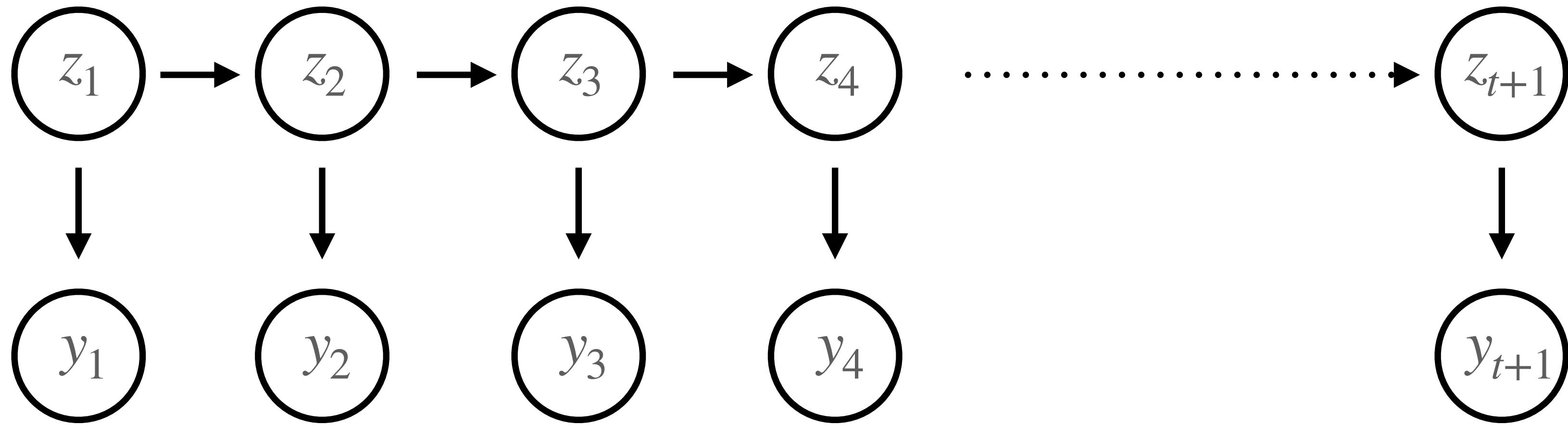


**LDS + Extensions, VAEs, SCA,  
Bi-cross-validation**

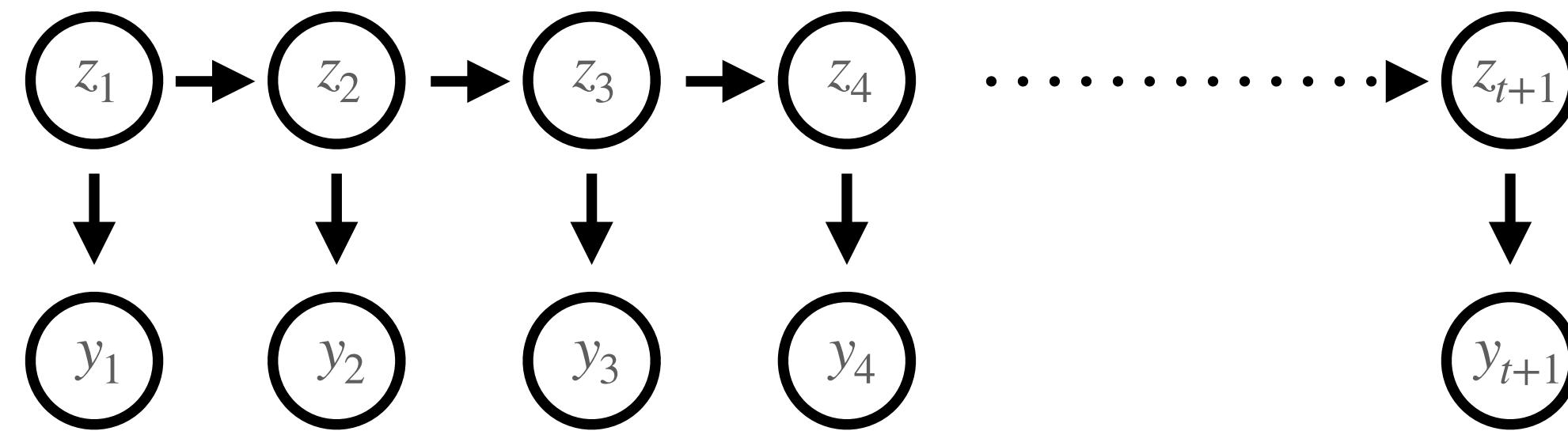
# Linear Dynamical Systems



$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

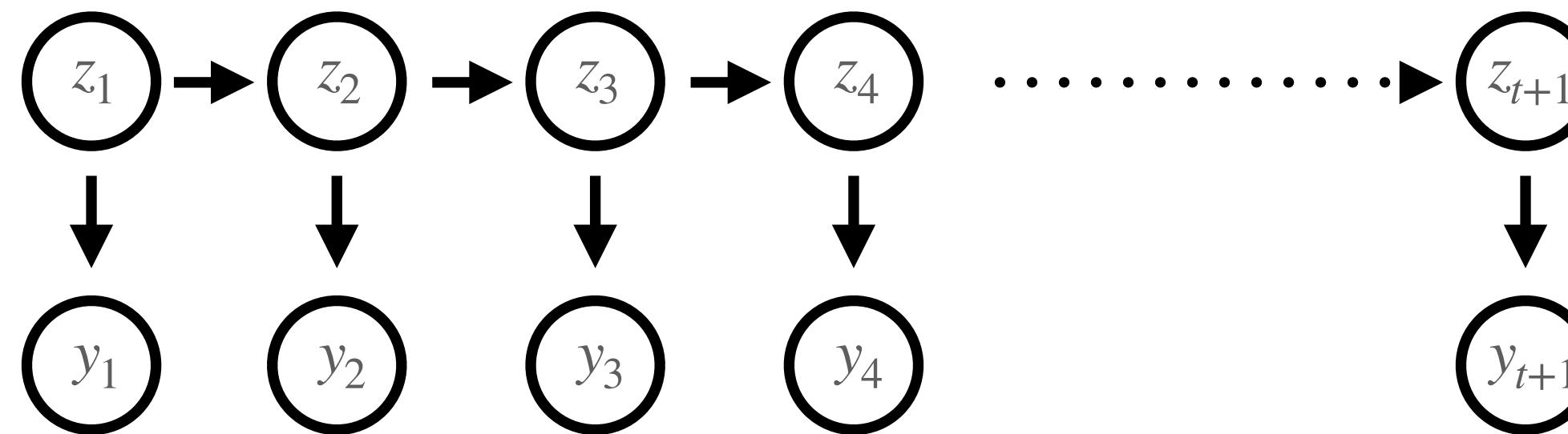
- Same graphical model as an HMM, but now  $z$  is continuous!
- Note the change in notation from  $x \rightarrow y$

# Linear Dynamical Systems



$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

# Linear Dynamical Systems

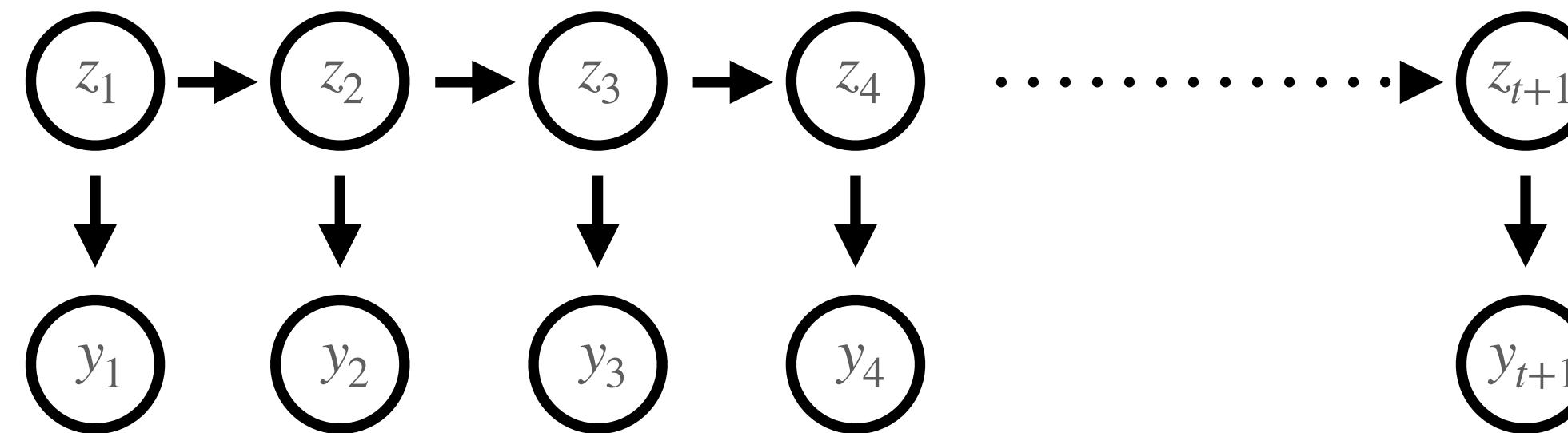


$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

Below the equation, three arrows point upwards towards the respective terms:

- An arrow points upwards from the term  $\mathcal{N}(z|0, \mathbf{I})$  to the initial state  $P(z_1)$ .
- An arrow points upwards from the term  $\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$  to the transition term  $p(z_t | z_{t-1})$ .
- An arrow points upwards from the term  $\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$  to the observation term  $p(y_t | z_t)$ .

# Linear Dynamical Systems



$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

Below the equation, three arrows point upwards to the corresponding terms:

- An arrow points to  $\mathcal{N}(z|0, \mathbf{I})$  under  $P(z_1)$ .
- An arrow points to  $\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$  under  $p(z_t | z_{t-1})$ .
- An arrow points to  $\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$  under  $p(y_t | z_t)$ .

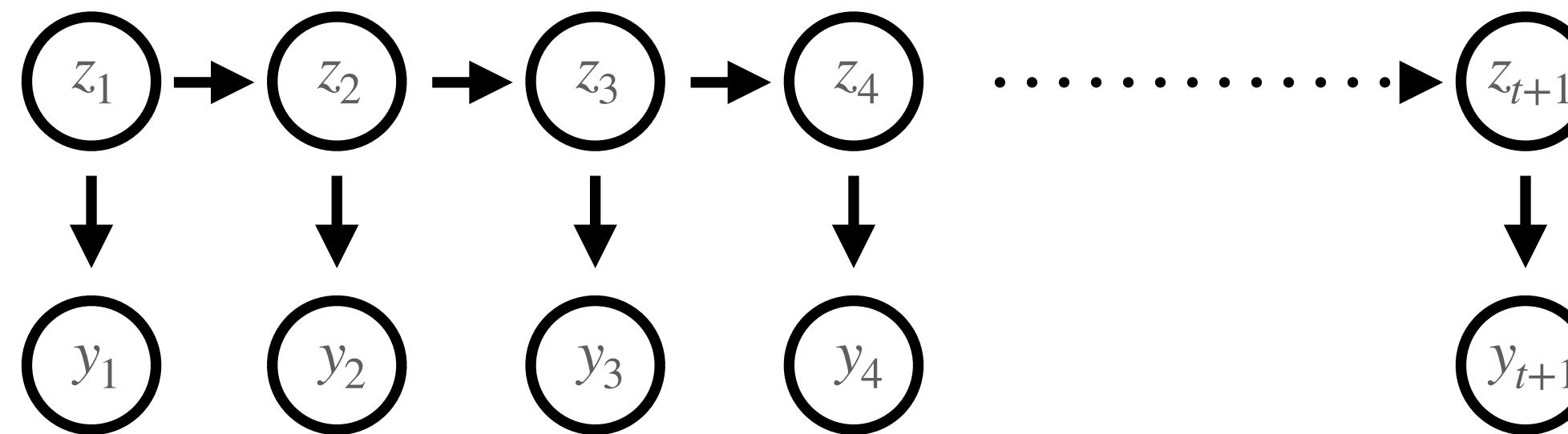
*Or more fully:*

$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t, \mathbf{Q}_t)$$

*Or more fully:*

$$\mathcal{N}(y_t | \mathbf{H}_t z_t + \mathbf{D}_t \mathbf{u}_t + \mathbf{d}_t, \mathbf{R}_t)$$

# Linear Dynamical Systems

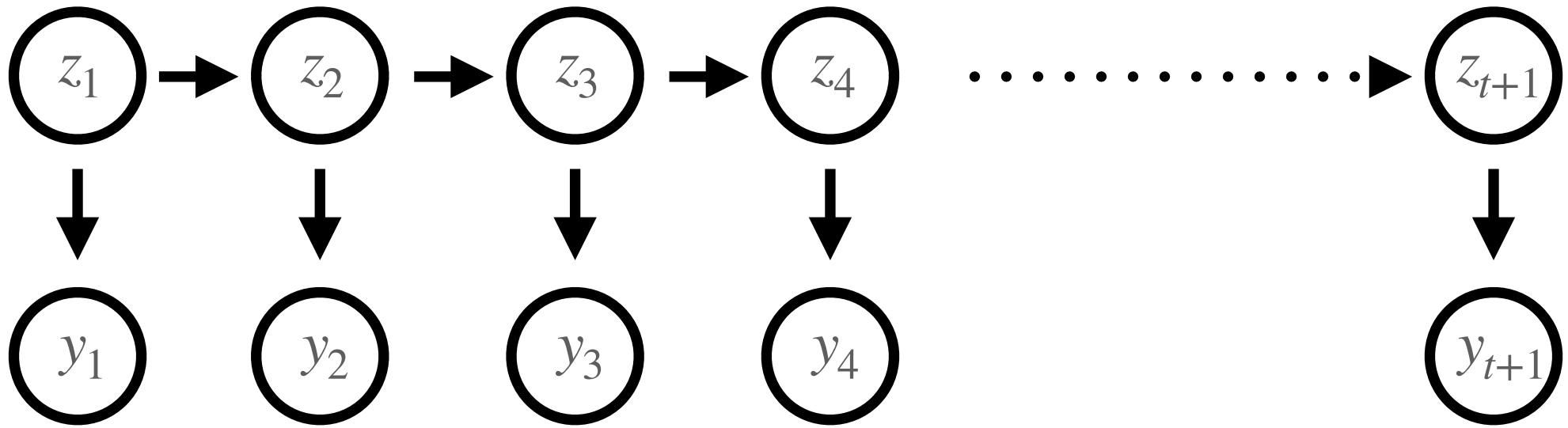


$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

$\mathcal{N}(z|0, \mathbf{I})$        $\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$        $\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$

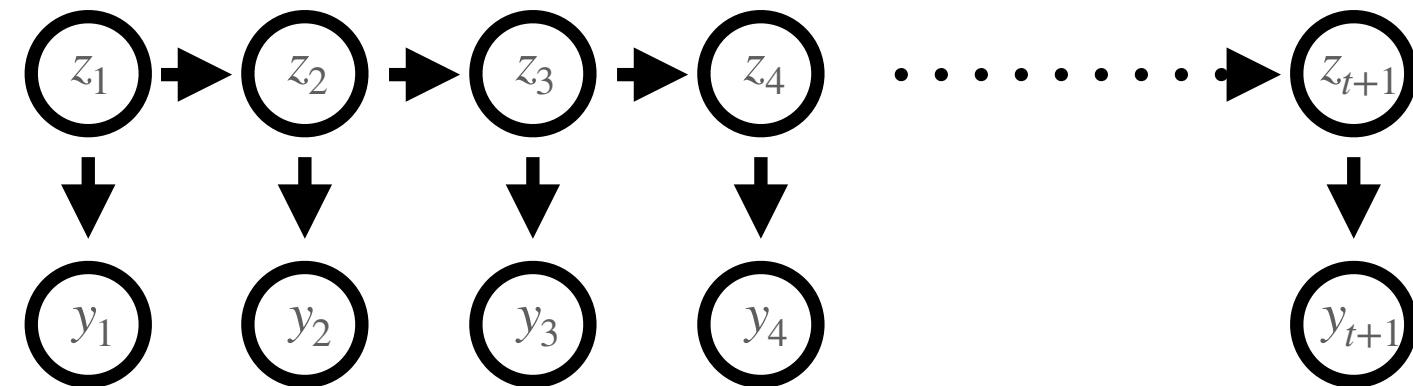
*Just like the Factor Analysis Model,  
but now with a dynamics term!*

# Simulating from an ~~HMM~~ LDS

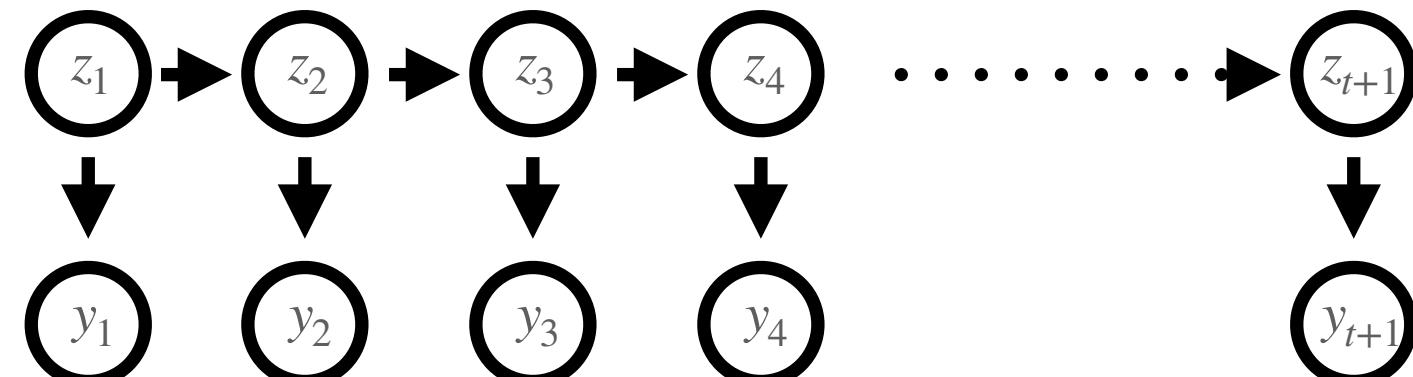


- Sample from  $P(z_1)$
- Sample from  $P(y_1 | z_1)$
- For all future time steps:
  - Sample  $P(z_{t+1} | z_t)$
  - Sample  $P(y_{t+1} | z_{t+1})$

# LDS: Inferring the Latent States with a Kalman Filter

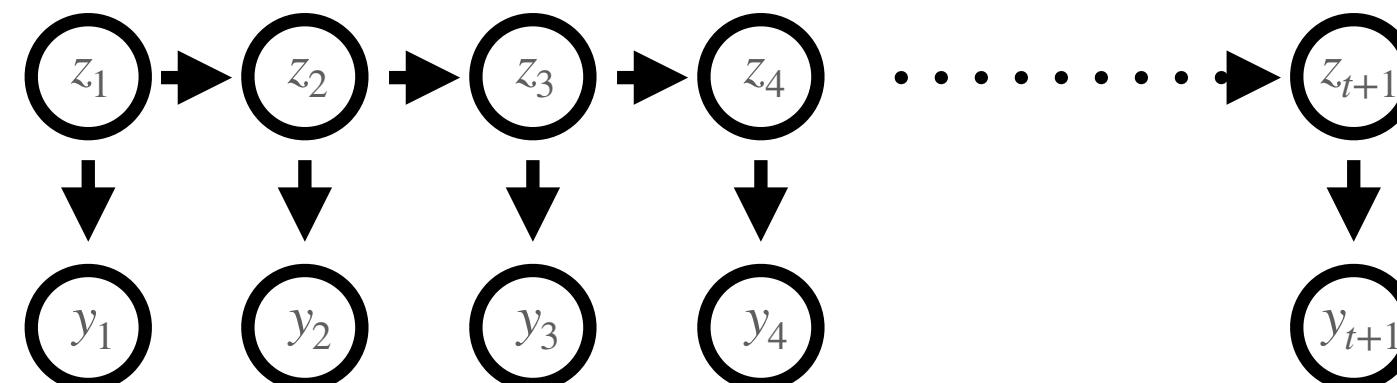


# LDS: Inferring the Latent States with a Kalman Filter



- Predict Step:
  - Predict forward in time with dynamics model
- Update Step:
  - Use observations to modify dynamics-only prediction from above

# LDS: Inferring the Latent States with a Kalman Filter

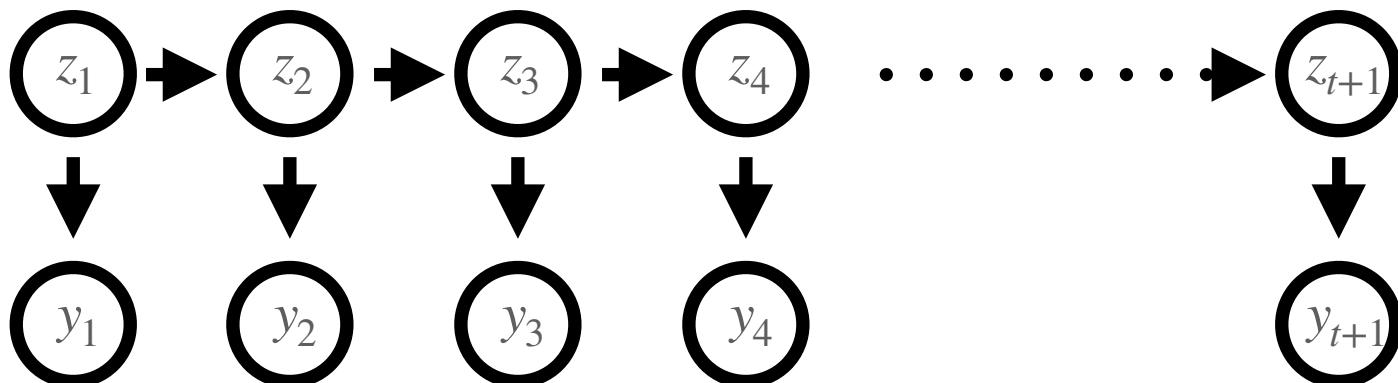


$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$$

$$\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$$

- Predict Step:
  - Predict forward in time with dynamics model
- Update Step:
  - Use observations to modify dynamics-only prediction from above

# LDS: Inferring the Latent States with a Kalman Filter



$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$$

$$\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$$

- Predict Step:

- Predict forward in time with dynamics model

$$p(z_t | y_{1:t-1}, u_{1:t}) = \mathcal{N}(z_t | \mu_{t|t-1}, \Sigma_{t|t-1})$$

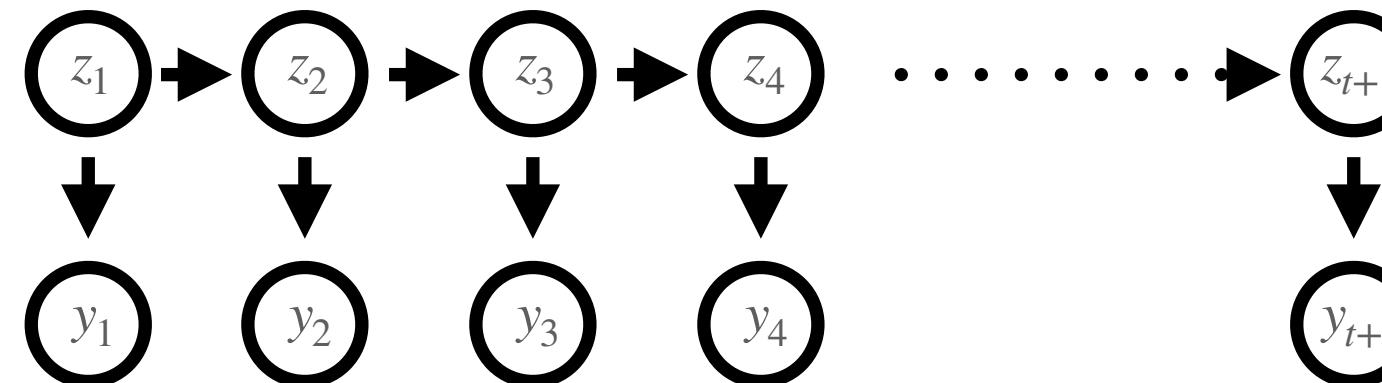
$$\mu_{t|t-1} = \mathbf{F}_t \mu_{t-1|t-1} + \mathbf{B}_t u_t + b_t$$

$$\Sigma_{t|t-1} = \mathbf{F}_t \Sigma_{t-1|t-1} \mathbf{F}_t^\top + \mathbf{Q}_t$$

- Update Step:

- Use observations to modify dynamics-only prediction from above

# LDS: Inferring the Latent States with a Kalman Filter



$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$$

$$\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$$

- Predict Step:

- Predict forward in time with dynamics model

$$p(z_t | y_{1:t-1}, u_{1:t}) = \mathcal{N}(z_t | \mu_{t|t-1}, \Sigma_{t|t-1})$$

$$\mu_{t|t-1} = \mathbf{F}_t \mu_{t-1|t-1} + \mathbf{B}_t u_t + b_t$$

$$\Sigma_{t|t-1} = \mathbf{F}_t \Sigma_{t-1|t-1} \mathbf{F}_t^\top + \mathbf{Q}_t$$

$$p(z_t | y_{1:t}, u_{1:t}) = \mathcal{N}(z_t | \mu_{t|t}, \Sigma_{t|t})$$

- Update Step:

- Use observations to modify dynamics-only prediction from above

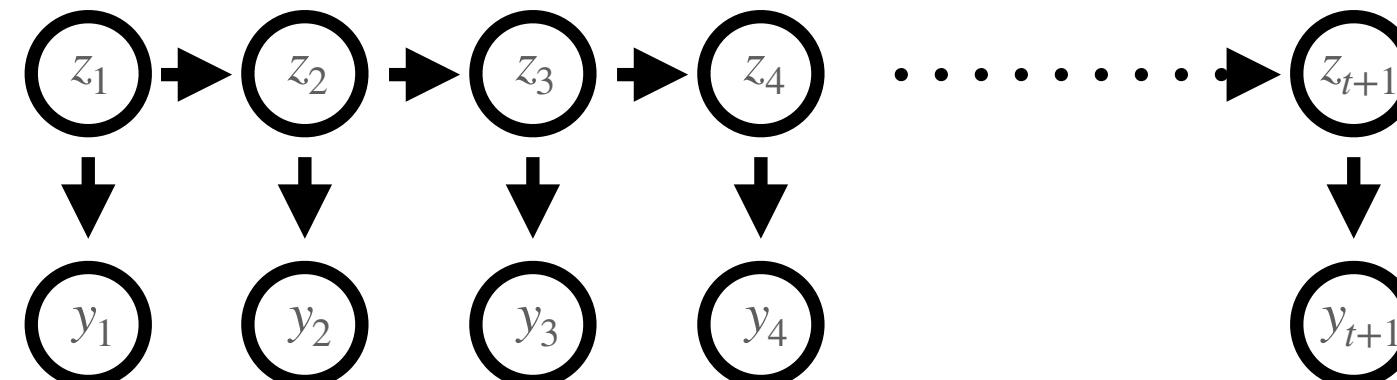
$$\hat{y}_t = \mathbf{H}_t \mu_{t|t-1} + \mathbf{D}_t u_t + d_t$$

$$\mathbf{S}_t = \mathbf{H}_t \Sigma_{t|t-1} \mathbf{H}_t^\top + \mathbf{R}_t$$

$$\mathbf{K}_t = \Sigma_{t|t-1} \mathbf{H}_t^\top \mathbf{S}_t^{-1}$$

$$\mu_{t|t} = \mu_{t|t-1} + \mathbf{K}_t (y_t - \hat{y}_t)$$

# LDS: Inferring the Latent States with a Kalman Filter



$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$$

$$\mathcal{N}(y_t | \mathbf{H}_t z_t, \mathbf{R}_t)$$

- Predict Step:

- Predict forward in time with dynamics model

$$p(z_t | y_{1:t-1}, u_{1:t}) = \mathcal{N}(z_t | \mu_{t|t-1}, \Sigma_{t|t-1})$$

$$\mu_{t|t-1} = \mathbf{F}_t \mu_{t-1|t-1} + \mathbf{B}_t u_t + b_t$$

$$\Sigma_{t|t-1} = \mathbf{F}_t \Sigma_{t-1|t-1} \mathbf{F}_t^\top + \mathbf{Q}_t$$

$$p(z_t | y_{1:t}, u_{1:t}) = \mathcal{N}(z_t | \mu_{t|t}, \Sigma_{t|t})$$

- Update Step:

- Use observations to modify dynamics-only prediction from above

$$\hat{y}_t = \mathbf{H}_t \mu_{t|t-1} + \mathbf{D}_t u_t + d_t$$

$$\mathbf{S}_t = \mathbf{H}_t \Sigma_{t|t-1} \mathbf{H}_t^\top + \mathbf{R}_t$$

Observations  
Noise

$$\mathbf{K}_t = \Sigma_{t|t-1} \mathbf{H}_t^\top \mathbf{S}_t^{-1}$$

Dynamics Noise →

$$\mu_{t|t} = \mu_{t|t-1} + \mathbf{K}_t (y_t - \hat{y}_t)$$

# LDS: Inferring the Latent States with a Kalman Smoother

- The Kalman Filter finds  $p(z_t | y_{1:t}, u_{1:t})$ 
  - Comparable to the “forward” algorithm for HMMs

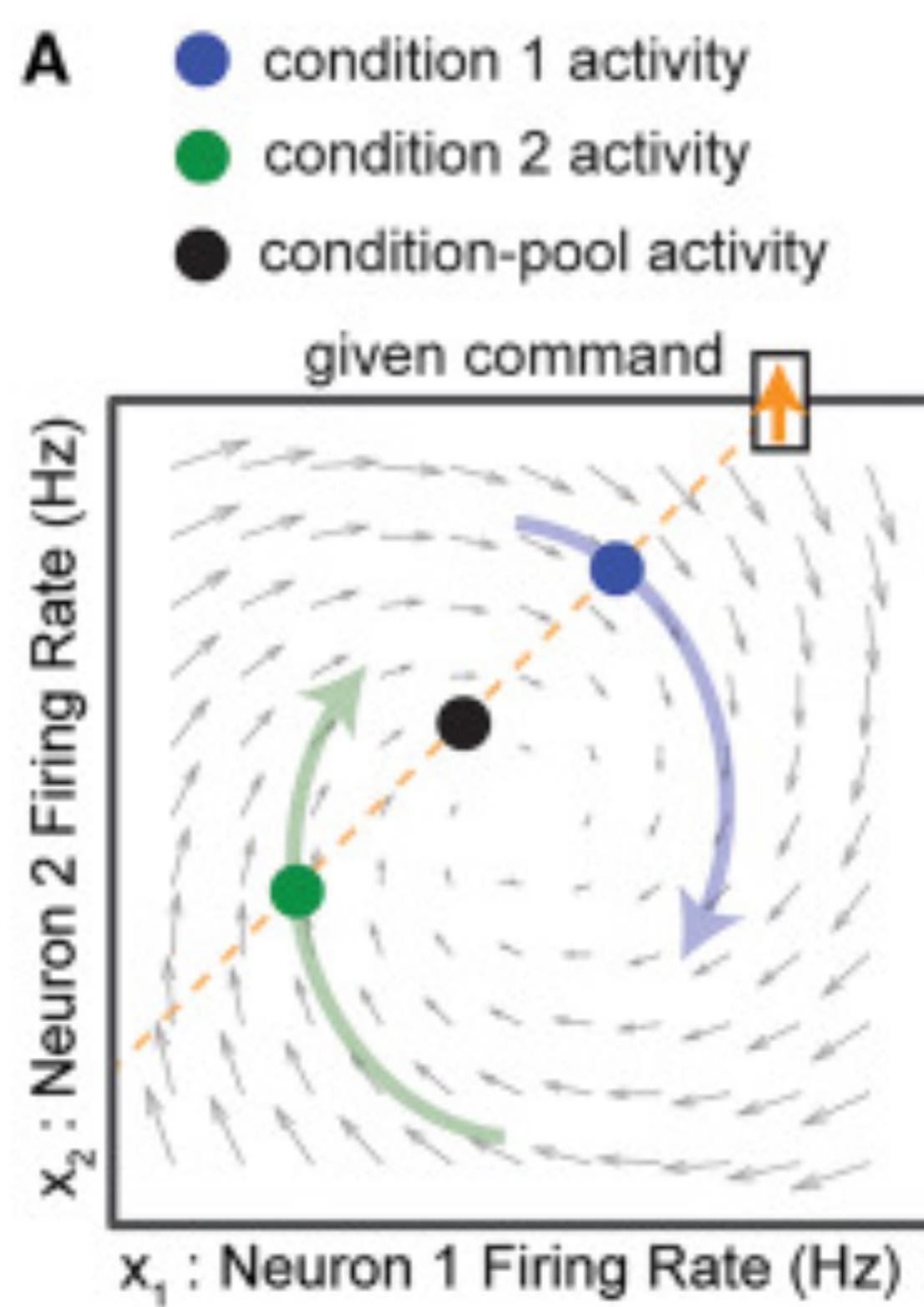
# LDS: Inferring the Latent States with a Kalman Smoother

- The Kalman Filter finds  $p(z_t | y_{1:t}, u_{1:t})$ 
  - Comparable to the “forward” algorithm for HMMs
- The Kalman Smoother finds  $p(z_t | y_{1:T}, u_{1:T})$ 
  - Also includes a calculation going backwards in time, and is comparable to the “forward-backward” algorithm for HMMs

# LDS: Model Fitting

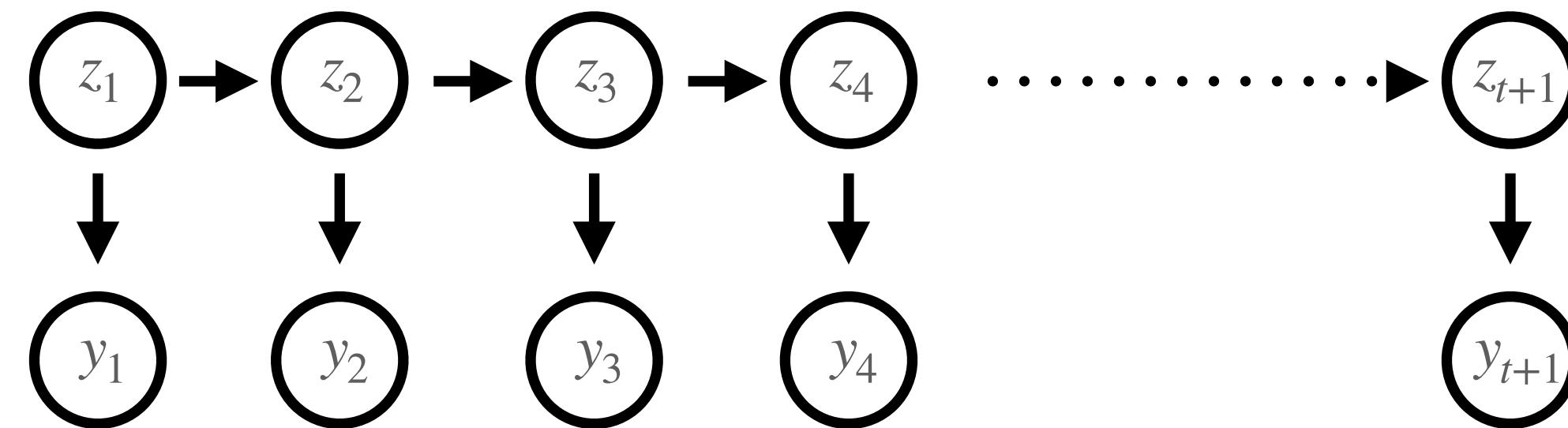
- Most standard is EM
  - E-step is Kalman Smoother
  - M-step optimizes parameters
    - Can learn interesting things about the dynamics too!
- Subspace Identification (SSID) is also used

## Invariant neural dynamics drive commands to control different movements



# Extensions

# Poisson Linear Dynamical Systems



$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

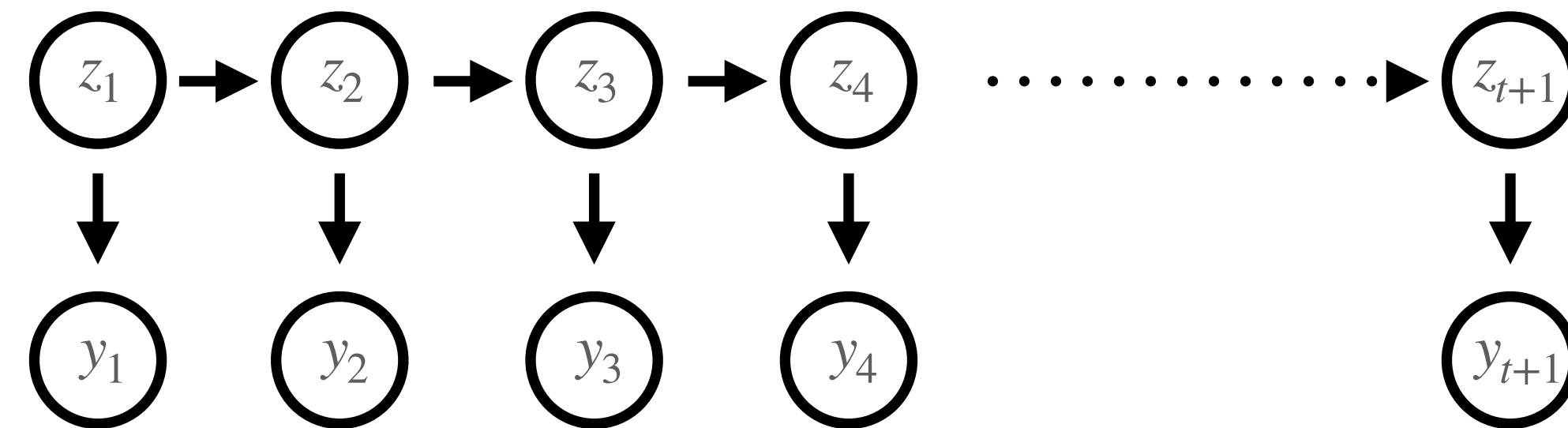
The diagram shows the probability distribution of the state  $z_t$  at time  $t$ . It consists of two parts: a prior distribution  $\mathcal{N}(z|0, \mathbf{I})$  and a conditional distribution  $\mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$ . Arrows point from these labels to the respective terms in the probability equation above.

$$\mathcal{N}(z|0, \mathbf{I}) \quad \mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t)$$

*Or more fully:*

$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t, \mathbf{Q}_t)$$

# Poisson Linear Dynamical Systems



$$P(z_{1:T}, y_{1:T}) = P(z_1) \prod_{t=2}^T p(z_t | z_{t-1}) \prod_{t=1}^T p(y_t | z_t)$$

*Or more fully:*

$$\mathcal{N}(z|0, \mathbf{I}) \quad \mathcal{N}(z_t | \mathbf{F}_t z_{t-1}, \mathbf{Q}_t) \quad \text{Poisson}(y_t | f(\mathbf{H}_t z_t))$$

*f=exp() or softplus()*

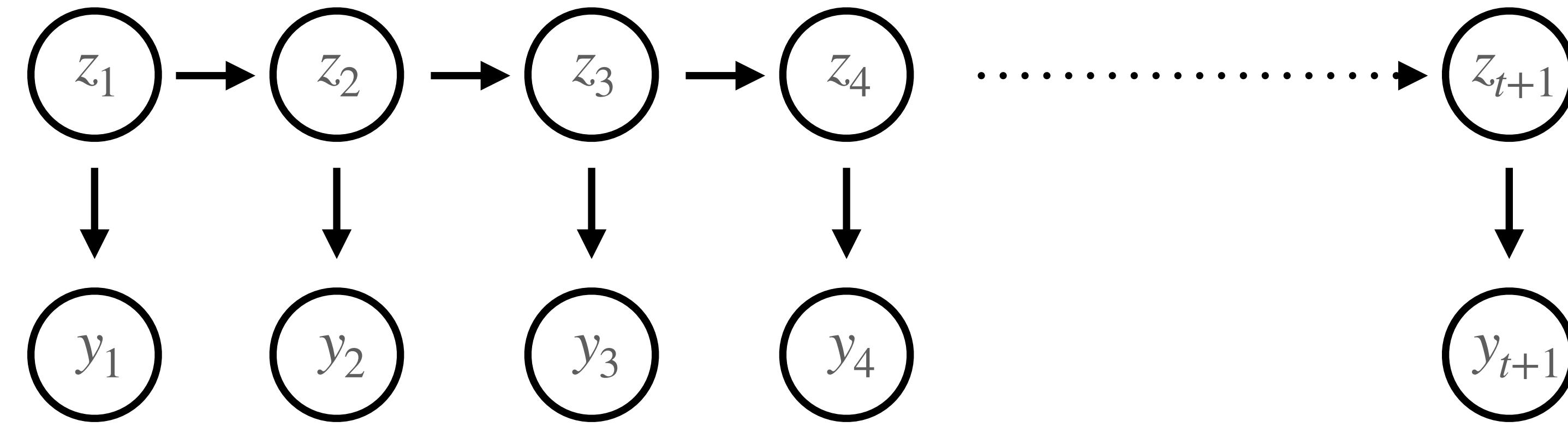
$$\mathcal{N}(z_t | \mathbf{F}_t z_{t-1} + \mathbf{B}_t u_t + \mathbf{b}_t, \mathbf{Q}_t)$$

$$\text{Poisson}(y_t | f(\mathbf{H}_t z_t + \mathbf{D}_t u_t + d_t))$$

# Switching Linear Dynamical Systems

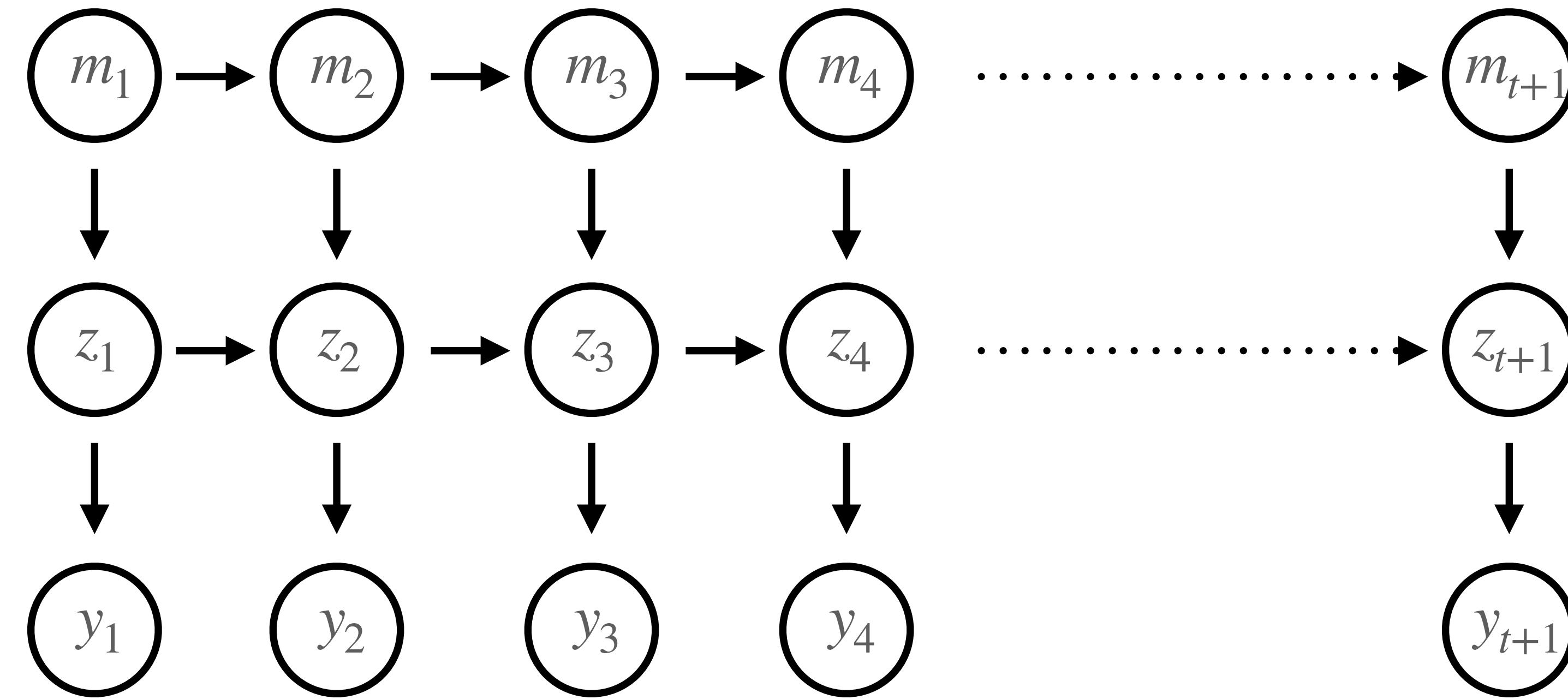
Continuous Latents

Observations  
(Neural Data)



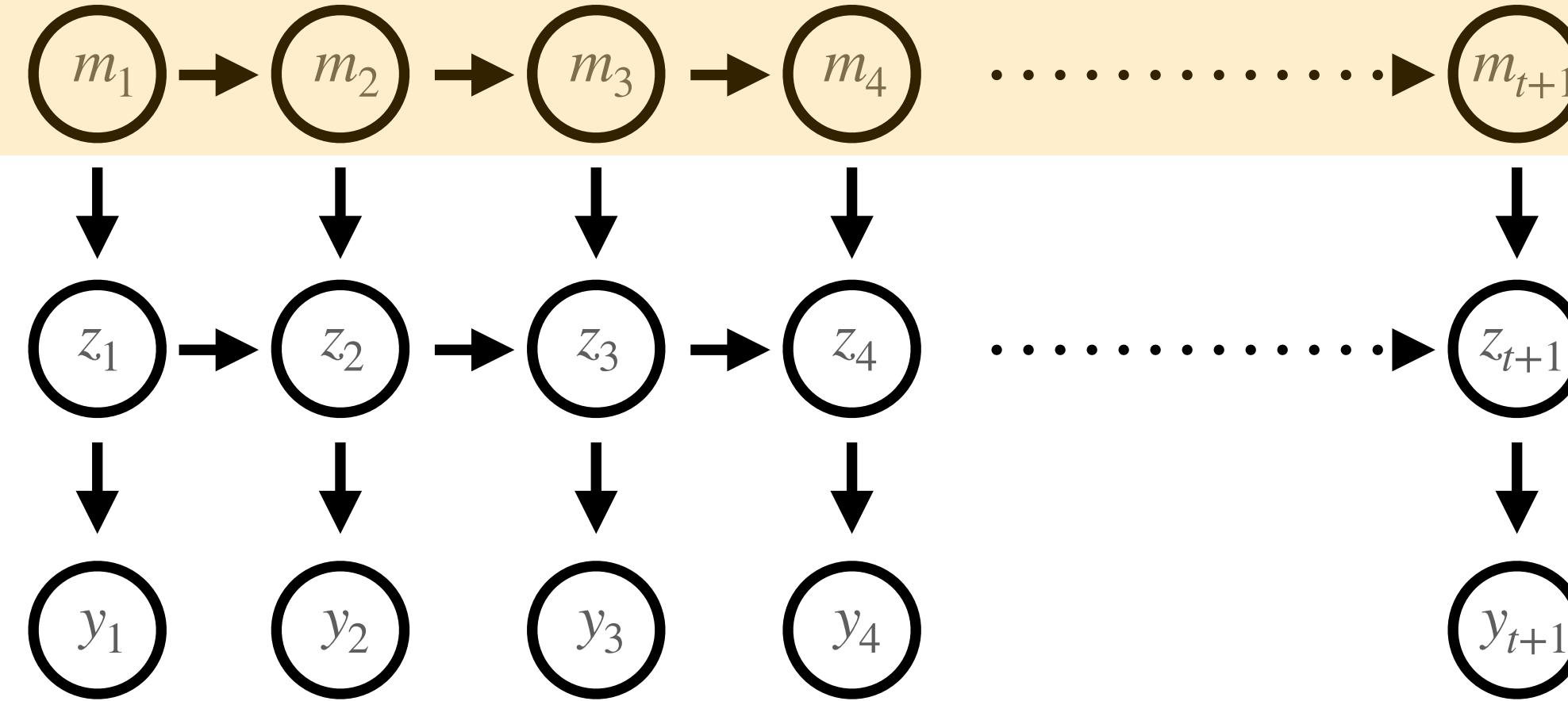
# Switching Linear Dynamical Systems

Discrete Latents  
Continuous Latents  
Observations  
(Neural Data)



# Switching Linear Dynamical Systems

Discrete Latents



Continuous Latents

Observations  
(Neural Data)

Discrete Latents

$$p(m_t = k | m_{t-1} = j) = A_{jk}$$

Continuous Latents

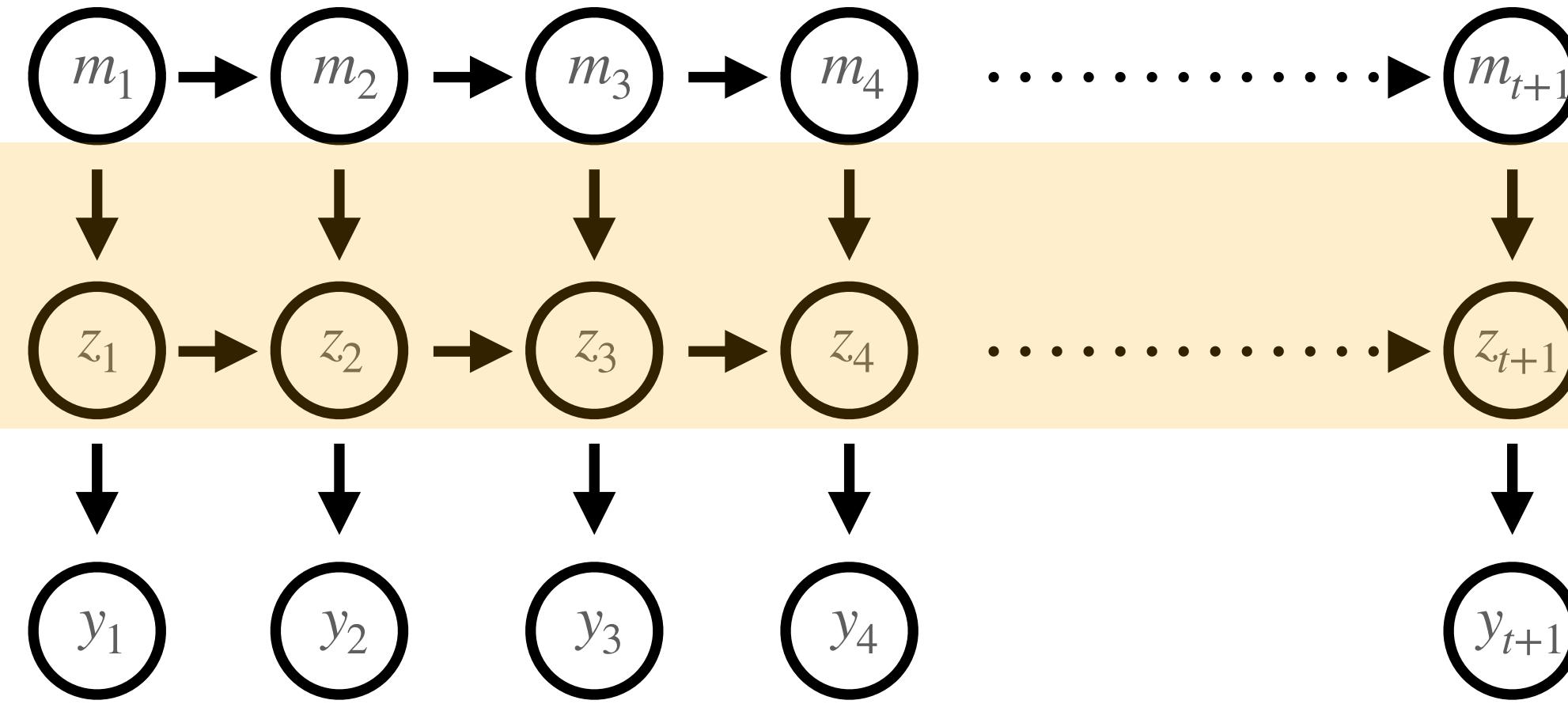
$$p(\mathbf{z}_t | \mathbf{z}_{t-1}, m_t = k, \mathbf{u}_t) = \mathcal{N}(\mathbf{z}_t | \mathbf{F}_k \mathbf{z}_{t-1} + \mathbf{B}_k \mathbf{u}_t + \mathbf{b}_k, \mathbf{Q}_k)$$

Observations  
(Neural Data)

$$p(\mathbf{y}_t | \mathbf{z}_t, m_t = k, \mathbf{u}_t) = \mathcal{N}(\mathbf{y}_t | \mathbf{H}_k \mathbf{z}_t + \mathbf{D}_k \mathbf{u}_t + \mathbf{d}_k, \mathbf{R}_k)$$

# Switching Linear Dynamical Systems

Discrete Latents



Continuous Latents

Observations  
(Neural Data)

Discrete Latents

$$p(m_t = k | m_{t-1} = j) = A_{jk}$$

Continuous Latents

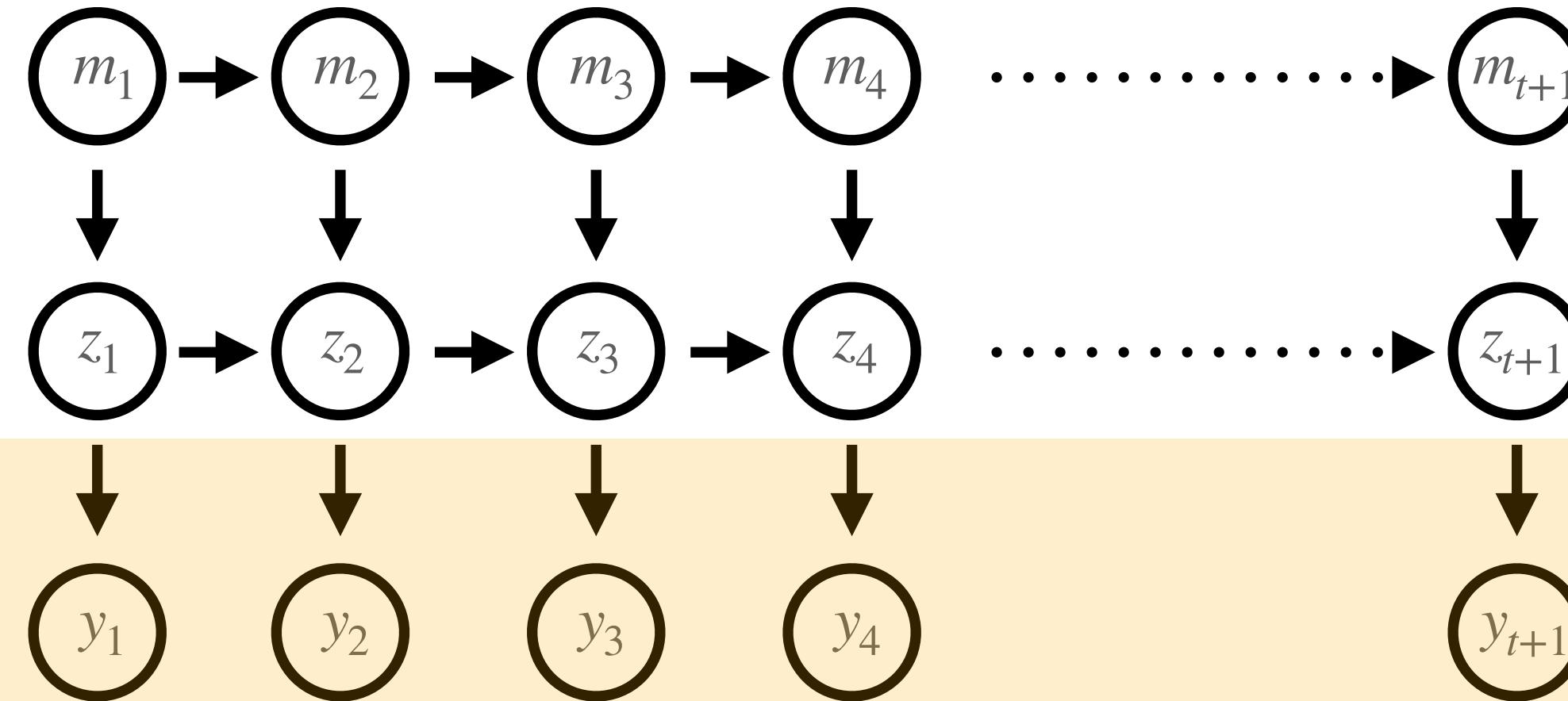
$$p(z_t | z_{t-1}, m_t = k, u_t) = \mathcal{N}(z_t | \mathbf{F}_k z_{t-1} + \mathbf{B}_k u_t + \mathbf{b}_k, \mathbf{Q}_k)$$

Observations  
(Neural Data)

$$p(y_t | z_t, m_t = k, u_t) = \mathcal{N}(y_t | \mathbf{H}_k z_t + \mathbf{D}_k u_t + \mathbf{d}_k, \mathbf{R}_k)$$

# Switching Linear Dynamical Systems

Discrete Latents



Continuous Latents

Observations  
(Neural Data)

Discrete Latents

$$p(m_t = k | m_{t-1} = j) = A_{jk}$$

Continuous Latents

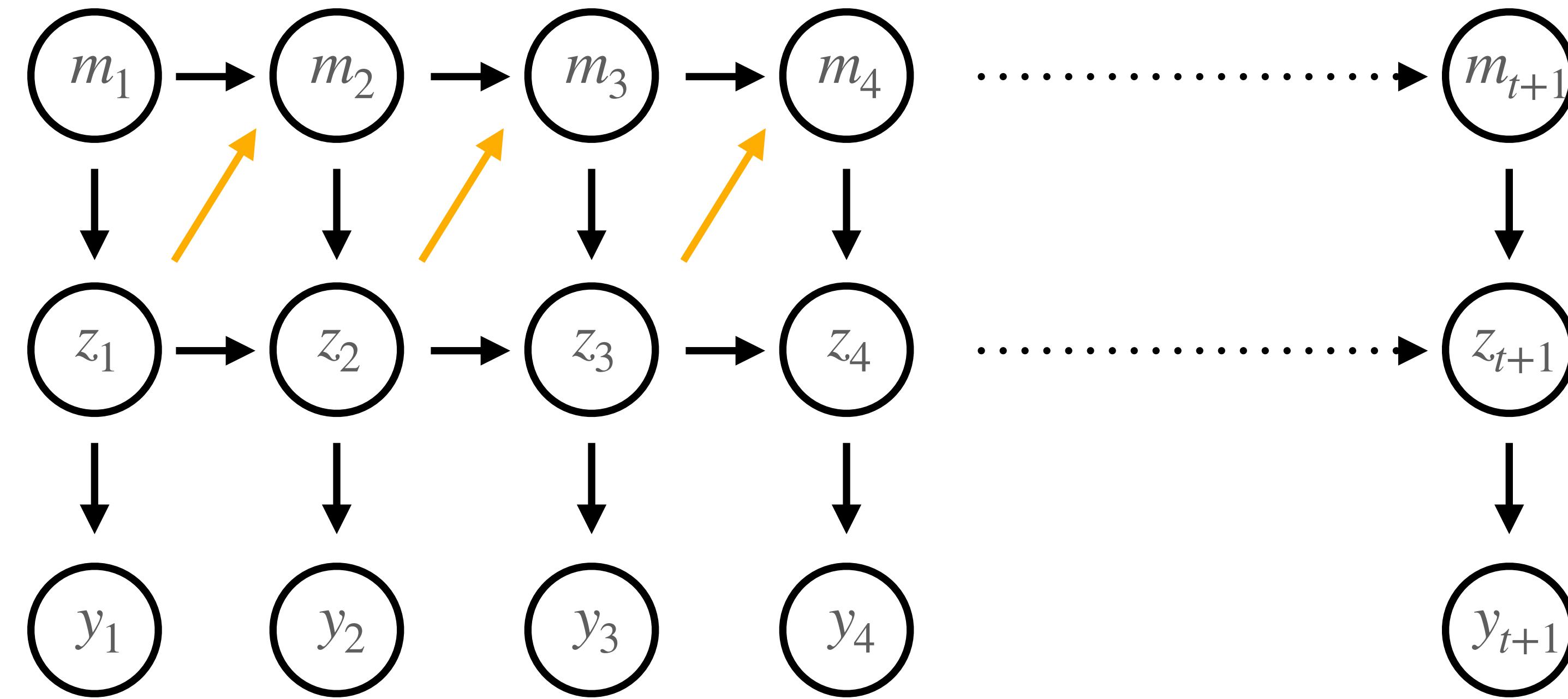
$$p(\mathbf{z}_t | \mathbf{z}_{t-1}, m_t = k, \mathbf{u}_t) = \mathcal{N}(\mathbf{z}_t | \mathbf{F}_k \mathbf{z}_{t-1} + \mathbf{B}_k \mathbf{u}_t + \mathbf{b}_k, \mathbf{Q}_k)$$

Observations  
(Neural Data)

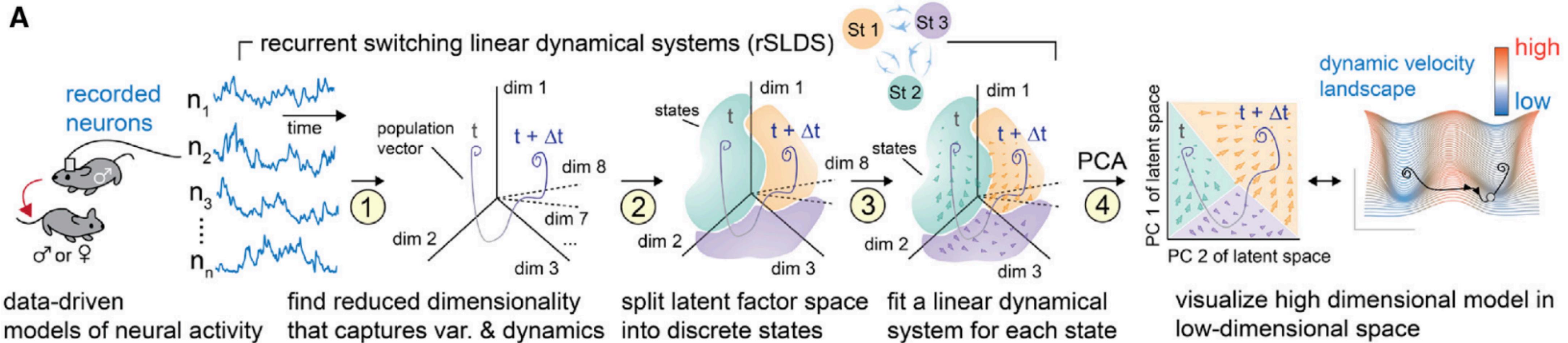
$$p(\mathbf{y}_t | \mathbf{z}_t, m_t = k, \mathbf{u}_t) = \mathcal{N}(\mathbf{y}_t | \mathbf{H}_k \mathbf{z}_t + \mathbf{D}_k \mathbf{u}_t + \mathbf{d}_k, \mathbf{R}_k)$$

# Recurrent Switching Linear Dynamical Systems

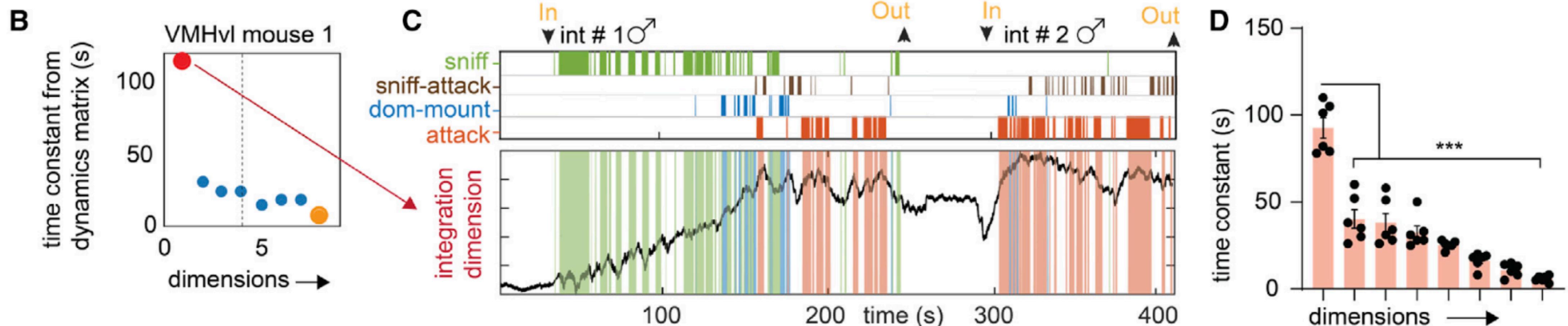
Discrete Latents  
Continuous Latents  
Observations  
(Neural Data)



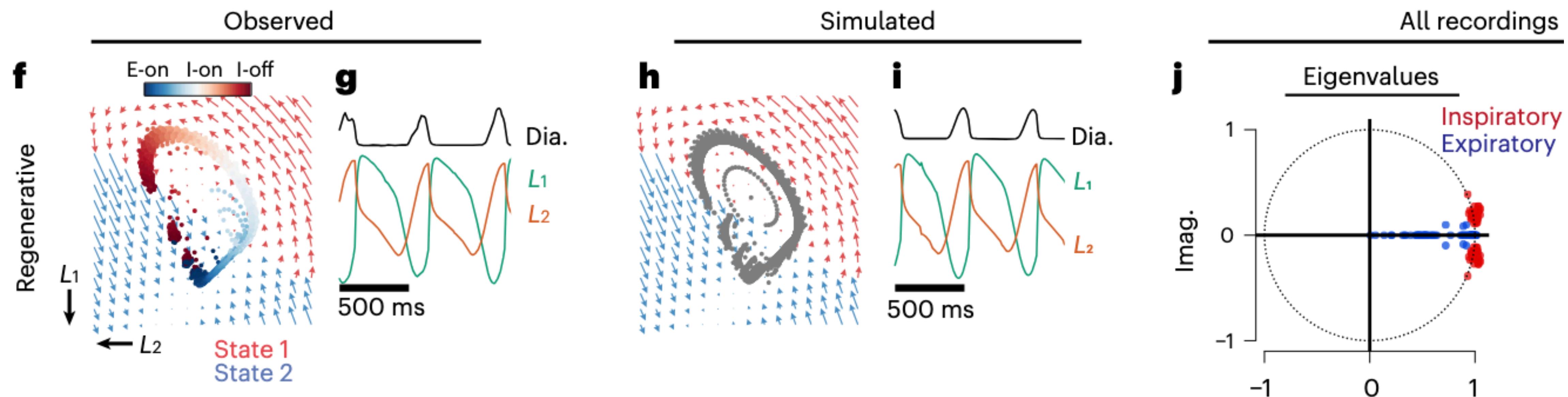
# An approximate line attractor in the hypothalamus encodes an aggressive state



VMHvl activity exhibits an integration dimension with ramping activity and persistence during interactions with males



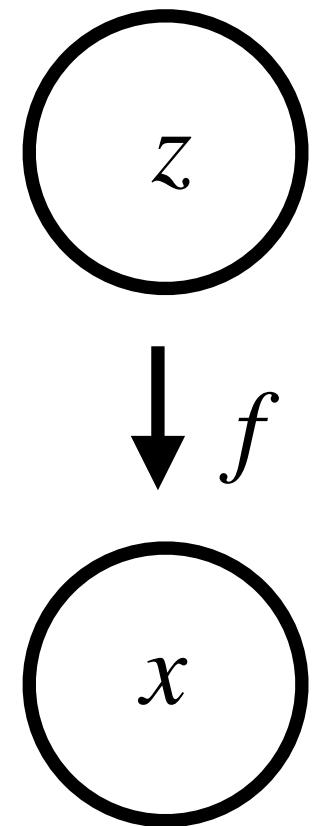
# Latent neural population dynamics underlying breathing, opioid-induced respiratory depression and gasping



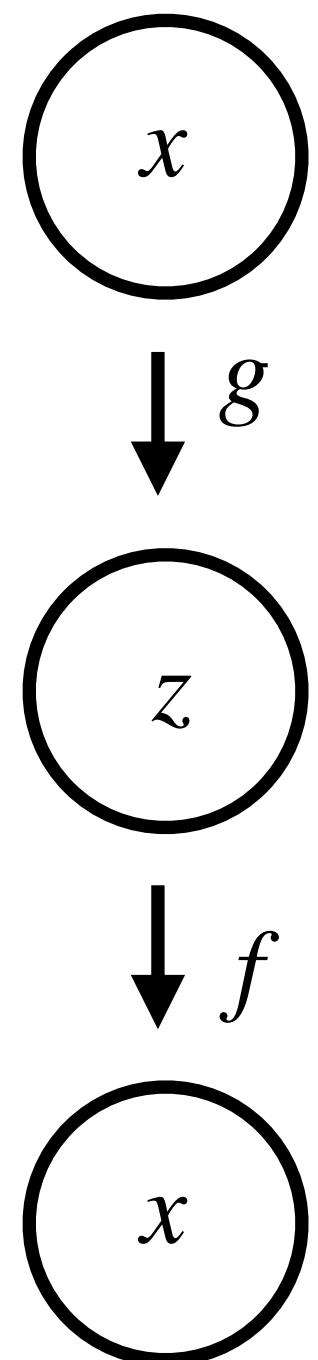
# **Variational Autoencoder (VAE)**

# Variational Autoencoder (VAE)

- Classically in latent variable models, we directly optimize  $\mathbf{z}$ 
  - E.g., let's say  $\mathbf{z}$  is a 1d Gaussian and  $\mathbf{x}$  has  $T$  time points, then we would optimize a mean and stdev. for all  $T$  time points.



# Variational Autoencoder (VAE)



- Classically in latent variable models, we directly optimize  $\mathbf{z}$ 
  - E.g., let's say  $\mathbf{z}$  is a 1d Gaussian and  $\mathbf{x}$  has  $T$  time points, then we would optimize a mean and stdev. for all  $T$  time points.
- VAEs use “amortized inference”, where  $\mathbf{z}$  is learned as a function of  $\mathbf{x}$ , so only the parameters of  $g$  (neural network parameters) are learned
  - In the above example, there would be separate neural networks trained to predict  $\mathbf{z}$ 's means and stdevs:  $\mu_t = g_\mu(x_t)$  and  $\sigma_t = g_\sigma(x_t)$

# VAEs vs. Standard Autoencoders

- Often perform similarly

# VAEs vs. Standard Autoencoders

- Often perform similarly
- VAEs are better generative models

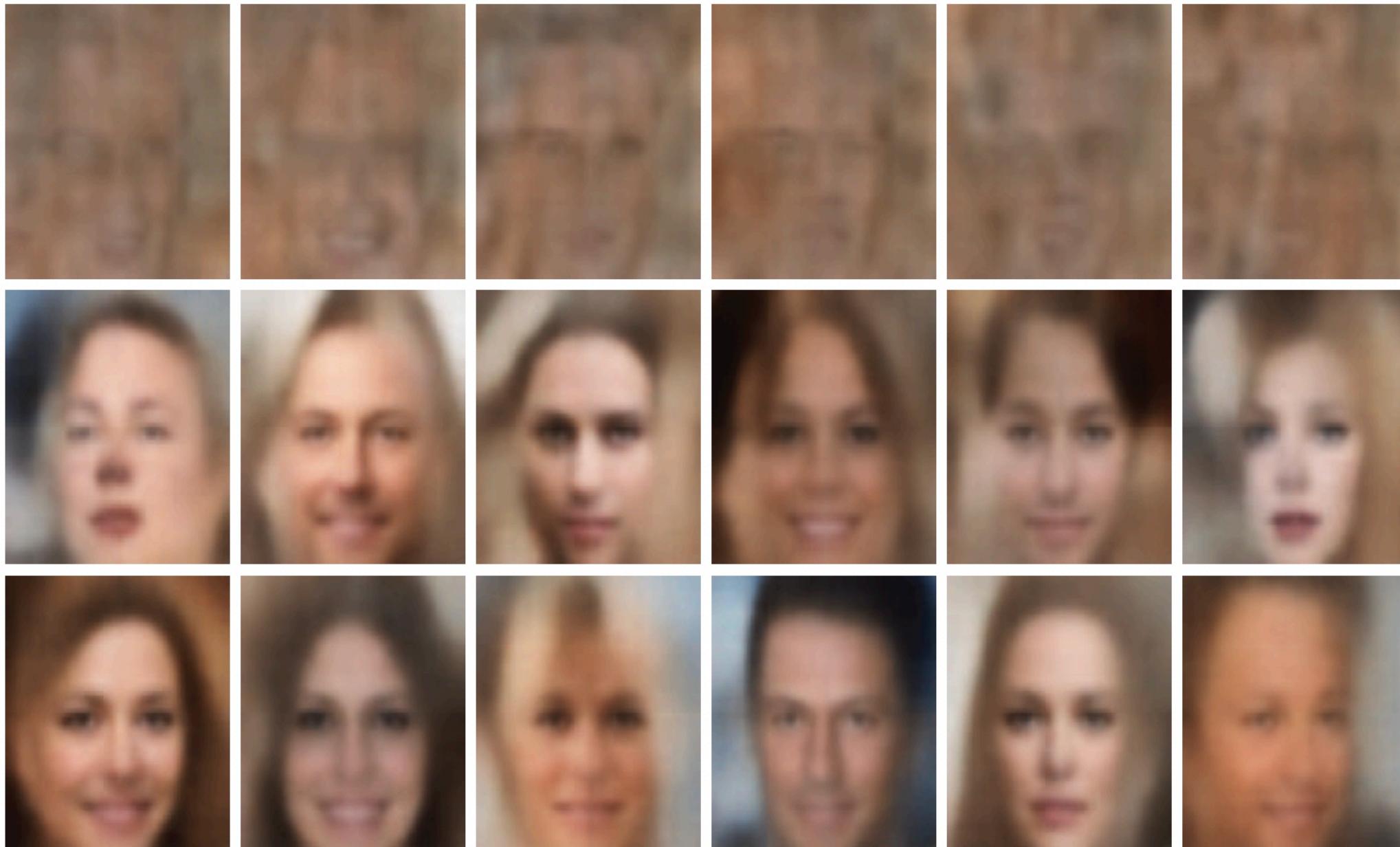
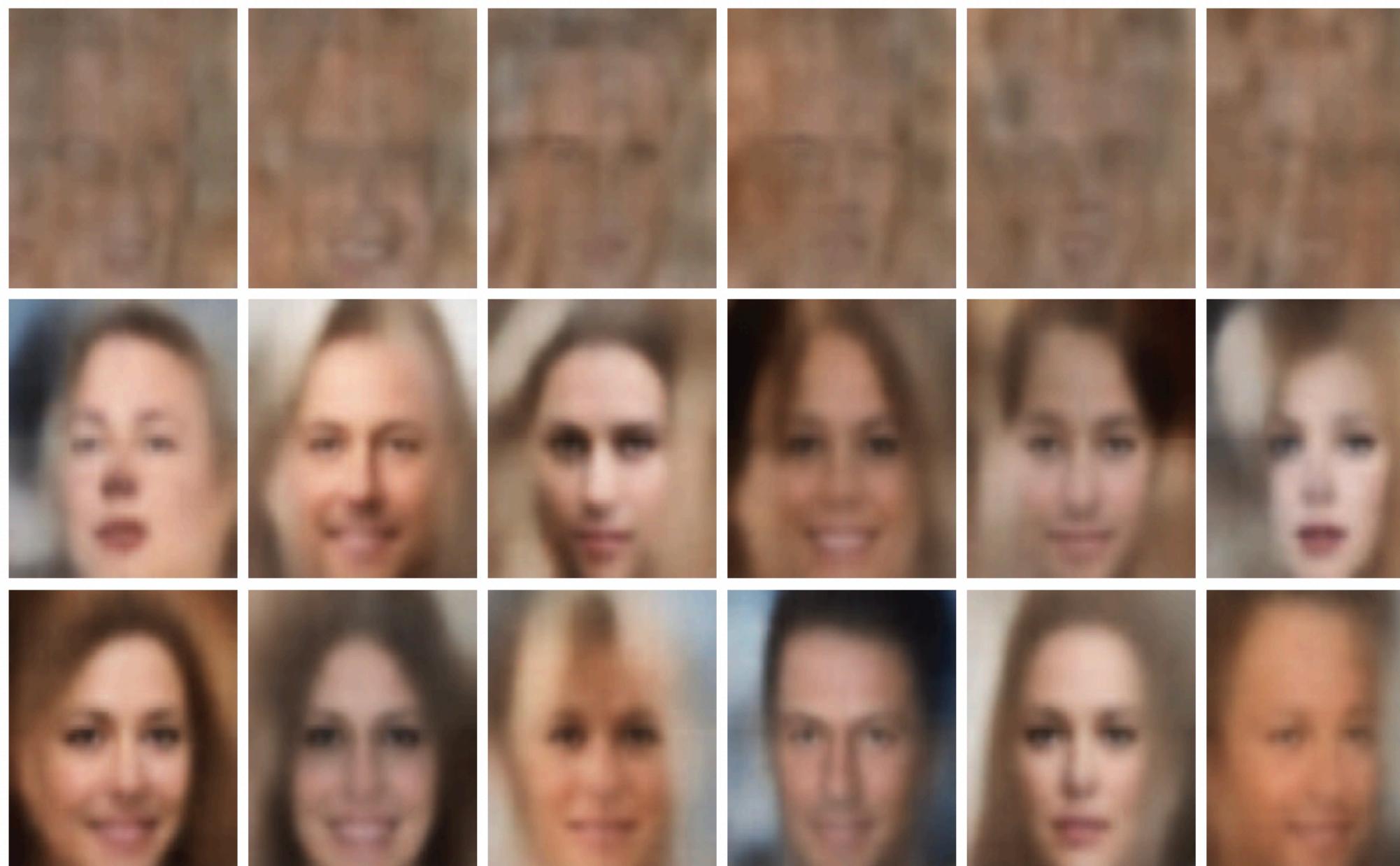


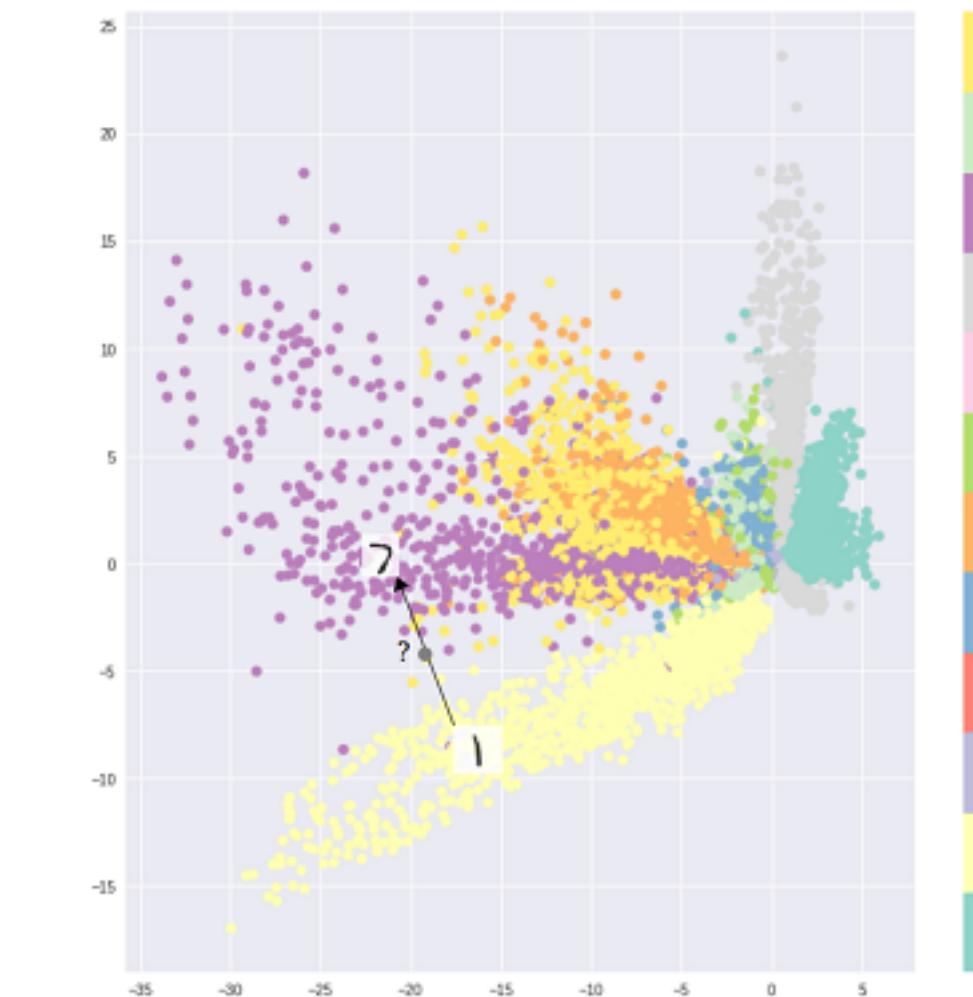
Figure 21.2: Illustration of unconditional image generation using (V)AEs trained on CelebA. Row 1: deterministic autoencoder. Row 2:  $\beta$ -VAE with  $\beta = 0.5$ . Row 3: VAE (with  $\beta = 1$ ). Generated by [celeba\\_vae ae comparison.ipynb](#).

# VAEs vs. Standard Autoencoders

- Often perform similarly
- VAEs are better generative models



AE Latents on MNIST



VAE Latents on MNIST

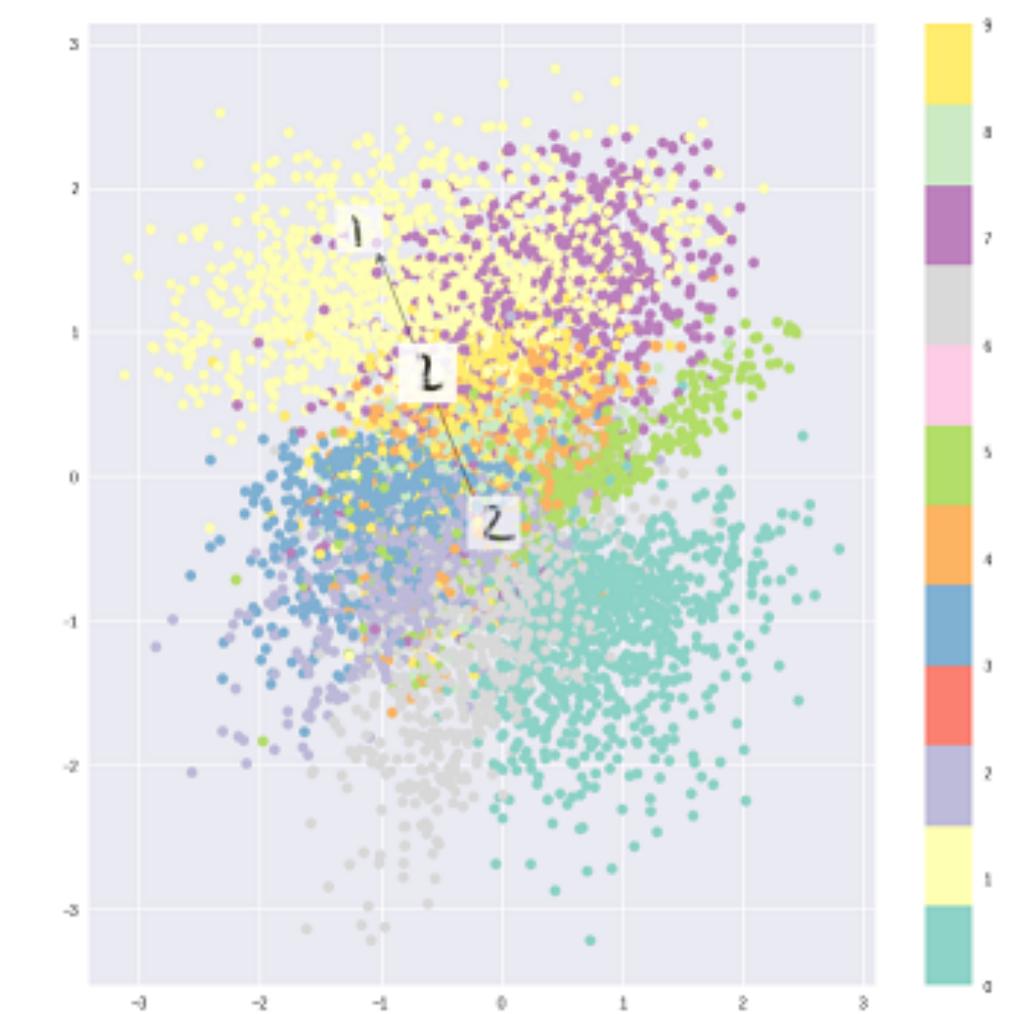


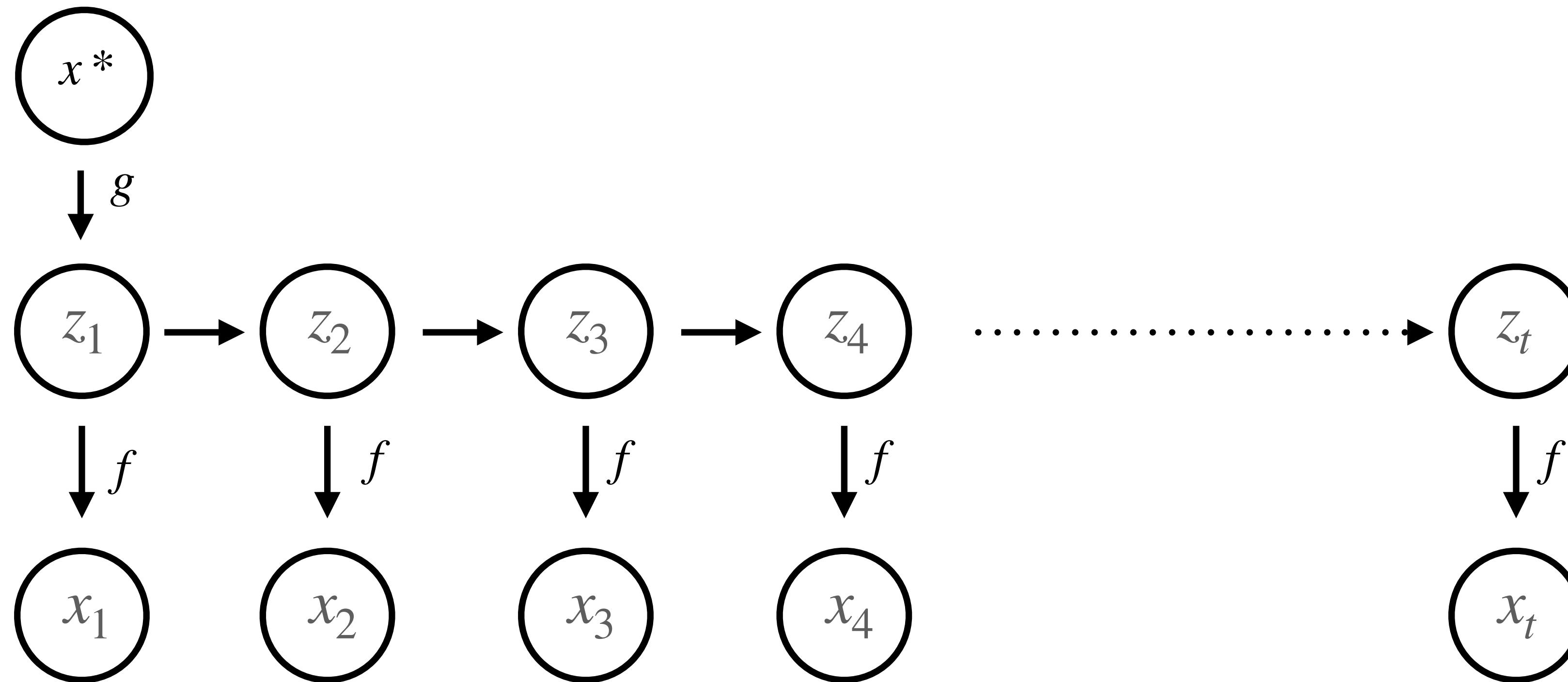
Figure 21.2: Illustration of unconditional image generation using (V)AEs trained on CelebA. Row 1: deterministic autoencoder. Row 2:  $\beta$ -VAE with  $\beta = 0.5$ . Row 3: VAE (with  $\beta = 1$ ). Generated by celeba vae ae comparison.ipynb.

# VAEs vs. Standard Autoencoders

- Often perform similarly
- VAEs are better generative models
- VAEs are flexible probabilistic models (can change prior, noise distributions, etc).

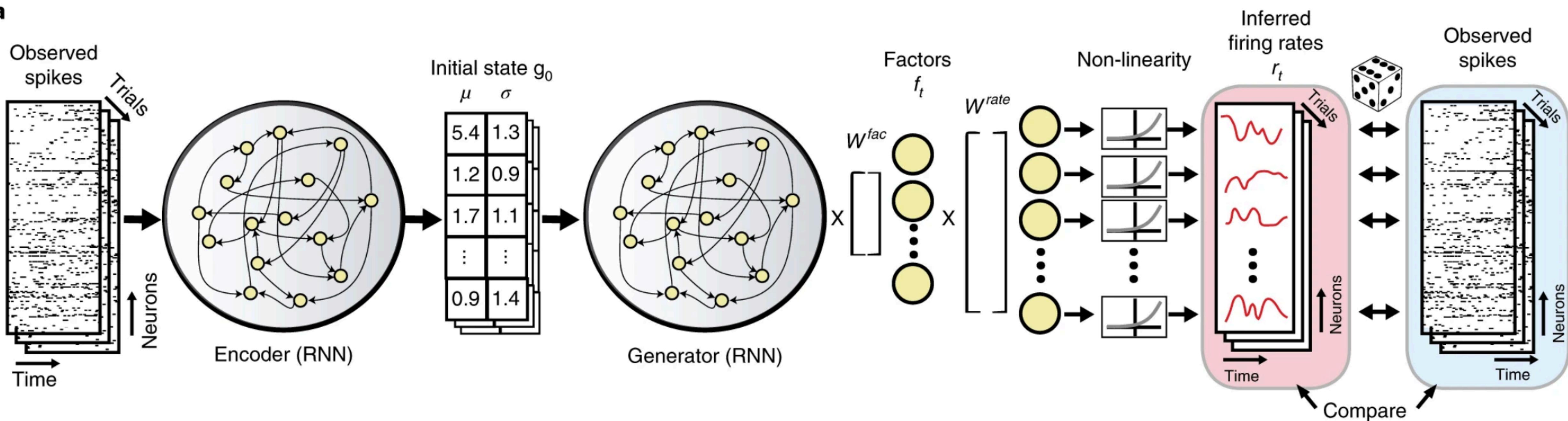
# Sequential Variational Autoencoder (sVAE)

- VAE where the latents have a recurrent structure (e.g. are an RNN)

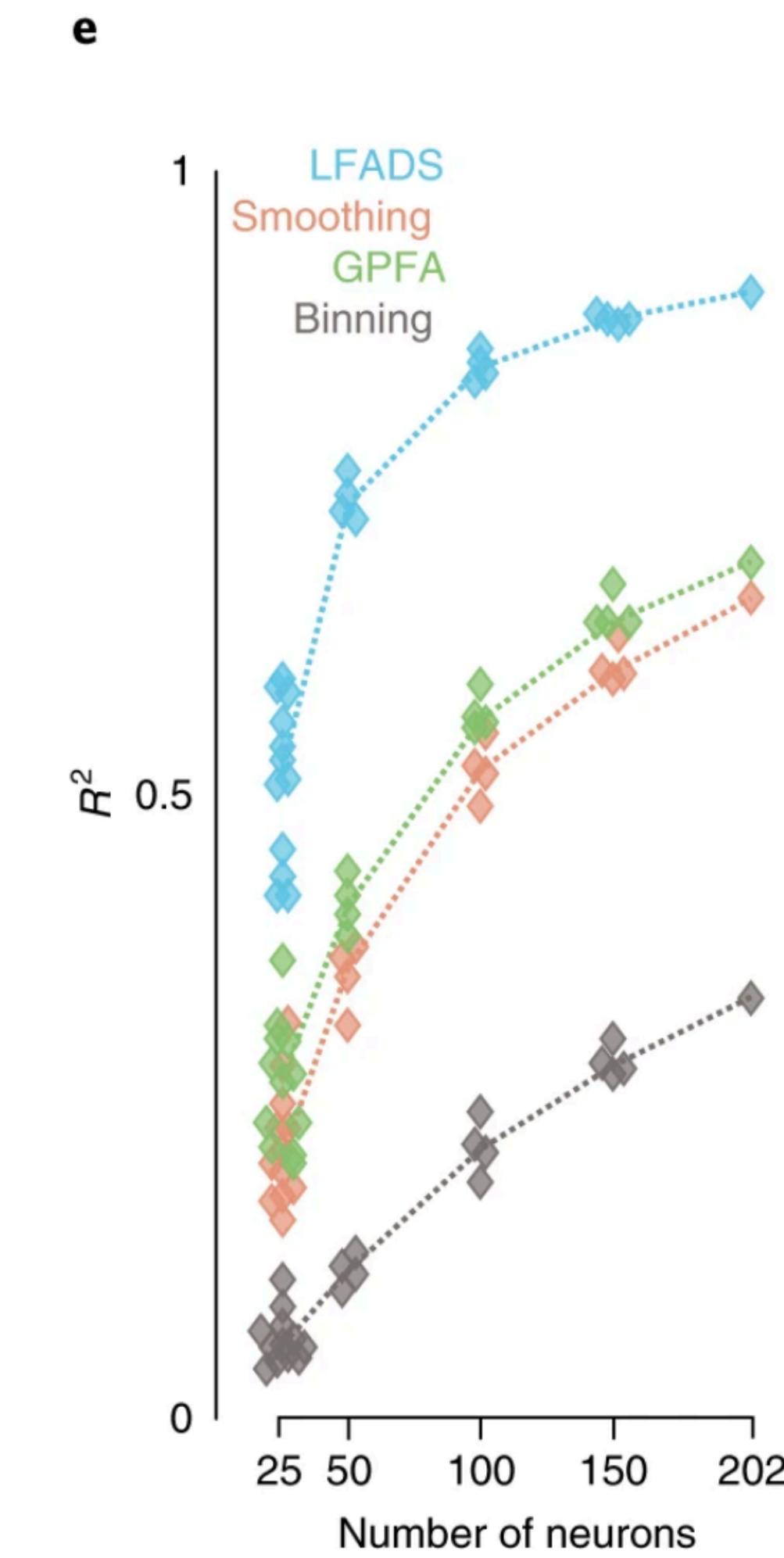
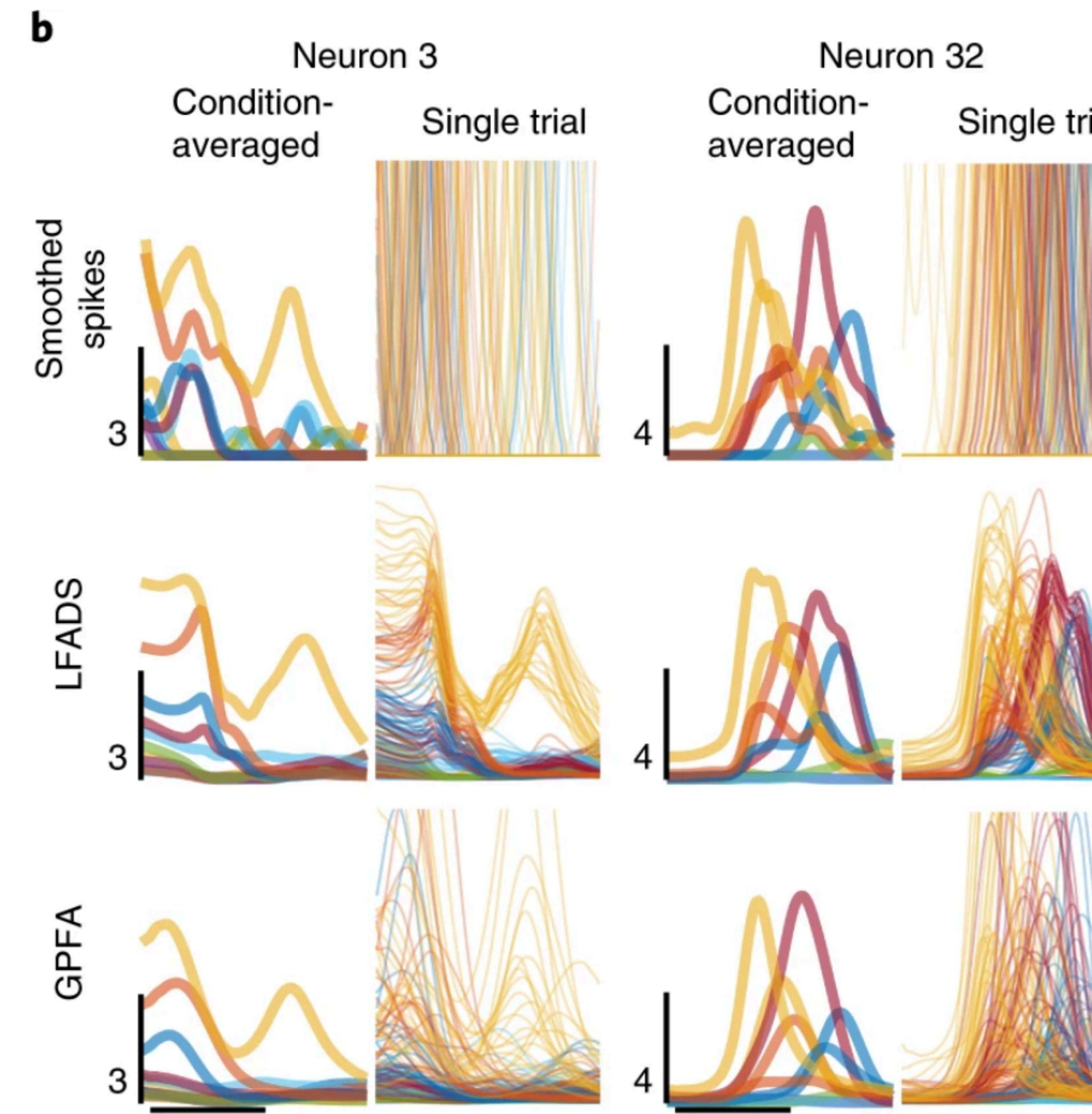


# LFADS

**a**



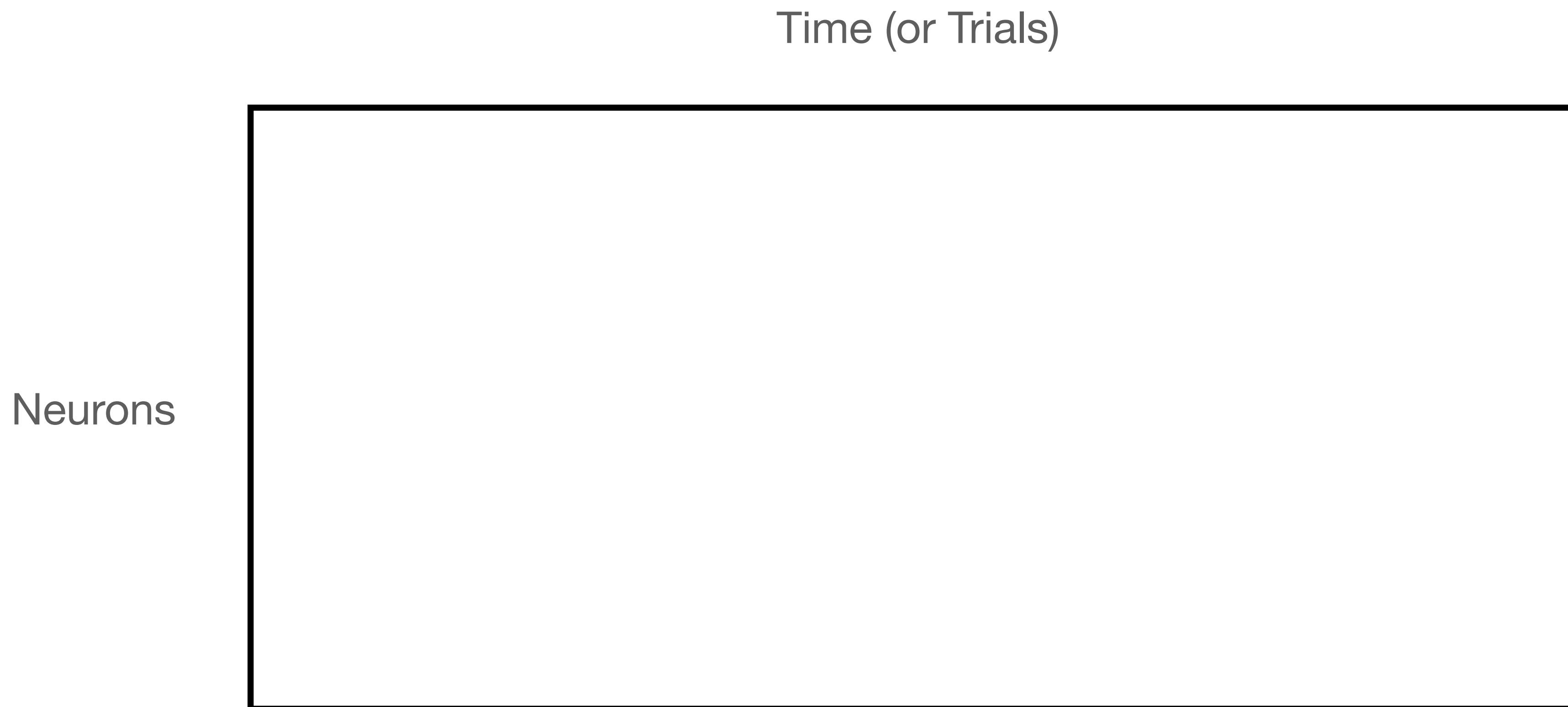
# LFADS



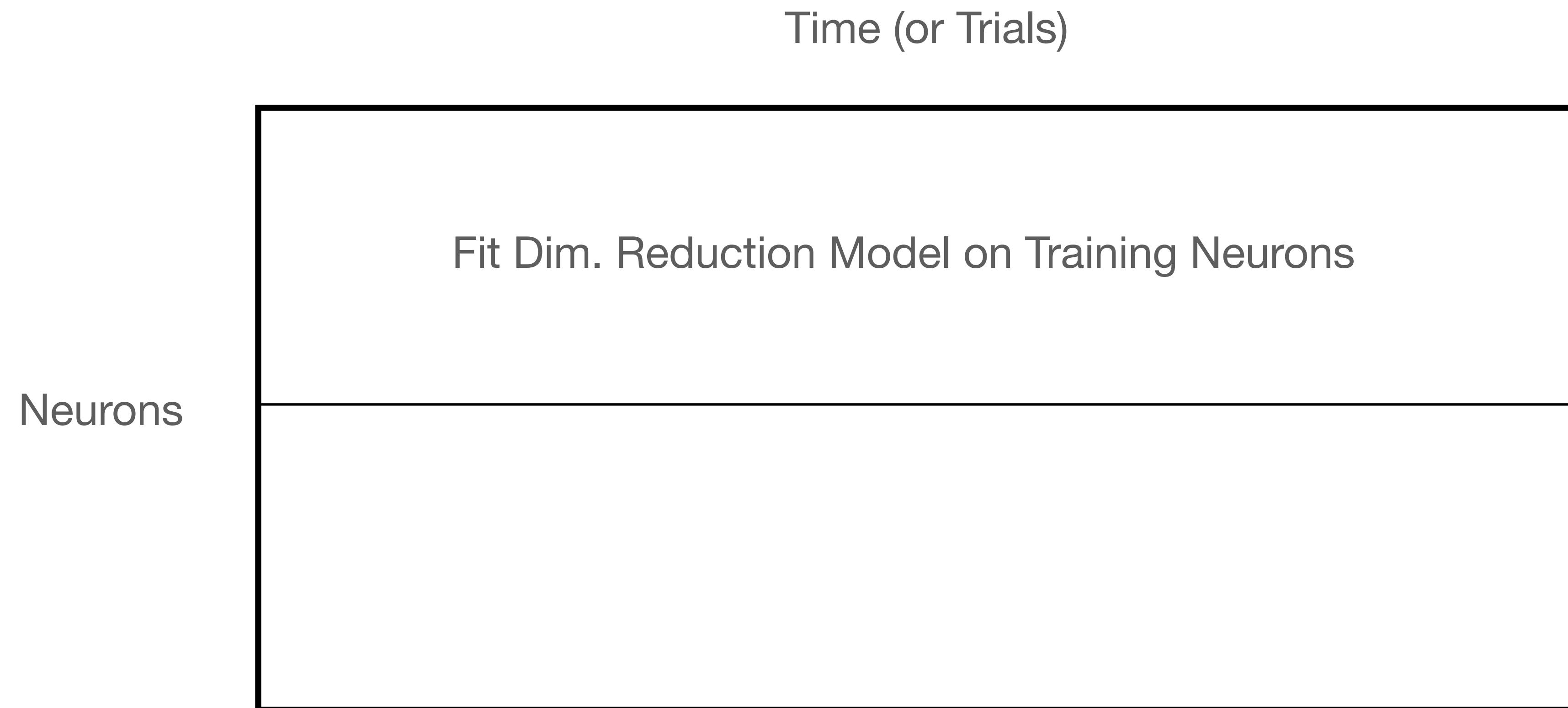
# Dimensionality in Linear and Nonlinear Models

- See *Notebook*

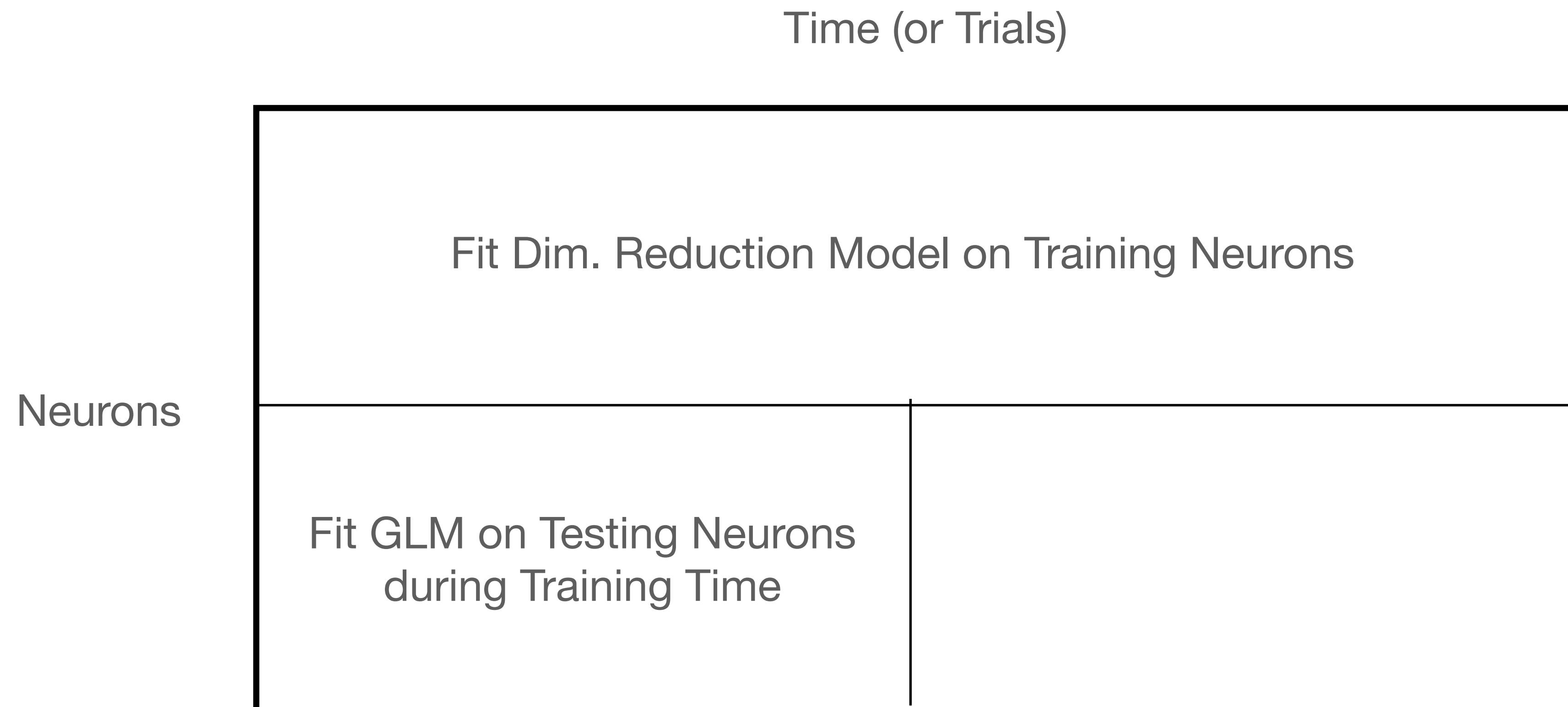
# Bi-cross-validation with Latent Variable Models + Dim. Reduction



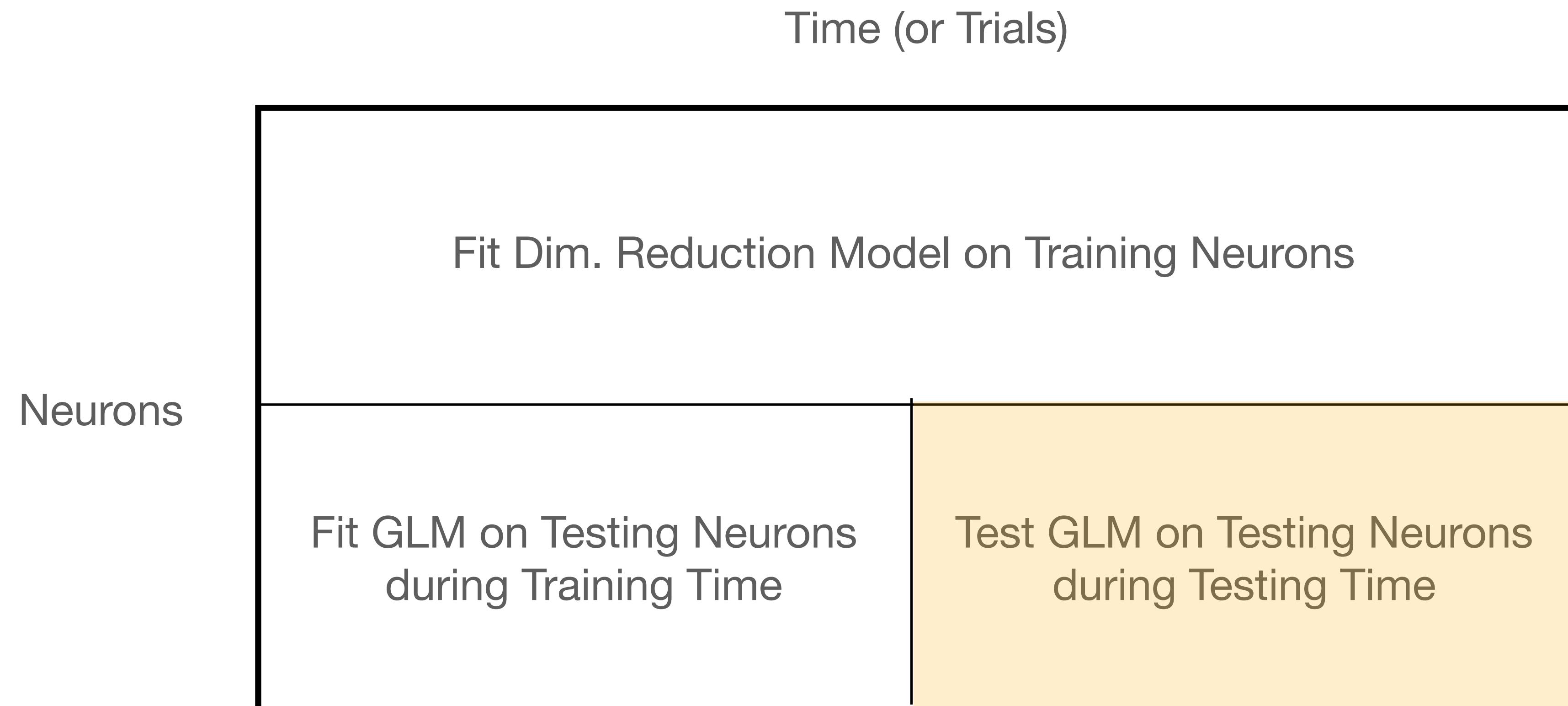
# Bi-cross-validation with Latent Variable Models + Dim. Reduction



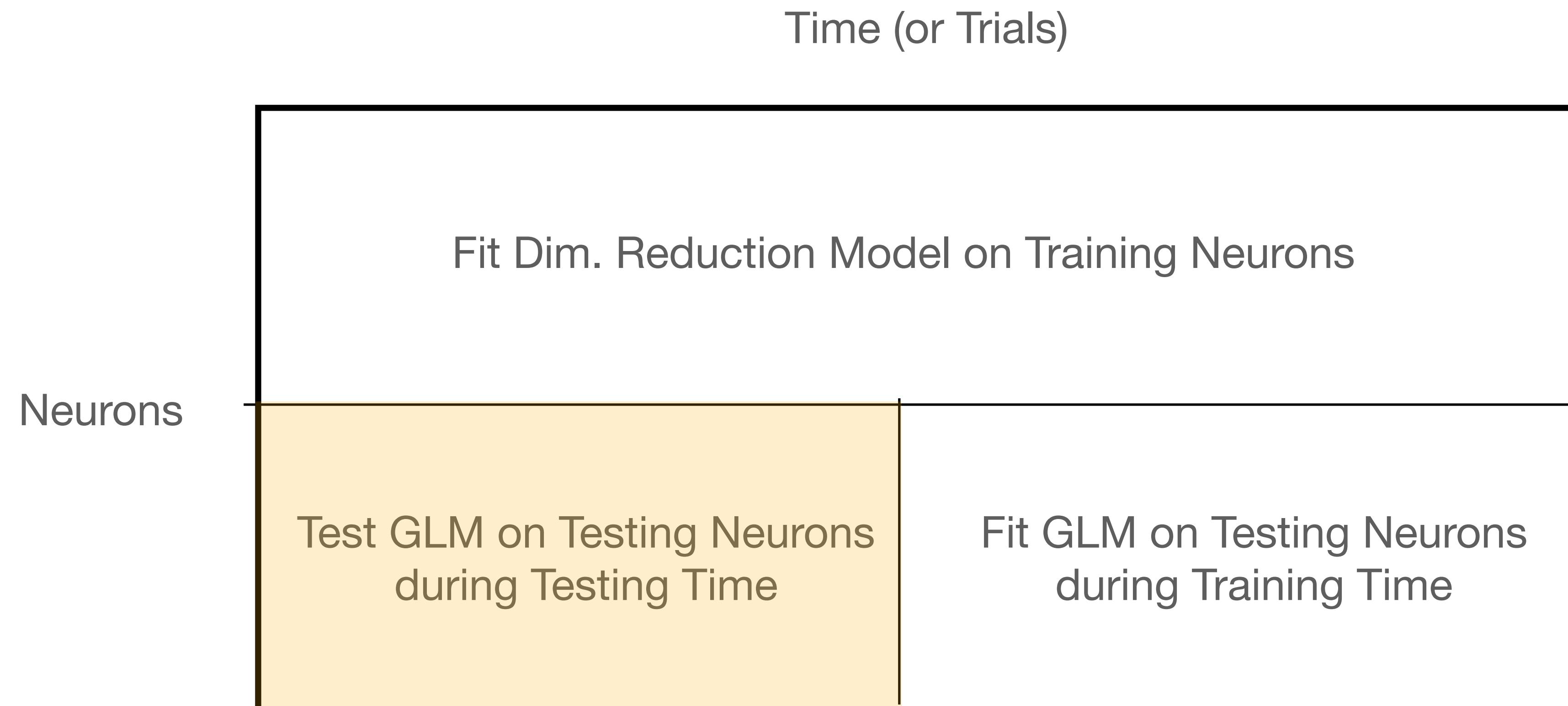
# Bi-cross-validation with Latent Variable Models + Dim. Reduction



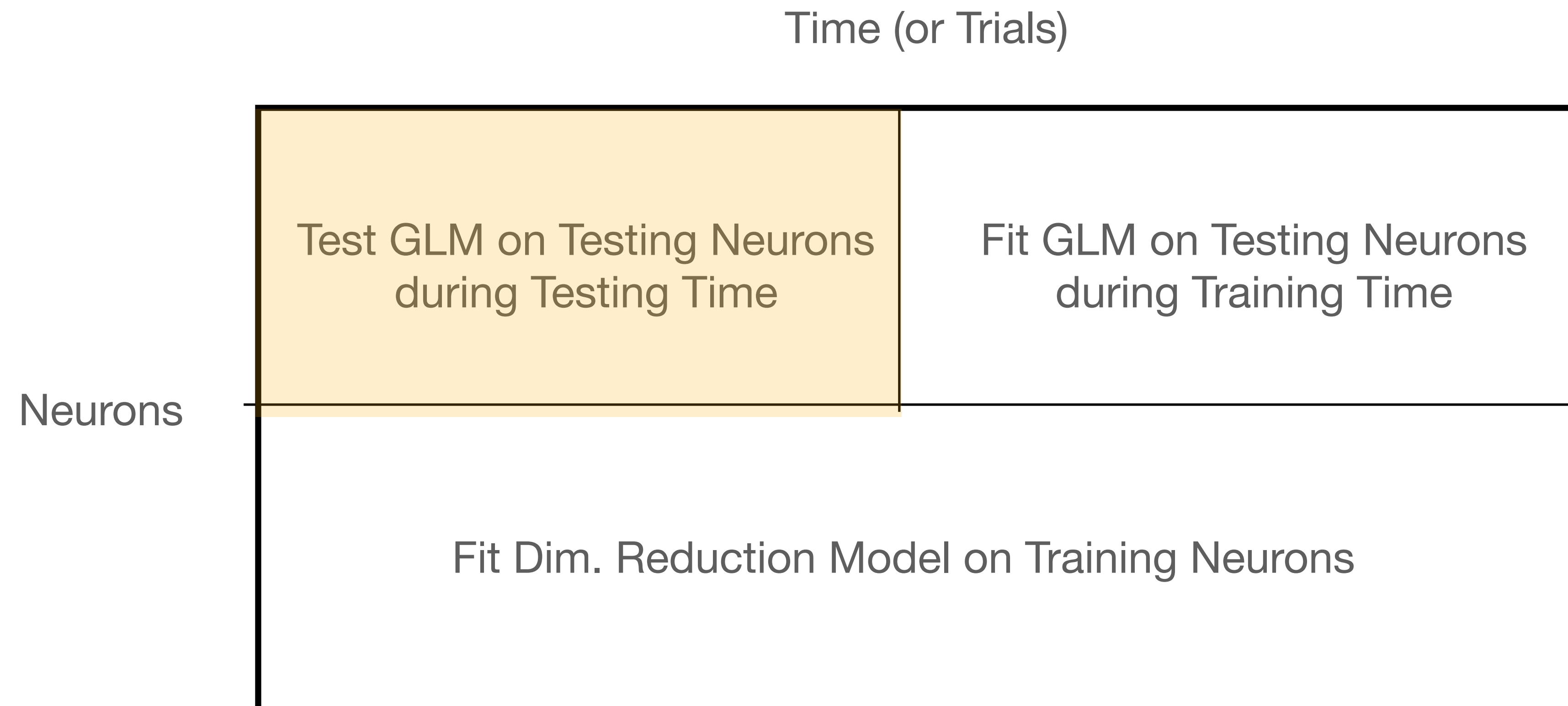
# Bi-cross-validation with Latent Variable Models + Dim. Reduction



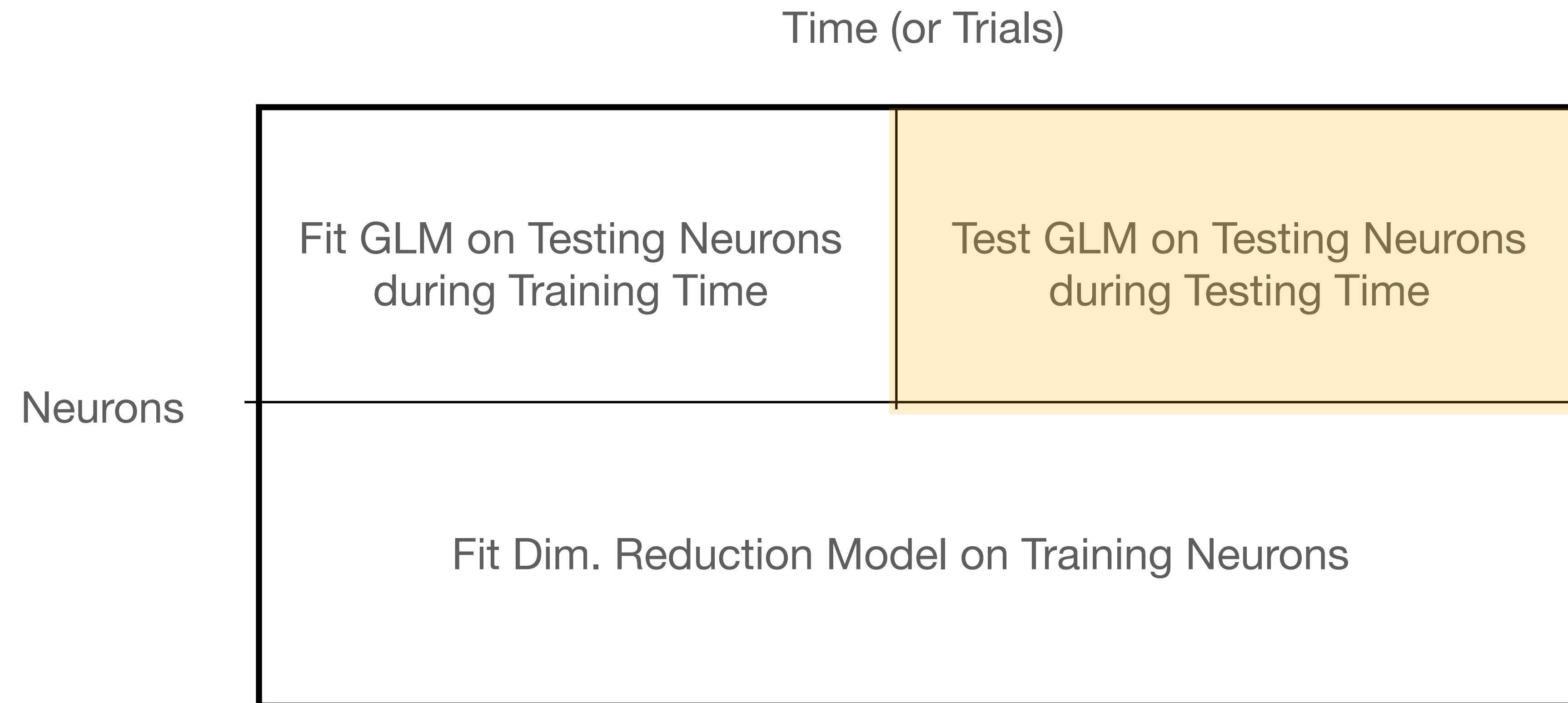
# Bi-cross-validation with Latent Variable Models + Dim. Reduction



# Bi-cross-validation with Latent Variable Models + Dim. Reduction

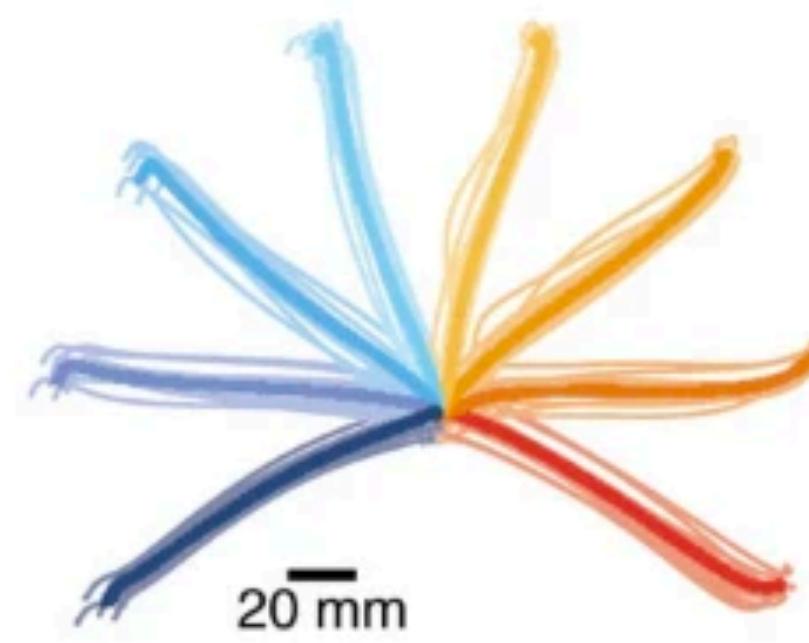
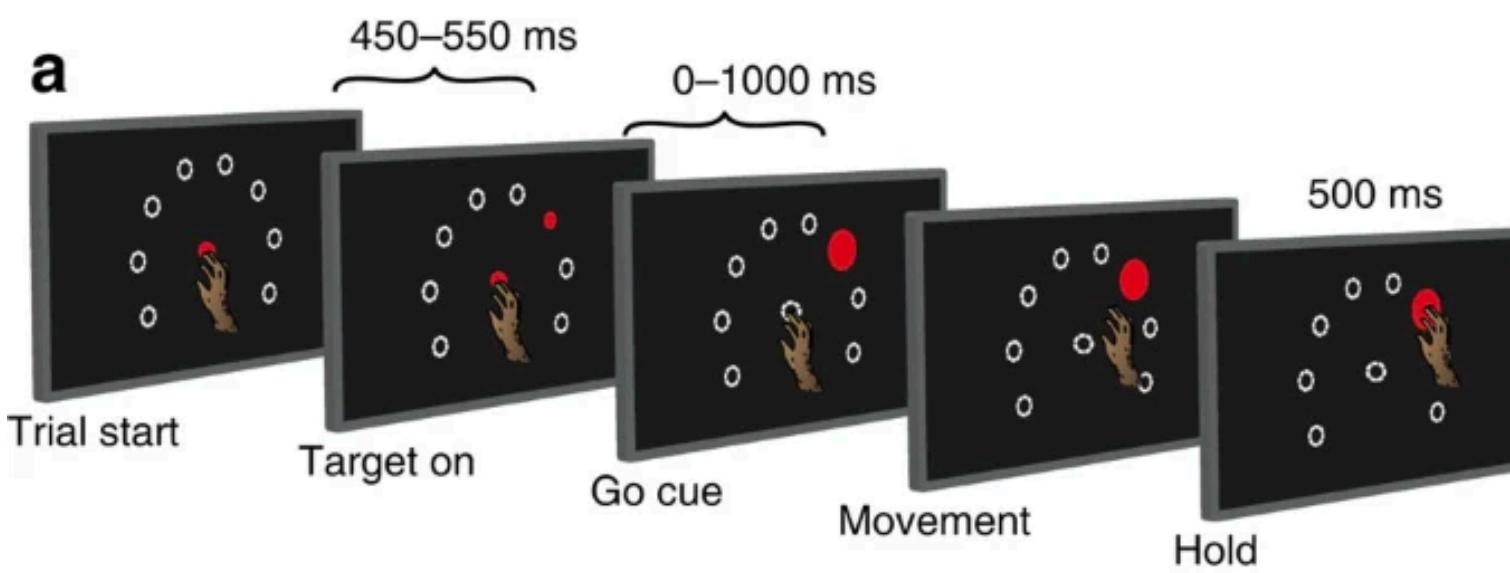


# Bi-cross-validation with Latent Variable Models + Dim. Reduction



# **Sparse Component Analysis**

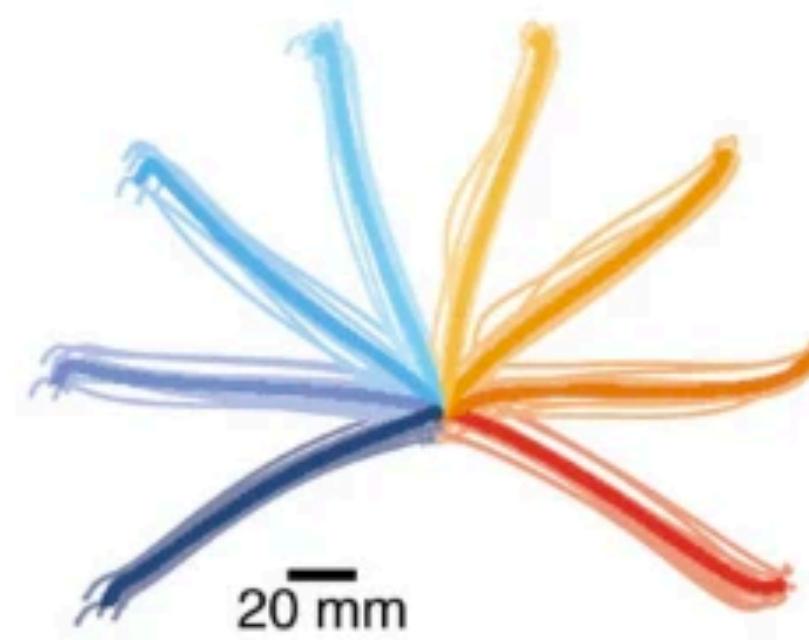
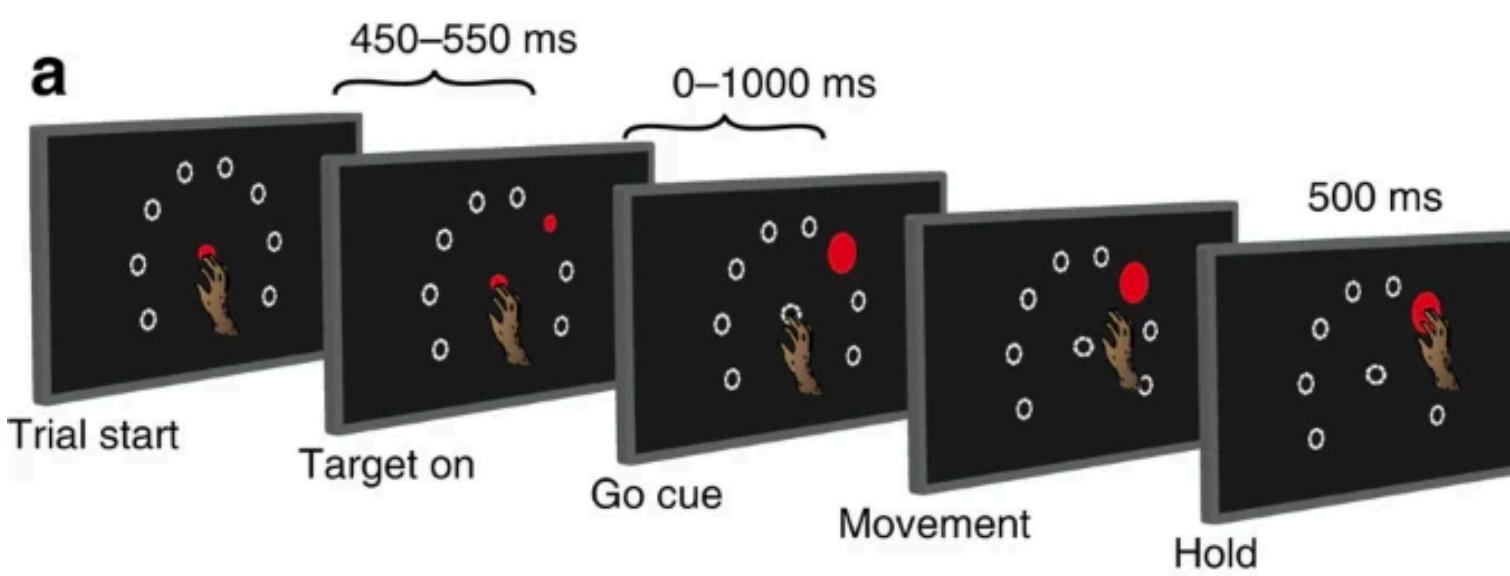
# How can we find interpretable structure when neurons' responses are so mixed?



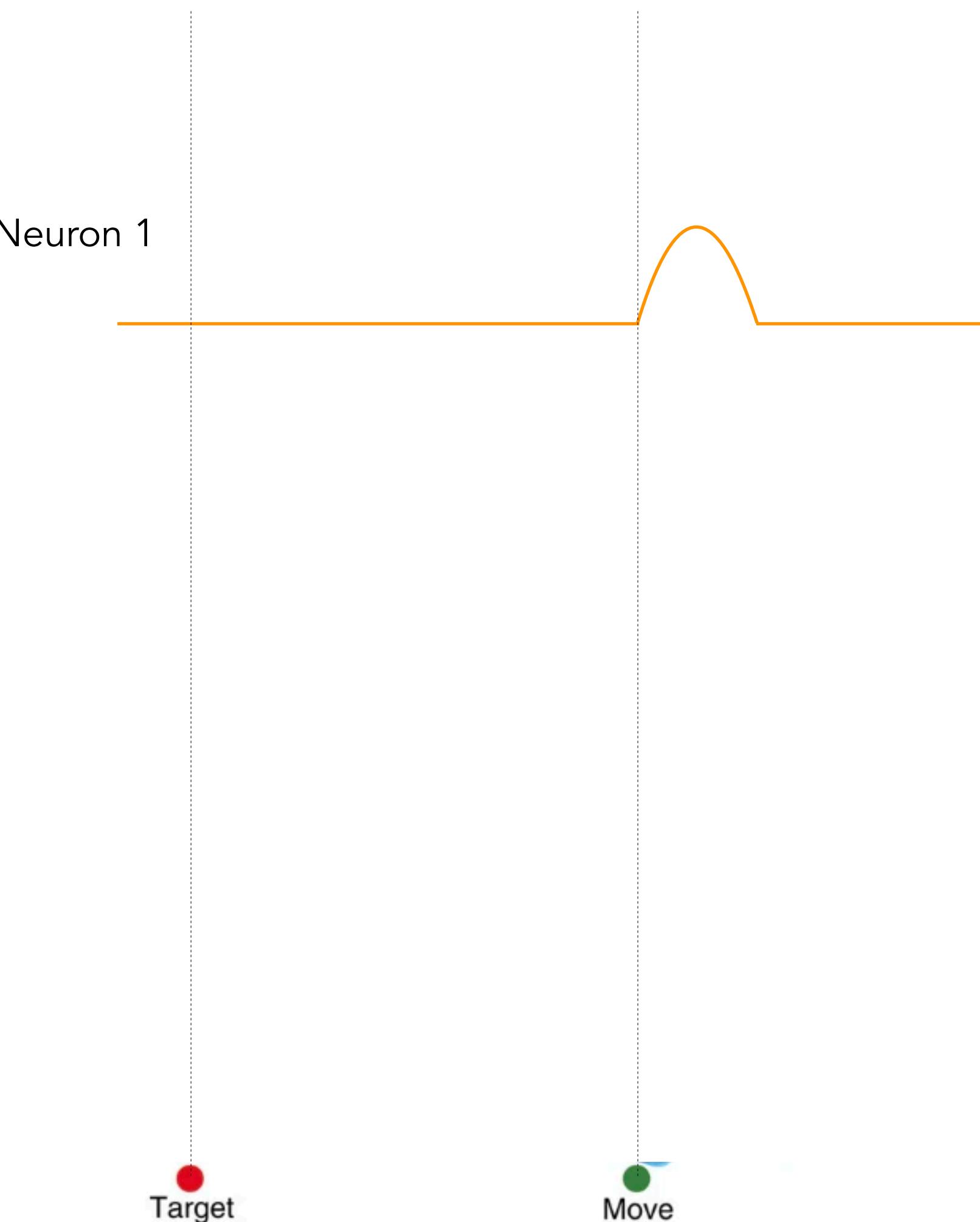
Elsayed et al 2016



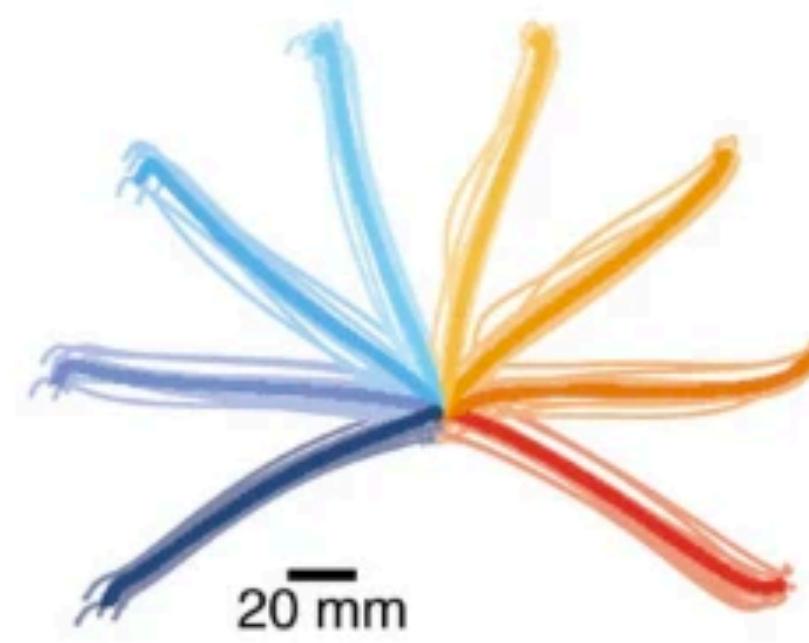
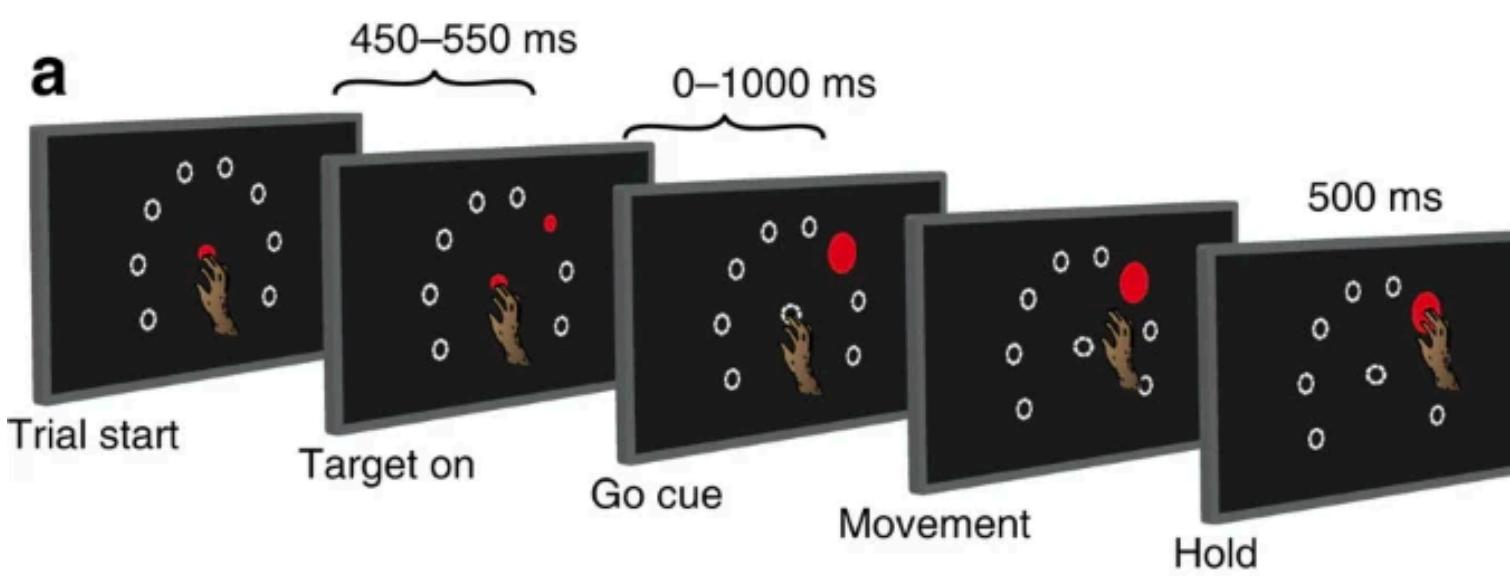
# How can we find interpretable structure when neurons' responses are so mixed?



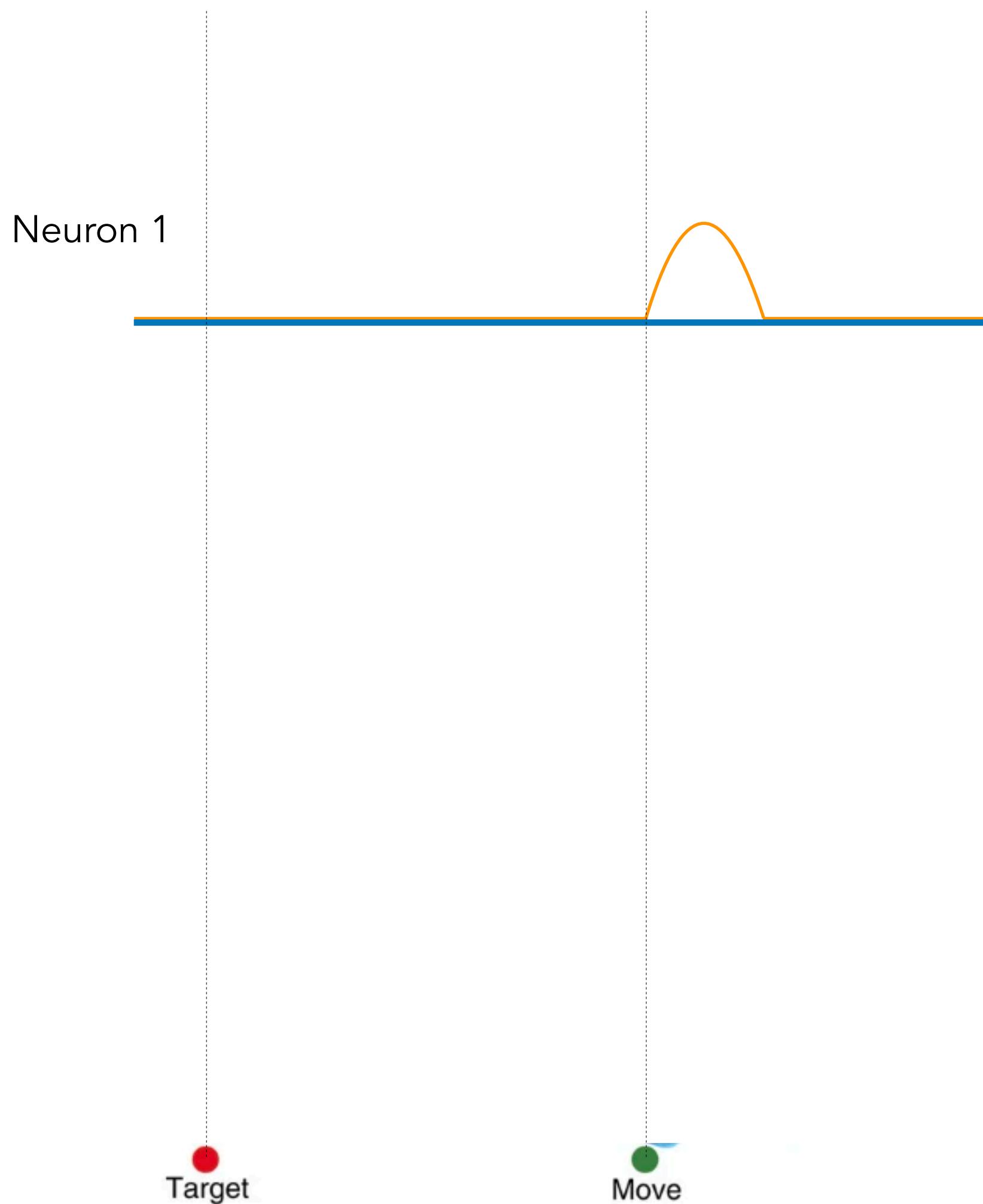
Elsayed et al 2016



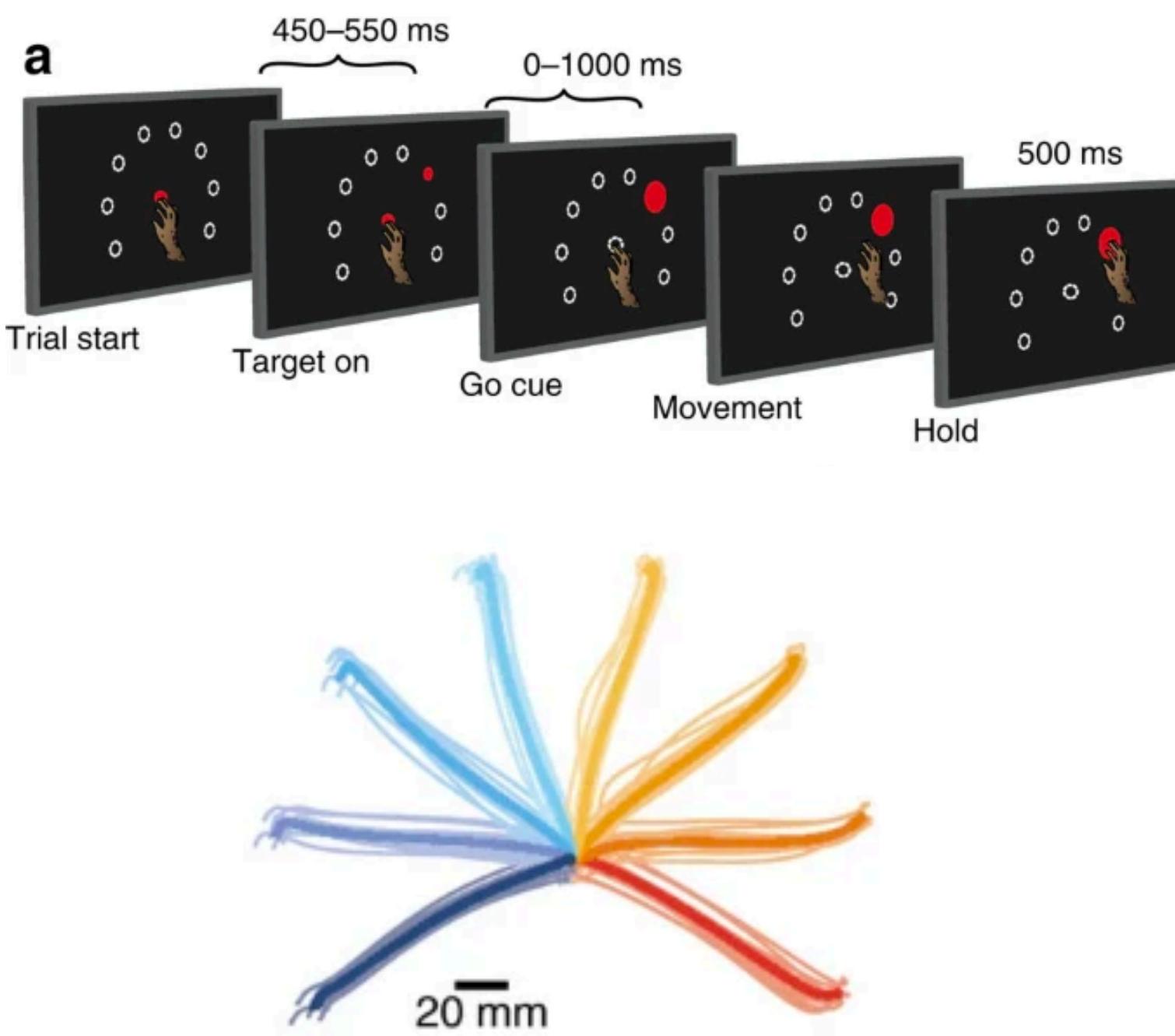
# How can we find interpretable structure when neurons' responses are so mixed?



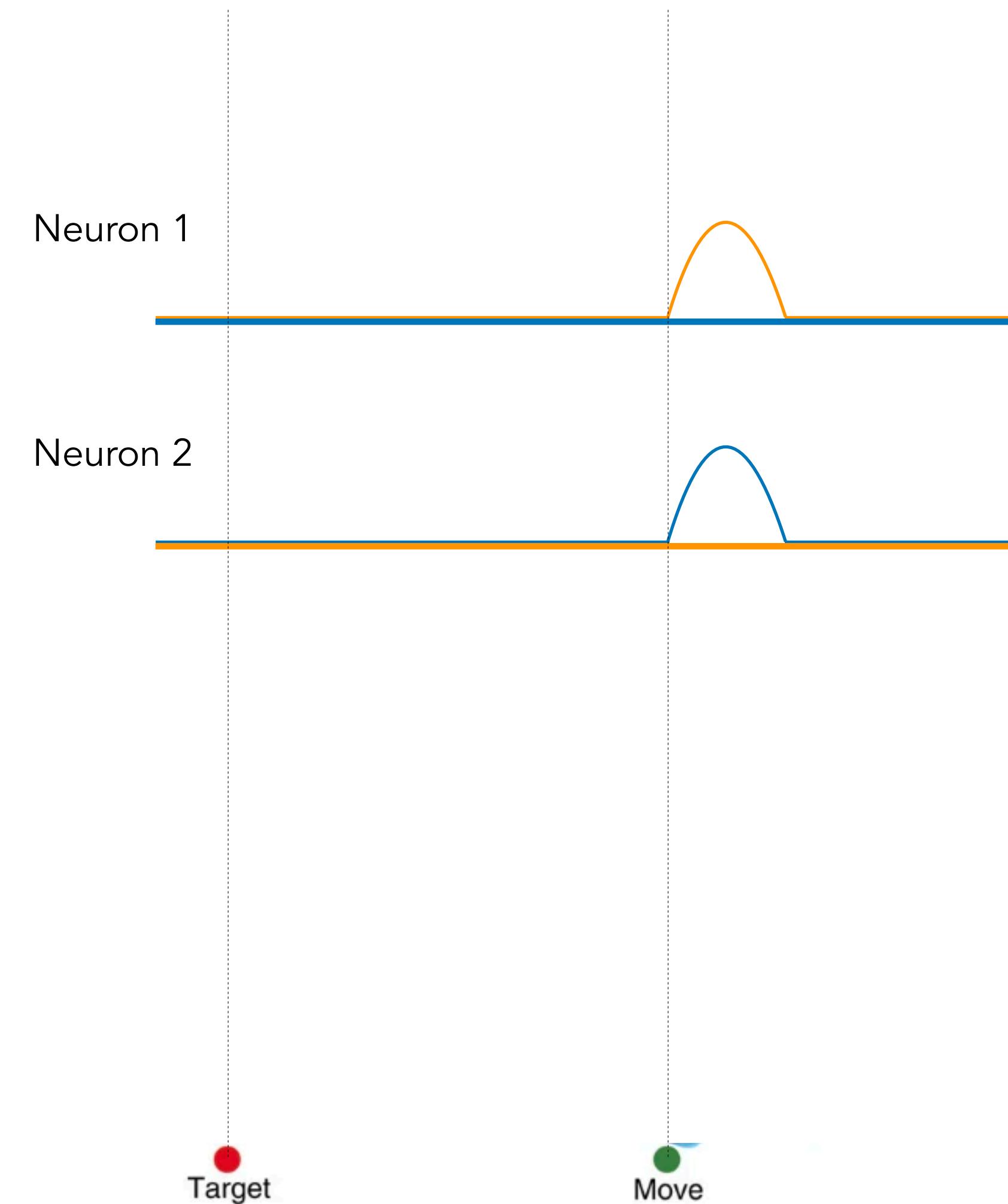
Elsayed et al 2016



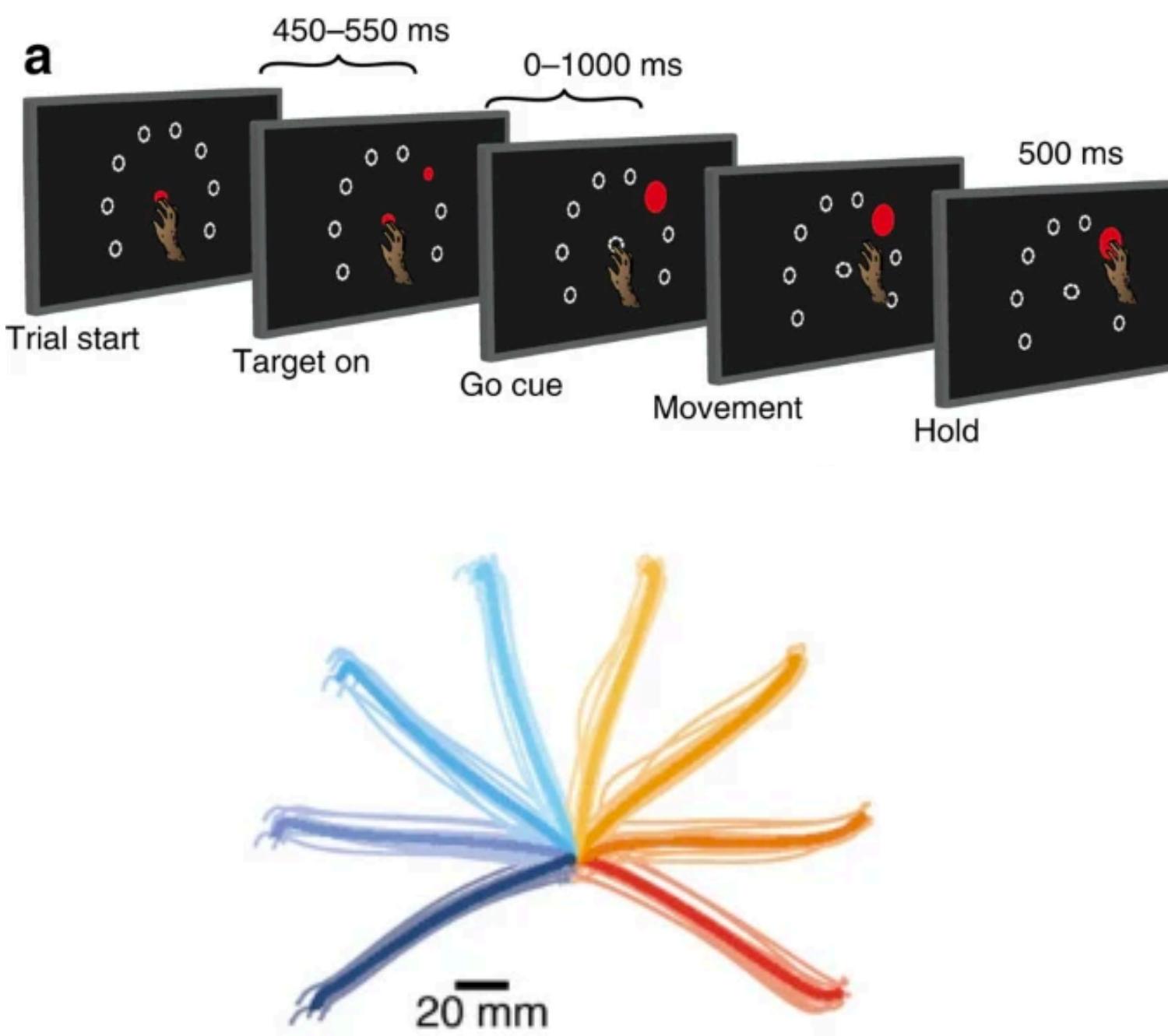
# How can we find interpretable structure when neurons' responses are so mixed?



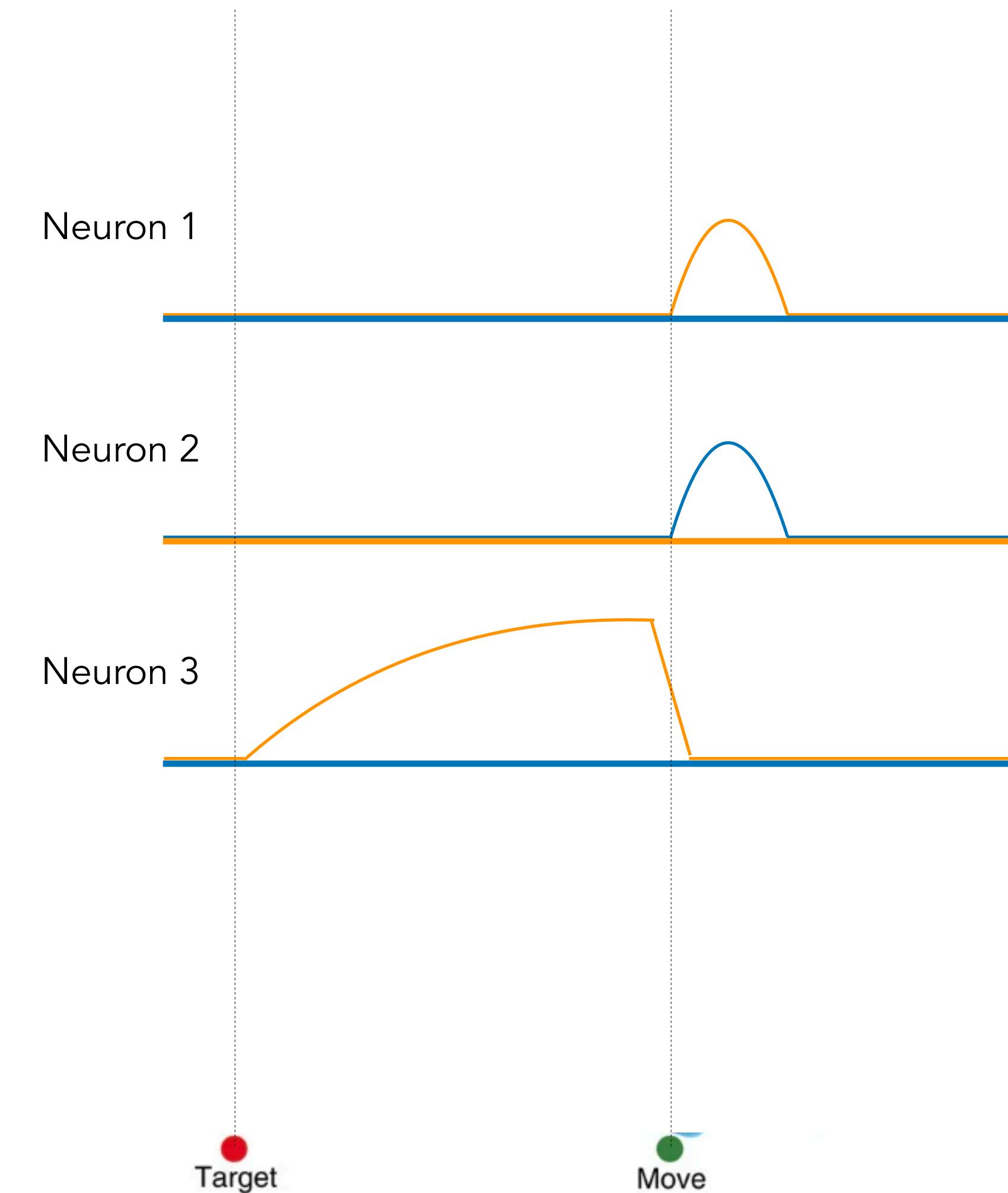
Elsayed et al 2016



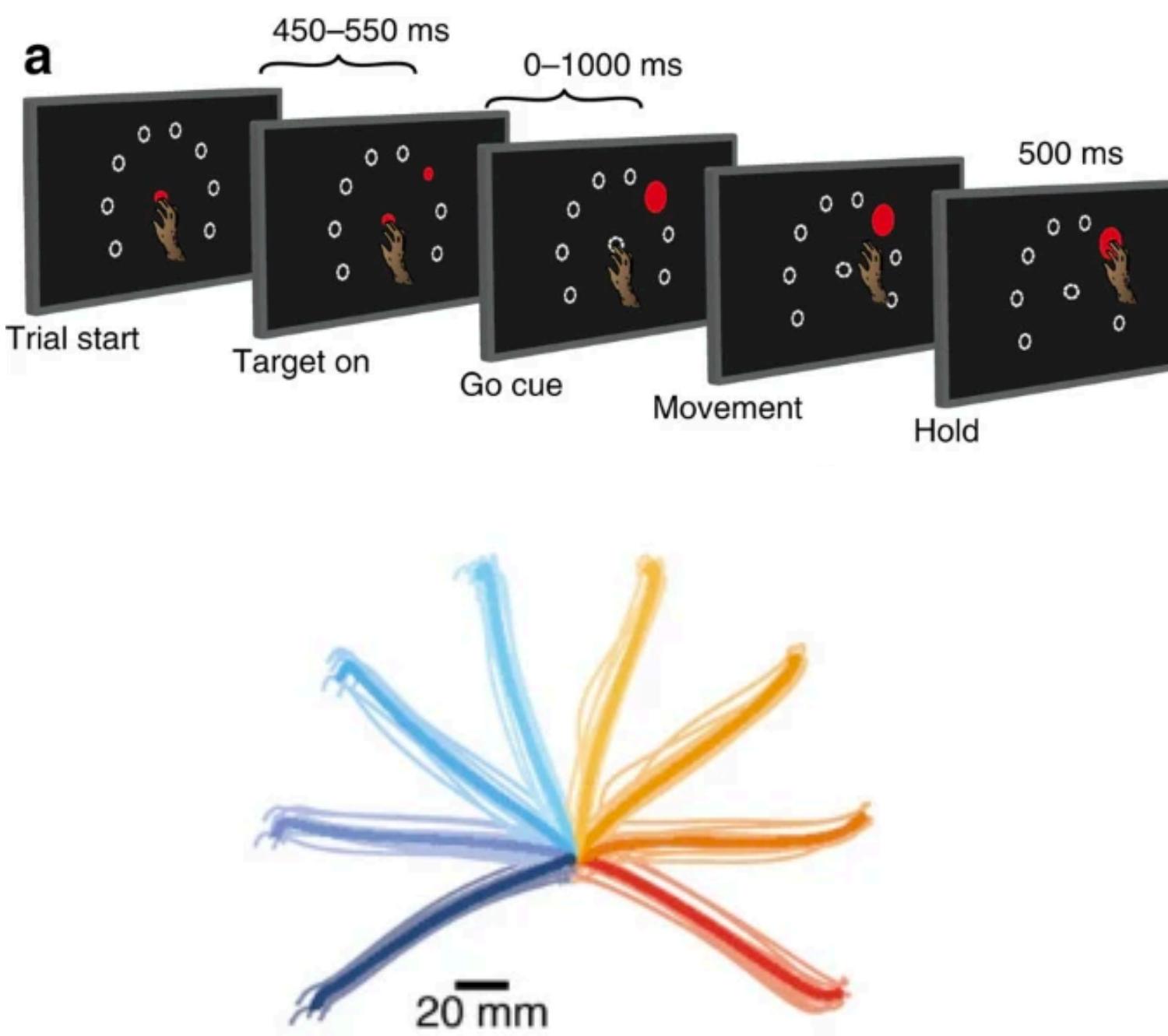
# How can we find interpretable structure when neurons' responses are so mixed?



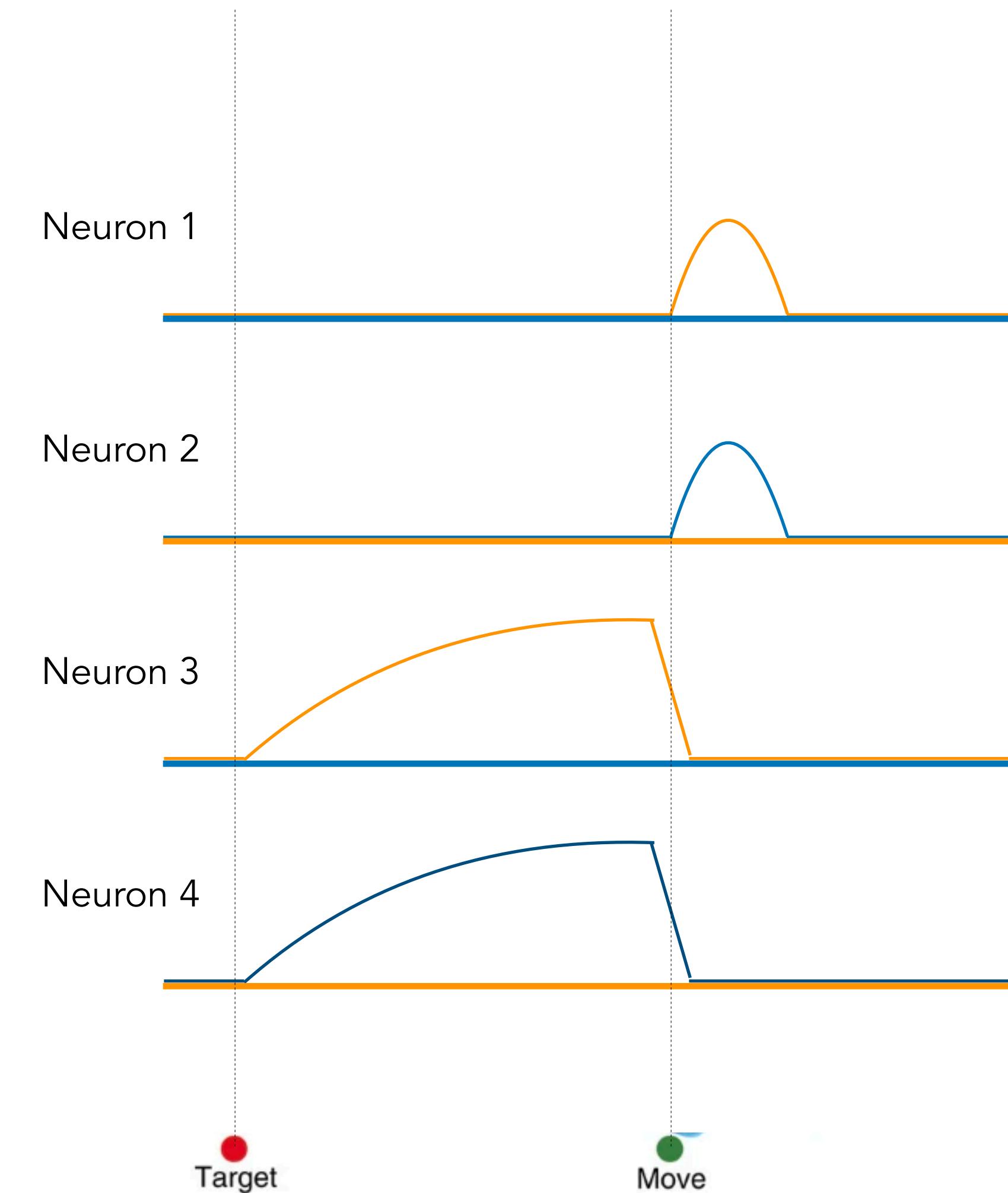
Elsayed et al 2016



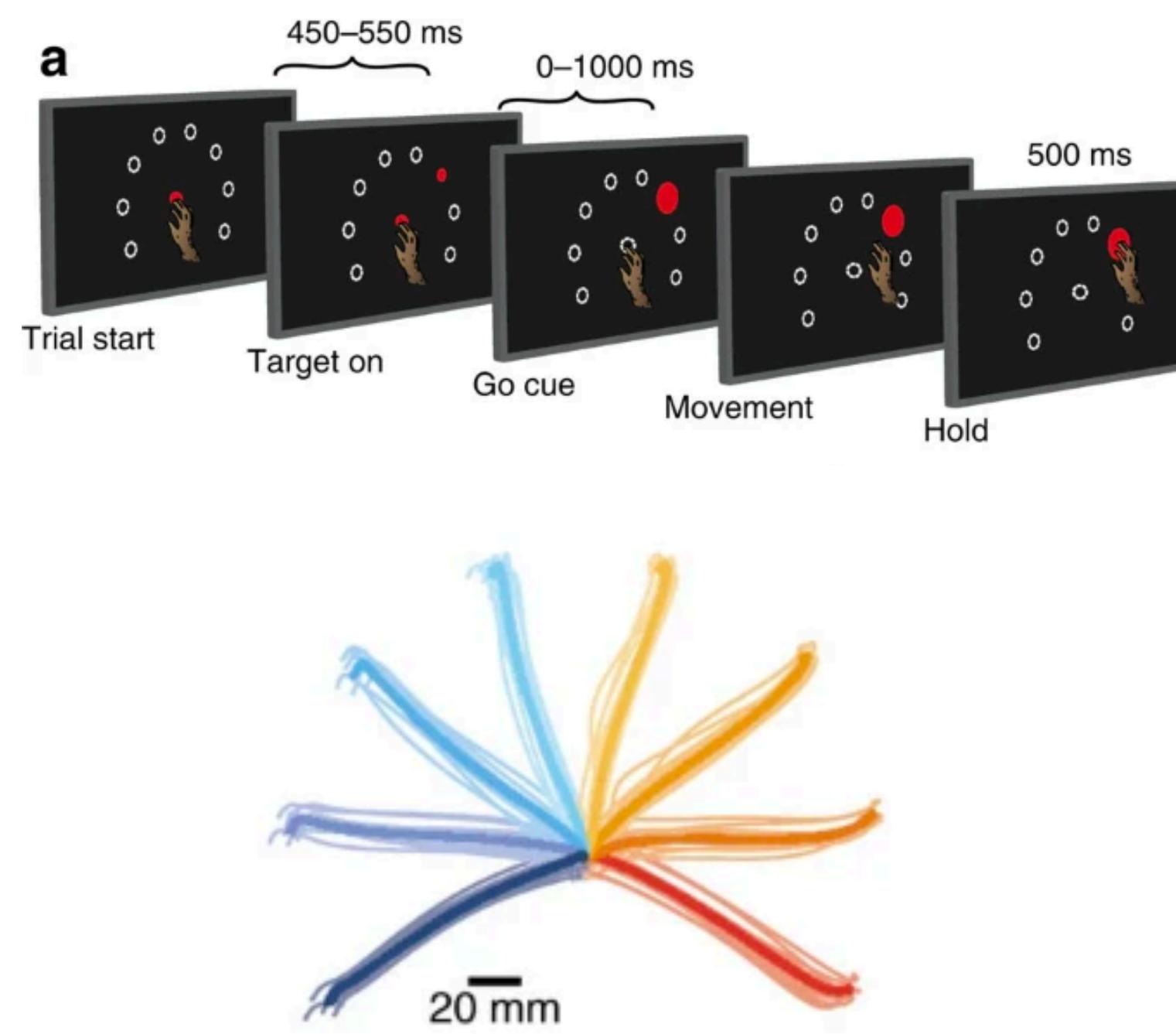
# How can we find interpretable structure when neurons' responses are so mixed?



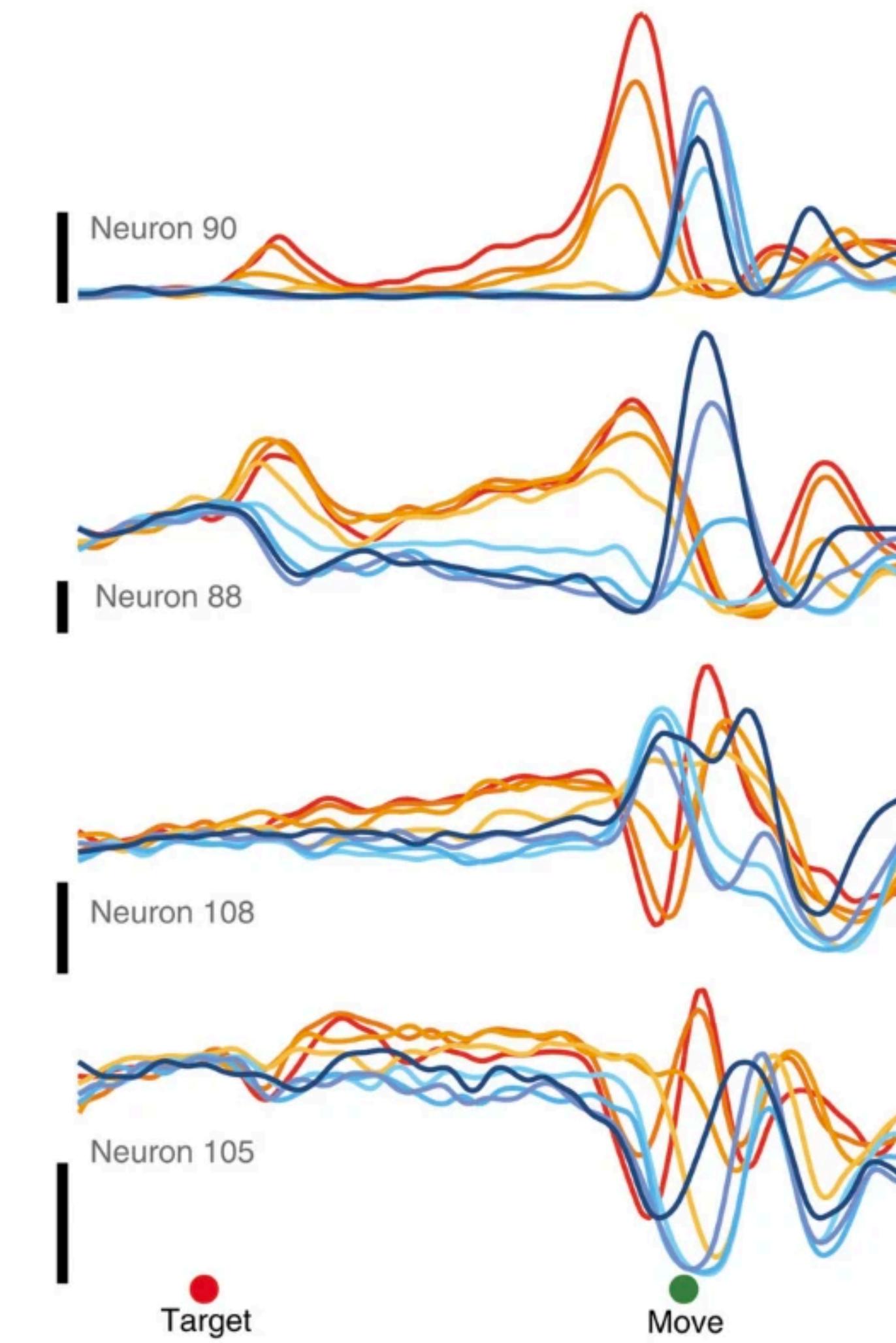
Elsayed et al 2016



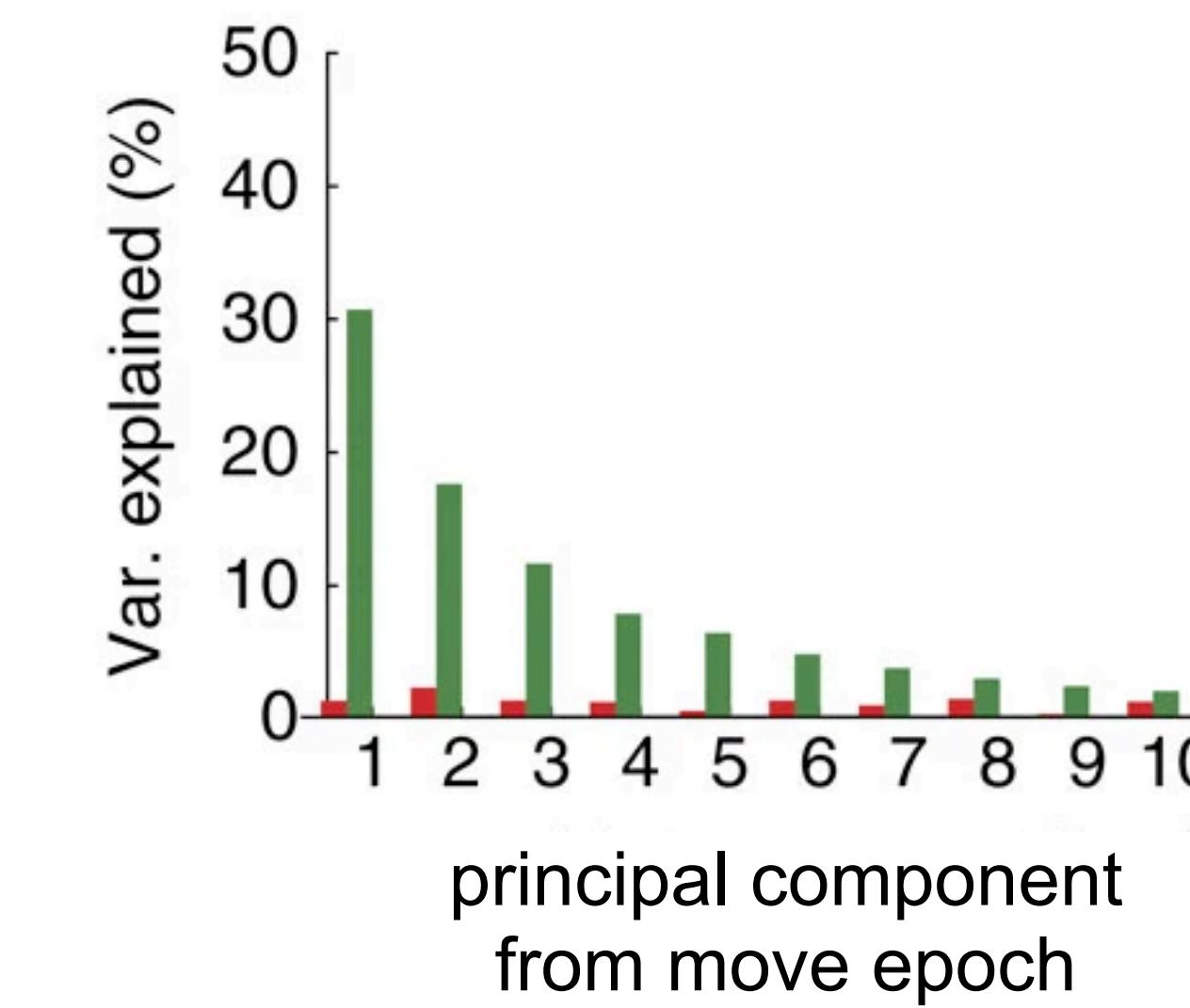
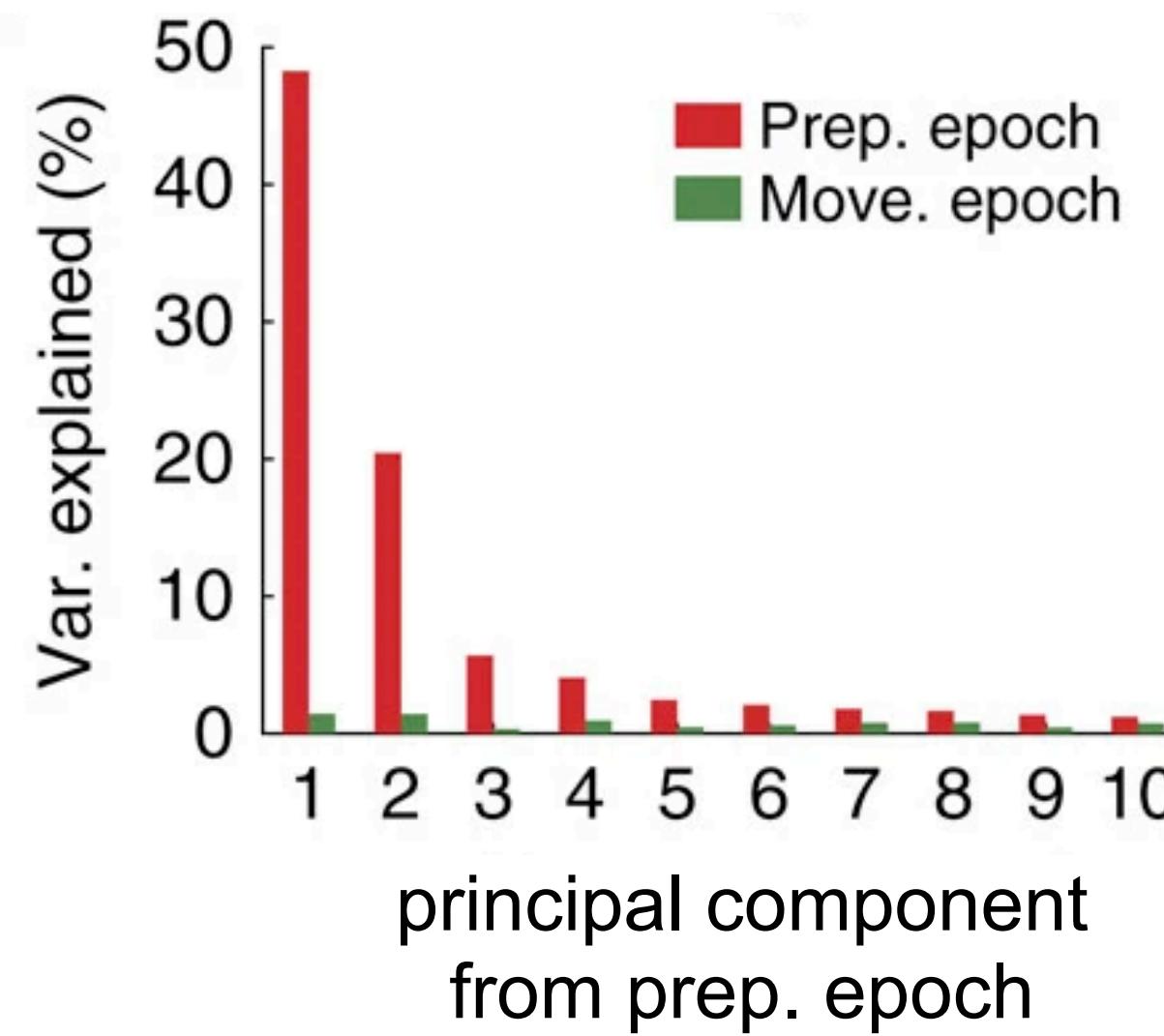
# How can we find interpretable structure when neurons' responses are so mixed?



Elsayed et al 2016

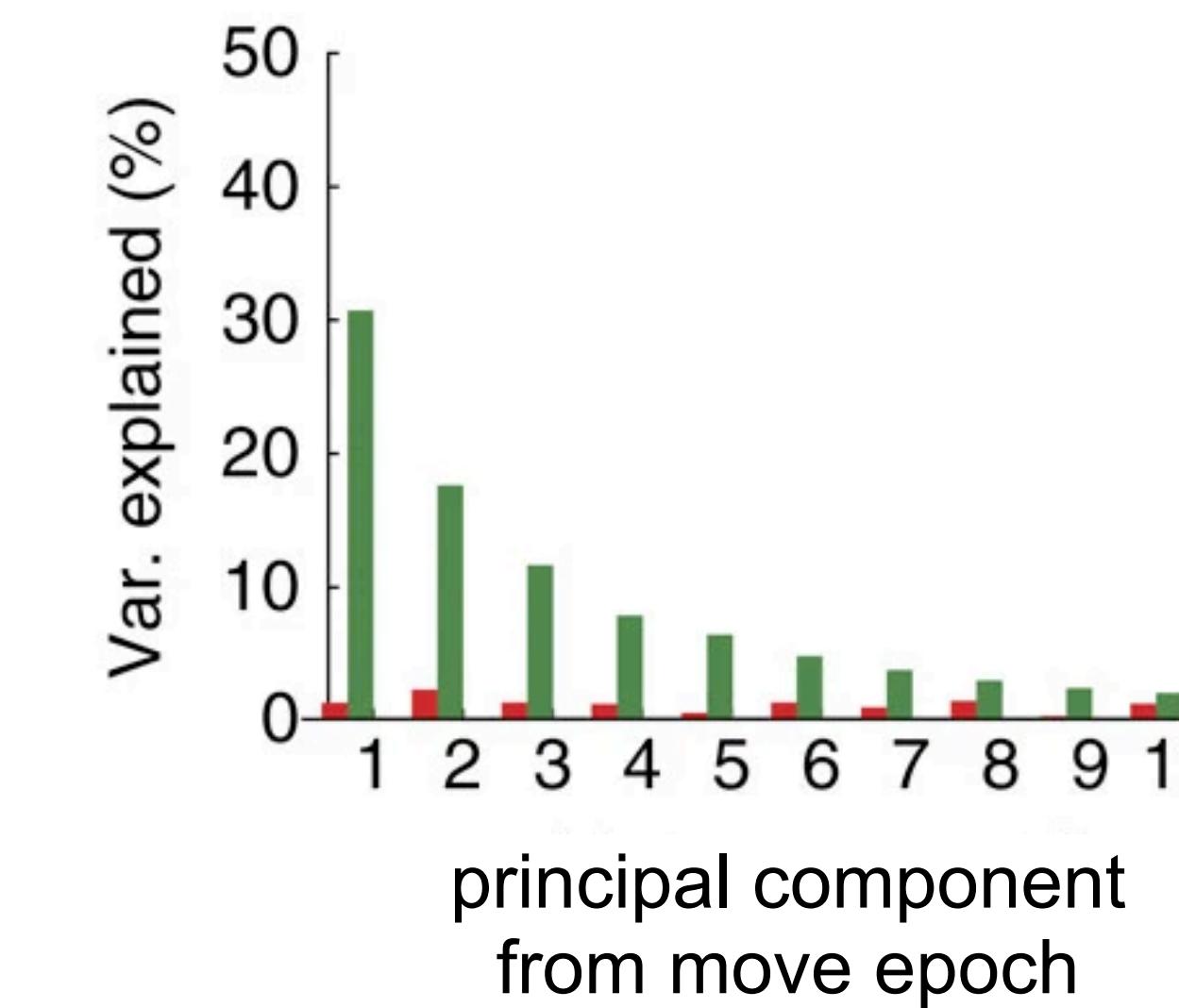
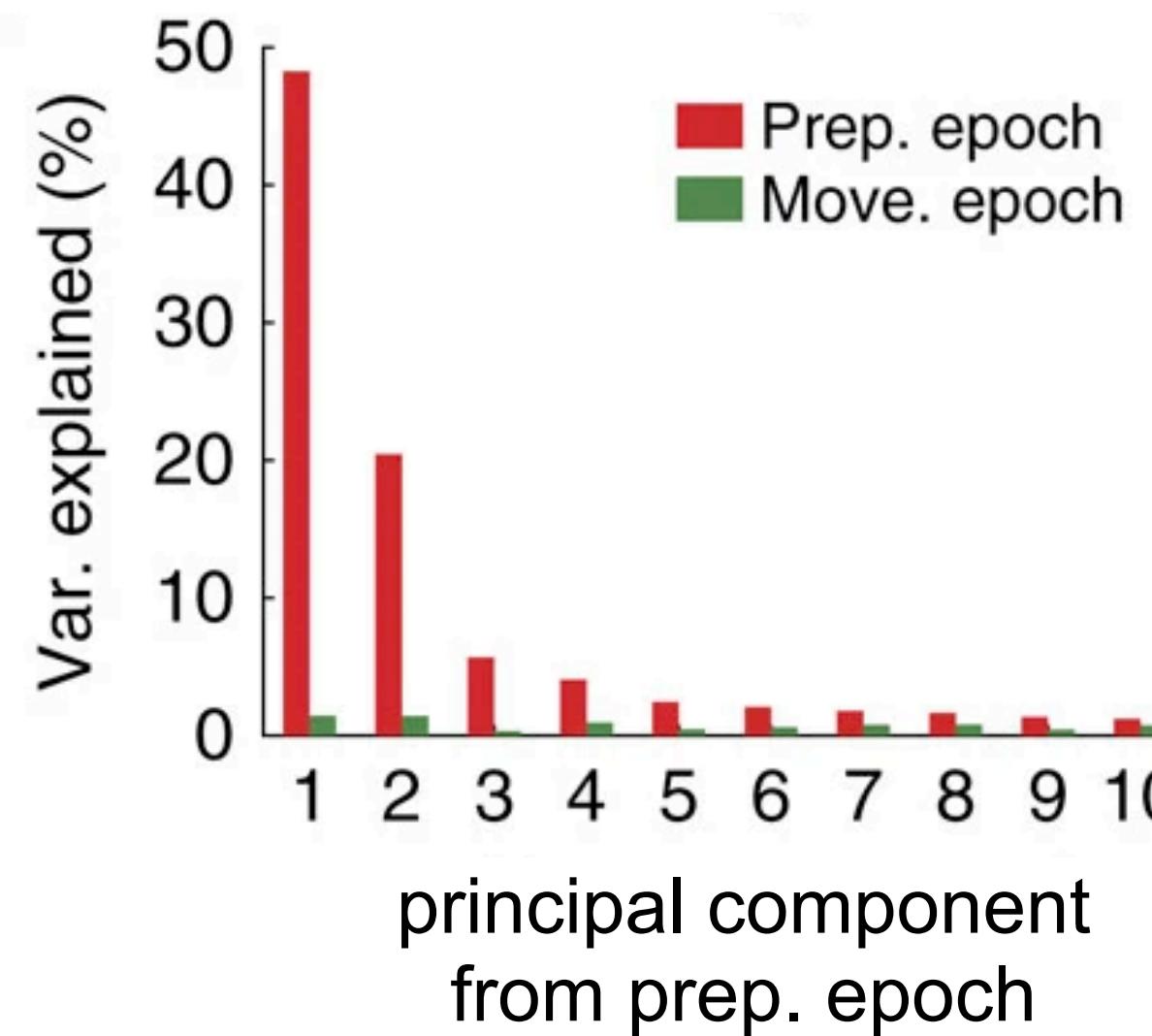


# Different computations occur in different dimensions



Elsayed et al 2016

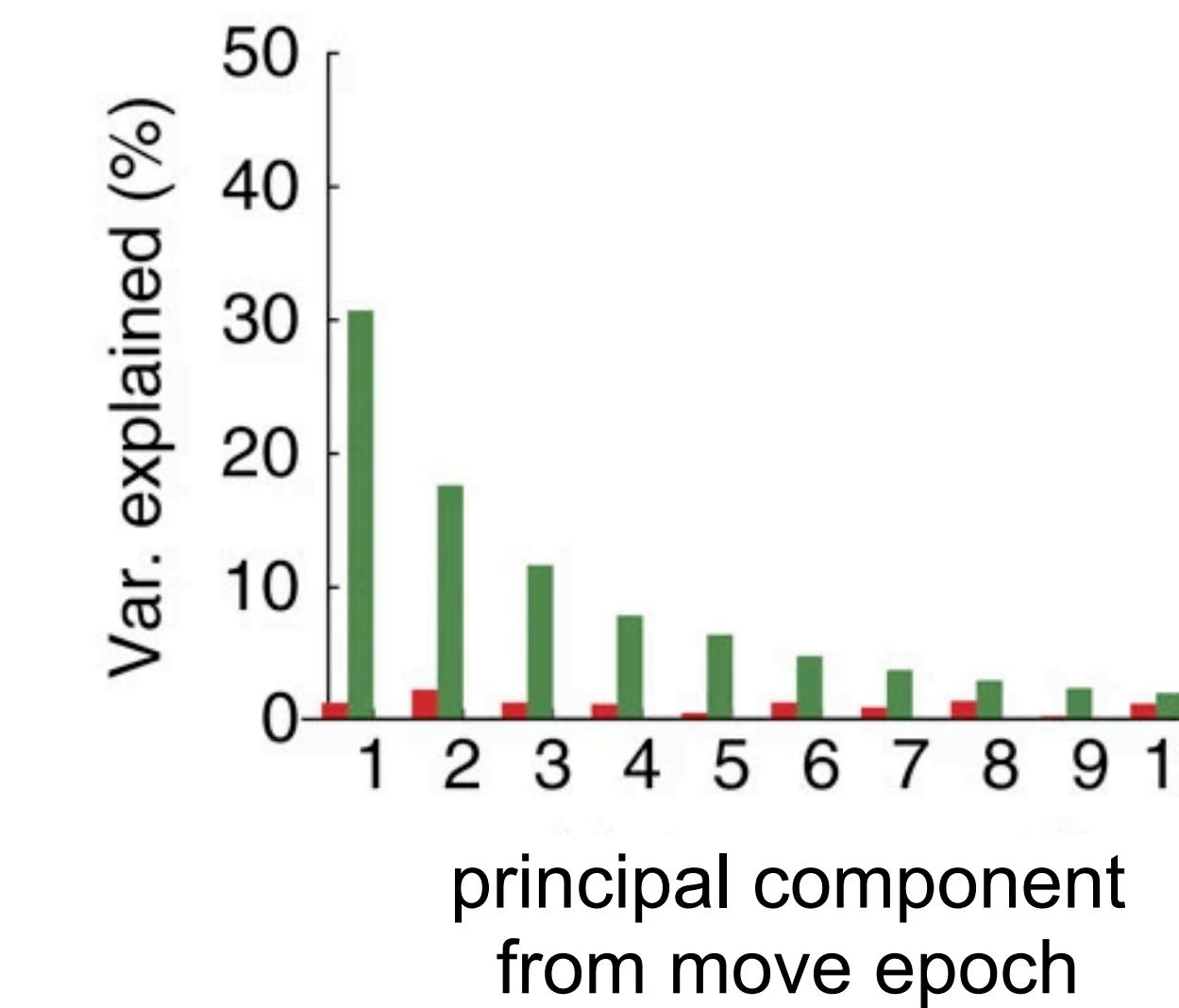
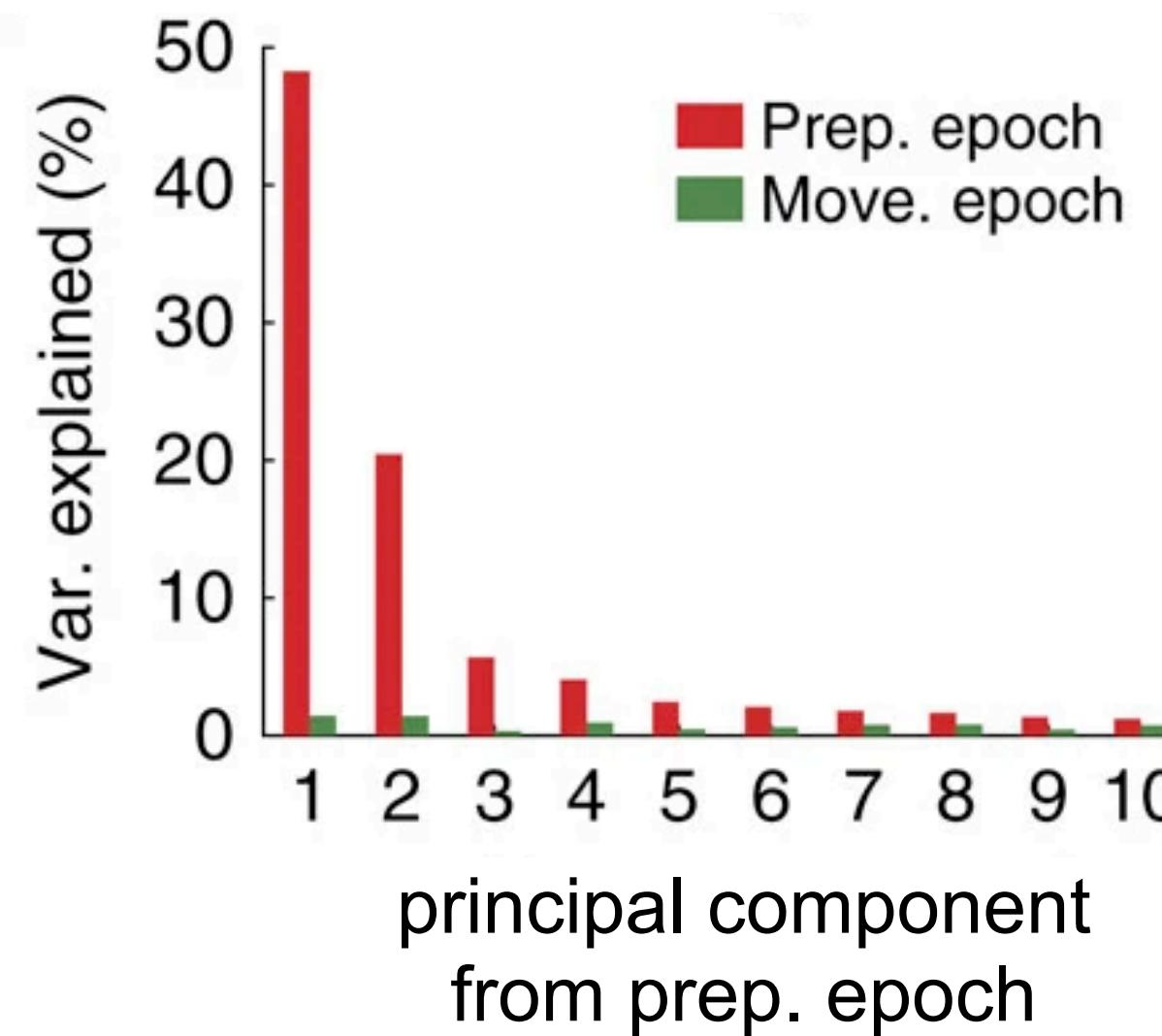
# Different computations occur in different dimensions



Elsayed et al 2016

- Working memory and motor preparation in prefrontal cortex (Tang et al 2020)
- Evaluation and selection in orbitofrontal cortex (Yoo et al 2020)

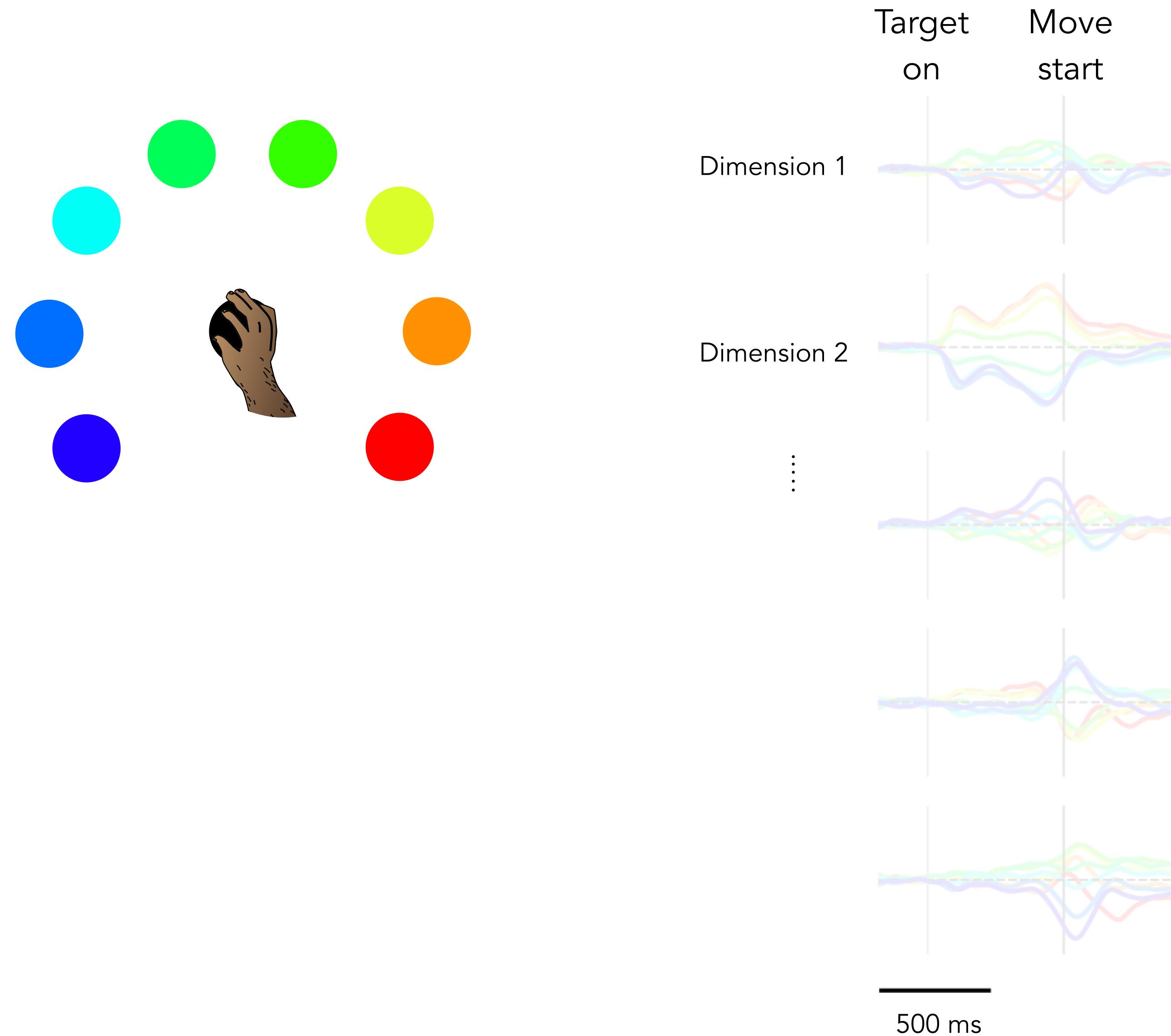
# Different computations occur in different dimensions



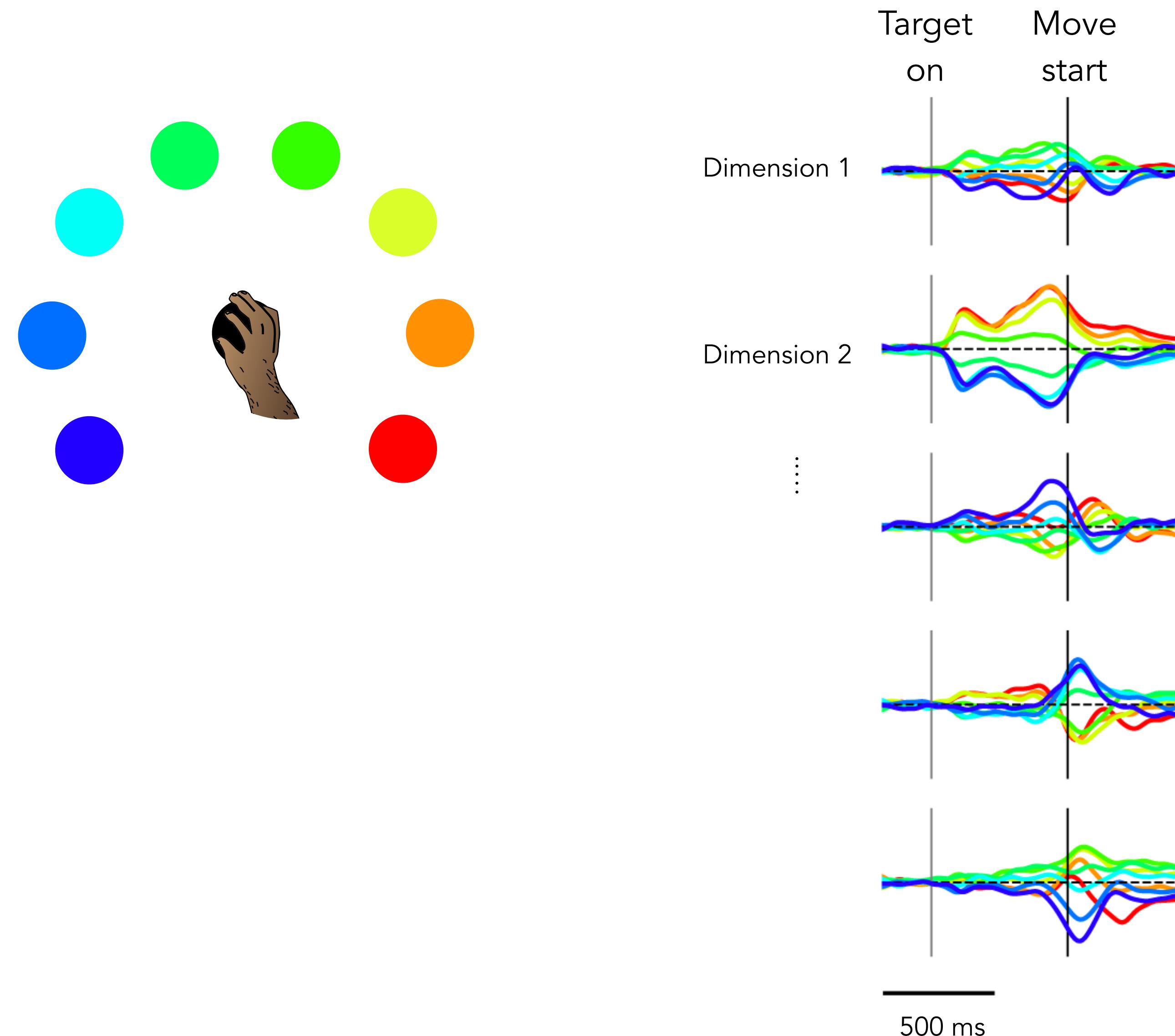
Elsayed et al 2016

- Need to know specific times of interest

# Mixing in the dimensions of PCA



# Mixing in the dimensions of PCA

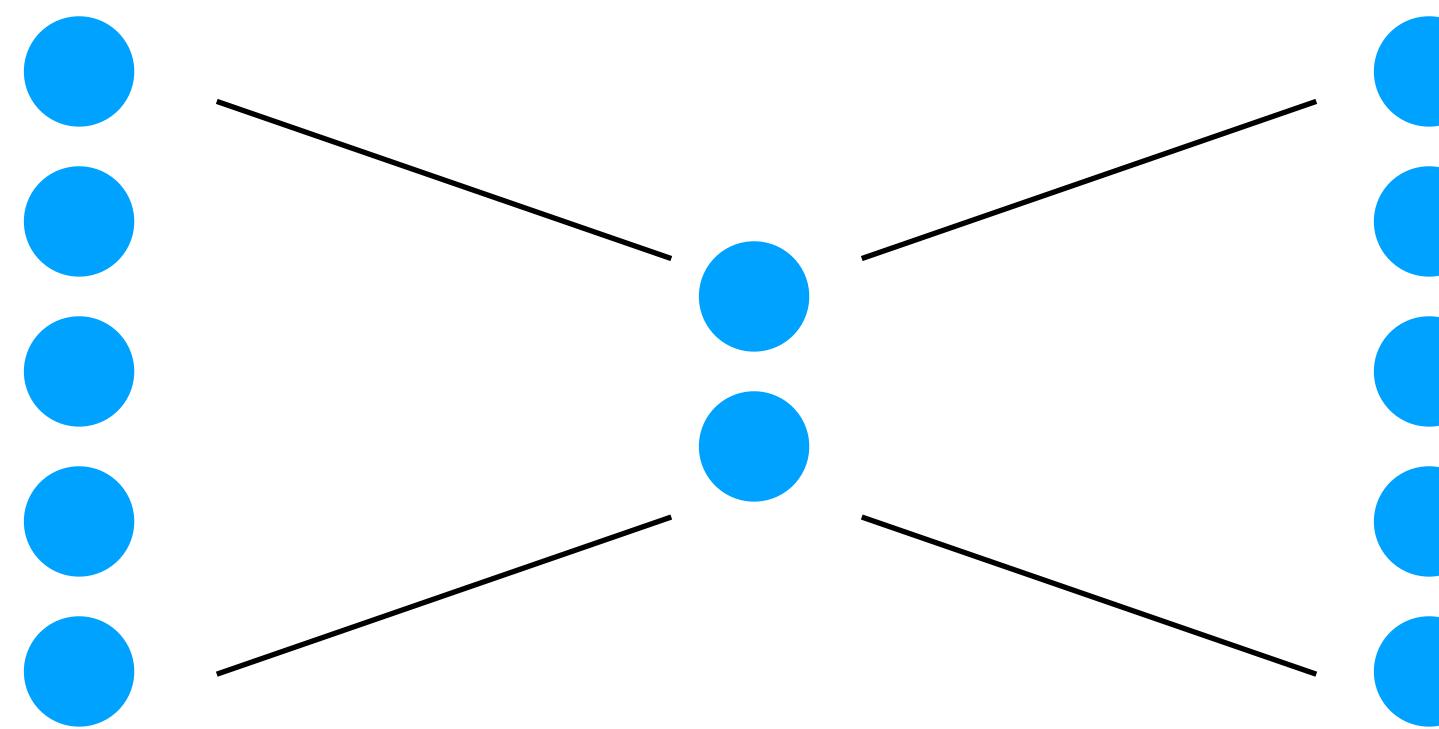


# Sparse Component Analysis (SCA)

- **Goal:** find low-D representation whose dimensions are only used at some times (i.e. used sparsely)

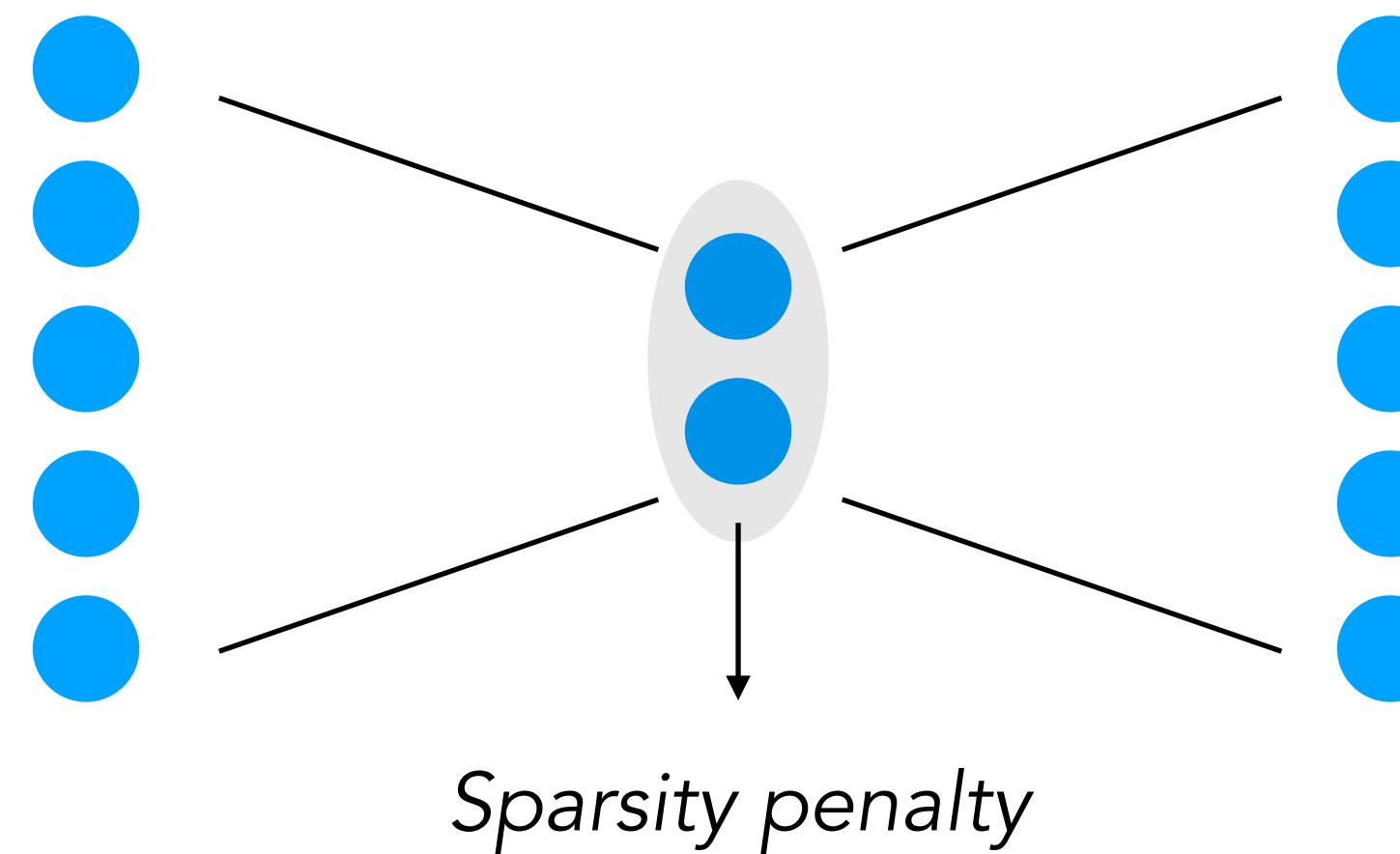
# Sparse Component Analysis (SCA)

- **Goal:** find low-D representation whose dimensions are only used at some times (i.e. used sparsely)



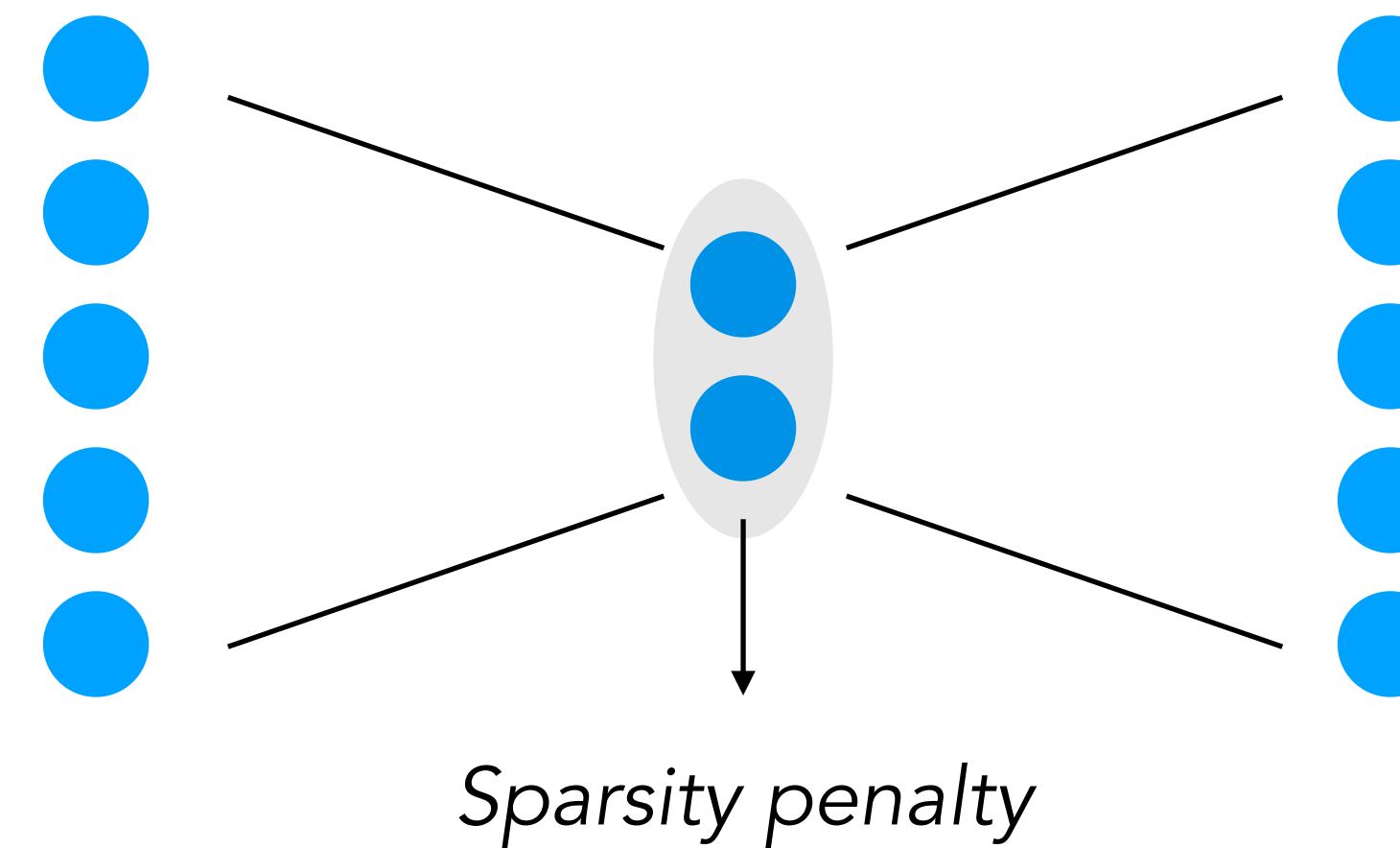
# Sparse Component Analysis (SCA)

- **Goal:** find low-D representation whose dimensions are only used at some times (i.e. used sparsely)



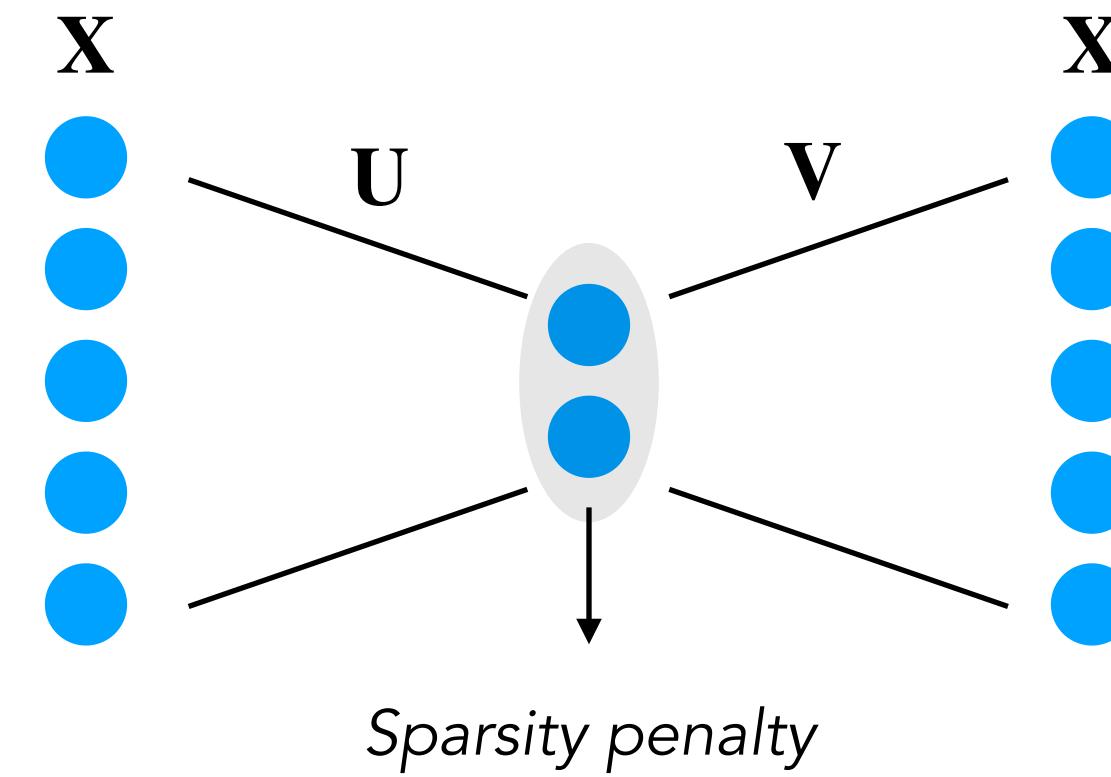
# Sparse Component Analysis (SCA)

- **Goal:** find low-D representation whose dimensions are only used at some times (i.e. used sparsely)



- Unsupervised (time epochs and behavior are not input into model)

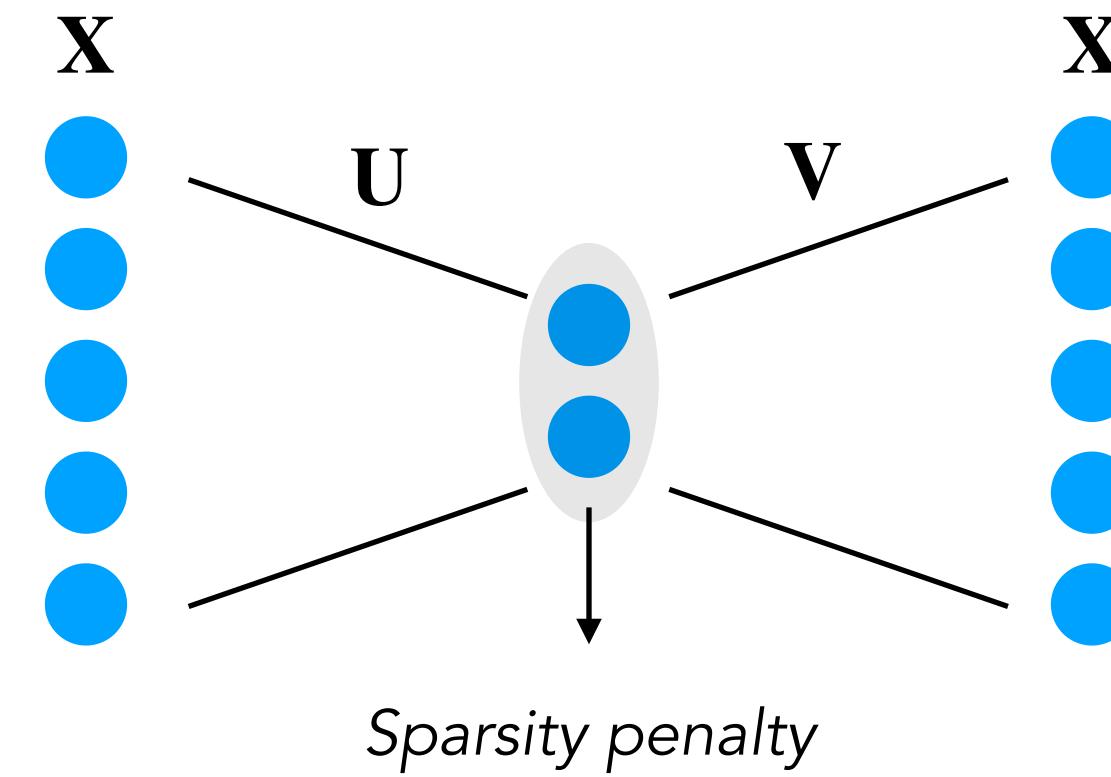
# Sparse Component Analysis (SCA)



Optimization:

$$\arg \min_{U,V} \left( \| \underbrace{(\mathbf{X} - \mathbf{X}UV)}_{\text{Reconstruction}} \|^2 \right)$$

# Sparse Component Analysis (SCA)



Optimization:

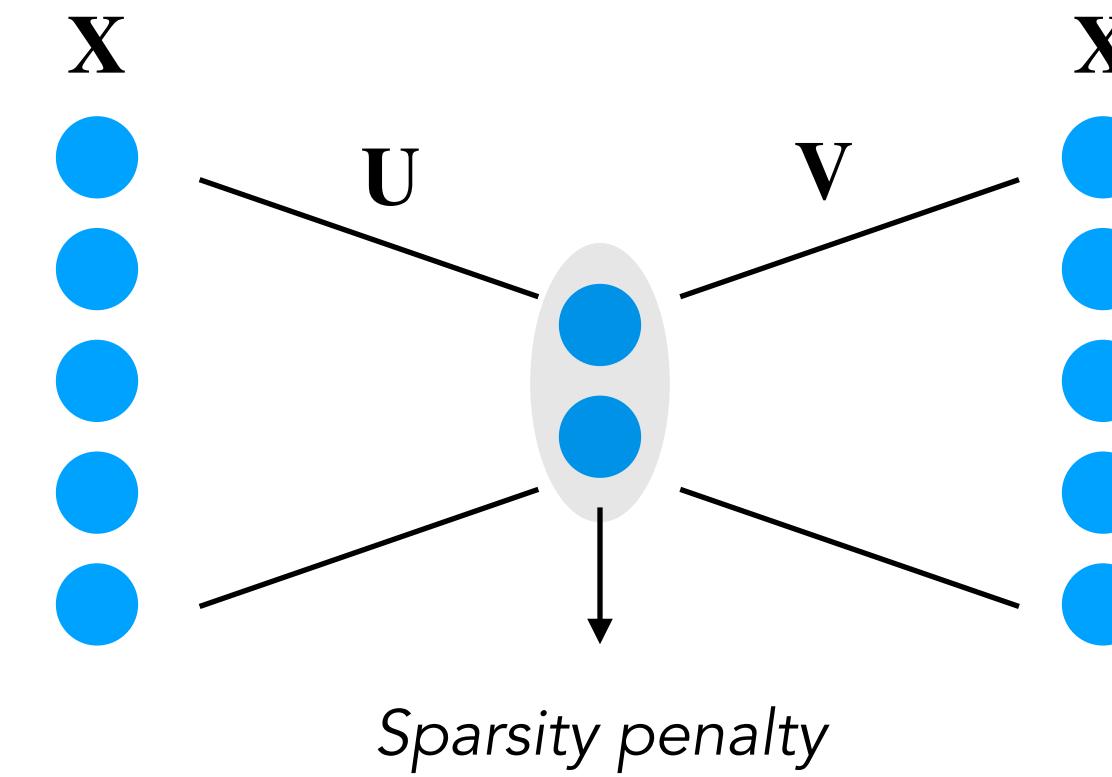
$$\arg \min_{U,V} \left( \| \cdot (\mathbf{X} - \mathbf{XUV}) \|^2 + \lambda \| \mathbf{XU} \|_1 \right)$$

[

Reconstruction

Sparsity

# Sparse Component Analysis (SCA)

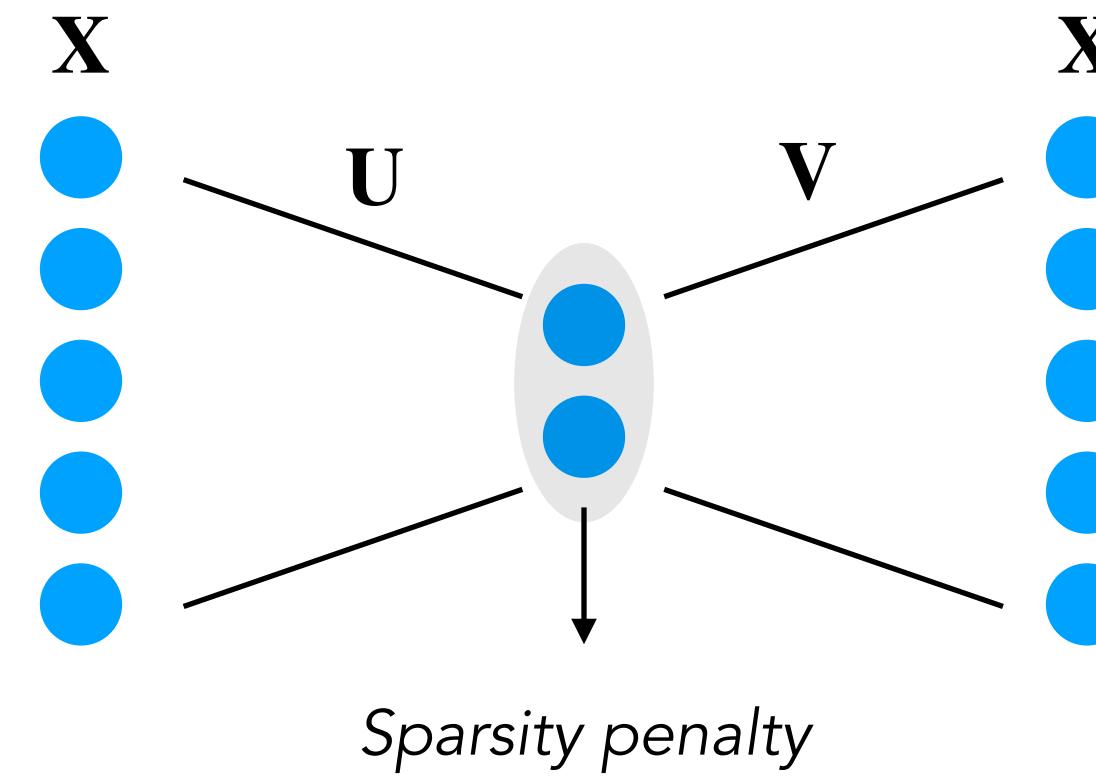


# Optimization:

$$\arg \min_{U,V} \left( \| (X - XUV) \|^2 + \lambda \| XU \|_1 + \gamma \| V^T V - I \| \right)$$


Reconstruction      Sparsity      Orthogonality

# Sparse Component Analysis (SCA)



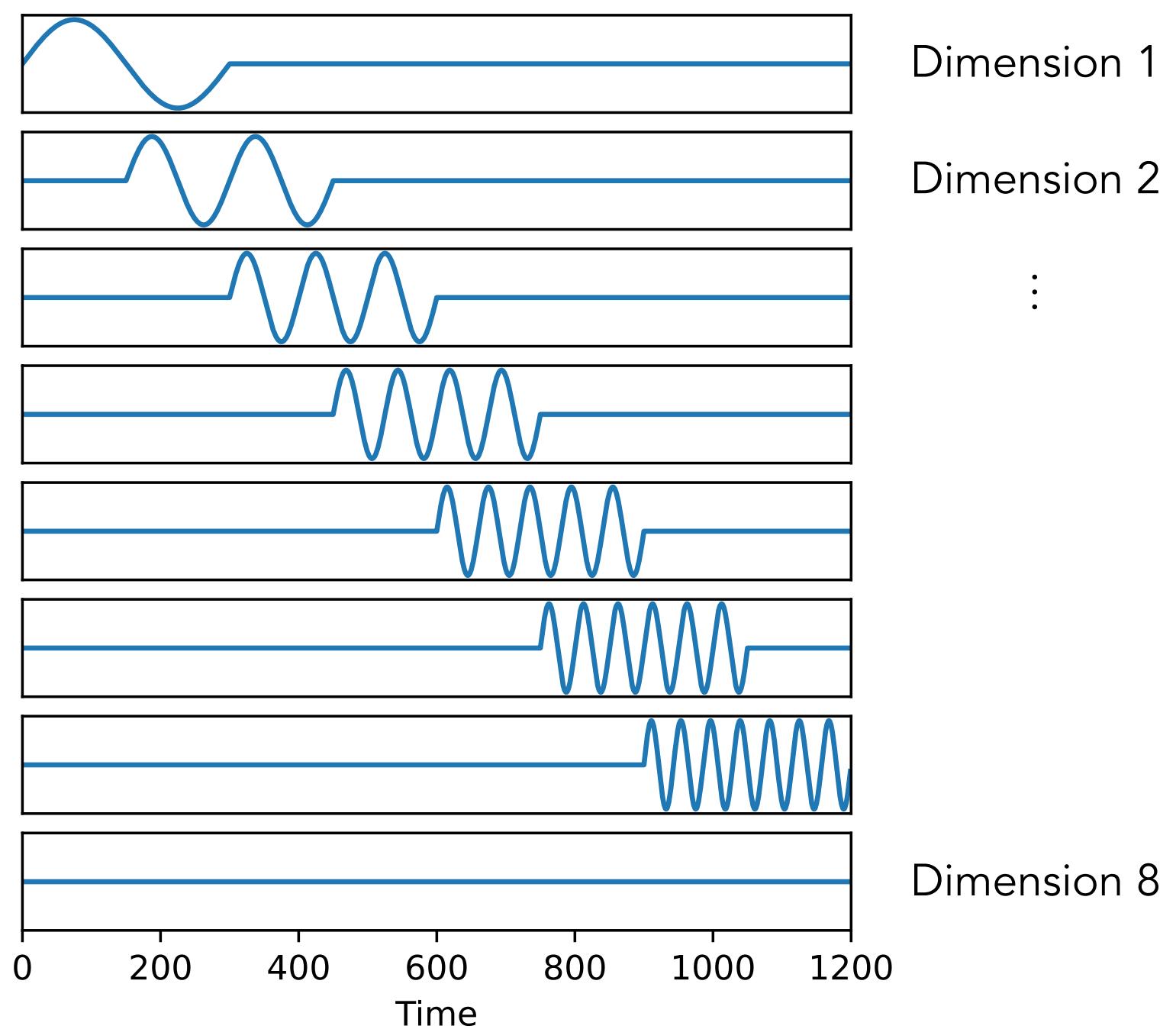
# Optimization:

$$\arg \min_{U,V} \left( \| W(X - XUV) \|^2 + \lambda \| XU \|_1 + \gamma \| V^T V - I \| \right)$$


Reconstruction      Sparsity      Orthogonality

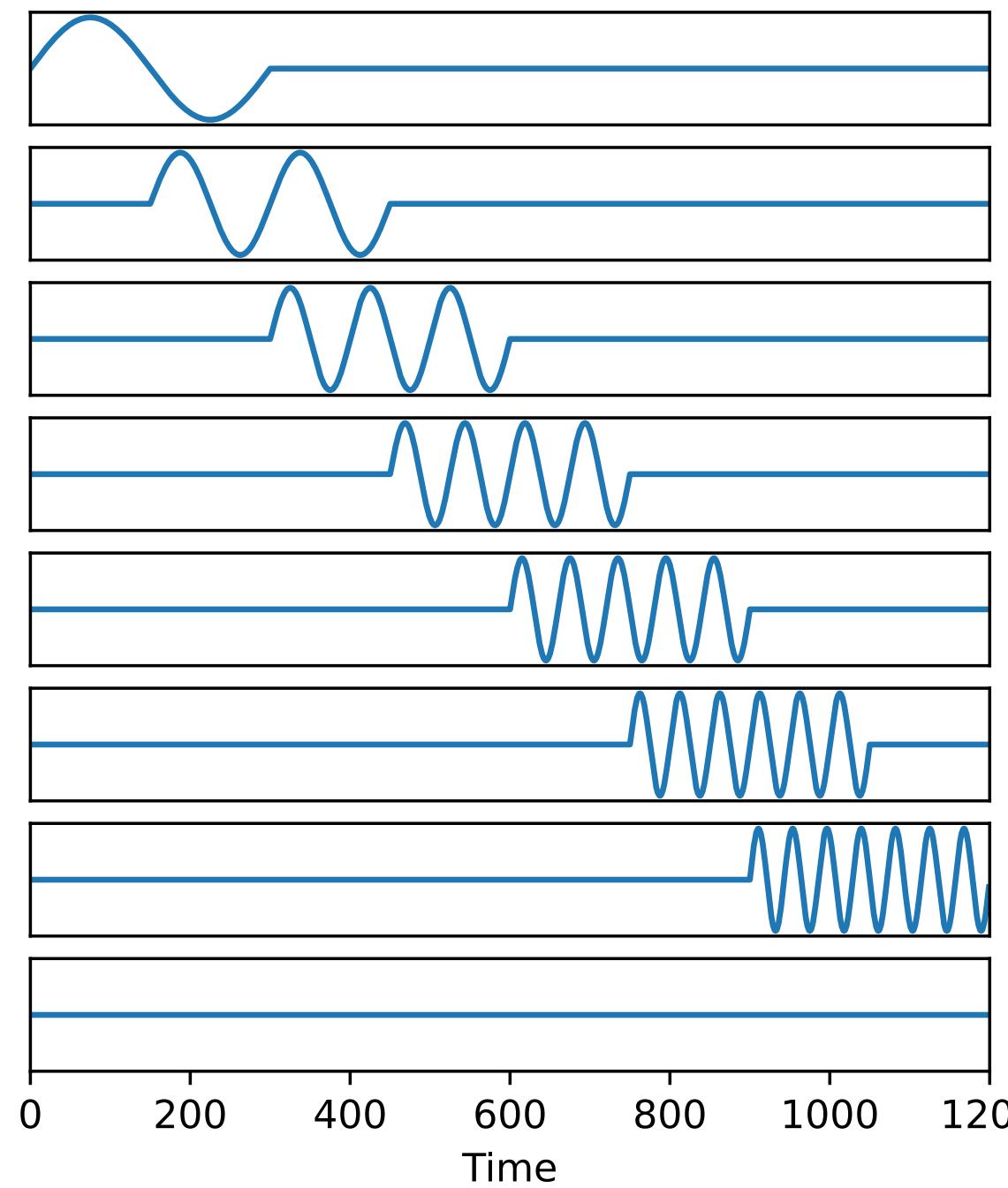
# Simulation example

**Ground Truth Dimensions**

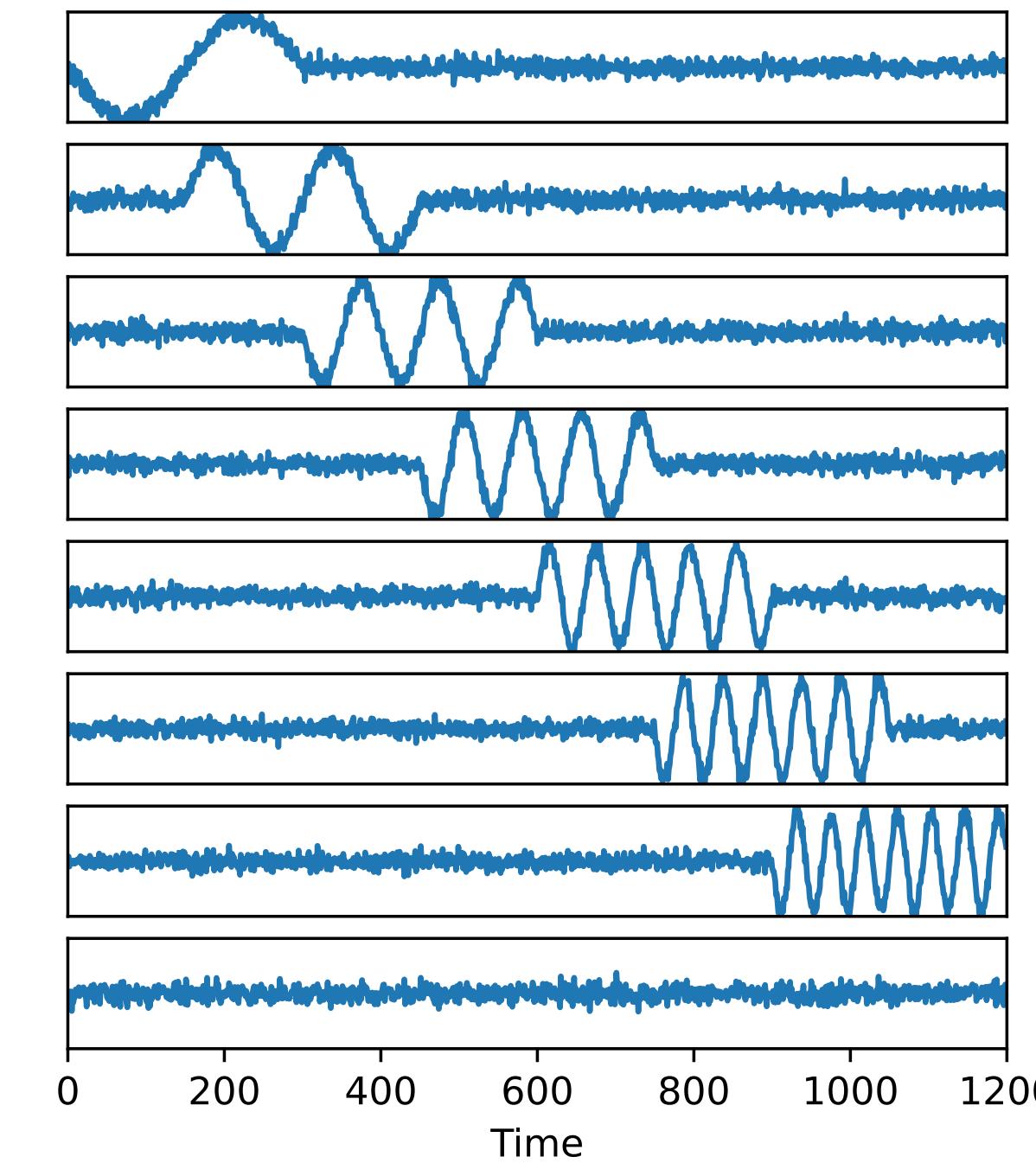


# Simulation example

Ground Truth Dimensions

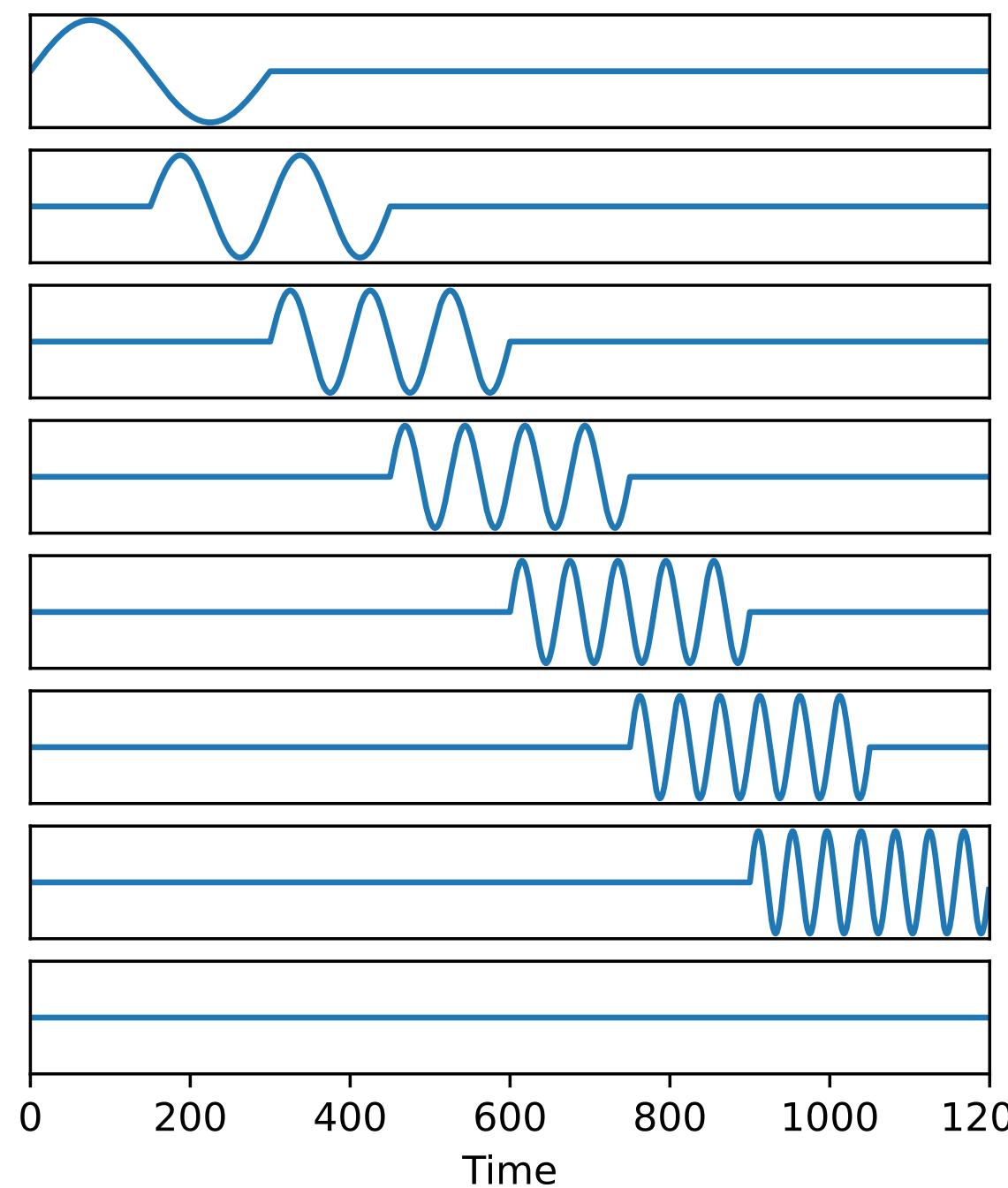


SCA Dimensions

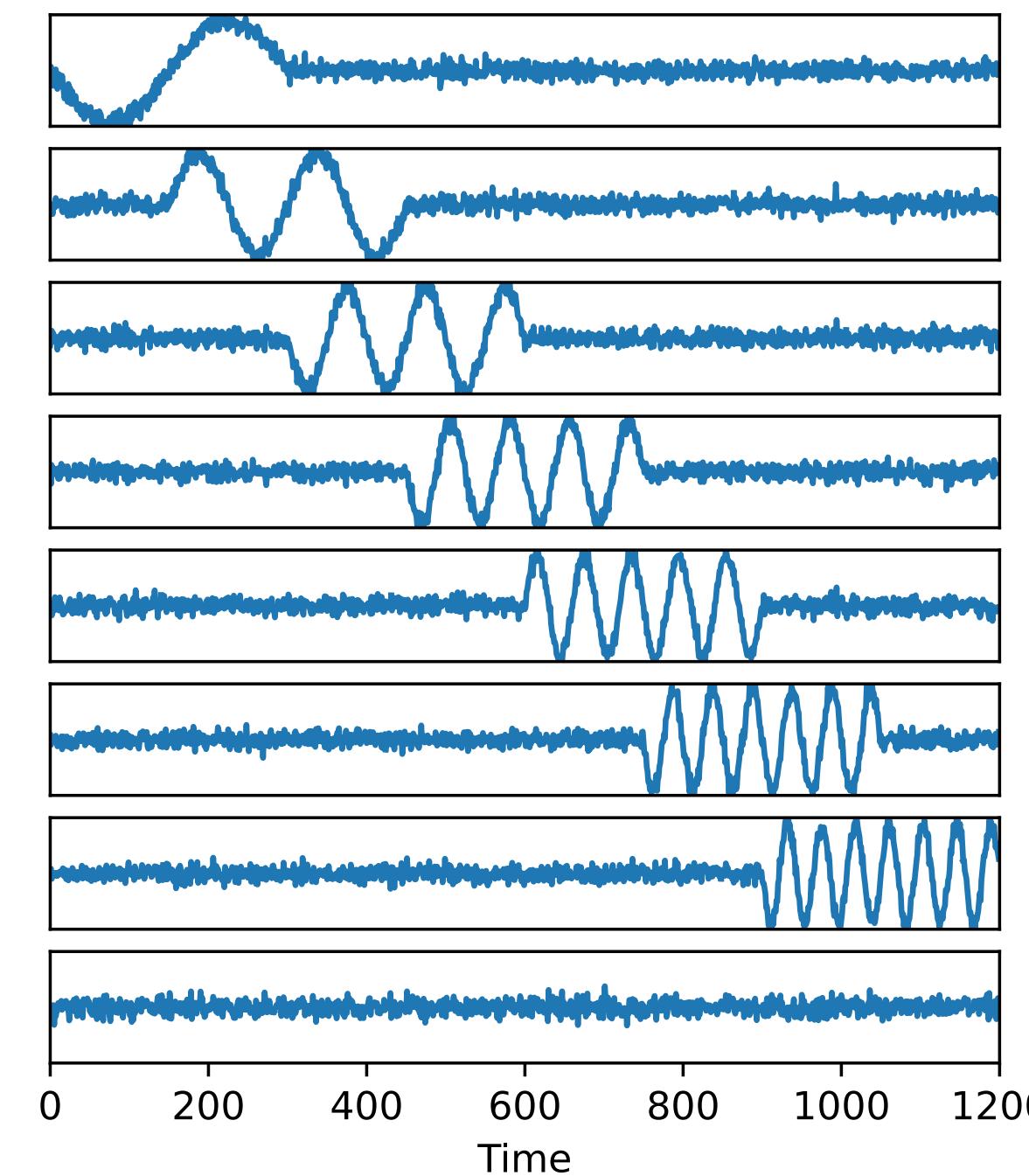


# Simulation example

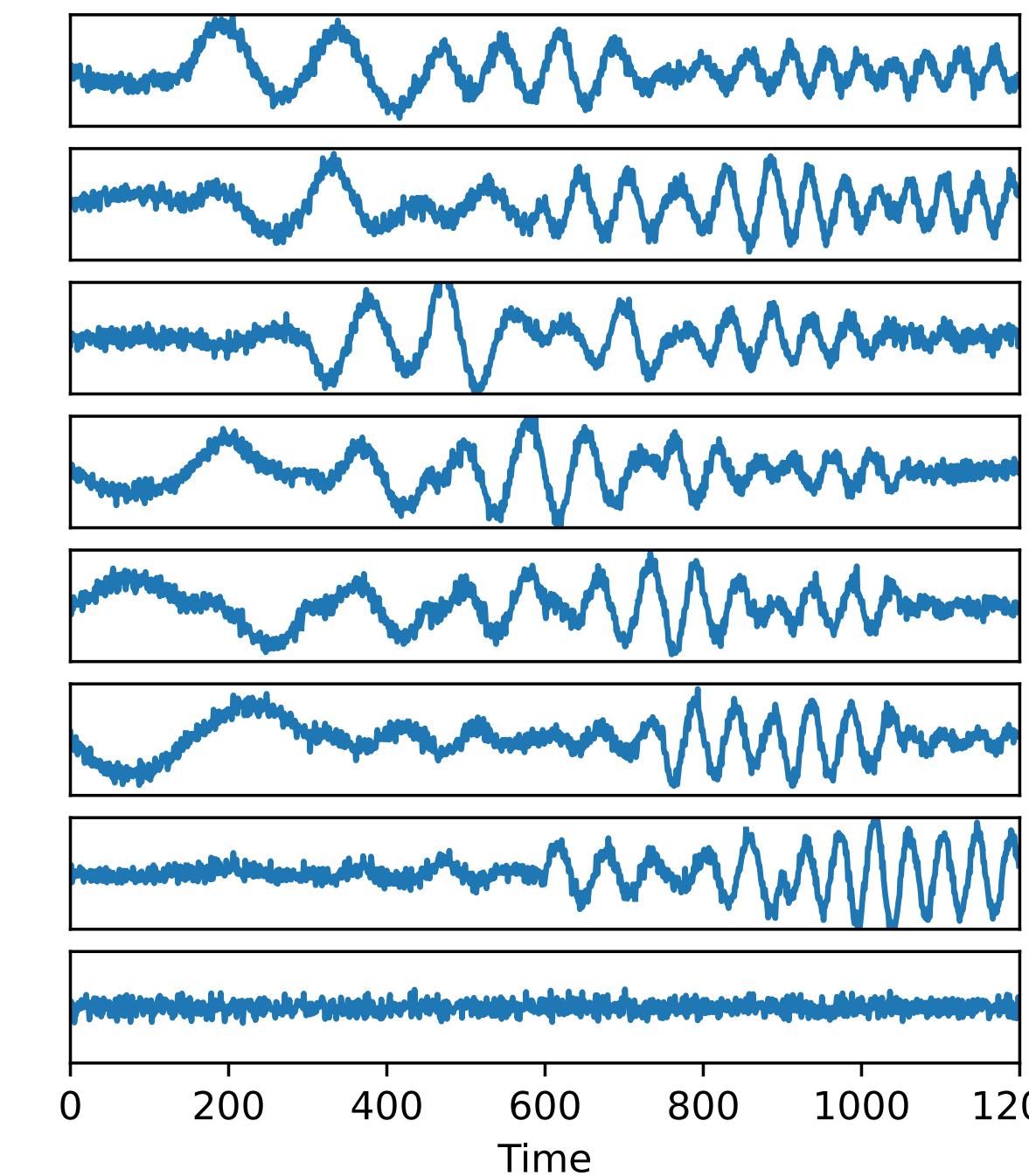
Ground Truth Dimensions



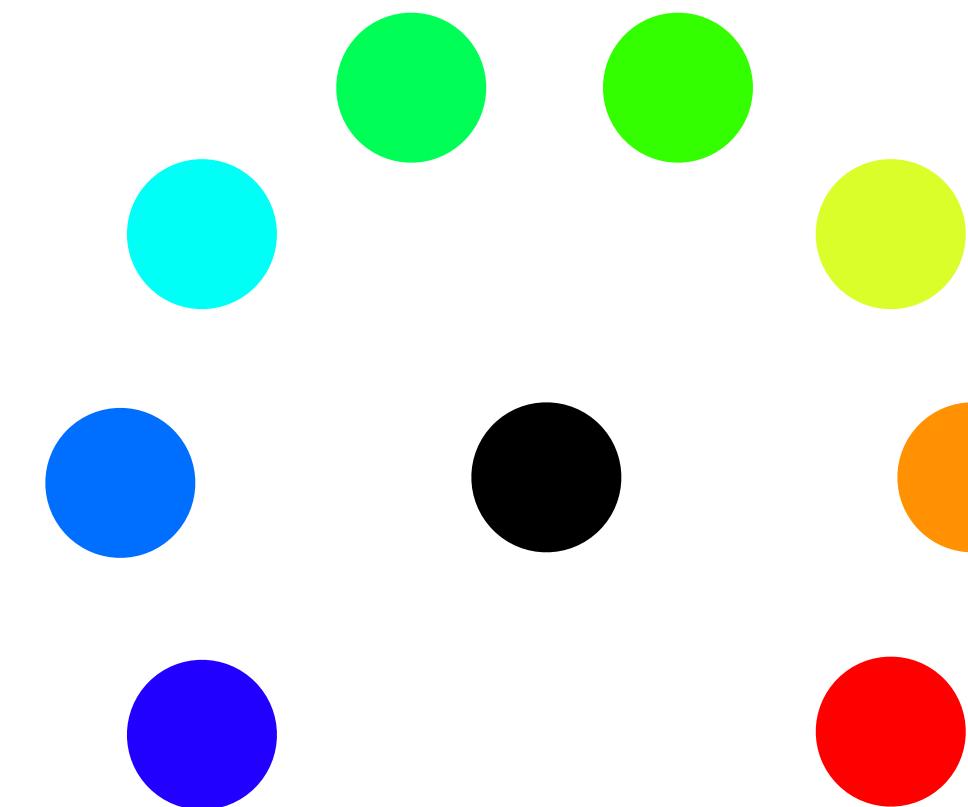
SCA Dimensions



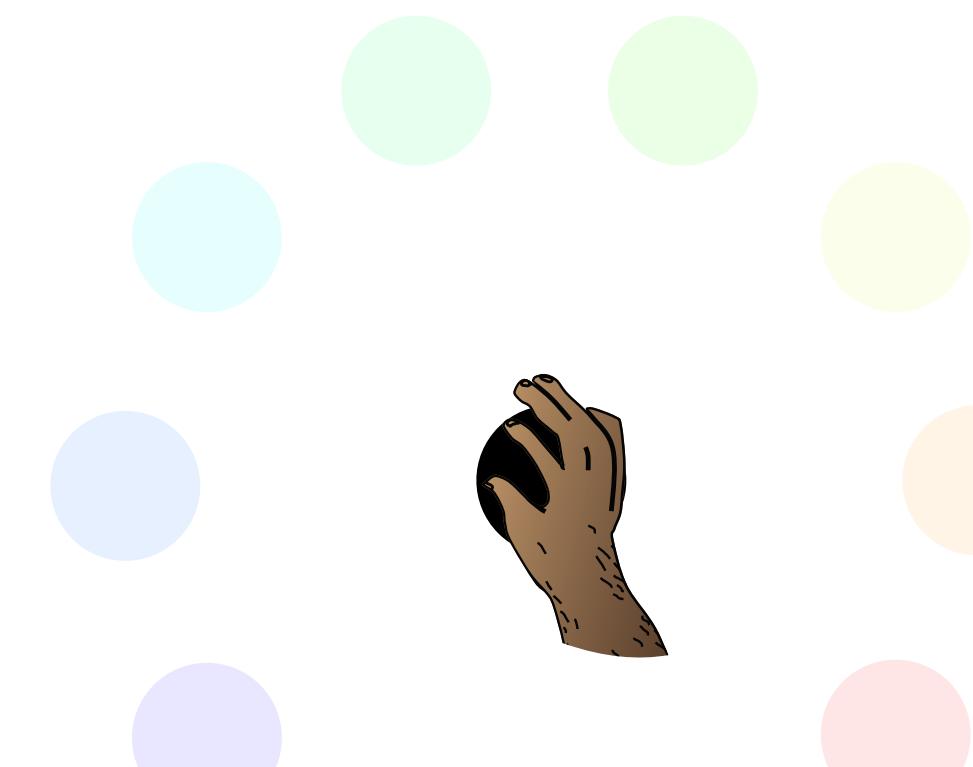
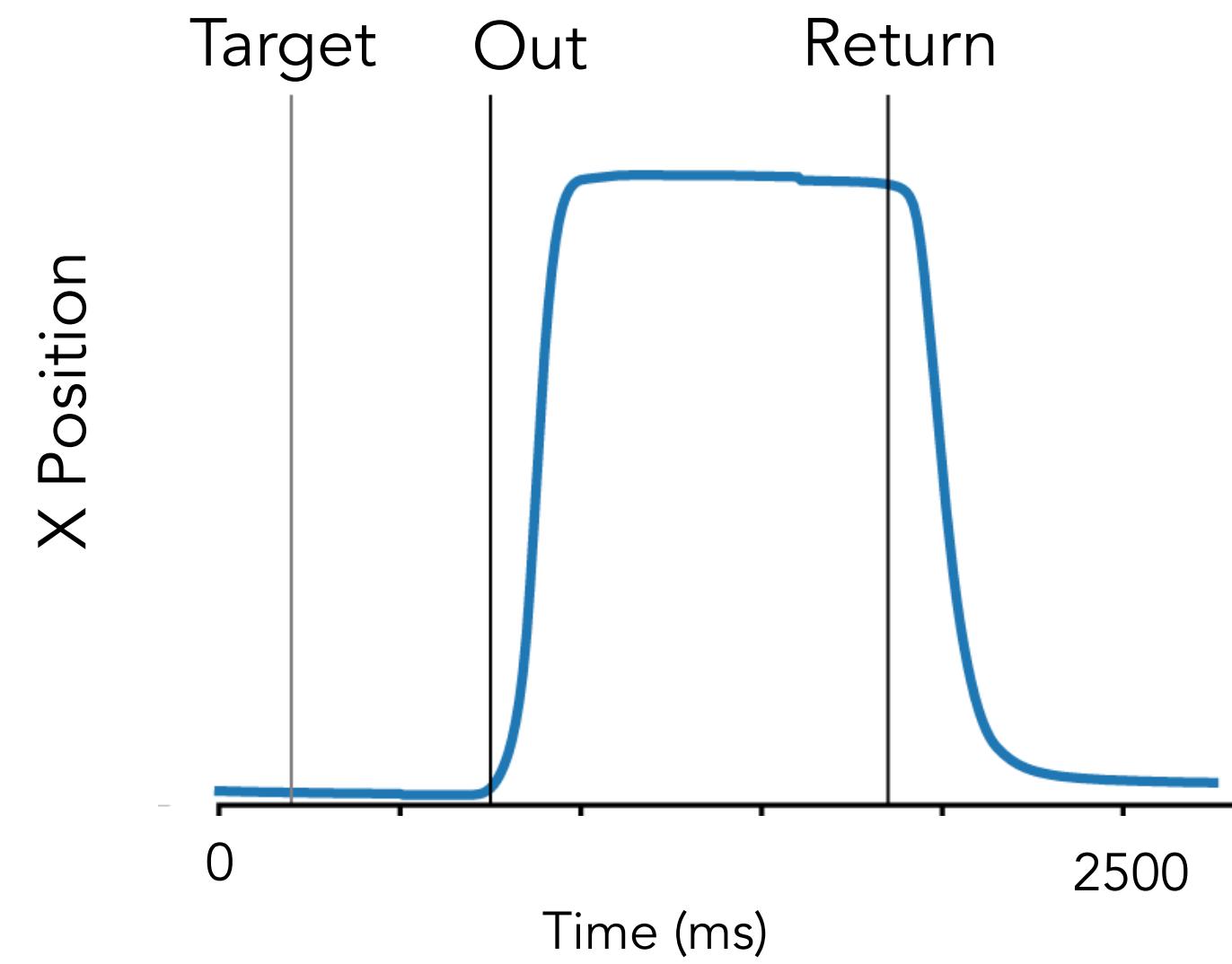
PCA Dimensions



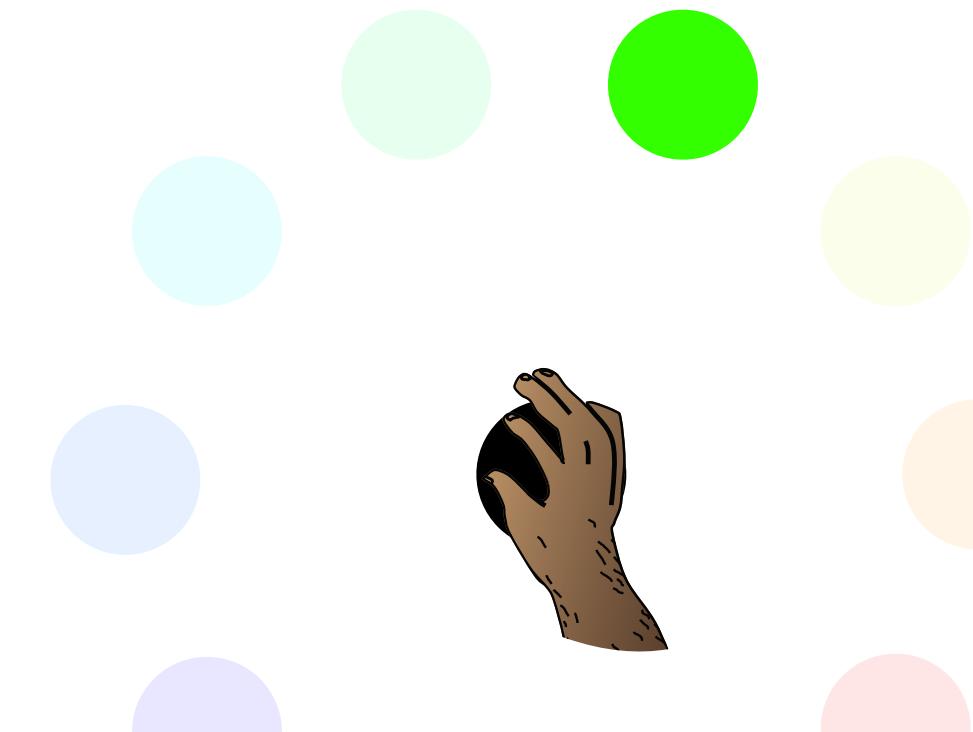
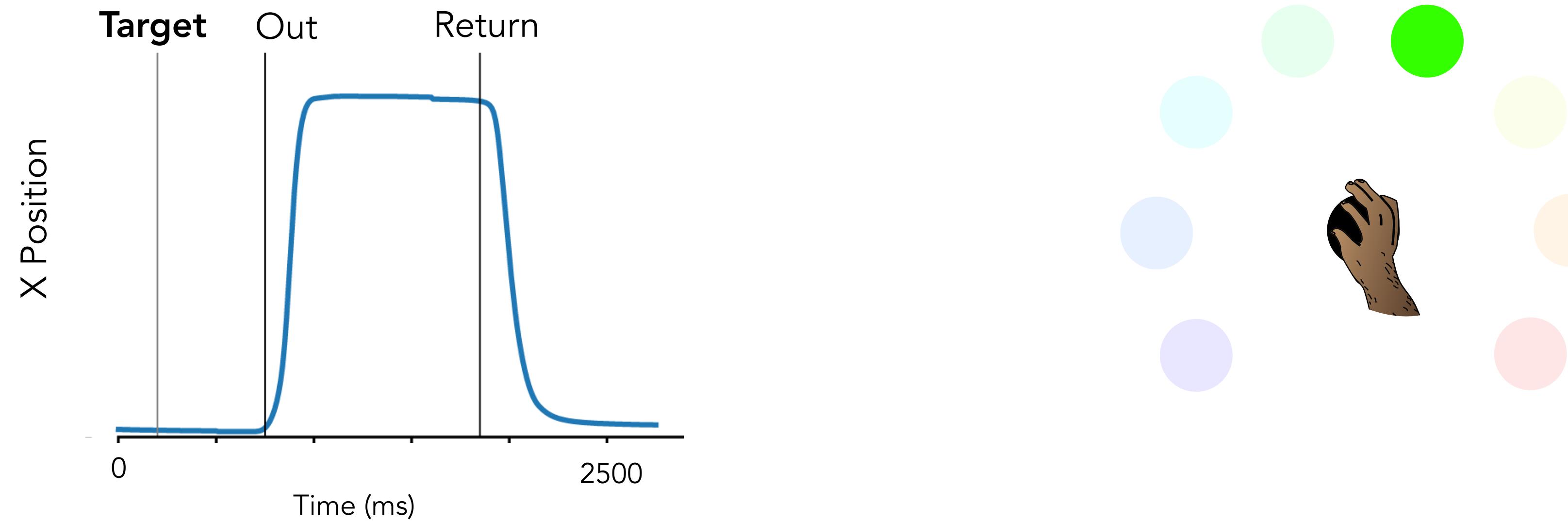
# Center-out and self-paced return task



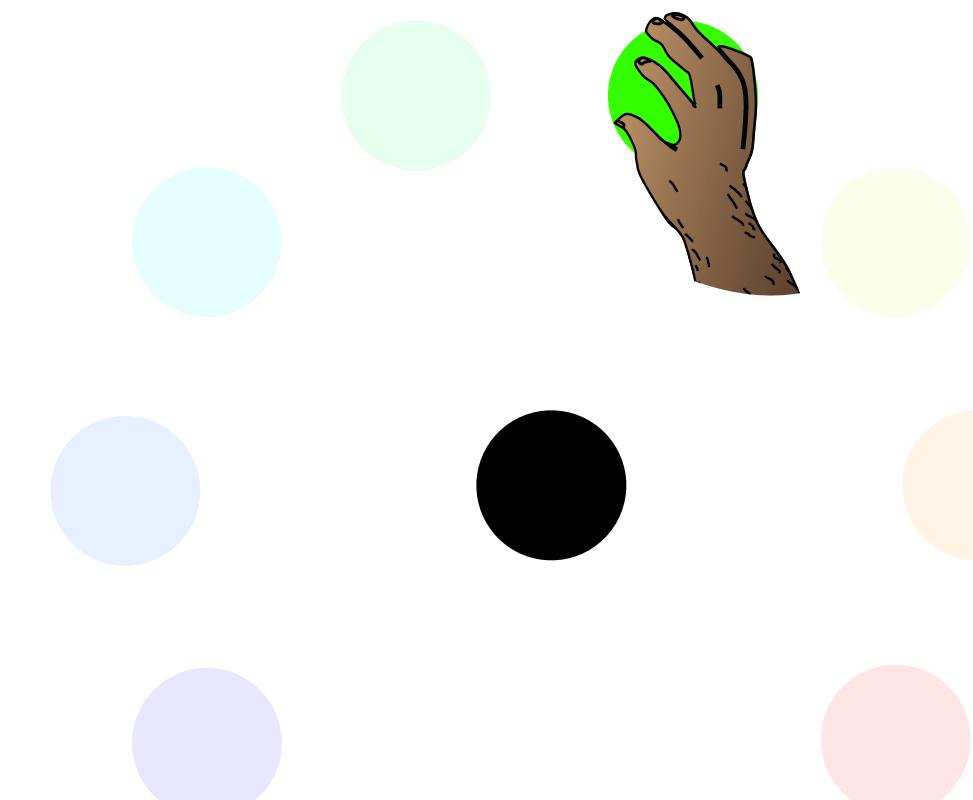
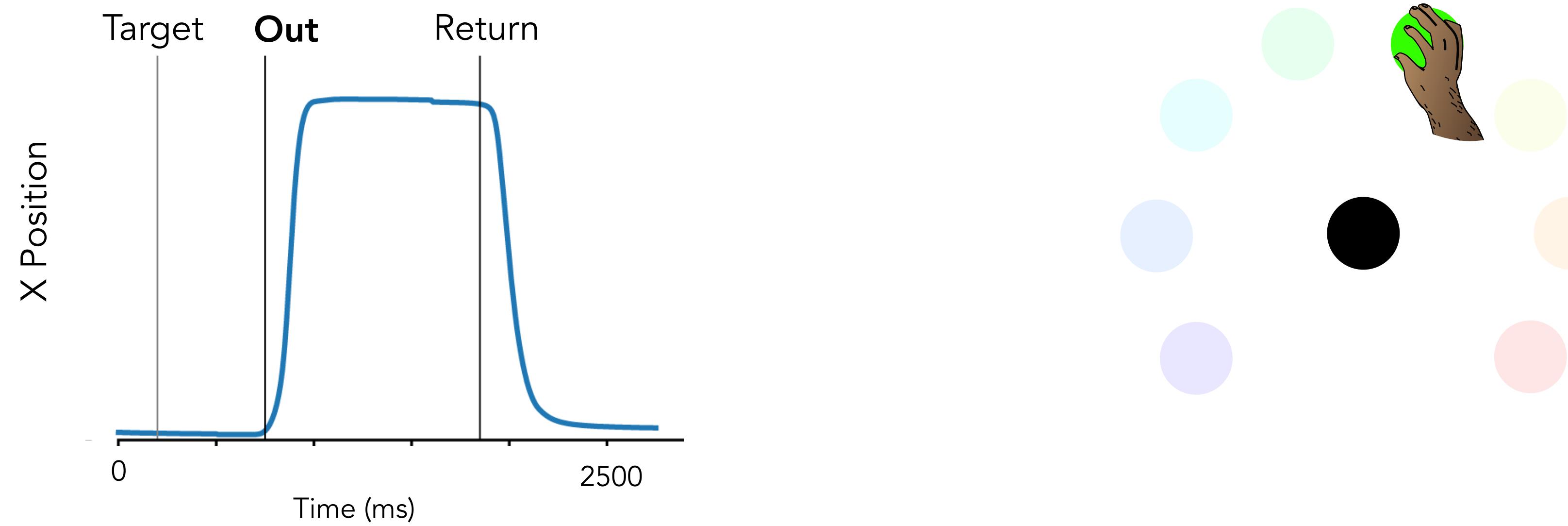
# Center-out and self-paced return task



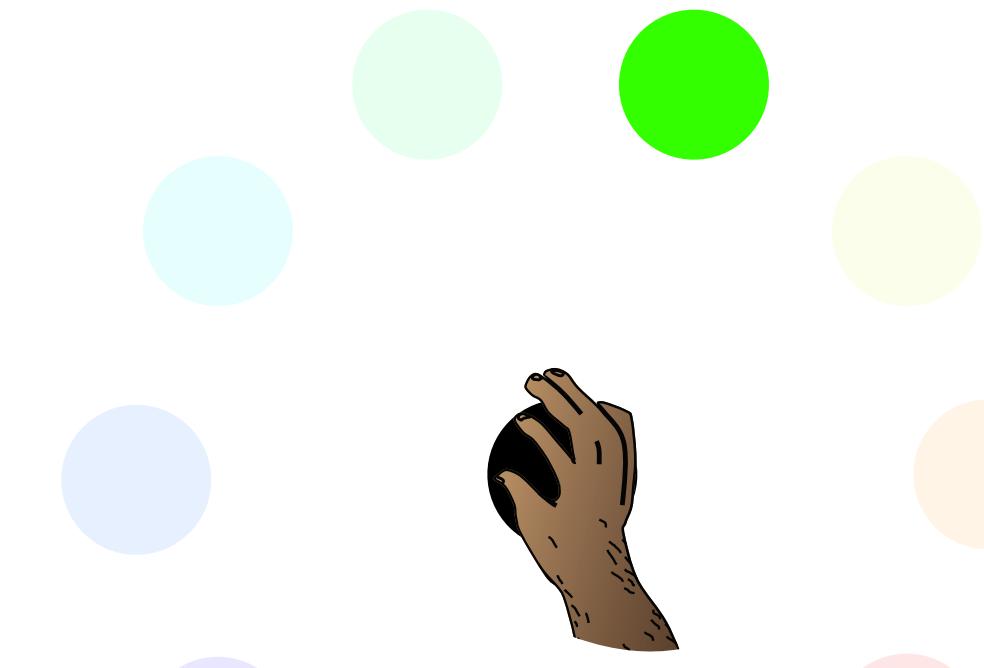
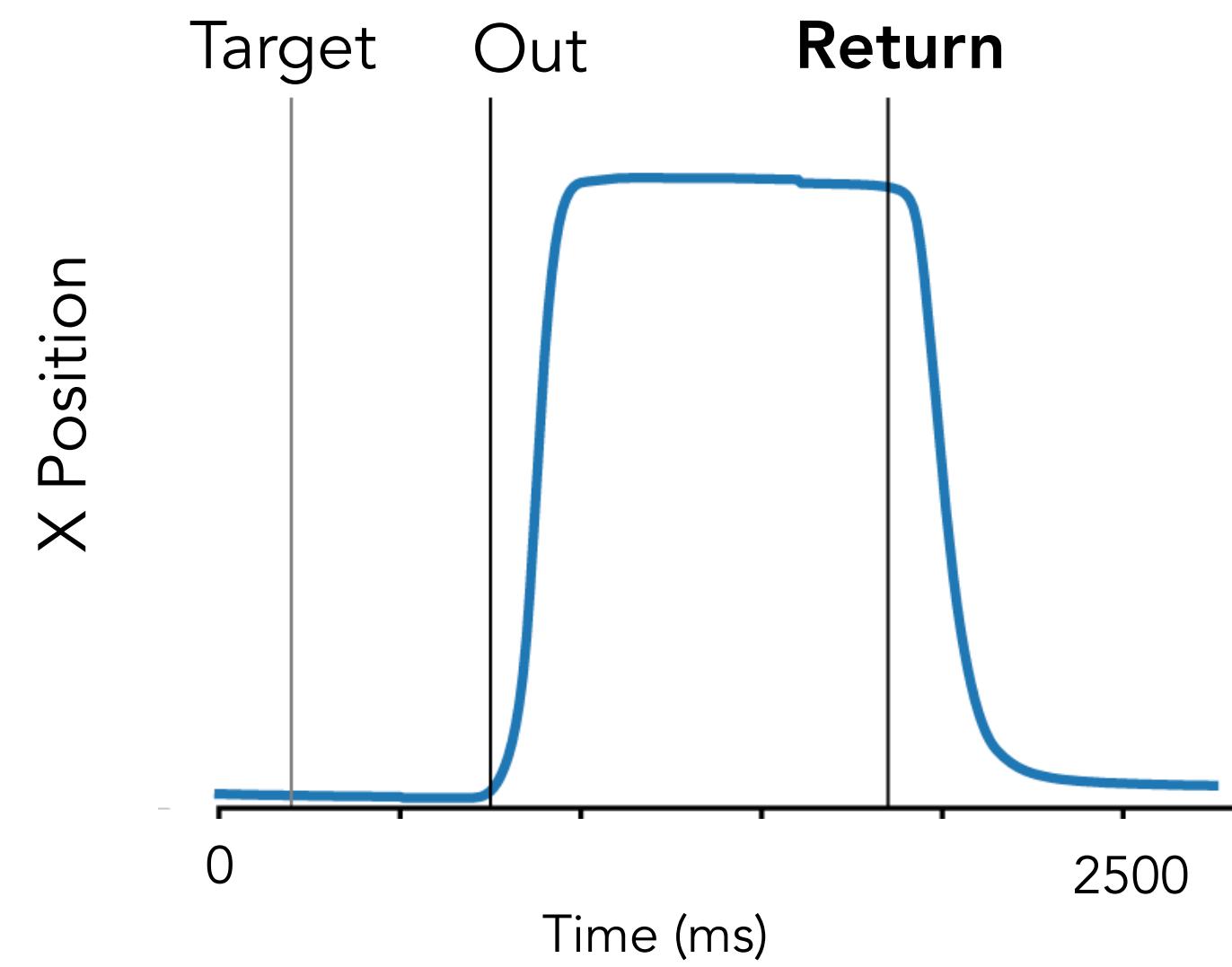
# Center-out and self-paced return task



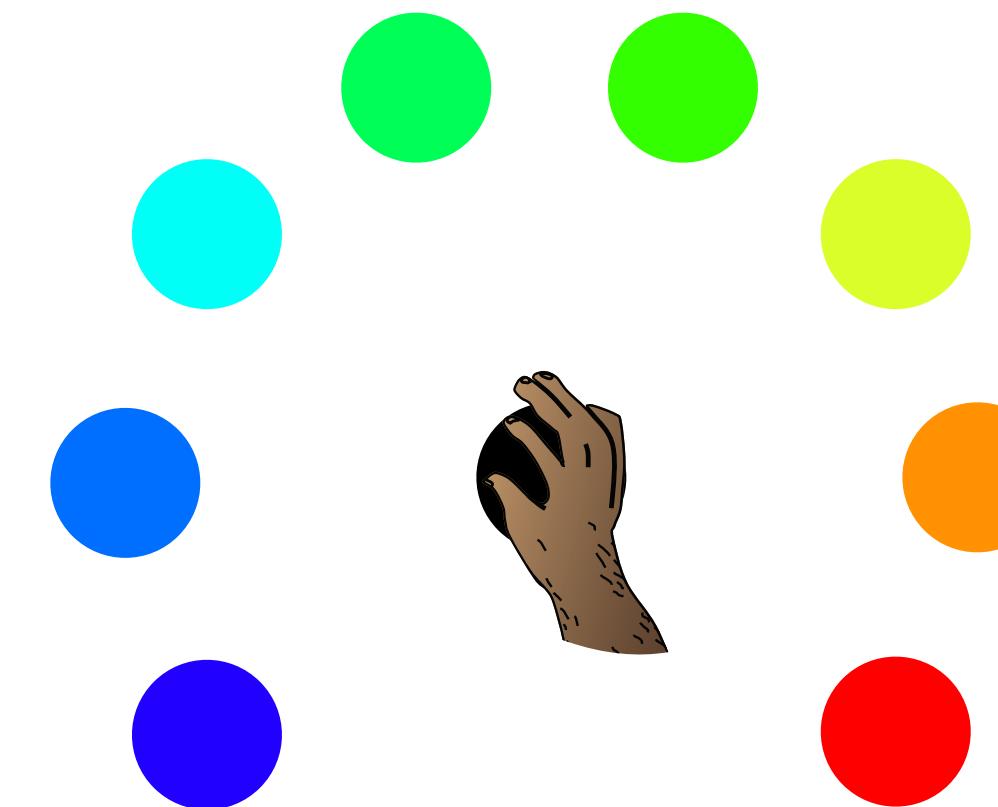
# Center-out and self-paced return task



# Center-out and self-paced return task

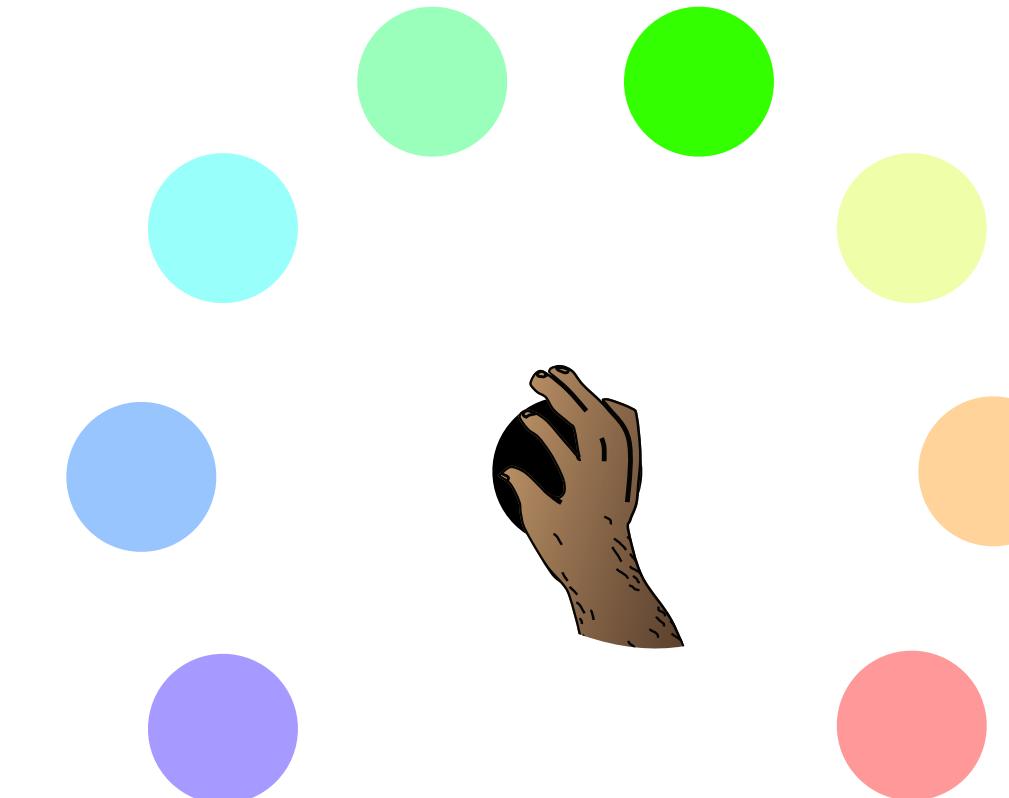
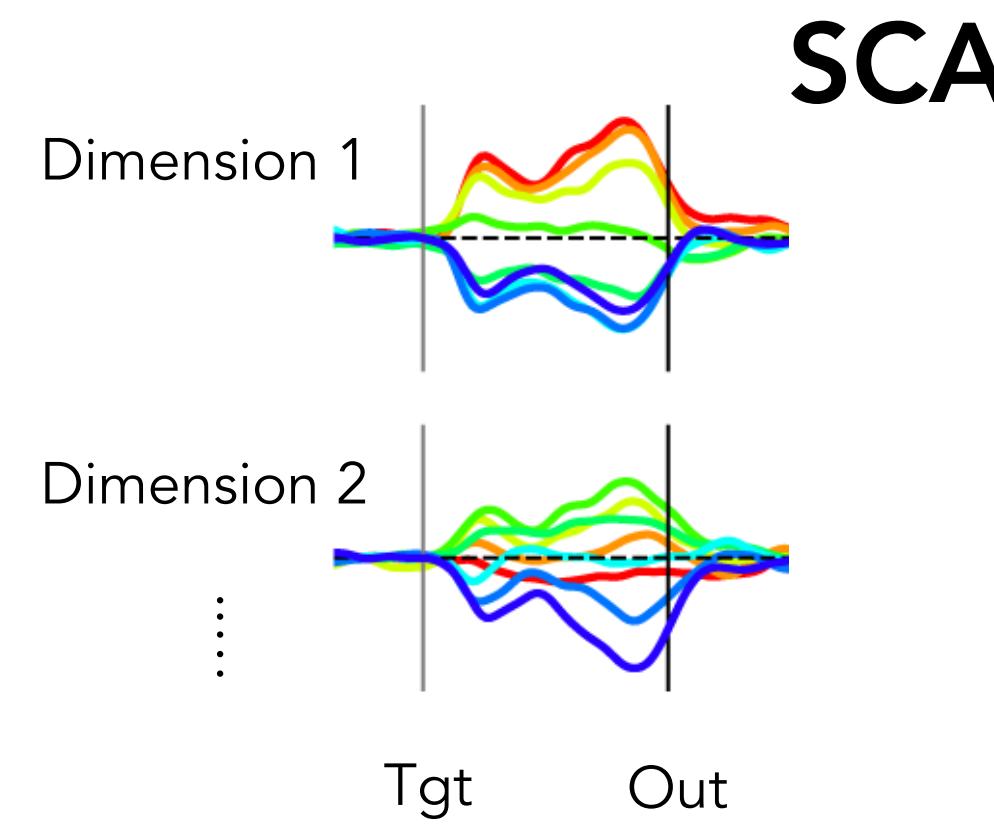


# SCA finds more sparse/interpretable dimensions



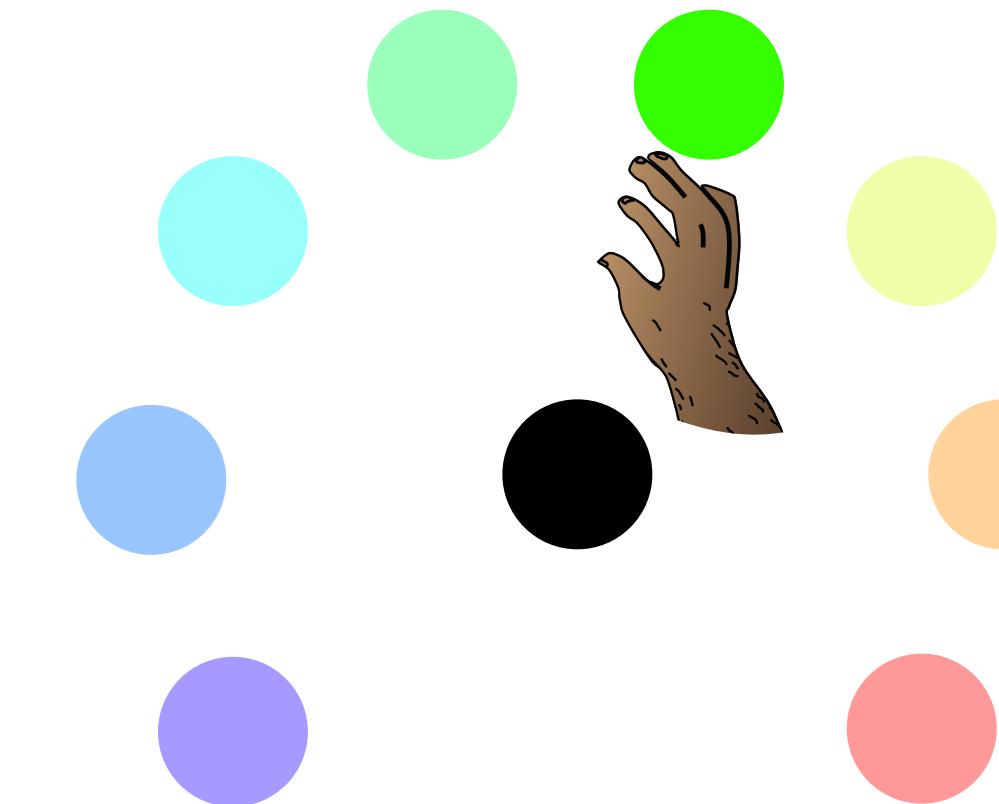
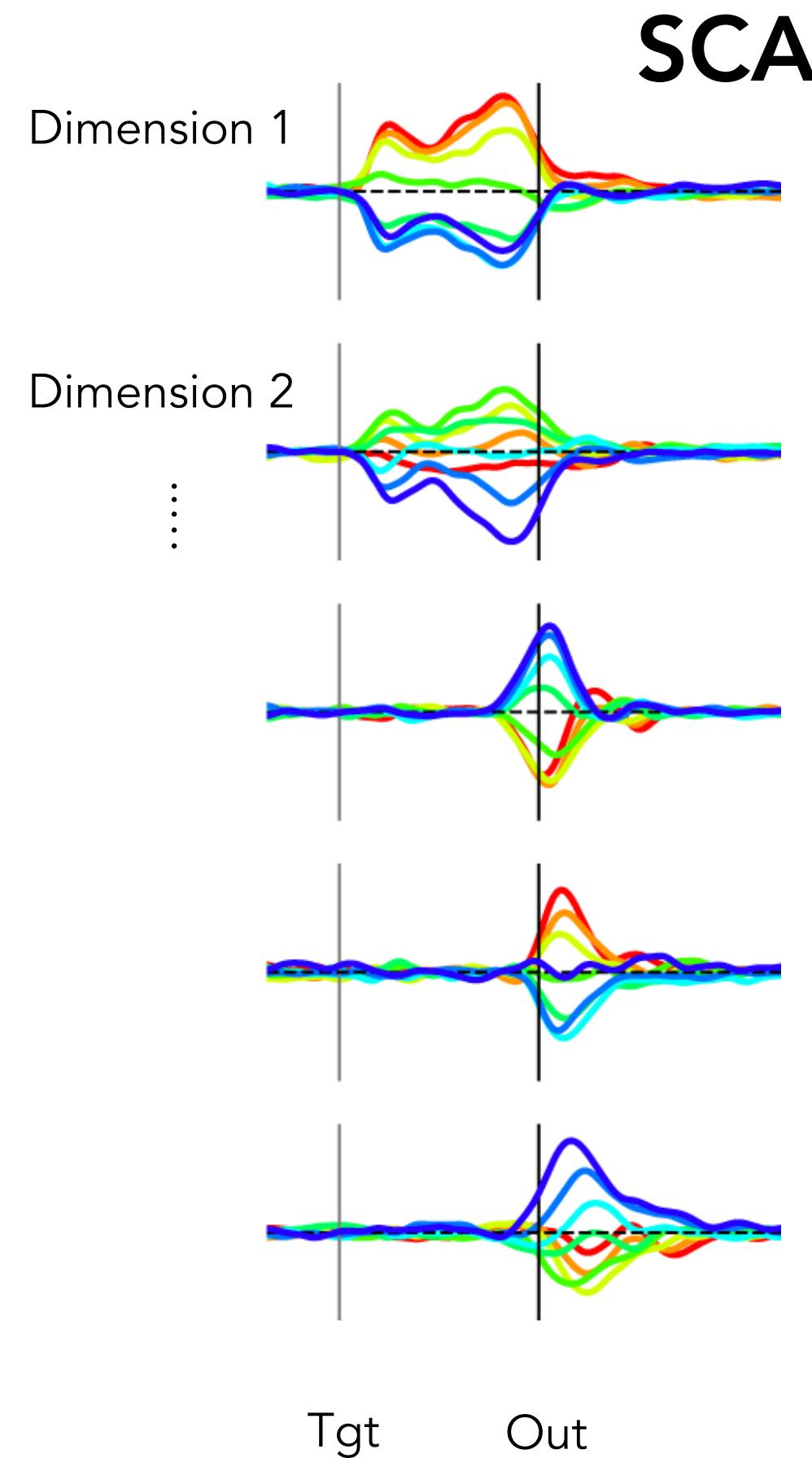
500 ms

# SCA finds more sparse/interpretable dimensions



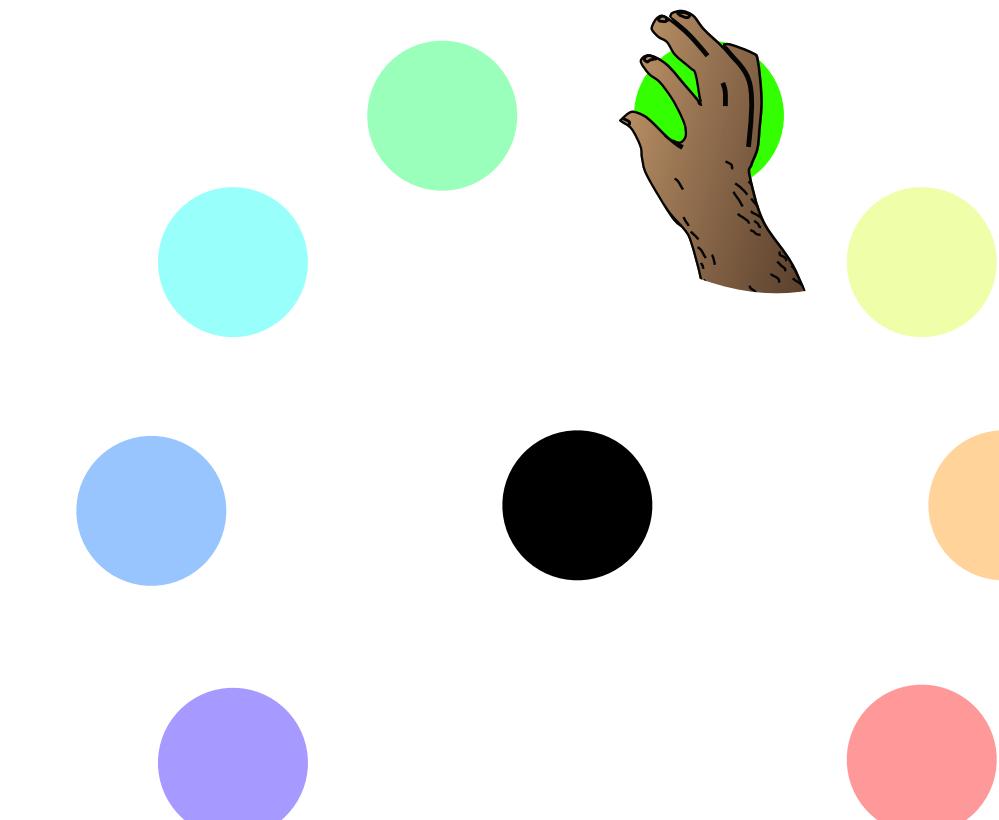
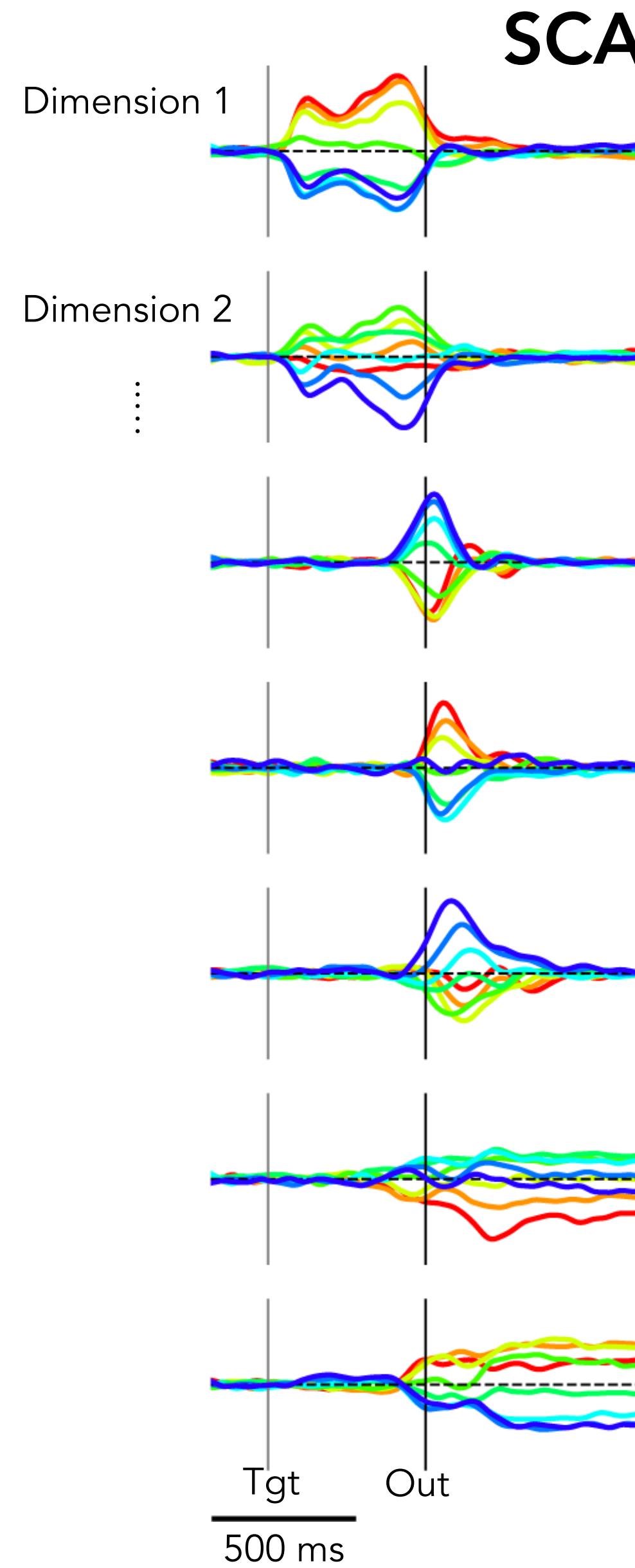
500 ms

# SCA finds more sparse/interpretable dimensions

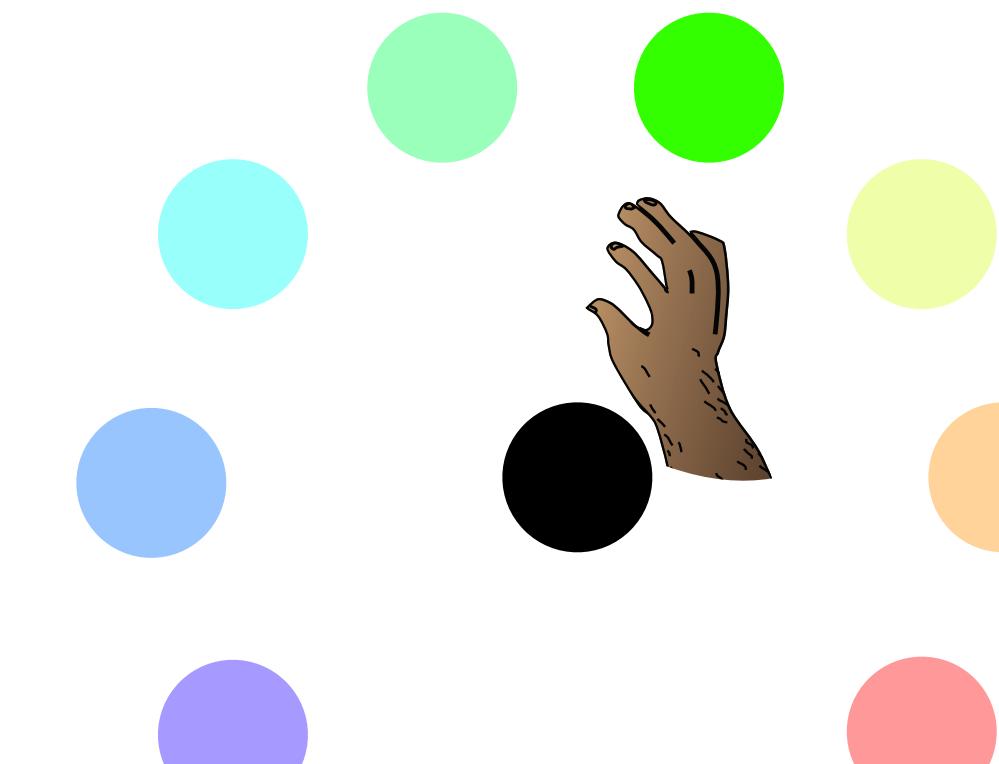
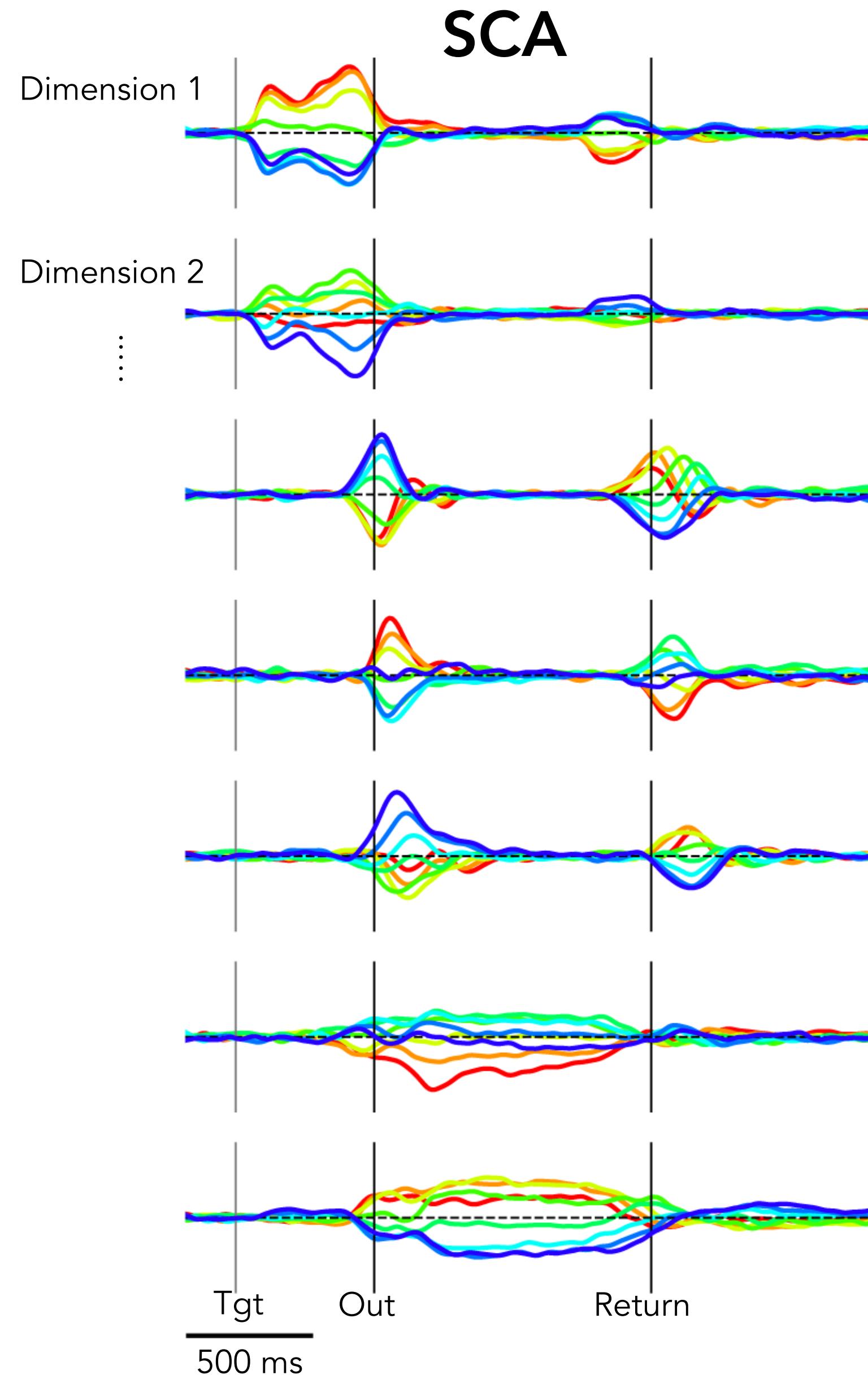


500 ms

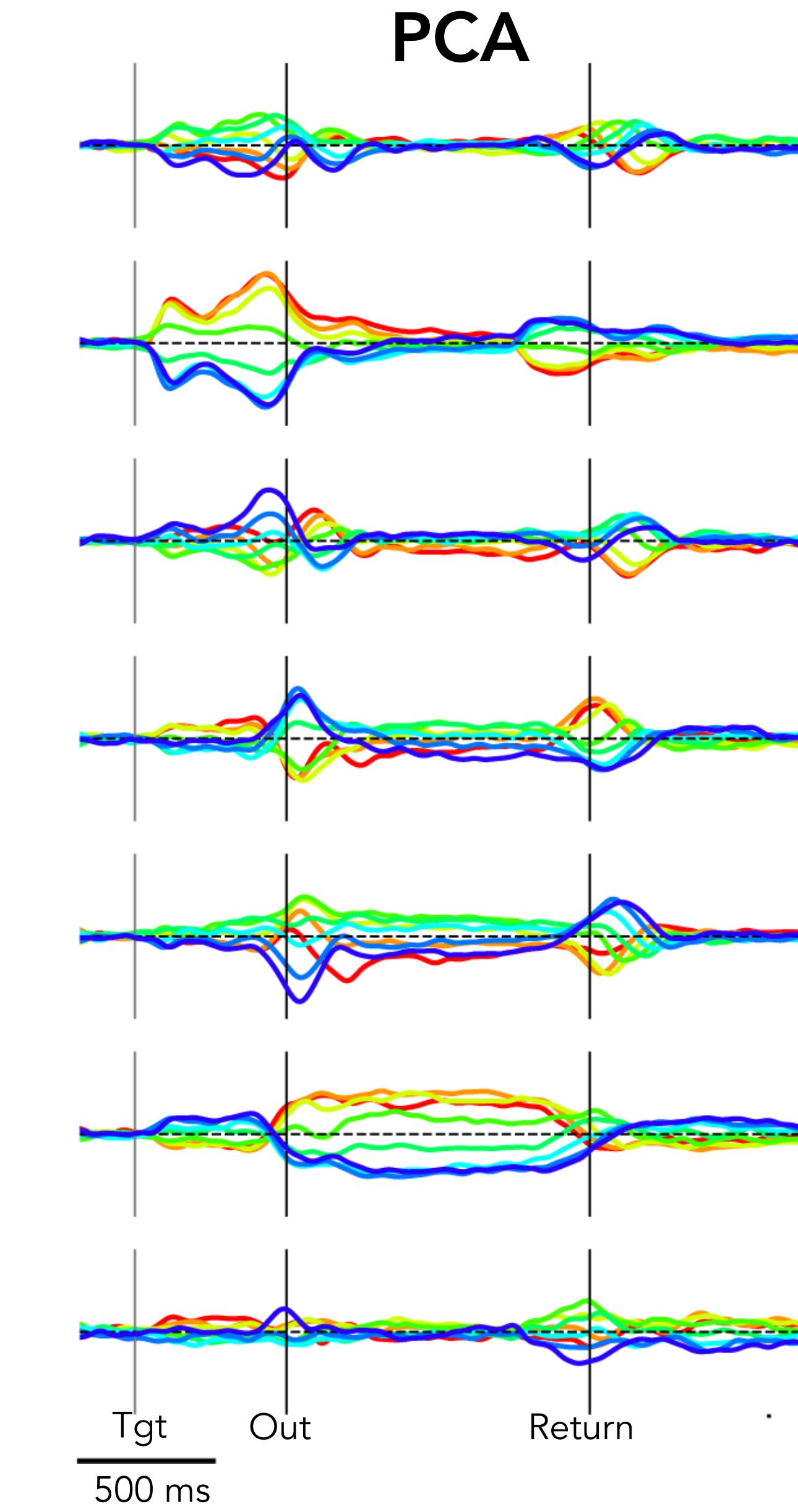
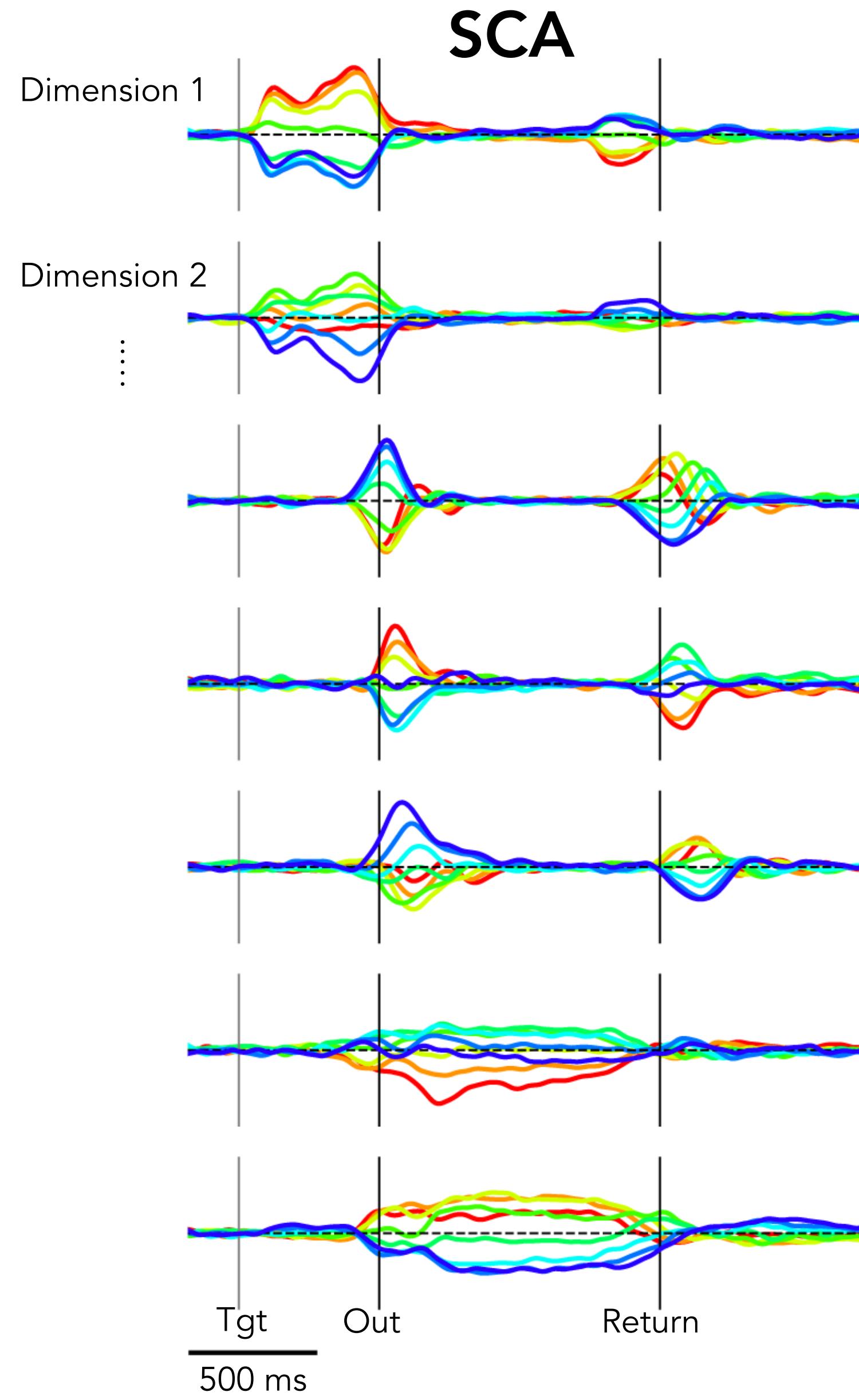
# SCA finds more sparse/interpretable dimensions



# SCA finds more sparse/interpretable dimensions



# SCA finds more sparse/interpretable dimensions



# Summary

- Find more interpretable dimensions in an unsupervised way!
- Code available at [github.com/glaserlab/sca](https://github.com/glaserlab/sca)
- Preprint available on *bioRxiv*



# Resources

- Probabilistic Machine Learning Book 2, Chaps. 7,10, 29
  - <https://probml.github.io/pml-book/book2.html>
- Good blog post on cross-validation:
  - <https://alexhwilliams.info/itsneuronalblog/2018/02/26/crossval/>