DRAFT: Discrete differential geometry in homotopy type theory

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Motivation

To use HoTT to study connections and explain their applicability to algebraic topology, via

- the Gauss-Bonnet theorem
- its vast generalization, Chern-Weil theory

Theorem (Gauss-Bonnet)

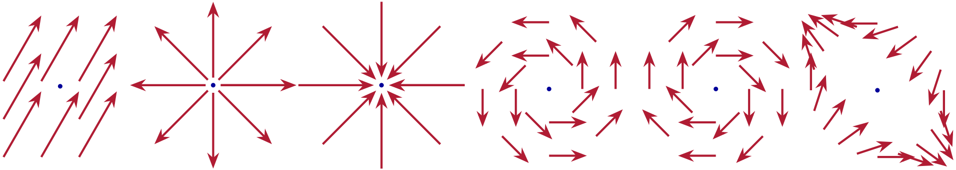
Let M be a compact 2-manifold without boundary, equipped with a Riemannian metric. Let K be the Gaussian curvature of M and let $\chi(M)$ be the Euler characteristic. Then

$$\frac{1}{2\pi}\int_M K\,dA = \chi(M).$$

Theorem (Poincaré-Hopf)

Let M be a compact smooth manifold without boundary. Let X be a vector field on M with isolated zeroes $x_1, \ldots, x_n$. Then

$$\sum_{i=1}^{n} \operatorname{index}_{x_i} = \chi(M).$$



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Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

HoTT background

1. Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-) Symmetry. `https://github.com/UniMath/SymmetryBook`.
2. Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023) Central H-spaces and banded types. arXiv:2301.02636
3. Scoccola, L. (2020) Nilpotent types and fracture squares in homotopy type theory, MSCS 30(5). arXiv:1903.03245

Discrete manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization procedure to turn them into homotopy pushouts

Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

- a set M_0 of $(n+1)$ vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \ldots, M_k]$ the k-skeleton of M.

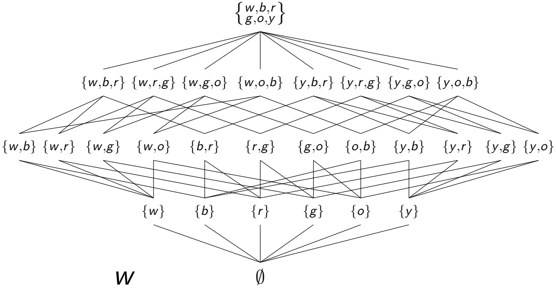
Simplicial complexes

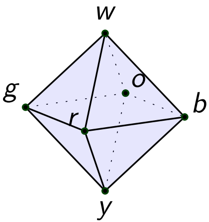
Example

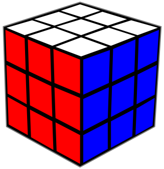
The complete simplex of dimension n, denoted $P(n)$, is the set $\{1, \ldots, n+1\}$ and its power set. The $(n-1)$-skeleton $P(n)_{\leq(n-1)}$ is denoted $\partial P(n)$ and will serve as a combinatorial $(n-1)$-sphere.

e.g., $\partial P(2)$ is an abstract triangle with 3 vertices and 3 edges (lacks its face).

Simplicial complexes







Here is a Hasse diagram of an abstract octahedron (vertices named for the colors on a Hungarian Cube)

Homotopy realization

We will realize simplicial complexes as pushouts.

The realization of a 0-dimensional complex M_0 is the set M_0.

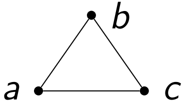
In particular the 0-sphere $\partial\Delta^1 \stackrel{\text{def}}{=} \partial P(1)$.

Homotopy realization

For a 1-dim complex $M \overset{\text{def}}{=} [M_0, M_1]$ form

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] \arrow[ur, Rightarrow, "h_1"'] & \mathbb{M}_1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

$$\{\{w\}, \{g\}\} \leftarrow \{\{w, g\}\} \times \{0, 1\} \rightarrow \{\{w, g\}\}$$

Next construct a 1-sphere $\partial\Delta^2 \overset{\text{def}}{=}$ 

as the realization of $\partial P(2)$:

$$\begin{tikzcd}
\partial P(2)_1 \times \partial\Delta^1 \arrow[r] \arrow[d] & \partial P(2)_1 \arrow[d] \\
\partial P(2)_0 \arrow[r] & \partial\Delta^2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$$\begin{tikzcd}
\{\{a,b\},\{b,c\},\{c,a\}\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{a,b\},\{b,c\},\{c,a\}\} \arrow[d] \\
\{\{a\},\{b\},\{c\}\} \arrow[r] & \partial\Delta^2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Homotopy realization

Then use $\partial\Delta^1, \partial\Delta^2$ to realize $M \overset{\text{def}}{=} [M_0, M_1, M_2]$.

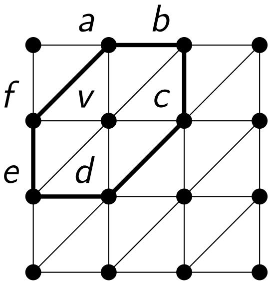
$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\overset{h_1}{\nearrow}"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\overset{h_2}{\searrow}"] & \mathbb{M}_2 \arrow[dl, phantom, "\urcorner", very near start] \\
& M_2 \times \partial\Delta^2 \arrow[u, "\mathbb{A}_1"] \arrow[r, "\mathrm{pr}_1"'] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

Homotopy realization

$*_{\mathbb{M}_1}, *_{\mathbb{M}_2}$ provide hubs. h_1, h_2 provide spokes.

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[dr, phantom, "\Searrow h_2"] & \mathbb{M}_2 \\
& M_2 \times \partial\Delta^2 \arrow[r, "\mathrm{pr}_1"'] \arrow[u, "\mathbb{A}_1"] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

Homotopy realization



The link of a vertex v in a 2-complex is the polygon of edges not containing v but whose union with v is a face.

This will be our model of the tangent space.

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$-sphere, i.e. its geometric realization is an $(n-1)$-sphere.

https://ncatlab.org/nlab/show/triangulation+theorem

Definition

Let G be a group with identity element e. A G-set is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$. If we have

$$\text{is\_torsor}(X, \phi) \stackrel{\text{def}}{=} ||X||_{-1} \times \prod_{x:X} \text{is\_equiv}(\phi(-, x) : (G, e) \to (X, x))$$

we say (X, ϕ) is a G-torsor. Denote the type of G-torsors by BG.

Lemma

Point BG at G_{reg}, the G-torsor G acting on itself on the right. Then $\Omega_{G_{\mathrm{reg}}} BG \simeq G$, so BG is a $\mathrm{K}(G, 1)$.

- $S^1 : \mathcal{U}$ is not an Aut S^1-torsor.
- It's a torsor for $(S^1 = S^1)_{(\mathrm{id})}$, the identity component.
- This omits the flip, the reversal of orientation.
- See the Buchholtz et. al. H-spaces paper for more.

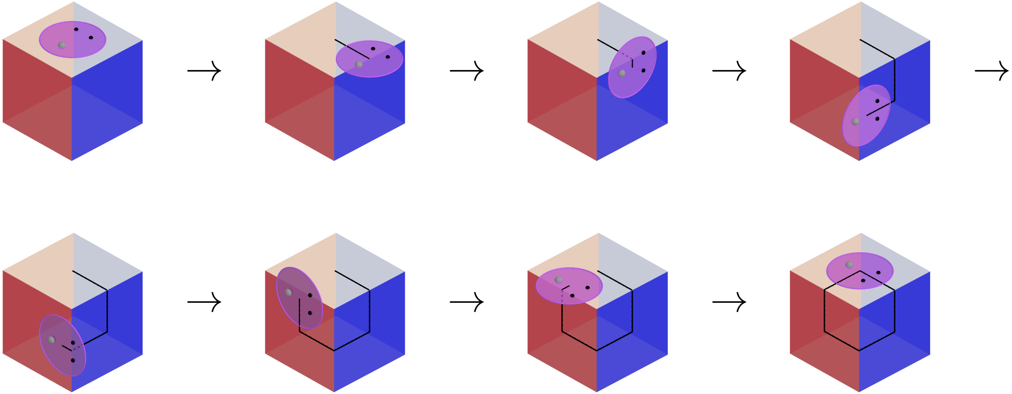
Definition

$\mathrm{EM}(G,n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G,n)) \overset{\text{def}}{=} \sum_{Y:\mathcal{U}} \|Y \simeq \mathrm{K}(G,n)\|_{-1}$

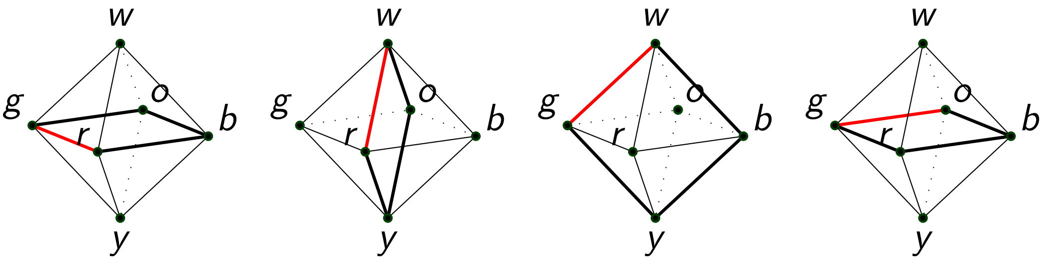
Definition

A $\mathrm{K}(G,n)$-bundle on a type M is a map $f : M \to \mathrm{EM}(G,n)$.

We further assume f factors through $\mathrm{K}(G,n+1)$ and so is principal.



Extend link from vertices to edges of the octahedron, by imagining tipping:



We obtain $\mathrm{tr}(\partial(wbr)) : Tw = Tw$.

Extend to the face wbr via homotopy $\flat(wbr) : \mathrm{id} = \mathrm{tr}(\partial(wbr))$.

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a cellular type and all the triangles commute in the diagram:

$$\begin{tikzcd}
\mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"'] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"] \\
 & & \mathcal{U} & &
\end{tikzcd}$$

- The map f_k is a *k-bundle* on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a *k-connection on the $(k-1)$-bundle f_{k-1}*.

Definition

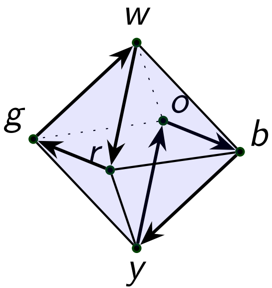
$$
\begin{tikzcd}
M_k \times \partial\Delta^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\Nearrow h_k"] & M_k \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r, "\imath_{k-1}"] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \arrow[dl, phantom, "\Swarrow"] \\
 & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{m_k\} \times \partial\Delta^k \arrow[r, "!"] \arrow[d, "\mathbb{A}_{k-1}"'] \arrow[dr, phantom, "\Swarrow \flat_k"] & \mathbf{1} \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}
$$

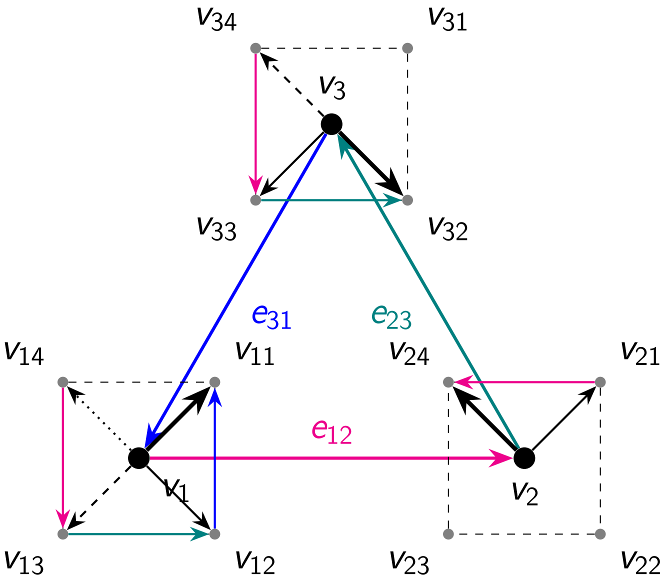
the filler $\flat_k$ is called a flatness structure for the face m_k, and its ending path is called curvature at the face m_k.

Vector fields

Let $T : \mathbb{M}_2 \to \mathsf{K}(\mathbb{Z}, 2)$ be an oriented tangent bundle on an oriented 2-dim cellular type

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It's a nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.





- $\partial F \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$
- We access pathovers asymmetrically: $X_{12} : T_{12} X_1 =_{T_2} X_2$
- $X(\partial F)$ is 3-sided inside a square
- To make a loop we cat with $\flat(\partial F)$

$$\mathrm{tr}_F \overset{\text{def}}{=} \mathrm{tr}(\partial F) \quad : Tm = Tm \qquad \text{holonomy}$$

$$\flat_F \overset{\text{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{Tm=Tm} \mathrm{tr}(\partial F) \qquad \text{flatness}$$

$$X_F \overset{\text{def}}{=} X(\partial F) \quad : \mathrm{tr}(\partial F)(X(m)) =_{Tm} X(m) \qquad \text{swirling}$$

Definition

The index of the vector field X on the face F is the integer

$$I_F^X \overset{\text{def}}{=} \Omega(\flat_F(X(m)) \cdot X_F) : \Omega(X(m) =_{Tm} X(m)).$$

$\mathrm{tr}_F \stackrel{\text{def}}{=} \mathrm{tr}(\partial F)$	$: Tm = Tm$	holonomy
$\flat_F \stackrel{\text{def}}{=} \flat(\partial F)$	$: \mathrm{id} =_{Tm=Tm} \mathrm{tr}(\partial F)$	flatness
$X_F \stackrel{\text{def}}{=} X(\partial F)$	$: \mathrm{tr}(\partial F)(X(m)) =_{Tm} X(m)$	swirling

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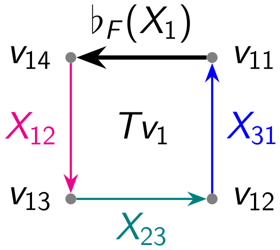

How do we make these happen?

$$\sum_F \flat_F \leftrightsquigarrow \int_M K \, dA$$

$$\sum_F I_F^X \leftrightsquigarrow \sum_{i=1}^n \mathrm{index}_{x_i}$$

$$? \leftrightsquigarrow \frac{1}{2\pi} \int_M K \, dA + \sum_{i=1}^n \mathrm{index}_{x_i} = 0$$

Observation 1: Use the torsor structure. If we choose $m : \mathbb{M}$ then $T_m = T_m$ acts on all fibers. We can define subtraction $T_i \times T_i \to (T_m = T_m)$.

Observation 2: Use the vector field. Given $X_i : T_i$ we can form subtraction $-X_i : T_i \to (T_m = T_m)$. $X_{ij} - X_j : T_{ji} X_i - X_j =_{T_m = T_m} 0$.

Observation 3: Use ap of addition. We can add $\alpha : a =_{\mathbb{C}(4)} 0$ and $\beta : b =_{\mathbb{C}(4)} 0$ to form $\alpha + \beta : (a + b) =_{\mathbb{C}(4)} 0$.

Together these remove the dependency. We can compute $\flat, I, X$ on each face independently and total them in $T_m = T_m$.

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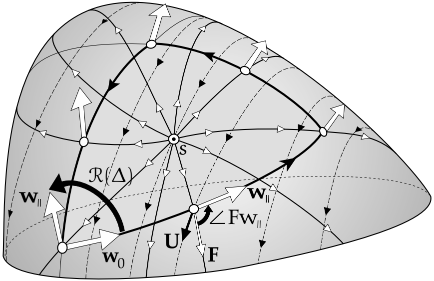
Classical proof



[26.2] *The difference $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ can be found by summing over the edges K_j the change $\Phi(K_j)$ in the illustrated angle $\angle F\mathbf{w}_{||}$, i.e., the rotation of $\mathbf{w}_{||}$ relative to* **F**.

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

Thank you.