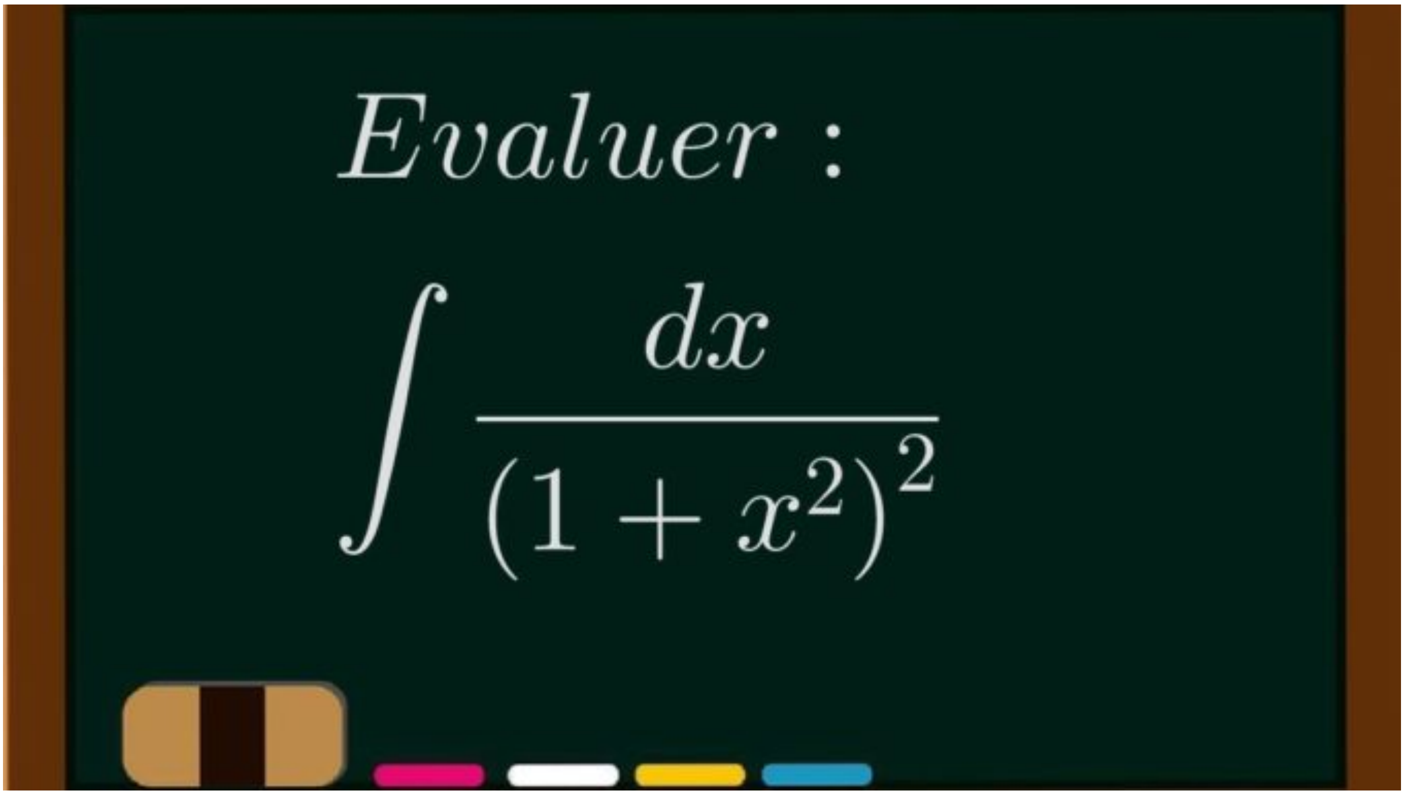
Ostrogradsky's method



This method is used to isolate the algebraic part in indefinite integral of the rational functions
$\int \frac{P(x)}{Q(x)} \mathrm{d}x = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}$

To find $Q_1(x)$:
This is the Greatest Common divisor of $Q(x)$ and its derivative.

To find $Q_2(x)$:
$Q_2(x) = \frac{Q(x)}{Q_1(x)}$

To find $P_1(x)$ and $P2_(x)$ we use partial fractions decomposition.

Finally we take the derivative of both sides.
We carry out the integration.

Evaluate: $\int \frac{\mathrm{d}x}{(1+x^2)^2}$

$P(x) = 1$
$Q(x) = (1+x^2)^2$
$Q'(x) = 4x(1+x^2)$

Greatest common divisor of $Q(x)$ and $Q'(x)$

$$Q_1(x) = (1 + x^2)$$

$$Q_2(x) = \frac{Q(x)}{Q_1(x)} = \frac{(1 + x^2)^2}{1 + x^2} = 1 + x^2$$

Now we put all together:

$$\int \frac{\mathrm{d}x}{(1 + x^2)^2} = \frac{Ax + B}{1 + x^2} + \int \frac{Cx + D}{1 + x^2}$$

Now we take the derivative of both sides:

$$\left(\int \frac{\mathrm{d}x}{(1 + x^2)^2} = \frac{Ax + B}{1 + x^2} + \int \frac{Cx + D}{1 + x^2}\right)'$$

We get:

$$\frac{\mathrm{d}x}{(1 + x^2)^2} = \left(\frac{Ax + B}{1 + x^2}\right)' + \frac{Cx + D}{1 + x^2}$$

$$\left(\frac{Ax + B}{1 + x^2}\right)' = \frac{A(1 + x^2) - 2x(Ax + B)}{(1 + x^2)^2}$$

$$\left(\frac{Ax + B}{1 + x^2}\right)' = \frac{A - 2Bx - Ax^2}{(1 + x^2)^2}$$

$$\frac{\mathrm{d}x}{(1 + x^2)^2} = \frac{A - 2Bx - Ax^2}{(1 + x^2)^2} + \frac{Cx + D}{1 + x^2}$$

Common denominator:

$$1 = (A - 2Bx - Ax^2) + (Cx + D)(1 + x^2)$$

$$1 = A - 2Bx - Ax^2 + Cx + D + Cx^3 + Dx^2$$

$$1 = (A + D) + (C - 2B)x + (-A + D)x^2 + Cx^3$$

Checking the equality:

$$A + D = 1 \Rightarrow D = 1 - A$$

$$C - 2B = 0 \Rightarrow C = 2B$$

$$C = 0$$

$$D - A = 0 \Rightarrow A = D$$

But:

$$A + D = 1 \Rightarrow 2A = 1 \Rightarrow A = D = \frac{1}{2}$$

$$C - 2B = 0 \Rightarrow C = 2B = 0$$

Putting it back:

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\int \frac{\mathrm{d}x}{1+x^2}$$

Finally:

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}x + C$$

Alternate methods:

Trigonometric substitution:

Evaluate: $\int \frac{\mathrm{d}x}{(1+x^2)^2}$

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Let:
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$x = \tan\theta$

$\mathrm{d}x = \sec^2\theta \mathrm{d}\theta$

$1 + x^2 = 1 + \tan^2\theta = sec^2\theta$

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \int \frac{sec^2\theta}{(\sec^2)^2\theta}\mathrm{d}\theta$$

$$\int \frac{sec^2\theta}{(\sec^2)^2\theta}\mathrm{d}\theta = \int \frac{\mathrm{d}\theta}{sec^2\theta} = \int \cos^2\theta \mathrm{d}\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = 1 + \cos 2\theta$$

$$\int \cos^2\theta \mathrm{d}\theta = \frac{1}{2}\int (1+\cos 2\theta)\mathrm{d}\theta$$

$$\int \cos^2\theta \mathrm{d}\theta = \frac{1}{2}\int \mathrm{d}\theta + \frac{1}{2}\int \cos(2\theta)\mathrm{d}\theta$$

$$\int \cos^2\theta \mathrm{d}\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

Back to x:

$x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\sin 2\theta = \frac{2x}{1+x^2}$

Finally:

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}x + C$$