# Numerical Astrophysics

**Assignment 2: ODEs**

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## Task 1

Given the initial value problem (IVP):

$$\frac{dy}{dt} = -2ty, \quad y(t=0) = 1 \tag{1}$$

the numerical solution between $t = 0$ and $t = 3$ was computed via explicit (forward Euler and Runge-Kutta 4) and implicit methods (backward Euler and Crank-Nicolson) whose results are plotted in figure 1.

Studying the individual performances of the methods revealed interesting behaviours. Although RK-4 behaves very erratically using 3 steps, it's already comparable to Crank-Nicolson at 6 steps and thus converges much fast than forward or backward Euler with increasing number of steps.

The comparison of backward and forward Euler shows that the implicit method performs better with fewer steps but as the number of steps increases, the convergence of Backward Euler to the analytic solution seems to slowly stagnate and at 30 steps both methods seem to provide similar results. Given the increased computational cost of BE, it only seems to be useful when using large stepsizes i.e. few number of steps.

It can be clearly seen that, given fewer steps i.e. bigger stepsizes, the implicit methods perform much better and provide a reasonably good approximation to the analytic solution. But as the number of steps increases, Runge-Kutta and Crank-Nicolson provide the best performance i.e. fastest convergence and highest accuracy with increasing number of steps.

The analytic solution of the IVP is given by:

$$y(t) = e^{-t^2} \tag{2}$$

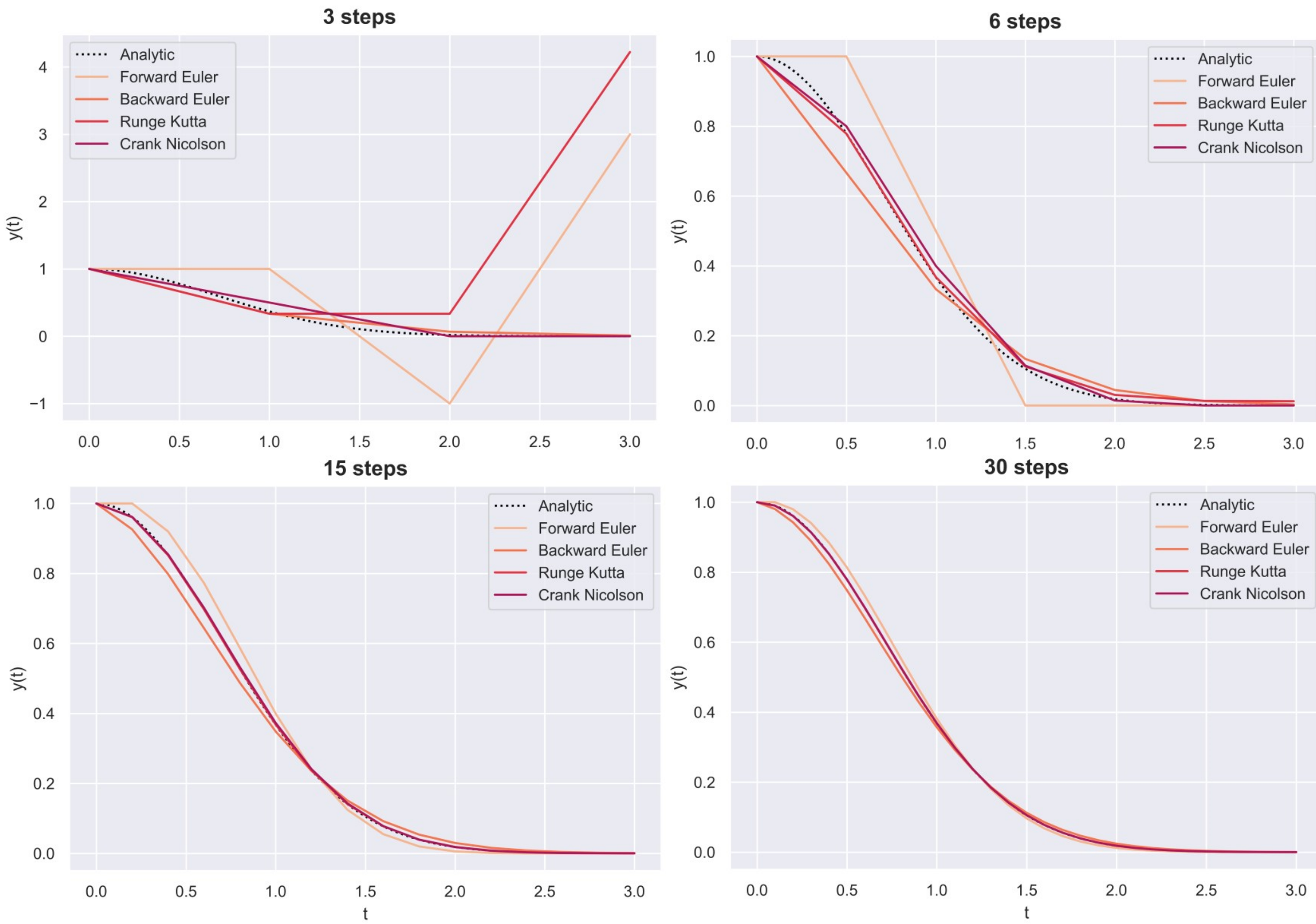


Figure 1: Convergence to analytic solution using different methods for increasing number of steps. Runge-Kutta and Crank-Nicolson performed the best.

## Task 2

Given the initial value problem (IVP):

$$\frac{dy}{dt} = 0.1y + \sin t, \quad y(t=0) = 1 \tag{3}$$

the numerical solution between $t = 0$ and $t = 20$ was computed via the Runge-Kutta-Fehlberg method which chooses the stepsize automatically based on the defined LTE-tolerance. In order to obtain the optimal LTE-tolerance the computation

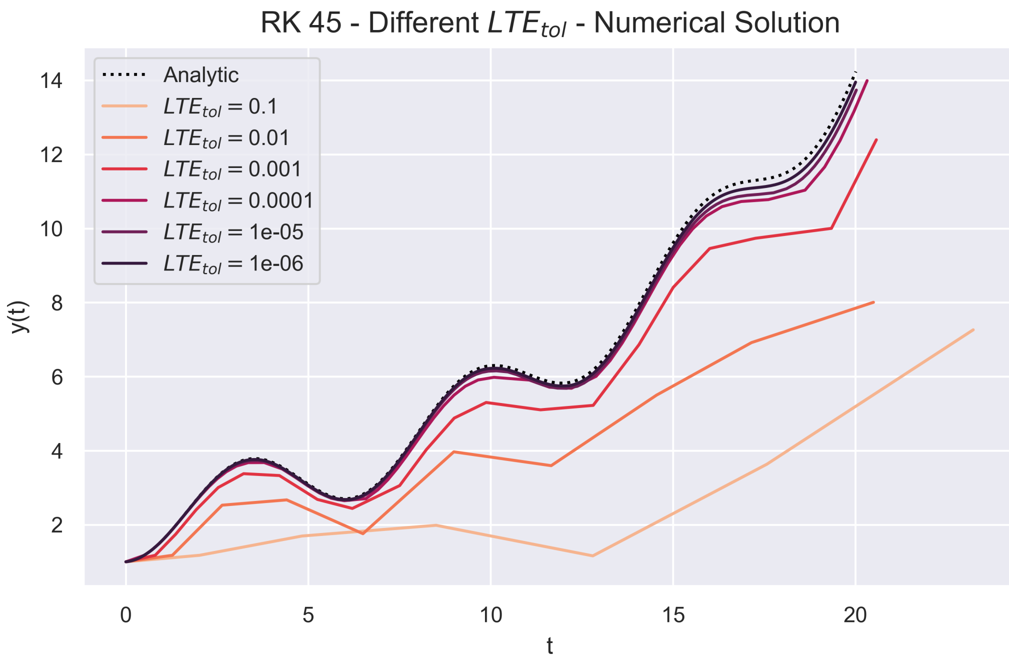


Figure 2: Examination of the optimal $LTE_{tol}$ using stepsize 0.5 as the initial value. The lower the tolerance the more accurate the solution.

was conducted using the values [1e-1, 1e-2, 1e-3, 1e-4, 1e-5, 1e-6].

The effect of the $LTE_{tol}$ on the numerical solution $y(t)$ can be examined in figure 2. As the $LTE_{tol}$ decreases, so does the GTE, such that the lowest tolerance leads to the best approximation to the analytic solution.

The analytic solution of the IVP is given by:

$$y(t) = 1.990099\ e^{0.1 \cdot t} - 0.0990099 \sin(t) - 0.990099 \cos(t) \tag{4}$$

Figure 3 compares the stepsizes with the respective time steps for all considered $LTE_{tol}$ values. This shows, that lower tolerances lead to smaller, more constrained stepsizes and thus to more accurate solutions. This of course implies increased computational cost due to a higher necessary number of steps during the computation.

A closer look at the plot reveals that the stepsizes start to decrease beyond the initial stepsize only with a tolerance lower than 1e-4, which corresponds nicely with the high accuracy of the lowest three LTE-values in figure 2.

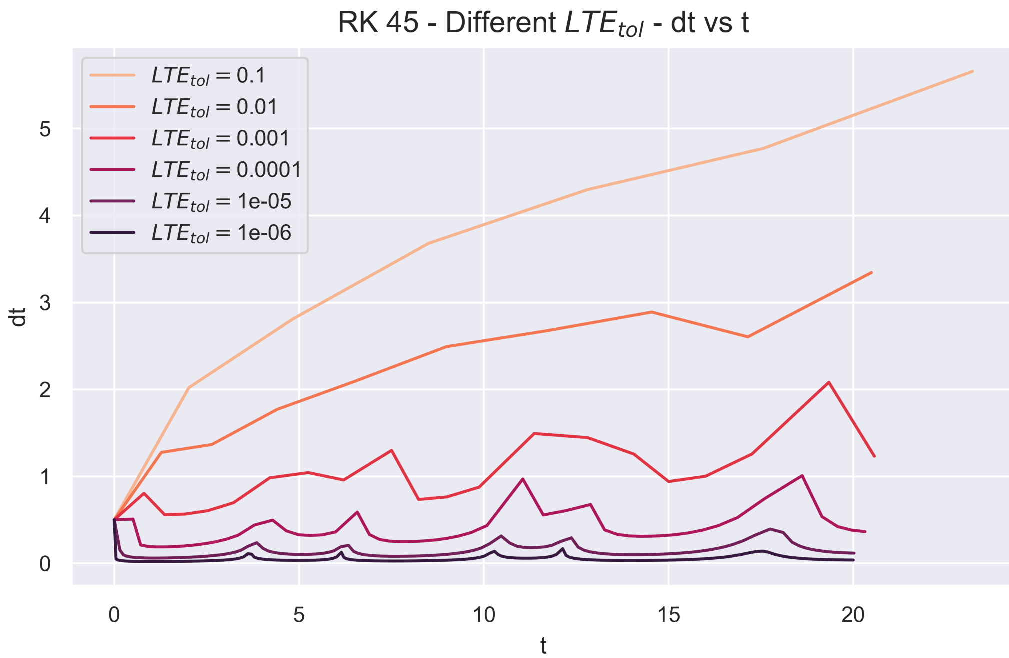


Figure 3: Comparison of stepsize dt to the corresponding time step t. The lower the tolerance the smaller the stepsize i.e. the more accurate the solution.

The resulting number of timesteps w.r.t. the LTE-values can be seen in figure 4. Here it is visible that lower tolerances lead to a rapidly increasing number of steps during the computation, which is the expected outcome.

Generally it can be seen that lower tolerances lead to more accurate results but also to an increased number of steps and thus a higher computational cost.

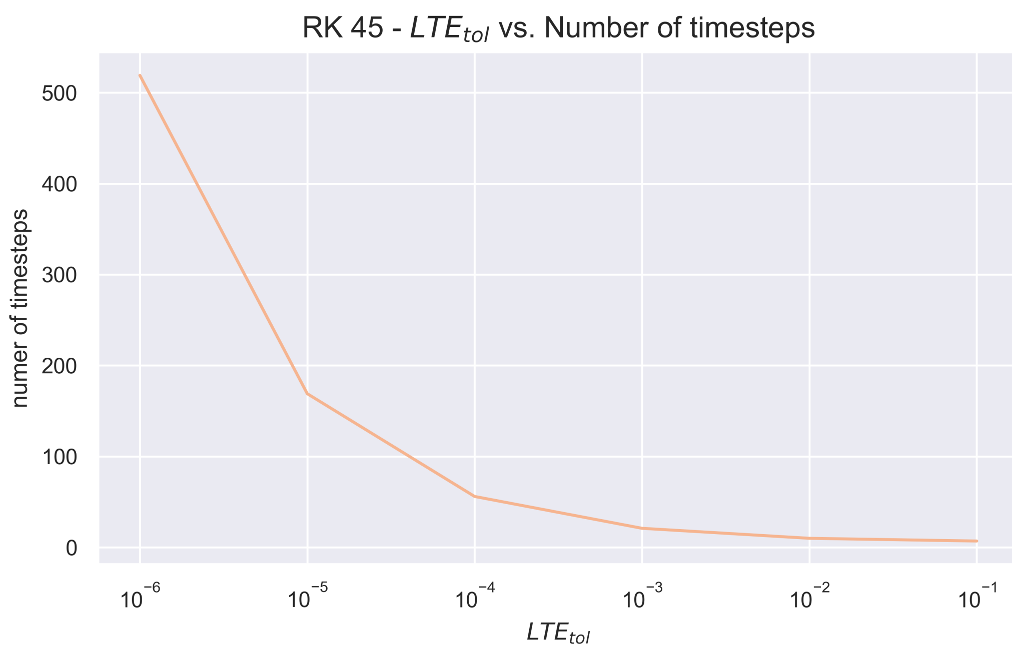


Figure 4: LTE-values vs the number of timesteps.