Modulation and Laplace

The following problems from Oppenheim are some of the suggested ones as practice problems during lectures 21 and 22.

8.18. Let $x[n]$ be a real-valued discrete-time signal whose Fourier transform $X(e^{j\omega})$ is zero for $\omega \geq \pi/4$. We wish to obtain a signal $y[n]$ whose Fourier transform has the property that, in the interval $-\pi < \omega \leq \pi$,

$$Y(e^{j\omega}) = \begin{cases} X(e^{j(\omega-\frac{\pi}{2})}), & \frac{\pi}{2} < \omega \leq \frac{3\pi}{4} \\ X(e^{j(\omega+\frac{\pi}{2})}), & -\frac{3\pi}{4} < \omega \leq -\frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}.$$

The system in Figure P8.18 is proposed for obtaining $y[n]$ from $x[n]$. Determine constraints that the frequency response $H(e^{j\omega})$ of the filter in the figure must satisfy for the proposed system to work.

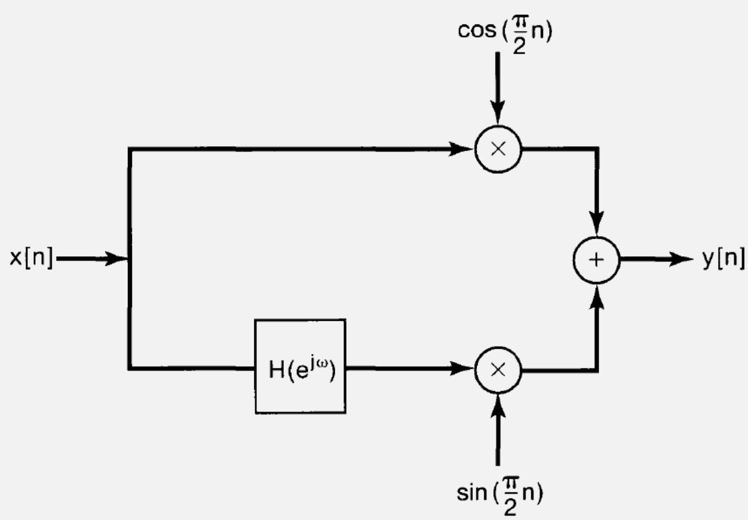
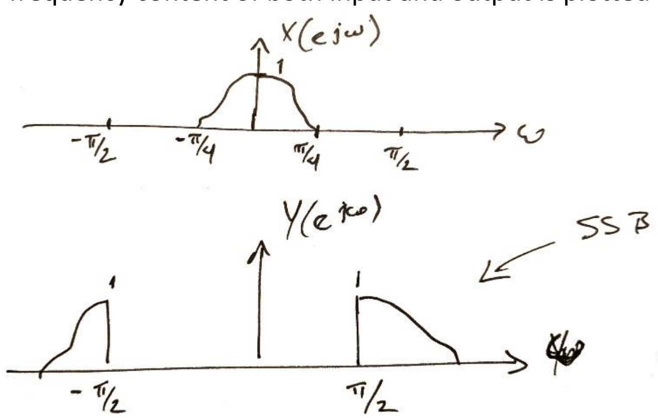


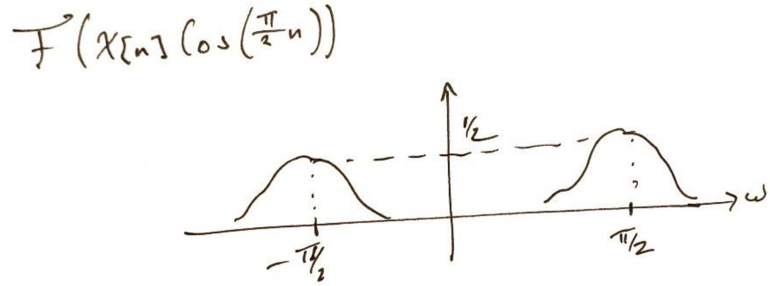
Figure P8.18

The frequency content of both input and output is plotted below.

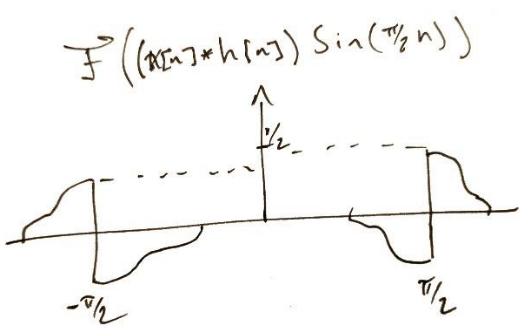


The goal of this system is single sideband (SSB) modulation.

The frequency spectrum of the cosine modulated signal is the following:

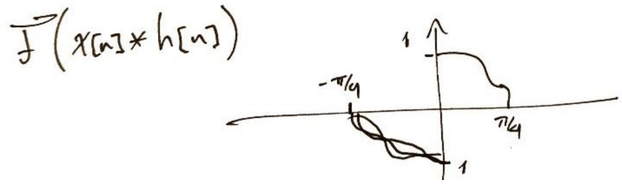


If we are to obtain the frequency spectrum of $Y(e^{j\omega})$, the signal being added to the cosine modulated one should have the following frequency spectrum



This way, when added together, the result will be the desired frequency spectrum.

In order to obtain the frequency spectrum above, the output of the filter $H(e^{j\omega})$ must look like this



Where $H\left(e^{j\omega}\right)$

$$H(e^{j\omega}) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}$$

8.32. Consider a discrete-time signal $x[n]$ with Fourier transform shown in Figure P8.32(a). The signal is amplitude modulated by a sinusoidal sequence, as indicated in Figure P8.32(b).

(a) Determine and sketch $Y(e^{j\omega})$, the Fourier transform of $y[n]$.

(b) A proposed demodulation system is shown in Figure P8.32(c). For what value of θ_c, ω_{lp}, and G will $\hat{x}[n] = x[n]$? Are any restrictions on ω_c and ω_{lp} necessary to guarantee that $x[n]$ is recoverable from $y[n]$?

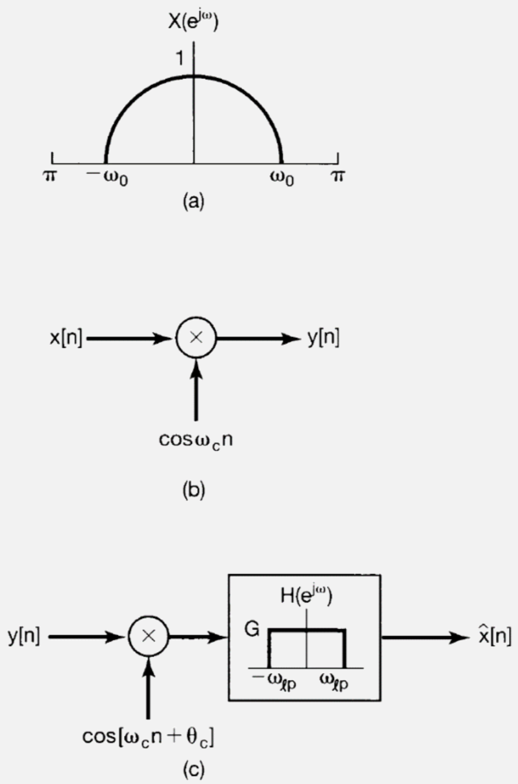


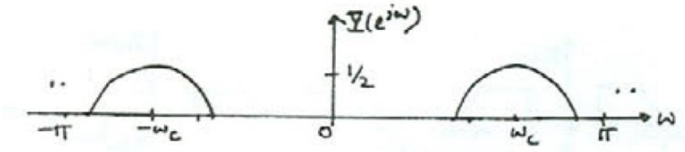
Figure P8.32

a) Convert to complex exponentials

$$y[n] = \frac{1}{2}x[n](e^{j\omega_c n} + e^{-j\omega_c n})$$

$$Y(e^{j\omega}) = \frac{1}{2}X(e^{j(\omega-\omega_c)}) + \frac{1}{2}X(e^{j(\omega+\omega_c)})$$

By making the following assumptions about $H(e^{j\omega})$: $\omega_c > \omega_{lp}$ and $\omega_c < \pi - \omega_{lp}$, we can sketch $Y(e^{j\omega})$ as



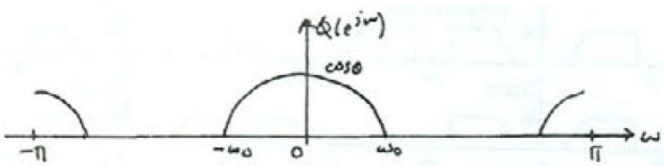
b) Let $q[n]$ be the signal that goes into the low-pass filter in the demodulator, and $c[n]$ the demodulating cosine, so that

$$Q(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

$$C(e^{j\omega}) = \pi\left[(e^{j\theta_c}\delta(\omega-\omega_c) + e^{-j\theta_c}\delta(\omega+\omega_c)\right]$$

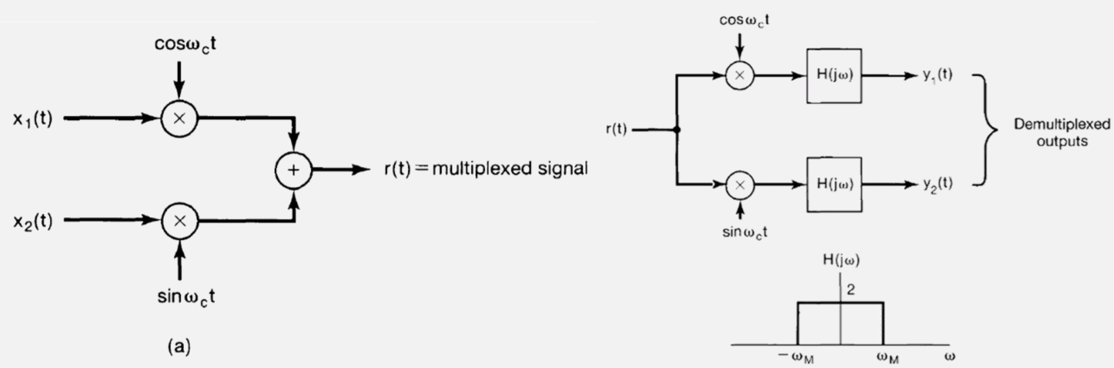
$$Q(e^{j\omega}) = \frac{1}{2}\left[e^{j\theta_c}Y(e^{j(\omega-\omega_c)}) + e^{-j\theta_c}Y(e^{j(\omega+\omega_c)})\right]$$

The spectrum of $Q(e^{j\omega})$ is



From where we find that $G = \frac{1}{\cos(\theta_c)}$, requiring that $\theta_c \neq 0$, and not be an odd multiple of $\pi/2$. The restrictions to ω_c and ω_{lp} are the ones we assumed for plotting $Y(e^{j\omega})$.

8.40. In Section 8.3, we discussed the use of sinusoidal modulation for frequency-division multiplexing whereby several signals are shifted into different frequency bands and then summed for simultaneous transmission. In the current problem, we explore another multiplexing concept referred to as *quadrature multiplexing*. In this multiplexing procedure, two signals can be transmitted simultaneously in the same frequency band if the two carrier signals are 90° out of phase. The multiplexing system is shown in Figure P8.40(a) and the demultiplexing system in Figure P8.40(b). $x_1(t)$ and $x_2(t)$ are both assumed to be band limited with maximum frequency ω_M, so that $X_1(j\omega) = X_2(j\omega) = 0$ for $|\omega| > \omega_M$. The carrier frequency ω_c is assumed to be greater than ω_M. Show that $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$.



Let the signals $q_1(t)$ and $q_2(t)$ be the signals $x_1(t)$ and $x_2(t)$ respectively after modulation. Their Fourier transforms are

$$Q_1(j\omega) = \frac{1}{2}X_1(\omega-\omega_c) + \frac{1}{2}X_1(\omega+\omega_c)$$
$$Q_2(j\omega) = \frac{1}{j2}X_2(\omega-\omega_c) - \frac{1}{j2}X_2(\omega+\omega_c)$$

The multiplexed signal $R(j\omega)$ is then

$$R(j\omega) = \frac{1}{2}X_1(\omega-\omega_c) + \frac{1}{2}X_1(\omega+\omega_c) - \frac{j}{2}X_2(\omega-\omega_c) + \frac{j}{2}X_2(\omega+\omega_c)$$

Let the signals $p_1(t) = r(t)\cos(\omega_c t)$ and $p_2(t) = r(t)\sin(\omega_c t)$. The Fourier transform of $p_1(t)$ is

$$P_1(j\omega) = \frac{1}{2}\left[\frac{1}{2}X_1(\omega-\omega_c-\omega_c) + \frac{1}{2}X_1(\omega+\omega_c-\omega_c) - \frac{j}{2}X_2(\omega-\omega_c-\omega_c) + \frac{j}{2}X_2(\omega+\omega_c-\omega_c)\right]$$
$$+\frac{1}{2}\left[\frac{1}{2}X_1(\omega-\omega_c+\omega_c) + \frac{1}{2}X_1(\omega+\omega_c+\omega_c) - \frac{j}{2}X_2(\omega-\omega_c+\omega_c) + \frac{j}{2}X_2(\omega+\omega_c+\omega_c)\right]$$

Given the cut-off frequency of the low-pass filter, we can ignore all the terms in $P_1(j\omega)$ that are outside the range of $[-\omega_M, \omega_M]$, i.e., all the terms with $\omega \pm 2\omega_c$, this will simplify our analysis. By doing that , it follows that

$$P_1(j\omega) = \frac{1}{2}\left[\frac{1}{2}X_1(j\omega) + \frac{1}{2}X_1(j\omega) - \frac{j}{2}X_2(j\omega) + \frac{j}{2}X_2(j\omega)\right]$$

$$P_1(j\omega) = \frac{1}{2}X_1(j\omega)$$

Thus, after filtering $p_1(t)$ we will obtain $x_1(t)$. The same procedure applies for $p_2(t)$.

9.2. Consider the signal

$$x(t) = e^{-5t}u(t-1),$$

and denote its Laplace transform by $X(s)$.

(a) Using eq. (9.3), evaluate $X(s)$ and specify its region of convergence.

(b) Determine the values of the finite numbers A and t_0 such that the Laplace transform $G(s)$ of

$$g(t) = Ae^{-5t}u(-t-t_0)$$

has the same algebraic form as $X(s)$. What is the region of convergence corresponding to $G(s)$?

a) The Laplace transform of $x(t)$ is as follows

$$X(s) = \int_{-\infty}^{\infty} e^{-5t}u(t-1)e^{-st}dt$$

$$X(s) = \int_{1}^{\infty} e^{-(5+s)t}dt$$

$$X(s) = \frac{e^{-(5+s)}}{s+5}, \quad \mathrm{Re}\{s\} > -5$$

b) We begin by obtaining the Laplace transform of $g(t)$

$$X(s) = A\int_{-\infty}^{\infty} e^{-5t}u(-t-t_0)e^{-st}dt$$

$$X(s) = \int_{-\infty}^{-t_0} e^{-(5+s)t}dt$$

$$X(s) = \frac{Ae^{-(5+s)t_0}}{s+5}, \quad \mathrm{Re}\{s\} < -5$$

Thus, $A = 1$, and $t_0 = -1$.

9.5. For each of the following algebraic expressions for the Laplace transform of a signal, determine the number of zeros located in the finite s-plane and the number of zeros located at infinity:

(a) $\dfrac{1}{s+1} + \dfrac{1}{s+3}$

(b) $\dfrac{s+1}{s^2-1}$

(c) $\dfrac{s^3-1}{s^2+s+1}$

For more information on zeros, check page 661 in Oppenheim.

a) Rearranging terms to

$$\frac{s+3+s+1}{(s+1)(s+3)} = \frac{s+4}{(s+1)(s+3)}$$

In finite s-plane: one zero at $s = -2$.

At infinite: one since order of denominator polynomial is higher than numerator.

b) Rearranging

$$\frac{s+1}{(s-1)(s+1)} = \frac{1}{s-1}$$

In finite s-plane: no zeros.

At infinite: one since order of denominator polynomial is higher than numerator.

c) Rearranging

$$\frac{(s-1)(s^2+s+1)}{s^2+s+1} = s-1$$

In finite s-plane: one zero at $s = 1$.

At infinite: No zeros since order of numerator polynomial is higher than denominator.

9.6. An absolutely integrable signal $x(t)$ is known to have a pole at $s = 2$. Answer the following questions:

(a) Could $x(t)$ be of finite duration?
(b) Could $x(t)$ be left sided?
(c) Could $x(t)$ be right sided?
(d) Could $x(t)$ be two sided?

a) No. For a finite length signal the ROC is the entire s-plane, so there can be no poles in the finite s-plane for a finite length signal.

b) Yes. Since the signal is absolutely integrable, ROC must include the $j\omega$-axis. Since $X(s)$ has a pole at $s = 2$, a valid ROC for this signal would be $Re\{s\} < 2$, which would correspond to a left-sided signal.

c) No. The same reasons that make it possible for this to be a left-sided signal also make it impossible for it to be a right sided signal.

d) Yes. It could be possible for another pole to exist so that the ROC becomes $\alpha < Re(s) < 2$, for $\alpha < 0$ so that it includes the $j\omega$-axis.

9.26. Consider a signal $y(t)$ which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) * x_2(-t+3)$$

where

$$x_1(t) = e^{-2t}u(t) \qquad \text{and} \qquad x_2(t) = e^{-3t}u(t).$$

Given that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \qquad \mathcal{R}e\{s\} > a,$$

use properties of the Laplace transform to determine the Laplace transform $Y(s)$ of $y(t)$.

We begin by finding the individual Laplace transforms for $x_1(t)$ and $x_2(t)$

$$x_1(t) = e^{-2t}u(t) \overset{L}{\longleftrightarrow} X_1(s) = \frac{1}{s+2}, \quad Re\{s\} > -2$$

$$x_2(t) = e^{-3t}u(t) \overset{L}{\longleftrightarrow} X_2(s) = \frac{1}{s+3}, \quad Re\{s\} > -3$$

Applying time scaling and shifting properties to $X_1(s)$ and $X_2(s)$ we get

$$x_1(t-2) \xleftrightarrow{L} e^{-2s} X_1(s) = \frac{e^{-2s}}{s+2}, \;\; Re\{s\} > -2$$

$$x_2(-t+3) \xleftrightarrow{L} e^{-3s} X_2(-s) = \frac{e^{-3s}}{3-s}, \;\; Re\{s\} > 3$$

By the convolution property, $Y(s)$ becomes

$$Y(s) = X_1(s) X_2(s) = \left[\frac{e^{-2s}}{s+2}\right]\left[\frac{e^{-3s}}{3-s}\right]$$