CPEN 416 / EECE 571O

Lecture 03: Single-qubit operations and projective measurement

Thursday 11 September 2025

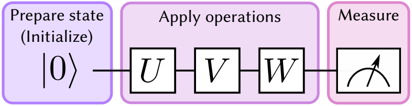
Announcements

- Practice quiz questions and practice assignment 1 available
- Tutorial 2 on Monday
- Quiz 2 on Tuesday (based on content from lectures 2 and 3)

Last time

We represented quantum computations in two new ways.

Quantum circuits:



Quantum functions (and QNodes) in PennyLane:

```
import pennylane as qml

dev = qml.device("default.qubit", wires=1)

@qml.qnode(dev)
def my_quantum_function():
    qml.Hadamard(wires=0)
    qml.PauliZ(wires=0)
    return qml.probs()
```

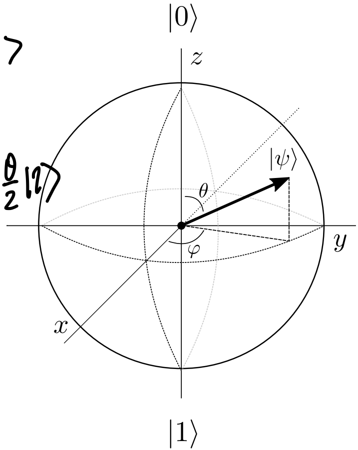
Last time

We parametrized single-qubit states with 2 real numbers, and plotted states in 3D space on the Bloch sphere.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\downarrow$

$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$

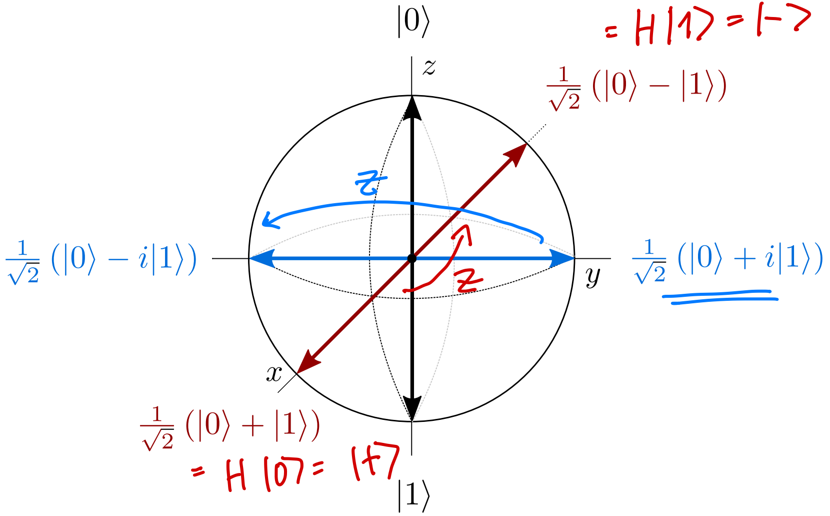


https://bloch.kherb.io/

Learning outcomes

1. Describe the behaviour of common single-qubit gates
2. compute the inner product between two quantum states
3. perform a projective measurement in the computational basis
4. apply basis rotations to perform a projective measurement in any basis

The Bloch sphere



Rotations: the Bloch sphere

Unitary operations rotate the state vector on the Bloch sphere.

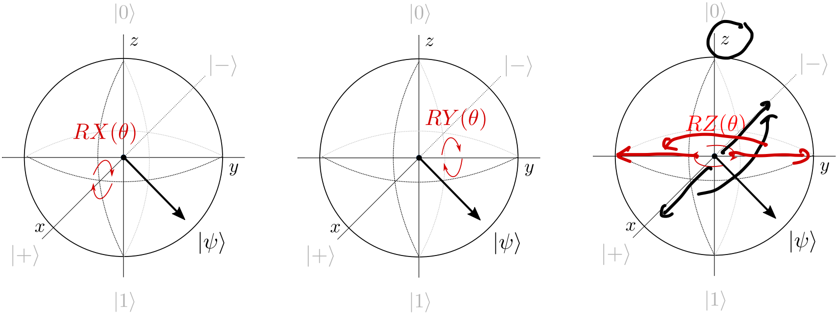
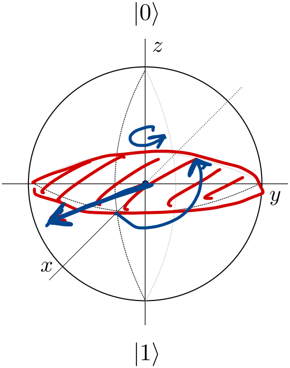


Image credit: Codebook node I.6

Z rotations (RZ)

$$RZ(\phi) = e^{-i\frac{\phi}{2}Z} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}$$



In PennyLane:

```
qml.RZ(phi, wires=wire)
```

Try at home: directly expand $e^{-i\frac{\phi}{2}Z}$ to obtain the matrix representation.

Z rotations (RZ)

$a|0\rangle + \underbrace{b e^{i\varphi}}_{\text{relative phase}} |1\rangle$

$Z = RZ(\pi) \sim RZ(-\pi)$

$$RZ(\phi) = e^{-i\frac{\phi}{2}Z} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}$$

$RZ(\phi)|0\rangle = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2} \\ 0 \end{pmatrix} = e^{-i\phi/2}|0\rangle$

Apply to a general state:

$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$

$RZ(\phi)|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cdot \underbrace{\left(RZ(\phi)|0\rangle\right)}_{e^{-i\phi/2}|0\rangle} + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}\underbrace{\left(RZ(\phi)|1\rangle\right)}_{e^{i\phi/2}|1\rangle}$

$= \cos\frac{\theta}{2} e^{-i\phi/2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi} e^{i\phi/2} |1\rangle$

$= e^{-i\phi/2}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi} e^{i\phi} |1\rangle\right)$

$\underset{\text{equiv.}}{\sim} \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i(\varphi+\phi)}|1\rangle$

ϕ'

S and T

Two other special cases: $\phi = \pi/2$, and $\phi = \pi/4$.

$$S = RZ(\pi/2) = \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$T = RZ(\pi/4) = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

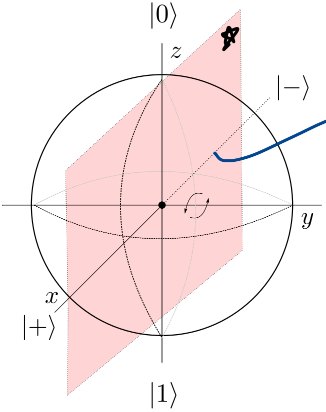
In PennyLane:

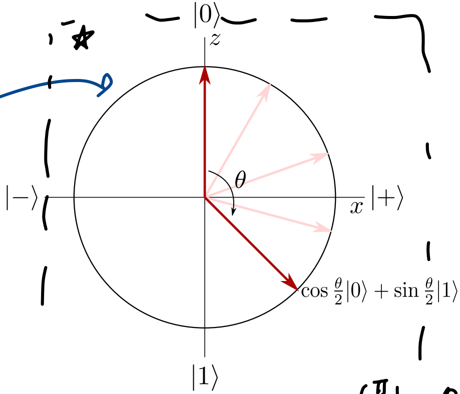
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi}|1\rangle$$
$$\downarrow$$
$$= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i(\varphi+\phi)}|1\rangle$$

```
qml.PauliZ(wires=wire)
qml.S(wires=wire)
qml.T(wires=wire)
```

S is part of a special group called the **Clifford group**.
T is used in universal gate sets for fault-tolerant QC.

Y rotations (RY)



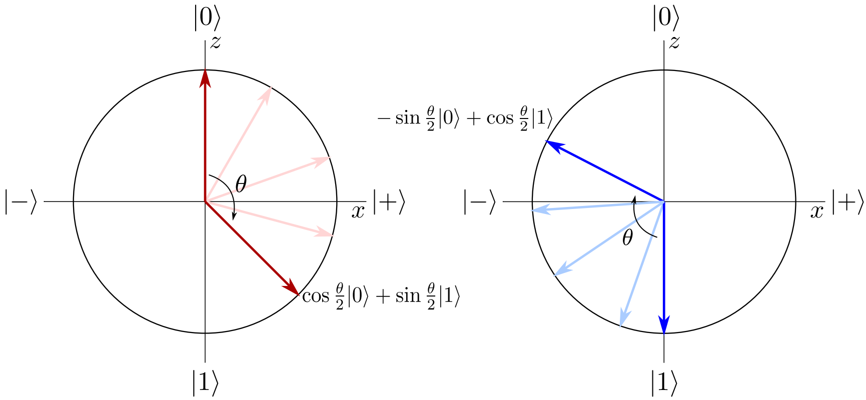


$RY(\pi)$: $\cos\left(\frac{\pi}{2}\right) : 0$

$\sin\left(\frac{\pi}{2}\right) : 1$

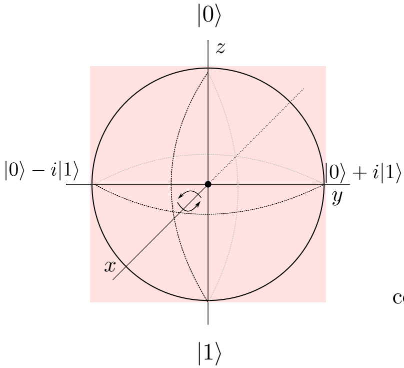
Y rotations (RY)

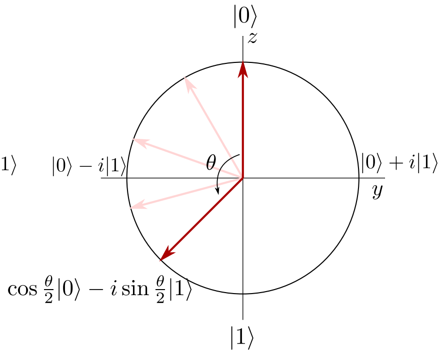
$$RY(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$



X rotations (RX)

$$RX(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$





Pauli rotations

These unitary operations are called **Pauli rotations**.

	Math	Matrix	Code	Special cases (θ)
RZ	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	`qml.RZ`	$Z(\pi), S(\pi/2), T(\pi/4)$
RY	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RY`	$Y(\pi)$
RX	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RX`	$X(\pi), SX(\pi/2)$

Pauli rotations

"quantum state prep."

Exercise: design a quantum circuit that prepares

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5\pi}{4}}|1\rangle$$

2 gates

Hint: you can return the state directly in PennyLane:

```
@qml.qnode(dev)
def some_circuit():
    # Gates...
    return qml.state()
```

v1. — $\boxed{RY(\pi/3)}$ — $\boxed{RZ(5\pi/4)}$ — $\boxed{Z}$

v2. — $\boxed{RY(\pi/3)}$ — $\boxed{T}$ —

$RY(\pi/3)|0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

$T \cdot RY(\pi/3)|0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}e^{i\pi/4}|1\rangle$

$-e^{\frac{i5\pi}{4}} = -e^{i(\pi+\pi/4)} = (-e^{i\pi})e^{i\pi/4} = e^{i\pi/4}$

Pauli rotations

Exercise: implement the circuit below in PennyLane.

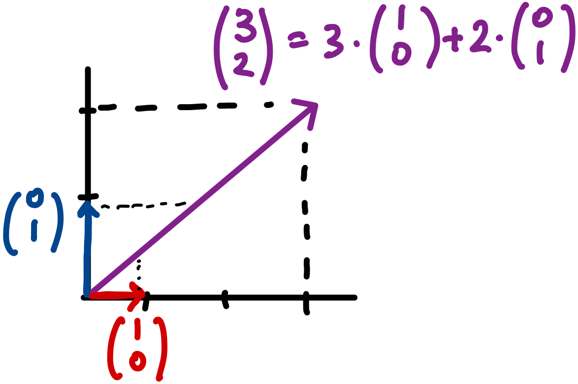
Run your circuit with two different values of θ and take 1000 shots.

How does θ affect the measurement outcome probabilities?

$$|\psi\rangle = \frac{1}{\sqrt{2}} e^{-i\theta/2} |0\rangle + \frac{1}{\sqrt{2}} e^{i\theta/2} |1\rangle$$

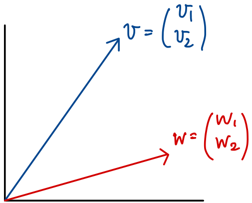
Inner products

We can now create any single-qubit quantum state: how do we *compare* them? Look to 2D space for intuition.



Inner products

The **inner product** (dot product) between two vectors quantifies how much overlap they have.



$$\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle = \vec{v}^T \cdot \vec{w} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= v_1 w_1 + v_2 w_2$$

$$= \sum_i v_i w_i$$

$$= |\vec{v}||\vec{w}| \cos\theta$$

Inner products

Take just one of these representations:

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$$

$$|v\rangle = v_1 |0\rangle + v_2 |1\rangle$$

$$|w\rangle = w_1 |0\rangle + w_2 |1\rangle$$

$$\boxed{v_i, w_i \in \mathbb{C}}$$

???

The inner product in Hilbert space is defined as

$$\left(|v\rangle^T\right)^* |w\rangle = \left(|v\rangle\right)^\dagger |w\rangle$$

To avoid cumbersome notation define the **bra** (*braket* notation):

$$\langle v| = \left(|v\rangle\right)^\dagger = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}$$

$$\langle v|w\rangle = v_1^* w_1 + v_2^* w_2$$

np.vdot(v, w)

Inner products

Exercise: compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

Start from here
on Tuesday.

Next time

Content: ⇒ first will cover projective meas. and basis rotations

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

Action items:

1. Prepare for quiz 2 (tutorial on Monday)
2. Work on practice Assignment 1

Recommended reading:

- For today: Codebook modules IQC, SQ; Nielsen & Chuang 1.3.3, 2.2.3-2.2.5
- For next class: Codebook module MQ; Nielsen & Chuang 4.3