Discrete differential geometry in homotopy type theory

Greg Langmead

June 12, 2024

Abstract

Homotopy type theory captures all the major concepts of differential geometry including forms, connections, curvature, and gauge theory. We show this by focusing on combinatorial manifolds, which are discrete in the sense of real cohesion[1], and drawing inspiration from the similarly young field of discrete differential geometry.

> "It is always ourselves we work on, whether we realize it or not. There is no other work to be done in the world." — Stephen Talbott, *The Future Does Not Compute*[2]

1   TODO

After meeting with SA,MA 5/1/2024

- Focus on one audience: HoTT folks. Don't intermix as much classic geometry.

- Section 1. Make explicit the triangle of DG, DDG, HoTT.

- Section 2. Sphere example and combinatorial manifolds, torus example.

- Section 3. Torsors and H-spaces

- Poincaré-Hopf and/or Gauss-Bonnet, even just for some examples. The curvature will be given by loops around a circle via $\Omega^2 f$ and the vector field will be integrated to an automorphism $M \to M$ and some integer from there.

$$
\begin{array}{ccc}
X : \prod\limits_{a:M} f(a) & \stackrel{\text{def}}{=} \sum\limits_{a:M} f(a) \xrightarrow{\ Ef\ } & ES^1 \\
& \downarrow{\scriptstyle \text{pr}_1}\ \ \diagup{\scriptstyle \exp} & \downarrow{\scriptstyle \text{pr}_1} \\
& M \xrightarrow{\ f\ } & BS^1 \\
& \downarrow{\scriptstyle \Omega^2} & \downarrow{\scriptstyle \Omega^2} \\
& \Omega^2 M \xrightarrow{\ \Omega^2 f\ } & \Omega^2 BS^1
\end{array}
$$

1

- Vector fields are sections of the disk bundle, which is the combinatorial neighborhood. We can define the circle bundle with our map to $\mathrm{BAut}_1(S^1)$ and add the base point to form the disk. Note that the requirement that every point have a link that is a simplicial sphere is called "combinatorial manifold" or "combinatorial triangulation".

- When is a vector field an automorphism?

- the disk bundle has a canonical section given by choosing the center point in each fiber. It has two other sections, one for each orientation of the circle.

- At a fixed point of a vector field therefore, we can compare the vector field to either of the canonical sections. Use this to map to S^1 or C_4 or a path in one of those. This will have a winding number which is our index.

- Prove that the total index is independent of a vector field that isn't the identity.

- Prove Poincaré-Hopf

2 Introduction: Discrete differential geometry

The observation that sparks the following discussion is this: if we can manage to reformulate differential geometry in discrete terms (i.e. finite, without infinitesimals) then we may also be able to construct it synthetically in homotopy type theory. Furthermore, if we do capture geometry in HoTT then there's a chance that it can become clearer and smaller. We would then have new tools, a new audience, and a new program to (re)explore geometry, gauge theory, low dimensional topology, and mathematical physics.

Applied mathematicians and computer scientists have been developing discrete differential geometry (DDG) for many years. The 2003 Ph.D. thesis of Anil Hirani [3] (see also the multi-author follow-up [4]) defines finite versions of vector fields, differential forms, the wedge product, the Hodge star, and several differential operators (exterior derivative, div, grad, curl, Laplace-Beltrami, Lie derivative). Hirani and others cite Whitney's 1957 book *Geometric Integration Theory*[5] which develops a theory of cochains by integrating smooth forms over chains. In 2004 Melvil Leok, Jerrold Marsden, and Alan Weinstein [6] defined discrete connections on principal bundles. This is probably the work most spiritually similar to this paper. The motivation for the above constructions was applied mathematics: modeling the differential equations of mechanics and fluid mechanics with the so-called "finite element" methods. The theory has been adopted and extended by the computer graphics community as well (see Keenan Crane's course notes [7] for a gentle survey).

The applied category theory community has begun to develop category theoretic foundations and software libraries to increase the reusability and compositionality of finite element methods in science and engineering problems. See for example recent work to bring discrete exterior calculus into the AlgebraicJulia library [8] [9].

For these classically-minded applied mathematicians DDG is defined on combinatorial manifolds such as simplicial complexes or polytopes, of any finite dimension. The 0-cells play the role of points, the 1-cells are path segments, and so on. They define n-forms as functions on the n-dimensional faces of the manifold into the real numbers, which is then extended by linearity to arbitrary n-

chains. Exterior differentiation is defined by Stokes theorem (which is no longer a theorem in this setting), by which we mean the following.

Definition 1. *(Exterior derivative in DDG.) Let ω be an $n-1$-form on a combinatorial manifold M, and let Ω be an n-face of M. Let ∂ be the boundary operator on faces. The exterior derivative d is defined by*

$$d\omega(\Omega) = \omega(\partial\Omega).$$

At this point a study of DDG would move into chain complexes, where the forms of different dimensions are combined through the grading. Grassman algebras would be introduced, to convey the dependence of forms on orientation. The Leibniz rule (product rule) would be explored. Defining connections and curvature require constructing a codomain that is group-like and then creating more definitions about transport and holonomy. At that point major theorems like Gauss-Bonnet and Chern-Weil would be available to prove.

We will take a different path. We will not define forms, complexes, or Grassman algebras at all. We will view ordinary HoTT functions out of a type as *synthetic discrete differential forms*. The codomain will be a central H-space, which combines the features of the real numbers (in that functions can be pointwise multiplied) and classifying spaces of groups (so that maps into the H-space can classify bundles). We will then merely *observe* the emergence of various aspects of geometry.

We won't be able to answer every question, so eventually we will stop and point to future directions.

3   Examples

3.1   The tangent bundle of the 2-sphere

We will define some higher inductive types to serve as domain and codomain of these motivating examples.

First we need a square, which will be a stand-in for a circle that can support a notion of a quarter-rotation.

Definition 2. *The higher inductive type C_4 (where C stands for "circle").*

$$C_4 : \mathsf{Type}$$
$$c_1, c_2, c_3, c_4 : C_4$$
$$c_1c_2 : c_1 = c_2$$
$$c_2c_3 : c_2 = c_3$$
$$c_3c_4 : c_3 = c_4$$
$$c_4c_1 : c_4 = c_1$$

We may also think of C_4 as the join of the two-element sets $\{c_1, c_3\} * \{c_2, c_4\}$.
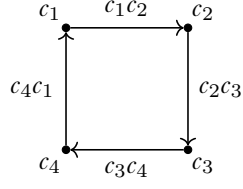
Figure 1: The HIT C_4 which is one of the types in $\mathrm{BAut}_1(S^1)$

Definition 3. *The HIT $\mathbb{O}_0$ is just 6 points, intended as the 0-skeleton of an octahedron, with vertices named after the colors on the faces of a Rubik's Cube.*

$$w, y, b, r, g, o : \mathbb{O}_0$$

Definition 4. *The HIT $\mathbb{O}_1$ is the 1-skeleton of an octahedron.*

$$w, y, b, r, g, o : \mathbb{O}_1$$
$$wb : w = b$$
$$wr : w = r$$
$$wg : w = g$$
$$wo : w = o$$
$$yb : y = b$$
$$yr : y = r$$
$$yg : y = g$$
$$yo : y = o$$
$$br : b = r$$
$$rg : r = g$$
$$go : g = o$$
$$ob : o = b$$

4

Definition 5. *The HIT $\mathbb{O}$ which has 8 faces:*

$$w, y, b, r, g, o : \mathbb{O}$$
$$wb : w = b$$
$$wr : w = r$$
$$wg : w = g$$
$$wo : w = o$$
$$yb : y = b$$
$$yr : y = r$$
$$yg : y = g$$
$$yo : y = o$$
$$br : b = r$$
$$rg : r = g$$
$$go : g = o$$
$$ob : o = b$$
$$wbr : wb \cdot br \cdot wr^{-1} = \mathsf{refl}_w$$
$$wrg : wr \cdot rg \cdot wg^{-1} = \mathsf{refl}_w$$
$$wgo : wg \cdot go \cdot wo^{-1} = \mathsf{refl}_w$$
$$wob : wo \cdot ob \cdot wb^{-1} = \mathsf{refl}_w$$
$$yrb : yr \cdot rb \cdot yb^{-1} = \mathsf{refl}_y$$
$$ygr : yg \cdot gr \cdot yr^{-1} = \mathsf{refl}_y$$
$$yog : yo \cdot og \cdot yg^{-1} = \mathsf{refl}_y$$
$$ybo : yb \cdot bo \cdot yo^{-1} = \mathsf{refl}_y$$
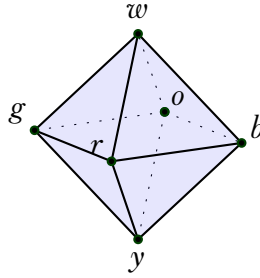

Figure 2: The HIT $\mathbb{O}$ which has 6 points, 12 1-paths, 8 2-paths.

We have obvious maps $\mathbb{O}_0 \xrightarrow{i_0} \mathbb{O}_1 \xrightarrow{i_1} \mathbb{O}$ that include each skeleton into the next-higher-dimensional skeleton.

Combinatorial spaces have a concept called the *link* of a vertex, which will be the main tool by which we connect with manifold theory. The vertices in the link are the vertices that are one edge

5

away from the given point, and the edges in the link are the edges connecting these. If the link of an n-dimensional combinatorial space is always a combinatorial $n - 1$-sphere, then we say the space is a *combinatorial triangulation*. We will look only at HITs that have a link that is merely equivalent to C_4.

Definition 6. *If we have* $X : \mathsf{Type}$ *then we define* $\mathrm{BAut}X \stackrel{\mathrm{def}}{=} \sum_{Y:\mathsf{Type}} ||X = Y||_{-1}$.

Denote by $abcd : \mathrm{BAut}C_4$ the HIT with vertices a, b, c, d and edges ab, bc, cd, da which clearly has various isomorphisms with C_4.

We can now define link on the 0-skeleton. Extending this later on to the 1-skeleton and 2-skeleton will take us into differential geometry!

Definition 7. $\mathsf{link} : \mathbb{O}_0 \to \mathrm{BAut}C_4$ *is given by induction:*

$$\mathsf{link}(w) = brgo$$
$$\mathsf{link}(y) = bogr$$
$$\mathsf{link}(b) = woyr$$
$$\mathsf{link}(r) = wbyg$$
$$\mathsf{link}(g) = wryo$$
$$\mathsf{link}(o) = wgyb$$

We chose these orderings for the vertices by standing at the given vertex and enumerating the link in clockwise order, starting from w if possible, else b.

Besides the link we also want to consider the 5-pointed object that includes the vertex itself and the edges connecting it to the vertices in the link. We will call such a shape an *xbox* since it is a square with both diagonals. We will denote xboxes by extending the square notation with a fifth letter to indicate the center of the xbox. For example, we can define an xbox C_4c as follows:

Definition 8. *The higher inductive type* C_4c *with center* c*, also denoted* $c_1 c_2 c_3 c_4 c$.

$$C_4c : \mathsf{Type}$$
$$c_1, c_2, c_3, c_4, c : C_4c$$
$$c_1 c_2 : c_1 = c_2$$
$$c_2 c_3 : c_2 = c_3$$
$$c_3 c_4 : c_3 = c_4$$
$$c_4 c_1 : c_4 = c_1$$
$$c_1 c : c_1 = c$$
$$c_2 c : c_2 = c$$
$$c_3 c : c_3 = c$$
$$c_4 c : c_4 = c$$

We can map our octahedron into the space of xboxes:

Definition 9. $\text{xbox} : \mathbb{O}_0 \to \mathrm{BAut}C_4c$ *is given by induction:*

$$\text{xbox}(w) = brgow$$
$$\text{xbox}(y) = bogry$$
$$\text{xbox}(b) = woyrb$$
$$\text{xbox}(r) = wbygr$$
$$\text{xbox}(g) = wryog$$
$$\text{xbox}(o) = wgybo$$

Finally we want to consider the 2-type that fills in the faces of the xbox. This is a contractible type we will call a *disk*.

Definition 10. *The higher inductive type $C_{4\text{disk}}$ with center c, also denoted $c_1c_2c_3c_4c_{\text{disk}}$.*

$$C_{4\text{disk}} : \mathsf{Type}$$
$$c_1, c_2, c_3, c_4, c : C_{4\text{disk}}$$
$$c_1c_2 : c_1 = c_2$$
$$c_2c_3 : c_2 = c_3$$
$$c_3c_4 : c_3 = c_4$$
$$c_4c_1 : c_4 = c_1$$
$$c_1c : c_1 = c$$
$$c_2c : c_2 = c$$
$$c_3c : c_3 = c$$
$$c_4c : c_4 = c$$
$$c_1c_2c : c_1c_2 \cdot c_2c \cdot c_1c^{-1} = \mathsf{refl}$$
$$c_2c_3c : c_2c_2 \cdot c_3c \cdot c_2c^{-1} = \mathsf{refl}$$
$$c_3c_4c : c_3c_2 \cdot c_4c \cdot c_3c^{-1} = \mathsf{refl}$$
$$c_4c_1c : c_4c_2 \cdot c_1c \cdot c_4c^{-1} = \mathsf{refl}$$

And we can map the octahedron into the space of disks:

Definition 11. $\text{disk} : \mathbb{O}_0 \to \mathrm{BAut}C_{4\text{disk}}$ *is given by induction:*

$$\text{xbox}(w) = brgow_{\text{disk}}$$
$$\text{xbox}(y) = bogry_{\text{disk}}$$
$$\text{xbox}(b) = woyrb_{\text{disk}}$$
$$\text{xbox}(r) = wbygr_{\text{disk}}$$
$$\text{xbox}(g) = wryog_{\text{disk}}$$
$$\text{xbox}(o) = wgybo_{\text{disk}}$$

The idea of such combinatorial manifolds is that each point has a designated local disk and local sphere (and xbox) in the same way that a smooth manifold has an atlas. We want to proceed now

to define bundles on this manifold. In HoTT that means constructing a map to a classifying space. The link map will be our map, but we have only defined it on $\mathbb{O}_0$.

Extending link to the 1-skeleton requires new choices. We will choose a map inspired by the tangent bundle of the sphere, using the transport we can intuitively see through the embedding of $\mathbb{O}$ in 3-dimensional space as in our pictures. If you focus for a moment just on a path $w \to b \to r$ and imagine rigidly tipping a moving equator along with a moving point, you can imagine a moving-equator point that starts at r and stays fixed when the north pole slides from w to b. When the north pole continues sliding from b to r the moving-equator point moves to g. Then it remains fixed when the north pole slides from r up to w. So all in all we "moved r to g." When we track all the points on the original w-equator we see that we performed the rotation we earlier named R.
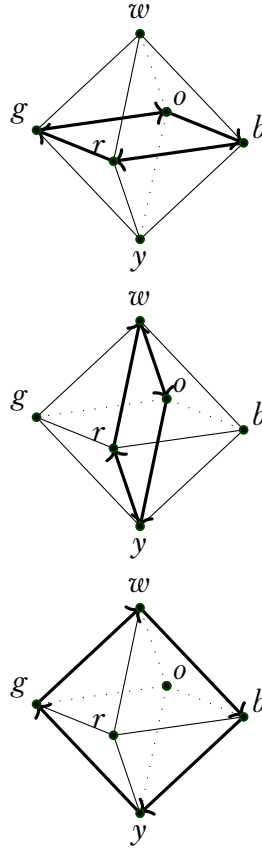

Figure 3: The equators for w, b, r.

In general we have:

Definition 12. *Define $T_1 : \mathbb{O} \to \mathrm{BAut}_1(S^1)$ on just the 1-skeleton by extending T_0 as follows: Transport away from w:*

- $T_1(wb) : [b, r, g, o] \mapsto [y, r, w, o]$ *(r, o fixed)*

- $T_1(wr) : [b, r, g, o] \mapsto [b, y, g, w]$ *(b, g fixed)*

- $T_1(wg) : [b, r, g, o] \mapsto [w, r, y, o]$

- $T_1(wo) : [b, r, g, o] \mapsto [b, w, g, y]$

Transport away from y:

- $T_1(yb) : [b, o, g, r] \mapsto [w, o, y, r]$
- $T_1(yr) : [b, o, g, r] \mapsto [b, y, g, w]$
- $T_1(yg) : [b, o, g, r] \mapsto [y, o, w, r]$
- $T_1(yo) : [b, o, g, r] \mapsto [b, w, g, y]$

Transport along the equator:

- $T_1(br) : [w, o, y, r] \mapsto [w, b, y, g]$
- $T_1(rg) : [w, b, y, g] \mapsto [w, r, y, o]$
- $T_1(go) : [w, r, y, o] \mapsto [w, g, y, b]$
- $T_1(ob) : [w, g, y, b] \mapsto [w, o, y, r]$

At this point we have defined a map on the 1-skeleton of $\mathbb{O}$.

Claim 1. T_1 *defines a principal circle bundle with connection over the 1-skeleton of $\mathbb{O}$.*

We now want to extend this map to all of $\mathbb{O}$ by providing values for the eight faces. Here we will be guided by the classical relationship between a connection and its curvature. The curvature is computed from the connection, it doesn't contain any new data. Classically the integral of curvature over a 2-cell is the holonomy given by transport around the boundary.

Definition 13. *Define $T_2 : \mathbb{O} \to \mathrm{BAut}_1(S^1)$ by extending T_1 as follows. We will send every clockwise triangle to R', the homotopy from* refl *to R:*

- $T_2(wbr) = R'$
- $T_2(wrg) = R'$
- $T_2(wgo) = R'$
- $T_2(ybo) = R'$
- $T_2(yrb) = R'$
- $T_2(ygr) = R'$
- $T_2(yog) = R'$
- $T_2(ybo) = R'$

3.2 Combinatorial manifolds

(This section is not quite off the ground.)

The combinatorial structure we have in mind is a nerve of a good open cover. What do we know about which smooth manifolds have such covers? While we're at it, let's survey all the

combinatorial-flavored spaces and survey what smooth manifolds are homotopy equivalent to which structures.

What topological manifolds are equivalent to a CW complex? The answer is the composition of a few results summarized by Allen Hatcher[1] (citing [10] and [11]):

> Every topological manifold has a handlebody structure except in dimension 4, where a 4-manifold has a handlebody structure if and only if it is smoothable. This is a theorem on page 136 of Freedman and Quinn's book "Topology of 4-Manifolds", with a reference given to the Kirby-Siebenmann book for the higher-dimensional case. It is then an elementary fact that an n-manifold with a handlebody structure is homotopy equivalent to a CW complex with one k-cell for each k-handle, so in particular there are no cells of dimension greater than n. At least in the compact case a manifold with a handlebody structure is in fact homeomorphic to a CW complex with k-cells corresponding to k-handles; see page 107 of Kirby-Siebenmann. This probably holds in the noncompact case as well, though I don't know a reference.

4 Central H-spaces

- introduce torsors

- show subtlety how BG doesn't classify stuff since it has extra properties

- draw
$$\begin{array}{ccccc} \sum_{a:M} U(f(x)) & \xrightarrow{:=} & P & \longrightarrow & EG & \longrightarrow & \mathcal{U}' \\ & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ & & M & \xrightarrow{f} & BG & \xrightarrow{U(\text{forget})} & \mathcal{U} \end{array}$$

- H-spaces paper result equating this to a universal cover of a component of the universe. (It should feel significant that $BS^1 \simeq \mathrm{BAut}_1(S^1)$.)

Central H-spaces are the classifying spaces for principal bundles on abelian groups. We won't be able to access the full theory for nonabelian groups just yet, but we hope that the theory of maximal tori and weights might bring even those within reach of the central H-space paradigm.

We will rely on the lovely paper by Buchholtz, Christensen, Flaten and Rijke [12].

Definition 14. *An H-space structure on a pointed type* (B, b) *consists of*

1. *A binary operation* $\mu : B \to B \to B$

2. *A left unit law* $\mu_l : \mu(\mathrm{pt}, -) = id_B$

3. *A right unit law* $\mu_r : \mu(-, \mathrm{pt}) = id_B$

4. *A coherence* $\mu_{lr} : \mu_l(\mathrm{pt}) =_{\mu(\mathrm{pt},\mathrm{pt})=\mathrm{pt}} \mu_r(\mathrm{pt})$

5. *A proof of left- and right- invertibility:* $\mu(a, -) : A \simeq A$, $\mu(-, b) : A \simeq A$

---

[1]https://mathoverflow.net/questions/201944/topological-n-manifolds-have-the-homotopy-type-of-n-dimensional-cw-complexes

Proposition 1. *([12] Prop 3.6) Let A be a pointed type. Then the following are equivalent:*

1. *A is central.*

2. *A is a connected H-space and $A \cdot \to A$ is a set.*

3. *A is a connected H-space and $A \simeq A$ is a set.*

This result will inform our study of the Leibniz rule: the analogue of the algebra of functions to $\mathbb{R}$ is:

Proposition 2. *For any type M and H-space B the type of maps $M \to B$ with base point the constant map is an H-space under pointwise multiplication.*

We will also be looking in detail at maps into the classifying space of S^1 bundles. The Buchholtz et al paper[12] describes this type in several helpful ways, summarized by:

Theorem 1. *For any central H-space A (such as S^1) the type of torsors of A is a delooping of A, and is equivalent to $\mathrm{BAut}_1(A) \overset{\text{def}}{=} \sum_{X:\mathcal{U}} ||X = A||_0$. This delooping is also a central H-space and so can be infinitely delooped.*

This means that we can form a sequence of deloopings $\mathbb{Z}$, S^1, $\mathrm{BAut}_1(S^1)$, ...

5   Why this is geometry

How can we double check that we are describing the intended theory of geometry? In this section we will enumerate a wishlist of facts that we believe characterize the subject, and then provide evidence for some of them.

Here are the translations that are covered in the current paper:

Connections are infinitesimal splittings of a principal bundle.	Paths in a sigma type are equivalent to a pair of paths.
Differentials satisfy the Leibniz (product) rule.	Horizontal composition in an H-space is performed in two steps.
Connections with 0-truncated groups are covering spaces with unique flat connection.	Transport around contractible loops is refl when fibers are sets.
The group of gauge transformations (bundle automorphisms) acts on the space of connections.	Homotopies of classifying maps respect the splitting of paths in sigma types.

And here are questions to explore in the future:

1. There's a notion of *tensorial* that holds for forms but not for connections.

2. Where is the Grassmannian structure of wedge product?

3. The Gauss-Bonnet theorem holds, relating the curvature of a 2-manifold to the Euler characteristic.

4. More generally, characteristic classes of bundles can be computed using a connection (Chern-Weil theory).

5.1   Higher maps contain connections

To hew close to the intended context of the term "connection" we will examine manifold-like types mapping into bundle-classifying-like types. The novelty here, compared to other HoTT investigations, is the focus on combinatorial types to stand in for manifolds.

In recent times it has been believed in the HoTT community that maps from a discrete type to a discrete classifying space can encode only the connections that a classical mathematician would call *flat* (zero curvature). In this context the word *discrete* means having the discrete topology, in the sense of cohesion[1]. This is not the case! We will show that if the codomain is a classifying space of S^1 or other group of homotopy dimension at least 1, then non-flat connections appear despite the type S^1 being topologically discrete. Another common meaning of the shorthand "discrete" is to indicate a 0-truncated type, i.e. a set, as opposed to a type with higher homotopical structures. We will show that indeed if the codomain classifies sets, which is the case for example with the classifying space $B\mathbb{Z}$, the delooping of $\mathbb{Z}$, then connections are flat. (The type we denote by $\mathbb{S}^1 \overset{\text{def}}{=} \{(x,y)|x^2 + y^2 = 1\}$ is a set and is not topologically discrete. We will not be discussing it at all in this paper.)

The DDG philosophy tells us to look at HITs that are polytope-like. A polytope M will have finitely many 0-dimesional (point) constructors $\{m_0^i\}$, finitely many 1-dimesnional constructors $\{m_1^{ij} : m_0^i = m_0^j\}$, and so on. Type families $f : M \to \mathcal{U}$ on this type specify where each of these constructors is sent. In DDG parlance, f restricted to the 0-dimensional constructors of M is a 0-form and f restricted to the 1-dimensional constructors (not the 1-skeleton but just the 1-dimensional parts, whatever that means in HoTT) is a 1-form, and so on.

A principal S^1 bundle is a family of S^1 torsors and so we will often be focusing on the function type $M \to \mathrm{BAut}_1(S^1)$. The novel claim here is that $M \to \mathrm{BAut}_1(S^1)$ contains more than just all the principal S^1-bundles: it also contains all the connections on all the bundles. Every connection is present, both curved and flat, because we have complete freedom to specify the images of the paths.

Classically, curvature is a property of the connection. It is computed either on infinitesimal loops, or on the infinitesimal surface bounded by the loop. In fact it is "the derivative of the connection" morally speaking. Getting into the details would wreck the simplicity we're going for[2]. We will look for curvature by examining f on 1-dimensional loops. If M is at least a 2-type and if we want to claim that f classifies a bundle with connection, then we will be required to map the 2-faces of M (the 2-dimesional constructors) to a path from refl to the image of a bounding loop. So at dimension 2 we will impose that constraint. Since $\mathrm{BAut}_1(S^1)$ is 2-truncated, f factors through the truncation map $M \to ||M||_2$ and so that's the top dimension.

There is an example at the end of this paper. For those who are best served by examples, do look at it and return to this point.

---

[2]I'm picking and choosing where to tell the story fully twice (see the next section) and where to simply look for motivation.

5.2 Connections as splittings

The classical story goes like this.

Definition 15. *The* vertical bundle VP *of a principal bundle* $\pi : P \to M$ *with Lie group* G *is the kernel of the derivative* $T\pi : TP \to TM$.

VP can be visualized as the collection of tangent vectors that point along the fibers. It should be clear that the group $\mathrm{Aut}P$ acts on VP: an automorphism $\phi : P \to P$ sends V_pP to $V_{\phi(p)}P$, where of course $\pi(p) = \pi(\phi(p))$.

Definition 16. *An* Ehresmann connection *on a principal bundle* $\pi : P \to M$ *with Lie group* G *is a splitting* $TP = VP \oplus HP$ *at every point of* P *into vertical and "horizontal" subspaces, which is preserved by the action of* $\mathrm{Aut}P$.

Being preserved by the action of $\mathrm{Aut}P$ means that the complementary horizontal subspaces in a given fiber of $\pi : P \to M$ are determined by the splitting at any single point in the fiber. The action of G on this fiber can then push the splitting around to all the other points.

The motivation for this definition is that we now have an isomorphism $T_p\pi : H_pP \to T_{\pi(p)}M$ between each horizontal space and the tangent space below it in M. This means that given a tangent vector at $x : M$ and a point p in $\pi^{-1}(x)$ we can uniquely lift the tangent vector to a horizontal vector at p. We can also lift vector fields and paths in this way. To lift a path $\gamma : [0, 1] \to M$ you must specify a lift for $\gamma(0)$ and then lift the tangent vectors of γ and prove that you can integrate the lift of that vector field upstairs in HP.

Then, armed with the lifting of paths one immediately obtains isomorphisms between the fibers of P. So the Ehresmann connection, the lifting of paths, and transport isomorphisms between fibers are all recapitulations of the structure that the connection adds to the bundle.

Moving now to HoTT, fix a type $M : \mathcal{U}$ and a type family $f : M \to \mathcal{U}$. Path induction gives us the transport isomorphism $\prod_{p:x=_M y} \mathrm{tr}(p) : f(x) = f(y)$. We can use this to define a type of *dependent paths*, also called *pathovers* or *paths over* a given path.

Definition 17. *With the above context and points* $a : f(x), b : f(y)$ *the type of* dependent paths over p *with endpoints* a, b *is denoted*

$$a \xrightarrow[p]{=} b.$$

By induction we can assume p *is* refl_a *in which case* $a \xrightarrow[p]{=} b$ *is* $a =_{f(x)} a$.

See [13] for more discussion of dependent paths (where they use the term "path over"), including composition, and associativity thereof.

We recall now the identity type of sigma types:

Theorem 2. *(HoTT book Theorem 2.7.2 [14]) If* $f : M \to \mathcal{U}$ *is a type family and* $w, w' : \sum_{x:M} f(x)$ *then there is an equivalence*

$$\mathsf{split} : (w = w') \simeq \sum_{p:\mathrm{pr}_1(w)=_M\mathrm{pr}_1(w')} [\mathrm{tr}(p)(\mathrm{pr}_2(w))] = \mathrm{pr}_2(w').$$

In particular, given $p : x =_M y$ and $w : f(x)$ we have $w \xrightarrow[p]{=} tr(p)(w) \simeq tr(p)(w) =_{f(y)} tr(p)(w)$ which has the term refl which we can call "the horizontal lift of p starting at w." We can imitate the classical definition of a connection by defining $\omega \overset{\text{def}}{=} \text{pr}_2 \circ \text{split}$, the projection onto the vertical component. And thus in HoTT we can see the equivalence of transport and lifting of paths into horizontal and vertical components.

5.3   The Leibniz (product) rule

The Leibniz rule for exterior differentiation states that if $f, g : M \to \mathbb{R}$ are two smooth functions to the real numbers then $d(fg) = f dg + g df$. Here fg is the function formed by taking the pointwise product of f and g. This is an interaction between multiplication in $\mathbb{R}$ and the action on vectors of smooth functions (the 1-forms df and dg).

To examine this situation in HoTT we need type-theoretic functions $f, g : M \to B$ from some type M to a central H-space B. Let $\mu : B \to B \to B$ be the H-space multiplication. How does μ act on paths? Suppose we have $a, a', b, b' : B$ and $p : a =_B a', q : b =_B b'$. Then we also have homotopies $\mu(p, -) : \mu(a, -) =_{B \to B} \mu(a', -)$ and $\mu(-, q) : \mu(-, b) =_{B \to B} \mu(-, b')$. Since $\mu(a, -) : B = B$ is an (unpointed) equivalence of B, and similarly for $\mu(b, -)$ and so on, this data assembles into the following diagram of higher groupoid morphisms:

$$
B \underset{\mu(a',-)}{\overset{\mu(a,-)}{\Longrightarrow}} \mu(p,-) \Big\Downarrow \; B \underset{\mu(-,b')}{\overset{\mu(-,b)}{\Longrightarrow}} \mu(-,q) \Big\Downarrow \; B
$$

And so the two homotopies can be horizontally composed to give a path

$$\mu(p, -) \star \mu(-, q) : \mu(a, b) = \mu(a', b').$$

Horizontal composition is given by

$$\mu(p, -) \star \mu(-, q) \overset{\text{def}}{=} (\mu(p, -) \cdot_r \mu(-, b)) \cdot (\mu(a', -) \cdot_l \mu(-, q))$$

where

$$\mu(p, -) \cdot_r \mu(-, b) : \mu(a, b) = \mu(a', b)$$

and

$$\mu(a', -) \cdot_l \mu(-, q) : \mu(a', b) = \mu(a', b')$$

are defined by path induction. See the HoTT book Theorem 2.1.6 on the Eckmann-Hilton argument[14].

We can recognize the process of using whiskering to form horizontal composition in the Leibniz rule.

Quick aside: moving from infinitesimal calculus to finite groupoid algebra actually involves two changes. The first is the change from vectors to paths, forms to functions and so on. But it's also

the case that tangent vectors have just the one basepoint, whereas paths have two endpoints. You can see this play out in this example, where a and a' were distinct points (and b and b').

The horizontal composition we build lives entirely in B and we didn't make use of M yet. The Leibniz rule will be a pointwise version of what's going on in B. Denote by $\mu \circ (f, g) : M \to B$ the map which sends $x \mapsto \mu(f(x), g(x))$.

Lemma 1. *Given $f, g : M \to B$ and $p : x =_M y$ then*

$$\mathsf{ap}(\mu \circ (f, g))(p) = \mu(f(p), -) \star \mu(-, g(p))$$
$$= [\mu(f(p), -) \cdot_r \mu(-, g(x))] \cdot [\mu(f(y), -) \cdot_l \mu(-, g(p))]$$
$$: \mu(f(x), g(x)) = \mu(f(y), g(y))$$

5.4 Covering spaces

If G is a 0-truncated group such as $\mathbb{Z}$ then the type of torsors (delooping) BG is 1-truncated. If $f : M \to BG$ is a type family then $\sum_{x:M} f(x)$ has fibers that are sets (G-torsors). So transport functions are set isomorphisms, and the transport of any contractible loop in M will be refl (the identity) of the fiber, which is what we mean by flat.

5.5 Gauge transformations

A *gauge transformation* is a term inherited from physics. It's an automorphism of a principal bundle $P \to M$, meaning a homeomorphism of P that commutes with the projection to M and so acts on each fiber. It is further required to be equivariant under the action of the group G, and so it's very similar to the act of multiplying each fiber by a continuously varying element of G.

In HoTT if the bundle is classified by $f : M \to \mathcal{U}$ then an automorphism is a homotopy $f \sim f$ and the group of gauge transformations is the loop space $\Omega_f(M \to \mathcal{U})$.

Recall that torsors have a physical interpretation as a quantity without a specified unit, such as mass, length, or time. When we choose a base point in a torsor it becomes the standard torsor G acting on itself (for example, the additive real numbers). A physicist is looking for properties or laws that are independent of such a choice. In the 20th century physicists further wondered about choices of units that vary from point to point, and began searching for laws that are invariant under this much larger space of transformations. And so they and we are led to explore quotienting by the action of the group of gauge transformations, and in particular the space of connections "mod gauge." In this scenario the base manifold M is spacetime, and a gauge transformation is a smoothly varying choice of gauge (units) at each point.

Gauge transformations act on connections. When we view connections as infinitesimal splittings of TP into vertical and horizontal sub-bundles, a gauge transformation that is constant in the neighborhood of a point will not change the splitting, it will just shift the fiber rigidly along itself, but one that is changing rapidly near a point will tilt the horizontal subspaces. So there are two effects: the effect of sliding along the fiber, and the effect of the rate of change of the gauge transformation. In classical geometry you'll see formulas like this:

Theorem 3. *Let $P \to M$ be a principal bundle and $A \in \Omega^1(M, \mathfrak{g})$ a connection 1-form on P. Suppose that $H \in \mathscr{G}(P)$ is a bundle automorphism. Then H^*A is a connection 1-form and in a neighborhood U of a point*

$x \in M$ we can write H as a function $H_U : U \to G$ where $H_U(x) \in G$ is a group element multiplying the fiber at x, and then we have

$$H^*A = \mathrm{Ad}_{H_U(x)^{-1}} \circ A + H_U^*(\mu_G)$$

where $\mu_G : \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form on G.

This theorem is a combination of Theorems 5.2.2 and 5.4.4 in the excellent recent book on gauge theory for mathematicians interested in physics by Mark Hamilton[15].

It's not so important to fully understand this formula because we will re-explain it in HoTT terms in a moment. But notice that H^*A (the action of the gauge transformation on the connection 1-form) has contributions from two terms (both of which are vertical — connections always map onto the vertical bundle). The first is the adjoint action at the specific point x. This is always what we expect when we shift the base point in a torsor and look at the resulting group (or in this case, the Lie algebra). The second term involves the Maurer-Cartan form, which is the derivative of subtraction in the group. It takes tangent vectors at $g : G$ to a tangent vector at the identity (the Lie algebra, denoted $\mathfrak{g}$) by differentiating the action of multiplication by g^{-1}. If we think in terms of finite-length paths, then imagine a path $p : g = g'$ and the function $(g^{-1} \cdot -)$. The function will act on the path to give a path $g^{-1} \cdot p : e = (g' \cdot g^{-1})$ that starts at the identity. So the Maurer-Cartan form shifts paths to start at the identity by subtracting off the start point. Our Maurer-Cartan term is the pullback of the Maurer-Cartan form by H which records how H acts infinitesimally, i.e. the contribution from the gauge transformation H that comes from the rapidity of change from point to point. This term will be large when H_U has a large derivative.

In HoTT the connection's parallel transport is visible as the transport function, and the horizontal-vertical splitting is visible in the decomposition of paths in the sigma type (total space) into pairs of paths. What is the effect of applying a homotopy $H : f \sim f$ on transport, and on splitting?

H is a family of fiber automorphisms: $H : \prod_{a:M} f(a) = f(a)$ which we can assemble into an equivalence $H' : \sum_{a:M} f(a) = \sum_{a:M} f(a)$ that acts fiberwise. We want to compute the action of $\mathsf{ap}(H')$ on the horizontal-vertical decomposition of paths from Theorem 2 by computing $\omega \circ \mathsf{ap}(H') = \mathrm{pr}_2 \circ \mathsf{split} \circ \mathsf{ap}(H')$.

Denote $\sum_{a:M} f(a)$ by P. We'll adopt a convention of roman letters for structures in M and Greek for those upstairs in P. Let $p : a =_M b$ be a path in the base and let $\pi : (a, \alpha) =_P (b, \beta)$ be a path in P over p. Then $\omega(\pi) : \mathrm{tr}_p(\alpha) = \beta$.

Now let's apply H. We have $\mathsf{ap}(H')(\pi) : (a, H(a)(\alpha)) =_P (b, H(b)(\beta))$ which is still a path over p. Applying ω we get

$$\omega(\mathsf{ap}(H')(\pi)) : \mathrm{tr}_p(H(a)(\alpha)) = (H(b)(\beta))$$

. Using the lemma below we can if we wish rewrite this as

$$\omega(\mathsf{ap}(H')(\pi)) : H(b) \left[\mathrm{tr}_p(\alpha) = \beta \right]$$

which uses only $H(b)$.

Lemma 2. *Given a function $f : M \to \mathcal{U}$ and homotopy $H : f \sim f$ the following square commutes and so in the type family we have $\mathrm{tr}(H(x) \cdot f(p)) = \mathrm{tr}(f(p) \cdot H(y))$.*

$$f(a) \overset{f(p)}{=\!=\!=} f(b)$$
$$H(a) \| \qquad \| H(b)$$
$$f(a) \underset{f(p)}{=\!=\!=} f(b)$$

6   Gauss-Bonnet, Vector Fields and Poincaré-Hopf theorem

To prove the Gauss-Bonnet theorem one needs a notion of *Euler characteristic* and a notion of curvature. To prove the Poincaré-Hopf theorem, one needs a notion of *the index of a vector field*, and again, a notion of curvature. To prove both and relate them it helps to equate the Euler characteristic to the index of a special vector field.

Definition 18. *A vector field is a section X of the disk bundle. A zero of X is a vertex x where $X(x) = x :$* $\mathsf{disk}(x)$*, the center of the disk.*

The following proofs follow the explanations given in [16]. But as we might hope, the proofs can be brought into HoTT in a much abbreviated form!

Note that X is a pointing of all the fibers. So a second vector field can be lifted to the classifying map to pointed types, which is contractible. That's really all that's going on in these classical proofs!

Definition 19. *The index of a vector field at a zero is given by transport composed with exp.*

Definition 20.

Bibliography

[1] M. Shulman, "Brouwer's fixed-point theorem in real-cohesive homotopy type theory," 2017.

[2] S. Talbott, *The Future Does Not Compute: Transcending the Machines in Our Midst*.   O'Reilly & Associates, 1995. [Online]. Available: https://books.google.com/books?id=KcXaAAAAMAAJ

[3] A. N. Hirani, "Discrete exterior calculus," Ph.D. dissertation, 2003.

[4] M. Desbrun, A. N. Hirani, M. Leok, and J. E. Marsden, "Discrete exterior calculus," 2005.

[5] H. Whitney, *Geometric Integration Theory*.   Princeton University Press, 1957. [Online]. Available: http://www.jstor.org/stable/j.ctt183q1bm

[6] M. Leok, J. E. Marsden, and A. D. Weinstein, "A discrete theory of connections on principal bundles," 2005.

[7] K. Crane, F. de Goes, M. Desbrun, and P. Schröder, "Digital geometry processing with discrete exterior calculus," in *ACM SIGGRAPH 2013 courses*, ser. SIGGRAPH '13.   New York, NY, USA: ACM, 2013. [Online]. Available: https://www.cs.cmu.edu/~kmcrane/ Projects/DDG/

[8] L. Morris, A. Baas, J. Arias, M. Gatlin, E. Patterson, and J. P. Fairbanks, "Decapodes: A diagrammatic tool for representing, composing, and computing spatialized partial differential equations," 2024.

[9] E. Patterson, A. Baas, T. Hosgood, and J. Fairbanks, "A diagrammatic view of differential equations in physics," *Mathematics in Engineering*, vol. 5, no. 2, p. 1–59, 2022. [Online]. Available: http://dx.doi.org/10.3934/mine.2023036

[10] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, ser. Annals of Mathematics Studies. Princeton University Press, 1977, no. 88, with notes by John Milnor and Michael Atiyah. MR:0645390. Zbl:0361.57004.

[11] M. H. Freedman and F. Quinn, *Topology of 4-Manifolds (PMS-39)*. Princeton University Press, 1990. [Online]. Available: http://www.jstor.org/stable/j.ctt7ztz2d

[12] U. Buchholtz, J. D. Christensen, J. G. T. Flaten, and E. Rijke, "Central h-spaces and banded types," 2023.

[13] M. Bezem, U. Buchholtz, P. Cagne, B. I. Dundas, and D. R. Grayson, "Symmetry," https://github.com/UniMath/SymmetryBook.

[14] Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[15] M. Hamilton, *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*, ser. Universitext. Springer International Publishing, 2017. [Online]. Available: https://books.google.com/books?id=8zVCDwAAQBAJ

[16] T. Needham, *Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts*. Princeton University Press, 2021. [Online]. Available: https://books.google.com/books?id=Mc0QEAAAQBAJ