CPEN 416 / EECE 571O

Lecture 03: Single-qubit operations and projective measurement

Thursday 11 September 2025

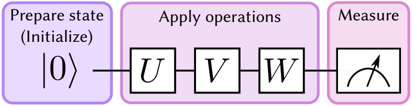
Announcements

- Practice quiz questions and practice assignment 1 available
- Tutorial 2 on Monday
- Quiz 2 on Tuesday (based on content from lectures 2 and 3)

Last time

We represented quantum computations in two new ways.

Quantum circuits:



Quantum functions (and QNodes) in PennyLane:

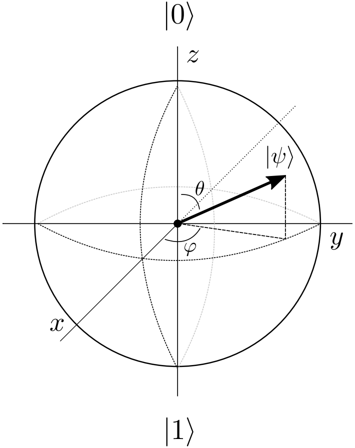
```
import pennylane as qml

dev = qml.device("default.qubit", wires=1)

@qml.qnode(dev)
def my_quantum_function():
    qml.Hadamard(wires=0)
    qml.PauliZ(wires=0)
    return qml.probs()
```

Last time

We parametrized single-qubit states with 2 real numbers, and plotted states in 3D space on the Bloch sphere.

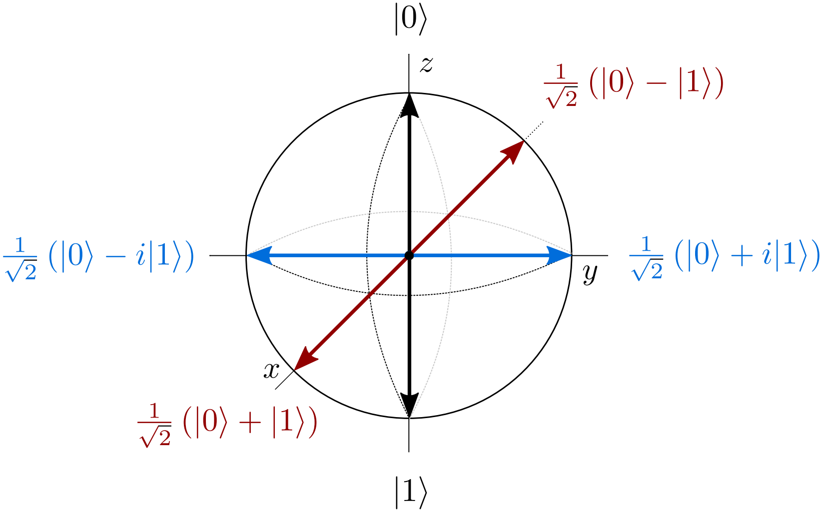


`https://bloch.kherb.io/`

Learning outcomes

1. Describe the behaviour of common single-qubit gates
2. compute the inner product between two quantum states
3. perform a projective measurement in the computational basis
4. apply basis rotations to perform a projective measurement in any basis

The Bloch sphere



Rotations: the Bloch sphere

Unitary operations rotate the state vector on the Bloch sphere.

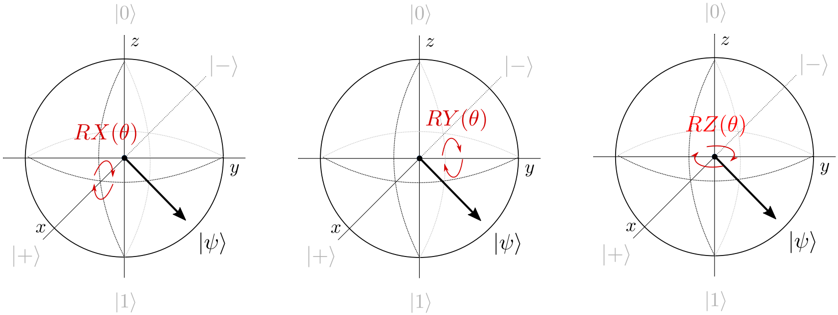
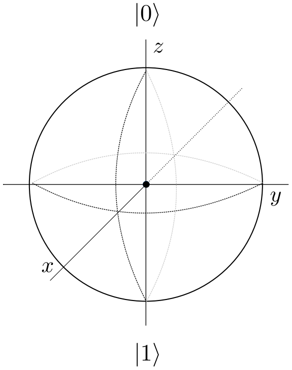


Image credit: Codebook node I.6

Z rotations (RZ)

$$RZ(\phi) = e^{-i\frac{\phi}{2}Z} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}$$



In PennyLane:

```
qml.RZ(phi, wires=wire)
```

Try at home: directly expand $e^{-i\frac{\phi}{2}Z}$ to obtain the matrix representation.

Z rotations (RZ)

$$RZ(\phi) = e^{-i\frac{\phi}{2}Z} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}$$

Apply to a general state:

S and T

Two other special cases: $\phi = \pi/2$, and $\phi = \pi/4$.

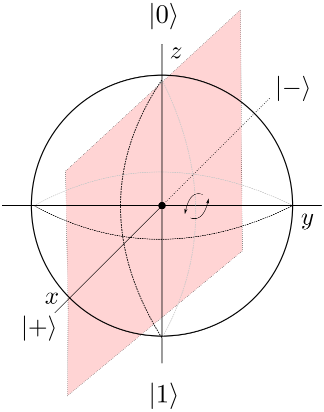
$$
\begin{aligned}
S &= RZ(\pi/2) = \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
T &= RZ(\pi/4) = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}
\end{aligned}
$$

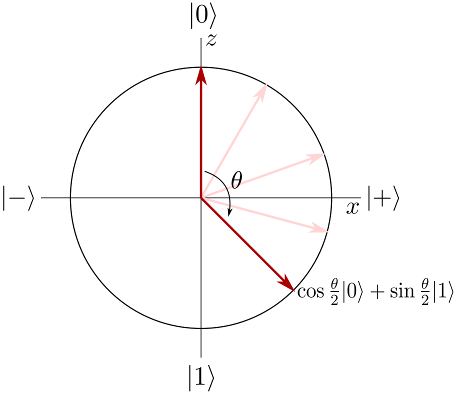
In PennyLane:

```
qml.PauliZ(wires=wire)
qml.S(wires=wire)
qml.T(wires=wire)
```

S is part of a special group called the **Clifford group**.
T is used in universal gate sets for fault-tolerant QC.

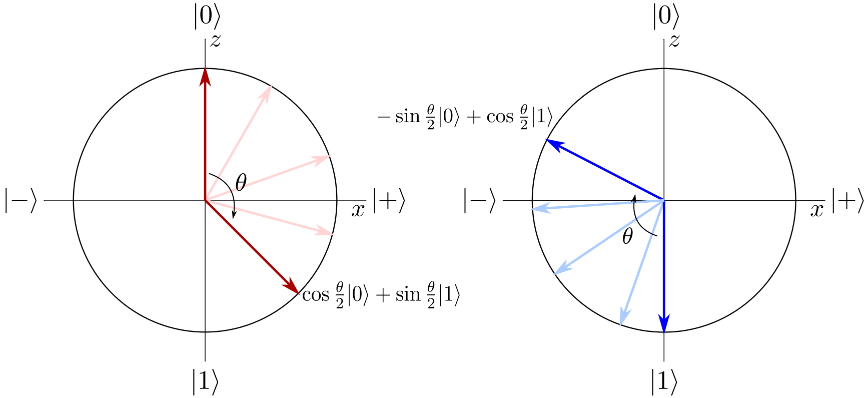
Y rotations (RY)





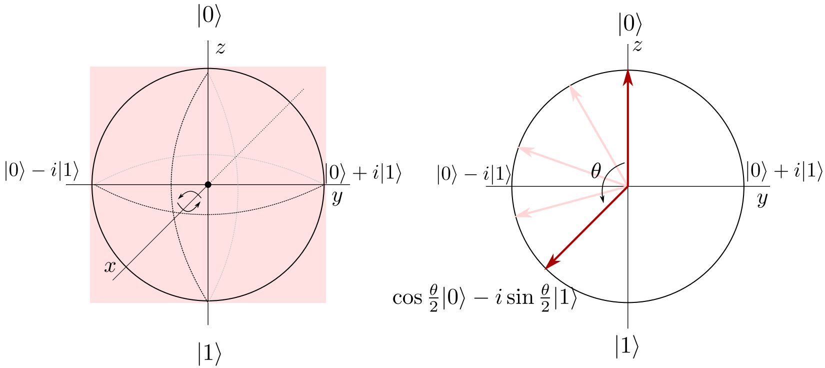
Y rotations (RY)

$$RY(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$



X rotations (RX)

$$RX(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$



Pauli rotations

These unitary operations are called **Pauli rotations**.

	Math	Matrix	Code	Special cases (θ)
RZ	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	`qml.RZ`	$Z(\pi), S(\pi/2), T(\pi/4)$
RY	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RY`	$Y(\pi)$
RX	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RX`	$X(\pi), SX(\pi/2)$

Pauli rotations

Exercise: design a quantum circuit that prepares

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5\pi}{4}}|1\rangle$$

Hint: you can return the state directly in PennyLane:

```
@qml.qnode(dev)
def some_circuit():
    # Gates...
    return qml.state()
```

Pauli rotations

Exercise: implement the circuit below in PennyLane.

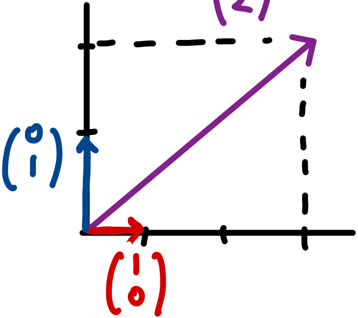
Run your circuit with two different values of θ and take 1000 shots.

How does θ affect the measurement outcome probabilities?

Inner products

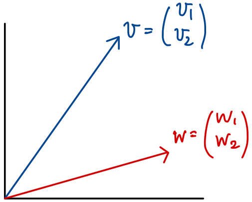
We can now create any single-qubit quantum state: how do we *compare* them? Look to 2D space for intuition.

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Inner products

The **inner product** (dot product) between two vectors quantifies how much overlap they have.



Inner products

Take just one of these representations:

The inner product in Hilbert space is defined as

To avoid cumbersome notation define the **bra** (*braket* notation):

Inner products

Exercise: compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

Inner products

Exercise: compute the inner product between all possible combinations of $|0\rangle$ and $|1\rangle$.

$\langle 0\vert 0\rangle$	
$\langle 0\vert 1\rangle$	
$\langle 1\vert 0\rangle$	
$\langle 1\vert 1\rangle$	

Orthonormal bases

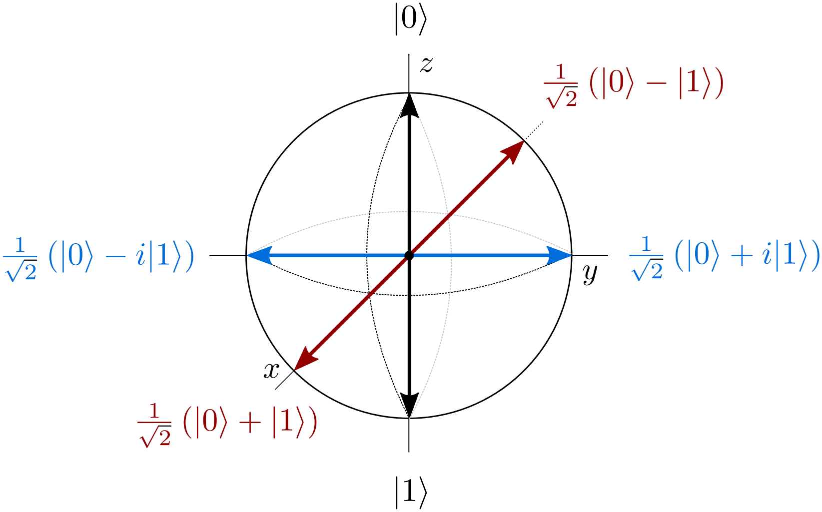
For a single qubit, a pair of states that are **normalized** and **orthogonal** constitute an **orthonormal basis** for the Hilbert space.

Exercise: do the states

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

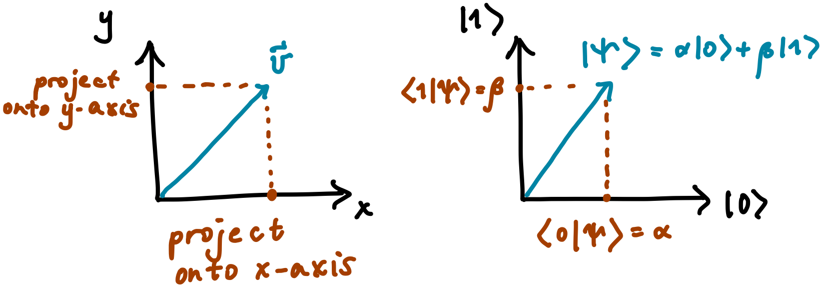
form an orthonormal basis?

Bases on the Bloch sphere



Projective measurements

Measurement is performed with respect to a basis; we perform **projections** to determine the overlap with a given basis state.



(Image for expository purposes only!)

Image credit: Xanadu Quantum Codebook I.9

Projective measurements

When we measure a qubit in state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of observing it in state $|\psi_i\rangle$ (outcome i) is

If we observe outcome i, the qubit *remains in* $|\psi_i\rangle$.

Example: Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. If we measure $|\psi\rangle$ in the computational basis,

Measurement in the computational basis

Exercise: what are the measurement outcome probabilities for

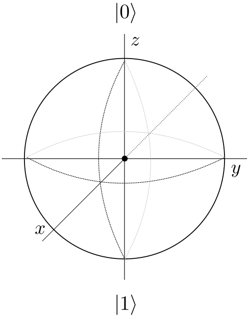
$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

in the computational basis?

Measurement in the computational basis

We know these are *not* the same state: there is a relative phase.

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



How to tell them apart with a measurement?

Basis rotations

So far we've seen 3 types of measurements in PennyLane:

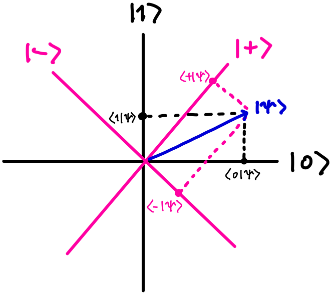
1. `qml.state()`
2. `qml.probs(wires=x)`
3. `qml.sample()`

But these all perform a computational basis measurement. This is also the only measurement allowed on most hardware.

How can we measure with respect to *different bases*?

Basis rotations

Recall unitary operations preserve length *and* angles between orthonormal quantum states (prove on practice A1!).



Projective measurements can be performed with respect to a different basis by applying a **basis rotation** to remap its states to computational basis states.

Image credit: Codebook node I.9

Basis rotations

Exercise: Suppose we want to measure in the "Y" basis. First, determine a sequence of gates that sends

$$
\begin{aligned}
|0\rangle \quad &\rightarrow \quad |p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \\
|1\rangle \quad &\rightarrow \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)
\end{aligned}
$$

Basis rotations

At the end of our circuit, apply the reverse (adjoint) of this transformation to rotate *back* to the computational basis.

If we measure and observe $|0\rangle$, we know the qubit was previously $|p\rangle$ in the Y basis (analogous result for $|m\rangle$).

Adjoints

PennyLane can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

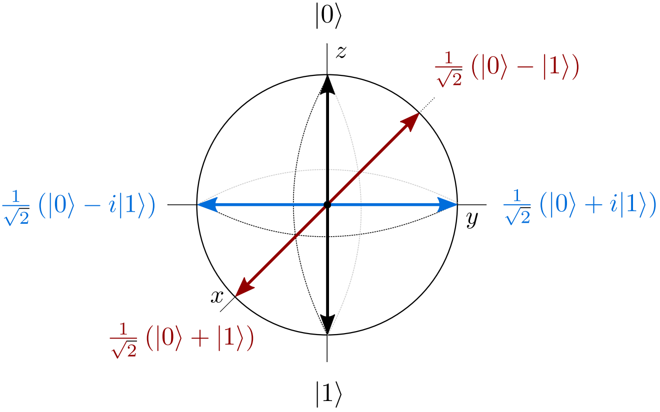
```
def some_function(phi):
    qml.RX(phi, wires=0)

def apply_adjoint(phi):
    qml.adjoint(qml.S)(wires=0)
    qml.adjoint(some_function)(phi)
```

`qml.adjoint` is a special type of function called a **transform**.

Projective measurements and the Bloch sphere

Projective measurement corresponds to a geometric projection onto the axis represented by a basis.



H-*RZ* exercise, revisited

Exercise: suppose we run the following circuit, but perform the measurement in the Y basis.

1. Using the Bloch sphere as intuition, what do you think the outcome probability of $|p\rangle$ looks like as a function of θ? Discuss with a neighbour.
2. Test your predictions in software.

Next time

Content:

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

Action items:

1. Prepare for quiz 2 (tutorial on Monday)
2. Work on practice Assignment 1

Recommended reading:

- For today: Codebook modules IQC, SQ; Nielsen & Chuang 1.3.3, 2.2.3-2.2.5
- For next class: Codebook module MQ; Nielsen & Chuang 4.3