Assignment - Hydrodynamics 1

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Part 1

The numeric solution to the advection problem was evaluated using three difference methods: *Central Difference, upwind* and *Lax-Wendroff*, over a grid with 100 points between x=0 and x=10 using the initial conditions:

$$q(x,0) = \begin{cases} 1.0 & \text{if } 2 \leq \text{x} \leq 4 \\ 0.1 & \text{otherwise} \end{cases}$$

and u = 0.5. Because the timestep must not be greater than the time it takes to travel the distance Δx with the speed u, the timestep is limited by:

$$\Delta t = c\frac{\Delta x}{u} \tag{1}$$

with $0 \leq c \leq 1$. Thus the maximum timestep is obtained if $c = 1$, i.e. $\Delta t_{max} = \frac{\Delta x}{u} = 0.\overline{202}$.

The results are stored in a M x N - matrix, where M is the number of grid points and N is the number of timesteps over the course of the calculation. Thus, a column of the matrix represents one single timestep and its row elements, the results for the grid points i.e. $q(x_i, t)$.
The code is written such that once executed, the results are plotted for all considered methods in order to get a valid comparison of the performance for the chosen parameters.

1. The analysis of the *Central Difference* method shows, that the results become increasingly unstable for increasing Δt. The first row of fig. 1 shows the behaviour using the respective Δt. With $c = 0.2$ the results already show very erratic behaviour as the shape of the initial "wave packet" is not conserved and is deformed substantially, which becomes even more apparent using $c = 0.6$ and $c = 1$. Given these results this method cannot be recommended for solving this problem.
2. The *upwind* method provides much better results, when comparing it to *central difference*. As Δt increases, the form of the wave packet is, although not perfect, well conserved and shows very stable results. Using the maximal Δt the results seem to conserve the form very well. By increasing Δt beyond the maximum allowed value, the results quickly diverge and become very unstable.
3. The *Lax-Wendroff* method seems to converge to the same results as the *upwind* method, as the form of the wave packets are conserved equally well for both methods. The provided results seem to be stable, independent of the respective Δt, despite showing some minor fluctuations.

Part 2

The same analysis as in task 1 was carried out again using four different flux-conservation schemes. Fig. 2 shows the performance comparison, with each row corresponding to the respective scheme. The Δt's were chosen as in fig. 1.

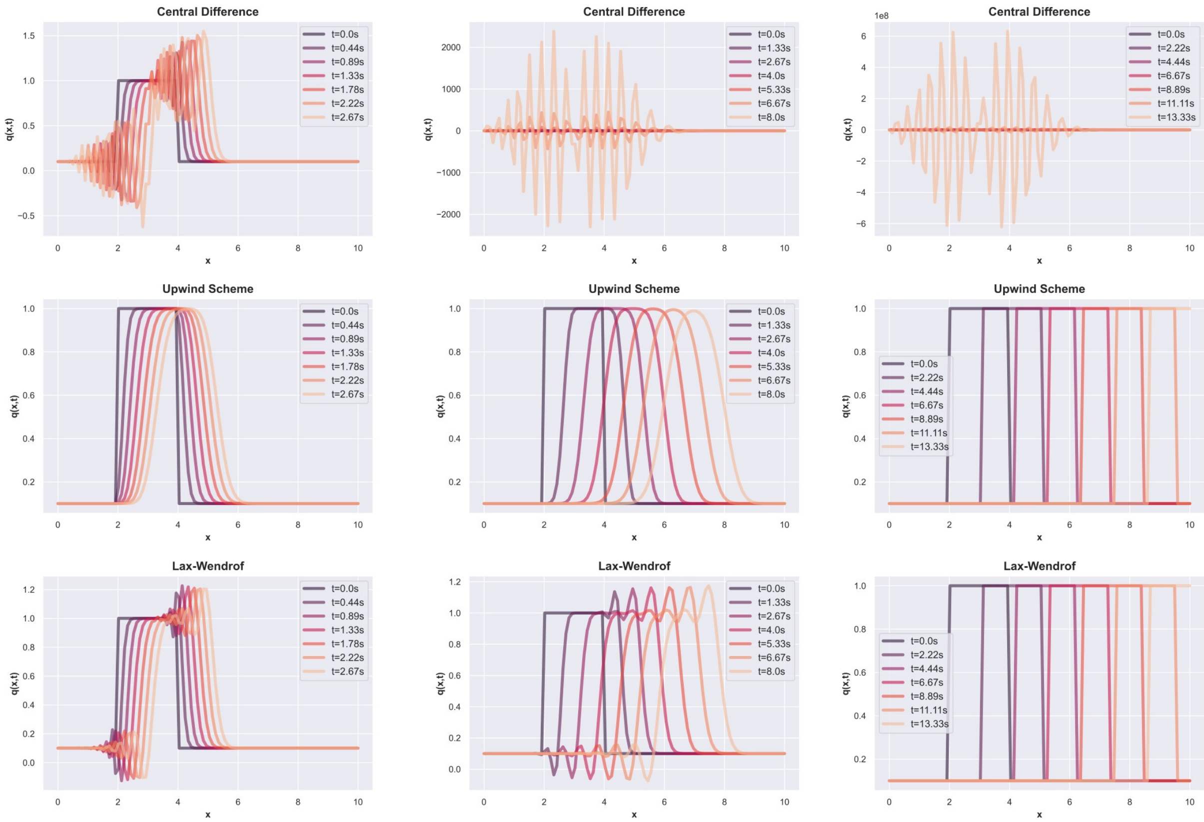

Figure 1: Comparison of the results provided by the three chosen methods. Each row corresponds to the results of the given method using three different Δt's. The first column was done using $c = 0.2$, the second $c = 0.6$ and the third $c = 1$. This serves the purpose of visually comparing the stability and the conservation of the form of the initial wave packet, achieved by each method at each Δt. The *Central Difference* method exhibits highly unstable results even at the smallest chosen Δt and becomes ever-more unstable as Δt increases. The results of the *upwind* method indicate much higher stability, as the form of the wave packet is sufficiently well conserved and reaches visually indistinguishable forms using $c = 1$. The *Lax-Wendroff* method, although appearing less stable than the *upwind* method, seems to converge to the same results using $c = 1$.

1. Due to the *donor-cell* and *upwind* method being mathematically equivalent, the results are also equivalent which can be seen be studying the first row of fig. 2. It shows the same behaviour with increasing performance i.e. the form of the wave packet being conserved, as Δt approaches the maxial Δt.

   Fromm's method uses the centered slope for the gradient and provides much better results, when comparing it to *central difference*. As Δt increases, the form of the wave packet, despite some minor fluctuations, is well conserved and shows very stable results. Again, using the maximal Δt the method seems to conserve the form very well.

   Although the *Beam-Warming* method shows stronger fluctuations compared to *donor-cell* or *Fromm*, the result using the maximal Δt appears to be equal.

   The flux-conservation variant of the *Lax-Wendroff* method seems to yield the same results as the variant used in fig. 1, as the forms of the wave packets are conserved equally for both methods.

2. The code was adapted to carry out the calculations for negative u-values by taking the q-value on the right instead of the left side of the considered grid point x_i when calculating the update q_i^{n+1}. This was done by increasing i by 1 in every method when computing $f_{i-1/2}^{n+1/2}$ and $f_{i+1/2}^{n+1/2}$.

   An example of this would be the following:

Donor-Cell method accepting negative and positive u-values

```
for n in range(N-1): # time steps
    for i in range(2,M-2): # grid points
        if u > 0:
            q[i,n+1] = q[i,n] + dt / dx * u*( q[i-1,n] - q[i,n] )
        elif u < 0:
            q[i,n+1] = q[i,n] + dt / dx * u*( q[i,n] - q[i+1,n] )
```

In the case of a negative u-value, $(q[i-1,n] - q[i,n])$ is changed to $(q[i,n] - q[i+1,n])$. The same applies to the other three methods. Additionally the c-value has to be negative as well in order to produce the correct results.

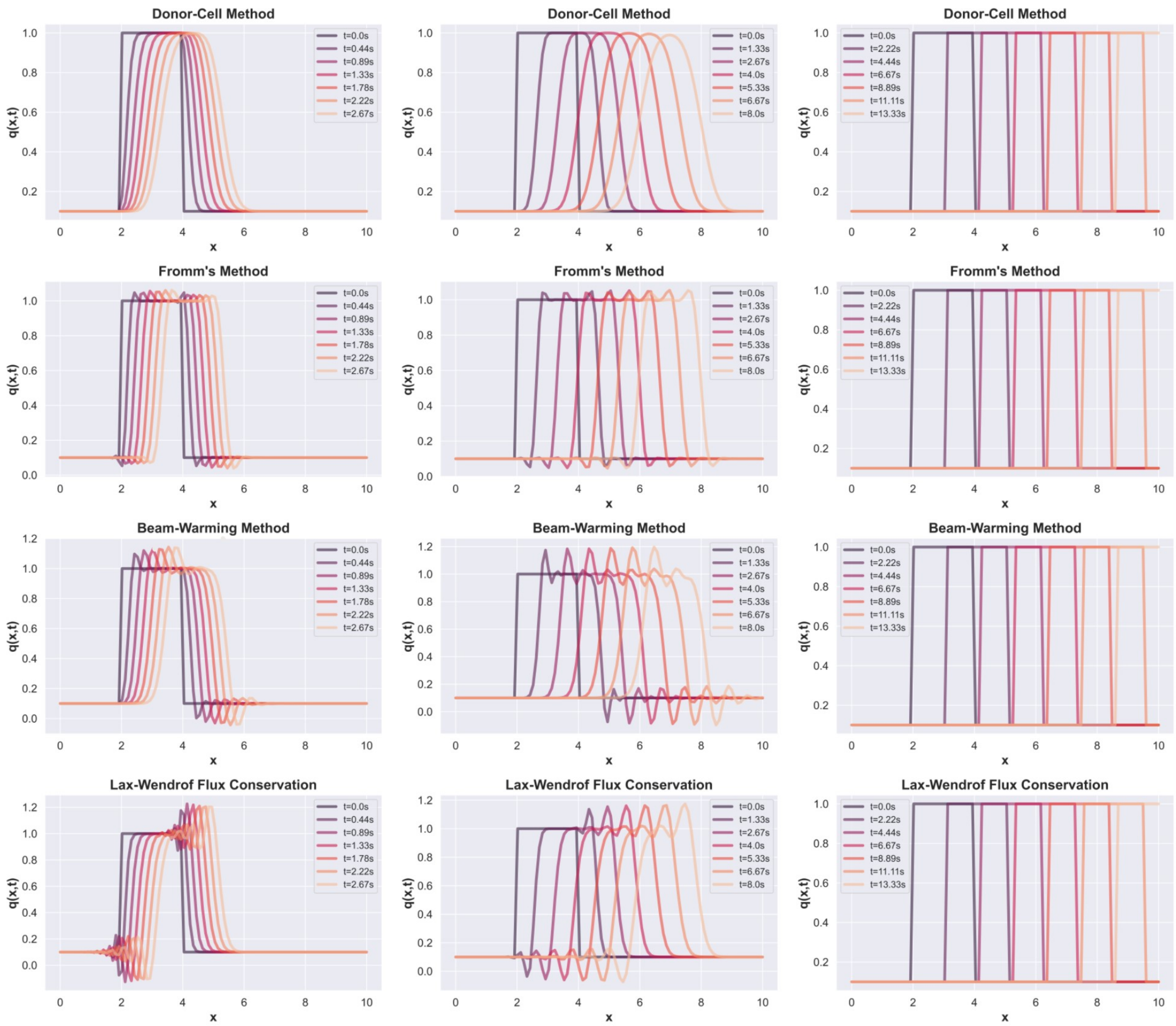

Figure 2: Performance comparison of all four flux-conservation schemes. As in fig. 1, the rows correspond the the respective method and the columns the to chosen c-value (0.2, 0.6, 1). Although the stability of the methods varies when using different c's, all four methods seem to converge to the same result as Δt reaches the maximal value. As such no clear "winner" can be declared.