Assignment - Hydrodynamics

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Part 1

The numeric solution to the advection problem was evaluated using three difference methods: *Central Difference, upwind* and *Lax-Wendroff*, over a grid with 100 points between x=0 and x=10 using the initial conditions:

$$q(x,0) = \begin{cases} 1.0 & \text{if } 2 \leq \text{x} \leq 4 \\ 0.1 & \text{otherwise} \end{cases}$$

and u = 0.5. Because the timestep must not be greater than the time it takes to travel the distance Δx with the speed u, the timestep is limited by:

$$\Delta t = c\frac{\Delta x}{u} \tag{1}$$

with $0 \leq c \leq 1$. Thus the maximum timestep is obtained if $c = 1$, i.e. $\Delta t_{max} = \frac{\Delta x}{u} = 0.\overline{202}$.

The results are stored in a M x N - matrix, where M is the number of grid points and N is the number of timesteps over the course of the calculation. Thus, a column of the matrix represents one single timestep and its row elements i.e. the results for the grid points $q(x_i, t)$.

> **Notice:** The code for this part and all the following parts was written such that, once executed, the results are plotted for all considered methods, in order to get a valid comparison of the performance for the chosen parameters.

1. The analysis of the *Central Difference* method shows, that the results become increasingly unstable for increasing Δt. The first row of fig. 1 shows the behaviour using the respective Δt. With $c = 0.2$ the results already show very erratic behaviour as the shape of the initial wave packet is not conserved and is deformed substantially, which becomes even more apparent using $c = 0.6$ and $c = 1$. Given these results this method cannot be recommended for solving this problem.

2. The *upwind* method provides much better results, when comparing it to *central difference*. As Δt increases, the form of the wave packet is, although not perfect, well conserved and shows very stable results. Using the maximal Δt the results seem to conserve the form very well. By increasing Δt beyond the maximum allowed value, the results quickly diverge and become very unstable.

3. The *Lax-Wendroff* method seems to converge to the same results as the *upwind* method, as the forms of the wave packets are conserved equally well for both methods. The provided results seem to be stable, independent of the respective Δt, despite showing some minor fluctuations.

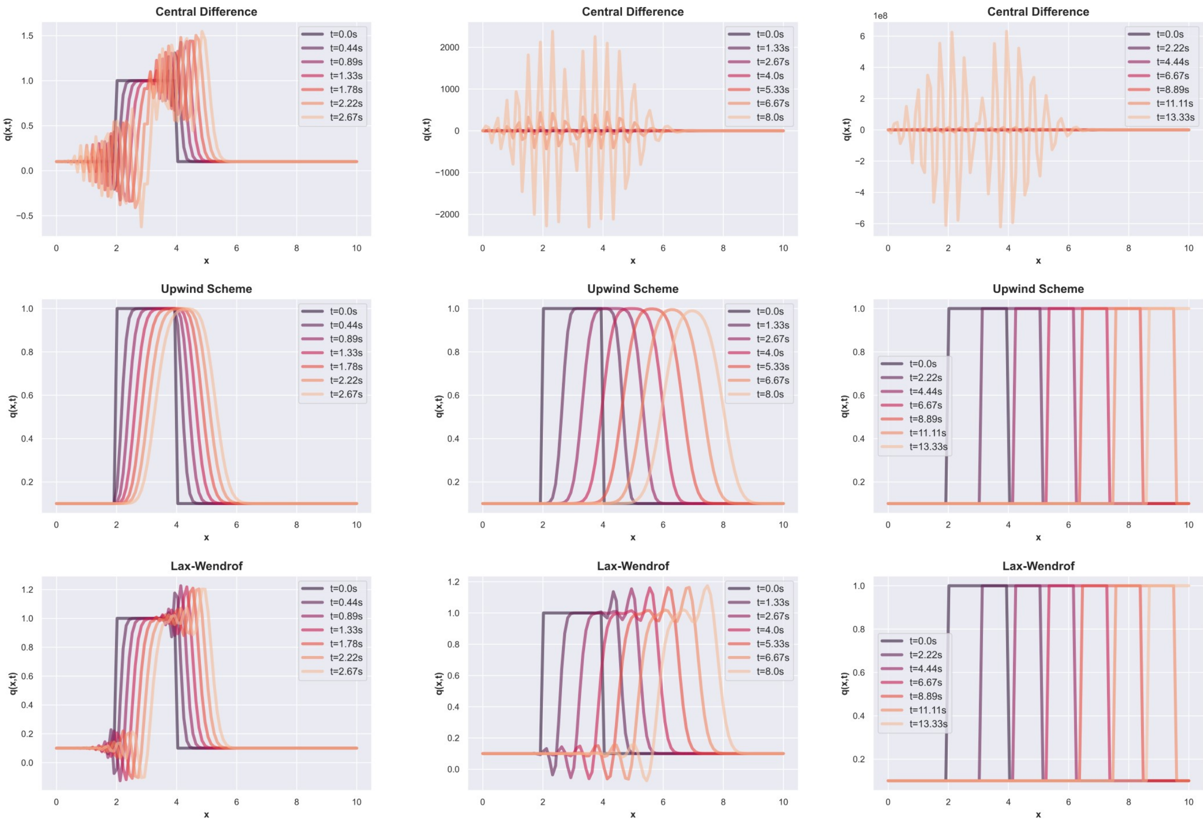

Figure 1: Comparison of the results provided by the three chosen methods. Each row corresponds to the results of the given method using three different Δt's. The first column was done using $c = 0.2$, the second $c = 0.6$ and the third $c = 1$. This serves the purpose of visually comparing the stability and the conservation of the form of the initial wave packet, achieved by each method at each Δt. The *Central Difference* method exhibits highly unstable results even at the smallest chosen Δt and becomes ever-more unstable as Δt increases. The results of the *upwind* method indicate much higher stability, as the form of the wave packet is sufficiently well conserved and reaches visually indistinguishable forms using $c = 1$. The *Lax-Wendroff* method, although appearing less stable than the *upwind* method, seems to converge to the same results using $c = 1$.

Part 2

The same analysis as in task 1 was carried out again using four different flux-conservation schemes. Fig. 2 shows the performance comparison, with each row corresponding to the respective scheme. The Δt's were chosen as in fig. 1.

1. Due to the *donor-cell* and *upwind* method being mathematically equivalent, the results are also equivalent which can be seen be studying the first row of fig. 2. It shows the same behaviour with increasing performance i.e. the form of the wave packet being conserved, as Δt approaches the maxial Δt.

   Fromm's method uses the centered slope for the gradient and provides much better results, when comparing it to *central difference*. As Δt increases, the form of the wave packet, despite some minor fluctuations, is well conserved and shows very stable results. Again, using the maximal Δt the method seems to conserve the form very well.

   Although the *Beam-Warming* method shows stronger fluctuations compared to *donor-cell* or *Fromm*, the result using the maximal Δt appears to be equal.

   The flux-conservation variant of the *Lax-Wendroff* method seems to yield the same results as the variant used in fig. 1, as the forms of the wave packets are conserved equally for both methods.

2. The code was adapted to carry out the calculations for negative u-values by taking the q-value on the right instead of the left side of the considered grid point x_i when calculating the update q_i^{n+1}. This was done by increasing i by 1 in every method when computing $f_{i-1/2}^{n+1/2}$ and $f_{i+1/2}^{n+1/2}$, respectively.

An example of this would be the following:

```
Donor-Cell method accepting negative and positive u-values

for n in range(N-1): # time steps
    for i in range(2,M-2): # grid points
        if u > 0:
            q[i,n+1] = q[i,n] + dt / dx * u*( q[i-1,n] - q[i,n] )
        elif u < 0:
            q[i,n+1] = q[i,n] + dt / dx * u*( q[i,n] - q[i+1,n] )
```

In the case of a negative u-value, $(q[i-1,n] - q[i,n])$ is changed to $(q[i,n] - q[i+1,n])$. The same applies to the other three methods. Additionally the c-value has to be negative as well in order to produce the correct results.

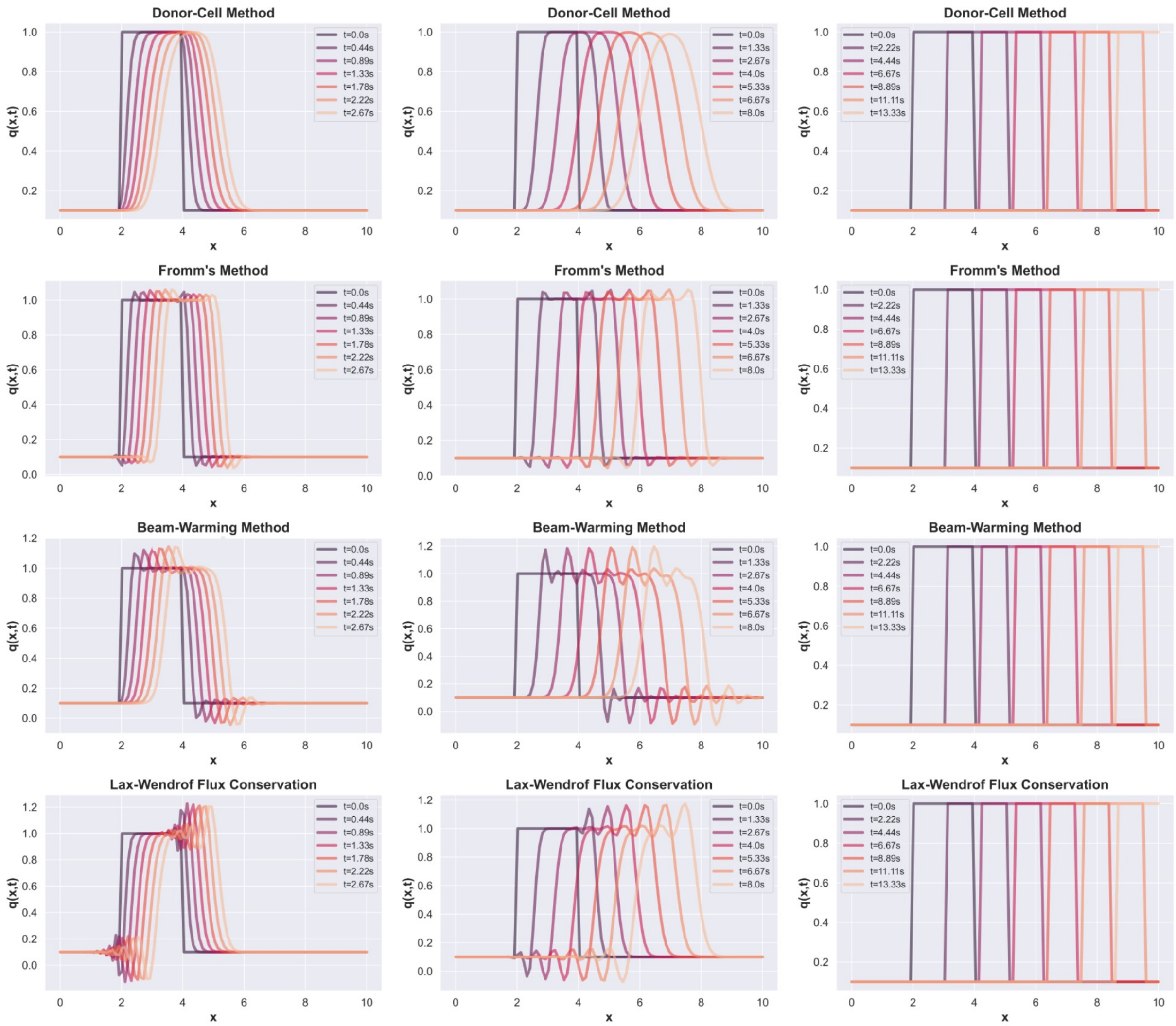

Figure 2: Performance comparison of all four flux-conservation schemes. As in fig. 1, the rows correspond to the respective method and the columns the to chosen c-value (0.2, 0.6, 1). Although the stability of the methods varies when using different c's, all four methods seem to converge to the same result as Δt reaches the maximal value. As such no clear "winner" can be declared.

Part 3

The Sod shock tube test was evaluated using two different methods: a Lax-Wendroff variant and the *Upwind* scheme, over a grid with 1000 points between x=0 and x=1 using the initial conditions taken from the assignment sheet.

The Δt for each timestep was determined by computing the minimum of all Δt's like such:

$$\Delta t = c \cdot min\left(\frac{\Delta x}{|u| + c_s}\right) \tag{2}$$

with $0 \leq c \leq 1$ and $c_s = \sqrt{\frac{\gamma P}{\rho}}$ for all cell centers.

> **Notice:** The results of the different quantities were each stored in a 2M+4 x 2N+1 - matrix, where M is the number of grid points and N is the number of timesteps over the course of the calculation. In this implementation the results of the full-steps, as well as the half-steps were stored i.e. the uneven indices of the timesteps n represent the half- and the even n's the full-steps, which effectively doubles the grid points. That's why $q^n_{i+\frac{1}{2}}$ becomes q^n_{i+1}, $q^n_{i-\frac{1}{2}}$ becomes q^n_{i-1} and q^n_{i+1} becomes q^n_{i+2}, while q^n_i stays q^n_i. It has been ensured, that only the full timesteps have been plotted in the results.

1. Fig. 3 shows the results of the evolved density, velocity and pressure using the *Lax-Wendroff* method. As the fluid at the boundaries expands into neighboring regions, all three quantities show substantial fluctuations at the boundaries. The black dotted line shows the initial state and the blue solid line the state after 200 timesteps using a Courant number of 0.97. The noticeable jumps at the right end of the region toward x=1, are most likely the product of a programming error that could not be identified, since the output should be symmetrical around x=0.5 and the left end showing no such jumps.

2. Fig. 4 shows the evolution of the system using the *upwind* method. In this case the blue line represents the state of the system after 500 steps using a Courant number of 0.2. These parameters were chosen to recreate the solution presented in the lecture. Comparing the two methods shows that the *upwind* method provides smoother results, as the fluctuations at the boundaries are not as pronounced as in the case of the *Lax-Wendroff* method. This may be due to the lower courant number i.e. smaller timesteps of the upwind scheme but due to the severe limitations that the *Lex-Wendroff* poses on the Courant number this cannot be resolved. However fig. 4 shows the same telling jumps on the right end of the region, which futher points to a coding error, as both method share the same programming framework.

Part 5

The numeric solution to the diffusion problem was evaluated using two different methods: the explicit *FTCS* and the implicit *BTCS*, over a grid with 100 points between x=0 and x=10 using the initial conditions:

$$q(x,0) = \begin{cases} 1.0 & \text{if } 3.5 \leq \text{x} \leq 6.5 \\ 0.1 & \text{otherwise} \end{cases}$$

with D = 1.0. The timestep was computed via:

$$\Delta t = c\frac{\Delta x^2}{2D} \tag{3}$$

with $0 \leq c \leq 1$.

The results were stored in a M x N - matrix, where M is the number of grid points and N is the number of timesteps over the course of the calculation. Thus, a column of the matrix represents one single timestep and its corresponding row elements i.e. the results for the grid points $q(x_i, t)$.

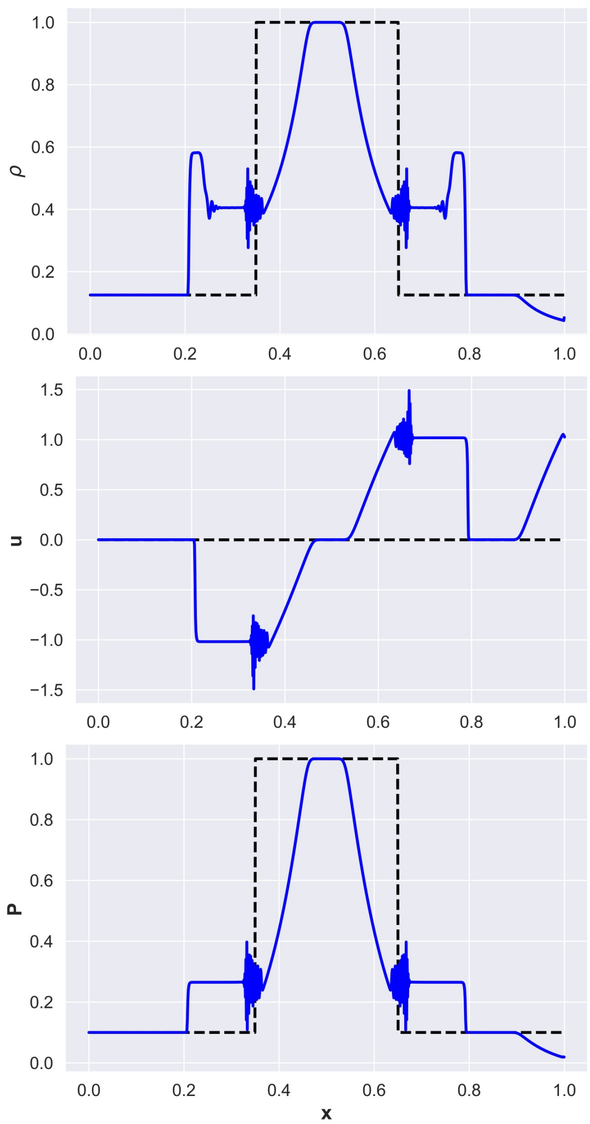

Figure 3: Sod shock tube test using the Lax-Wendroff scheme. Results show pronounced fluctuations of the three measured quantities at the boundaries, where the fluid expands into neighboring regions. Features unexpected jumps at the right end of the considered region, that most likely are a product of an unidentified programming error. Black dotted line presents the initial state and the blue line the state of the system after 200 timesteps with c=0.97.

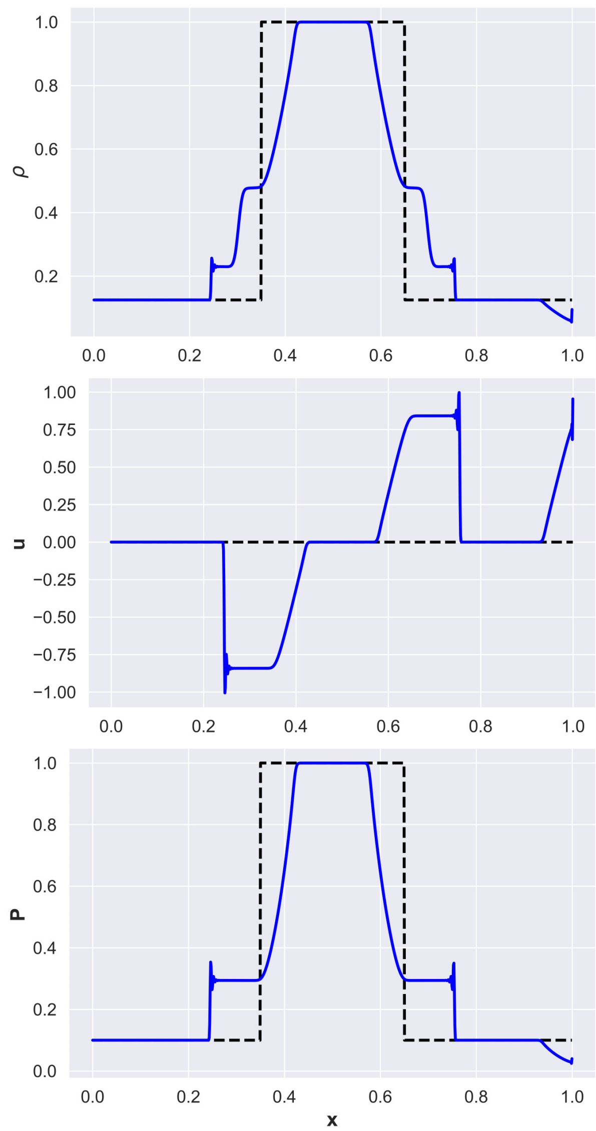

Figure 4: Sod shock tube test using the upwind scheme. Results show much smoother evolution of the three measured quantities at the boundaries, where the fluid expands into neighboring regions. Also features unexpected jumps at the right end of the considered region, that most likely are a product of an unidentified programming error. Black dotted line presents the initial state and the blue line the state of the system after 500 timesteps with c=0.2.

1. The analysis of the *FTCS* method shows, that the results become increasingly unstable for increasing Δt, similar to the *Central Difference* method of the first part of this assignment. The first row of fig. 1 shows the behaviour using the respective Δt. With $c = 0.5$ the results look very similar compared to the BTCS method as no obvious fluctuations are visible. Using c=1 leads to visible instabilities of the results. It is quite obvious that the chosen Δt is too large and thus the results becoming inaccurate. The considerable influence of Δt is obvious with c=1.5 as the results diverge quickly and show huge fluctuations.

2. The *BTCS* method provides much more stable results compared to *FTCS*. As Δt increases, the results appear to be virtually unaffected. Only minor differences could be identified and are only visible when looking at the results numerically, not visually. Due to the unconditional stability of this method, the results don't show the same unstable behaviour as Δt is increased beyond c=1.

   As stated in the assignment sheet, the values at the boundaries where kept constant and show the expected *unrealistic* values.

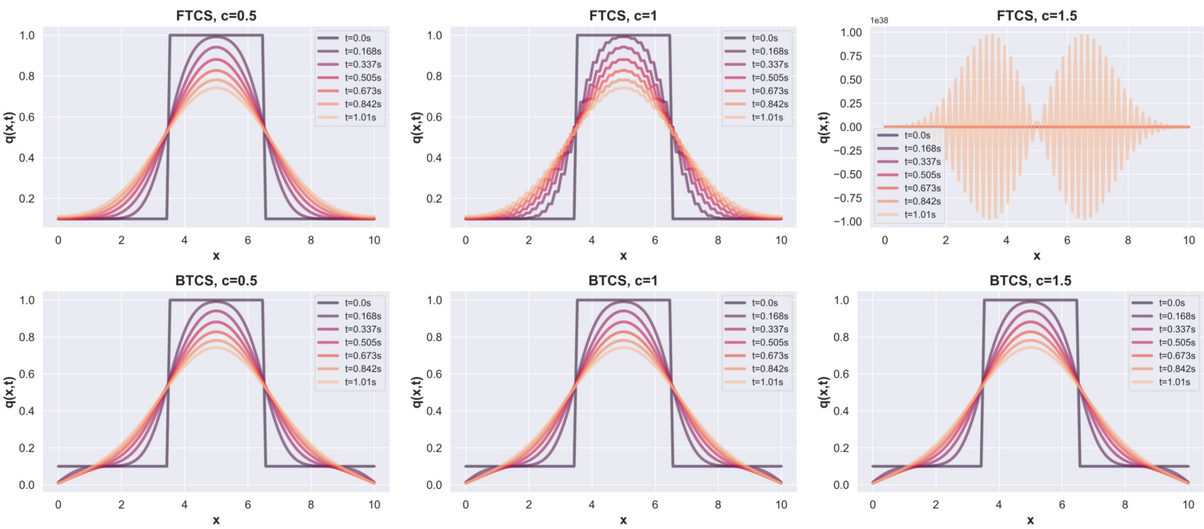

Figure 5: Comparison of the results provided by the two chosen methods. Each row corresponds to the results of the given method using three different Courant numbers (0.5, 1, 1.5) and thus different Δt's. The first column was done using $c = 0.5$, the second with $c = 0.1$ and the third with $c = 1.5$. This serves the purpose of visually comparing the stability achieved by each method at each Δt. As Δt increases, the *FTCS* method exhibits increasingly unstable results and completely diverges with c>1. The results of the *BTCS* method indicate much higher stability, as the results seem to be virtually unaffected by the changing Δt. The boundary values exhibit the expected unrealistic values and where kept constant during the computation.