DRAFT: Discrete differential geometry in homotopy type theory

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Motivation

To use HoTT to study connections and explain their applicability to algebraic topology, via

- the Gauss-Bonnet theorem
- its vast generalization, Chern-Weil theory

Theorem (Gauss-Bonnet)

Let M be a compact 2-manifold with metric, without boundary. Let K be the Gaussian curvature of M and let $\chi(M)$ be the Euler characteristic. Then

$$\frac{1}{2\pi}\int_M K dA = \chi(M).$$

Theorem (Poincaré-Hopf)

Let M be a compact smooth manifold without boundary. Let X be a vector field on M with isolated zeroes $x_1, \ldots, x_n$. Then

$$\sum_{i=1}^{n} \mathrm{index}_{x_i} = \chi(M).$$

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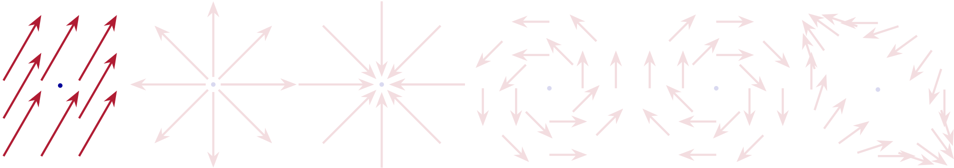

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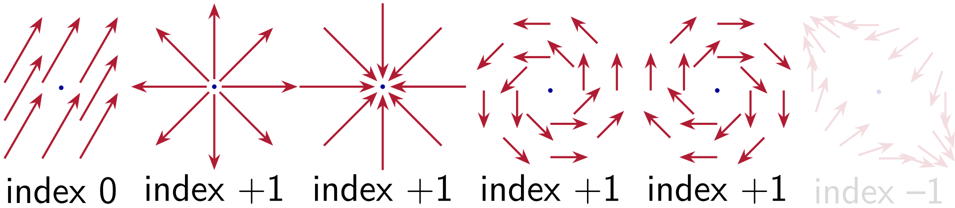

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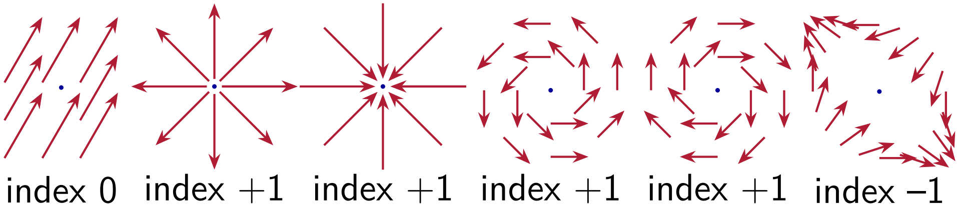


Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

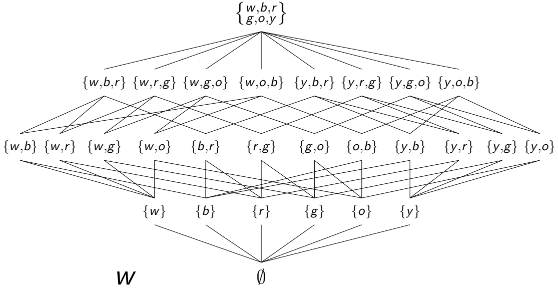
HoTT background

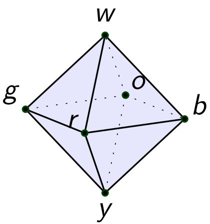
1. Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-) Symmetry. `https://github.com/UniMath/SymmetryBook`.
2. Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023) Central H-spaces and banded types. arXiv:2301.02636
3. Scoccola, L. (2020) Nilpotent types and fracture squares in homotopy type theory, MSCS 30(5). arXiv:1903.03245

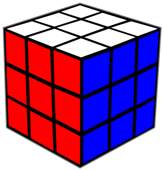
Discrete manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization functor to turn them into homotopy pushouts

Simplicial complexes







A Hasse diagram of a simplicial complex (vertices named for the colors on a Hungarian Cube)

Higher realization

Let $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$ be a simplicial complex. Form a type $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \to \mathbb{M}_1 \to \mathbb{M}_2$ with pushouts.

$$\begin{tikzcd} M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\ M_0 = \mathbb{M}_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \mathbb{M}_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\partial\Delta^1$ is a 0-sphere: the set $\{a, b\}$.

Higher realization

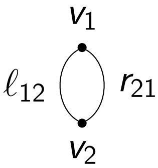
Definition

An *n*-gon $\mathbb{C}(n)$ is the realization of a complex $C(n)$:

$$C(n)_0 = \{v_1, \ldots, v_n\}$$
$$C(n)_1 = \{e_1 = \{v_1, v_2\}, \ldots, e_{n-1} = \{v_{n-1}, v_n\}, e_n = \{v_n, v_0\}\}$$

Toss in two non-complexes

$$\mathbb{C}(1) \stackrel{\text{def}}{=} S^1, \quad \mathbb{C}(2) \stackrel{\text{def}}{=}$$



Higher realization

Then push out faces $\circ$ from a 2-dim type.

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[dr, phantom, "\Searrow h_2"] & \mathbb{M}_2 \\
& M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"'] \arrow[u, "\mathbb{A}_1"] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

$\partial\Delta^2$ is a 1-sphere, i.e. $\mathbb{C}(3)$.

Higher realization

Lemma

$\mathbb{C}(2) \simeq \mathbb{C}(1)$ *and in general* $\mathbb{C}(n) \simeq \mathbb{C}(n-1)$.

Corollary

All n-gons are equivalent to S^1.

Higher realization

$*_{\mathbb{M}_1}, *_{\mathbb{M}_2}$ provide hubs. h_1, h_2 provide spokes.

$$\begin{tikzcd}
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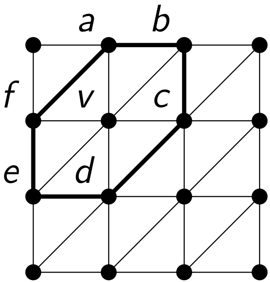
& M_2 \times \partial\Delta^2 \arrow[u, "\mathbb{A}_1"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\Searrow h_2"] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}$$

Higher realization



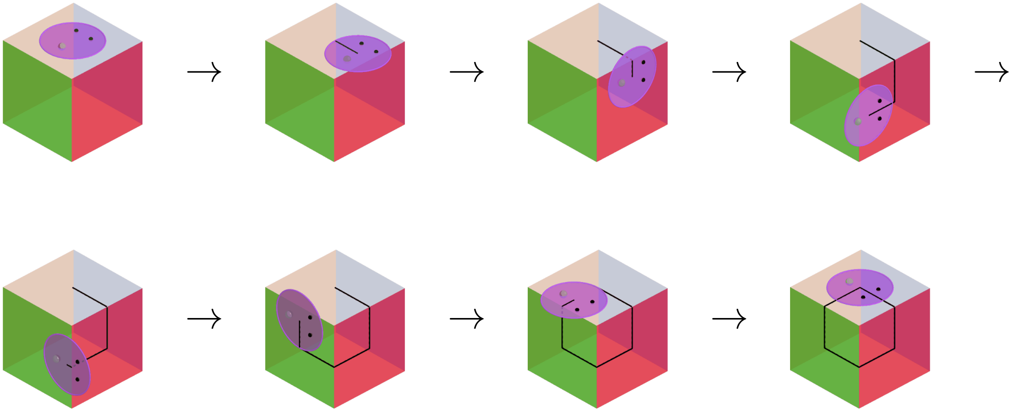
The link of a vertex v in a 2-complex is the polygon of edges not containing v but whose union with v is a face.

This will be our model of the tangent space.

Theorem (Whitehead (1940))

Every smooth manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is the geometric realization of an $(n-1)$-sphere.

`https://ncatlab.org/nlab/show/triangulation+theorem`



Rotation

Let $R : [abcd] \to [abcd]$ send $a \mapsto b, b \mapsto c, c \mapsto d, d \mapsto a$.

Extend R to edges.

Lemma

$[\![R]\!] : [\![abcd]\!] \to [\![abcd]\!]$ is homotopic to the identity, i.e. we have $\prod_{x:[\![abcd]\!]} x = [\![R]\!](x)$.

Proof.

Use edges. $\square$

Definition

Let G be a group with identity element e. A G-set is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If we have

$$\text{is\_torsor}(X, \phi) \stackrel{\text{def}}{=} ||X||_{-1} \times \prod_{x:X} \text{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

we say (X, ϕ) is a G-torsor. Denote the type of G-torsors by BG.

Lemma

Point BG at G_{reg}, the G-torsor G acting on itself on the right. Then $\Omega_{G_{\mathrm{reg}}} BG \simeq G$, so BG is a $\mathrm{K}(G, 1)$.

- $S^1 : \mathcal{U}$ is not an Aut S^1-torsor.
- It's a torsor for $(S^1 = S^1)_{(\mathsf{id})}$, the identity component.
- This omits the flip, the reversal of orientation.
- See the Buchholtz et. al. H-spaces paper for more.

Definition

$$\mathrm{EM}(G,n) \stackrel{\mathrm{def}}{=} \mathsf{BAut}(\mathrm{K}(G,n)) \stackrel{\mathrm{def}}{=} \sum_{Y:\mathcal{U}} \|Y \simeq \mathrm{K}(G,n)\|_{-1}$$

Definition

A $\mathrm{K}(G,n)$-bundle on a type M is a map $f : M \to \mathrm{EM}(G,n)$.

We further assume f factors through $\mathrm{K}(G,n+1)$ and so is principal.

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a cellular type and all the triangles commute in the diagram:

$$\begin{tikzcd} \mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"'] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"] \\ & & \mathcal{U} & & \end{tikzcd}$$

- The map f_k is a *k-bundle* on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a *k-connection on the $(k-1)$-bundle f_{k-1}*.

Definition

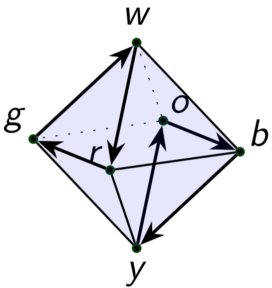
$$\begin{tikzcd}
M_k \times \partial\Delta^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\overset{h_k}{\Nearrow}"] & M_k \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r, "\imath_{k-1}"] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \arrow[dl, phantom, "\Swarrow"] \\
 & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{m_k\} \times \partial\Delta^k \arrow[r, "!"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\overset{\flat_k}{\Swarrow}"] & \mathbf{1} \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

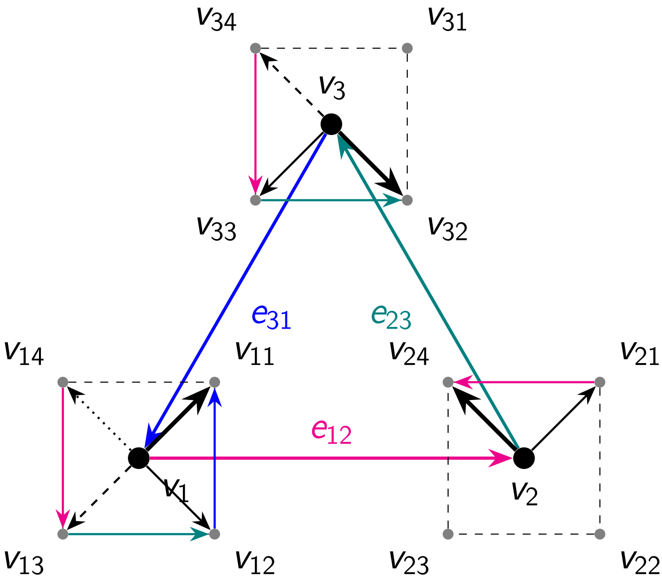
the filler $\flat_k$ is called a flatness structure for the face m_k, and its ending path is called curvature at the face m_k.

Vector fields

Let $T : \mathbb{M}_2 \to \mathrm{K}(\mathbb{Z}, 2)$ be an oriented tangent bundle on an oriented 2-dim cellular type

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It's a nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.





- $\partial F \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$
- We access pathovers asymmetrically: $X_{12} : T_{12} X_1 = X_2$
- $X(\partial F)$ is 3-sided inside a square
- To make a loop we cat with $\flat(\partial F)$

$$\mathrm{tr}_F \overset{\text{def}}{=} \mathrm{tr}(\partial F) \quad : Tm = Tm \qquad \text{holonomy}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{Tm = Tm} \mathrm{tr}(\partial F) \qquad \text{flatness}$$

$$X_F \overset{\text{def}}{=} X(\partial F) \quad : \mathrm{tr}(\partial F)(X(m)) =_{Tm} X(m) \qquad \text{swirling}$$

Definition

The index of the vector field X on the face F is the integer

$$I_F^X \overset{\text{def}}{=} \Omega(\flat_F(X(m)) \cdot X_F) : \Omega(X(m) =_{Tm} X(m)).$$

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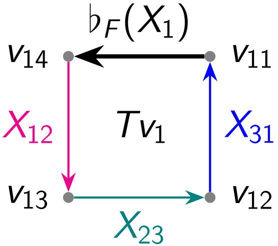
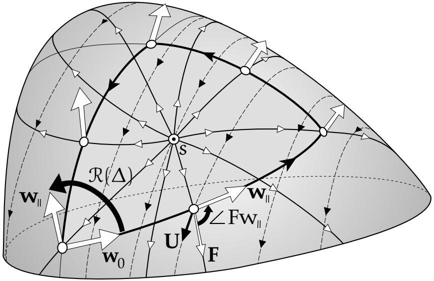


Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi\,\mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F w_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

- The classical proof is discrete-flavored.
- "$\angle F w_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.