DRAFT: Discrete differential geometry in homotopy type theory

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Motivation

To use HoTT to study connections and explain their applicability to algebraic topology, via

- the Gauss-Bonnet theorem
- its vast generalization, Chern-Weil theory

Theorem (Gauss-Bonnet)

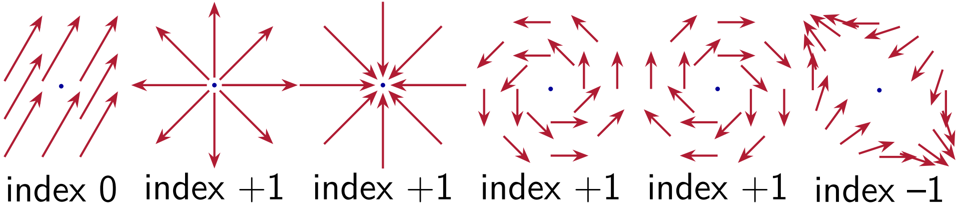
Let M be a compact 2-manifold without boundary, equipped with a Riemannian metric. Let K be the Gaussian curvature of M and let $\chi(M)$ be the Euler characteristic. Then

$$\frac{1}{2\pi}\int_M K\,dA = \chi(M).$$

Theorem (Poincaré-Hopf)

Let M be a compact smooth manifold without boundary. Let X be a vector field on M with isolated zeroes $x_1, \ldots, x_n$. Then

$$\sum_{i=1}^{n} \mathrm{index}_{x_i} = \chi(M).$$



Plan

- Manifolds
- Classifying maps
- Connections and curvature
- Theorems

HoTT background

1. Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-) Symmetry. `https://github.com/UniMath/SymmetryBook`.
2. Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023) Central H-spaces and banded types. arXiv:2301.02636
3. Scoccola, L. (2020) Nilpotent types and fracture squares in homotopy type theory, MSCS 30(5). arXiv:1903.03245

Discrete manifolds in HoTT

- Recall the classical theory of simplicial complexes
- Define a realization procedure to construct types

Simplicial complexes

Definition

An abstract simplicial complex M of dimension n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

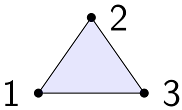
- a set M_0 of $(n+1)$ vertices
- sets M_k of subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

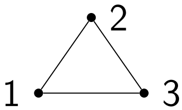
We call the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \ldots, M_k]$ the k-skeleton of M.

Simplicial complexes

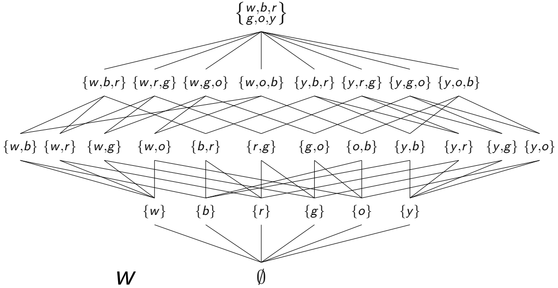
Example

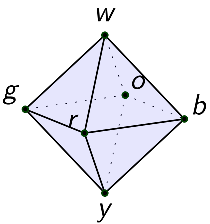
The complete simplex of dimension n, denoted $P(n)$, is the set $\{1, \ldots, n+1\}$ and its power set. The $(n-1)$-skeleton $P(n)_{\leq(n-1)}$ is denoted $\partial P(n)$ and will serve as a combinatorial $(n-1)$-sphere.

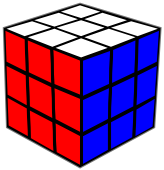
e.g., $P(2)$ is 

, $\partial P(2)$ is 

Simplicial complexes







Here is a Hasse diagram of an abstract octahedron (vertices named for the colors on a Hungarian Cube)

Homotopy realization: dimension 0

We will realize simplicial complexes as pushouts.

The realization of a 0-dimensional complex M_0 is the set M_0.

In particular the 0-sphere $\partial\Delta^1 \stackrel{\text{def}}{=} \partial P(1)$.

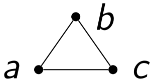
Homotopy realization: dimension 1

For a 1-dim complex $M \overset{\text{def}}{=} [M_0, M_1]$ form

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

$$\{\{w\}, \{g\}\} \leftarrow \{\{w, g\}\} \times \{0, 1\} \rightarrow \{\{w, g\}\}$$

Homotopy realization: dimension 1

Next construct a 1-sphere $\partial\Delta^2 \overset{\text{def}}{=}$ 

:

$$
\begin{tikzcd}
\partial P(2)_1 \times \partial\Delta^1 \arrow[r] \arrow[d] & \partial P(2)_1 \arrow[d] \\
\partial P(2)_0 \arrow[r] & \partial\Delta^2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

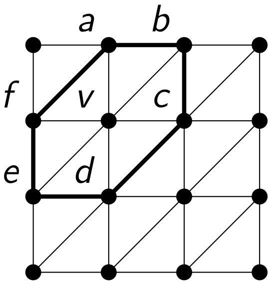
$$
\begin{tikzcd}
\{\{a,b\},\{b,c\},\{c,a\}\} \times \{0,1\} \arrow[r] \arrow[d] & \{\{a,b\},\{b,c\},\{c,a\}\} \arrow[d] \\
\{\{a\},\{b\},\{c\}\} \arrow[r] & \partial\Delta^2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Homotopy realization: dimension 2

To realize $M \stackrel{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\Delta^1, \partial\Delta^2$:

$$
\begin{tikzcd}
M_1 \times \partial\Delta^1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_0"'] \arrow[dr, phantom, "\Nearrow h_1"] & M_1 \arrow[d, "*_{\mathbb{M}_1}"] & \\
M_0 = \mathbb{M}_0 \arrow[r] & \mathbb{M}_1 \arrow[r] \arrow[dr, phantom, "\Searrow h_2"] & \mathbb{M}_2 \\
& M_2 \times \partial\Delta^2 \arrow[u, "\mathbb{A}_1"] \arrow[r, "\mathrm{pr}_1"'] & M_2 \arrow[u, "*_{\mathbb{M}_2}"']
\end{tikzcd}
$$

Homotopy realization



The link of a vertex v in a 2-complex is the polygon of edges not containing v but whose union with v is a face.

This will be our model of the tangent space.

Smoothness

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a combinatorial manifold: a simplicial complex of dimension n such that the link is a combinatorial $(n-1)$-sphere, i.e. its geometric realization is an $(n-1)$-sphere.

`https://ncatlab.org/nlab/show/triangulation+theorem`

What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of torsors, called principal bundles.

Torsors
Definition

- Let G be a group with identity element e.
- A *G-set* is a set X equipped with a homomorphism $\phi : (G, e) \to \mathrm{Aut}(X)$.
- If we have a proof of

$$\mathrm{is\_torsor}(X, \phi) \stackrel{\mathrm{def}}{=} ||X||_{-1} \times \prod_{x:X} \mathrm{is\_equiv}(\phi(-, x))$$

  we say (X, ϕ) is a *G-torsor*. Denote the type of G-torsors by BG.
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

Facts

- $\Omega(BG, G_{\text{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.
- BG is connected.
- Previous 2 $\implies$ BG is a $\mathrm{K}(G, 1)$.
- $\mathrm{ev}(e) : (G_{\text{reg}} =_{BG} X) \to X$ is an equivalence.
- See the Buchholtz et. al. H-spaces paper for more.

A connected component of $\mathcal{U}$?

Definition

The type of Eilenberg-Mac Lane spaces $\mathrm{EM}(G, n)$ is the connected component of $\mathrm{K}(G, n)$:

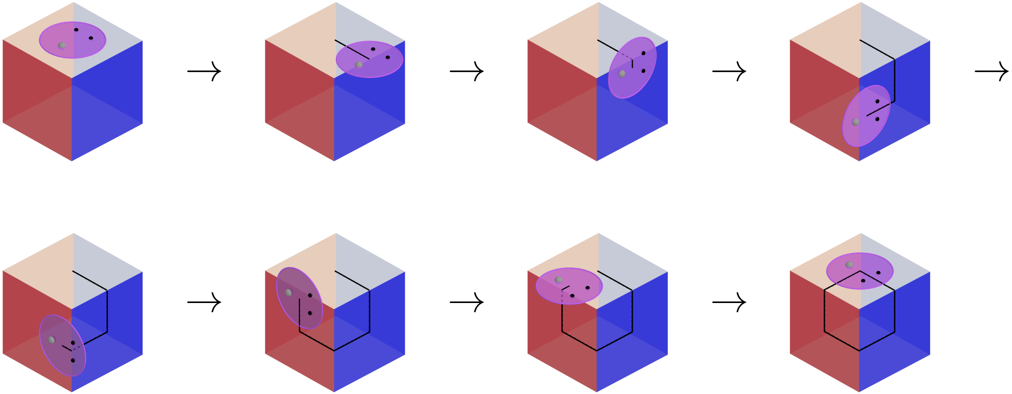
$$\mathrm{EM}(G, n) \stackrel{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n)) \stackrel{\text{def}}{=} \sum_{Y:\mathcal{U}} ||Y \simeq \mathrm{K}(G, n)||_{-1}$$

It is a property of a map $f : A \to \mathrm{EM}(G, n)$ to factor through $\mathrm{K}(G, n+1)$. See the Scoccola paper.

Coincidences of 2 dimensions

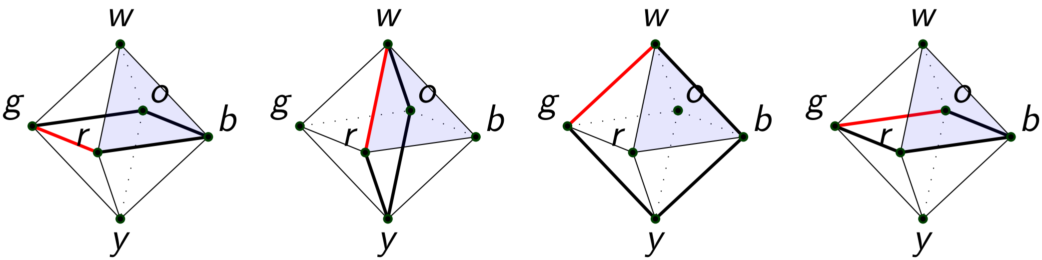
- S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$ since $\Omega(S^1, \mathrm{base}) \simeq \mathbb{Z}$.
- So $\mathrm{EM}(\mathbb{Z}, 1)$ is a type of mere circles.
- But $S^1 =_{\mathrm{EM}(\mathbb{Z},1)} S^1$ contains an order 2 flip, so $\not\simeq S^1$.
- For a map $f : A \to \mathrm{EM}(\mathbb{Z}, 1)$ to factor through $\mathrm{K}(\mathbb{Z}, 2)$, it must somehow avoid flips.
- This deserves to be called orientability.
- $\mathrm{link} : \mathbb{M}_0 \to \mathrm{EM}(\mathbb{Z}, 1)$ is a great starting point.

What we hope to capture and explain



$T : \mathbb{M} \to \mathrm{EM}(\mathbb{Z}, 1)$ extending link

We define T on edges by imaginging tipping:



$\mathrm{tr}(\partial(wbr)) : Tw = Tw$ is clockwise rotation by one notch.

We define T on the face wbr by the shortest homotopy $T(wbr) : \mathrm{id} = \mathrm{tr}(\partial(wbr))$.

Definition

If $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is a realization and all the triangles commute in the diagram:

$$\begin{tikzcd} \mathbb{M}_0 \arrow[r, "\imath_0"] \arrow[drr, "f_0"] & \mathbb{M}_1 \arrow[r, "\imath_1"] \arrow[dr, "f_1"] & \mathbb{M}_2 \arrow[r, "\imath_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "\imath_{n-1}"] & \mathbb{M}_n \arrow[dll, "f_n"] \\ & & \mathcal{U} & & \end{tikzcd}$$

- The map f_k is a *k-bundle* on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a *k-connection on the $(k-1)$-bundle f_{k-1}*.

Definition (cont.)

$$
\begin{tikzcd}
M_k \times \partial\Delta^k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbb{A}_{k-1}"'] & M_k \arrow[d, "*_{\mathbb{M}_k}"] \\
\mathbb{M}_{k-1} \arrow[r, "\imath_{k-1}"] \arrow[ur, phantom, "\Nearrow h_k"] \arrow[dr, "f_{k-1}"'] & \mathbb{M}_k \arrow[d, "f_k"] \arrow[ul, phantom, very near start, "\lrcorner"] \\
& \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{F\} \times \partial\Delta^2 \arrow[r, "!"] \arrow[d, "\mathbb{A}_{k-1}"'] & \mathbf{1} \arrow[d, "*_{\mathbb{M}_k}"] \arrow[dl, phantom, "\swarrow \flat_F"] \\
\mathbb{M}_{k-1} \arrow[r] & \mathcal{U}
\end{tikzcd}
$$

the filler $\flat_F$ is called a flatness structure for the face F, and its ending path (the holonomy around the boundary) is called the k-curvature at the face F.

With these definitions we have now achieved one of our main goals.

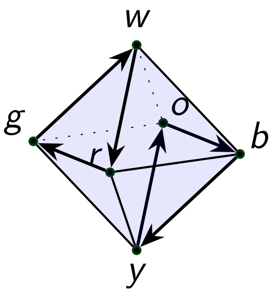
Without a definition of Euler characterisitc we can't prove Gauss-Bonnet.

But once we add vector fields there is a lot more to say.

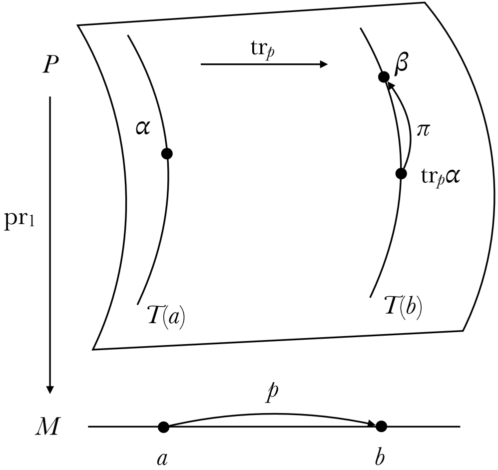
Vector fields

Let $T : \mathbb{M}_2 \to \mathrm{K}(\mathbb{Z}, 2)$ be an oriented tangent bundle on a 2-dim cellular type

- A vector field is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It's a nonvanishing vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.



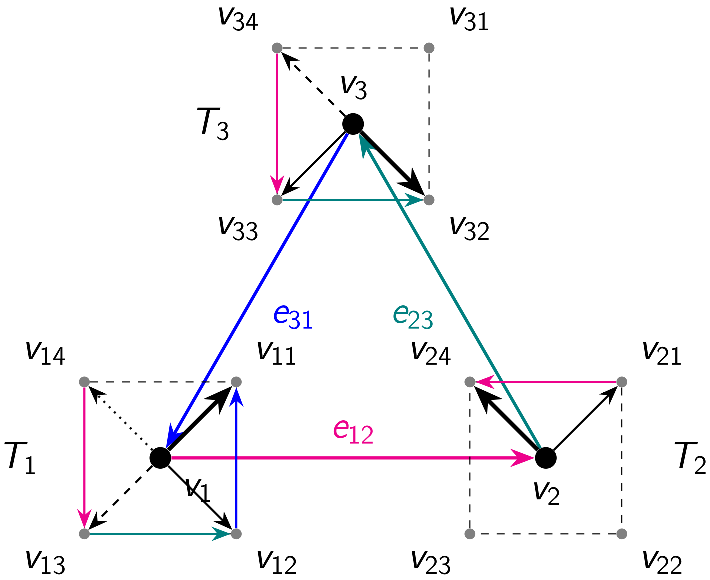
Pathovers



- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display it.
- Dependent functions map paths to pathovers (apd).

Building up a triangle-over

$$
\begin{tikzcd}
T_1 \arrow[r, "T_{21}"] & T_2 \arrow[r, "T_{32}"] & T_3 \arrow[r, "T_{13}"] & T_1 \\
 & & & T_{13}T_{32}T_{21}X_1 \\
 & & T_{32}T_{21}X_1 & T_{13}T_{32}X_2 \arrow[u, equal, "T_{13}T_{32}X_{12}:"] \\
 & T_{21}X_1 & T_{32}X_2 \arrow[u, equal, "T_{32}X_{12}:"] & T_{13}X_3 \arrow[u, equal, "T_{13}X_{23}:"] \\
X_1 & X_2 \arrow[u, equal, "X_{12}:"] & X_3 \arrow[u, equal, "X_{23}:"] & X_1 \arrow[u, equal, "X_{31}:"]
\end{tikzcd}
$$



- $\partial F \stackrel{\text{def}}{=} e_{12} \cdot e_{23} \cdot e_{31}$.
- tr thins out arrows.
- X on a path is drawn in the path's color.
- $X(\partial F)$ traces 3 sides of a square.

$$\begin{aligned}
\mathrm{tr}_F &\stackrel{\text{def}}{=} \mathrm{tr}(\partial F) && : T_1 =_{K(\mathbb{Z},2)} T_1 && \text{holonomy} \\
\flat_F &\stackrel{\text{def}}{=} \flat(\partial F) && : \mathrm{id} =_{T_1 = T_1} \mathrm{tr}_F && \text{flatness} \\
X_F &\stackrel{\text{def}}{=} X(\partial F) && : \mathrm{tr}_F(X_1) =_{T_1} X_1 && \text{swirling}
\end{aligned}$$

Definition

The index of the vector field X on the face F is the integer

$$I_F^X \stackrel{\text{def}}{=} \Omega(\flat_F(X_1) \cdot X_F) : \Omega(X_1 =_{T_1} X_1).$$

$$\begin{tikzcd}
v_{14} \arrow[d, "X_{12}"'] \arrow[dr, phantom, "Tv_1"] & v_{11} \arrow[l, "\flat_F(X_1)"'] \\
v_{13} \arrow[r, "X_{23}"'] & v_{12} \arrow[u, "X_{31}"']
\end{tikzcd}$$

$$\mathrm{tr}_F \stackrel{\text{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{\mathrm{K}(\mathbb{Z},2)} T_1 \qquad \text{holonomy}$$

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How do we make these happen?

$$\sum_F \flat_F \leftrightsquigarrow \int_M K \, dA$$

$$\sum_F I_F^X \leftrightsquigarrow \sum_{i=1}^n \mathrm{index}_{x_i}$$

$$? \leftrightsquigarrow \frac{1}{2\pi} \int_M K \, dA - \sum_{i=1}^n \mathrm{index}_{x_i} = 0$$

Observation 1: Use the torsor structure. If we choose $m : \mathbb{M}$ then $T_m = T_m$ acts on all fibers. We can define subtraction $T_i \times T_i \to (T_m = T_m)$.

Observation 2: Use the vector field. Given $X_i : T_i$ we can form subtraction $-X_i : T_i \to (T_m = T_m)$. $X_{ij} - X_j : T_{ji}X_i - X_j =_{T_m = T_m} 0$.

Observation 3: Use ap of addition. We can add $\alpha : a =_{\mathbb{C}(4)} 0$ and $\beta : b =_{\mathbb{C}(4)} 0$ to form $\alpha + \beta : (a + b) =_{\mathbb{C}(4)} 0$.

Together these remove the dependency. We can compute $\flat, I, X$ on each face independently and total them in $T_m = T_m$.

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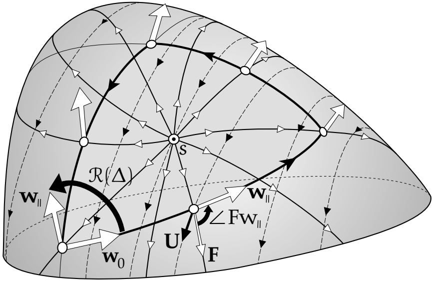
Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi\, \mathcal{I}_{\mathrm{F}}(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(\mathrm{K}_j)$ *in the illustrated angle* $\angle \mathrm{F}\mathbf{w}_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Thank you.