Informatics for astronomers

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N-body code – Physical model

Fundamental considerations

The ODE(s) describing the dynamics of some physical system are called *equation(s) of motion*. In the case of an N-body system one needs to describe the motion of each individual body by individual ODEs (for their respective position vectors). For this purpose we have to express the forces acting on each individual body and insert them into Newton II: $\vec{F} = m\,\vec{a} = m\,\ddot{\vec{r}}$.

The force acting on one of these point masses m_1 caused by one of the other point masses m_2 is

$$\vec{F} = -\frac{Gm_1m_2\vec{r}}{r^3} \;, \tag{1}$$

where G is the gravitational constant, $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative vector pointing from m_2 to m_1, and $r = |\vec{r}|$ is the masses' distance.

System of ODEs for 2 bodies

In the case of only 2 gravitationally interacting bodies (m_1 and m_2) each of the 2 bodies is gravitationally attracted only by the other body. Therefore the equation of motion (from Newton II) for m_1 is

$$-\frac{Gm_1m_2\vec{r}}{r^3} = m_1\,\ddot{\vec{r}}_1 \;, \tag{2}$$

with $\vec{r} = \vec{r}_1 - \vec{r}_2$.

This system of 3[1] second-order ODEs can be rewritten into a system of 6 first-order ODEs, by means of the definition $\dot{\vec{r}}_1 = \vec{v}_1$. Using this (and rearranging) we end up with

$$\begin{pmatrix} \dot{\vec{r}}_1 \\ \dot{\vec{v}}_1 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \\ -\frac{Gm_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \end{pmatrix} \tag{3}$$

as equations of motion for m_1.

[1] In three-dimensional space.

Together with those for m_2 we finally end up with a system of 12 coupled first-order ODEs (for $\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2$), describing the dynamics of 2 interacting point masses in three-dimensional space:

$$\begin{pmatrix} \dot{\vec{r}}_1 \\ \dot{\vec{v}}_1 \\ \dot{\vec{r}}_2 \\ \dot{\vec{v}}_2 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \\ -\frac{Gm_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \\ \vec{v}_2 \\ -\frac{Gm_1}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) \end{pmatrix} \tag{4}$$

System of ODEs for 3 bodies

In a system of 3 point masses, each one feels the gravitational pull of the other two. Therefore these forces (vectors) have to be added up and we end up with

$$\begin{pmatrix} \dot{\vec{r}}_1 \\ \dot{\vec{v}}_1 \\ \dot{\vec{r}}_2 \\ \dot{\vec{v}}_2 \\ \dot{\vec{r}}_3 \\ \dot{\vec{v}}_3 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \\ -\frac{Gm_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) - \frac{Gm_3}{|\vec{r}_1-\vec{r}_3|^3}(\vec{r}_1 - \vec{r}_3) \\ \vec{v}_2 \\ -\frac{Gm_1}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) - \frac{Gm_3}{|\vec{r}_2-\vec{r}_3|^3}(\vec{r}_2 - \vec{r}_3) \\ \vec{v}_3 \\ -\frac{Gm_1}{|\vec{r}_3-\vec{r}_1|^3}(\vec{r}_3 - \vec{r}_1) - \frac{Gm_2}{|\vec{r}_3-\vec{r}_2|^3}(\vec{r}_3 - \vec{r}_2) \end{pmatrix}, \tag{5}$$

which are 18 coupled first-order ODEs (in 3 dimensions).

System of ODEs for N bodies

In the general case of N bodies in three-dimensional space we end up with a system of 6N coupled first-order ODEs for $\vec{r}_1, \vec{v}_1, ..., \vec{r}_N, \vec{v}_N$.