Abstract

We identify connections, curvature, and gauge transformations within the structures of homotopy type theory. Whereas most classical treatments of these structures rely entirely on infinitesimal definitions, there is an equivalent discrete story of which the infinitesimal version is a limit, analogous to the relationship between smooth paths and tangent vectors, or between de Rham and Čech cohomology. We will show how to identify the elements of discrete gauge theory, provide some evidence that this is what we have found, and use it to prove some results from the 20th century mathematics of gauge theory that depend only on homotopy types.

1 Introduction

Following David Jaz Myers in [1], we define a cover as follows:

Definition 1. *A map $\pi : E \to B$ is a* cover *if it is $\int_1$-étale and its fibers are sets.*

Recall that π being $\int_1$-étale means that the naturality square

$$\begin{tikzcd} E \arrow[r, "(-)^{\int_1}"] \arrow[d, "\pi"'] & \int_1 E \arrow[d, "\int_1 \pi"] \\ B \arrow[r, "(-)^{\int_1}"'] & \int_1 B \end{tikzcd}$$

is a pullback, which means among other things that each corresponding fiber of the vertical maps is equivalent.

As David proves in [1], the type of $\int_1$-étale maps into B is equivalent to the function type $\int_1 B \to \mathsf{Type}_{\int_1}$. Since a cover has the further condition that the fibers are sets this implies

Lemma 1. *The type of covers over B is equivalent to $\int_1 B \to \mathsf{Type}_{\int_0}$.*

In homotopy type theory we pass easily between the vertical picture (maps into B) and the horizontal picture (classifying maps of B into some type of fibers). But when we unpack this a little bit we find important classical stories. If we equip B with a basepoint $*_B$ then a map $f : \int_1 B \cdot\!\!\to \mathsf{Type}_{\int_0}$ is an action of the group $\int_1 B$ on the set $f(*_B)$, which is the fiber over $*_B$.

Let us move up a dimension:

Definition 2. *A map $\pi : E \to B$ is a* 2-cover *if it is $\int_2$-étale and its fibers are groupoids.*

Lemma 2. *The type of 2-covers over B is equivalent to $\int_2 B \to \mathsf{Type}_{\int_1}$. Further, since $\mathsf{Type}_{\int_1}$ is 2-truncated, the type of 2-covers of B is equivalent to $\int B \to \mathsf{Type}_{\int_1}$.*

This is the type we will examine.

Let S^1 be the homotopical circle. This is a 1-type and its loop space is $\mathbb{Z}$. We can deloop S^1 by forming its type of torsors. This is equivalent to $\mathrm{BAut}_1 S^1 \stackrel{\text{def}}{=} \sum_{(X:\mathsf{Type})} ||X = S^1||_0$. $\mathrm{BAut}_1(S^1)$ is a 2-type. (see [2]).

Since $S^1 : \mathsf{Type}_{\int_1}$ we see that $\mathrm{BAut}_1(S^1)$ is a subtype of $\mathsf{Type}_{\int_1}$. Which is to say, among the type of 2-covers of B are those whose fibers are all S^1-torsors.

If we believe in the shape operator, then we can design our own discrete types that implement known combinatorial versions of spaces, and discard the original smooth and continuous spaces completely. Mike Shulman proves in [3] that the shapes of the smooth circle and sphere are S^1 and S^2 respectively.

One compelling candidate for a general type of combinatorial spaces are *higher polytopes*. Polytopes are simply posets satisfying some extra conditions that make them a generalization of polyhedrons to arbitrary dimension. The poset records the information about the set of vertices, edges, faces, and higher faces and their containment. An explicit grading is assigned to each element of the poset to specify its dimension. By using the higher-dimensional constructors of higher inductive types, we can import a polytope into homotopy type theory as a discrete type that intuitively captures the homotopy type of any manifolds we are interested in!

Fix a map $f : B \cdot\!\!\to \mathrm{BAut}_1(S^1)$ for some discrete type B with basepoint $*_B$. Then $f(*_B)$ is some S^1-torsor, and the pointedness of f means we also have a path $*_f : f(*_B) = *_{\mathrm{BAut}_1(S^1)}$. Loops $p, q : *_B = *_B$ are mapped by $\mathsf{ap}(f)$ to $\mathsf{ap}(f)(p)$ and $\mathsf{ap}(f)(q)$. Let's drop the "ap" and just use the notation f, viewing it as a higher functor. Then $f(p)$ and $f(q)$ are automorphisms of the torsor $f(*_B)$, i.e. elements of the loop space $\Omega\mathrm{BAut}_1(S^1) = S^1$ (up to conjugation by $*_f$). Suppose now there is a 2-path $H : p = q$. Then $f(H) : f(p) = f(q)$ is a homotopy between these two automorphisms. That is, a path in $\Omega\mathrm{BAut}_1(S^1) = S^1$. So the fact that S^1 is a 1-type means we have somewhere nontrivial to map H. If instead of S^1 we had started with a discrete 0-type such as $\mathbb{Z}$ as our group, then $f(H)$ would be a proposition.

Bibliography

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[3] M. Shulman, "Brouwer's fixed-point theorem in real-cohesive homotopy type theory," 2017.