1 Parameterization a Beta Distribution by its Mode and Shape Parameter

Here I briefly jot down the new parameterization we will use for the Beta distribution. Before, we had been parameterization a beta distribution by its mean and a dispersion parameter related to its variance. We modelled the mean using a linear model. However, the drawback was that when the variance of a beta distribution is large, its mean and mode are not close to each other. What we really wanted was to model the mode of a Beta distribution using a linear model. With this parameterization found from literature [1], we can now do that.

1.1 Beta Distribution Facts

- $p(x) \sim x^{\alpha-1}(1-x)^{\beta-1}$
- mode $= \frac{\alpha-1}{\alpha+\beta-2}$
- unimodal if both $\alpha > 1$ and $\beta > 1$

1.2 New Parameterization

Let m be the desired mode of the beta distribution, and $s > -1$ be a shape parameter. They show a new parameterization $Beta'(m, s)$ which is equivalent to a (old parameterization) $Beta(\alpha, \beta)$ distribution where

$$\alpha = 1 + sm \tag{1}$$
$$\beta = 1 + s(1-m) \tag{2}$$

First, we can check that the mode of a $Beta'(m, s)$ distribution is indeed m. The mode of this distribution is:

$$\frac{\alpha - 1}{\alpha + \beta - 2} = \frac{sm}{sm + s(1-m)} = m \tag{3}$$

1.3 Unimodality

They also claim that a $Beta'(m, s)$ distribution is unimodal iff $s > 0$. This is true because m is between 0 and 1, and plugging $s > 0$ into equation 1 and 2 leads to $\alpha > 1$ and $\beta > 1$.

References

[1] Dorp, J. On Some Elicitation Procedures for Distributions with Bounded Support with Applications in Pert. Page 2.