1 Prior Predictive Distribution for $\tilde{\mu}^c$

Here, we study the prior predictive distribution of $\tilde{\mu}^c$, the underlying 'true' value of the parameter c for a patient with covariate vector $\tilde{x}$. The goal of this section is to see how our prior belief on what $\tilde{\mu}^c$ varies depends on the hyperparameters. We first describe the log-normal distribution, because it turns out this is the distribution that $\tilde{\mu}^c$ follows a log-normal distribution in the prior.

1.1 Log-Normal Distribution

A log-normal distribution is a continuous distribution defined on the open interval $(0, \infty)$. A random variable X follows a log-normal distribution if the transformed random variable $\log(X)$ follows a normal distribution. An equivalent definition makes the parameterization of a log-normal distribution clear:

Definition 1. If $Y \sim \text{Normal}(\mu, \sigma^2)$, then the transformed random variable $X = \exp(Y)$ follows a Log-Normal(μ, σ^2) distribution.

Thus, a Log-Normal distributed random variable X is parameterized using the mean and standard deviation of the normal distribution that $\log(X)$ follows.

1.1.1 Key Properties

The density of a Log-Normal(μ, σ^2) distribution is:

$$f_X(x; \mu, \sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} \exp -\frac{(\log(x) - \mu)^2}{2\sigma^2}; x > 0 \tag{1}$$

The log-normal distribution is unimodal regardless of the choice of parameters. This is suitable for our poses. There are analytical formulas for the mean and mode; the mean is $\exp(\mu + \frac{\sigma^2}{2)}$, and the mode is $\exp(\mu - \sigma^2)$. Like in the case of the logit-normal distribution, neither the mean nor mode are constant as σ^2 increases, for fixed μ. Also, the mode and mean exhibit opposite trends. Finally, as one would expect, the variance is increasing in σ^2.

1.1.2 Log-Normality of $\tilde{\mu}^c$ in the Prior

The only hyperparameter that $\tilde{\mu}^c$ depends on is c^c. The argument for the log-normality of $\tilde{\mu}^c$ is exactly analogous to that for the logit-normality of $\tilde{\mu}^a$. The prior predictive distribution of μ^c, $\mu^c|c^c$, is distributed $g^c(\mu^{c*}_{pop} + B^c\tilde{x})$. $B^c \sim N(0, c^c I)$, and so $B^c\tilde{x} \sim N(0, \tilde{\sigma}^c)$ where $\tilde{\sigma^c} := \tilde{x}'(c^c I)\tilde{x} = c^c \sum_{j=1}^{k} \tilde{x}_j^2$. Thus $\mu^{c*}_{pop} + B^c\tilde{x} \sim N(\mu^{c*}_{pop}, \tilde{\sigma}^c)$ and thus $\tilde{\mu}^c \sim \text{log-normal}(\mu^{c*}_{pop}, \tilde{\sigma}^c)$