Introductory Real Analysis Course 1 (27 September 2022)

Spherical neighborhood of a point $x_0 \in A \subset \mathbb{R}$

$B_{x_0}(\varepsilon) = \{x \in A : \ |x - x_0| < \varepsilon\} = (-\varepsilon + x_0, x_0 + \varepsilon)$

$\varepsilon < x - x_0 < \varepsilon$

$-\varepsilon + x_0 < x < x_0 + \varepsilon$

$B_{x_0}(\varepsilon) = I_{x_0}(\varepsilon)$

Example

$A = [0, 1)$ is a neighborhood of $x_0 = \frac{1}{3}$

$B_{\frac{1}{3}}(\frac{1}{100}) = \left(\frac{1}{3} - \frac{1}{100}, \frac{1}{3} + \frac{1}{100}\right) \subset [0, 1)$

We can say that A is a neighborhood of $x_0 = \frac{1}{3}$

Definition

$x_0 \in A' =$ (is the set of all limit points of A)

if $\forall X_{x_0}$ we have that $U_{x_0} \cap A \neq \emptyset$

Example

$A = (0, 1]$

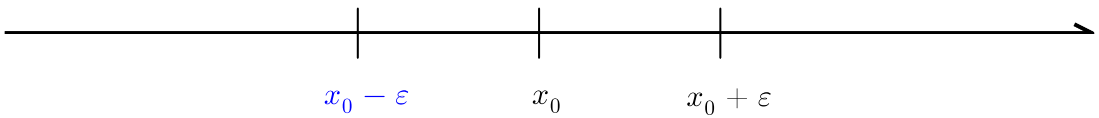
$\forall U_0 \cap A \neq ø$ or $\forall U_0 \cap A \neq \{0\}$ or $\forall U_0 \cap A \neq \{x_0\}$

We will have:

$0 \notin A$ or $0 \in A^{'}$

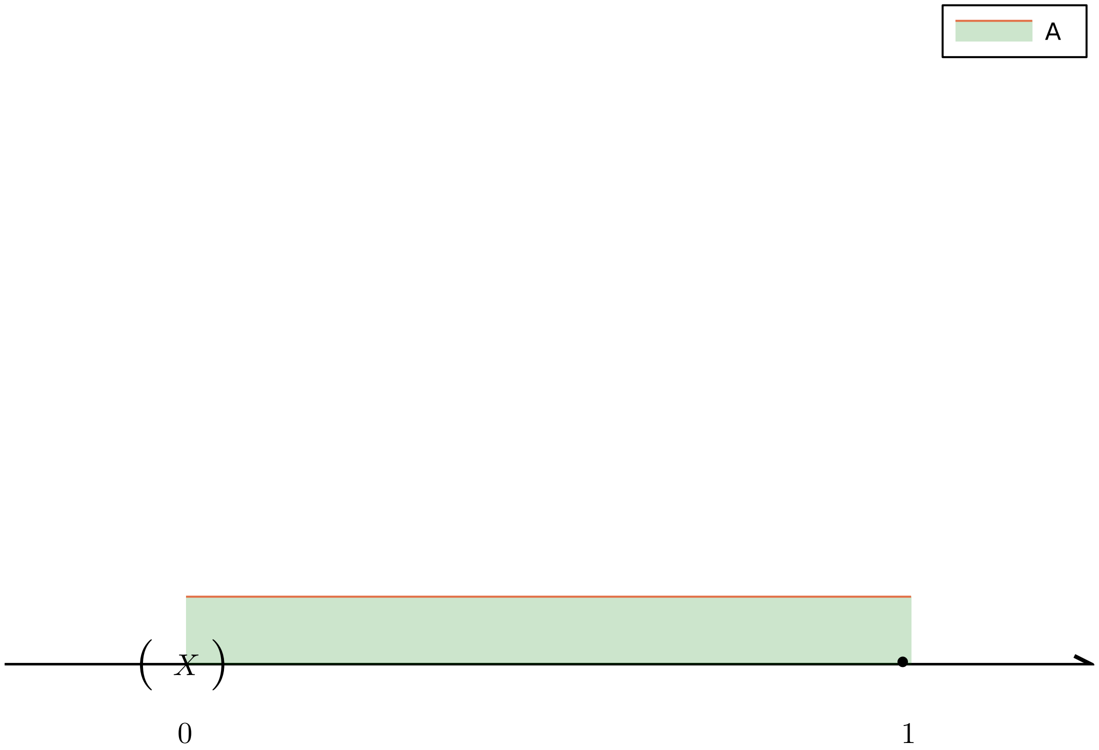
In [50]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)

annotate!([(0.5,0, (L"|", 17, :black))])
annotate!([(0.6,0, (L"|", 17, :black))])
annotate!([(0.7,0, (L"|", 17, :black))])
annotate!([(0.5,-0.1, (L"x_{0} - \varepsilon", 10, :blue))])
annotate!([(0.7,-0.1, (L"x_{0} + \varepsilon", 10, :black))])
annotate!([(0.6,-0.1, (L"x_{0}", 10, :black))])
```

Out[50]:



In [51]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)
plot!(f,0.4,0.8, fill=(0, 0.2, :green), label="A")

annotate!([(0.38,0, (L"(", 17, :black))])
annotate!([(0.4,0, (L"X", 10, :black))])
annotate!([(0.42,0, (L")", 17, :black))])
annotate!([(0.8,-0.01, (L"\cdot", 33, :black))])
annotate!([(0.4,-0.1, (L"0", 10, :black))])
annotate!([(0.8,-0.1, (L"1", 10, :black))])
```

Out[51]:



Example

$A = \{\frac{1}{n}\}_{n \in \mathrm{N}}$

$A^{'}$?

$\ln_{n \to +\infty} \frac{1}{n} = 0 \leftrightarrow$

$$\forall \varepsilon > 0 \ \exists \bar{n}_{\varepsilon}: \ \forall n > \bar{n}_{\varepsilon}: \ \left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$n > \bar{n}_a$

$0 < \frac{1}{n} < \varepsilon$

0 is the only limit part of A: $A^{'} = \{0\}$

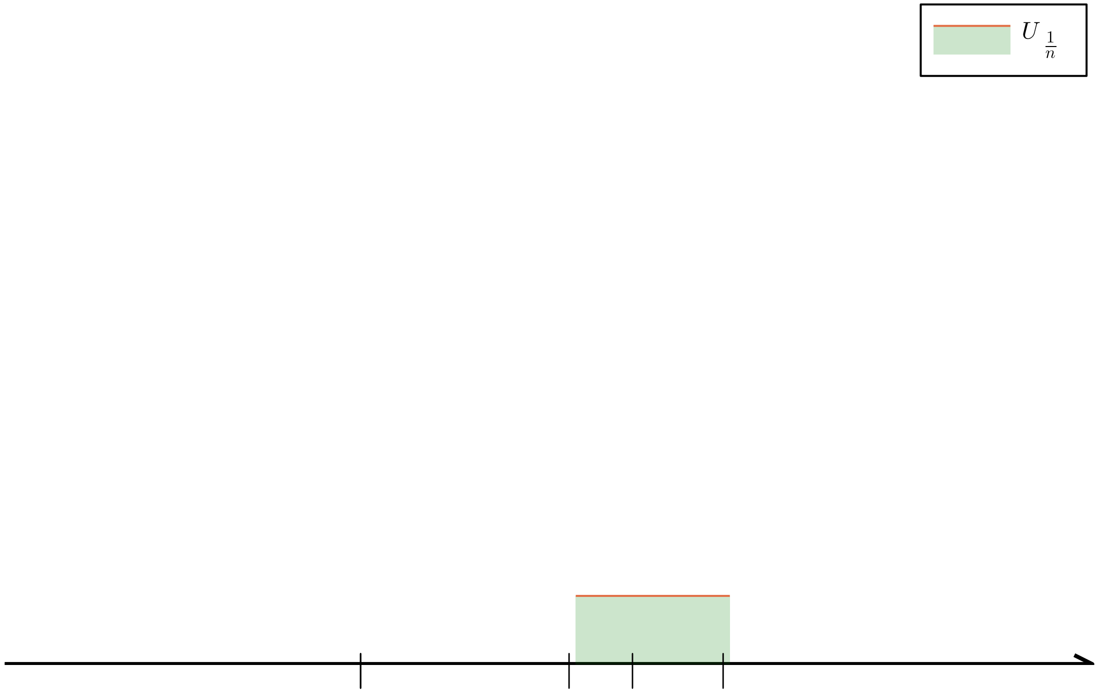
Why no other limit parts?

In [52]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)

annotate!([(0.58,0, (L"(", 17, :black))])
annotate!([(0.6,0, (L"X", 10, :black))])
annotate!([(0.615,0, (L"|", 12, :red))])
annotate!([(0.62,0, (L")", 17, :black))])
annotate!([(0.65,-0.01, (L"|", 12, :black))])
annotate!([(0.67,-0.01, (L"|", 12, :black))])
annotate!([(0.7,-0.01, (L"|", 12, :black))])
annotate!([(0.8,-0.01, (L"|", 12, :black))])
annotate!([(0.6,0.1, (L"0", 10, :black))])
annotate!([(0.615,-0.1, (L"\frac{1}{n}", 10, :red))])
annotate!([(0.65,-0.1, (L"\frac{1}{4}", 10, :black))])
annotate!([(0.67,-0.1, (L"\frac{1}{3}", 10, :black))])
annotate!([(0.7,-0.1, (L"\frac{1}{2}", 10, :black))])
annotate!([(0.8,-0.1, (L"1", 10, :black))])
```

Out[52]:

In [53]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)
plot!(f,0.615,0.7, fill=(0, 0.2, :green), label=L"U_{\frac{1}{n}}")

annotate!([(0.5,-0.01, (L"|", 12, :black))])
annotate!([(0.615,-0.01, (L"|", 12, :black))])
annotate!([(0.65,-0.01, (L"|", 12, :black))])
annotate!([(0.7,-0.01, (L"|", 12, :black))])
annotate!([(0.5,-0.01, (L"|", 12, :black))])
annotate!([(0.5,-0.1, (L"0", 10, :black))])
annotate!([(0.615,-0.1, (L"\frac{1}{m+1}", 10, :black))])
annotate!([(0.65,-0.1, (L"\frac{1}{m}", 10, :black))])
annotate!([(0.7,-0.1, (L"\frac{1}{m-1}", 10, :black))])
```

Out[53]:



Example

$A = \{\frac{(-1)^n}{n}\}$

$\{0\} = A^{'}$

Limit of Functions

Example

$$f(x)\begin{cases} 1 & x = 1 \\ x + 3 & x \neq 1 \end{cases}$$

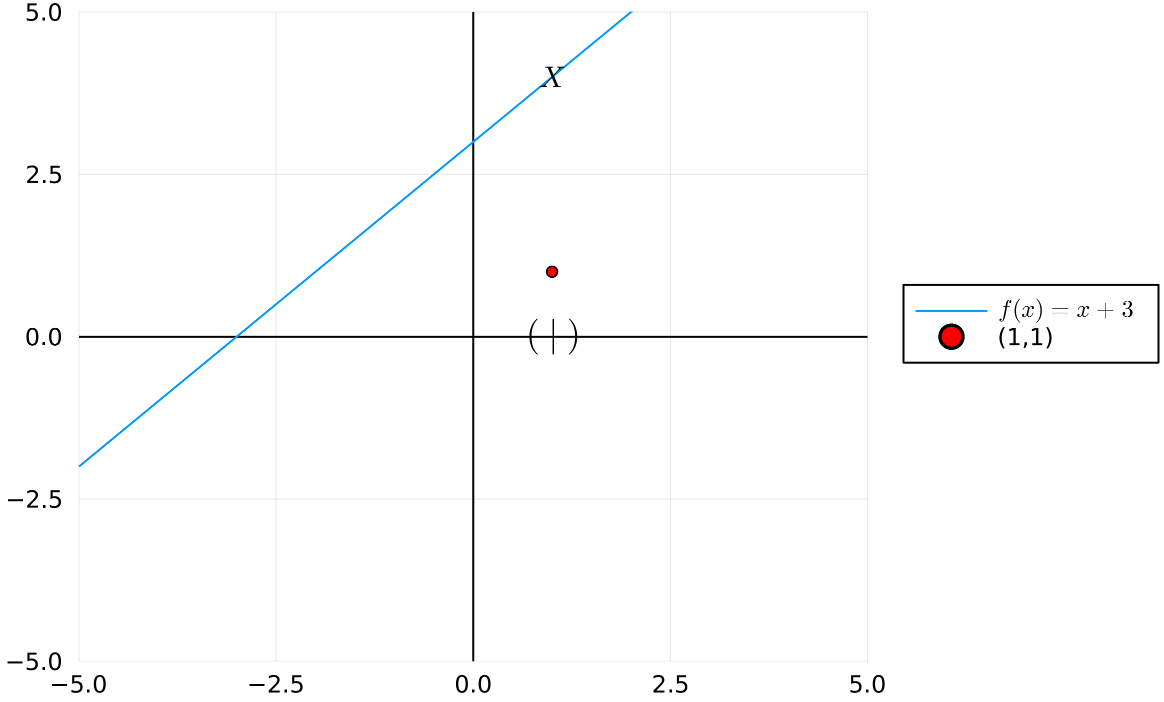
$f: \mathrm{R} \to \mathrm{R}$

$x \to 1^{\pm} \quad f(x) \to 4$

$\lim_{x \to 1} f(x) = 4$

In [54]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = x+3

plot(f,-5,5, xlims=(-5,5), ylims=(-5,5),
    bottom_margin = 10mm, label=L"f(x) = x+3", framestyle = :zerolines,
    legend=:outerright)

# Scatter plots
scatter!([1], [1], color = "red1", label="(1,1)", markersize = 3)

annotate!([(1,f(1), (L"X", 10, :black))])
annotate!([(0.8,0, (L"(", 12, :black))])
annotate!([(1.1,0, (L"|", 12, :black))])
annotate!([(1.3,0, (L")", 12, :black))])
```

Out[54]:



Example

$f(x) = \frac{1}{x^2}$

$D_+ = \mathrm{R} \setminus \{0\} = \{x \neq 0\}$

$x \rightarrow 0 \quad f(x) \rightarrow \frac{1}{0^+} = +\infty$

$x \rightarrow +\infty \quad f(x) \rightarrow \frac{1}{+\infty} = 0$

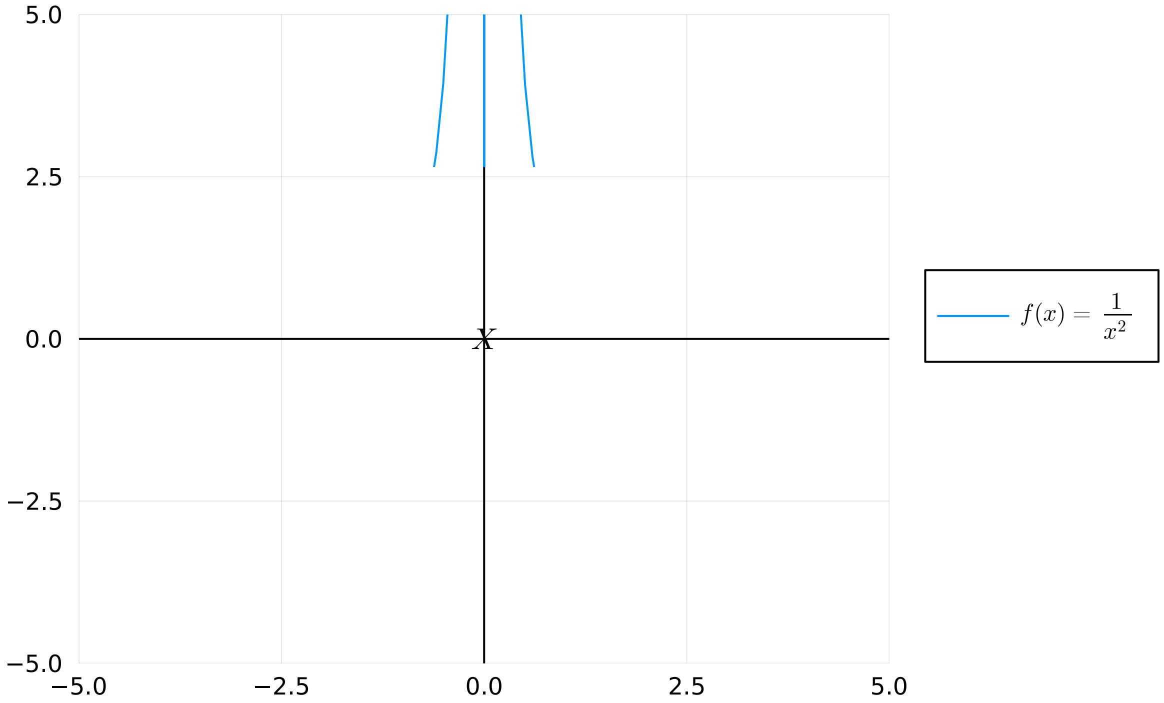
$x \rightarrow -\infty \quad f(x) \rightarrow \frac{1}{+\infty} = 0$ because $(-\infty)^2 = +\infty$

$f(x)$ even:

$f(x) = f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2}$

In [55]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 1/(x^(2))

plot(f,-5,5, xlims=(-5,5), ylims=(-5,5),
    bottom_margin = 10mm, label=L"f(x) = \frac{1}{x^{2}}", framestyle = :zerolines,
    legend=:outerright)

annotate!([(0,0, (L"X", 10, :black))])
```

Out[55]:



Example

$f(x) = \frac{1}{x^2} : [1, 3] \to \mathrm{R}$

$f(1) = 1$

$f(\frac{1}{2}) = 4$

Example

$f(x) = \frac{1}{x^2}\ :\ [-2, 0) \cup (0, +2] \to \mathrm{R}$

$x \to 0^{\pm} \quad \lim_{x \to 0^{\pm}} \frac{1}{x^2} = +\infty$

Definition

$f : D \subseteq \mathrm{R} \to \mathrm{R}$

$x_0 \in D^{'}$

$\lim_{x \to x_0} f(x) = l \quad l \neq \pm\infty$

if and only if:

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(x_0, \varepsilon) > 0 : \quad \forall x \in D \text{ with } 0 < |x - x_0| < \delta \to |f(x) - l| < \varepsilon$$

$x \neq x_0$

$l - \varepsilon < f(x) < l + \varepsilon$

$x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

Equivalent Definition of Limit by Neighborhoods

$\lim_{x \to x_0} f(x) = l \leftrightarrow \forall V_l \ \exists U_{x_0} : \quad \forall x \in D \cap U_{x_0} \setminus \{x_0\} \to f(x) \in V_l$

Example

$f(x) \quad D_f = \mathrm{R}$

$\lim_{x \to 1}(3x + 1) = 4$

We have to check that

$\forall \varepsilon > 0 \; \exists \delta > 0 : x \in \mathrm{R}$ with $0 < |x - 1| < \delta = \frac{\varepsilon}{3}$

thus

$$|f(x) - l| < \varepsilon$$
$$|3x + 1 - 4| < \varepsilon$$

Fix $\varepsilon > 0$

$$|3x - 3| < \varepsilon$$
$$3|x - 1| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{3} = \delta$$

Example

$\lim_{x \to 2} \frac{x}{x+1} = \frac{2}{3}$

$D_f = \{x \neq -1\}$

$\forall \varepsilon > 0 \; \exists \delta > 0$:

$\forall x \subset D_f$ with $0 < |x - 2| < \delta \to \left| \frac{1}{x+1} - \frac{2}{3} < \varepsilon \right|$

Fix $\varepsilon > 0$

$$\left|\frac{3x - 2x - 2}{3(x + 1)}\right| = c\varepsilon$$
$$\frac{|x - 2|}{3|x + 1|} < \varepsilon$$

For $x > 1 \rightarrow |x + 1| = x + 1 > 2$ and $\delta = 6\varepsilon$:

$$\frac{1}{|x + 1|} = \frac{1}{2}$$
$$\frac{|x - 2|}{3|x + 1|} < \frac{|x - 2|}{6} < \varepsilon$$
$$\frac{|x - 2|}{6} < \frac{6\varepsilon}{6}$$
$$|x - 2| < 6\varepsilon$$

Note:

If you make a mistake and a contradiction appears

$\lim_{x \rightarrow 2}\frac{x}{x+1} = 2$

$D_f = \{x \neq -1\}$

$x > -1$

Fix ε

$$\left|\frac{x}{x+1} - 2\right| < \varepsilon$$

$$\left|\frac{x-2x-2}{|x+1|}\right| < \varepsilon$$

$$\frac{|-(x+2)|}{|x+1|} < \varepsilon$$

$$(|x| = |-x|)$$

$$(x \to 2 \text{ we can assume } x > 1)$$

$$|x+2|$$

Introductory Real Analysis Course 2 (September 2022)

$\lim_{x \to 1} 3^{x+2} = 27$

$D_f = \mathrm{R}$

$\forall \varepsilon > 0 \, \exists \delta > 0: \ \forall x \in \mathrm{R}$ with

$0 < |x-1| < \delta \to |3^{x+2} - 27| < \varepsilon$

$$|3^{x+2} - 27| < \varepsilon$$

$$3^2 |3^x - 3| < \varepsilon$$

$$|3^x - 3| < \frac{\varepsilon}{9}$$

$$-\frac{\varepsilon}{9} < 3^x - 3 < \frac{\varepsilon}{9}$$

$$-\frac{\varepsilon}{9} + 3 < 3^x < 3 + \frac{\varepsilon}{9}$$

- Interesting case:

$-\frac{\varepsilon}{9} + 3 > 0$

$-\varepsilon + 27 > 0$

if $\varepsilon < 27$

If $\varepsilon = 27$ then $-\frac{\varepsilon}{9} + 3 < 3^x = 0$ is true.

If $\varepsilon > 27$ then $-\frac{\varepsilon}{9} + 3 < 0 < 3^x$ is true.

Take $\varepsilon < 27$, then

$$\begin{aligned}
\log_3\left(-\frac{\varepsilon}{9} + 3\right) &< \log_3 3^x &&< \log_3\left(3 + \frac{\varepsilon}{9}\right) \\
\log_3\left(-\frac{\varepsilon}{9} + 3\right) &< x \log_3 3 &&< \log_3\left(3 + \frac{\varepsilon}{9}\right) \\
\log_3\left(-\frac{\varepsilon}{9} + 3\right) &< x &&< \log_3\left(3 + \frac{\varepsilon}{9}\right)
\end{aligned}$$

We will have:

$\log_3\left(-\frac{\varepsilon}{9} + 3\right) < 1$

$\log_3\left(3 + \frac{\varepsilon}{9}\right) > 1$

The inequality:

$\log_3\left(-\frac{\varepsilon}{9} + 3\right) < x < \log_3\left(3 + \frac{\varepsilon}{9}\right)$

is a neighborhood of $x_0 = 1$

$$3-\frac{\varepsilon}{9}<3$$

$$\log_3\left(3-\frac{\varepsilon}{9}\right)<\log_3 3=1$$

Assume $x>1$, and $x\to 1^+$ means that $x-1>0$

$$\log_3\left(-\frac{\varepsilon}{9}+\varepsilon\right)-1<x-1<\log_3\left(3+\frac{\varepsilon}{9}\right)-1$$

$$x-1<\log_3 3\left(1+\frac{\varepsilon}{27}\right)-1$$

$$x-1<\log_3 3+\log_3\left(1+\frac{\varepsilon}{27}\right)-1$$

$$x-1<\log_3\left(1+\frac{\varepsilon}{27}\right)$$

with $\log_3\left(1+\frac{\varepsilon}{27}\right)=\delta_1$

Similarly, if $x<1:\ x\to 1^-$ you find δ_2 and then choose $\delta=\min(\delta_1,\delta_2)$

- Optional: in the more complex case to find the exact δ, it is sufficient to find a neighborhood of x_0

Definition

$f\colon D\subseteq \mathrm{R}\to \bar{\mathrm{R}}$

$x_0\in D'$

$\lim_{x\to x_0} f(x)=+\infty$

$\{-\infty\} \cup \mathrm{R} \cup \{+\infty\} = \bar{\mathrm{R}}$

$\forall M > 0\ \exists \delta = \delta(x_0, M) > 0:\ \forall x \in D$ with $0 < |x - x_0| < \delta \rightarrow f(x) > M$

an equivalent statement with neighborhood for $f(x) > M$ is $f(x) \in (M, +\infty)$

$V_{+\infty} := (\alpha, +\infty) \quad \alpha in \mathrm{R}$

$V_{-\infty} := (-\alpha, \beta) \quad \beta in \mathrm{R}$

$\forall V_{+\infty}\ \exists U_{x_0}:\ \forall x \in (U_{x_0} \cap D) \setminus \{x_0\} \rightarrow f(x) \in V_{+\infty}$

Example

$\lim_{x \to 1} \frac{1}{(x-1)} = \infty$

is a wrong notation, because

$\lim_{x \to 1^+} \frac{1}{(x-1)}$ has a plus sign.

$\lim_{x \to 1^-} \frac{1}{(x-1)}$ has a minus sign.

Example

$\lim_{x \to 1} \frac{1}{(x-1)^2} = \left(\frac{1}{0^+}\right) = +\infty$

$\forall M > 0\ \exists \delta > 0:\ \forall x \in \mathrm{R} \setminus \{1\}$

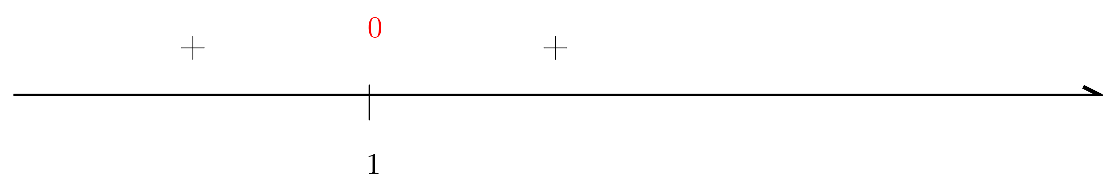
$D_f = \{\mathrm{R} \setminus \{1\}\}$

$$0 < |x - 1| < \delta \to \frac{1}{(x-1)^2} > M$$

$$(\text{since} x \neq 1 \; (x-1)^2 > 0)$$

In [56]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)

annotate!([(0.5,-0.1, (L"1", 10, :black))])
annotate!([(0.5,-0.01, (L"|", 12, :black))])
annotate!([(0.5,0.1, (L"0", 10, :red))])
annotate!([(0.4,0.07, (L"+", 12, :black))])
annotate!([(0.6,0.07, (L"+", 12, :black))])
```

Out[56]:



Definition

$f: D \subseteq \mathrm{R} \to \bar{\mathrm{R}}$

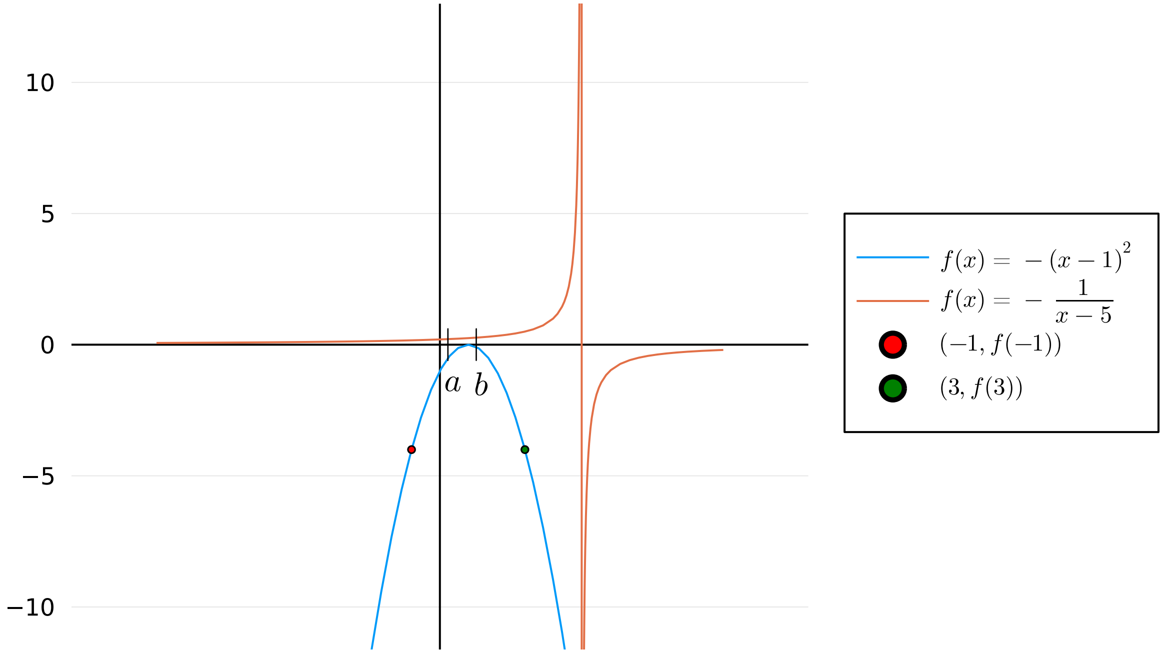
$x_0 \in D^{'}$

$\lim_{x \to x_0} f(x) = -\infty$

$\forall M > 0 \ \exists \delta = \delta(M, x_0) > 0: \ \forall x \in D \text{ with } 0 < |x - x_0| < \delta \to f(x) < -M$

In [57]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = -(x-1)^(2)
g(x) = -1/(x-5)

plot(f,-10,10, xticks=false, xlims=(-13,13), ylims=(-33,33),
    bottom_margin = 10mm, label=L"f(x) = -(x-1)^{2} ", framestyle = :zerolines,
    legend=:outerright)
plot!(g,-10,10, xticks=false, xlims=(-13,13), ylims=(-13,13),
    bottom_margin = 10mm, label=L"f(x) = - \frac{1}{x-5} ", framestyle = :zerolines,
    legend=:outerright)

scatter!([-1], [f(-1)], color = "red", label=L"(-1,f(-1))", markersize = 2)
scatter!([3], [f(3)], color = "green", label=L"(3,f(3))", markersize = 2)

annotate!([(0.5,0, (L"|", 11, :black))])
annotate!([(1.5,0, (L"|", 11, :black))])
annotate!([(0.5,-1.5, (L"a", 11, :black))])
annotate!([(1.5,-1.5, (L"b", 11, :black))])
```

Out[57]:



Example

Give the definition with neighborhoods

$\lim_{x \to 1^-} \left(e^{\frac{1}{(x-1)}} \right) = -\infty$

$e^x D_f = \{x \neq 1\}$

$x \rightarrow x - 1 \xrightarrow{1\,1} |x-1| \xrightarrow{\frac{1}{x}} \frac{1}{|x-1|} \xrightarrow{e} e^{\frac{1}{|x-1|}} \xrightarrow{-} -e^{\frac{1}{|x-1|}}$

$\forall M > 0 \; \exists \delta > 0 : \; \forall x \in D_{\frac{1}{|x-1|}}$ with $0 < |x-1| < s \rightarrow -e < -M$

Fix $M > 0$

- $\log_a x$ is a strictly increasing function for $a > 1$
- $\log_a x$ is a strictly decreasing function for $0 < a < 1$

$\ln() \rightarrow e^{\frac{1}{|x-1|}} > M$

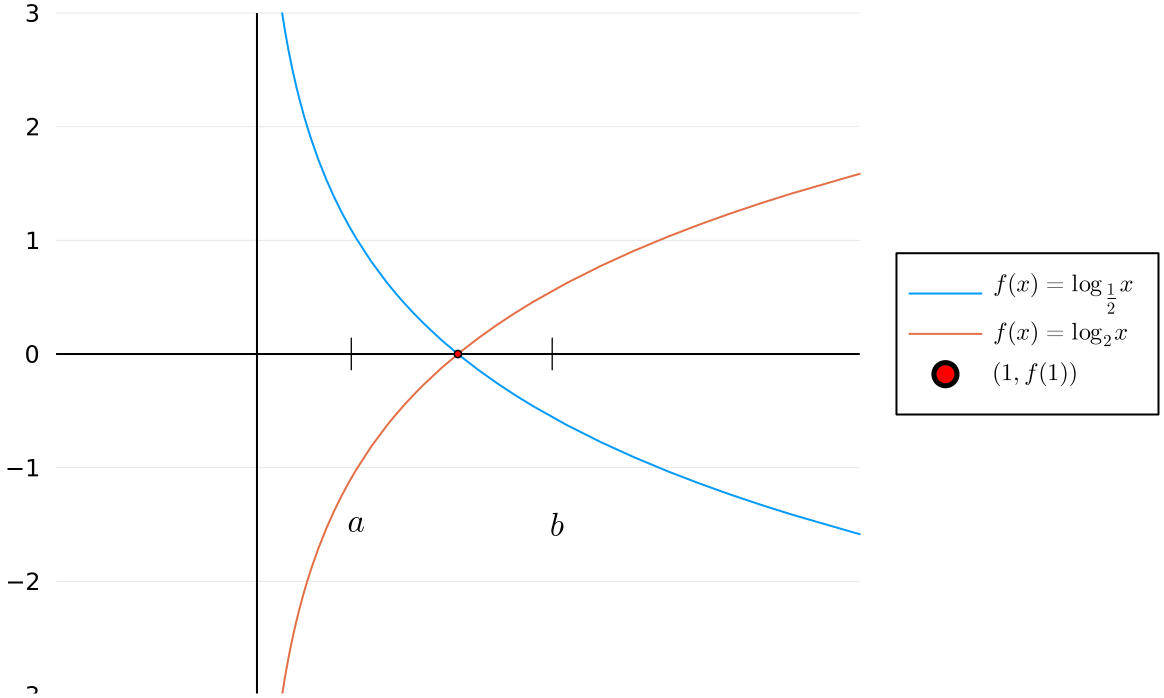
If $M \leq 1$ then $\ln() \rightarrow \frac{1}{|x-1|} > \ln M$

$\frac{1}{|x-1|} > 0$

If $M > 1$ then $|x-1| < \frac{1}{\ln \; M} = \delta$

In [58]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = log(1/2,x)
g(x) = log(2,x)

plot(f,-10,10, xticks=false, xlims=(-1,3), ylims=(-3,3),
    bottom_margin = 10mm, label=L"f(x) = \log_{\frac{1}{2}}x ", framestyle = :zerolines,
    legend=:outerright)
plot!(g,-10,10, xticks=false, xlims=(-1,3), ylims=(-3,3),
    bottom_margin = 10mm, label=L"f(x) = \log_{2}x ", framestyle = :zerolines,
    legend=:outerright)

scatter!([1], [f(1)], color = "red", label=L"(1,f(1))", markersize = 2)

annotate!([(0.5,0, (L"|", 11, :black))])
annotate!([(1.5,0, (L"|", 11, :black))])
annotate!([(0.5,-1.5, (L"a", 11, :black))])
annotate!([(1.5,-1.5, (L"b", 11, :black))])
```

Out[58]:



Definition

$\lim_{x \to +\infty} f(x) = +\infty / -\infty$

Definition

$\lim_{x \to -\infty} f(x) = +\infty / -\infty$

Recall

$p \rightarrow q \leftrightarrow -q \rightarrow -p$

Proof by contradiction:

$p \rightarrow q \leftrightarrow p \wedge -q \rightarrow -p$

On left side: p is the hypothesis and q is the theorem.

Deny this: $p \wedge -q \rightarrow -p$

"Bridge over Troubled Water Theorem"

Let $f : D \subseteq \mathrm{R} \rightarrow \mathrm{R}$

$x_0 \in D'$ then

$\lim_{x \to x_0} f(x) = l \leftrightarrow \forall \{x_n \subset_{n \in \mathrm{N}} D \; x_n \neq x_0\}$ with

$\lim_{n \to +\infty} x_n = x_0$ we have

$\lim_{n \to +\infty} f(x_n) = l$

Proof

Hypothesis:

1) $\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall x \in D$ with $0 < |x - x_0| < \delta \ \rightarrow \ |f(x) - l| < \varepsilon$

2) $x_n \ \rightarrow \ x_0 \ \leftrightarrow \ \forall \delta_1 > 0 \ \exists \bar{n} : \forall n > \bar{n} \ \rightarrow \ |x_n - x_0| < \delta_1$

Theorem:

$\forall \varepsilon > 0 \ \exists \tilde{n} : \forall n > \tilde{n} = 1$

$|f(x_n) - l| < \varepsilon$

Fix arbitrarily $\varepsilon > 0$ by

1) $\exists \delta > 0$ by

2) For this $\delta > 0 \ \exists \bar{n} : \ \forall n > \bar{n} \ \exists \delta > 0 \ |x_n - x_0| < \delta$ thus by 1) $|f(x_n) - l| < \varepsilon$

Hypothesis $\forall x_n \ \rightarrow \ x_0 \quad x_n \neq x_0$

$\forall \varepsilon > 0 \ \exists \bar{n} : \ \forall n > \bar{n} \ \rightarrow \ |f(x_n) - l| < \varepsilon$

Theorem:

$\lim_{x \rightarrow x_0} f(x) = l$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall x \in D$ with

$0 < |x - x_0| < \delta \ \rightarrow \ |f(x) - l| < \varepsilon$

Introductory Real Analysis Course 3 (30 September 2022)

To (The "Bridge Theorem")

Let $f : D \subseteq \mathrm{R} \ \rightarrow \ \mathrm{R}$ and $x_0 \in D^{'}$ then an equivalent

$\alpha = \ \lim_{x \rightarrow x_0} f(x) = l \ \leftrightarrow \ \forall \{x_n\} \subset D : x_n \neq x_0$ and $x_n \ \rightarrow \ x_0$ thus

$\lim_{n \to +\infty} f(x_n) = l$

Example

if $f(x) = x^2 - e^x$

$x_n = \frac{1}{n} \to 0$ for $n \to +\infty$

$f(x_n) = \frac{1}{n^2} - e^{\frac{1}{n}}$ (it is a sequence)

$f(x_n) \to -1$ for $n \to \infty$

To prove $\alpha \to \beta$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall x \in D$ with $0 < |x - x_0| < \delta \to |f(x) - l| < \varepsilon$

we also know

$x_n \to x_0 \leftrightarrow \forall - б > б \ \exists \bar{n} : \ \forall n > \bar{n} \to |x_n - x_0| < б$

Then for β

$\leftrightarrow \forall \varepsilon > 0 \ \exists \tilde{n} : \ \forall n > \tilde{n} \to |f(x_n) - l| < \varepsilon$

Choose $б = \delta \to \exists \tilde{n} : \ \forall n > \tilde{n} \to |x_n - x_0| < \delta \to |f(x_n) - l| < \varepsilon$

To prove $\beta \to \alpha$

By contradiction

$(p \to q \leftrightarrow -q \to -p)$

or

$(\beta) \to (\alpha) \leftrightarrow -(\alpha) \to -(\beta)$

Then

(α)

$\forall \varepsilon > 0 \ \exists \delta > 0: \ \forall x \in D$ with $0 < |x - x_0| < \delta \rightarrow |f(x) - l| < \varepsilon$

-(α)

$\exists \varepsilon > 0 \ \forall \delta > 0: \ \exists x_0 \in D$ with $0 < |x - x_0| < \delta \rightarrow |f(x) - l| \geq \varepsilon$

Choose $\left(\{\delta_n\} = \{\frac{1}{n}\} \right)$

$$\begin{aligned} \delta_1 = 1 \ \exists x_1 \ \rightarrow \ |x_1 - x_0| < 1 \\ |f(x_1) - l| \geq \varepsilon \\ \\ \delta_1 = \frac{1}{2} \ \exists x_2 \ \rightarrow \ |x_2 - x_0| < \frac{1}{2} \\ |f(x_2) - l| \geq \varepsilon \\ \vdots \\ \delta_n = \frac{1}{n} \ \exists x_n \ \rightarrow \ |x_n - x_0| < \frac{1}{n} \\ |f(x_n) - l| \geq \varepsilon \end{aligned}$$

Now we have $x_n \rightarrow x_0$ since $0 < |x_n - x_0| < \frac{1}{n}$ with $|f(x_n - l| \geq \varepsilon$

Contradiction with hypothesis (β)

We can use the "Bridge Theorem" to prove that a limjt does not exist

Example

$\lim_{x \rightarrow 0^+} \cos\left(\frac{1}{x}\right)$

$D_f = \{x \neq 0\}$

We will prove that this limit does not exist presenting two different sequences converging to 0^+ on which f has different limits

$x_n = \frac{1}{2\pi n} \to 0^+ (n \to +\infty) \quad y_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \to 0^+ (n \to +\infty)$

From x_n:

$\cos(x_n) = \cos(2\pi n) = 1$

From y_n:

$\cos(y_n) = \cos(\frac{\pi}{2} + 2\pi n) = 0$

thus $\nexists \lim_{x \to 0^+} \cos\left(\frac{1}{x}\right)$

Theorem: Uniqueness of a Limit

If the limit

$$\lim_{x \to x_0} f(x)$$

exists, it is unique

Proof

First case, x_0 is finite in $D_f^{'}$ and $\lim_{x \to x_0} f(x) = l$

By contradiction, assume that

1) $\lim_{x \to x_0} f(x) = l_1$ and

2) $\lim_{x \to x_0} f(x) = l$

at the same time with $l_1 \neq l$.

If 1) holds $\leftrightarrow \ \forall \varepsilon > 0 \ \exists \delta_1 > 0: \ \forall x \in D_f$ with $0 < |x - x_0| < \delta_1 \ \to \ |f(x) - l_1| < \varepsilon$

If 2) holds $\leftrightarrow \ \forall \varepsilon > 0 \ \exists \delta_2 > 0: \ \forall x \in D_f$ with $0 < |x - x_0| < \delta_2 \ \to \ |f(x) - l_1| < \varepsilon$

Now I choose $\varepsilon = \frac{|l_1 - l|}{2} \ \bar{\delta} = \ \min \{\delta_1, \delta_2\}$

$\forall x \in D_f$ with $0 < |x - x_0| < \bar{\delta} \ \to \ |l_1 - l| = |l_1 - f(x) + f(x) - l| \leq |l_1 - f(x)| + |f(x) - l| < 2\varepsilon = 2\frac{|l_1 - l|}{2}$

Contradiction!

Example

Prove the second case

$l_1 = +\infty$ and l finite.

Theorem 3: Local Sign

Let $f: D \subseteq \mathrm{R} \ \to \ \mathrm{R}$ and $x_0 \in D'$

If $\exists \ \lim_{x \to x_0} f(x) = l$ finite or infinite.

Then $\exists U_{x_0}$ such that $\forall x \in U_{x_0} \setminus \{x_0\}$

$f(x)$ has the same sign of l.

Proof

1) For the first limit, $l = +\infty \ \leftrightarrow \ \lim_{x \to x_0} f(x) = +\infty \ \leftrightarrow \ \forall M > 0 \ \exists \delta > 0: \ \forall x \in D_f$ with $0 < |x - x_0| < \delta \ \to \ f(x) > M$

2) Fix $\bar{M} > 0$ then consider

$U_{x_0} = (x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \ \rightarrow \ f(x) > \bar{M} > 0$

3) For the second limit l finite > 0

since

$\lim_{x \rightarrow x_0} f(x) = l > 0$

take $\varepsilon = \frac{l}{2} \ \rightarrow \ \exists \delta > 0: \ \forall x \in D_f$ with $0 < |x - x_0| < \delta \ \rightarrow \ |f(x) - l| < \frac{l}{2}$

$$-\frac{l}{2} + l < f(x) < \frac{l}{2} + l$$
$$0 < \frac{l}{2} < f(x) < \frac{3}{2}l$$

$U_{x_0} = (x_0 - \delta_1, x_0 + \delta) - \{x_0\}$

Example

The other cases: $l = -\infty, \ l < 0$ finite.

Theorem: Partial Reverse of Theorem "Local Sign"

Assume that $\exists \ \lim_{x \rightarrow x_0} f(x) = l$ (finite or infinite).

If $\exists$ a neighborhood U_{x_0} of x_0 such that $f(x) > 0 (f(x) < 0) \ \forall x \in U_{x_0} \setminus \{x_0\}$

then $l \geq 0$ or $+\infty$ ($l \leq 0$ or $-\infty$)

Proof

Case $f(x) > 0$, l finite

By contradiction assume $l < 0$.

Choose $\varepsilon \to \frac{-l}{2} \to \exists \delta > 0: \forall x \in D_f$ with $0 < |x - x_0| < \delta \to |f(x) - l| < -\frac{l}{2}$

$$|f(x) - l| < -\frac{l}{2}$$

$$+\frac{l}{2} + l < f(x) < -\frac{l}{2} + l$$

$$3\frac{l}{2} < f(x) < \frac{l}{2} < 0$$

$f(x) < 0$ with $U_{x_0}^{'\{x_0\}}$

Contradiction.

Example

Case $l = +\infty$

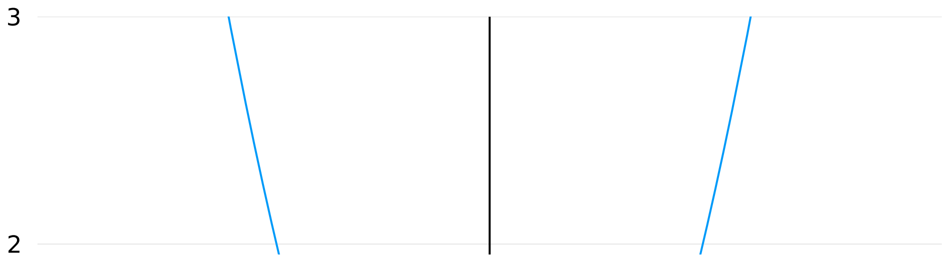
Note

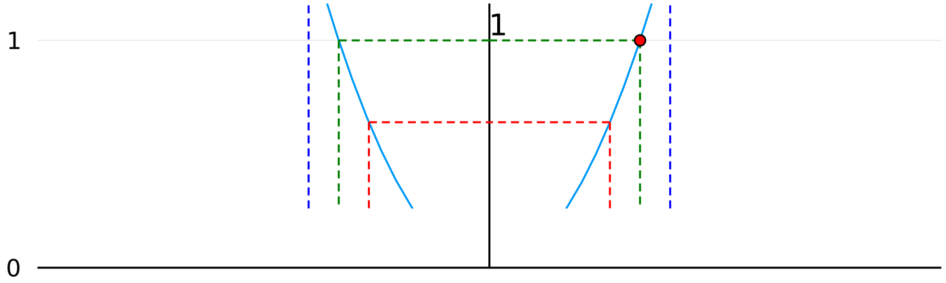
$$f(x) \begin{cases} x^2 \text{ for } x \neq 0 \\ 1 \text{ for } x = 0 \end{cases}$$

$f(x) > 0$

In [59]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) =  (x^2)
plot(f,-3,3, xlims=(-3,3), xticks = false, ylims=(0,3),
    bottom_margin = 10mm, label=L"x^{2}", framestyle = :zerolines,
    legend=:outerright)

plot!([0.73,0.73],[0.05,0], label="", linecolor=:red)
plot!([1.3,1.3],[0.05,0], label="", linecolor=:red)

plot!([1,0],[f(1),f(1)], label="", linecolor=:green, linestyle=:dash)
plot!([1,1],[f(1),0], label="", linecolor=:green, linestyle=:dash)

plot!([-1,0],[f(-1),f(-1)], label="", linecolor=:green, linestyle=:dash)
plot!([-1,-1],[f(-1),0], label="", linecolor=:green, linestyle=:dash)

plot!([1.2,0],[f(1.2),f(1.2)], label=L"1 + \varepsilon", linecolor=:blue, linestyle=:dash)
plot!([1.2,1.2],[f(1.2),0], label="", linecolor=:blue, linestyle=:dash)

plot!([-1.2,0],[f(-1.2),f(-1.2)], label="", linecolor=:blue, linestyle=:dash)
plot!([-1.2,-1.2],[f(-1.2),0], label="", linecolor=:blue, linestyle=:dash)

plot!([0.8,0],[f(0.8),f(0.8)], label=L"1 - \varepsilon", linecolor=:red, linestyle=:dash)
plot!([0.8,0.8],[f(0.8),0], label="", linecolor=:red, linestyle=:dash)

plot!([-0.8,0],[f(-0.8),f(-0.8)], label="", linecolor=:red, linestyle=:dash)
plot!([-0.8,-0.8],[f(-0.8),0], label="", linecolor=:red, linestyle=:dash)

scatter!([1.0], [f(1.0)], color = "red", label="", markersize = 3)

annotate!([(0.06,1.07, ("1", 10, :black))])
```

Out[59]:





exists and finite, $x_0 \in D_f^{'}$.

Then f us bounded in a neighborhood of $x_0 : U_{x_0} \setminus \{x_0\}$

Theorem: Algebra of Limits

holds for "standard cases".

Example: The undetermined forms in Limits $\frac{\pm\infty}{\pm\infty}$

$$\lim_{x \to -\infty} \frac{x^2 + x - 5x^2}{2x^2 + 1} = \lim_{x \to -\infty} \frac{x^2\left(1 + \frac{1}{x} - \frac{5}{x^2}\right)}{x^2\left(2 + \frac{1}{x^2}\right)}$$

$$= \frac{1}{2}$$

$$\lim_{x \to +\infty} \frac{x^2 + x}{x^2 + 1} = 1$$

$$\begin{aligned}\lim_{x \to +\infty} \frac{x^3+1}{-x+2} &= \lim_{x \to +\infty} \frac{x^3\left(1+\frac{1}{x^3}\right)}{x\left(-1+\frac{2}{x}\right)} \\ &= \frac{+\infty}{-1} \\ &= -\infty\end{aligned}$$

Recall

$\mathrm{R} \to \alpha > 0$

$\to$ (increasing order of infinity)

$\ln n,\ n^\alpha,\ e^n,\ n!,\ n^n$

Example

$$\lim_{n \to +\infty} \frac{e^n}{n^2} = +\infty$$

$$\lim_{n \to +\infty} \frac{n!}{e^n} = 0$$

$$\begin{aligned}\lim_{x \to +\infty} \frac{e^{x^2}+x^2+\ln x+1}{x^{10}+2x} &= \frac{+\infty}{+\infty} \\ &= +\infty\end{aligned}$$

$$\begin{aligned}\lim_{x \to +\infty} e^{\frac{x+1}{x^2-2}} &= e^0 \\ &= 1\end{aligned}$$

$$\lim_{x \to +\infty} \frac{x+1}{x^2-2} = 0$$

Recall

$$\lim_{x \to x_0} g(x) = l^m$$

if $g(x) \to l \quad g(x) > 0 \to l \geq 0$

$f(x) \to m$

$x \to x_0$

Example

$$\lim_{x \to 2^+} \frac{-3}{x-2} = \frac{-3}{0^+} = -\infty$$

$D_f = \{x \neq 2\}$

$x - 2 > 0$

$x > 2$

$x = 2$ is a vertical asymptote.

$$\lim_{x \to 2^-} \frac{-3}{x-2} = \frac{-3}{0^-} = +\infty$$

$$\lim_{x \to +\infty} \frac{-3}{x-2} = 0$$

$$\lim_{x \to -\infty} \frac{-3}{x-2} = 0$$

Note

$$\lim_{x \to 0^+} x \ln x = 0$$

Undetermined form $0 \cdot (-\infty)$

$$\lim_{x \to 0^+} \ln x = -\infty$$

By using algebraic manipulation:

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{+\infty}$$

Introductory Real Analysis Course 4 (3 October 2022)

Numerical Series

A finite sum of real numbers is well-defined by the algebraic properties of $\mathbb{R}$ but in order to make sense f an infinite series we need to consider its convergence.

Definition

Let $\{a_n\}$ be a sequence of real numbers. The series $\sum_{n=1}^{\infty} a_n$ converges to a sum $S \in \mathbb{R}$.

If the sequence $\{S_n\}$ of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges to S as $n \to +\infty$.

Otherwise the series does not converge.

If we consider $\sum_{n=0}^{\infty} (-1)^n$

$S_{2k} = 1$

$S_{2k+1} = 0$

S_{2k} and S_{2k+1} are sequence of the partial sums.

This is the alternating series $\nexists \lim_k S_k$.

Example

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{+\infty} a_k$$

Mengoli's series.

We can write

$$a_k = \frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\begin{aligned} S_n = a_1 + \cdots + a_n &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

$S_n \to 1 \quad n \to +\infty$

Telescoping Series

Let $\{a_n\}$ be a telescoping sequence, this means that $\exists \{b_n\}$ such that $a_n = b_n - b_{n+1}\ \forall n \geq 1$.

If $\{a_n\}$ is a telescoping sequence.

then

$\sum_{n=0}^{+\infty} a_n$ is a telescoping series.

In this case $\sum_{n=0}^{+\infty} a_n$ converges $\leftrightarrow \lim_n b_n = l \in \mathrm{R}$

For this type of series we have

$$S_n = \sum_{k=0}^{n} b_k - b_{k+1} = b_0 - b_{n+1}$$

If $\{b_n\}$ is convergent, the sum of the series is convergent.

Cauchy Criterion

Let $\{s_n\}$ be a sequence.

Then $\{s_n\}$ is convergent if and only if $\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ such that $\forall n, m \geq N(\varepsilon) \ |s_n - s_m| < \varepsilon$

Theorem: Cauchy Criterion for Series

Let $\sum_{n=0}^{+\infty} a_n$ be a series.

Then $\sum_{n=0}^{+\infty} a_n$ is convergent $\leftrightarrow \ \forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ such that

$\forall p \geq N(\varepsilon) \ \forall q \geq 0$

$|a_p + a_{p+1} + \cdots + a_{p+q}| < \varepsilon$

Necessary Condition

To get convergence.

If $\sum_{n=0}^{+\infty} a_n = s \in \mathrm{R}$ (if the series is convergent), then $\lim_n a_n = 0$

It is necessary but not sufficient

To get the convergence.

Example

If $a_n = \frac{1}{n}$ then $\lim_n \frac{1}{n} = 0$ but the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent.

Series of Positive Terms

$$\sum_{n=0}^{+\infty} a_n,\ a_n \geq 0 \quad \forall n \geq 0$$

then the sequence $S_n = \sum_{k=0}^{n} a_k$ is monotone increasing because

$$S_n = S_{n-1} + a_n \geq S_{n-1} \quad \forall n$$

There are two possibilities

$$\lim_n S_n \begin{matrix} s \in \mathrm{R} \\ +\infty \end{matrix}$$

this means that $\sum_{n=0}^{+\infty} a_n$ cannot be indetermined.

Theorem: Comparison

Let $\{a_n\}$ and $\{b_n\}$ be two sequences with positive terms. Assume that there exists $n_0 \in \mathrm{N}$ such that $(0 <)a_n \leq b_n \quad \forall n \geq n_0$.

Then, if the series $\sum_{k=0}^{+\infty} b_k$ is convergent, then also $\sum_{k=0}^{+\infty} a_k = +\infty$ is convergent.

Similarly, if $\sum_{k=0}^{+\infty} = +\infty$ then also $\sum_{k=1}^{+\infty} b_k = +\infty$

Example

Let us define $a_n = q^n, n \in \mathrm{N}, q \in \mathrm{R}$ (the geometric progression with ratio q.

If $q \neq 1$,

$\S_n = 1 + q + q^2 + \cdots + q^n = \frac{1-q^{n+1}}{1-q}$

If $q = 1,\ S_n = n$

Taking the limit $n \to +\infty$ we get

$$\lim_{n} S_n \begin{cases} \frac{1}{1-q} & |q| < 1 \\ +\infty & q \geq 1 \\ \nexists & q \leq -1 \end{cases}$$

For this reason the geometric series with set q:

$$\sum_{n=0}^{+\infty} q^n \text{ is} \begin{cases} \text{convergent with sum} \frac{1}{1-q} \text{ if } |q| < 1 \\ \text{divergent } q \geq 1 \\ \text{undetermined } q \leq< -1 \end{cases}$$

Example

Let us consider $a_n = \frac{1}{n!}$

This is a sequence of positive terms and hence $\{S_n\}$ is strictly monotone increasing sequence.

We want to prove that $\sum_{n=0}^{+\infty} \frac{1}{n!}$ is convergent.

We want to find some $\{b_n\}$ such that $b_n \geq 0 \ \forall n$ and there exists some $n_0 \in \mathrm{N}$ such that

$\frac{1}{n!} \leq b_n \quad \forall n \geq n_0$

We choose $b_n = \frac{1}{2^n}$

$$\lim_{n} \frac{2^n}{n!} = 0$$

then $\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ such that $\forall n \geq N(\varepsilon)$: $\left|\frac{2^n}{n!} - 0\right| < \varepsilon$

If we take $\varepsilon = 1$ then there is some $N(1)$ such that $\forall n \geq N(1)$

$$\frac{2^n}{n!} = \left|\frac{2^n}{n!} - 0\right| < 1$$

$$2^n < n!$$

$$\frac{1}{n!} < \frac{1}{2^n} = b_n \quad \forall n \geq N(1)$$

We can apply the comparison theorem to deduce that $\sum_{n=1}^{+\infty}\frac{1}{n!}$ is convergent.

($\sum_{n=0}^{+\infty}\frac{1}{2^n}$ is the geometric series with ratio $\frac{1}{2}$)

Example

The series $\sum_{n=1}^{+\infty}\frac{1}{n^\alpha}$ with $\alpha \in \mathrm{R},\ \alpha \leq 1$ is divergent.

If $\alpha = 1$ $\sum_{n=1}^{+\infty}\frac{1}{n}$ is divergent.

If $\alpha < 1 \rightarrow \frac{1}{n} < \frac{1}{n^\alpha} \rightarrow \sum_{n=1}^{+\infty}\frac{1}{n^2}$ divergent (Comparison Theorem).

Example

$\sum_{n=1}^{+\infty}\frac{1}{n^\alpha}$ $\alpha \in \mathrm{R}$ $\alpha \geq 2$ is convergent.

If $\alpha = 2$, we can compare it with Mengoli's series

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)}$$

Applying the comparison theorem is convergent.

If $\alpha > 2 \quad \frac{1}{n^\alpha} \leq \frac{1}{n^2} \ \rightarrow$ the series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ is convergent.

Example: Definitely Positive Series

$\sum_{n=0}^{\infty} \frac{n^2 - 10n}{n^2 + 1}$ this is positive if $n \geq 10$

$$\frac{n^2 - 10n}{n^2 + 1} = \frac{n^2\left(1 - \frac{10}{n}\right)}{n^4\left(1 + \frac{1}{n^4}\right)} = \frac{1}{n^2}\frac{\left(1 - \frac{10}{n}\right)}{\left(1 + \frac{1}{n^4}\right)} \leq \frac{1}{n^2}$$

Comparison theorem $\rightarrow$ the series is convergent.

Corollary

Let $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ be two series with positive terms. If $a_n \; b_n$ the two series have the same behavior.

Proof

$$\lim_{n} \frac{a_n}{b_n} = 1$$

$\varepsilon > 0 \ \exists N = N(\varepsilon)$ such that $\forall n \geq N \left| \frac{a_n}{b_n} - 1 \right| < \varepsilon$

Choosing $\varepsilon = \frac{1}{2}$

$$-\frac{1}{2} + 1 < \frac{a_n}{b_n} < \frac{1}{2} + 1$$

$$\frac{1}{2} < \frac{a_n}{b_n} < \frac{3}{2}$$

$$\frac{1}{2}b_n < a_n < \frac{3}{2}b_n$$

Theorem: Root Test

Let $\sum_{n=0}^{+\infty} a_n$ be a series with positive terms.

If there exist l, $0 \le l < 1$ and an index N such that $\sqrt[n]{a_n} \le l$ for $n \ge N$, then the series is convergent.

If $\sqrt[n]{a_n} \ge 1$ for infinitely many values of n, the series is divergent.

Proof

$\sqrt[n]{a_n} \le l$ holds $\forall n \ge N$, pne has:

$a_n \le l^n$. We can compare

$\sum_{n=0}^{+\infty} a_n$ with $\sum_{n=0}^{+\infty} l^n$ (the geometric series with ration l).

We get the convergence with $0 \le l < 1$

If $\sqrt[n]{a_n} \ge 1$ $a_n \ge 1$, the necessary condition is not verified $\rightarrow$ $\sum_{n=0}^{+\infty} a_n$ is divergent.

Corollary

Let $\sum_{n=1}^{+\infty} a_n$ a series with positive terms.

If the limit $\lim_n \sqrt[n]{a_n} = l$ exists then

- If $l > 1$ the series diverges
- If $l < 1$ the series converges

Proof

Let $0 \leq l < 1$

$\forall \varepsilon > 0, \;\; \exists N = N(\varepsilon)$ such that if $n \geq N(\varepsilon)$ one has

$$\sqrt[n]{a_n} < l + \varepsilon$$

if $\varepsilon = \frac{1}{2} - \frac{l}{2} \quad \sqrt[n]{a_n} < l + \frac{1}{2} - \frac{l}{2} < 1 \quad \forall n \geq N$ the series converges.

- $l > 1$, then for infinitely many values of n $\sqrt[n]{a_n} > 1$ (the necessary condition is not verified)

Example

$\sum_{n=1}^{\infty} \frac{a^n}{n^n}$ with $a \geq 0$

$$\lim_n \sqrt[n]{\frac{a^n}{n^n}} = \lim_n \frac{a}{n} \to 0$$

The series converges for the root test.

Theorem: The Ratio Test

Let $\sum_{n=0}^{+\infty} a_n$ a series with positive terms ($a_n > 0$).

If there exists $l,\ 0 < l < 1$, such that

$$\frac{a_{n+1}}{a_n} \leq l$$

then the series is convergent.

If there exists N such that $a_{n+1} \geq a_n\ \forall n \geq N$ the series diverges.

Proof

$n \geq 0$

$$\begin{aligned} &a_1 \leq la_0 \\ &a_2 \leq la_1 \leq l^2 a_0 \\ &\quad\vdots \\ &a_n \leq la_{n-1} \leq \cdots \leq l^n a_0 \end{aligned}$$

We can compare our series with the geometric one $\rightarrow$ it converges.

If $a_{n+1} \geq a_n$ definitely, the necesary condition is not verified.

Corollary

Let $\sum_{n=0}^{+\infty} a_n$ be a series with positive terms with $(a_n > 0)$. If there exists the limit $\lim_n \frac{a_{n+1}}{a_n} = l$

- $l < 1$ the series converges

Introductory Real Analysis Course 5 (5 October 2022)

$f: D \subset \mathrm{R} \rightarrow \mathrm{R}$

D domain

Im $f = \{y \in \mathbb{R} : \exists x \in D \text{ with } f(x) = y\}$

Im f = Range of $f = R(f)$

Im $f \subseteq \mathbb{R}$

Definition

f is bounded if $\exists M \geq 0$

$|f(x)| \leq M \quad \forall x \in D$

$-M \leq f(x) \leq M$

- If f is bounded from above it has a supremum ($\sup_D f = L$)

which is $\sup$ of Im f / as the least upper bound of Im f

(Im f subset of $\mathbb{R}$)

$\sup$ = least upper bound

- If f is bounded from below it has an infimum ($\inf_D f = l$)

which is $\inf$ of Im f as the greatest lower bound

Definition

f is said unbounded from above if we deny this statement:

$\exists \beta : \ \forall x \in D \ f(x) \leq \beta$ (f bounded from above)

We deny the statement above with:

$\forall \beta \in \mathbb{R} \ \exists x_\beta : f(x_\beta) > \beta$

Question: f unbounded from above

$\mathrm{R}:\ \lim_{x \to x_0} f(x) = +\infty$

Counterexample

Consider $f(x) = x\cos(x)$ for $x \to \infty$

If $\exists\ \lim_{x \to x_0} f(x) = +\infty \quad \to f$ is unbounded

For $x \to +\infty$ then

$\forall M > 0\ K_M > 0: \forall x > K_M\ f(x) > M$

$x \in D$

For any bound you set (M) on y-axis with K_M on x-axis you find x such that $f(x) > M$

$x > K_M$

$x = k_{M+1}$

Characterization of $\sup_D f(x) = L$

1) $f(x) \leq L \quad \forall x \in D$

2) $\forall \varepsilon > 0\ \exists \bar{x} \in D: f(\bar{x}) > L - \varepsilon$

Example

$f(x): x^2: (1, 2) \to \mathrm{R}$

Im $f: (1, 4)$

Check:

1) $x^2 \leq 4$

Find the roots of

$x^2 - 4 = 0$

$x = \pm 2$

$$x^2 - 4 \leq 0$$
$$-2 \leq x \leq 2$$

$x \in (1, 2)$

2) $\forall \varepsilon > 0 \ \exists \bar{x} \in (1, 2) : \bar{x}^2 > 4 - \varepsilon$

if ε is big then

$\bar{x}^2 > 4 - \varepsilon$ (will always true for $\varepsilon = 4, 5, 6, \ldots$)

$$(\bar{x})^2 - (\varepsilon - 4) > 0$$
$$\bar{x} > \sqrt{\varepsilon - 4}$$

if $(\bar{x})^2 < -\sqrt{4 - \varepsilon} \cup \bar{x} > \sqrt{4 - \varepsilon}$

with $4 - \varepsilon$ smaller than 2

- If $\sup$ belong to the set $\rightarrow$ becomes $\max$
- If $\inf$ belong to the set $\rightarrow$ becomes $\min$

Characterization of $\inf f(x) = l$

1) $\forall x \in D \ f(x) \geq l$

2) $\forall \varepsilon > 0 \ \exists \tilde{x} \in D : f(\tilde{x}) < l + \varepsilon$

Example

$f : [0, 2] \rightarrow \mathrm{R}$

$$f(x)\begin{cases} 2x+1 & x \in [0,1) \\ 2(2-x) & x \in [0,3) \end{cases}$$

$\lim_{x \to 1^-} f(x) = 3$

$\text{Im}\, f = [0, 3)$

$0 = f(2)$

- 3 is a limit point.

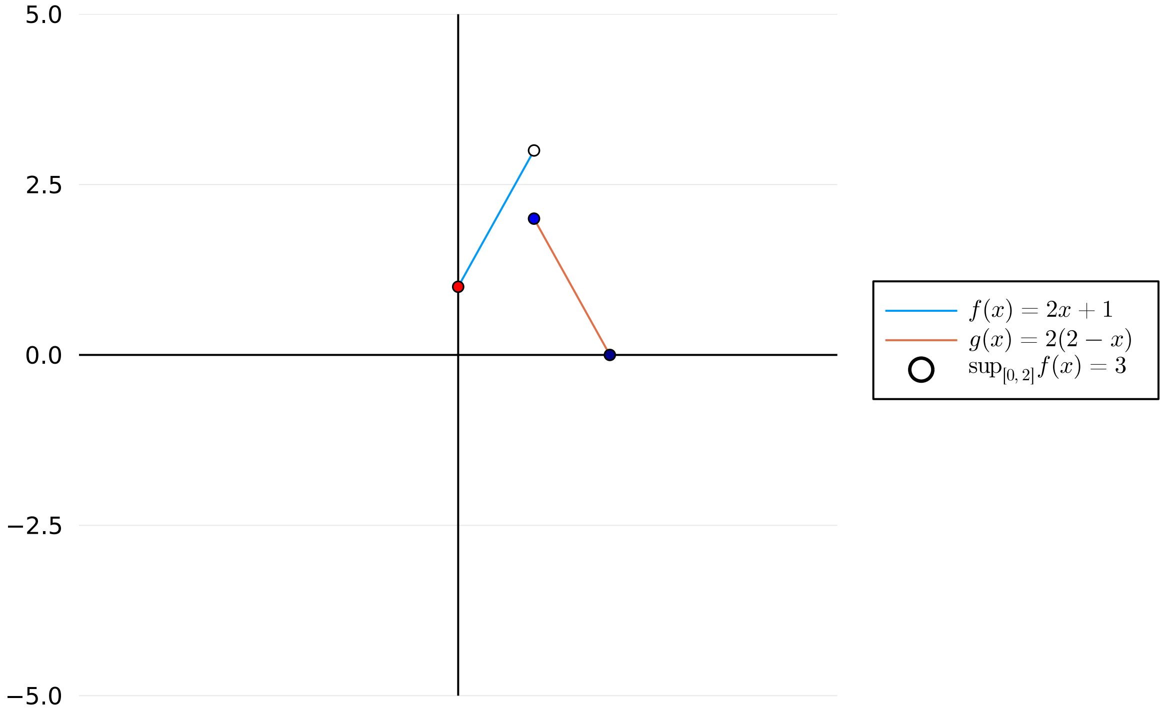
$\sup_{[0,2]} f(x) = 3$

- 3 is not a $\max_{[0,2]} f(x)$

$\inf_{[0,2]} f(x) = 0$

In [60]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 2x+1
g(x) = 2*(2-x)

plot(f,0,1, xticks=false, xlims=(-5,5), ylims=(-5,5),
    bottom_margin = 10mm, label=L"f(x) =  2x+1", framestyle = :zerolines,
    legend=:outerright)
plot!(g,1,2, xticks=false, xlims=(-5,5), ylims=(-5,5),
    bottom_margin = 10mm, label=L"g(x) =  2(2-x)", framestyle = :zerolines,
    legend=:outerright)

# Scatter plots
scatter!([0], [f(0)], color = "red1", label="", markersize = 3)
scatter!([1], [f(1)], color = "white", label=L"\sup_{[0,2]} f(x)=3", markersize = 3)
scatter!([1], [g(1)], color = "blue2", label="", markersize = 3)
scatter!([2], [g(2)], color = "blue4", label="", markersize = 3)
```

Out[60]:



Definition

- if $\inf_D f(x)$ belongs to Im f it is called minimum ($\min_D f = m$)
- if $\sup_D f(x)$ belongs to Im f it is called maximum ($\max_D f = M$)

Example

$f(x) = \frac{1}{x} : \ (0, 1] \ \rightarrow \ \mathrm{R}$

find l, L, m, M if exist

$f(1) = 1$

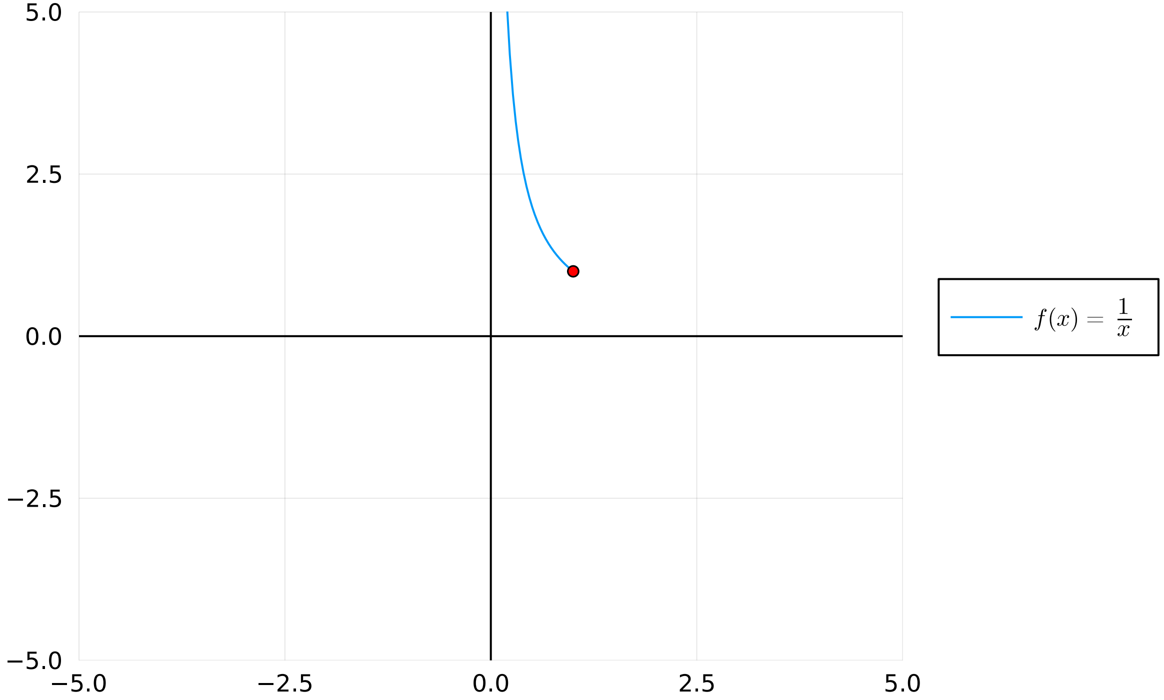
$$\lim_{x \to 0^+} \frac{1}{x} = \ +\infty$$
$$\inf_I f = 1 = \min_I f$$
$$\sup_I f = \ +\infty$$

it has no $\max$ and Im $f = [1, \ +\infty)$

Example

In [61]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 1/x

plot(f,0,1, xlims=(-5,5), ylims=(-5,5),
    bottom_margin = 10mm, label=L"f(x) = \frac{1}{x}", framestyle = :zerolines,
    legend=:outerright)

# Scatter plots
scatter!([1], [f(1)], color = "red1", label="", markersize = 3)
```

Out[61]:



Example

$f(x) = \frac{1}{x} : [1, +\infty) \to \mathrm{R}$

$I_1 = [1, +\infty)$

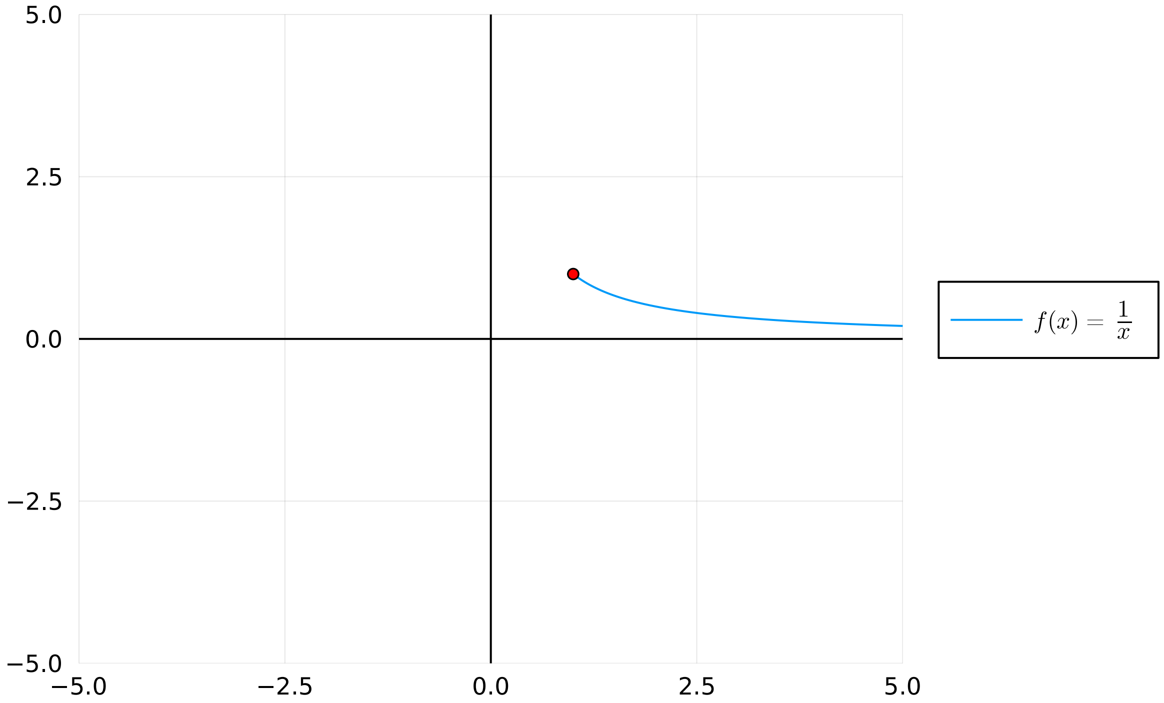
$$\lim_{x \to +\infty} f(x) = 0$$
$$\operatorname{Im} f = (0, 1]$$
$$\inf_{I_1} = 0$$

0 is not a minimum for f since if it was then I should have for some x.

$\frac{1}{x} = 0$ (impossible)

In [62]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 1/x

plot(f,1,5, xlims=(-5,5), ylims=(-5,5),
    bottom_margin = 10mm, label=L"f(x) = \frac{1}{x}", framestyle = :zerolines,
    legend=:outerright)

# Scatter plots
scatter!([1], [f(1)], color = "red1", label="", markersize = 3)
```

Out[62]:



Example

Find the extrema of

$f(x) = \frac{x}{x^2+1}$

over $\mathbb{R}$

$$f(-x) \begin{cases} f(x) \text{ even } \rightarrow \text{ cosine} \\ -f(x) \text{ odd } \rightarrow \text{ sine} \\ \text{neither} \end{cases}$$

Example

Assumed all is defined

a) $\inf_D (f+g) \geq \inf_D f + \inf_D g$

b) $\sup_D (f+g) \leq \sup_D f + \sup_D g$

Example

Find extrema of

$f(x) = \frac{1}{(x-3)^3} : [0, +\infty) - \{3\} \rightarrow \mathbb{R}$

One-Sided Limits

Definition

$\lim_{x \rightarrow x_0^+} f(x) = l \leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in D \cap (x_0, x_0 + \delta) \rightarrow |f(x) - l| < \varepsilon$

$f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$

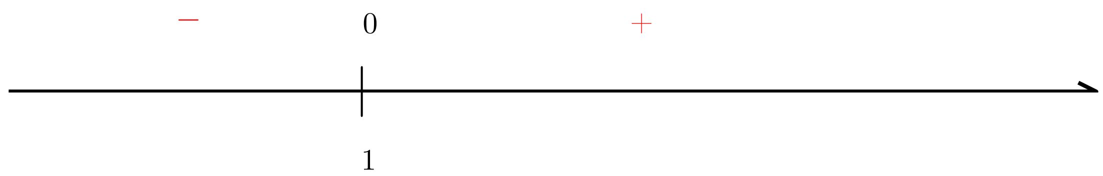
$\exists \cup_{x_0}^{+} : \forall x \subset \{D \cap \cup_{x_0}^{+}\} - \{x_0\} \rightarrow$ right neighbor

Example

$$\lim_{x \to 1^-} \frac{1}{x-1} = \left(\frac{1}{0^-}\right) = -\infty$$

In [63]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)
#plot!(f,0.55,0.7, fill=(0, 0.2, :green), label="")

annotate!([(0.5,0, (L"|", 17, :black))])
annotate!([(0.5,-0.1, (L"1", 10, :black))])
annotate!([(0.4,0.1, (L"-", 10, :red))])
annotate!([(0.5,0.1, (L"0", 10, :black))])
annotate!([(0.65,0.1, (L"+", 10, :red))])
```

Out[63]:



$\forall M > 0 \ \exists \delta > 0 : \ \forall x \in D \cap (x_0 - \delta, x_0) \ \rightarrow \ f(x) < \ -M$

$$\frac{1}{x-1} < \ -M$$

$$\frac{1}{1-x} > M$$

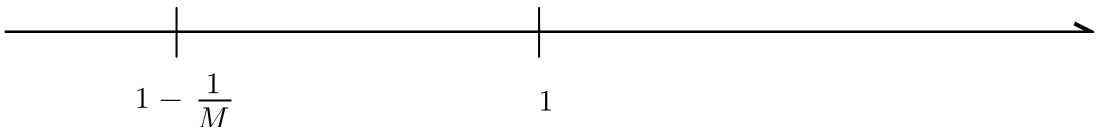
$$1 - x < \frac{1}{M}$$

$$1 > x > 1 - \frac{1}{M}$$

with $\delta = \frac{1}{M}$

In [64]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label="", bottom_margin = 10mm)
#plot!(f,0.55,0.7, fill=(0, 0.2, :green), label="")

annotate!([(0.4,0, (L"|", 17, :black))])
annotate!([(0.6,0, (L"|", 17, :black))])
annotate!([(0.4,-0.1, (L"1 - \frac{1}{M}", 10, :black))])
annotate!([(0.6,-0.1, (L"1", 10, :black))])
```

Out[64]:



Introductory Real Analysis Course 6 (10 October 2022)

$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$

$\alpha \geq 2$ convergent

$\alpha \leq 1$ divergent

what happens when $1 < \alpha < 2$?

Theorem: Cauchy Condensation Test

Let $\{a_n\}$ be a sequence with positive terms and decreasing.

$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$

Then $\sum_n a_n$ converges if and only if

$\sum_n 2^n a_{2^n}$ converges.

Proof

Let $\{s_n\}$ be the sequence of partial sums of $\sum_n a_n$ and $\{S_n\}$ be the sequence of partial sums of $\sum_n 2^n a_{2^n}$

$$
\begin{aligned}
a_0 &\leq a_0 \\
a_1 &\leq a_0 \\
a_2 + a_3 &\leq a_2 + a_2 = 2a_2 \\
a_4 + a_5 + a_6 + a_7 &\leq 4a_4 \\
&\vdots \\
a_{2^n} + a_{2^n+1} + a_{2^n+2} + \cdots + a_{2^{n+1}-1} &\leq 2^n a_{2^n}
\end{aligned}
$$

Then, summing $s_{2^{n+1}-1} \leq a_0 + S_n$

Analogously

$$a_0 \geq \frac{1}{2}a_0$$

$$a_1 + a_2 \geq \frac{1}{2}2a_2$$

$$a_3 + a_4 \geq \frac{1}{2} \cdot 4a_4$$

$$\vdots$$

$$a_5 + a_6 + a_7 + a_8 \geq \frac{1}{2}8a_8$$

$$\vdots$$

$$a_{2^{n+1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} \geq \frac{1}{2}2^n a_{2^n}$$

Then, summing $s_{2n} \geq \frac{1}{2}S_n$

$\{S_n\}$ is bounded $\leftrightarrow$ $\{s_n\}$ is bounded. The two series have the same behavior.

Example

$\sum_{n=1}^{+\infty}\frac{1}{n^\alpha}$ is convergent with $\alpha > 1$

$\frac{1}{n^2}$ is decreasing, we can apply the Cauchy condensation test.

$$2^n a_{2^n} = 2^n\frac{1}{(2^n)^\alpha} = (2^{1-\alpha})^n = \sum_{n=1}^{+\infty}(2^{1-\alpha})^n$$

$\sum_{n=1}^{+\infty}(2^{1-\alpha})^n$ is the geometric series with ratio $2^{1-\alpha}$, this is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

The same happens for $\sum_{n=1}^{+\infty}\frac{1}{n^2}$

Absolutely Convergent Series

Definition

The series $\sum_n a_n$ converges absolutely if

$\sum_n |a_n|$ converges.

Definition

We say that $\sum_n a_n$ converges conditionally if $\sum_{n=1}^{+\infty} |a_n|$ diverges.

Theorem

If $\sum_n |a_n|$ converges $\rightarrow$ $\sum_n a_n$ is convergent.

Proof

$\forall p, q \geq 0$

$|a_p + a_{p+1} + \cdots + a_{p+q}| \leq |a_p| + |a_{p+1}| + \cdots + |a_{p+1}|$

The convergence of $\sum_n |a_n|$ $\rightarrow$

$\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ such that $|a_p + |a_{p+1}| + \cdots + |a_{p+q}| < \varepsilon$ $\forall p \geq N$, $\forall q \geq 0$.

This is true also for the left-hand side:

$|a_p + a_{p+1} + \cdots + a_{p+q}| < \varepsilon$

The Cauchy test implies the convergence of $\sum_n a_n$.

Example

The series $\sum_n \frac{(-1)^n}{n^\alpha}$, $\alpha > 1$ is convergent.

$\left|\frac{(-1)^n}{n^\alpha}\right| = \frac{1}{n^\alpha} \rightarrow \sum_n \frac{(-1)^n}{n^\alpha}$ is convergent because $\sum_n \left|\frac{(-1)^n}{n^\alpha}\right|$ is convergent.

Alternating Series

An alternating series is one in which successive terms of the sequence have opposite signs.

Theorem: Leibniz

Let

$$\sum_{n=0}^{+\infty} (-1)^n a_n \quad \text{with } a_n > 0 \ \forall n \in \mathrm{N}$$

If

1) the sequence $\{a_n\}$ is decreasing

2) $\lim_n a_n = 0$

The series $\sum_{n=0}^{+\infty} (-1)^n a_n$ is convergent.

Proof

From 1)

$$s_n = a_0 - a_1 + a_2 - \cdots + \ldots (-1)^n a_n$$
$$s_{2n+2} = s_{2n} - (a_{2n+1} + a_{2n+2} \leq s_{2n}$$
$$\text{(because } a_{2n+1} \geq a_{2n+2}\text{)}$$
$$s_{2n+1} = s_{2n-1} + (a_{2n} - a_{2n+1}) \geq s_{2n-1}$$

The sequence of partial sums with even indexes is decreasing.

The sequence of partial sums with odd indexes is increasing.

Furthermore,

$$s_{2n} - s_{2n+1} = a_{2n+1} \quad (*)$$

The sequence $\{s_{2n}\}$ is bounded from below $\rightarrow$ it converges.

S is the limit

$S = \inf_n\{s_{2n}\}$

From 2)

$\lim_n a_n = 0$ and from (*) we deduce that $\{s_{2n+1}\}$ converges to S.

Then the series $\sum_{n=0}^{+\infty}(-1)^n a_n$ converges and S is the sum.

Example

The series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges both for Leibniz.

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is conditionally convergent because $\sum_{n=1}^{+\infty}\frac{1}{n} = \sum_{n=1}^{+\infty}\left|\frac{(-1)^{n+1}}{n}\right|$ diverges.

Exercises

1) $\sum_{n=1}^{\infty}\frac{n}{n+1}$

$\lim_n \frac{n}{n+1} = 1$

The series does not converge because the necessary condition is not fulfilled $\lim_n a_n = 0$

2) $\sum_{n=1}^{\infty} \frac{\log \ n}{n}$

With the Comparison theorem:

$\frac{\log \ n}{n} \geq \frac{1}{n}$

The series diverges.

3) $\sum_{n=1}^{\infty} \frac{3n^2+1}{n^2+n+1}$

with the Comparison theorem:

$\frac{3n^2+1}{n^4+n+1} \leq \frac{3n^2+1}{n^4} = \frac{3}{n^2} + \frac{1}{n^4}$ the series is convergent.

$\frac{3}{n^2}$ is a generalized harmonic series $\sum \frac{1}{n^{\alpha}}$ with $\alpha \geq 2$.

4) $\sum_{n=1}^{\infty} \frac{5n-1}{3n^2+2}$

$\frac{5n-1}{3n^2+2} = \frac{5n}{3n^2+2} - \frac{1}{3n^2+2}$

$\frac{1}{n}$ harmonic series, then it diverges.

5) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

with the Root test

$\lim_n \sqrt[n]{\frac{n}{2^n}} = \lim_n \frac{\sqrt[n]{n}}{2} = \frac{1}{2}$

$l = \frac{1}{2}$

If the result from the root test we obtain $l < 1$ then the series converges.

6) $\sum_{n}^{\infty}\frac{n^n}{2^n \cdot n!} = \sum_{n=1}^{\infty}a_n$

$a_{n+1} = \frac{(n+1)^{n+1}}{2^{n+1}\cdot(n+1)!}$

Ratio test

$$\begin{aligned}
\lim_n \frac{a_{n+1}}{a_n} &= \lim_n \frac{(n+1)^{(n+1)}}{2^{n+1}\cdot(n+1)!}\cdot\frac{2^n\cdot n!}{n^n} \\
&= \lim_n \frac{(n+1)^n(n+1)}{2^n\cdot 2\cdot(n+1)n!}\cdot\frac{2^n\cdot n!}{n^n} \\
&= \lim_n \frac{1}{2}\left(\frac{n+1}{n}\right)^n \\
&= \lim_n \frac{1}{2}\left(1+\frac{1}{n}\right)^n \\
&= \frac{e}{2}
\end{aligned}$$

$\frac{e}{2} > 1$ thus the series is divergent.

7) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\log\,(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\log\,(n+2)}$

Use the Leibniz test:

$\lim_n \frac{1}{\log\,(n+2)} = 0$

The sequence $\{\frac{1}{\log\,(n+2)}\}$ is decreasing, thus the series is convergent.

8) $\sum_{n=1}^{\infty}\frac{\sin\,n+\cos\,n}{n^3}$

$\frac{|\sin\ n + \cos\ n|}{n^3} \leq \frac{2}{n^3}$

With the Comparison test:

$\sum_{n=1}^{+\infty} \frac{1}{n^{\alpha}}$ when $\alpha \geq 2$ it converges.

9) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

With the Comparison test:

$\frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}$, with $\alpha \geq 2$ thus the series converges.

10) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

With the Comparison test:

$\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{1}{2n}$, with $\alpha < 2$ thus the series diverges.

$\frac{n}{n^2+1}\ \frac{1}{n}$

11) $\sum_{n=1}^{\infty} \sqrt[n]{n}$

The necessary condition $\lim\ a_n = 0$ is not verified. Thus, the series diverges.

12) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

With the Ratio test

$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \cdot \lim_n \frac{2^n \cdot 2 \cdot n!}{(n+1) \cdot n! \cdot 2^n}$$
$$= \lim_n \frac{2}{n+1} = 0$$

We have $l = 0$ thus the series converges.

13) $\sum_{n=1}^{\infty} \frac{2^n}{n^5}$

with the Root test:

$$\lim_n \sqrt[n]{\frac{2^n}{n^5}} = \lim_n \frac{2}{\sqrt[n]{n^5}} = 2$$

$(\sqrt[n]{n})^5 = \sqrt[n]{n^5}$, with $l > 1$ the series diverges.

14) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

with the Ratio test:

$$\lim_n \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2}$$
$$= \lim_n \frac{(n+1)^2 \cdot n!}{(n+1) \cdot n! \cdot n^2}$$
$$= 0$$

$l < 1$ thus the series converges.

15) $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n-1}\right)^n$

with the Root test:

$$\begin{aligned}\lim_n \sqrt[n]{\left(\frac{n+1}{3n-1}\right)^n} &= \lim_n \frac{n+1}{3n-1} \\ &= \frac{n\left(1+\frac{1}{n}\right)}{3n\left(1-\frac{1}{3n}\right)} \\ &= \frac{1}{3}\end{aligned}$$

16) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n+1}$

With the Leibniz, check:

i) $\lim_n \frac{1}{2n+1} = 0$

ii) $\frac{1}{2n+1}$ is a decreasing series

thus the series is convergent.

17) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n+1}$

With the Leibniz, check:

i) $\lim_n \frac{n}{n^2+1} = 0$

ii) $\frac{n}{n^2+1}$ is a decreasing series

thus the series is convergent.

$$\frac{n+1}{(n+1)^2+1} \leq \frac{n}{n^2+1}$$
$$\frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} \leq 0$$
$$\frac{(n+1)(n^2+1) - n[(n+1)^2+1]}{((n+1)^2+1)(n^2+1)} \leq 0$$
$$\frac{n^3+n+n^2+1-n[n^2+2n+1+1]}{[(n+1)^2+1](n^2+1)} \leq 0$$
$$\frac{n^3+n+n^2+1-n^3-2n^2-2n}{[(n+1)^2+1](n^2+1)} \leq 0$$
$$\frac{-n^2-n+1}{[(n+1)^2+1](n^2+1)} \leq 0$$

18) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

with the Root test:

$$\lim_n \frac{1}{\sqrt[n]{n^2}} = 1$$

In this case the Root test fails, because $l = 1$ but we know that the series converges.

19) $\sum_{n=1}^{\infty} \frac{\sin\ n}{n^2}$

$\sin(n)$ could be negative. Thus, let's study the series of absolute values.

$$\sum_{n=1}^{\infty} \left| \frac{\sin\ n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin\ n|}{n^2}$$

$$|\sin n| \leq 1$$

with the Comparison theorem:

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ the series is absolutely convergent.

Limits of Functions from $\mathbf{R}$ to $\mathbf{R}$

Let $X \subseteq \mathrm{R}$ and $f: X \to \mathrm{R}$. The aim of the limit notion is to describe the behavior of the function f " close to" an accumulation point of X.

Definition

Let $f: X \to \mathrm{R}$ and $x_0 \in \mathrm{R}^* = \mathrm{R} \cup \{\pm\infty\}$ an accumulation point of X, let $l \in \mathrm{R}^*$, we say that the limit of $f(x)$, for x that tends to x_0 is l, and we write

$\lim_{x \to x_0} f(x) = l$

if $\forall V$ neighborhood of l, it is possible to find a neighborhood U of x_0 such that $f(x) \in V$ if $x_0 \neq x \in U \cap X$

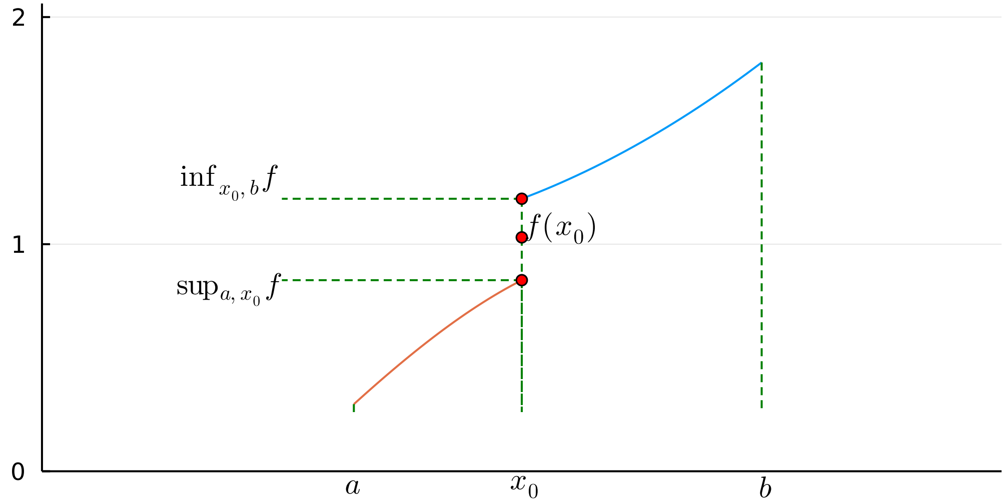
Monotone Functions

The aim is to find sufficient conditions in order to guarantee the possibility to perform the limit operation. In this sense, the calss of monotone

In [65]:

```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) =  (0.2x^2 + 1)
g(x) = sin(x)

plot(f,1,2, xlims=(-1,3), xticks = false, ylims=(0,3),
    bottom_margin = 10mm, label="",
    legend=:outerright)
plot!(g,0.3,1, xlims=(-1,3), xticks = false, ylims=(0,3),
    bottom_margin = 10mm, label="",
    legend=:outerright)

plot!([0.73,0.73],[0.05,0], label="", linecolor=:red)
plot!([1.3,1.3],[0.05,0], label="", linecolor=:red)

plot!([1,0],[f(1),f(1)], label="", linecolor=:green, linestyle=:dash)
plot!([1,1],[f(1),0], label="", linecolor=:green, linestyle=:dash)

plot!([1,0],[g(1),g(1)], label="", linecolor=:green, linestyle=:dash)
plot!([1,1],[g(1),0], label="", linecolor=:green, linestyle=:dash)

# Draw vertical lines
plot!([0.3,0.3],[g(0.3),0], label="", linecolor=:green, linestyle=:dash)
plot!([2,2],[f(2),0], label="", linecolor=:green, linestyle=:dash)

scatter!([1.0], [f(1.0)], color = "red", label="", markersize = 3)
scatter!([1.0], [g(1.0)], color = "red", label="", markersize = 3)
scatter!([1.0], [1.03], color = "red", label="", markersize = 3)

annotate!([(-0.22,1.27, (L"\inf_{x_{0},b} f", 10, :black))])
annotate!([(-0.22,0.8, (L"\sup_{a, x_{0}} f", 10, :black))])

annotate!([(0.3,-0.07, (L"a", 10, :black))])
annotate!([(1.02,-0.07, (L"x_{0}", 10, :black))])
annotate!([(2.02,-0.07, (L"b", 10, :black))])
annotate!([(1.17,1.07, (L"f(x_{0})", 10, :black))])
```

Out[65]:



$_\varepsilon, x_0)$.

This proves that $\lim_{x \to x_0} f(x)$

2) If $l = +\infty$, we get:

$\forall k > 0, \ \exists x_k \in (a, b)$ such that $f(x_k) > k$, it results $f(x) \geq f(x_k) > k \ \forall x \in (x_k, b)$, this proves the limit $\lim_{x \to b^-} f(x) = +\infty$

Remark

When $l = +\infty, \ x_0 = b$, because if $x_0 < b$ we would have $\sup_{(a, x_0)} f \leq f(x_0)$ because f is increasing.

Introductory Real Analysis Course 7 (11 October 2022)

Limits for Monotone Functions

Example

Let $\alpha \neq 0, x_0 \in \mathbb{R}^+$. The following relations hold

$\lim_{x \to x_0} x^\alpha = x_0^\alpha$

$$\lim_{x \to 0^+} x^\alpha \begin{cases} 0 & \alpha > 0 \\ +\infty & \alpha < 0 \end{cases}$$

$$\lim_{x \to +\infty} x^\alpha \begin{cases} +\infty & \alpha > 0 \\ 0 & \alpha < 0 \end{cases}$$

$f: x \to x^\alpha$ defined on R^+

Example

Let $a > 0,\ a \neq 1,\ x_0 \in \mathrm{R}^+$. We have

$\lim_{x \to x_0}$ a^{x} = a^{x_{0}}

$$\lim_{x \to +\infty} a^x \begin{cases} +\infty & \alpha > 1 \\ 0 & \alpha < 1 \end{cases}$$

$$\lim_{x \to -\infty} a^x \begin{cases} 0 & \alpha > 1 \\ +\infty & \alpha < 1 \end{cases}$$

Those exponential functions are monotone.

Example

Let $a > 0, a \neq 1,\ x_0 \in \mathrm{R}^+$. We have

$\lim_{x \to x_0} \log_a x = \log_a x_0$

$$\lim_{x \to 0^+} \log_a x \begin{cases} -\infty & \alpha > 1 \\ +\infty & \alpha < 1 \end{cases}$$

$$\lim_{x \to +\infty} \log_a x \begin{cases} +\infty & \alpha > 1 \\ -\infty & \alpha < 1 \end{cases}$$

Remark

We use the theorem about limits of monotone functions and the example above in order to solve

$$\lim_{x \to x_0} f(x)^{g(x)}$$

($f(x)$ is definitely positive)

$$f(x)^{g(x)} = a^{g(x)\ \log_a\ f(x)} \quad a^{\log_a\ f(x)^{g(x)}} = a^{g(x)\ \log_a\ f(x)}$$

$$\lim_{x \to x_0} g(x) \log_a f(x) = l \quad (\infty - \infty, \frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 0^0, 1^\infty, \infty^0)$$

$$\lim_{x \to x_0} f(x)^{g(x)} = a^l$$

$0^0, 1^\infty, \infty^0 \ \rightarrow \ 0 \cdot \infty$

Example

We would like to compute the following limit

$\lim_{x \to 0^+} x^x$ and $\lim_{x \to +\infty} x^x$

We choose $a > 1$

1) $\lim_{x \to 0^+} x \log_a x$

2) $\lim_{x \to +\infty} x\log_a x$

thus, it is of form $0 \cdot \infty$ and the limit is $+\infty$

$\lim_{x \to 0^+} a^{x \log_a\ x}$

$$\lim_{x \to 0^+ x \log_a\ x = 0} \quad \lim_{x \to 0^+} x^x = 1$$

Infinitesimal and Infinite Functions

Let f and g be two functions defined in a neighborhood of $x_0 \in \mathrm{R}^*$.

Definition

We say that the function $f(x)$ is infinitesimal in x_0 if $\lim_{x \to x_0} f(x) = 0$

We say that the function $f(x)$ is infinite in x_0 if

$$\lim_{x \to x_0} f(x) = +\infty \text{ (or } -\infty) \ (\infty)$$

Example

$\sin x,\ x^{\frac{1}{3}},\ \log_2(1+x)$, these functions are infinitesimal for $x \to 0$.

- $\frac{1}{\sqrt{x}},\ 2^{-x}$ are infinitesimal for $x \to +\infty$
- $\log(1-x)$ is infinite for $x \to 1$

If we have two infinitesimal functions f and g for $x \to x_0$, it is very useful to make a comparison between them.

If $g \neq 0$ definitely for $x \to x_0$, we have 4 possibilities

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \begin{cases} 1.0 \\ 2.l \text{ finite number } l \neq 0 \\ 3.(\infty, -infty)(\infty) \\ 4.\nexists \end{cases}$$

We say that if

1. holds: f is an infinitesimal function of greater order with respect to g
2. holds: f has the same infinitesimal order of g

3. holds: f is an infinitesimal function of lower order with respect to g
4. holds: f and g are not comparable. $\lim_{x \to +\infty} \sin x \frac{2^{-x}}{2^{-x}} \nexists$

If f and g are infinite functions for $x \to x_0$ we have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \begin{cases} 1.0 \\ 2.l\ l \neq 0 \text{ finite} \\ 3.(\infty, -infty)(\infty) \\ 4.\nexists \end{cases}$$

If

1. holds: f is an infinite function with lower order with respect to g
2. holds: f and g have the same order of infinity
3. holds: f is an infinite function with greater order with respect to g
4. holds: f and g are not comparable

Example

- x and $\sin x$ are infinitesimal of the same order for $x \to 0$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

- x^2 and $1 - \cos(2x)$ are infinitesimal of the same order.

If f_1 is an infinitesimal function of greater order with respect to f for $x \to x_0$, we can say that f_1 is neglectable with respect to f.

If f, f_1, g, g_1 are infinitesimal functions for $x \to x_0$, if f_1 is an infinitesimal greater than f and f_1 is an infinitesimal greater than $g \to$

$$\lim_{x \to x_0} \frac{f + f_1}{g + g_1} = \lim_{x \to x_0} \frac{f(\left(1 + \frac{f_1}{f}\right)}{g\left(1 + \frac{g_1}{g}\right)} = \lim_{x \to x_0 \frac{f}{g}}$$

$$\lim_{x \to x_0} \frac{f_1}{f} = 0 \quad \lim_{x \to x_0 \frac{g_1}{g}} = 0$$

Example

$$\lim_{x \to 0} \frac{x^2 + x^4 + x^5}{x + x^7 + x^8} = \lim_{x \to 0} \frac{x^2}{x} = 0$$

$$\lim_{x \to 0} \frac{x^2(1 + x^2 + x^3)}{x(1 + x^6 + x^7)} \quad f = x^2 \; g = x$$

Example

$$\lim_{x \to 0} \frac{3x + (\sin x)^2 + 2x^4}{x^3 - x} = \lim_{x \to 0} \frac{3x}{-x} = -3$$

or

$$\lim_{x \to 0} \frac{x\left(3 + \frac{(\sin\ x)^2}{x} + 2x^3\right)}{x(x^2 - 1)} = \lim_{x \to 0} \frac{3x}{-x} = -3$$

we know that $\lim_{x \to 0} \frac{\sin\ x}{x} = 1$

If f_1 is an infinite function for $x \to x_0$ of lower order with respect to f, we say that f_1 is neglectable.

Example

$$\lim_{x \to -\infty} \frac{x^2 + 1 + |x|^{\frac{1}{2}}}{3x - x^3} = \lim_{x \to -\infty} \frac{x^2}{-x^3} = \lim_{x \to -\infty} -\frac{1}{x} = 0$$

Landau' Symbols

1) If $g \neq 0$ definitely for $x \to x_0$, then

$$f(x) = o(g(x)) \quad \text{if} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = o$$

o is a letter number 15 in the 'abc' alphabet. It is small o for the equation above.

for example, f is an infinitesimal function of greater order with respect to g for $x \to x_0$.

Moreover, $f(x) = o(1)$ means that f is an infinitesimal function for $x \to x_0$

$$\lim_{x \to x_0} \frac{f(x)}{1} = o$$

When $\lim_{x \to x_0} f(x) = l$, if $l \in \mathrm{R}$, it implies $f(x) = l + o(1)\ x \to x_0$.

2) The symbol $\mathcal{O}(.)$ ("big O")

If $g(x) \neq 0$ definitely for $x \to x_0$, then

$$f(x) = \mathcal{O}(g(x))$$

means that $\frac{f(x)}{g(x)}$ is definitely bounded for $x \to x_0$

3) The symbol (asymptote to …)

if $g(x) \neq 0$ definitely for $x \to x_0$, then

$f(x)\ g(x)$ means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$

Remark

If $f\ f_1$ and $g\ g_1$ for $x \to x_0$ then

$$\frac{f}{g}\ \frac{f_1}{g_1}$$

this could be useful when we compute limits.

4) The symbol $\asymp$ (the function "has the same order of ...")

If $g(x) \neq 0$ definitely $x \to x_0$, then $f(x) \asymp g(x)$ means that $\exists m$ and M, $0 < m < M$, such that for $x \to x_0$ definitely.

$m|g(x)| \leq |f(x)| \leq M|g(x)|$

In particular, $f(x) \asymp 1$ means that $f(x)$ and $\frac{1}{f(x)}$ are bounded in a neighborhood of x_0.

Asymptotes

It is interesting in the following case:

$$f(x) = ax + b + o(1) \quad x \to +\infty \quad (\text{or } -\infty) \quad (*)$$

This means that f behaves like a first order polynomial when $x \to +\infty\ (-\infty)$.

In this case $y = ax + b$ is called asymptote for the function f.

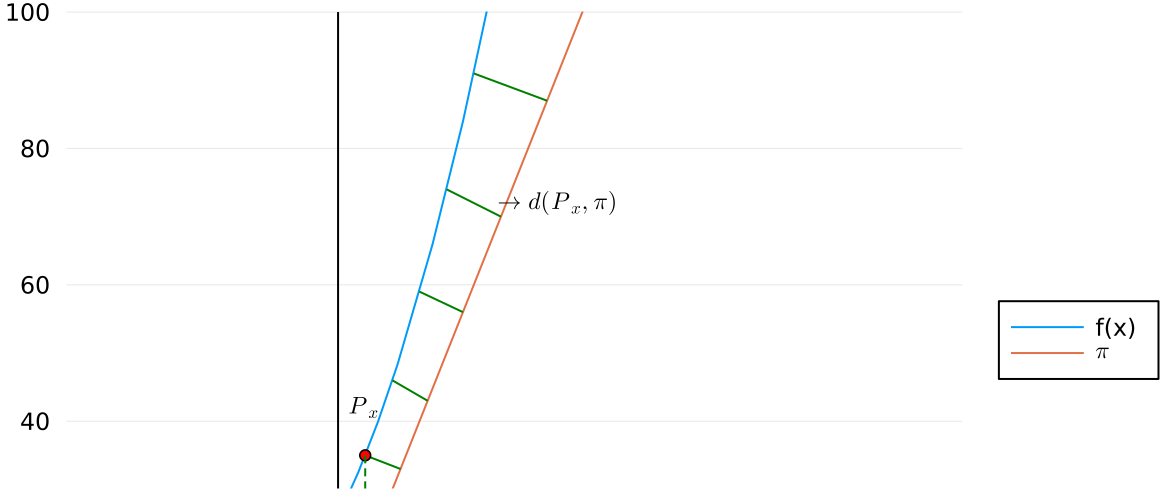
The geometric interpretation is the following:

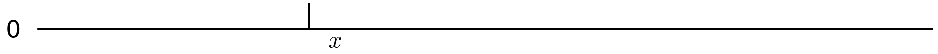
$\pi:\ y = ax + b$ and P_x a point with coordinates $(x, f(x))$.

If $d(P_x, \pi)$ indicates the distance between P_x and π one has:

In [59]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = (x+4)^(2) +10
g(x) = 10x + 10

plot(f,-8,30, xlims=(-10,23), xticks = false, ylims=(0,100),
    bottom_margin = 10mm, label="f(x)", framestyle = :zerolines,
    legend=:outerright)
plot!(g,0,30, xlims=(-10,23), xticks = false, ylims=(0,100),
    bottom_margin = 10mm, label=L"\pi", framestyle = :zerolines,
    legend=:outerright)

plot!([1,2.3],[f(1),g(2.3)], label="", linecolor=:green)
plot!([2,3.3],[f(2),g(3.3)], label="", linecolor=:green)
plot!([3,4.6],[f(3),g(4.6)], label="", linecolor=:green)
plot!([4,6],[f(4),g(6)], label="", linecolor=:green)
plot!([5,7.7],[f(5),g(7.7)], label="", linecolor=:green)

scatter!([1.0], [f(1.0)], color = "red", label="", markersize = 3)
plot!([1,1],[f(1),0], label="", linecolor=:green, linestyle=:dash)

annotate!([(1,42, (L"P_{x}", 8, :black))])
annotate!([(1,-2, (L"x", 8, :black))])
annotate!([(8,72, (L"\rightarrow d(P_{x},\pi)", 8, :black))])
```

Out[59]:





Introductory Real Analysis Course 8 (12 October 2022)

Exercises

1) $\ln(x^2 - x)$

check the roots and the sign notations:

$$x^2 - x > 0$$
$$x(x-1) > 0$$
$$x_1 = 0 \quad x_2 = 1$$

$D = \{x \in \mathrm{R} \,|\, x \in [0;1]\} = \mathrm{R} \setminus [0;1] = (-\infty, 0) \cup (1, +\infty)$

2) $\frac{\sqrt{x^2-3x-4}}{x-5}$

check the roots and the sign notations:

$$x - 5 \neq 0 \rightarrow x \neq 5$$
$$x^2 - 3x - 4 \geq 0$$
$$(x+1)(x-4) \geq 0$$

$D = \{x \in \mathrm{R} : x \leq -1,\ x \geq 4,\ x \neq -5\} = (-\infty, -5) \cup (-5, -1] \cup [4, +\infty)$

Exercises

1) Let $f : x \rightarrow \mathrm{R}; x_0, l \in \mathrm{R}$, the opposite statement is

$$\exists \varepsilon_0 > 0\ \forall \delta > 0\ \exists \bar{x} \in X \quad 0 < |\bar{x} - x_0| < \delta \rightarrow |f(\bar{x}) - l| > \varepsilon_0$$

2) Let $f : \mathrm{R} \rightarrow \mathrm{R}$ opposite to $\lim_{x \rightarrow -\infty} f(x) = +\infty$ is equivalent to

$$\exists \mu_0 :\ \forall N > 0\ \exists \bar{x} \in X : \bar{x} < -N\, f(x) < \mu_0$$

Example

1) Calculate

$$\lim_{x \to 2} \frac{x^3 - 7x + 6}{x^3 - 5x^2 + 2x + 8}$$

First use the $lim_{x \to 2}$ as the divider, then divide the numerator and denominator with the divider.

$$\begin{array}{rl}
x^3 - 7x + 6 & \\
- & \\
x^3 - 2x^2 & (x-2)(x^2) \\
\text{– – – – – – – – – – – – – – – – –} & \\
2x^2 - 3x & \\
- & \\
2x^2 + 4x & (x-2)(2x) \\
\text{– – – – – – – – – – – – – – – – –} & \\
-3x + 6 & \\
- & \\
-3x + 6 & (x-2)(-3) \\
\text{– – – – – – – – – – – – – – – – –} & \\
0 &
\end{array}$$

Thus, we get that $x^3 - 7x + 6$ divided by $x - 2$ is $x^2 + 2x - 3$.

Now to the denominator:

$$
\begin{array}{rl}
x^3 - 5x^2 + 2x + 8 & \\
- & \\
x^3 - 2x^2 & (x-2)(x^2) \\
\hline
-3x^2 + 2x & \\
- & \\
-3x^2 + 6x & (x-2)(-3x) \\
\hline
-4x + 8 & \\
- & \\
-4x + 8 & (x-2)(-4) \\
\hline
0 &
\end{array}
$$

Thus, we get that $x^3 - 5x^2x + 2x + 8$ divided by $x - 2$ is $x^2 - 3x - 4$.

Then we will have

$$
\begin{aligned}
\lim_{x \to 2} \frac{x^3 - 7x + 6}{x^3 - 5x^2 + 2x + 8} &= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x - 3)}{(x-2)(x^2 - 3x - 4)} \\
&= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x - 3)}{(x-2)(x^2 - 3x - 4)} \\
&= \lim_{x \to 2} \frac{x^2 + 2x - 3}{x^2 - 3x - 4} \\
&= \frac{2^2 + 2(2) - 3}{2^2 - 3(2) - 4} \\
&= \frac{5}{-6} = -\frac{5}{6}
\end{aligned}
$$

Julia Codes to Find Common Factors in Polynomial

In [66]:
```
using SymPy
@syms x
p = (1, 2, 3, 4, 5) # 1 + 2x + 3x^2 + 4x^3 + 5x^4
evalpoly(x, p)
```

Out[66]: $x(x(x(5x + 4) + 3) + 2) + 1$

In [67]:
```
using SymPy

apart((x^4 + 2x^2 + 5) / (x-2))
```

Out[67]: $x^3 + 2x^2 + 6x + 12 + \frac{29}{x-2}$

In [68]:
```
# Find factors of a polynomial
using SymPy

p = 2x^4 + x^3 - 19x^2 - 9x + 9
factor(p)
```

Out[68]: $(x - 3)(x + 1)(x + 3)(2x - 1)$

In [69]:
```
# Find factors of a polynomial
using SymPy

p = 3x^3 + 2x^2 - x - 4
factor(p)
```

Out[69]: $(x - 1)\left(3x^2 + 5x + 4\right)$

In [70]:
```
# The answer is a vector of values
# when substituted in for the free variable x produce 0

using SymPy

solve(x^2 + 2x - 3)
```

Out[70]:

$$\left[\begin{matrix}-3\\1\end{matrix}\right]$$

In [71]:
```
using SymPy

@syms a b c
solve(a*x^2 + b*x + c, x)
```

Out[71]:

$$\left[\begin{matrix}\frac{-b-\sqrt{-4ac+b^{2}}}{2a}\\\frac{-b+\sqrt{-4ac+b^{2}}}{2a}\end{matrix}\right]$$

In [72]:
```
# Solve with complex roots

using SymPy

@syms a
@syms b::real
c = symbols("c", positive=true)
solve(a^2 + 1)     # works, as a can be complex
```

Out[72]:

$$\left[\begin{matrix}-i\\i\end{matrix}\right]$$

In [73]:
```
using SymPy

p = x^2 - 2

rts = solve(p)
prod(x-r for r in rts)
```

Out[73]:

$$\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

In [74]:
```julia
using SymPy

@syms x
solve(x^4 - 2x - 1)
```

Out[74]:

$$\left[\begin{array}{l}
-\frac{\sqrt{-\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}{2}-\frac{\sqrt{-\frac{4}{\sqrt{-\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}-2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}+\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}}{2}\\
-\frac{\sqrt{-\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}{2}+\frac{\sqrt{-\frac{4}{\sqrt{-\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}-2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}+\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}}{2}\\
\frac{\sqrt{-\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}{2}+\frac{\sqrt{-2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}+\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+\frac{4}{\sqrt{-\frac{2}{3\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}+2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}}}}}{2}\\
\sqrt{-2\sqrt[3]{\frac{1}{4}+\frac{\sqrt{129}}{36}}+\frac{2}{3\sqrt[3]{\frac{1}{4}+\cdots}}+\frac{4}{\sqrt{\cdots}}} \cdots
\end{array}\right]$$

$$\left. - \frac{\sqrt{\cdots}}{2} + \frac{\sqrt{- \frac{2}{3 \sqrt[3]{\frac{1}{4} + \frac{\sqrt{129}}{36}}} + 2 \sqrt[3]{\frac{1}{4} + \frac{\sqrt{129}}{36}}}}{2} \right]$$

In [75]:
```julia
# Third- and fourth-degree polynomials can be solved in general

using SymPy

@syms a b c d x
rts = solve(a*x^3 + b*x^2 + c*x + d, x)
```

Out[75]:

$$\left[\begin{array}{c} -\frac{-\frac{3c}{a}+\frac{b^2}{a^2}}{3\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}-\frac{\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}{3}-\frac{b}{3a} \\ -\frac{-\frac{3c}{a}+\frac{b^2}{a^2}}{3\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}-\frac{\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}{3}-\frac{b}{3a} \\ -\frac{-\frac{3c}{a}+\frac{b^2}{a^2}}{}-\frac{\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}{3}-\frac{b}{3a} \end{array}\right]$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}$$

In [76]:
```
rts[1]
```

Out[76]:

$$-\frac{-\frac{3c}{a}+\frac{b^2}{a^2}}{3\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}-\frac{\sqrt[3]{\frac{\sqrt{-4\left(-\frac{3c}{a}+\frac{b^2}{a^2}\right)^3+\left(\frac{27d}{a}-\frac{9bc}{a^2}+\frac{2b^3}{a^3}\right)^2}}{2}+\frac{27d}{2a}-\frac{9bc}{2a^2}+\frac{b^3}{a^3}}}{3}-\frac{b}{3a}$$

In [77]:
```
# Numerically find roots

using SymPy

rts = solve(x^5 - x + 1)
N.(rts)
```

Out[77]:
```
5-element Vector{Number}:
                      -1.167303978261418684256045899854842180720560371525489039140082449275651903429536
 -0.18123244446987538 - 1.0839541013177107im
 -0.18123244446987538 + 1.0839541013177107im
   0.7648844336005847 - 0.35247154603172626im
   0.7648844336005847 + 0.35247154603172626im
```

In [78]:
```
# Numerically find roots example 2

using SymPy

ex = x^7 -3x^6 +  2x^5 -1x^3 +  2x^2 + 1x^1  - 2
solve(ex)
```

Out[78]:

$$\begin{bmatrix} 1 \\ 2 \\ \operatorname{CRootOf}\left(x^5 - x - 1, 0\right) \\ \operatorname{CRootOf}\left(x^5 - x - 1, 1\right) \\ \operatorname{CRootOf}\left(x^5 - x - 1, 2\right) \\ \operatorname{CRootOf}\left(x^5 - x - 1, 3\right) \\ \operatorname{CRootOf}\left(x^5 - x - 1, 4\right) \end{bmatrix}$$

Introductory Real Analysis Course 9 (17 October 2022)

Continuous Function

Let $f: D \subset \mathrm{R} \to \mathrm{R}$

Definition

$x_0 \in D$ is an isolated point for D if $\exists U_{x_0}: U_{x_0} \cap D = \{x_0\}$

Definition: Continuous function

f is said to be continous in $x_0 \in D$ if

1) x_0 is an isolated point for D

$f : D = [0, 1) \cup \{2\} \rightarrow \mathrm{R}$

$\{2\}$ is an isolated point for D.

$\exists U_2 : U_2 \cap D = \{2\}$

2) $x_0 \in D' \cap D$ and $\exists \lim_{x \rightarrow x_0} f(x) = f(x_0)$

f in the example above is a continuous function in its domain.

Notes

The statement 2) is also equivalent to

(a) $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, x_0) > 0 : \ \forall x \in D$ with $|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \varepsilon$

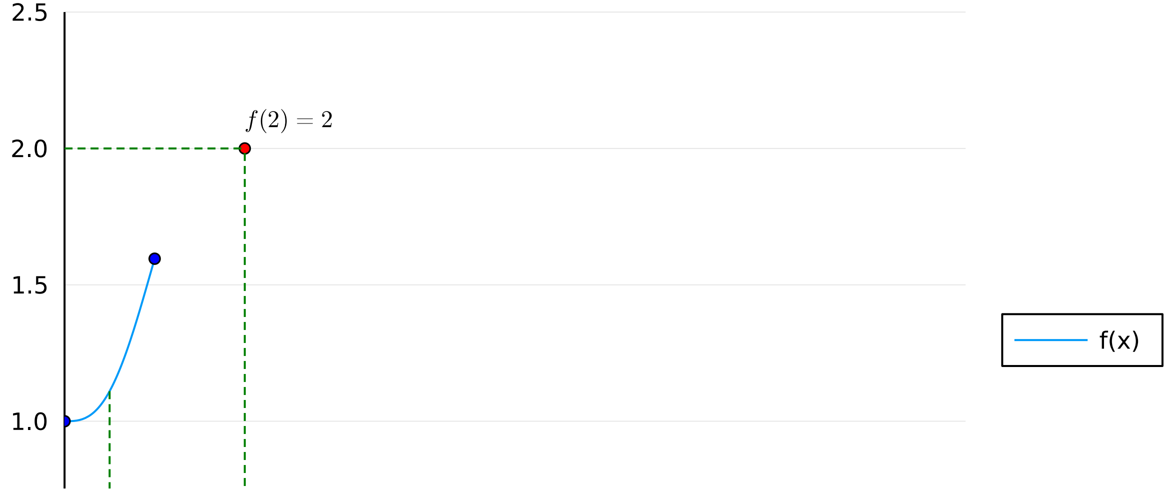
(b) $\forall V_{f(x_0)} \ \exists U_{x_0} : \ \forall x \in U_{x_0} \cap D \rightarrow f(x) \in V_{f(x_0)}$

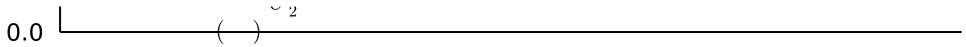
$(f(U_{x_0} \cap D) \subset V_{f(x_0)})$

(c) $\forall x_n \rightarrow x_0 \rightarrow \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$

In [95]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = (sin.(x))^(3) + 1

plot(f,0,1, xlims=(0,10), xticks = false, ylims=(0,2.5),
    bottom_margin = 10mm, label="f(x)", framestyle = :zerolines,
    legend=:outerright)

scatter!([0], [f(0)], color = "blue", label="", markersize = 3)
scatter!([1], [f(1)], color = "blue", label="", markersize = 3)

scatter!([2.0], [0], color = "red", label="", markersize = 3)
scatter!([2.0], [2], color = "red", label="", markersize = 3)

plot!([0.5,0.5],[f(0.5),0], label="", linecolor=:green, linestyle=:dash)
plot!([0,2],[2,2], label="", linecolor=:green, linestyle=:dash)
plot!([2,2],[0,2], label="", linecolor=:green, linestyle=:dash)

annotate!([(0.5,-1, (L"x", 8, :black))])
annotate!([(1.8,0, (L"(", 8, :black))])
annotate!([(2.2,0, (L")", 8, :black))])
annotate!([(2.5,0.1, (L"U_{2}", 8, :black))])
annotate!([(2.5,2.1, (L"f(2)=2", 8, :black))])
```

Out[95]:





Example

$f: [0, 3] \to \mathrm{R}$

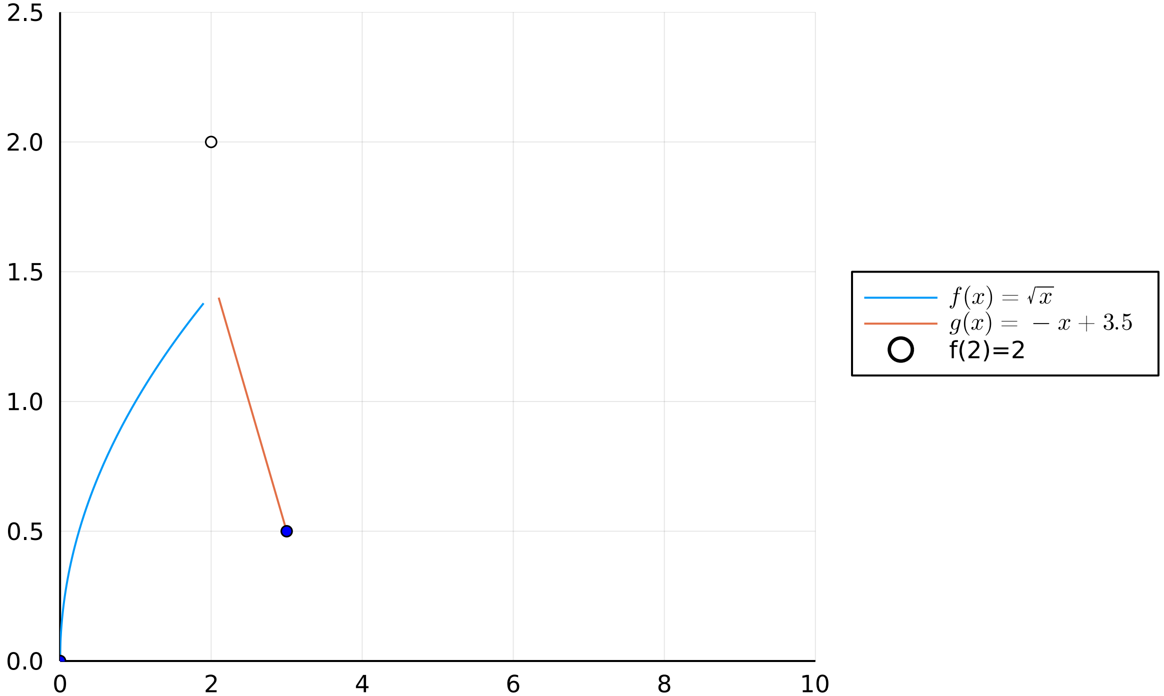
$\lim_{x \to 2^{\pm}} f(x) = 1 \neq f(2) = 2$

- f not continuous in $x = 2$

see the graph below.

In [11]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = sqrt.(x)
g(x) = -x+3.5

plot(f,0,1.9, xlims=(0,10), ylims=(0,2.5),
    bottom_margin = 10mm, label=L"f(x)= \sqrt{x}", framestyle = :zerolines,
    legend=:outerright)
plot!(g,2.1,3, xlims=(0,10), ylims=(0,2.5),
    bottom_margin = 10mm, label=L"g(x)=-x+3.5", framestyle = :zerolines,
    legend=:outerright)

scatter!([2], [2], color = "white", label="f(2)=2", markersize = 3)
scatter!([0], [f(0)], color = "blue", label="", markersize = 3)
scatter!([3], [g(3)], color = "blue", label="", markersize = 3)
```

Out[11]:



Example

$f: [1, 3] \rightarrow \mathrm{R}$

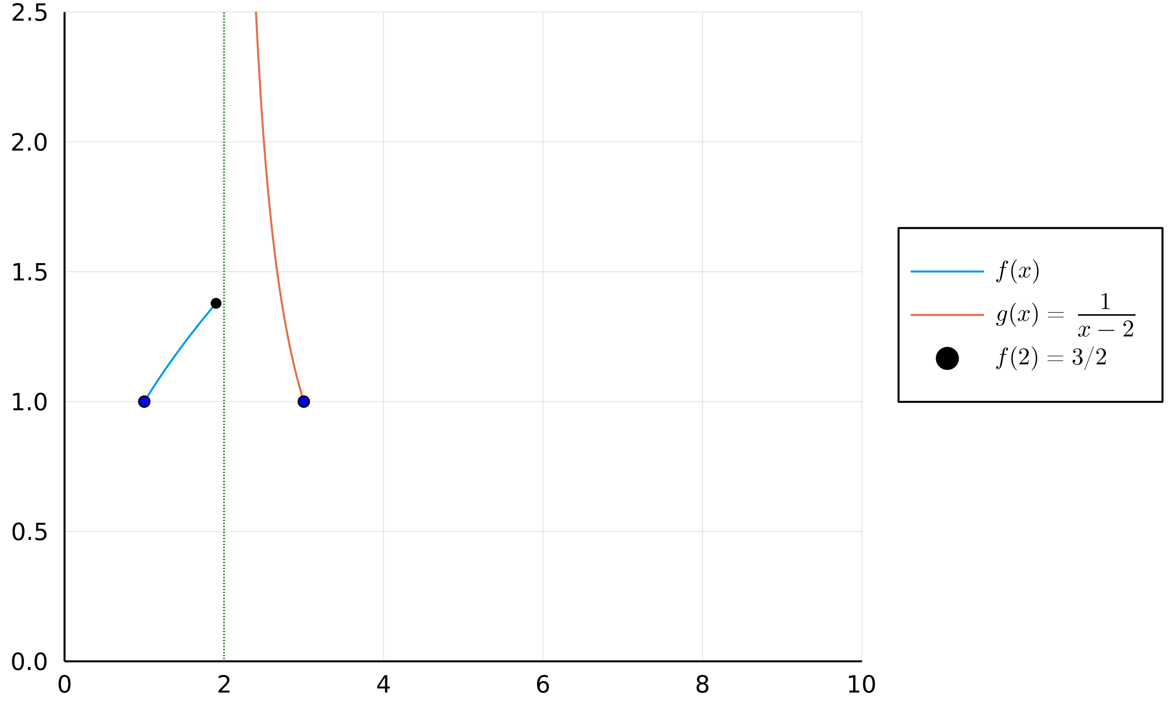
$$\lim_{x \to 2^-} f(x) = \frac{3}{2} = f(2) \quad \lim_{x \to 2^+} f(x) = +\infty$$

- f is not continuous in $x = 2$
- f is continuous from the left in $x = 2$

see the graph below.

In [21]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = sqrt.(x)
g(x) = 1/(x-2)

plot(f,1,1.9, xlims=(0,10), ylims=(0,2.5),
    bottom_margin = 10mm, label=L"f(x)", framestyle = :zerolines,
    legend=:outerright)
plot!(g,2.1,3, xlims=(0,10), ylims=(0,2.5),
    bottom_margin = 10mm, label=L"g(x) = \frac{1}{x-2}", framestyle = :zerolines,
    legend=:outerright)
plot!([2], seriestype="vline", color=:green, label="", linestyle=:dot)

scatter!([1], [f(1)], color = "blue", label="", markersize = 3)
scatter!([1.9], [f(1.9)], color = "black", label=L"f(2)=3/2", markersize = 3)
scatter!([3], [g(3)], color = "blue", label="", markersize = 3)
```

Out[21]:



Example

$f : (1, 3) \to \mathrm{R}$

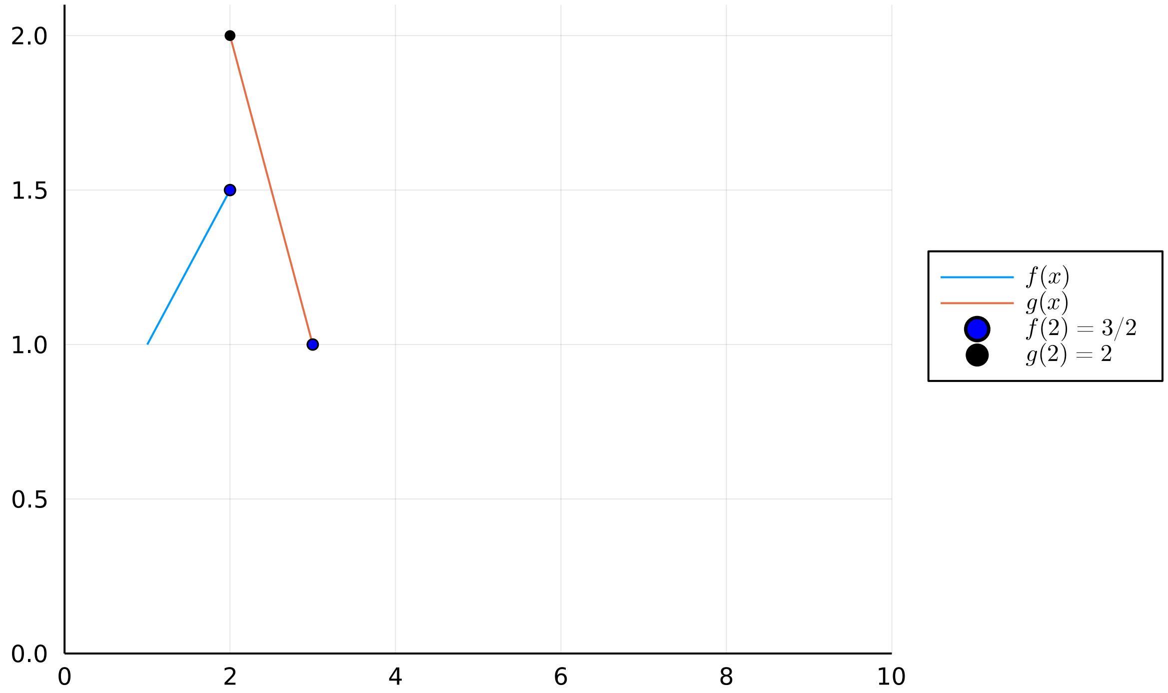
$$\lim_{2^-} f(x) = \frac{3}{2} \neq \lim_{x \to 2^+} f(x) = 2 = f(2)$$

f is not continuous in $x = 2$ but it is continuous from the right.

see the graph below.

In [49]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = (x/2+1/2)
g(x) = -x+4

plot(f,1,2, xlims=(0,10), ylims=(0,2.1),
    bottom_margin = 10mm, label=L"f(x)", framestyle = :zerolines,
    legend=:outerright)
plot!(g,2,3, xlims=(0,10), ylims=(0,2.1),
    bottom_margin = 10mm, label=L"g(x)", framestyle = :zerolines,
    legend=:outerright)

scatter!([2], [f(2)], color = "blue", label=L"f(2)=3/2", markersize = 3)
scatter!([2], [g(2)], color = "black", label=L"g(2)=2", markersize = 3)
```

Out[49]:



We say that f is continuous in D if it is continuous in every part of D.

If f is continuous in $x_0 \in D \cap D'$ we can write

1) $\lim_{x \to x_0} f(x) = f(x_0)$

or

2) $\lim_{h \to 0} f(x_0 + h) = f(x_0)$

$x_0 + h = x$

$h = x - x_0$

or

$$\lim_{h \to 0} \leftrightarrow \lim_{x \to x_0}$$

3) $\lim_{x \to x_0} [f(x) - f(x_0)] = 0$

Continuity of Some Elementary Functions

Example

$f(x) = \sin x : \mathrm{R} \to \mathrm{R}$ is continuous in any $x_0 \in \mathrm{R}$ that is:

1) $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x : \mathrm{R}$ with

$|x - x_0| < \delta \to |\sin x - \sin x_0| < \varepsilon$

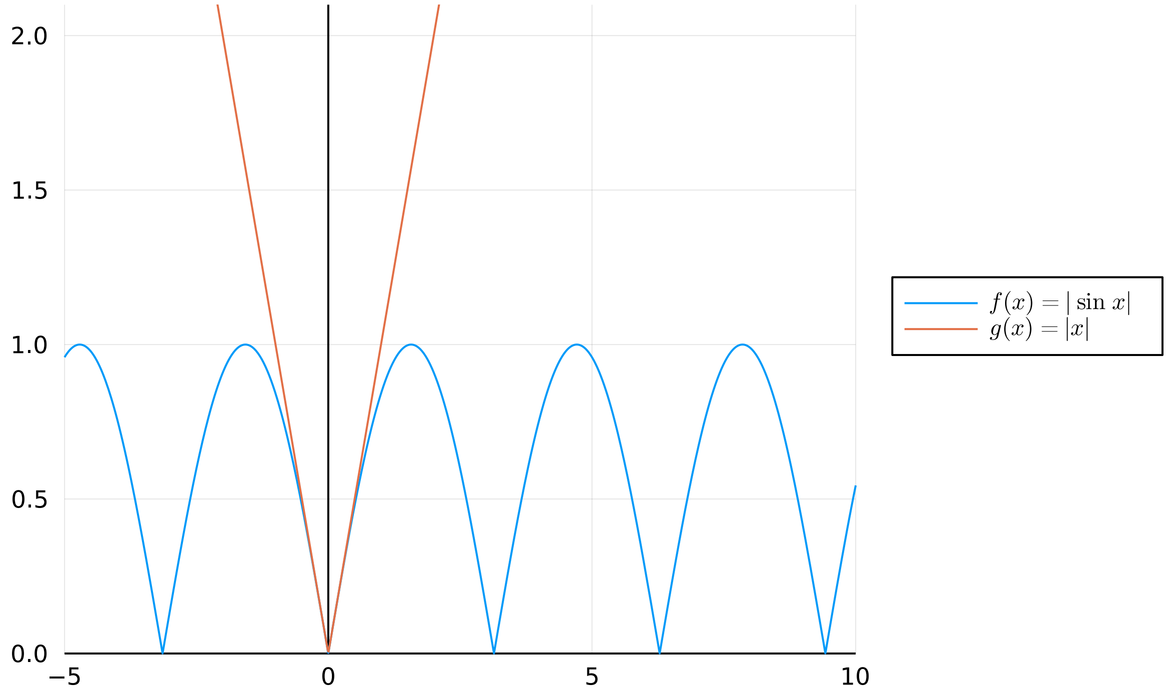
we know that $|\sin x| \leq |x|$

$$\Big| \; (x - x_0) \quad (x + x_0) \Big| \quad \Big| \; (x - x_0) \Big| \; \Big| \; (x + x_0) \Big| \quad \Big| \; (x - x_0) \Big| \quad |x - x_0|$$

In [52]:

```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = abs(sin.(x))
g(x) = abs(x)

plot(f,-5,10, xlims=(-5,10), ylims=(0,2.1),
    bottom_margin = 10mm, label=L"f(x) = |\sin \ x|", framestyle = :zerolines,
    legend=:outerright)
plot!(g,-5,10, xlims=(-5,10), ylims=(0,2.1),
    bottom_margin = 10mm, label=L"g(x) = |x|", framestyle = :zerolines,
    legend=:outerright)
```

Out[52]:



Example

$\cos x$ is continuous in R

$$\cos x - \cos x_0 = -2\, sin\left(\frac{x + x_0}{2}\right) \sin\left(\frac{x - x_0}{2}\right)$$

Example

$e^x : \mathrm{R} \to \mathrm{R}$ is continuous take $x_0 \in \mathrm{R}$

$$\lim_{x \to x_0} e^x = e^{x_0} \leftrightarrow \lim_{x \to x_0} e^x - e^{x_0} = 0$$

$$\lim_{x \to x_0} e^x \frac{e^{x_0}}{e^{x_0}} = 0$$

$$\lim_{x \to x_0} e^{x_0}\left(e^x e^{x_0} - 1\right) = 0$$

$$\lim_{x \to x_0} e^{x_0}(x - x_0) \cdot \left(\frac{e^{x - x_0} - 1}{x - x_0}\right) = 0$$

$\lim_{y \to 0} \frac{e^y - 1}{y} = 1$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}} = e$$

Prove, by $\varepsilon - \delta$ definition of continuity starting from $\forall \varepsilon > 0$

$|e^x - e^{x_0}|$

Theorems on Continuous Functions

From definition of limits it follows

Propositionz: The constant functions are continuous, the sum or difference of continuous function is continuous, the product of continuous functions is continuous and if $f(x)$ and $g(x)$ are continuous $\frac{f(x)}{g(x)}$ is continuous where $g(x) \neq 0$, the composition of continuous functions is continuous.

Example

$\tan x = \frac{\sin x}{\cos x}$ is continuous

if $x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

Example

$f(x) = \cos(\ln x)$

$x \to \ln x \to \cos(\ln x)$

$(0, +\infty) \to \mathbb{R} \to \mathbb{R}$

Since $\cos(x)$ is continuous in $\mathbb{R}$ and $\ln(x)$ is continuous in $(0, +\infty)$, thus composition is continuous.

Example

$$f(x)\begin{cases} (e^{\frac{1}{x}} + 2)^{-1} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where f is continuous, thus $D_f = \mathbb{R}$.

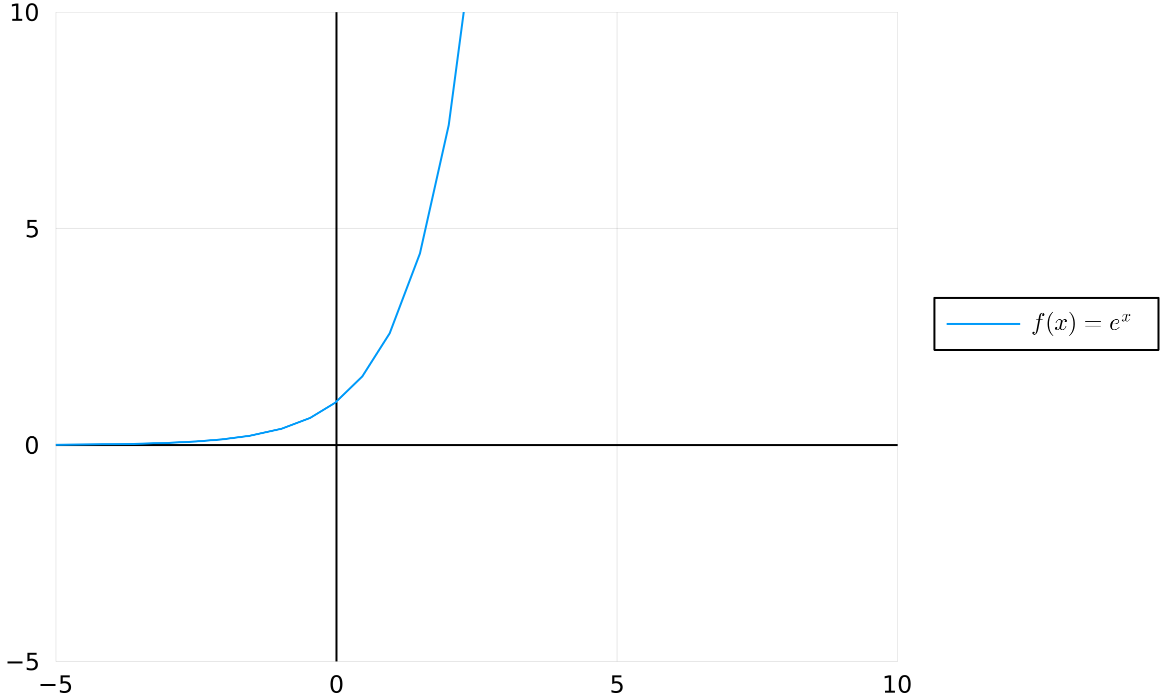
We only have to check if f is continuous in $x_0 = 0$

$$\lim_{x \to 0^+} \frac{1}{e^{\frac{1}{x}} + 2} = 0 = f(0)$$

$$\lim_{x \to 0^-} \frac{1}{e^{\frac{1}{x}} + 2} = \frac{1}{2} \neq 0$$

In [56]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = exp.(x)

plot(f,-5,10, xlims=(-5,10), ylims=(-5,10),
    bottom_margin = 10mm, label=L"f(x) = e^{x}", framestyle = :zerolines,
    legend=:outerright)
```

Out[56]:



Example

$$f(x)\begin{cases} \frac{x^3-3x^2+3x-9}{\sqrt{x-3}} & x>3 \\ 17 & x\leq 3 \end{cases}$$

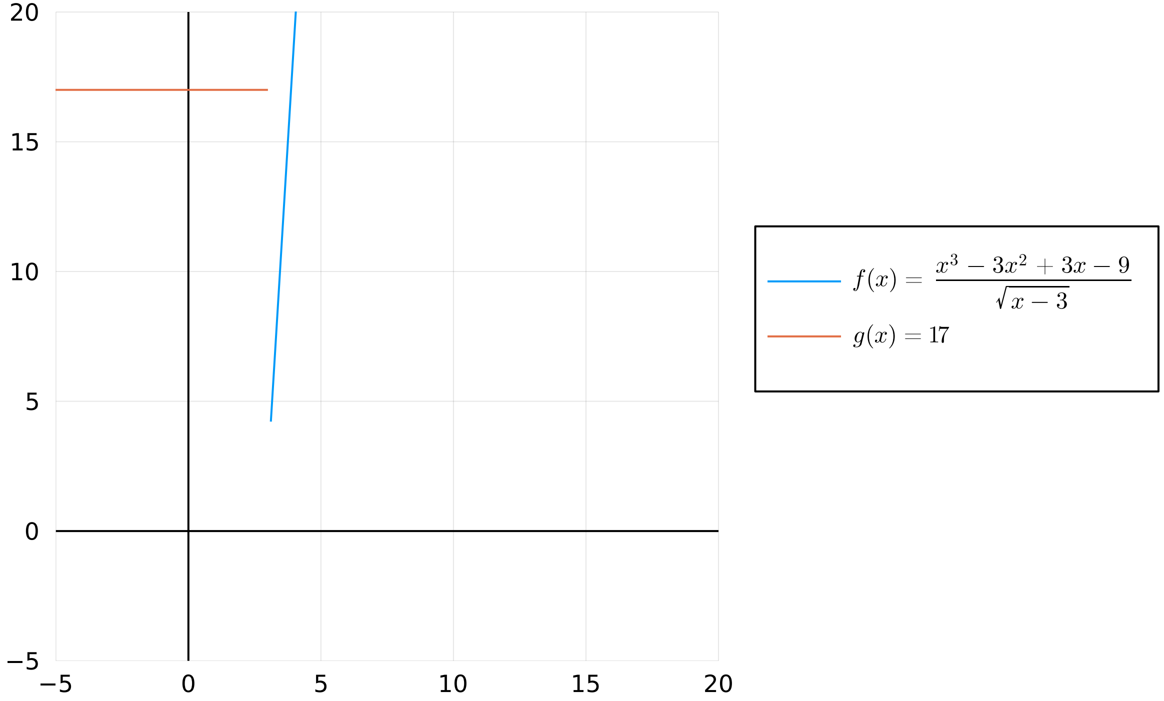
study the continuity of $f(x)$

To be continuous we will check the limit of function at $x = 3$

$$\lim\ f(x) = 17 \qquad \lim\ f(x) = \infty$$

In [1]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = (x^(3) - 3x^(2) + 3x - 9)/(sqrt(x-3))
g(x) = 17

plot(f,3,20, xlims=(-5,20), ylims=(-5,20),
    bottom_margin = 10mm, label=L"f(x) = \frac{x^{3} - 3x^{2} + 3x - 9}{\sqrt{x-3}}", framestyle = :zerolin
    legend=:outerright)

plot!(g,-10, 3, xlims=(-5,20), ylims=(-5,20),
    bottom_margin = 10mm, label=L"g(x) = 17", framestyle = :zerolines,
    legend=:outerright)
```

Out[1]:



Theorem 1: Permanence of sign

Let $f: D \subseteq \mathrm{R} \to \mathrm{R}$, continuous in $x_0 D \cap D'$, if $f(x_0) > 0$ ($f(x_0) < 0$) then $\exists U_{x_0}: \forall x \in U_{x_0} \cap D$ we have $f(x) > 0$ ($f(x) < 0$).

Theorem 2: On the Zeros of a Continuous Functions

Let $f: [a, b] \to \mathrm{R}$ continuous with $f(a) \cdot f(b) < 0$, then there exists at least one $\bar{x} \in (a, b)$ such that $f(\bar{x}) = 0$

Proof

Consider the case $f(a) > 0$ and $f(b) < 0$

$c_0 = \frac{a+b}{2}$ if $f(c_0) = 0$ you are done.

If $f(c_0) \neq 0$ then:

1) In the case of $f(c_0) < 0$

take

$[a_1, b_1] = [a, c_0]$

$$c_1 = \frac{a + c_0}{2} = \frac{a_1 + b_1}{2}$$

if $f(c_1) < 0$ take

$[a_2, b_2] = [a, c_1]$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{a + c_1}{2}$$

if $f(c_2) < 0$ take

$[a_3, b_3] = [c_2, b_2]$

till $f(c_n) = 0$ or you have a sequence of intervals.

$[a, b] > [a_1, b_1] > [a_2, b_2] > \cdots > [a_n, b_n]$

$$|b_n - a_n| = \frac{|b-a|}{2^{n+1}} \to 0 \quad (***)$$

as $n \to \infty$

$a \le a_1 \le a_2 \le \cdots \le a_n \le b_n \le b_{n-1} \le \cdots \le b$

So we have two bounded monotonic sequences $\{a_n\}$ increasing and $\{b_n\}$ decreasing.

$$\exists \alpha \in (a, b): \quad a_n \to \alpha \quad (*) \exists \beta \in (a, b): \quad b_n \to \beta \quad (**)$$

we have to prove that $\alpha = \beta$ and that this common value coincides with $\bar{x}$ which is the zero of the function $f(x)$, indeed

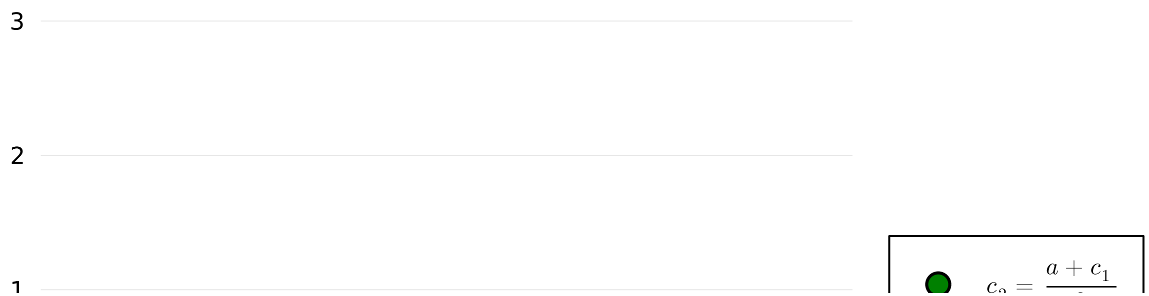
$$|\alpha - \beta| = |\alpha - a_n + a_n - b_n + b_n - \beta| \le |\alpha - a_n| + |a_n - b_n| + |b_n - \beta|$$

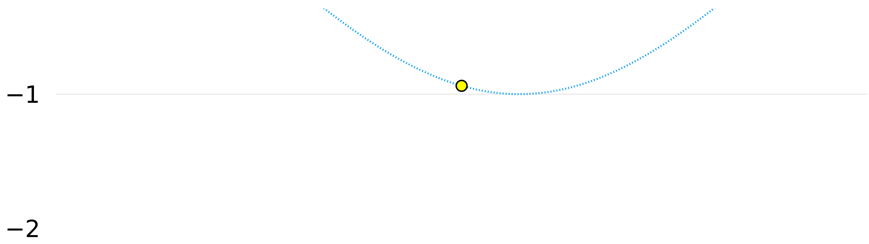
- $|\alpha - a_n| \to 0$
- $|a_n - b_n| \to 0$
- $|b_n - \beta| \to 0$

thus $\alpha = \beta$

In [35]:
```
using Plots, LaTeXStrings
gr()

a, b = 5, 10
g(x) = sin.(x+pi)

plot(g, a, b; legend=:outerright, label="", framestyle=:zerolines,
    xlims = (5,10), xticks=false,
    ylims = (-2,3), yticks = -2:1:3,
    linestyle=:dot, size=(600, 360))

scatter!([5.2], [g(5.2)], color = "red", label="", markersize = 3)
scatter!([5.775], [g(5.775)], color = "green", label=L"c_{2} = \frac{a+c_{1}}{2}", markersize = 3)
scatter!([6.35], [g(6.35)], color = "blue", label=L"c_{1} = \frac{a + c_{0}}{2}", markersize = 3)
scatter!([7.5], [g(7.5)], color = "yellow", label=L"c_{0} = \frac{a+b}{2}", markersize = 3)
scatter!([9.8], [g(9.8)], color = "red", label="", markersize = 3)

annotate!([(5.2,-0.2, (L"a", 8, :black))])
annotate!([(5.2,0, (L"|", 8, :black))])

annotate!([(5.775,-0.2, (L"c_{2}", 8, :black))])
annotate!([(5.775,0, (L"|", 8, :black))])

annotate!([(6.35,-0.2, (L"c_{1}", 8, :black))])
annotate!([(6.35,0, (L"|", 8, :black))])

annotate!([(7.5,-0.2, (L"c_{0}", 8, :black))])
annotate!([(7.5,0, (L"|", 8, :black))])

annotate!([(9.8,-0.2, (L"b", 8, :black))])
annotate!([(9.8,0, (L"|", 8, :black))])
```

Out[35]:





we set $\bar{x} = \alpha = \beta$.

Now we prove that $f(\bar{x}) = 0$

$$\lim_{n \to \infty} f(a_n) = f(\bar{x}) \geq 0$$

$$\lim_{n \to \infty} f(b_n) = f(\bar{x}) \leq 0$$

$a_n \to \bar{x}$

$b_n \to \bar{x}$

f is continuous. Then by continuity and by uniqueness of limit we have

$f(\bar{x}) = 0$

Theorem 3: Intermediate Values

Let $f: I \to \mathrm{R}$, I interval ((a, b), or $(a, b]$, or $[a, b)$, or $[a, b]$).

f continuous in I, then f takes all the values between $l \ \inf_I f(x)$ and $L = \sup_I f(x)$

Proof

Take any t such that $l < t < L$, we want to prove that $\exists \bar{x} \in I$ such that $f(\bar{x}) = t$.

Since $l = \inf_I f(x)$ and $t > l$ by the 2nd property of $\inf$:

$\exists x_1 \in I: f(x_1) < t$

If $l = \inf_I f$ then

1) $\forall x \in I \quad f(x) \geq l$

2) $\forall \varepsilon > 0 \; \exists \bar{x} \in I : f(\bar{x}) < l + \varepsilon$

since $L = \sup_I f(x)$ and $t < L$ by the 2nd property of $\sup$

$\exists x_2 \in I : f(x_2) > t$

thus we have

$l \leq f(x_1) < t < f(x_2) \leq L$

now define

$\psi : [x_1, x_2] \to \mathbb{R}$ continuous

$$\begin{aligned} \psi(x) &= f(x) - t \\ \psi(x_1) &= f(x_1) - t < 0 \\ \psi(x_2) &= f(x_2) - t > 0 \end{aligned}$$

by Theorem 2 of zeros, $\exists \bar{x} \in [x_1, x_2] \subset I$ such that $\psi(\bar{x}) = 0$

$\leftrightarrow f(\bar{x}) - t = 0$

$f(\bar{x}) = t$

Theorem 4: Continuous Functions map Intervals in Intervals

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ continuous, then $f(I)$ is on interval.

Proof

By theorem 3,

$$f(I) = (\inf_{I} f, \sup_{I} f)$$

Introductory Real Analysis Course 10 (18 October 2022)

Continuity

Let $f(x)$ be a function defined in the interval I of $\mathbb{R}$ and $x_0 \in I$.

We say that $f(x)$ is continuous in x_0, if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Exercises

1) If the function f is continuous

$$f(x) = \frac{1}{x-2} \quad f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$$

Is the function continuous in 2?

The function is not defined in 2, thus we cannot say that it is continuous.

2) Let $f(x)$ be defined in $(0, 2)$

$$f(x) \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x < 2 \end{cases} \quad \lim_{x \to 1} x = 1 = f(1)$$

which is the set of discontinuity points of f?

This set is empty because the function is continuous in $(0, 2)$.

3) We have to study the continuity of the function defined on $\mathbb{R}$

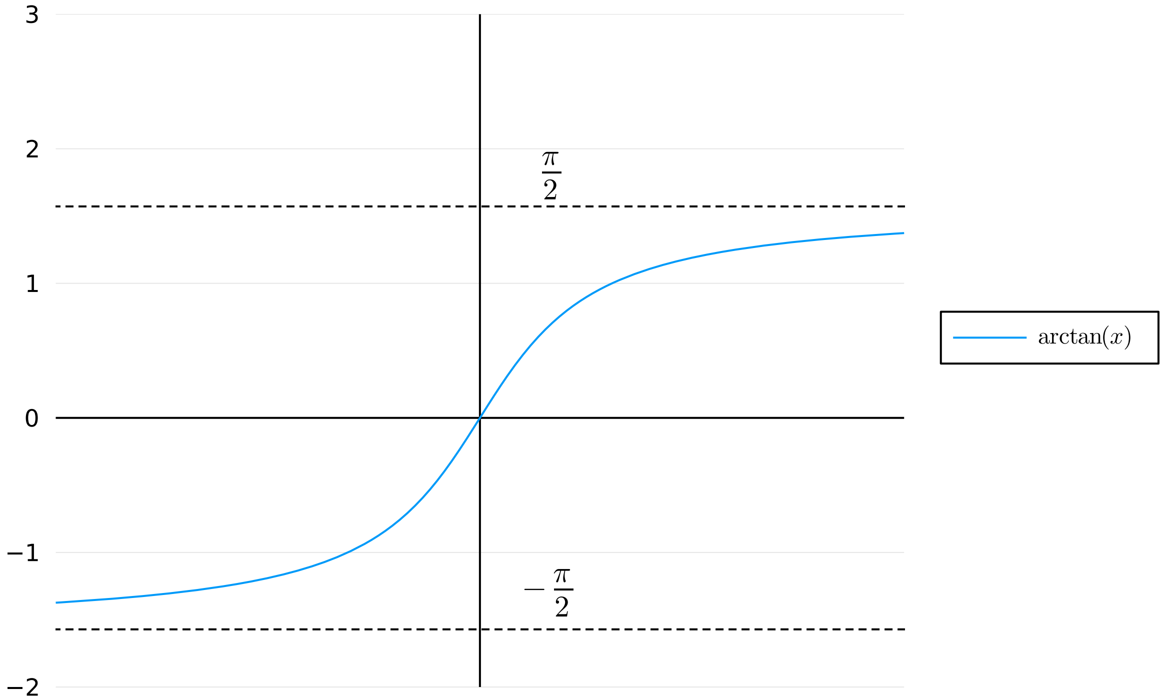
$$f(x)\begin{cases} atan\ \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is $f(0) = \lim_{x \to 0} f(x)$

$$\lim_{x \to 0^+} atan\frac{1}{x} = \frac{\pi}{2} \quad \lim_{x \to 0^-} atan\frac{1}{x} = -\frac{\pi}{2}$$

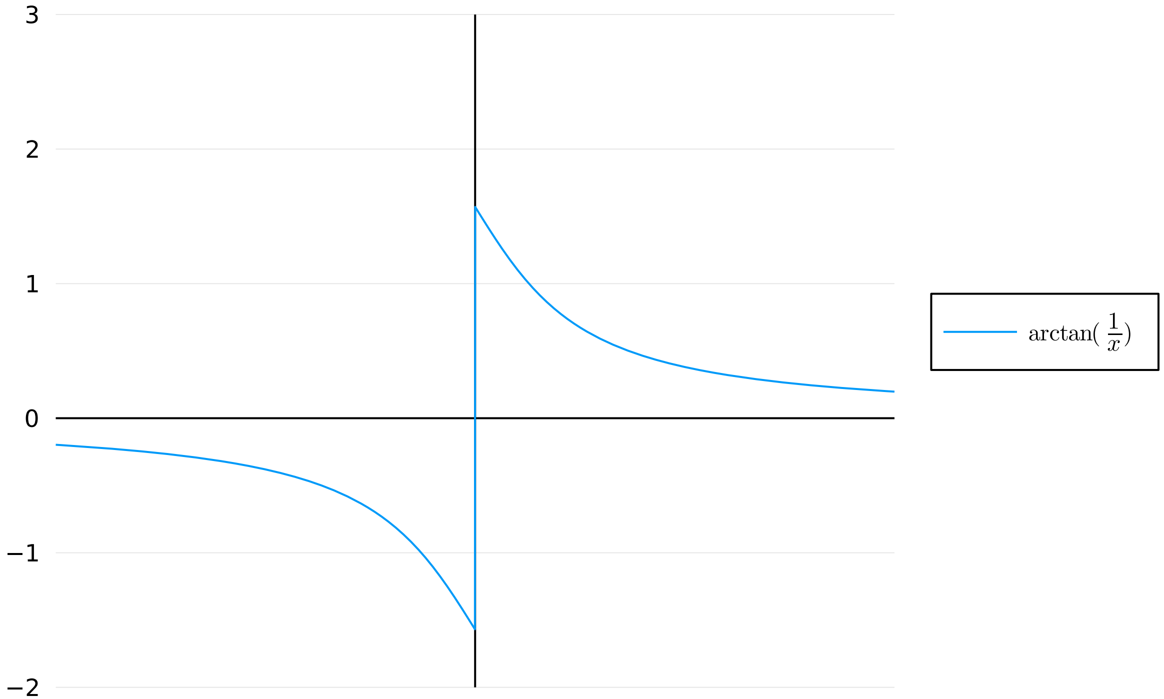
In [48]:
```
using Plots, LaTeXStrings
gr()

a, b = -5, 5
g(x) = atan.(x)

plot(g, a, b; legend=:outerright, label=L"\arctan(x)", framestyle=:zerolines,
    xlims = (-5,5), xticks=false,
    ylims = (-2,3), yticks = -2:1:3,
    size=(600, 360))

plot!([22/14], seriestype="hline", linestyle=:dash, color=:black, label="")
plot!([-22/14], seriestype="hline", linestyle=:dash, color=:black, label="")

annotate!([(0.83,1.8, (L"\frac{\pi}{2}", 10, :black)),
           (0.83,-1.3, (L"- \frac{\pi}{2}", 10, :black))])
```

Out[48]:



In [50]:
```
using Plots, LaTeXStrings
gr()

a, b = -5, 5
g(x) = atan.(1/x)

plot(g, a, b; legend=:outerright, label=L"\arctan(\frac{1}{x})", framestyle=:zerolines,
    xlims = (-5,5), xticks=false,
    ylims = (-2,3), yticks = -2:1:3,
    size=(600, 360))
```

Out[50]:



Exercise

4) We have to study the function

$$f(x)\begin{cases} \sin x \cdot 2^{\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x \to 0^+} \sin x \, 2^{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\sin x}{x} \cdot x \, 2^{\frac{1}{x}} = \lim_{x \to 0^+} \frac{2^{\frac{1}{x}}}{\frac{1}{x}} = +\infty$$

Zero is a discontinuity point.

5) We have to say if these functions can be prolonged by continuity

$$f_1(x) = e^{-\frac{1}{x^2}} \quad \mathrm{R} \setminus \{0\}$$

$$\bar{f}_1 = \begin{cases} f_1 & x \in \mathrm{R} \setminus \{0\} \\ ? & x = 0 \end{cases}$$

$$\lim_{x \to 0} e^{-\frac{1}{x^2}} = 0 \rightarrow \bar{f}_1 = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

thus $\bar{f}_1(x)$ is continuous.

6) $f_2(x) = \frac{\sin \ x}{x} = 1 \quad \mathrm{R} \setminus \{0\}$

$$f_2(x) = \frac{\sin x}{x} = 1 \quad \rightarrow \quad \bar{f}_2(x) = \begin{cases} \frac{\sin \ x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

7) Find a and b such that the function $f(x)$ defined on R by

$$f(x)\begin{cases} \sin x & x \le -\frac{\pi}{2} \\ a\sin x + b & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \cos x & x \ge \frac{\pi}{2} \end{cases}$$

is continuous on $\mathbb{R}$.

$$\lim_{x \to -\frac{\pi}{2}^-} \sin x = -1$$

$$\lim_{x \to -\frac{\pi}{2}^+} a\sin x + b = -a + b$$

$$\lim_{x \to -\frac{\pi}{2}^-} a\sin x + b = a + b$$

$$\lim_{x \to -\frac{\pi}{2}^+} \cos = 0$$

From the limit above we obtain:

$$\begin{cases} -a + b = -1 \\ a + b = 0 \end{cases}$$

thus $b = -\frac{1}{2}$ and $a = \frac{1}{2}$.

8) Let us prove that the function $f(x)$ defined by

$$f(x)\begin{cases} \frac{\ln\ (1+x)}{x} & x \in (-1, 0) \\ \frac{e^x - 1}{x} & \in (0, 1) \end{cases}$$

is continuous in $(-1, 1) \setminus \{0\}$ and it admits on a continuous extension on $(-1, 1)$, but not on $[-1, 1]$.

$$\lim_{x \to 0^-} \frac{\ln(1+x)}{x} = 1 \quad \lim_{x \to 0^+} \frac{e^x - 1}{x} = 1$$

$$f(x) \quad \text{on } (-1, 1)\begin{cases} \frac{\ln\ (1+x)}{x} & x \in (-1, 0) \\ 1 & x = 0 \\ \frac{e^x - 1}{x} & x \in (0, 1) \end{cases}$$

$$\lim_{x \to -1} \frac{\ln(1+x)}{x} = +\infty \quad \lim_{x \to 1} \frac{e^x - 1}{x} = e - 1$$

9) We have to find values of $\alpha \in \mathrm{R}$ such that the function

$$f(x)\begin{cases} \sqrt{x} + 1 & x \geq 0 \\ \sin(x+\alpha) & x < 0 \end{cases}$$

is continuous.

$$f(0) = 1 \quad \lim_{x \to 0^-} \sin(x+\alpha) = \sin(x)\cos(\alpha) + \cos(x)\sin(\alpha) = \sin(\alpha)$$

because $\lim_{x \to 0^-}\sin(x) \approx= 0$ and $\lim_{x \to 0^-}\cos(x) \approx= 1$

hence

$\sin(\alpha) = 1$

$\alpha = \frac{\pi}{2} + 2k\pi,\ k \in \mathrm{Z}$

we use $\frac{\pi}{2}$ to represents the degree which will resulting to $\sin(\alpha) = \sin(\frac{\pi}{2}) = 1$ and add $+2k\pi$ because sine is a periodic function.

10) Let us study the continuity in the point $x_0 = 0$ of the function

$$f(x)\begin{cases} \frac{x}{|\alpha x|} & x < 0 \\ \beta - 1 & x = 0 \\ e^{\beta x}\,(\beta x - x) & x > 0 \end{cases}$$

$\alpha \neq 0$ and $\beta \in \mathrm{R}$.

$$\lim_{x \to 0^-} \frac{x}{|\alpha x|} = \lim_{x \to 0^-} \frac{1}{|\alpha|}\frac{x}{-x} = -\frac{1}{|\alpha|}$$

$$\lim_{x \to 0^+} e^{\beta x}(2\beta - x) = 2\beta$$

to make the function continuous we have to have this condition fulfilled:

$2\beta = \beta - 1 = -\frac{1}{|\alpha|}$

if we choose $\beta = -1$, then

$$-2 = -\frac{1}{|\alpha|}$$

$$|\alpha| = \frac{1}{2}$$

$$\alpha = \pm\frac{1}{2}$$

Minimum / Maximum - Inf/Sup

1) Let us compute $\inf$ and $\sup$ of the function on $\mathbb{R}$ defined by $f(x) = \frac{1}{1+4x^2}$.

Let us verify that $f(x)$ has a maximum but not a minimum.

$$0 < \frac{1}{1+4x^2} \leq 1 \quad \forall x \in \mathbb{R}$$

- we notice $f(0) = 1 \rightarrow$ we can say that 1 is the maximum of f.
- 0 is the $\inf$ of f.

Let us consider $0 < \varepsilon < 1$, the inequality $\frac{1}{1+4x^2} < \varepsilon$ it is verified when

$$\begin{aligned} 1 &< \varepsilon(1+4x^2) \\ 1 &< \varepsilon + 4\varepsilon x^2 \\ \frac{1-\varepsilon}{4\varepsilon} &< x^2 \\ |x| &> \sqrt{\frac{1-\varepsilon}{4\varepsilon}} \end{aligned}$$

2) Let us compute $\inf$ and $\sup$ of the function defined on $\mathbb{R}$

$$f(x) = \frac{e^x - 1}{e^x + 1} = \frac{e^x}{e^x+1} - \frac{1}{e^x+1}$$

$-1 < \frac{e^x-1}{e^x+1} < 1$ the function is bounded.

The $\sup$ is 1 because $0 < \varepsilon < 2$ the inequality $\frac{e^x-1}{e^x+1} > 1 - \varepsilon$ is verified if $x > \log\frac{2-\varepsilon}{\varepsilon}$.

You can verify that -1 is the $\inf$.

Limit

1)

$$\lim_{x \to +\infty} 2^{\frac{1-x^2}{x+2}} = 0$$

because

$$\lim_{x \to +\infty} \frac{1-x^2}{x+2} = -\infty$$

2)

$$\lim_{x \to -\infty} e^{x\sqrt{-x}} = 0$$

3)

$$\lim_{x \to +\infty} e^{\frac{\sin 3x}{x}} = 1$$

4)

$$\lim_{x \to +\infty} \frac{\sin 3x}{x} = 0$$

$\sin 3x$ is bounded, $-1 < \sin x < 1$.

5)

$$\begin{aligned}\lim_{x \to +\infty} \left(\frac{1}{x}\right)^x &= \lim_{x \to +\infty} e^{x \ln\left(\frac{1}{x}\right)} \\ &= \lim_{x \to +\infty} e^{-x \ln(x)} \\ &= 0\end{aligned}$$

In [13]:
```
# SymPy can compute symbolic limits with the limit function
# For lim x -> x_{0} f(x)

using SymPy
@vars x y z

#limit(sin(x)/x, x, 0)

# a pair can be used to indicate the limit:
limit(sin(x)/x, x=>0)

# limit((pi/2-x-acos(x))/x^3, x=>0)
```

Out[13]: 1

Julia Codes

To compute limit from the right or the left:

$$\lim_{x \to 0^+} \frac{1}{x}$$

or

$$\lim_{x \to 0^-} \frac{1}{x}$$

In [15]:
```
using SymPy
@vars x y z

limit(1/x, x => 0, "+")
#limit(1/x, x, 0, '-')
```

Out[15]: ∞

Introductory Real Analysis Course 12 (19 October 2022)

Example

$f(x) = x^2 : (0, 1) \to \mathrm{R}$ is it uniformly continuous?

We have to verify that $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 : \ \forall x_1, x_2 \in (0, 1)$ with $|x_1 - x_2| < \delta \to |(x_1^2) - (x_2)^2| < \varepsilon$.

1) Fix ε.

2) Is there exist a $\delta > 0$

3)

$|(x_1)^2 - (x_2)^2| = |x_1 - x_2||x_1 + x_2| < 2|x_1 - x_2| < \varepsilon$

$0 < x_1, x_2 < 1$

it is enough to have $|x_1 - x_2| < \frac{\varepsilon}{2} = \delta$

indeed.

$|(x_1)^2 - (x_2)^2| < 2|x_1 - x_2| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$

Example

Given $f(x) = x^2 : \mathrm{R} \to \mathrm{R}$ prove that f is not uniformly continuous.

1) Fix $\varepsilon = 1$

2) Take $x_1 \in \mathrm{R}$ and $x_2 = x_1 - \frac{\delta}{2}$

3)

$$|x_1 - x_2| = \left|x_1 - x_1 + \frac{\delta}{2}\right| = \frac{\delta}{2} < \delta$$

$$\left|x_1^2 - (x_1 - \frac{\delta}{2})^2\right| = \left|x_1^2 - x_1^2 + 2x_1\frac{\delta}{2} - \frac{\delta^2}{4}\right|$$

$$= \left|\delta\left(x_1 - \frac{\delta}{4}\right)\right| < 1 = \varepsilon$$

If x becomes very big or very small, $\left|\delta\left(x_1 - \frac{\delta}{4}\right)\right|$ becomes bigger than 1.

Example

$f(x) = \sqrt{x} : [0, +\infty] \rightarrow \mathrm{R}$ prove that f is uniformly continuous.

It holds $\forall x_1, x_2 \in \mathrm{R}_0{}^+$

$$|\sqrt{x_1} - \sqrt{x_2}| \leq \sqrt{|x_1 - x_2|}$$

call the above $(*)$

- $\alpha = \frac{1}{2}$
- Particular case of Holderian function $\exists c > 0$ thus

$$|f(x_1) - f(x_2)| \leq c|x_1 - x_2|^\alpha \quad 0 < \alpha < 1$$

Remember that $\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0: \ \forall x_1, x_2 \in [0, +\infty]$ with

$|x_1 - x_2| < \delta \rightarrow |\sqrt{x_1} - \sqrt{x_2}| < \varepsilon$

- Fix arbitrarily $\varepsilon > 0$, since

$|\sqrt{x_1} - \sqrt{x_2}| \leq \sqrt{|x_1 - x_2|} < \varepsilon$

Then by $(*)$:

$\delta = \varepsilon^2$

$\sqrt{|x_1 - x_2|} < \sqrt{\varepsilon^2} = \varepsilon$

if $|x_1 - x_2| < \varepsilon^2$

Example

Prove $f(x) = \sqrt{x} : [1, +\infty) \to \mathrm{R}$ is uniformly continuous without using $(*)$

Proposition (Hunter p. 127 Prop 7.24)

A function $f : D \subset \mathrm{R} \to \mathrm{R}$ is not uniformly continuous on D if and only if there exists a $\varepsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in D such that

$$\lim_{n \to +\infty} |x_n - y_n| = 0 \quad \text{and} \quad |f(x_n) - f(y_n)| > \varepsilon_0$$

Example

$f(x) = \frac{1}{x} : (0, 1] \to \mathrm{R}$ is not uniformly continuous.

1) Take $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$

2) Then

$$\lim_{n \to +\infty} \left| \frac{1}{n} - \frac{1}{n+1} \right| = \lim_{n \to +\infty} \left| \frac{n+1-n}{n(n+1)} \right| = 0$$

and

$$|f(x_n - f(y_n)| = |n - n - 1| = 1 \quad (1 > \frac{1}{2})$$

3) Take $\varepsilon_0 = \frac{1}{2}$

Theorems about Uniform Continuity

1) Theorem Weierstrass

If $f: K \subset \mathrm{R} \to \mathrm{R}$ is continuous and K is compact then f is uniformly continuous.

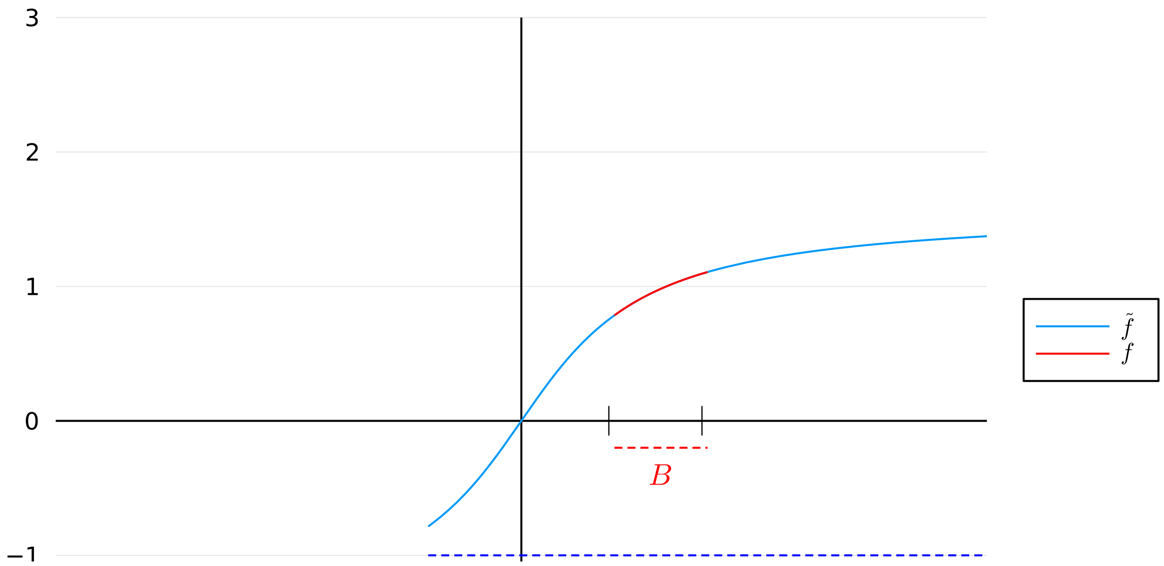
2) If $f: D \subset \mathrm{R} \to \mathrm{R}$ is uniformly continuous and D is bounded, then f is bounded on D.

3) (Extension Theorem) Let $f: D \subset \mathrm{R} \to \mathrm{R}$ then f is uniformly continuous in D if and only if f is the restriction to D of a function $\tilde{f}$ uniformly continuous in $\bar{D}$

From the figure below we can see that:

In [11]:
```
using Plots, LaTeXStrings
gr()

a, b = -1, 5
g(x) = atan.(x)

plot(g, a, b; legend=:outerright, label=L"\tilde{f}", framestyle=:zerolines,
    xlims = (-5,5), xticks=false,
    ylims = (-2,3), yticks = -2:1:3,
    size=(600, 360))

plot!(g, 1,2; legend=:outerright, label=L"f",
    linecolor=:red, framestyle=:zerolines,
    xlims = (-5,5), xticks=false,
    ylims = (-2,3), yticks = -2:1:3)

plot!([1,2],[-0.2,-0.2], label="", linecolor=:red, linestyle=:dash)
plot!([-1,5],[-1,-1], label="", linecolor=:blue, linestyle=:dash)

annotate!([(1,0, (L"|", 10, :black)),
           (2,0, (L"|", 10, :black)),
           (2.5,-1.5, (L"A", 10, :blue)),
           (1.5,-0.4, (L"B", 10, :red))])
```

Out[11]:



4) Let $f: D \subset \mathrm{R} \rightarrow \mathrm{R}$ uniformly continuous in D then there exists finite $\lim_{x \rightarrow x_0} f(x)\ \forall x_0 \in D'$

$(x_0) \neq \pm \infty$

Example: Application of Theorem 1 and Theorem 3

If $f(x) = x \sin\left(\frac{1}{x}\right): (0, 1] \rightarrow \mathrm{R}$, then is it uniformly continuous?

f is continuous in $(0, 1]$

$$\lim_{x \rightarrow 0^+} x \sin\left(\frac{1}{x}\right) \quad x \in (0, 1]$$

$$\tilde{f}(x) \begin{cases} x \sin\left(\frac{1}{x}\right) & x \in (0, 1] \\ 0 \quad x = 0 \end{cases}$$

$\bar{D} = [0, 1]$

$$\lim_{x \rightarrow 0^+} \tilde{f}(x) = 0 = \tilde{f}(0)$$

so $\tilde{f}: [0, 1] \rightarrow \mathrm{R}$ is a continuous function in a compact interval thus

1) By Theorem Weierstrass $\tilde{f}$ is uniformly continuous then by Theorem 3 f is uniformly continuous.

Example

Check whether $f(x) = \frac{1}{x^2 - 4}$ is uniformly continuous in $(-1, 2)$

Differentiable Functions

Let $I \subset \mathbb{R}$ be an interval, and $f: I \rightarrow \mathbb{R}$ is said to be differentiable in $x_0 \in \dot{I} = \{x \in I: \ \exists\ B_r(x) \subset I, r > 0\}$ if there exists finite the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if this limit exists, we call it $f^{'}(x_0)$ which is the derivative of f in x_0.

So if f is differentiable in x_0 we write

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f^{'}(x_0)$$

or we can write

$$\lim_{h \to x_0} \frac{f(x_0 + h) - f(x_0)}{h} = f^{'}(x_0)$$

where we have set $h = x - x_0$ and $x = h + x_0$.

Example

check whether $f(x) = e^x$ is differentiable in $\mathbb{R}$.

Let $x_0 \in \mathbb{R}$ then

$$\lim_{h \to 0} \frac{e^{x_0 + h} - e^{x_0}}{h} = \lim_{h \to 0} e^{x_0} \frac{(e^h - 1)}{h} = e^{x_0}$$

because $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$f(x) = a^x = e^{\ln\ a^x} = e^{x \ln a}$

$x \ \rightarrow \ x \cdot c \ \rightarrow \ e^{cx}$

$D(e^{cx}) = e^{cx} \cdot c$

and

$D(a^x) = a^x \cdot \ln$

Example

Check whether the function differentiable or not

$$f(x) = \ln x \quad D_f = \{x > 0\}$$

with $f(x_0), x_0 > 0$

thus

$$\lim_{h \to 0} \frac{\ln(x_0 + h) - \ln(x_0)}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x_0+h}{x_0}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x_0}\right)}{h}$$

$$\text{it is in the indeterminate form of } \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x_0}\right)}{h \, \frac{x_0}{x_0}}$$

$$= \frac{1}{x_0}$$

$D(\ln x) = \frac{1}{x}$

Example

$$D(\log_a x) = D\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} D(\ln x) = \frac{1}{\ln a}\frac{1}{x}$$

with $a > 0$, for the $\frac{\ln\ x}{\ln\ a}$ we will have $a \neq 1$.

Then we will ahve

$D(K \cdot f(x)) = K \cdot D(f(x))$

Geometrical Interpretation of Derivatives

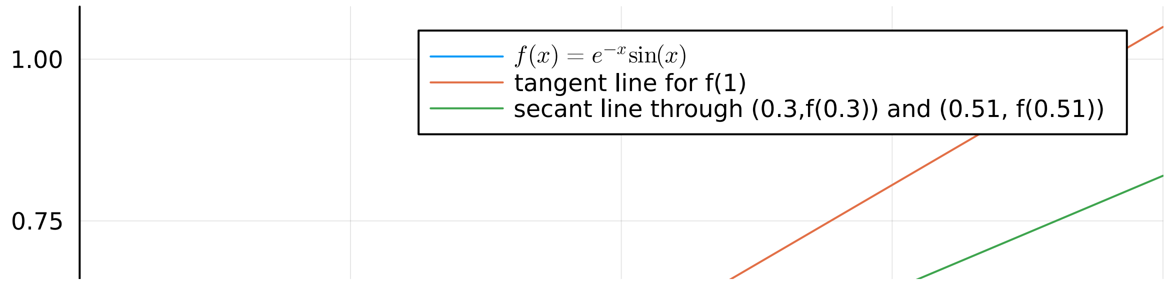
s is the secant line through $A = (x_0, f(x_0))$ on α $\beta = (x_0 + h, f(x_0 + h))$

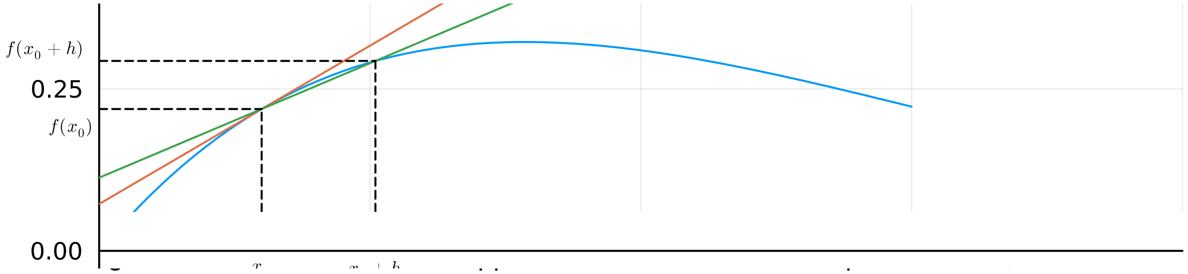
$$m_s = \frac{f(x_0 + h) - f(x_0)}{h} = \tan \alpha_s$$

$$m_t = \lim \frac{f(x_0 + h) - f(x_0)}{h} = f^{'}(x_0) = \tan\alpha_t$$

In [80]:
```
using MTH229
using Plots, LaTeXStrings

#f(x) = 10/(1+x^2) - 10*exp(-(1/2)*x^2)
# tangent in MTH229 package
tangent(f, c) = x -> f(c) + f'(c)*(x-c)
# secant in MTH229 package
sec_line(h) = secant(f, c, c+h)

f(x) = exp(-x)*sin(x)
c = 0.3

plot(f, 0, 1.5, xlims=(-1,1),
    label=L"f(x) = e^{-x} \sin(x)", framestyle=:zerolines)
plot!(tangent(f, c), xlims=(0,2), label="tangent line for f(1)")
# Plot secant line through (0.3, f(0.3)) and (0.51, f(0.51))
# 0.51 -> x + c = 0.3 + 0.21
plot!(sec_line(0.21), label="secant line through (0.3,f(0.3)) and (0.51, f(0.51))")
#plot!(f', label=L"f'(x)")

plot!([0,0.3],[f(0.3),f(0.3)], label="", linecolor=:black, linestyle=:dash)
plot!([0,0.51],[f(0.51),f(0.51)], label="", linecolor=:black, linestyle=:dash)

annotate!([(-0.051,0.19, (L"f(x_{0})", 6, :black))])
annotate!([(-0.1,0.31, (L"f(x_{0} + h)", 6, :black))])

plot!([0.3,0.3],[0,f(0.3)], label="", linecolor=:black, linestyle=:dash)
plot!([0.51,0.51],[0,f(0.51)], label="", linecolor=:black, linestyle=:dash)

annotate!([(0.3,-0.03, (L"x_{0}", 6, :black))])
annotate!([(0.51,-0.03, (L"x_{0} + h", 6, :black))])
```

Out[80]:





oom in to the graph it might appear to look straight, and that straightness would have a slope that would match the tangent line.

The notation is

$$f^{'}(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

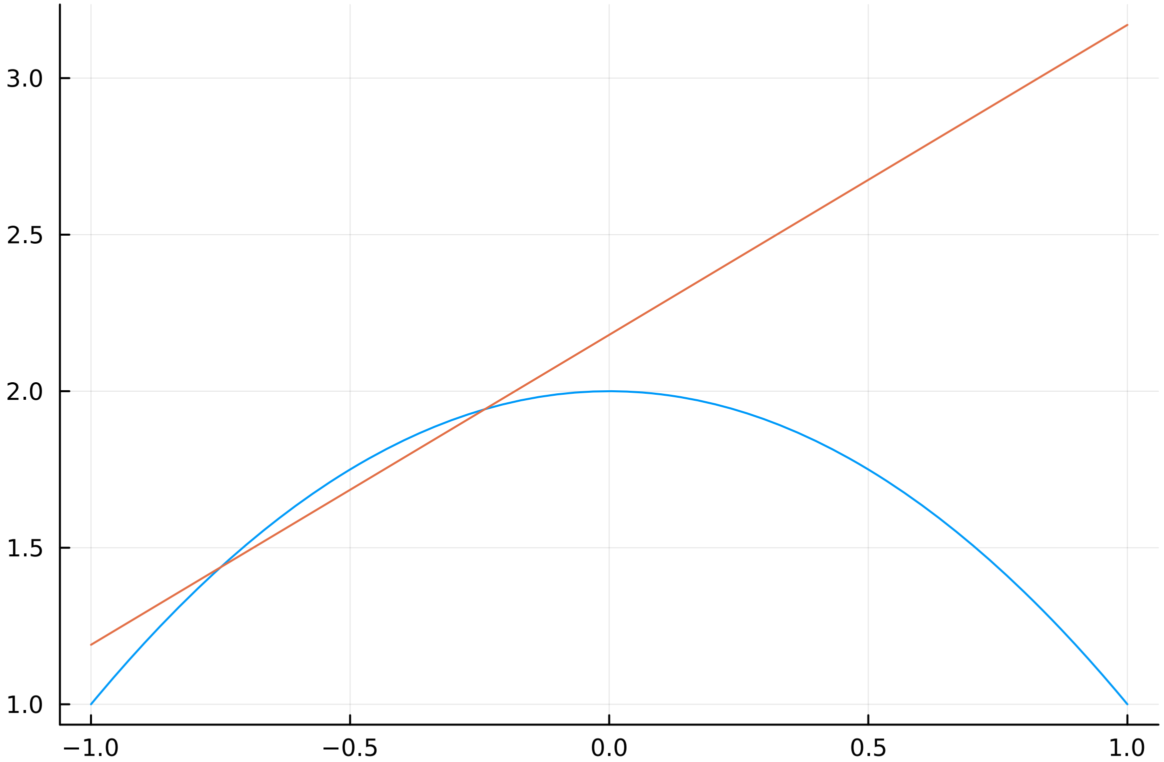
it is a derivative at a point.

The tangent line is most easily expressed in terms of the point-slope formula for a line where the point is $(c, f(c))$ and the slope is $f^{'}(c)$

$$y = f(c) + f^{'}(c) \cdot (x - c)$$

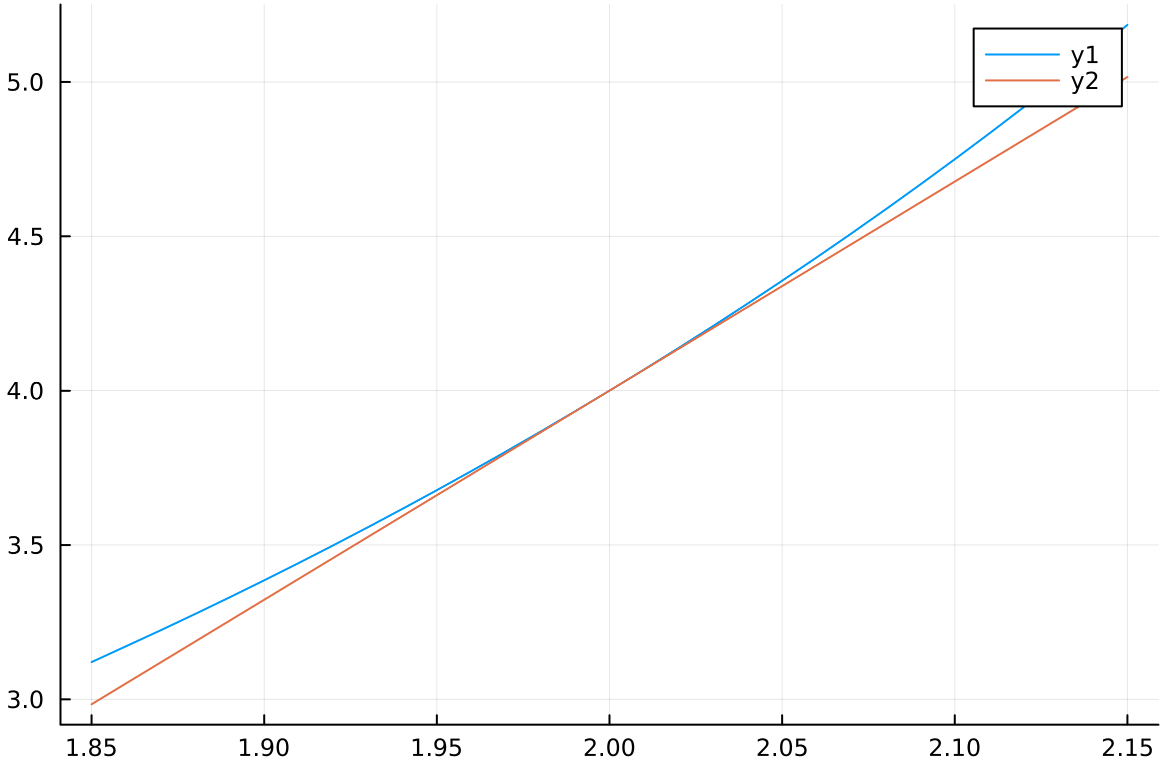
In [6]:
```
# intuitively, the tangent line is the best straight-line approximation to a function near the point (c,f(c)
using MTH229
using Plots

f(x) = 2 - x^2
c = -0.75
# secant in MTH229 package
sec_line(h) = secant(f, c, c+h)
plot(f, -1, 1, legend=false)
plot!(sec_line(0.51))
```

Out[6]:



In [10]:
```
# the slope of the tangent line will be approximated using a numeric derivative
using MTH229
using Plots

f(x) = x^x
c = 2; h = 0.0001
m = ( f(c + h) - f(c) ) / h
tangent_line(x) = f(c) + m * (x - c)

# To compare the different
f(2.1) - tangent_line(2.1)

plot([f, tangent_line], 1.85, 2.15)
```

Out[10]:

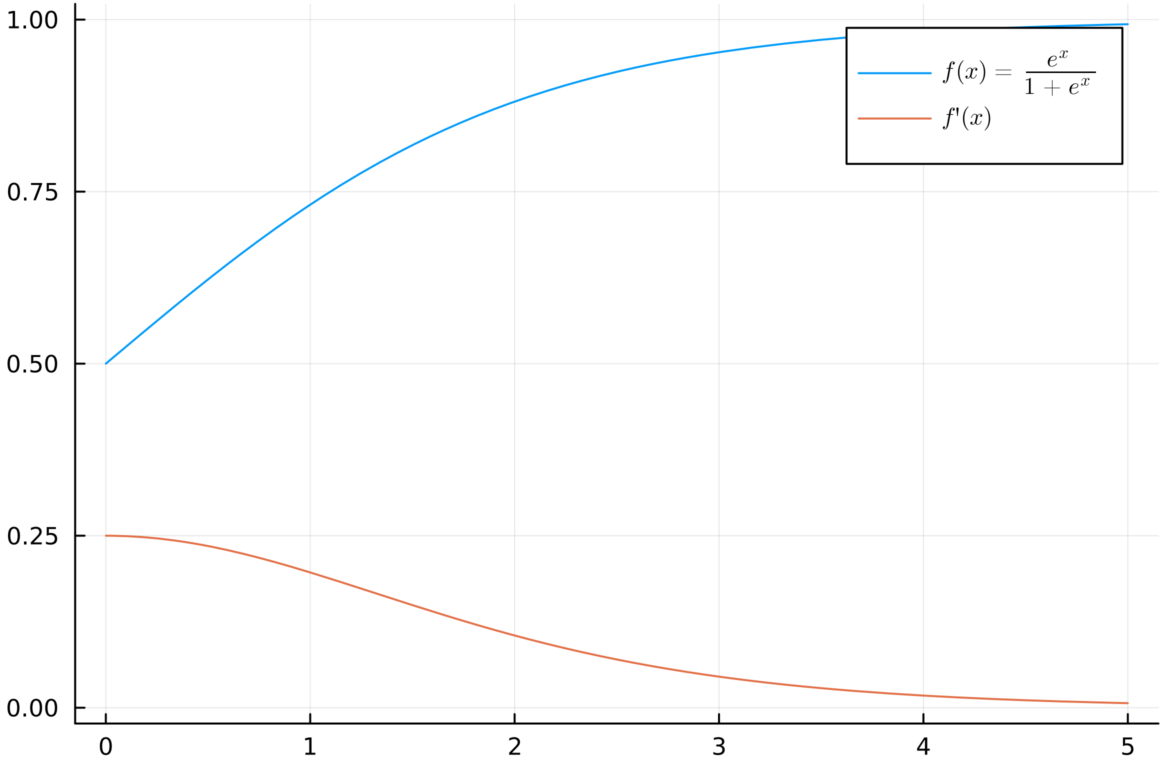


Julia Codes to Plot Derivative of a Function

In [11]:

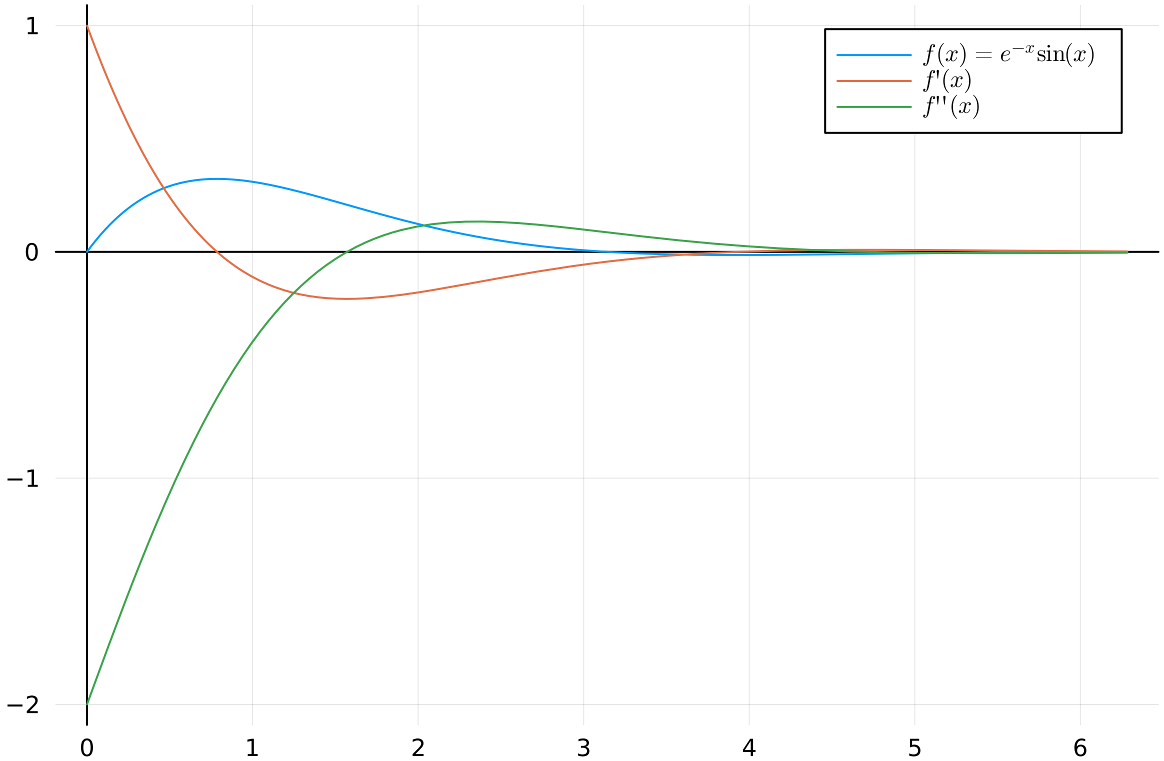
```
# define a function to find the derivative at a point using the “forward difference quotient”
using MTH229
using Plots, LaTeXStrings

forward_difference(f, x0, h) = (f(x0 + h) - f(x0))/h
Df(f; h=1e-8) = x -> forward_difference(f, x, h)

f(x) = exp(x)/(1 + exp(x))
plot(f, 0, 5, label=L"f(x) = \frac{e^{x}}{1 + e^{x}}")
plot!(Df(f), label=L"f'(x)")
```

Out[11]:



In [19]:
```
using MTH229
using Plots, LaTeXStrings

#f(x) = 10/(1+x^2) - 10*exp(-(1/2)*x^2)
f(x) = exp(-x)*sin(x)

plot(f, 0, 2pi, label=L"f(x) = e^{-x} \sin(x)",
    framestyle=:zerolines)
plot!(f', label=L"f'(x)")
plot!(f'', label=L"f''(x)")
```

Out[19]:

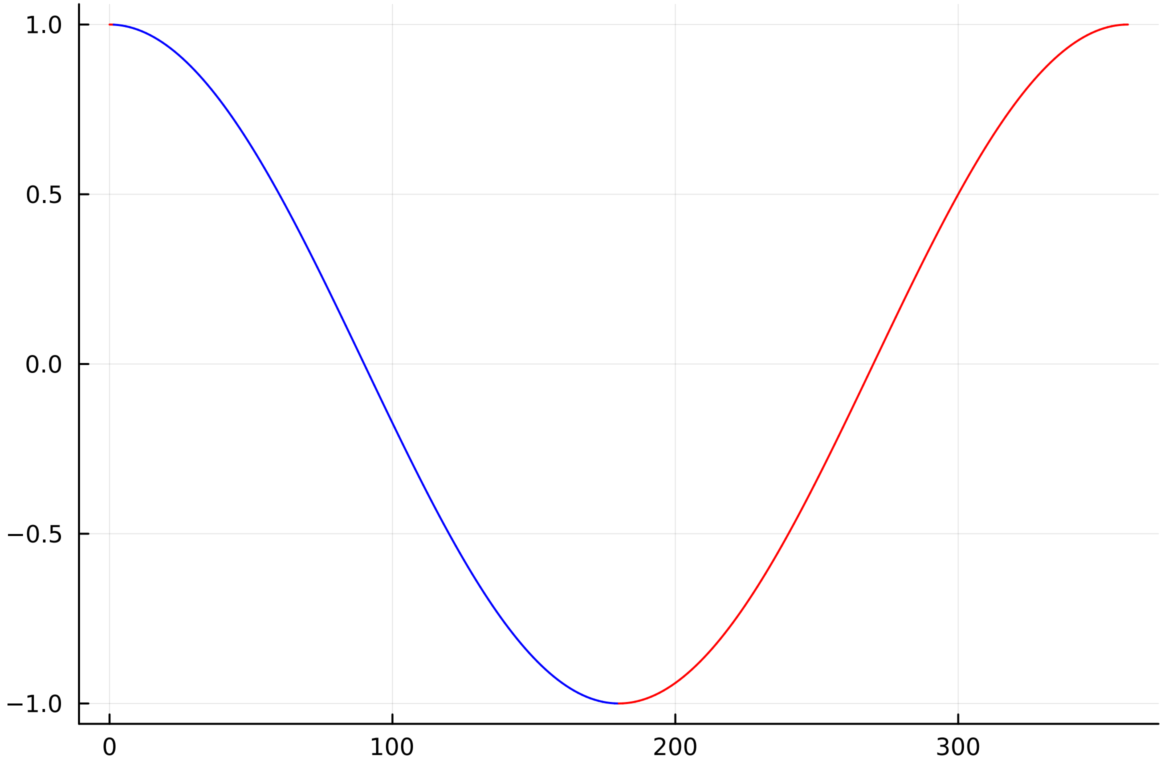


In [12]:
```
# find the roots numerically
using MTH229
using Plots

f(x) = x^3 -5x + 4
fp(x) = Df(f)(x)
#plot(fp, -5, 5)
find_zeros(fp, -10, 10)
```

Out[12]:
```
2-element Vector{Float64}:
 -1.290994449095253
  1.2909944355487426
```

In [13]:
```
# We wish to indicate on the graph where fp(x) > 0.
# We can do this by defining a function that is when that is the case and NaN otherwise
using MTH229
using Plots

f(x) = cosd(x)          # using degrees
fp(x) = Df(f)(x)                # use default h
plotif(f, fp, 0, 360)  # second color when fp > 0
```

Out[13]:



Differentiability implies Continuity

If a function is differentiable then it is continuous, not the other way around, if a function is continuous it does not guarantee that it is differentiable.

For example: function $y = |x|$ is continuous but not differentiable in $x_0 = 0$

Proposition

If f is differentiable in $x_0 \in D_f$ then f is continuous in x_0.

Proof

Assume f is differentiable in x_0 then there exist a finite

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Then we want to prove that

$$\lim_{x \to x_0} f(x) = f(x_0) \leftrightarrow \lim_{x \to x_0} [f(x) - f(x_0)] = 0$$

$$\begin{aligned} \lim_{x \to x_0} [f(x) - f(x_0)] &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right)(x - x_0) \\ &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \cdot \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0 \end{aligned}$$

with $f'(x_0)$ is finite.

Example

Let $c \subset \mathrm{R}$ be a constant then $D(c) = 0$

$f(x) = c : \mathrm{R} \to \mathrm{R}$

$\forall x_0 \in \mathrm{R}$ we will have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c - c}{x - x_0} = \lim_{x \to x_0} 0 = 0$$

Theorem: Basic Algebraic Properties of Derivatives

Let $f, g : I \rightarrow \mathrm{R}$ two differentiable funcitons in $x\dot{I}$ and $K \in \mathrm{R}$ be a constant.

1) $(f \pm g)' = (f)' \pm (g)'$

2) $(kf)' = k(f)'$

For 1) and 2) differentiation is a linear operation

3) $(f \cdot g)' = f' \cdot g + f \cdot g'$

4) $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

$g \neq 0$

4') $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

$g \neq 0$

Proof

4) $x_0 \in I$

$$\begin{aligned}
\lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0} &= \lim_{x \to x_0} \frac{f(x) \cdot g(x_0) - f(x_0) \cdot g(x)}{g(x) \cdot g(x_0)(x - x_0)} \\
&= \frac{1}{g(x_0)^2} \left[\lim_{x \to x_0} \frac{[f(x) - f(x_0)]g(x_0)}{x - x_0} - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right] \\
&= \frac{1}{g(x_0)^2} \left[\left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \cdot g(x_0) - \left(\lim_{x \to x_0} \frac{[g(x) - g(x_0)]}{x - x_0} \right) f(x_0) \right] \\
&= \frac{f^{'}(x_0) \cdot g(x_0) - g^{'}(x_0) + f(x_0)}{(g(x_0))^2}
\end{aligned}$$

Introductory Real Analysis Course 13 (24 October 2022)

Example

$f : (0, 1] \to \mathbb{R}$

$$f(x) = \sin\left(\frac{1}{x}\right)$$

f is continuous on $(0, 1]$ but it is not uniformly continuous on $(0, 1]$.

Proposition

A function $f : A \to \mathbb{R}$ is not uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and sequences $(x_n), (y_n)$ in A such that

$$\lim_{n} |x_n - y_n| = 0 \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0 \ \forall n \in \mathbb{N}$$

- We define $x_n, y_n \in (0, 1]$ for $n \in \mathbb{N}$ by

$$x_n = \frac{1}{2n\pi} \quad y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

then

$|x_n - y_n| \rightarrow 0$ but

$$|f(x_n) - f(y_n)| = |\sin(2n\pi) - \sin(2n\pi + \frac{\pi}{2})| = 1 \quad \forall n \in \mathbb{N}$$

Let $f : (a, b) \rightarrow \mathbb{R}$ such that there exists the derivative in x, then

$$\varepsilon(h) = \frac{f(x+h) - f(x)}{h} - f^{'}(x) \quad f^{'}(x_0) = \lim_{x \to x_0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is infinitesimal with h.

If $h \neq 0$ then

$$f(x+h) - f(x) = f^{'}(x)h + h\varepsilon(h)$$

Theorem: Derivative of Composite Functions

Let $f : (a, b) \rightarrow \mathbb{R}, \; g : (c, d) \rightarrow \mathbb{R}$ with $f(a, b) \subseteq (c, d)$.

If f is such that there exists the derivative in x and g such that there exists the derivative in $f(x)$, the composite function $w = g \circ f$ is such that there exists the derivative in x and the following formula holds

$$w^{'}(x) = g^{'}(f(x)) \cdot f^{'}(x)$$

Proof

f is differentiable in x and g is differentiable in $y = f(x)$, then $\forall k$ sufficiently small

$$g(y+k) - g(y) = g^{'}(y)k + k\varepsilon(k)$$

with $\varepsilon(k) \rightarrow 0$ if $k \rightarrow 0$.

Let us write $k = f(x+h) - f(x)$, we get that when $h \to 0$,

$$k \to 0 \quad \text{and} \quad \frac{k}{h} \to f'(x)\left(\frac{k}{h} = \frac{f(x+h) - f(x)}{h}\right)$$

As a consequence if $h \to 0$ then $\varepsilon(k) \to 0$.

Dividing $g(y+k) - g(y) = g'(y)k + k\varepsilon(k)$ by h, being $y + k = f(x+h)$

$$\frac{g(f(x+h)) - g(f(x))}{h} = \frac{k}{h}[g' f(x) + \varepsilon(k)]$$

then passing to the limit for $h \to 0$ we get

$$w'(x) = g'(f(x)) \cdot f'(x)$$

Example

1) Let $f(x) = \sin x$ and $g(y) = y^3$

then

$$w(x) = g(f(x)) = (\sin x)^3$$
$$w'(x) = 3\sin^2 x \cos x$$

2) $f(x) = x^4 + 3x^2 + 1$ and $g(y) = \log y$

$$w(x) = g(f(x)) = \log(x^4 + 3x^2 + 1)$$
$$w'(x) = \frac{1}{x^4 + 3x^2 + 1}(4x^3 + 6x)$$

Derivative for the Inverse Function

Theorem

Let $f: (a, b) \to \mathrm{R}$, continuous and strictly monotone.

If f is differentiable in $x_0 \in (a, b)$ and $f^{'}(x_0) \neq 0$ then the inverse function $g = f^{-1}$ is differentiable in $y_0 = f(x_0)$ and the following formula holds

$$g^{'}(y_0) = \frac{1}{f^{'}(x_0)}$$

Proof

$$\frac{g(y) - g(y_0)}{y - y_0}, \quad y \neq y_0$$

If $g(y) = x$ and $g(y_0) = x_0$, we have by definition of inverse function $y = f(x)$ and $y_0 = f(x_0)$ with $x \neq x_0$, then we can write

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \quad (*)$$

f is continuous in x_0 and g too; so if $y \to y_0, \ x = g(y) \to g(y_0) = x_0$

Passing to the limit for $y \to y_0$ in $(*)$, we get the

$$g^{'}(y_0) = \frac{1}{f^{'}(x_0)}$$

Example

Let $f(x) = x + e^x$, f is continuous and increasing on $\mathbb{R}$. Let $g = g(y)$ be the inverse function. Let us compute $g^{'}(1)$.

We apply the formula $y_0 = 1$ and $x_0 = 0$

$g^{'}(1) = \frac{1}{g^{'}(0)} = \frac{1}{2}$

$g(y) = f^{-1}(x)$

$g^{'}(1)$

We have to notice that it is impossible to find an explicit expression of g; because you should get this from $y = e^x + x$

$y = 1$

for

$1 = e^x + x \rightarrow x = 0$

$y_0 = 1$

$x_0 = 0$

Example

Let $f(x) = \text{im } x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let us compute the derivative of $\sin^{-1}$.

We write $y = \sin x,\ f'(x) = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$

with $\cos^2 x + \sin^2 x = 1 \rightarrow \cos^2 x = 1 - \sin^2 x$

$$\begin{aligned} g'(y) &= \frac{1}{f'(y)} \\ &= \frac{1}{\sqrt{1 - y^2}} \\ \frac{d}{dy}\sin^{-1}(y) &= \frac{1}{\sqrt{1 - y^2}} \end{aligned}$$

Example

Let $f(x) = \cos x$ on $[0, \pi]$. Let us compute the derivative of $g = \cos^{-1} \cdot g(y)$

We write

$$y = \cos x, \quad f'(x) = -\sin x = -\sqrt{1 - \cos^2 x} = -\sqrt{1 - y^2}$$

we get

$$g'(y) = \frac{1}{f'(y)} = -\frac{1}{\sqrt{1-y^2}}$$

Example

$$f(x) = \tan(x) \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \quad g = \tan^{-1} \quad g(y) = \tan^{-1}(y)$$

Let us write $y = \tan(x)$, we will have

$$f'(x) = 1 + (\tan x)^2 = 1 + y^2$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{1+y^2} \quad \frac{d}{dy}\tan^{-1}(y) = \frac{1}{1+y^2}$$

Example

$f(x) = a^x \quad (a > 0, a \neq 1)\ g = \log_1$

$$g'(y) = \frac{1}{y \ln a}$$

If $f : (a, b) \to \mathrm{R}$ is differentiable in x then we can write $f(x+h) - f(x) = f'(x)h + h\varepsilon(h)$ where $\varepsilon(h) \to 0$ with h.

The increment $\delta f := f(x+h) - f(x)$ is written like the sum of two sums

$$f'(x)h$$

that is linear in h. And

$$h\varepsilon(h) = o(h)$$

is infinitesimal of order greater than h if $h \to 0$.

We introduce the following:

Definition

Let us suppose that given $f: (a, b) \to \mathrm{R}$ and a point $x \in (a, b)$, there exists a real number A such that, for all h such that $x + h \in (a, b)$, one has

$$\delta f = A \cdot h + o(h) \quad h \to 0$$

We say that f is differentiable in x and the linear part Ah is called differential of f in x and we write it $df(x)$.

If we divide $\delta f = Ah + o(h)$ by h and passing to the limit as $h \to 0$

$$\frac{\delta f}{h} = \frac{Ah}{h} + o(h) \quad \delta f = f(x + h) - f(x)$$

$A = df$

$$df(x) = f^{'}(x)h$$

Fundamental Theorems

Theorem: Fermat

Let $f: (a, b) \to \mathrm{R}$ and $x_0 \in (a, b)$. If there exists the derivative of f in x_0 and f has a local extreme in x_0 then $f^{'}(x_0) = 0$.

Proof

Let x_0 be a local maximum point. There exists I of x_0, such that $I \subset (a, b)$, and $f(x_0) \geq f(x) \ \forall x \in I$.

Let us consider the

$$\Phi(x) = \frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0$$

For $x \in I$, we have:

(1) $x > x_0 \to \Phi(x) \leq 0$

(2) $x < x_0 \rightarrow \Phi(x) \geq 0$

For (1),

$$\lim_{x \to x_0^+} \Phi(x) = f'_+(x_0) \leq 0$$

For (2),

$$\lim_{x \to x_0^-} \Phi(x) = f'_-(x_0) \geq 0$$

The derivative in x_0 exists, we can conclude $f'_+(x_0) = f'_-(x_0) = 0$.

Analogously when x_0 is a local minimum point.

Definition

The points in which the function f has a null / zero derivative are called critical points.

Theorem: Rolle

Let $f: [a, b] \rightarrow \mathrm{R}$, such that:

1) f is continuous on $[a, b]$

2) f is differentiable in (a, b)

3) $f(a) = f(b)$

Then there exists $c \in (a, b)$ such that $f'(c) = 0$

Proof

From the Weierstrass theorem we know that f has a maximum and a minimum; let $x_0, x_1 \in [a, b]$ such that

$$f(x_0) = M = \max_{[a,b]} f \quad f(x_1) = m = \min_{[a,b]} f$$

- If $M = m$ then f is a constant and $f'(x) = 0 \ \forall x \in (a, b)$
- If $M > m$ from 3) at least one between x_0 and x_1 is an interval point. In that point the derivative exists and by 2) the derivative must be zero because of the Fermat theorem.

Theorem: Cauchy

Let $f, g : [a, b] \to \mathbb{R}$, such that

1) f, g continuous on $[a, b]$

2) f, g differentiable on (a, b)

then there exists $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

Introductory Real Analysis Course 14 (25 October 2022): Differentiable Functions

Definition

Given $f : D \subseteq \mathbb{R} \to \mathbb{R}$

$x_0 \in D$ is said to be a point of local or relative minimum (maximum) if $\exists U_{x_0} : \ \forall x \in U_{x_0}$, then

$$f(x) \geq f(x_0) \quad (f(x)) \leq f(x_0)$$

$f(x_0)$ is called local minimum (local maximum).

Definition

$x_0 \in D$ is said to be a point of global or absolute minimum if

$$\forall x \in D \quad f(x) \geq f(x_0)$$

($f(x_0)$ is a global minimum).

x_0 is said to be a part of global or absolute maximum if

$$\forall x \in D \quad f(x) \leq f(x_0)$$

($f(x_0)$ is a global maximum)

Theorem: Fermat

- (There exists a local maximum or local minimum for differentiable functions)

Let $f\colon D \subseteq \mathrm{R} \rightarrow \mathrm{R}$ and $x_0 \in D$ a point of local maximum or minimum, if f is differentiable in x_0, then

$$f^{'}(x_0) = 0$$

x_0 is a stationary point.

Proof

Without restriction of generality assume that x_0 is a point of local minimum $\rightarrow \exists\ U_{x_0} \subset D\colon\ \forall x \in U_{x_0}\, f(x) \geq f(x_0)$

consider

$$x > x_0 \quad \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f^{'}_{+}(x_0) \geq 0$$

now consider

$$x < x_0 \quad \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f^{'}_{-}(x_0) \leq 0$$

since $f^{'}(x_0)$ exists then

$$f'_{+}(x_0) = f'_{-}(x_0) \rightarrow f'(x_0) = 0$$

- Notes: The condition $f'(x_0) = 0$ is not sufficient to the existence of local maximum or minimum.

Counter-example

$$f(x) = x^3$$
$$f'(x) = 3x^2$$

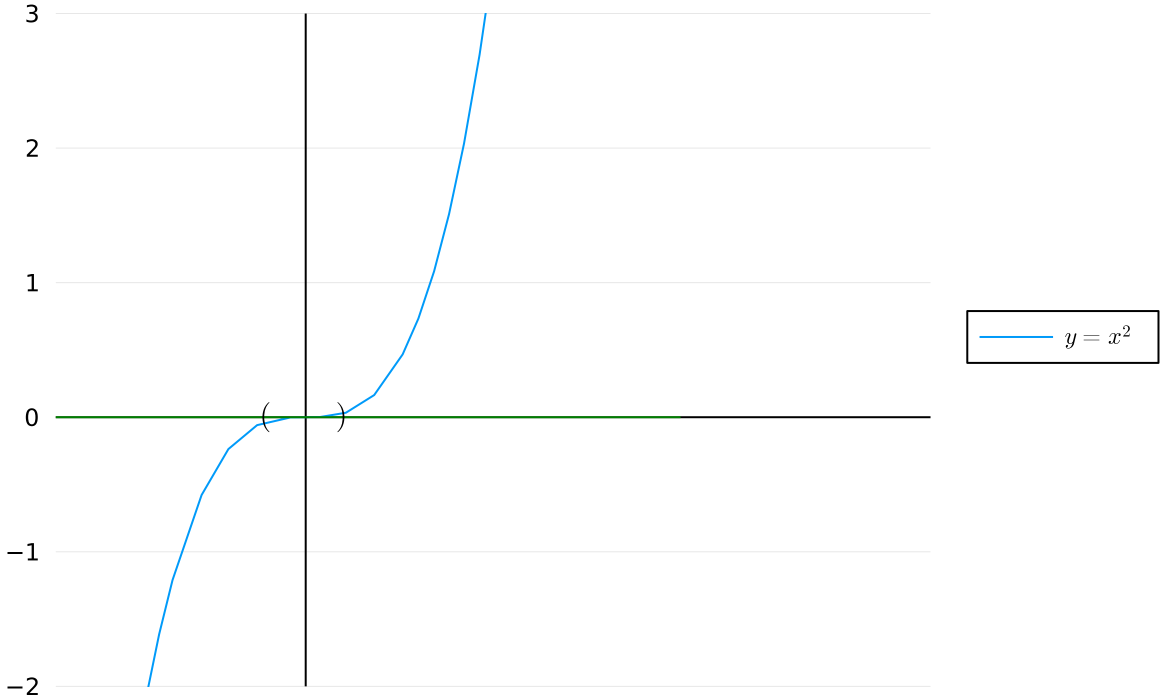
with $f'(x) \geq 0$,

$x = 0,\ f'(0) = 0,\ f(0) = 0.$

- Notes: If you have a differentiable function such that $f'(x_0) = 0$, you need to check the sign of the derivative nearby to establish if it is a local maximum or minimum.

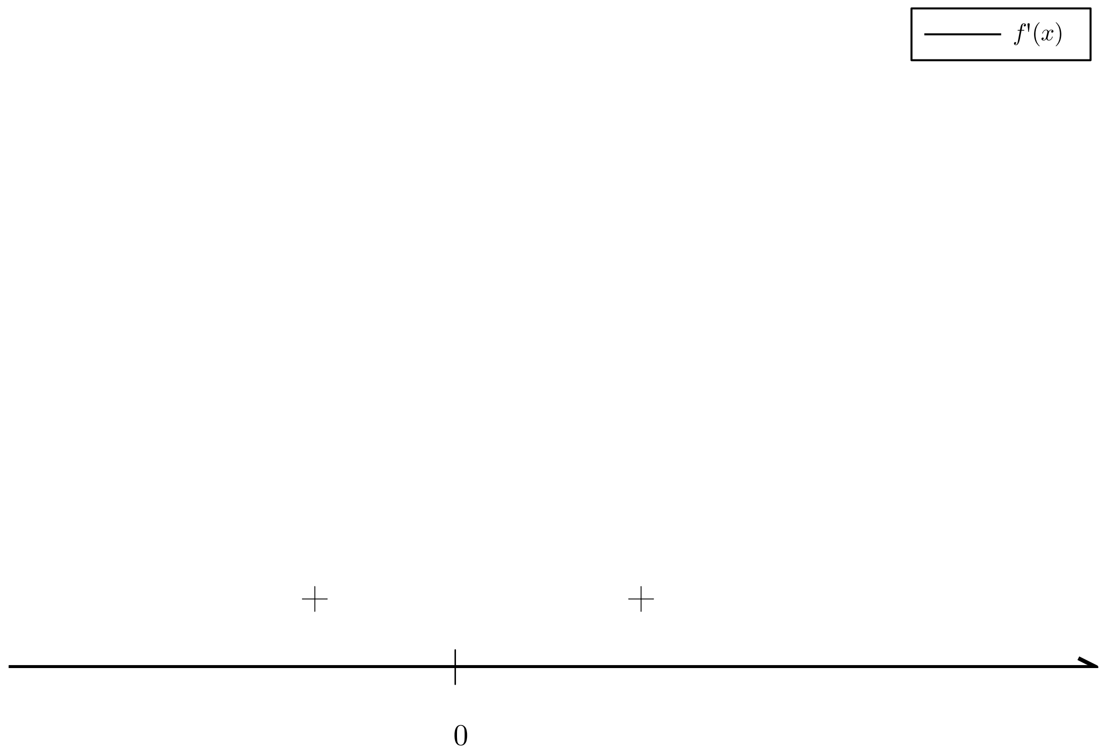
In [13]:
```
using Plots, LaTeXStrings
gr()

a, b = -2, 5
g(x) = x^3

plot(g, a, b; legend=:outerright, label=L"y = x^{2}", framestyle=:zerolines,
    xlims = (-2,5), xticks=false,
    ylims = (-2,3), yticks = -2:1:3,
    size=(600, 360))

plot!([-2,3],[0,0], label="", linecolor=:green)

annotate!([(-0.3,0, (L"(", 10, :black)),
           (0.3,0, (L")", 10, :black))])
```

Out[13]:



In [19]:
```
using Plots, LaTeXStrings, Plots.PlotMeasures

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label=L"f'(x)", bottom_margin = 10mm)

annotate!([(0.55,0, (L"|", 12, :black))])
annotate!([(0.47,0.1, (L"+", 13, :black))])
annotate!([(0.65,0.1, (L"+", 13, :black))])
annotate!([(0.55,-0.1, (L"0", 10, :black))])
```

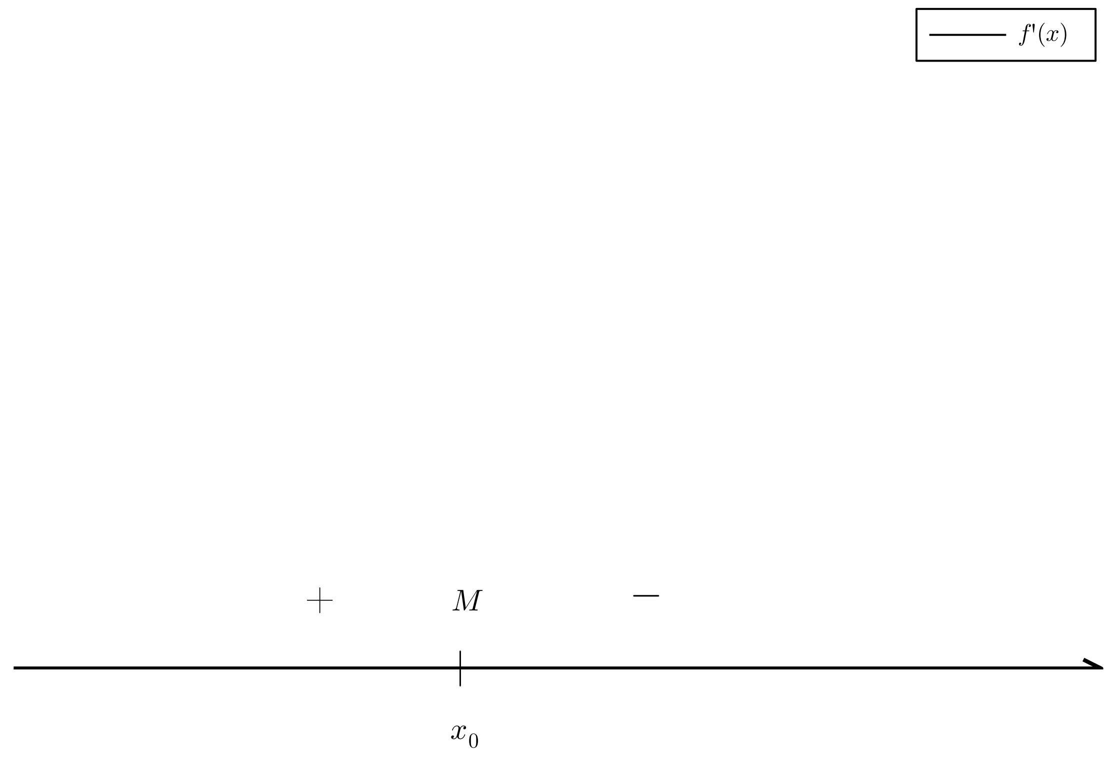
Out[19]:



- If the sign is $+$ then f is increasing near that point
- If the sign is $-$ then f is decreasing near that point.

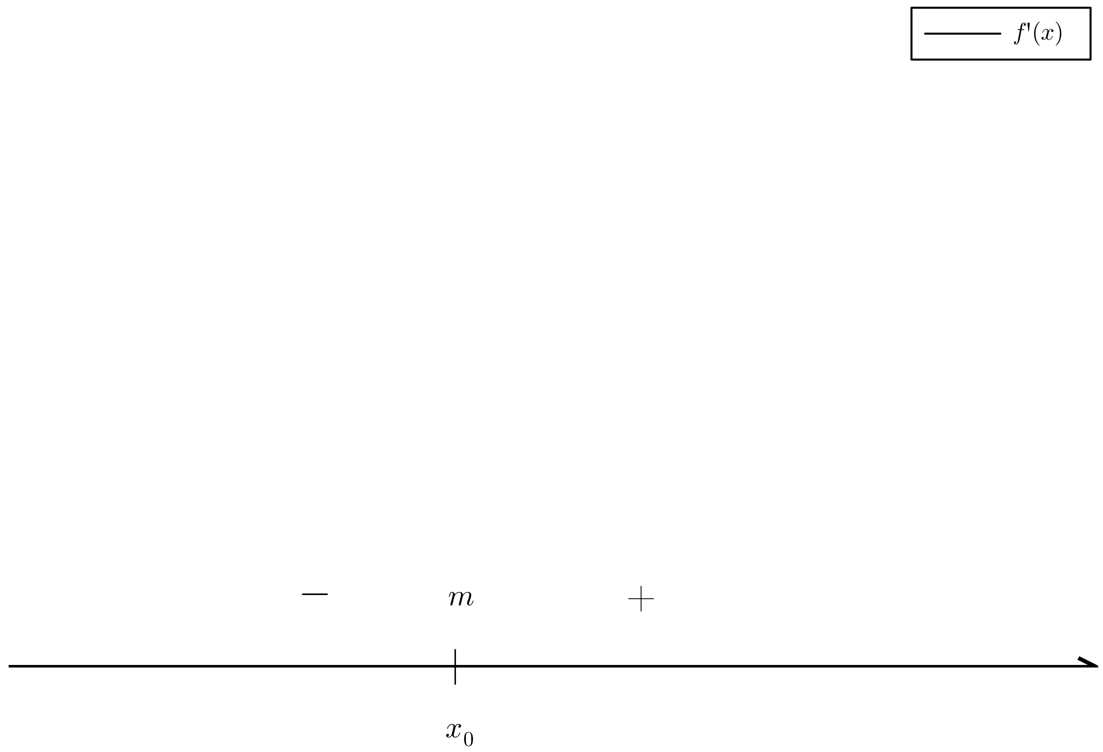
In [18]:
```
# M is a local maximum
using Plots, LaTeXStrings, Plots.PlotMeasures

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label=L"f'(x)", bottom_margin = 10mm)

annotate!([(0.55,0, (L"|", 12, :black))])
annotate!([(0.47,0.1, (L"+", 13, :black))])
annotate!([(0.65,0.1, (L"-", 13, :black))])
annotate!([(0.55,-0.1, (L"x_{0}", 10, :black))])
annotate!([(0.55,0.1, (L"M", 10, :black))])
```

Out[18]:



In [17]:
```
# m is a local minimum
using Plots, LaTeXStrings, Plots.PlotMeasures

f(x) = 0.1

plot([0.3,0.9],[0,0],arrow=true,color=:black,linewidth=2, xticks=false, yticks=false,
     ylims=(0,1), showaxis=false, label=L"f'(x)", bottom_margin = 10mm)

annotate!([(0.55,0, (L"|", 12, :black))])
annotate!([(0.47,0.1, (L"-", 13, :black))])
annotate!([(0.65,0.1, (L"+", 13, :black))])
annotate!([(0.55,-0.1, (L"x_{0}", 10, :black))])
annotate!([(0.55,0.1, (L"m", 10, :black))])
```

Out[17]:



Question

Where and/or when we can look for a local minimum or maximum?

Let $f: [a, b] \to \mathrm{R}$ (K is compact).

1) If f is continuous by Weierstrass Theorem $\to$ there exists maximum and minimum values (global)

$$f(x) \in [m, M]$$

If $f: \mathrm{R} \to \mathrm{R}$, f continuous and differentiable, you do not know if the absolute maximum and minimum exist.

Example

$y = e^x$

Where we can find global maximum and minimum for $f: [a, b] \to \mathrm{R}$, f continuous

a) at the ends of the domain

b) in points of $[a, b]$ in which f is not differentiable

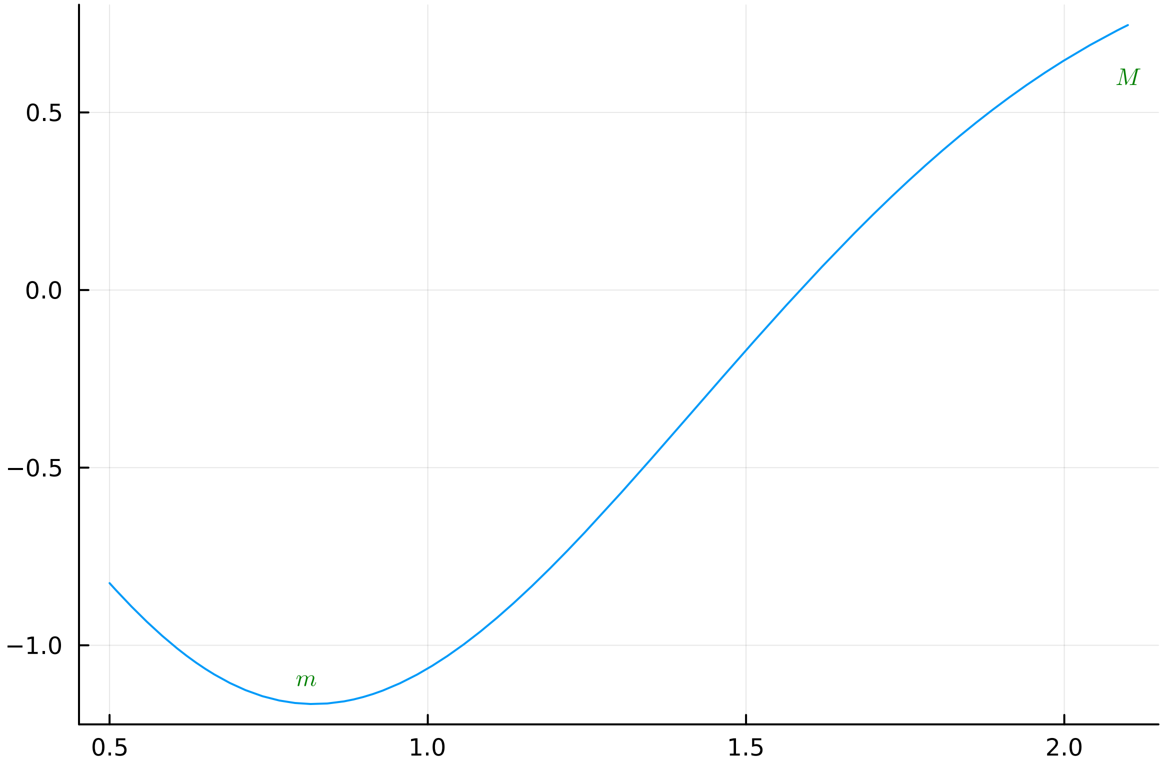
x_0 is an angular point if $f'_-(x_0) \neq f'_+(x_0)$

c) If f is differentiable in (a, b) check where $f'(x_0) = 0$

- Note:$D = [1, s] \cup [8, 10]$ it is compact

In [87]:
```
using MTH229
using Plots

f(x) = 10/(1+x^2) - 10*exp(-(1/2)*x^2)

plot(f, .5, 2.1, legend=false)

annotate!([(2.1,0.6, (L"M", 8, :green))])
annotate!([(0.81,-1.1, (L"m", 8, :green))])

scatter!([6], [f(6)], color = "blue", label="", markersize = 3)

plot!([3.5,3.5],[f(3.5),0], label="", linecolor=:red, linestyle=:dash)
```

Out[87]:



Theorem: Rolle

Let $f: [a, b] \rightarrow \mathrm{R}$ continuous in $[a, b]$ and differentiable in (a, b) with $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f^{'}(c) = 0$

Proof

1) Assume that the minimum and the maximum value (that exist by Weierstrass Theorem), are taken in the ends of the domain

for example $f(a) = m$ and $f(b) = M$ (or viceversa).

$$f(a) = f(b) \rightarrow m = M \rightarrow f \text{ constant } \rightarrow f^{'}(x) = 0$$

2) One of the maximum or minimum points are interval.

For example$f(x_0) = m, \ a < x_0 < b$

by hypothesis f is differentiable in x_0.

By Fermat' Theorem:

$f^{'}(x_0) = 0$

so one can take $c = x_0$

Theorem: Lagrange

Let $f: [a, b] \rightarrow \mathrm{R}$, f continuous in $[a, b]$, then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f^{'}(c)(b - a)$$

Proof

Apply Rolle's Theorem to

$$g(x) = f(x) - (x - a)\frac{f(b) - f(a)}{b - a}$$

Example: Applications of Lagrange' Theorem

- Note: if f is constant on $[a, b] \rightarrow f^{'}(x) = 0$

1) Let f be a differentiable function on (a, b) if $f^{'}(x) = 0 \ \forall x \in (a, b)$ then f is constant.

Proof

$\forall x_1, x_2 \in (a, b) \ \rightarrow \ \exists c : f^{'}(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \ \rightarrow \ f(x_1) = f(x_2)$

2) Let f differentiable in (a, b) then f is increasing (or decreasing) if and only if $f^{'}(x) \geq 0$ ($f^{'}(x) \leq 0$)

Proof

Take any $x_1 < x_2$ by Lagrange's Theorem:

$$\exists c \in (x_1, x_2): \quad \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f^{'}(c) \geq 0 \quad \rightarrow f(x_1) - f(x) \leq 0 \quad f(x_1) < f(x_2)$$

with $x_1 - x_2 < 0$

and vice versa.

Assume f is increasing

$$\lim_{x_1 \rightarrow x_2} \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f^{'}(x_2) \geq 0$$

if $x_1 < x_2$

Proposition

3) Let $f: (a, b) \rightarrow \mathrm{R}$ continuous and differentiable in $(a, b) \setminus \{x_0\}$. If there exists

$$\lim_{x \to x_0} f^{'}(x) = l \quad \text{(finite)}$$

then f is differentiable in x_0 and $f^{'}(x_0) = l$

Application

given

$$\begin{cases} g_1(x) & x > x_0 \end{cases}$$

Example for the 1st Test (Midterm)

1) Compute

$$\lim_{x \to 4} \frac{x - \sqrt{3x + 4}}{4 - x}$$

it is of the form of $\frac{0}{0}$

2) Compute

$$\lim_{x \to +\infty} \left(\frac{x^2 + 5x + 1}{x^2 + x} \right)^x$$

it is of the form of $1^{+\infty}$

Introductory Real Analysis Course 15 (26 October 2022)

Theorem: de L'Hopital

Let $-\infty \leq a < b \leq +\infty$ and $f, g : (a, b) \to \mathrm{R}$ such that

a) $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$ (or $+\infty$ or $-\infty$)

b) f, g are differentiable in (a, b) with $g(x) \neq 0$ in (a, b) with $g(x) \neq 0$ in (a, b)

c) $\exists \lim_{x \to a^+} \frac{f^{'}(x)}{g^{'}(x)} = L \in \bar{\mathrm{R}}$ or (R^*)

then

$$\exists \lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

analogous for $\lim_{x \to b^-}$

Application

Example

$$\begin{aligned}
\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right) &= \lim_{x \to 0^+} \frac{\sin x - x \cos x}{x \sin x} \quad \left(\text{in } \frac{0}{0} \text{ form}\right) \\
&= \lim_{x \to 0^+} \frac{\cos x - \cos x + x \sin x}{\sin x + x \cos x} \cdot \frac{\sin x}{\sin x} \\
&= \lim_{x \to 0^+} \frac{x}{1 + \frac{x}{\sin\ x} \cos x} \\
&= \frac{0}{2} = 0
\end{aligned}$$

Example

$$\lim_{x \to 0^+} x \ln x \quad \text{in } 0 \cdot (-\infty) \text{ form}$$

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\\ &= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\\ &= \lim_{x\to 0^+} -x = 0\end{aligned}$$

It holds that $\lim_{x\to 0^+} x(\ln x)^\alpha = 0$ for $\alpha > 0$

Higher Order Derivatives

If one knows that $f^{'}(x)$ exists in (a, b), one can ask if there exists $f^{''}(x)$.

We can try to compute

$f^{''}(x) = (f^{'}(x))^{'}$ if $f^{'}(x)$ is continuous.

If $\exists f^{''}(x)$ and it is continuous, we can try to compute

$f^{'''}(x) = (f^{''}(x))^{'}$

We define $f \in C^n(a, b)$ if $\exists\ f^{(k)}(x) \quad \forall 0 \leq k \leq n$ (where by $f^{(0)}(x) = f(x)$) and they are continuous functions, till $f^{(n)}(x)$.

- Note: $C^n(a, b)$ is a vector space

indeed if $f, g \in C^n(a, b)$

$$f + g \in C^n(a, b) \quad D^{(n)}(f + g) = D^{(n)}(f) + D^n(g)$$

- if $k \in \mathrm{R}$ then $k \cdot f \in C^n(a, b)$
- if $f, g \in C^n(a, b)$ then $(f \cdot g) \in C^n(a, b)[f(x) \cdot g(x)]$

Theorem: Leibniz's Formula

If $f, g \in C^n(a, b)$ then $f \cdot g \in C^n(a, b)$ and it holds

$$D^n(f \cdot g)(x) = \sum_{h=0}^{n} \binom{n}{h} D^{(h)} f(x) \cdot D^{(n-h)} \cdot g(x)$$

(Proof by induction on n and by properties of binomial coefficients)

Convex Functions

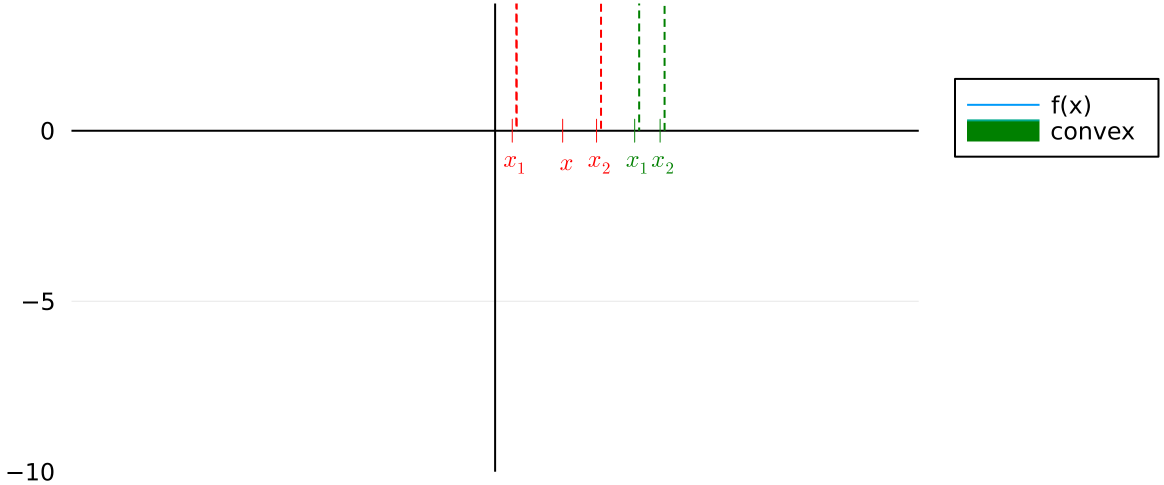
Definition

Let $f\colon (a, b) \to \mathrm{R}$, it is said convex (concave) if $\forall x_1, x_2 \in (a, b)$ and $\forall \alpha \in [0, 1]$ it holds

1) For convex

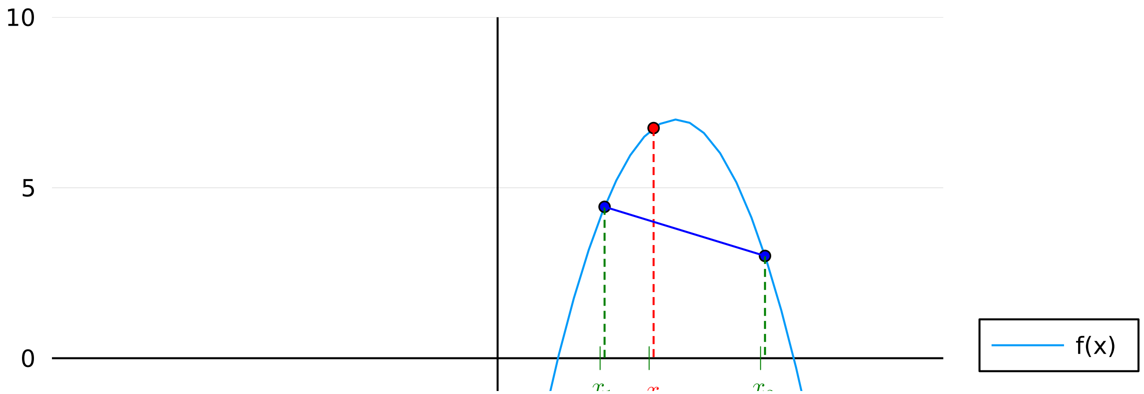
In [35]:
```
# Convex example

using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = (x-3)^(2) + 2x

plot(f,-10,10, xlims=(-10,10), xticks = false, ylims=(-10,10),
    bottom_margin = 10mm, label="f(x)", framestyle = :zerolines,
    legend=:outerright)

scatter!([2.5], [f(2.5)], color = "blue", label="", markersize = 3)
scatter!([0.51], [f(0.51)], color = "blue", label="", markersize = 3)

scatter!([1.7], [f(1.7)], color = "red", label="", markersize = 3)

plot!([0.51,2.5],[f(0.51),f(2.5)], label="", linecolor=:red)
plot!([0.51,0.51],[f(0.51),0], label="", linecolor=:red, linestyle=:dash)
plot!([2.5,2.5],[f(2.5),0], label="", linecolor=:red, linestyle=:dash)

plot!([0.5,0.5],[f(0.5),0], label="", linecolor=:red, linestyle=:dash)
plot!(f,3.4,4, fill=(6.5, 7, :green), label="convex")

plot!([3.4,3.4],[f(3.4),0], label="", linecolor=:green, linestyle=:dash)
plot!([4,4],[f(4),0], label="", linecolor=:green, linestyle=:dash)

annotate!([(0.51,-1, (L"x_{1}", 8, :red))])
annotate!([(2.5,-1, (L"x_{2}", 8, :red))])
annotate!([(1.7,-1, (L"x", 8, :red))])
annotate!([(0.51,0, (L"|", 8, :red))])
annotate!([(1.7,0, (L"|", 8, :red))])
annotate!([(2.5,0, (L"|", 8, :red))])

annotate!([(3.4,-1, (L"x_{1}", 8, :green))])
annotate!([(4,-1, (L"x_{2}", 8, :green))])
annotate!([(3.4,0, (L"|", 8, :green))])
annotate!([(4,0, (L"|", 8, :green))])
```

Out[35]:


0
−5
−10
x_1
x
x_2
x_1
x_2
f(x)
convex

In [47]:
```
# Concave example

using Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = -(x-3)^(2) + 2x

plot(f,-10,10, xlims=(-10,10), xticks = false, ylims=(-10,10),
    bottom_margin = 10mm, label="f(x)", framestyle = :zerolines,
    legend=:outerright)

scatter!([2.4], [f(2.4)], color = "blue", label="", markersize = 3)
scatter!([3.5], [f(3.5)], color = "red", label="", markersize = 3)
scatter!([6], [f(6)], color = "blue", label="", markersize = 3)

plot!([3.5,3.5],[f(3.5),0], label="", linecolor=:red, linestyle=:dash)
plot!([2.4,2.4],[f(2.4),0], label="", linecolor=:green, linestyle=:dash)
plot!([6,6],[f(6),0], label="", linecolor=:green, linestyle=:dash)

plot!([2.4,6],[f(2.4),f(6)], label="", linecolor=:blue)

annotate!([(2.4,-1, (L"x_{1}", 8, :green))])
annotate!([(2.4,0, (L"|", 8, :green))])
annotate!([(3.5,-1, (L"x", 8, :red))])
annotate!([(6,-1, (L"x_{2}", 8, :green))])
annotate!([(3.5,0, (L"|", 8, :green))])
annotate!([(6,0, (L"|", 8, :green))])
```

Out[47]:

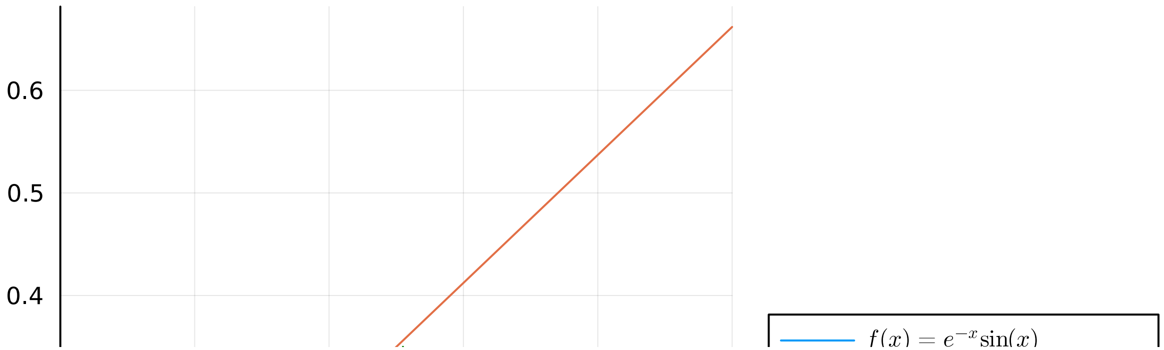


Let $f:(a,b)$ ... differentiable in (a,b) then

- f is convex $\leftrightarrow$ $\forall x_0, x_1 \in (a,b) \quad f(x_1) \geq f(x_0) + f'(x_0)(x - x_0)$
- f is concave $\leftrightarrow$ $\forall x_0, x_1 \in (a,b) \quad f(x_1) \leq f(x_0) + f'(x_0)(x - x_0)$

In [103]:
```
# Concave example

using MTH229, Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = exp(-x)*sin(x)

# tangent in MTH229 package
tangent(f, c) = x -> f(c) + f'(c)*(x-c)

#f(x) =
c = 0.21

plot(f, 0, 1, xlims=(-1,1),
    label=L"f(x) = e^{-x} \sin(x)", framestyle=:zerolines,
     legend=:outerright)
plot!(tangent(f, c), xlims=(0,1), label="tangent line for f(0.21)")

scatter!([0.21], [f(0.21)], color = "blue", label="", markersize = 3)
scatter!([0.51], [f(0.51)], color = "blue", label="", markersize = 3)

plot!([0.21,0.21],[f(.21),0], label="", linecolor=:green, linestyle=:dash)
plot!([0.51,0.51],[0.35,0], label="", linecolor=:green, linestyle=:dash)

annotate!([(0.21,-0.02, (L"x_{0}", 8, :green))])
annotate!([(0.21,0, (L"|", 8, :green))])
annotate!([(0.51,-0.02, (L"x_{1}", 8, :green))])
annotate!([(0.51,0, (L"|", 8, :green))])
```

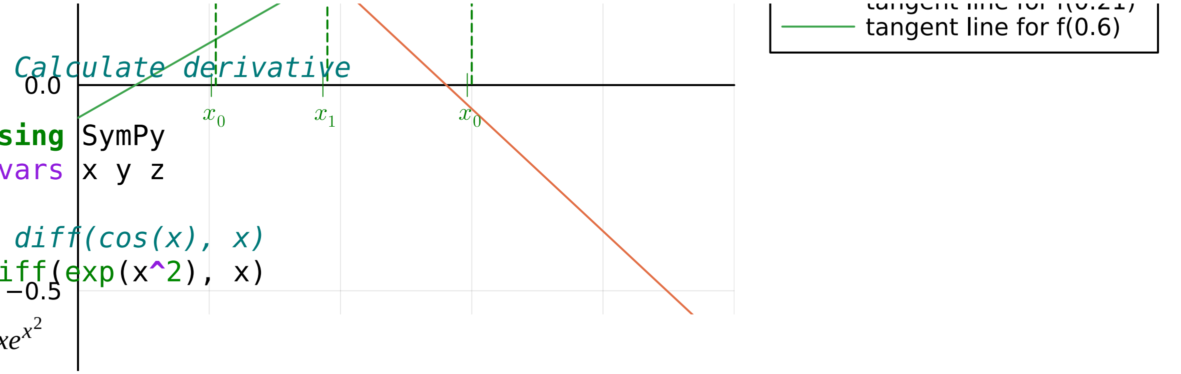
Out[103]:




0.1

In [120]:
```
# Convex Example

using MTH229, Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = 1-exp(-x)*sin(3x)

# tangent in MTH229 package
tangent(f, c) = x -> f(c) + f'(c)*(x-c)

#f(x) =
c = 0.21
d = 0.6

plot(f, 0, 1, xlims=(-1,1),
    label=L"f(x) = 1 - e^{-x} \sin(3x)", framestyle=:zerolines,
     legend=:outerright)
plot!(tangent(f, c), xlims=(0,1), label="tangent line for f(0.21)")
plot!(tangent(f, d), xlims=(0,1), label="tangent line for f(0.6)")

scatter!([0.21], [f(0.21)], color = "blue", label="", markersize = 3)
scatter!([0.6], [f(0.6)], color = "blue", label="", markersize = 3)

plot!([0.21,0.21],[f(.21),0], label="", linecolor=:green, linestyle=:dash)
plot!([0.6,0.6],[f(0.6),0], label="", linecolor=:green, linestyle=:dash)
plot!([0.38,0.38],[f(0.38),0], label="", linecolor=:green, linestyle=:dash)

annotate!([(0.21,-0.08, (L"x_{0}", 8, :green))])
annotate!([(0.21,0, (L"|", 8, :green))])
annotate!([(0.6,-0.08, (L"x_{0}", 8, :green))])
annotate!([(0.6,0, (L"|", 8, :green))])
annotate!([(0.38,-0.08, (L"x_{1}", 8, :green))])
annotate!([(0.38,0, (L"|", 8, :green))])
```

Out[120]:



In [108]:
```
# Calculate derivative

using SymPy
@vars x y z

# diff(cos(x), x)
diff(exp(x^2), x)
```

Out[108]: $2xe^{x^2}$

In [7]:
```
# diff can take multiple derivatives at once.
# both of the following find the third derivative of x^4
using SymPy
@vars x y z

#diff(x^4, x, x, x)
diff(x^4, x, 3)
```

Out[7]: $24x$

Theorem: There Exists the Second Derivative f''

Let $f: (a, b) \to \mathrm{R},\ f \in (a, b)$, then

- f is convex $\leftrightarrow f''(x) \geq 0 \quad \forall x \in (a, b)$
- f is convex $\leftrightarrow f''(x) \leq 0 \quad \forall x \in (a, b)$

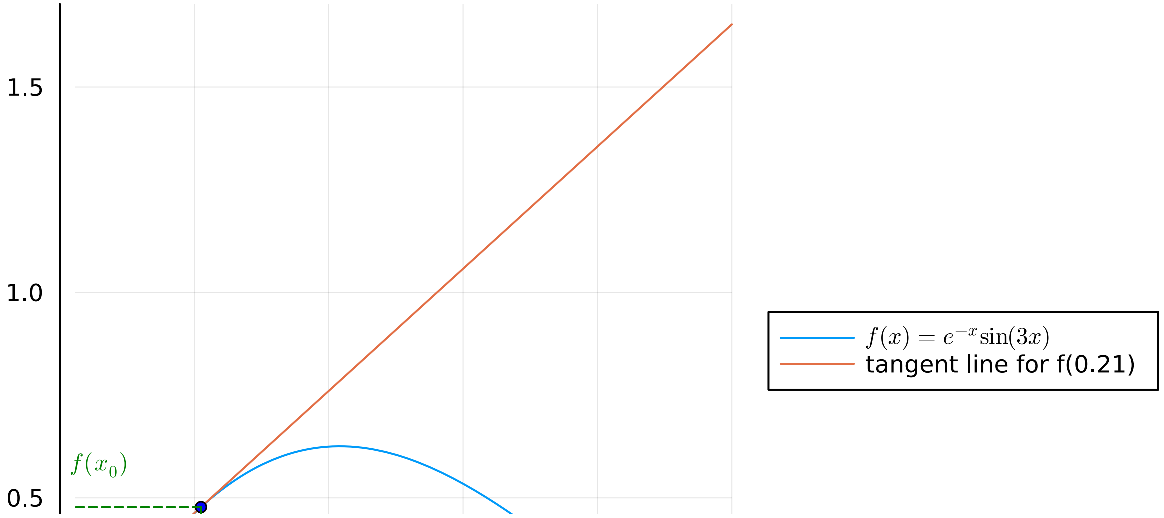
Tangent Line

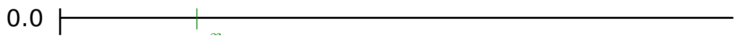
Equation of the tangent line is a point in which it is differentiable.

Take $f: D \subseteq \mathrm{R} \to \mathrm{R}$ and let $x_0 \in \dot{D}$ be a point in which $\exists f'(x_0)$ (f is differentiable in x_0).

In [131]:
```
using MTH229, Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = exp(-x)*sin(3x)

# tangent in MTH229 package
tangent(f, c) = x -> f(c) + f'(c)*(x-c)

#f(x) =
c = 0.21

plot(f, 0, 1, xlims=(-1,1),
    label=L"f(x) = e^{-x} \sin(3x)", framestyle=:zerolines,
     legend=:outerright)
plot!(tangent(f, c), xlims=(0,1), label="tangent line for f(0.21)")

scatter!([0.21], [f(0.21)], color = "blue", label="", markersize = 3)

plot!([0.21,0.21],[f(0.21),0], label="", linecolor=:green, linestyle=:dash)
plot!([0.21,0],[f(0.21),f(0.21)], label="", linecolor=:green, linestyle=:dash)

annotate!([(0.06,0.58, (L"f(x_{0})", 8, :green))])
annotate!([(0.24,-0.06, (L"x_{0}", 8, :green))])
annotate!([(0.21,0, (L"|", 8, :green))])
```

Out[131]:





$p = (x_0, f(x_0))$

$m_t = \tan(\alpha_t) = f^{'}(x_0)$

- α_t is an angle made by the tangent line with the x-axis.

Then the equation of the tangent line is

$$y - f(x_0) = f^{'}(x_0)(x - x_0)$$

the equation above is called the general equation of a line. Can also be writtne as

$$p_0 = (x_0, y_0) \quad y - y_0 = m_t(x - x_0)$$

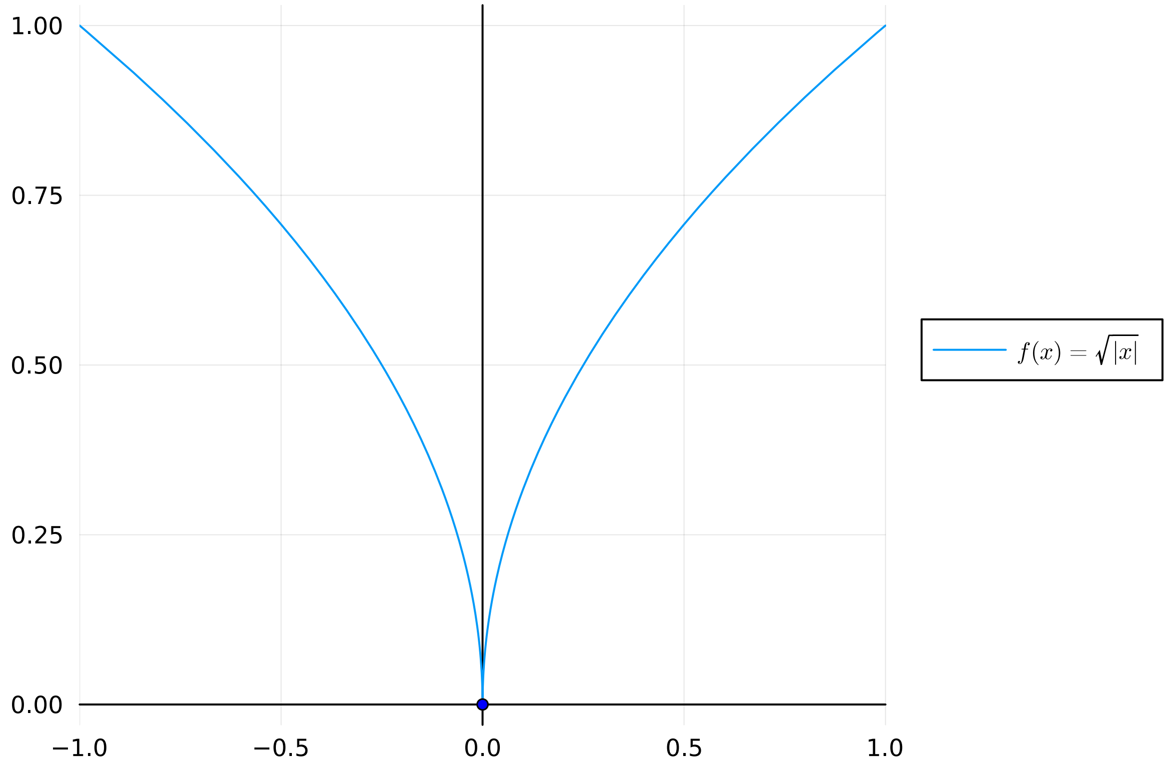
Angular Points

f is not differentiable in x_0

The point where f is not differentiable in x_0 is called the cusp point.

In [136]:
```
using MTH229, Plots, LaTeXStrings, Plots.PlotMeasures
gr()

f(x) = sqrt(abs(x))

plot(f, -1, 1, xlims=(-1,1),
    label=L"f(x) = \sqrt{|x|}", framestyle=:zerolines,
     legend=:outerright)
scatter!([0], [f(0)], color = "blue", label="", markersize = 3)
```

Out[136]:



Example

$y = \sqrt{x}$

$y = f(x)$is defined in $[0, +\infty)$

$$y' = \frac{1}{2\sqrt{x}}$$

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty$$

Example

$y = \sqrt{|x|}$

$x_0 = 0$ is a cusp point.

Compute $\lim_{x \to 0^+} f'(x)$ to know the limit position of the tangent line.

Infinitesimum (Sequel)

Definition

If we have $\lim_{x \to x_0} f(x), g(x) = 0$

and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

this means that $f(x) = o(g(x))$ for $x \to x_0$

Example

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \cdot x = 0$$

with $(1 - \cos x) = o(x)$

$$\lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2(1 + \cos x)} = \frac{1 - \cos^2 x}{2x^2}$$
$$\lim_{x \to 0} \frac{\sin^2 x}{2x^2} = \frac{1}{2} \cdot 1$$

Example

$$\lim_{x \to +\infty} \frac{\ln\left(1 + \frac{1}{\ln x}\right)}{\frac{1}{x^2}} = \lim_{x \to +\infty} x^2 \ln\left(1 + \frac{1}{\ln x}\right)$$
$$= \lim_{x \to +\infty} \ln\left[\left(1 + \frac{1}{\ln x}\right)^{x^2}\right]$$
$$= \lim_{x \to +\infty} \ln\left(\left[\left(1 + \frac{1}{\ln x}\right)^{\ln x}\right]^{\frac{x^2}{\ln x}}\right)$$
$$= +\infty$$

because we know that $\frac{x^2}{\ln x} \to +\infty$ and $\lim_{x \to +\infty}\left(1 + \frac{1}{\ln x}\right)^{\ln x} = e$ thus $e^{+\infty} \to +\infty$

- Note:

$$f(x) = o(1) \quad x \to x_0 \leftrightarrow \lim_{x \to x_0} f(x) = 0 \quad (*)$$

then if f is continuous in x_0 we can write $\lim_{x \to x_0}[f(x) - f(x_0)] = 0$.

By $(*)$ one can write

$$f(x) - f(x_0) = o(1) \quad x \to x_0$$

and if f is differentiable in x_0, we can write

$$\lim_{x \to x_0}\left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right] = 0$$

by $(*)$ one can write

$$\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) = o(1) \quad x \to x_0$$

for $x \neq x_0$

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(1)(x - x_0) \quad x \to x_0$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(1)(x - x_0) \quad x \to x_0$$

since

$$\lim_{x \to x_0} \frac{o(1)(x - x_0)}{(x - x_0)} = 0$$

Introductory Real Analysis Course 16 (2 November 2022)

∃δ>0: ∀ ↔ø

Introductory Real Analysis Course 17 (7 November 2022)

∃δ>0: ∀ ↔ ø

In [8]:

```
# To compute an indefinite integral, use the integrate function.
# There are two kinds of integrals, definite and indefinite

using SymPy
@vars x y z

integrate(cos(x), x)
```

Out[8]: $\sin(x)$

In [9]:

```
# To compute a definite integral,
# pass the argument (integration_variable, lower_limit, upper_limit)

using SymPy
@vars x y z

integrate(exp(-x), (x, 0, oo))
```

Out[9]: 1

In [11]:

```
# To compute multiple definite integrals

using SymPy
@vars x y z

integrate(exp(-x^2 - y^2), (x, -oo, oo), (y, -oo, oo))
```

Out[11]: π

Important Limits of Functions

For indetermined form: 1^{∞}

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{f(x) \to \pm\infty} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = e$$

For indetermined form: $\left[\frac{0}{0}\right]$

(1)

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

$$\lim_{f(x) \to 0} \frac{a^{f(x)} - 1}{f(x)} = \ln a$$

(2)

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{f(x) \to 0} \frac{e^{f(x)} - 1}{f(x)} = 1$$

(3)

$$\lim_{x \to 0} \frac{\log_a(1 + x)}{x} = \frac{1}{\ln a}$$

$$\lim_{f(x) \to 0} \frac{\log_a(1 + f(x))}{f(x)} = \frac{1}{\ln a}$$

(4)

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{f(x) \to 0} \frac{\ln(1+f(x))}{f(x)} = 1$$

(5)

$$\lim_{x \to 0} \frac{(1+x)^k - 1}{x} = k$$

$$\lim_{f(x) \to 0} \frac{(1+f(x))^k - 1}{f(x)} = k$$

(6)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{f(x) \to 0} \frac{\sin(f(x))}{f(x)} = 1$$

(7)

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{f(x) \to 0} \frac{1 - \cos(f(x))}{[f(x)]^2} = \frac{1}{2}$$

(8)

$$\lim_{x \to 0} \frac{\tan(x)}{x} = 1$$

Appendix

Packages

1. CalculusWithJulia
2. Plots
3. SymPy (SymPy can quickly find a factorization, even for quite large polynomials with rational or integer coefficients)
4. MTH229 (A very great package that can be used to determine limit, secant line, tangent line, derivative, and more)

Learning Source:

1. https://docs.juliahub.com/CalculusWithJulia/AZHbv/0.0.8/precalc/polynomial_roots.html (https://docs.juliahub.com/CalculusWithJulia/AZHbv/0.0.8/precalc/polynomial_roots.html)
2. Introductory Real Analysis Class Courses (2022/2023) at Universita degli Studi dell'Aquila

a. Professor Rosella Colomba Sampalmieri

b. Professor Felisia Angela Chiarello

c. Professor Kateryna Stiepanova

3. https://docs.juliahub.com/SymPy/KzewI/1.0.28/Tutorial/calculus/ (https://docs.juliahub.com/SymPy/KzewI/1.0.28/Tutorial/calculus/)
4. https://docs.juliahub.com/CalculusWithJulia/AZHbv/0.0.5/differentiable_vector_calculus/scalar_functions_applications.html (https://docs.juliahub.com/CalculusWithJulia/AZHbv/0.0.5/differentiable_vector_calculus/scalar_functions_applications.html)
5. https://mth229.github.io/derivatives.html (https://mth229.github.io/derivatives.html)

In [2]:
```
# To activate project designated for Real Analysis
# Create an empty folder named RealAnalysis that is in one folder with this notebook
import Pkg
Pkg.activate("RealAnalysis")
```

```
  Activating project at `~/LasthrimProjection/JupyterLab/RealAnalysis`
```

In [2]:
```
]st
```

```
      Status `~/LasthrimProjection/JupyterLab/RealAnalysis/Project.toml`
  [a2e0e22d] CalculusWithJulia v0.1.0
  [ebaf19f5] MTH229 v0.2.11
  [91a5bcdd] Plots v1.35.3
  [24249f21] SymPy v1.1.7
```

In [ ]: