Discrete differential geometry in homotopy type theory

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Thank you!

Appendix: Conjectural dictionary

Conjectionary

Homotopy realization likely amounts to the shape operator.

> *[G]iven a “cell complex” presentation of a classical topological space, if we can convert it into both a specification for a HIT and a colimit decomposition of that space that issufficiently “cofibrant”, then $\int$ will preserve that colimit and take the space to the HIT.*
>
> — Mike Shulman, *Brouwer’s Fixed Point Theorem in Real-Cohesive HoTT*

Conjectionary

Homotopy realization (and/or $\int$) may provide the following relationships.

Classical octahedron	Homotopy realization, e.g. $\mathbb{O}$
Classical combinatorial manifold M	$\int M$
Derivative	ap
Leibniz rule for $f, g : M \to \mathbb{R}$	Given H-space $(A, *)$, $f, g : X \to A$, $p : x =_X y$ then $\mathrm{ap}(f * g)(p) = \mathrm{ap}(f)(p) * (ga) \cdot (fb) * \mathrm{ap}(g)(p)$. Because in $A \times A$, $(fp, gp) = (fp, \mathrm{refl}) \cdot (\mathrm{refl}, gp)$
The sphere is not flat, as a **pointwise** statement	Nontrivial flatness on each face

Conjectionary

Connections being "affine", and not (quite) 1-forms	$T_{ji} : T_i = T_j$ being a torsor and not (quite) a group
Space of connections for a given P is contractible.	Two extensions to $\mathbb{O}_1$...

Conjectionary

Maurer-Cartan form.	Hmm, consider the trivial connection on $M \times G$ or $G \to *$.
Gauge transformations acting on connections and maybe functions (YM) of connections.	
The based gauge group acts freely on connections.	

Conjectionary

Characteristic classes.	$BS^1 \to B^n\mathbb{Z}$.
Chern-Weil theory.	$\mathbb{O} \xrightarrow{T} BS^1 \to B^n\mathbb{Z}$.
Hopf fibration.	$\mathbb{O} \xrightarrow{?} \mathsf{EM}(\mathbb{Z}, 1)$.
Zeros of X = Poincare dual of the Euler class.	