Weak 2-randomness

Daniel Osherson and Scott Weinstein, *Recognizing Strong Random Reals*, 2008

Greg Langmead, November 23, 2020

Summary

- A random real should be impossible for us to **compute**
  - Not just directly, but also with machines that have the right bits as **limiting values** ("**computably approximable**")
- or to **computably recognize** as it streams in bit by bit
  - with a machine that maps $2^{<\omega} \to \{\text{Yes}, \text{No}\}$
- These operations motivate "strong M-L" tests
  - These are M-L tests but where the decreasing measure is not bounded by 2^{-n} (hence the name "strong 1-randomness")
  - These are exactly the Π_2^0 sets of measure 0 (hence the name "weak 2-randomness")

Weak 2-random $\hookrightarrow$ ML random $\hookrightarrow$ Computably random $\hookrightarrow$ Schnorr random $\hookrightarrow$ Weak 1-random

Review: Borel hierarchy

- Σ^0_1: effectively open; unions of prefix-free cylinders
- Π^0_1: effectively closed; complement of Σ^0_1
- Σ^0_2: countable union of uniformly Π^0_1 sets (WLOG: nested)
- Π^0_2: countable intersection of uniformly Σ^0_1 sets (WLOG: nested)

Computably approximable

aka *limiting recursive*

- Computable sequence $P : \mathbb{N} \to \{0,1\}$. It emits bit *j,* marked final*,* and terminates.
- The *limiting* version converges to a bit: $P : \mathbb{N} \to 2^{\omega}$, s.t. $\lim_{n\to\infty} P(j)_n$ exists for all j.
  - Epistemic Note: as observers at finite time n we don't get to know if the limit is achieved

$\exists$ a computably approximable ML random

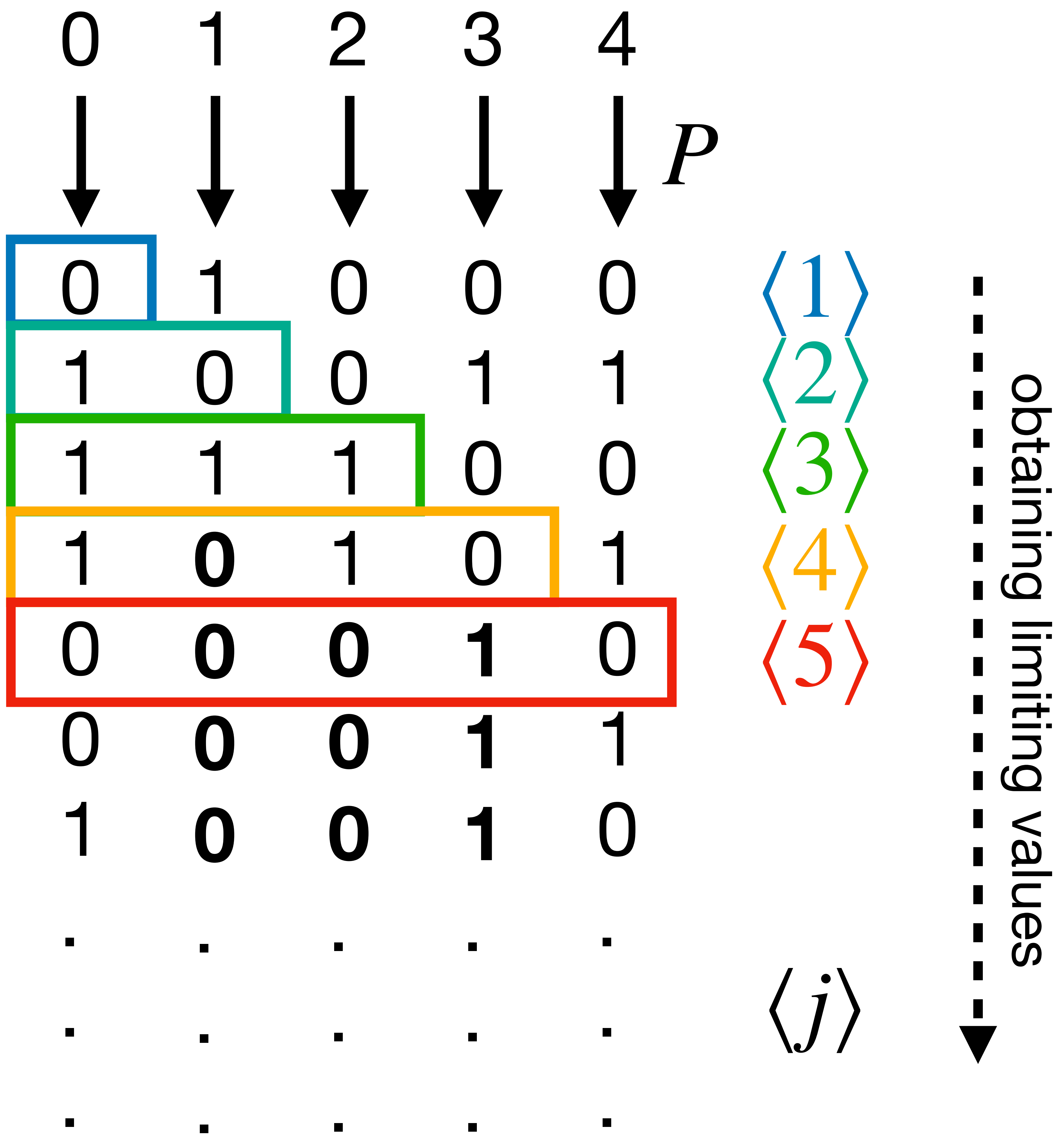
- Chaitin's $\Omega = \sum_{\sigma \in \mathrm{dom}\,\mathcal{U}} 2^{-|\sigma|}$ is ML-random number
    - approximable from below by $\Omega_s = \sum_{\mathcal{U}(\sigma)[s]\downarrow} 2^{-|\sigma|}$
- $\mathcal{U}$ is a universal prefix-free compression machine implementing Kolmogorov complexity.
- So Ω is a left-c.e. real
- Hence computably approximable (each digit can be computed in finite steps)
- Such computability should be non-random, so let's tighten the definition.

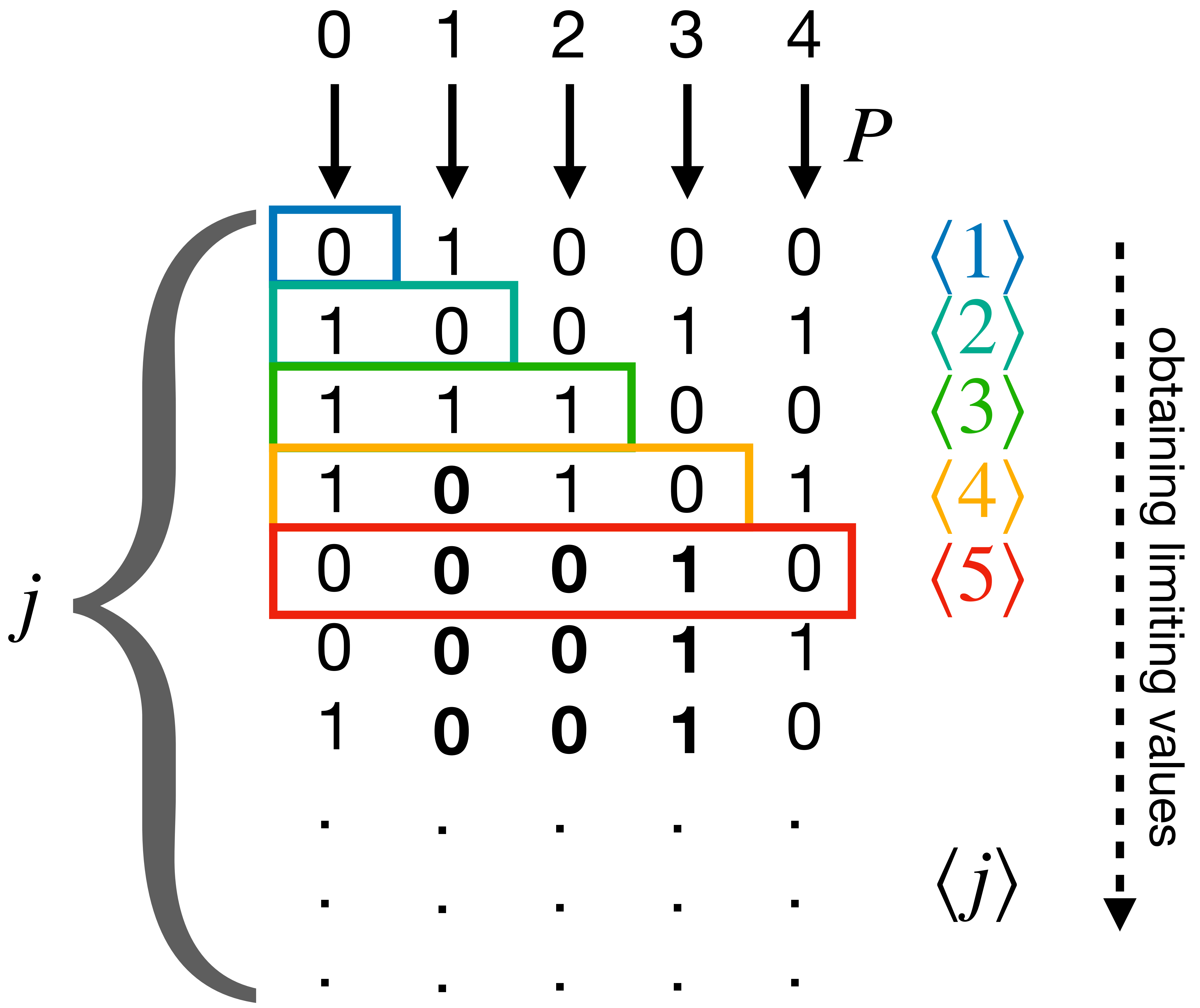
Generalized ML tests

- Uniformly Σ^0_1 sets $\{U_n\}$ with $\mu\left(\bigcap_n U_n\right) = 0$
    - drops the condition $\mu\left(U_n\right) \leq 2^{-n}$ from ML randomness
    - WLOG: U_n are nested, and each is a union of cylinders $[W_n]$ for some c.e. collection of prefix-free strings $W_n \subset 2^{<\omega}$
    - O&W name the uniform enumeration f, so f_i is an enumeration of W_i
    - equivalently $\lim_{n\to\infty} \mu(U_n) = 0$
    - **a Π^0_2 set of measure 0**
- hence: there are *more* such tests than ML-randomness, and *fewer* randoms
- also called: strong 1-randomness, weak 2-randomness

Weak 2-randoms are not computably approximable

- weak 2-random $\implies$ no computable approximation
- contrapositive: if x is c.a., i.e.
  $\exists P \lim_{n\to\infty} P(i)_n = x_i \implies x \in \bigcap_i [W_{f(i)}]$
- Given P,
  - Define $\langle j \rangle$, a sequence of length j, by $\langle j \rangle(i) = P(i)_j$
    - call this a "P-output": we have run P for j steps and j is also the length
  - $W_{f(n)} = \{\sigma \in 2^{<\omega} : (|\sigma| = n)$ and $\exists j > n\,.\,\sigma \subset \langle j \rangle\}$
    - sequences of length n that are prefixes of P-outputs





$[W_{f(n)}] \supseteq [W_{f(n+1)}]$

Being a prefix of some $\langle j \rangle$ means your prefixes are as well.

If P limit-computes x then

$\forall n \, \exists j \, . \, x[n] = \langle j \rangle [n].$

Eventually P gets the first n bits of x right.

If $y \neq x$ then

$\forall n \, \exists j \, . \, \forall k \geq j \, . \, y[n] \neq \langle k \rangle [n]$

if you differ from x at some bit, then eventually so does $\langle k \rangle$.

$\therefore \bigcap_n [W_{f(n)}] = \{x\}$ and so f is a GML test

Weak 2-tests = computable learners

- Computable learner: $L : 2^{<\omega} \to \{\text{Yes}, \text{No}\}$
- L **recognizes** $x \in 2^\omega$ if $\exists S \, . \, \mu(S) = 0$, $x \in S$, and s.t. $y \in S \iff L(y[n]) = \text{Yes}$ infinitely often
  - Note: this means $\{L(y[n])\}$ does not converge to No (might or might not converge to Yes)
  - It's like saying "I never stop resonating with x and its friends in S"
  - Note: later we'll consider learners that converge to Yes
  - Note: having a whole set S that the learner recognizes instead of just x is confusing!
- Claim: strong ML test $\iff L$

Given a learner L for a set $X \ni x$

$W_{f(n)} = \{b \in 2^{<\omega} \,.\, \exists c \subset b, |\,c\,| \geq n, L(c) = \text{Yes}\}$

descending chain because such c works for level $n-1$ as well.

$\bigcap_n [W_{f(n)}] = S$ so f is a GML test.

Given a GML test f

Enumerate the elements of $W_{f(n)} : W_{f(n),1}, W_{f(n),2}, \cdots$

these are each single strings w/ two parameters

Define $g : 2^{<\omega} \to \mathbb{N}$ by

$g(b) = \max(\{j \,.\, \exists \sigma \subset b \,.\, \sigma \in \text{some } W_{f(j),\leq|b|}\})$

length of longest prefix seen in an expanding set

L will detect jumps in g:

For $\beta \in \{0,1\}$ define $L(b\beta) = \text{Yes}$ if $g(b\beta) > g(b)$, else No.

oh look, I see a longer string in one of the GML test sets I'm visiting

Strong recognition

- The learner L strongly recognizes x if it converges to Yes (is Yes cofinitely often in O&W's terminology).
- We will prove this is equivalent to weak 1-randomness, a.k.a. Kurtz randomness.
- A number is Kurtz random iff it is in every measure-1 Σ^0_1 set.
- This is the largest set of randoms in the zoo.

Given a strong learner L for a set $Y \ni x \Longrightarrow x$ is not Kurtz random

$Y_m = \{b \in 2^\omega . \forall n > m, L(b[n]) = \text{Yes}\}$ ← recognized by L starting at bit m

$\bigcup_m Y_m$ is everything recognized by L

and $x \in \text{some } Y_m$, call it Y_k. ← Y, Y_m measure 0 by hypothesis

$S_m = \{b \in 2^\omega . |b| > m \text{ and } L(b) = \text{No}\}$

$[S_m] = 2^\omega - Y_m$. ← not recognized by L at bit m

S_k is a Kurtz test and $x \notin S_k$.

Not Kurtz random $\implies$ strongly recognized

Given S with $x \notin [S]$, $\mu([S]) = 1$.

Enumerate the elements of S:
$S_1, S_2, \ldots$

$$L(b) = \begin{cases} \text{No} & \exists \sigma \subset b \,.\, \exists i \,.\, \sigma \in S_{\leq i} \\ \text{Yes} & \text{otherwise} \end{cases}$$

← No means "in $[S]$" because to be in S means being in some S_n

$\{n : L(y[n]) = \text{Yes}\}$ is cofinite iff $y \notin [S]$, hence includes x.

Here's the Deal

We have finite minds and computing machines are good models of this.

"Random" should mean "numbers we cannot identify with our minds."

If we strongly recognize a number, it is definitely non-random. But we can weaken this and say that even more numbers are non-random.

To weakly recognize a number you have to answer "Yes" infinitely many times.

This is "minimally infinite" and so is the right demarcation line.

A couple counterpoints

- Computable learners might spend 10 million years between Yeses.
- Do we really recognize such a number in practical terms?
- Maybe it's OK for these to be called "random" since they are not super-accessible to us and our computers.