DRAFT: Discrete differential geometry in homotopy type theory

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Motivation

To use HoTT to study connections and explain their applicability to algebraic topology, via

- the Gauss-Bonnet theorem
- its vast generalization, Chern-Weil theory

Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

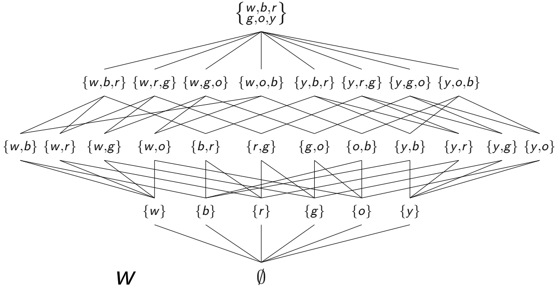
HoTT background

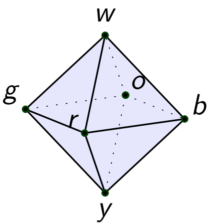
1. Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-) Symmetry. `https://github.com/UniMath/SymmetryBook`.
2. Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023) Central H-spaces and banded types. arXiv:2301.02636
3. Scoccola, L. (2020) Nilpotent types and fracture squares in homotopy type theory, MSCS 30(5). arXiv:1903.03245

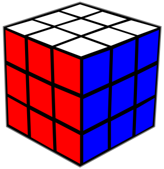
Discrete manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization functor via higher inductive types (pushouts)

Simplicial complexes







A Hasse diagram of a simplicial complex (vertices named for the colors on a Hungarian Cube)

Higher realization

Form a pushout of edges to create a 1-type.

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \arrow[dl, phantom, "\Uparrow h_1 \quad \lrcorner"] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1
\end{tikzcd}
$$

Higher realization

Definition

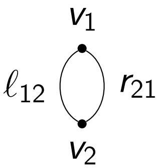
An n-**gon** $\mathbb{C}(n)$ is the realization of a complex $C(n)$:

$$C(n)_0 = \{v_1, \ldots, v_n\}$$

$$C(n)_1 = \{e_1 = \{v_1, v_2\}, \ldots, e_{n-1} = \{v_{n-1}, v_n\}, e_n = \{v_n, v_0\}\}$$

Toss in the non-complexes

$$\mathbb{C}(1) \overset{\text{def}}{=} S^1, \quad \mathbb{C}(2) \overset{\text{def}}{=}$$



Higher realization

Lemma

$\mathbb{C}(2) \simeq \mathbb{C}(1)$ *and in general* $\mathbb{C}(n) \simeq \mathbb{C}(n-1)$.

Corollary

All n-gons are equivalent to S^1.

Higher realization

Then push out maps from a 1-type triangle to from a 2-dim type.

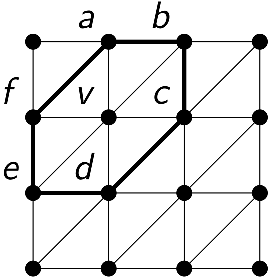
$$
\begin{tikzcd}
 & M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[r, phantom, shift right=6, "\overset{h_2}{\nearrow}"] & M_2 \arrow[d, "*_{\mathbb{M}_2}"] \\
\mathbb{M}_0 \arrow[r] \arrow[r, phantom, shift right=6, "\overset{h_1}{\searrow}"] & \mathbb{M}_1 \arrow[r] \arrow[ur, phantom, very near end, "\lrcorner"] & \mathbb{M}_2 \\
M_1 \times \partial\Delta^1 \arrow[u, "\mathbb{A}_0"] \arrow[r, "\mathrm{pr}_1"'] & M_1 \arrow[u, "*_{\mathbb{M}_1}"'] \arrow[ul, phantom, very near end, "\ulcorner"] &
\end{tikzcd}
$$

Higher realization

$*_{\mathbb{M}_1}, *_{\mathbb{M}_2}$ provide hubs. h_1, h_2 provide spokes.

$$\begin{tikzcd}
 & M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_1"'] \arrow[dr, phantom, "h_2 \Rightarrow"] & M_2 \arrow[d, "*_{\mathbb{M}_2}"] \\
\mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] & \mathbb{M}_2 \\
M_1 \times \partial\Delta^1 \arrow[u, "\mathbb{A}_0"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "h_1 \Rightarrow"] & M_1 \arrow[u, "*_{\mathbb{M}_1}"'] &
\end{tikzcd}$$

Higher realization



The **link** of a vertex v in an n-complex is the $(n-1)$-subcomplex of faces not containing v but whose union with v is a face.

This will be our model of the tangent space.

Rotation

Let $R : [abcd] \to [abcd]$ send $a \mapsto b, b \mapsto c, c \mapsto d, d \mapsto a$.

Extend R to edges.

Lemma

$[\![R]\!] : [\![abcd]\!] \to [\![abcd]\!]$ is homotopic to the identity, i.e. we have $\prod_{x : [\![abcd]\!]} x = [\![R]\!](x)$.

Proof.

Use edges. $\square$

Definition

Let G be a group with identity element e. A G-**set** is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If we have

$$\mathsf{is\_torsor}(X, \phi) \stackrel{\text{def}}{=} ||X||_{-1} \times \prod_{x:X} \mathsf{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

we say (X, ϕ) is a G-**torsor**. Denote the type of G-torsors by BG.

Lemma

Point BG at G_{reg}, the G-torsor G acting on itself on the right. Then $\Omega_{G_{\mathrm{reg}}} BG \simeq G$, so BG is a $\mathrm{K}(G, 1)$.

Definition

$\mathrm{EM}(G, n) \stackrel{\text{def}}{=} \mathsf{BAut}(\mathrm{K}(G, n)) \stackrel{\text{def}}{=} \sum_{Y:\mathcal{U}} \|Y \simeq \mathrm{K}(G, n)\|_{-1}$

Definition

A $\mathrm{K}(G, n)$-**bundle** on a type M is a map $f : M \to \mathrm{EM}(G, n)$.

We further assume f factors through $\mathrm{K}(G, n+1)$ and so is principal.

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a cellular type and all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"'] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"] \\
 & & \mathcal{U} & &
\end{tikzcd}$$

- The map f_k is a k-**bundle** on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-**connection on the** $(k-1)$-**bundle** f_{k-1}.

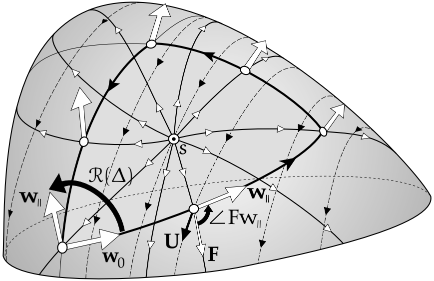
Definition

If $\mathbb{M}$ is the realization of a simplicial complex and we have

$$
\begin{tikzcd}
M_k \times \partial\Delta^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[r, phantom, bend right, "h_k \, \Nearrow"'] & M_k \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r, "\imath_{k-1}" {name=I}] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \arrow[dl, phantom, "\Swarrow"] \\
& \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{m_k\} \times \partial\Delta^k \arrow[r, "!"] \arrow[d, "\mathbb{A}_{k-1}"'] & \mathbf{1} \arrow[d, "*_{\mathbb{M}_k}"] \arrow[dl, Rightarrow, "\flat_k"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}
$$

then we say the filler $\flat_k$ is called a **flatness structure for the face** m_k, and its ending path is called **curvature at the face** m_k.

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F\mathbf{w}_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: from Tristan Needham, *Visual Differential Geometry and Forms*

- The classical proof is discrete-flavored.
- "$\angle F w_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.