Graphical Models and Image Processing **61,** 363–367 (1999)
Article ID gmip.1999.0506, available online at http://www.idealibrary.com on IDEAL®

Unification of Distance and Volume Optimization in Surface Simplification

Dongryeol Kim, Jinsoo Kim, and Hyeong-Seok Ko[1]

Seoul National University Human Animation Center, School of Electrical Engineering, Seoul National University, Seoul 151-742, Korea
E-mail: drkim@math.snu.ac.kr, jskim@math.snu.ac.kr, ko@graphics.snu.ac.kr

Received March 29, 1999; revised August 9, 1999; accepted August 23, 1999

A popular method for simplifying a surface is to repeatedly contract an edge into a vertex and take concomitant actions. In such edge contraction algorithms, the position of the new vertex plays an important role in preserving the original shape. Two methods among them are distance optimization and volume optimization. Even though the two methods were independently developed by different groups and were regarded as two different branches, we found that they are unifiable. In this paper we show that they can be expressed with the same formula, and the only differences are in the weights. We prove that volume optimization is actually a distance optimization weighted by the area of triangles adjacent to the contracted edge.

Key Words: unification; surface simplification; distance optimization; volume optimization.

1. INTRODUCTION

In real-time or interactive applications, models with millions of polygons are burdensome even with fast graphics hardwares. Therefore simplification of surfaces has been the subject of a great deal of research. Simplification algorithms can be divided into three main categories according to the strategies employed [4]:

- vertex clustering [8, 10],
- vertex decimation [1, 3, 11, 12], and
- edge contraction [2, 4–7, 9].

Simplification algorithms based on iterative edge contraction replace an edge with a vertex, remove two triangles that shared the edge, and adjust the remaining triangles to

[1] To whom correspondence should be addressed: SNU Human Animation Center, School of Electrical Engineering, Seoul National University, Seoul 151-742, Korea. Fax: +82-2-882-4656.

1077-3169/99 $30.00

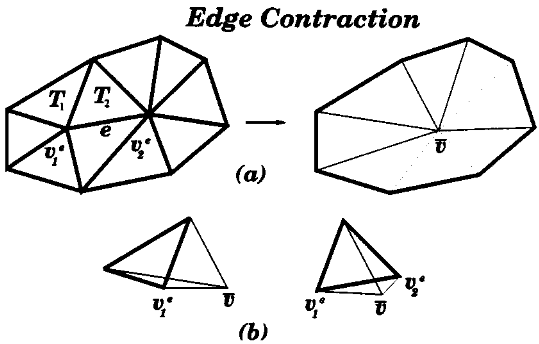

FIG. 1. Edge contraction. (a) Edge e is contracted into a vertex $\bar{v}$. (b) The two drawings from left to right are associated with the triangles T_1 and T_2, respectively. They are tetrahedra showing the volume change during the procedure. For example, in the left one, the triangle drawn with thick lines coincides with T_1 and is the base of the tetrahedron. The thin lines now form the volume change associated with T_1 due to the new vertex position $\bar{v}$.

reflect the new vertex position on a closed manifold surface as shown in Fig. 1a. As a result, three edges and one vertex are reduced per contraction.

Algorithms based on iterative edge contraction are getting attention these days since the new vertex position can be controlled for retaining the geometry of the original model and retriangulation is not needed. Several variations of edge contraction have been proposed based on which edge should be collapsed first and where to place the new vertex. Among the variations, two techniques are of interest: distance optimization and volume optimization.

Distance optimization [4, 9] is to minimize the distance from an object vertex (of the simplified model) to the nearby planes of the original model. Ronfard and Rossignac [9] measured the error at a vertex by the maximum distance between the vertex and the planes. Instead of the maximum distance, Garland and Heckbert [4] used the sum of the squared distances, along with a memory-computation efficient algorithm called "quadric error metrics" to accumulate the cost of contraction at an object vertex as the simplification progresses. Cohen *et al.* [2] presented an algorithm which computes the planar projections to use a piecewise linear mapping. They chose a vertex in two dimensions and determined the position of the vertex by minimizing the error in the remaining third direction.

Volume optimization is to control the position of the new vertex so that the local geometry resembles the original shape by minimizing the sum of the squares of the volume differences formed between the old and new triangles (Fig. 1b). After Guéziec [5] introduced the idea, Lindstrom and Turk [7] added boundary optimization condition to minimize the change of the area enclosed by the boundary and also adopted triangle shape optimization to avoid singularity by placing the new vertex nearby the center of surrounding triangles.

The above distance and volume optimizations have been developed independently and seem to be two different methods. Since they have their own merits, one might think of combining the two approaches. If Q and V are the objective functions that are minimized in distance optimization and volume optimization, respectively, then minimization of $\alpha Q + \beta V$ would produce in between the two individual optimizations. Controlling the values of α and β, one might get a more desirable result than when either of the optimizations is done alone.

However, our investigation reveals that such combination has little meaning since the two methods turn out to be unifiable. We found the optimization goal Q and V can be expressed by the same formulation, and the only differences are in the weights of the terms appearing in the formula. In fact we will prove that volume optimization is a distance optimization weighted by the area of triangles adjacent to the contracted edge. Hence simply combining the two methods produces no significant effect.

2. UNIFICATION OF DISTANCE AND VOLUME OPTIMIZATION

For an edge e, let v_1^e and v_2^e be the two vertices. For a vertex v, let $I(v)$ represent the set of indices of the triangles adjacent to v, and let $I(e) = I(v_1^e) \cup I(v_2^e)$. Let v_{j1}, v_{j2}, and v_{j3} be the vertices of a triangle T_j, and let P_j be the plane containing the triangle T_j.

Ronfard and Rossignac [9], and Garland and Heckbert [4] made use of the vertex-to-plane distances to locate the new vertex. Let $n_j = (n_{jx}, n_{jy}, n_{jz})$ be the outward unit normal vector of the plane P_j, and let c_j be the signed distance between the origin and the plane, depending on which side of the plane P_j the origin lies. Then the signed distance from a point $\overline{v} = (\overline{v}_x, \overline{v}_y, \overline{v}_z)$ to this plane is given by

$$d(\overline{v}, P_j) = n_j \cdot \overline{v} + c_j, \tag{1}$$

where $\cdot$ represents the inner product. Let $D_j(\overline{v}) = d(\overline{v}, P_j)$. To determine the new vertex $\overline{v}$ for the contracted edge e, we minimize the error metric

$$Q_e(\overline{v}) = \sum_{j \in I(e)} D_j(\overline{v})^2,$$

which is the sum of squared distances from $\overline{v}$ to the planes of triangles adjacent to v_1^e or v_2^e. (Garland and Heckbert [4] used $Q'_e(\overline{v}) = Q_{v_1^e}(\overline{v}) + Q_{v_2^e}(\overline{v})$, where $Q_v(\overline{v}) = \sum_{j \in I(v)} D_j(\overline{v})^2$.)

Another way to preserve the geometrical shape during simplification is to minimize the variance of volume differences between the object and original models. Let $V_j(\overline{v}) = V(\overline{v}, v_{j1}, v_{j2}, v_{j3})$ be the volume of the tetrahedron consisting of $\overline{v}$ and T_j as in Fig. 1b. Since

$$V_j(\overline{v}) = \tfrac{1}{6}\{((v_{j1} - v_{j2}) \times (v_{j1} - v_{j3})) \cdot \overline{v} - \det(v_{j1}, v_{j2}, v_{j3})\}, \tag{2}$$

the variance of the volume differences multiplied by 36 is

$$\begin{aligned} V_e(\overline{v}) &= 36 \sum_{j \in I(e)} V_j(\overline{v})^2 \\ &= \overline{v} \sum_{j \in I(e)} \left(N_j^T N_j\right) \overline{v}^T - 2 \sum_{j \in I(e)} \det(v_{j1}, v_{j2}, v_{j3})\, N_j \cdot \overline{v} + \sum_{j \in I(e)} \det(v_{j1}, v_{j2}, v_{j3})^2, \end{aligned}$$

where $N_j = (v_{j1} - v_{j2}) \times (v_{j1} - v_{j3})$ and the magnitude of N_j is twice the area of T_j. In [7], Lindstron and Turk imposed an extra constraint that the volume should be constant. When there are multiple solutions, triangle shape optimization was done.

In summary, the distance optimization and volume optimization minimize

$$Q_e(\overline{v}) = \sum_{j \in I(e)} D_j(\overline{v})^2 \tag{3}$$

and

$$V_e(\overline{v}) = 36 \sum_{j \in I(e)} V_j(\overline{v})^2, \tag{4}$$

respectively.

Here we claim that each $V_j(\overline{v})$ is a multiple of $D_j(\overline{v})$ by a constant factor. For that, let us compare Eqs. (1) and (2). The first term of Eq. (2) is $\|N_j\|$ times that of Eq. (1). Therefore we need to check whether the second term, $-\det(v_{j1}, v_{j2}, v_{j3})$, of Eq. (2) is also $\|N_j\|$ times c_j, the second term of Eq. (1). Since the plane equation of P_j is $n_j \cdot x + c_j = 0$ and v_{j1} is on P_j,

$$-c_j = n_j \cdot v_{j1} = \frac{(v_{j1} - v_{j2}) \times (v_{j1} - v_{j3})}{\|(v_{j1} - v_{j2}) \times (v_{j1} - v_{j3})\|} \cdot v_{j1} = \frac{\det(v_{j1}, v_{j2}, v_{j3})}{\|N_j\|}.$$

Hence we obtain

$$V_j(\overline{v}) = \frac{1}{6}\|N_j\| D_j(\overline{v}).$$

Therefore Eq. (4) can be written as

$$V_e(\overline{v}) = \sum_{j \in I(e)} \|N_j\|^2 D_j(\overline{v})^2. \tag{5}$$

Thus minimization of V_e is a weighted l^2-minimization of D_j's, in which the weight of D_j is given by the area of its corresponding triangle T_j up to a constant factor.

3. CONCLUSION

Even though the above unification of distance and volume optimization is valid in general, the unification does not exactly work for the specific algorithms proposed in [4, 7] due to the following reasons. First, Garland and Heckbert's quadric error metric [4] is an approximation: the quadric is slightly different from Eq. (3). Instead of summing over $I(e)$, they summed over $I(v_1^e)$ and $I(v_2^e)$ separately, resulting in duplicate summation over $I(v_1^e) \cap I(v_2^e)$. Once $\overline{v}$ is determined, the quadric error metric for $\overline{v}$ has to be given. They simplified this reduction process by adding two previous objective functions $Q_{v_1^e}$ and $Q_{v_2^e}$, instead of updating $Q_{\overline{v}}$ by calculating $\sum_{j \in I(\overline{v})} D_j(\hat{v})^2$. Second, Lindstrom and Turk's algorithm [7] is not simply a volume optimization, but also includes volume preservation and triangle shape optimization.

In the above, we have compared two methods, distance optimization and volume optimization for surface simplification. The difference between them is only in the weights. Let us suppose that we are to minimize the following equation:

$$G_e(\overline{v}) = \alpha Q_e(\overline{v}) + \beta V_e(\overline{v}). \tag{6}$$

Then it represents a Lagrangian formulation for two objective functions or a constrained optimization problem. Apparently, by adjusting α and β the importance of each objective function can be controlled. However, Eq. (6) can be rewritten as

$$G_e(\overline{v}) = \sum_{j \in I(e)} (\alpha + \beta'_j) D_j(\overline{v})^2, \tag{7}$$

where $\beta'_j = \beta \|((v_{j1} - v_{j2}) \times (v_{j1} - v_{j3}))\|^2$. From Eq. (7) we can conclude that the simple combination of two optimization methods such as in Eq. (6) is just a particular case of the weighted l^2-minimization of D_j's. Noting that the weight of D_j in Eq. (7) is proportional to the area of each triangle up to an additive constant, in order to produce a significantly different result, we might have to put other things into the weight than the area of the triangles.

ACKNOWLEDGMENTS

This work has been supported by Creative Research Initiatives of the Korean Ministry of Science and Technology. The work was also partially supported by ASRI (Automation and Systems Research Institute), Seoul National University. The second author gratefully acknowledges the support of the Research Institute of Mathematics, Global Analysis Research Center, and Basic Science Research Institute.

REFERENCES

1. A. Ciampalini, P. Cignoni, C. Montani, and R. Scopigno, Multiresolution decimation based on a global error, in *The Visual Computer*, Vol. 13, No. 5, pp. 228–246, Springer-Verlag, Berlin/New York, 1997.
2. J. Cohen, D. Manocha, and M. Olano, Simplifying polygonal models using successive mappings, in *IEEE Visualization'97 Proceedings, 1997*, pp. 395–402.
3. J. Cohen, A. Varshney, D. Manocha, G. Turk, H. Weber, P. Agarwal, F. Brooks, and W. Wright, Simplification envelopes, in *SIGGRAPH'96 Proceedings, 1996*, pp. 119–128.
4. M. Garland and P. S. Heckbert, Surface simplification using quadric error metrics, in *SIGGRAPH'97 Proceedings, 1997*, pp. 209–216.
5. A. Guéziec, Surface simplification inside a tolerance volume, in *Second Annual International Symposium on Medical Robotics and Computer Aided Surgery, 1995*, pp. 132–139.
6. H. Hoppe, Progressive mesh, in *SIGGRAPH'96 Proceedings, 1996*, pp. 99–108.
7. P. Lindstrom and G. Turk, Fast and memory efficient polygonal simplification, in *IEEE Visualization'98 Conference Proceedings, 1998*, pp. 279–286.
8. D. Luebke and C. Erikson, View-dependent simplification of arbitrary polygonal environments, in *SIGGRAPH'97 Proceedings, 1997*, pp. 199–208.
9. R. Ronfard and J. Rossignac, Full-range approximation of triangulated polyhedra, in *Computer Graphics Forum, Eurographics'96 Proceedings, 1996*, Vol. 15, No. 3, pp. 67–76.
10. J. Rossignac and P. Borrel, Multi-resolution 3D approximations for rendering complex scenes, in *Modeling in Computer Graphics: Methods and Application, 1993*, pp. 455–465.
11. W. J. Schroeder, J. A. Zarge, and W. E. Lorensen, Decimation of triangle meshes, in *SIGGRAPH'92, Proceedings, 1992*, pp. 65–70.
12. M. Soucy and D. Laurendeau, Multiresolution surface modeling based on hierarchical triangulation, *Comput. Vision Image Understand.* **63**, 1996, 1–14.