makes the analogy [18] between a fundamental, asymptotically free theory, and a differential equation. The former contains infinitesimally small distance information about a theory, as we said before, and so the analogy with information about a function's derivatives is very close. The ability to describe a theory at any energy scale is, then, analogous to solving the differential equation. Nonabelian gauge theories are therefore candidate "problems," and the challenge is to find their "solutions." Donaldson theory is asymptotically free because it is an $SU(2)$ gauge theory. Its correlation functions are the Donaldson polynomials, and the challenge in terms of renormalization is to compute these correlation functions at all energies, and *solve Donaldson theory*. Mathematics has no basis for even stating this challenging problem. Donaldson theory is not a question at all in its mathematical presentation. It is simply a construction that yields interesting information. But physics goes further, and places it at one end of the real numbers, near infinity, because that is where it fits in terms of energy scale. Of what picture is this the limit? No one has addressed this question. What has been addressed, though, is what lies at the other end of the line, at zero energy. The answer: Seiberg-Witten invariants.

Abelian gauge theories are the opposite of asymptotically free: the coupling parameter vanishes for *low* energy (the $t \to -\infty$ limit in the above theorem). These are called *infrared free* theories. Infrared free theories are, then, the candidates for the solutions to asymptotically free ones. They are well-behaved as perturbation series only near zero energy. Seiberg-Witten theory is a theory of a $U(1)$ gauge field coupled to a spinor, and so is an abelian gauge theory. As such it is infrared free, and is one of a host of candidates for solutions to Donaldson theory. In fact, Seiberg and Witten show that it is exactly the right solution.

4.2. **Overview of the physics proof.** To describe the physics proof of the conjecture would be too vast an undertaking for this paper, and so we merely sketch it, pointing out some of the deep issues that will need confronting. First of all, the result does not involve the twisted theory we have considered here, but rather the $N = 2$ Minkowski space version. However, Witten's conjecture clearly involves assuming that the flow to low energy commutes with the twisting operation, as one simply twists the low energy physical theory to obtain the Seiberg-Witten invariants. The good news then is that we have learned all the mathematical tools needed to twist the low energy physical theory in the preceding sections of this paper. The bad news is that it may not be possible to cast a workable parallel of the flow to low energy in twisted terms.

The central issue to sort out, however, is the concept of the *quantum vacuum.* In classical physics, the minima of the action form the *classical vacuum manifold.* The action principal states that a physical system will assume one of the configurations from this space. In quantum physics, a system can assume any of the states in the whole field space, but the path integral weighs minima of the action with much higher probability. To turn the crank of quantum field theory, one must select a classical vacuum and write the action in coordinates that perturb from this state. The *quantum* vacuum for this theory is then a state in the theory's Hilbert space that has no particles, or is invariant under the entire Poincaré group. This indirect definition makes it hard to quantify, especially because it is difficult or impossible to actually construct this Hilbert space! Certain details of the theory influence whether the quantum vacuum is unique or not. In the case of our $SU(2)$ theory, the classical vacuum manifold is the complex plane modulo the action of $z \mapsto z^2$, with the resulting cone singularity at the origin. It is believed that the quantum vacuum manifold is this same space, but with a different metric and other properties. This is a guess. There is currently no way to ascertain the validity of this, even physically. However, if one assumes a whole lot about physical theories and what sort of perturbations of classical vacua are permitted to arise from quantization, then it is the simplest guess. There are singularities on the quantum vacuum manifold as well, but there are two of them, and neither of them is at $z = 0$. The whole of the matter revolves around proving what this manifold is, what metric it has, and what the monodromies of the coupling parameter around the singularities are. Other details are important too, however. Seiberg and Witten claim that there are "BPS states" in each theory of the quantum manifold, and that the mass of these states is different in each theory. The two singularities are the two points where this mass vanishes. That is what is making the metric blow up at those points, they claim. A BPS state is by definition a vector in the Hilbert space that is annihilated by *half* of the eight supersymmetry operators (which act on the Hilbert space), so this object, which is some sort of soliton, should be quite tractable mathematically. Perhaps this is the way to access a mathematical theory about quantum vacua and the conjecture.

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