ELEC 221 Lecture 05
CT Fourier series

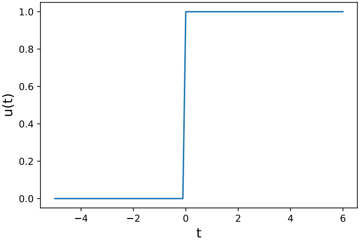
Thursday 19 September 2024

Announcements

- Assignment 1 due tonight at 23:59
- Quiz 3 Tuesday
- Assignment 2 released next week
- Tutorial Assignment 2 on Monday

Last time

We defined the CT unit step, $u(t)$



And the CT unit impulse function, $\delta(t)$:

Last time

We introduced the CT convolution integral,

If we know the impulse response of an LTI system, we can determine its response to any other signal.

Last time

We saw how properties of the impulse response are related to the system's properties as a whole.

(**Memory**) For a system to be memoryless,

(**Invertibility**) If a system with impulse response $h(t)$ is invertible, there exists an *inverse system* with impulse response $h_i(t)$ s.t.

(Analogous for DT case)

(**Stability**) For a system to be stable, the impulse response $h(t)$ must be *absolutely integrable*, i.e.,

is finite.

(**Causality**) The impulse response $h(t)$ of a causal system must have the property

(Analogous for DT case)

Today

Learning outcomes:

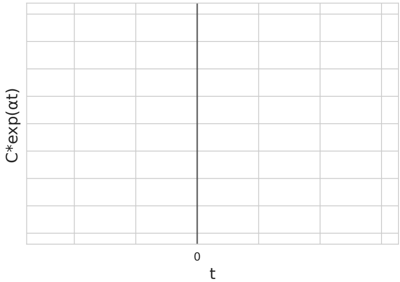
- Use the convolution integral to show that complex exponentials are eigenfunctions of LTI systems
- Define and compute the system function (frequency response) of an LTI system
- Express a periodic CT signal as a Fourier series, and compute its Fourier coefficients

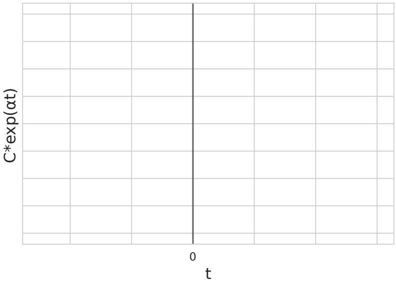
Review: complex exponential functions

Most general form:

where α can be real or complex.

Case: both C and α are real-valued.





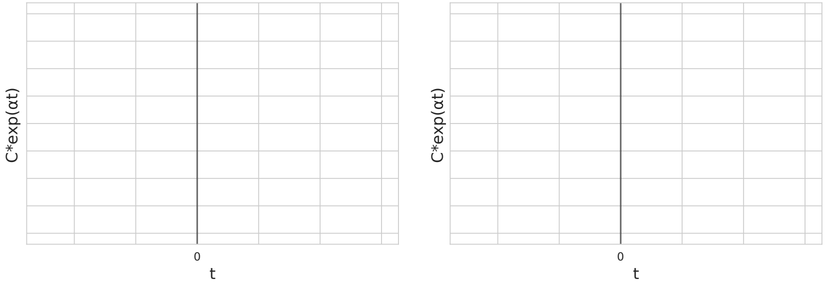
Case: α is complex. Most generally, we can write

Euler's relation allows us to write

As a result,

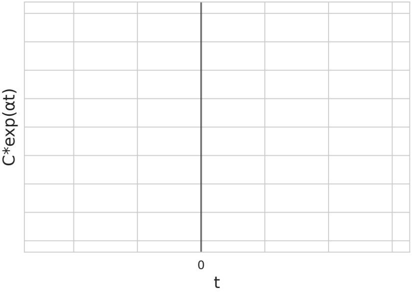
Review: complex exponential functions

Case: $\alpha = j\omega$. Then,

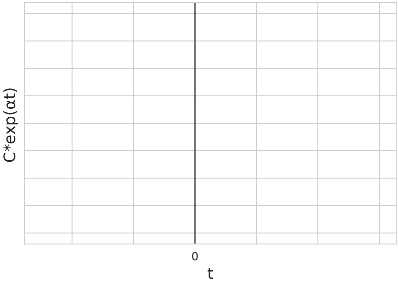


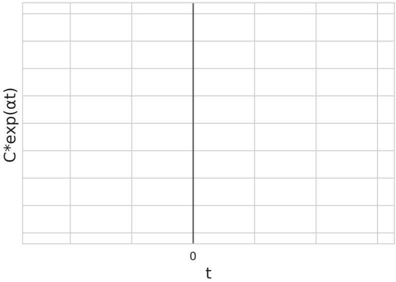
Review: complex exponential functions

$x(t) = Ce^{j\omega t}$ is periodic:



Case: $\alpha = r + j\omega$. Then,





LTI systems and complex exponential functions

Recall the convolution integral:

What happens when $x(t)$ is a complex exponential signal?

Write $x(t) = e^{st}$. Then:

To summarize:

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is the **system function**.

...so what?

Recall that for **LTI** systems,

If all $x_i(t)$ are complex exponential signals, and

then

The response is a superposition **of those same signals**, scaled by the system function.

System functions

Consider the limited set of signals[1]:

$x(t)$ has frequency ω and period $T = 2\pi/\omega$.

When such signals are input into an LTI system, the system function, $H(s) = H(j\omega)$, is called the *frequency response*.

[1] We will see the general case at the end of the course.

Exercise: frequency response

Suppose we have access to a system with impulse reponse

1. What is the frequency response, $H(j\omega)$, of the system?

2. What is the output of the system given input

3. What is the output of the system given input

Exercise: frequency response

Suppose we have access to a system with impulse reponse

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Exercise: frequency response

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Exercise: frequency response

Suppose we have access to a system with impulse reponse

3. What is the output of the system given input

The Fourier series

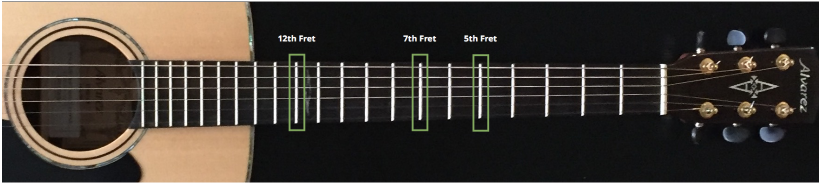
How can we express arbitrary signals using complex exponentials?

Consider

It also has an infinite number of cousins called *harmonics*.

ω is the **fundamental frequency**.

(Yes, these harmonics.)



We can create a superposition of all harmonics,

This signal is also periodic with period $T = 2\pi/\omega$.

Given $x(t)$ with period T[2], we can write it as a **Fourier series**.

The c_k (complex numbers) are its **Fourier coefficients**.

[2] There are some additional conditions; we will cover this next class

Usually dealing with $x(t)$ that is always *real*.

This means .

We can leverage this to express $x(t)$ in a different way.

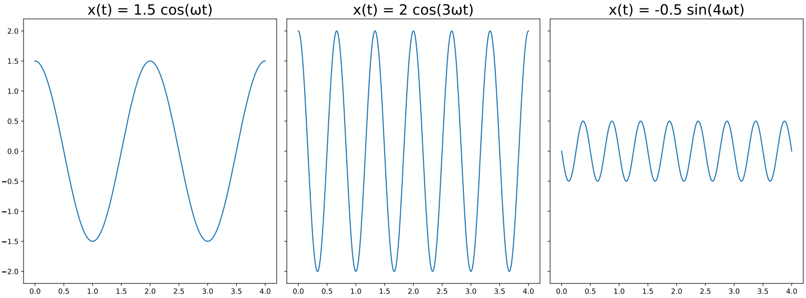
Apply Euler's relation:

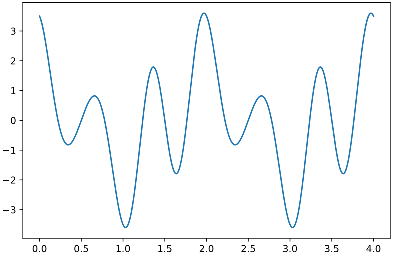
With a bit of work (i.e., in assignment 1!), you can find

where a_0, a_k, b_k are expressed in terms of the c_k.

Note: sometimes a_0 is written $a_0/2$ for convenience

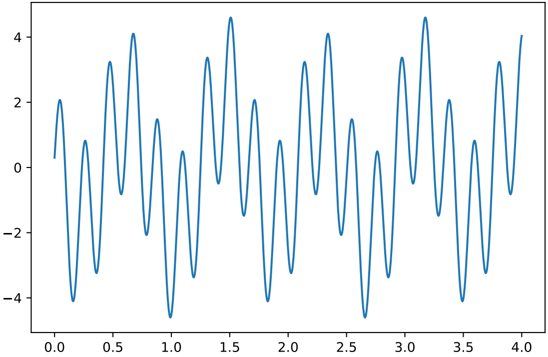
Example





Going backwards

What if we are given a signal like this:



Can we determine the c_k?

Evaluating Fourier coefficients

The $e^{jk\omega t}$ are **basis functions** and have **orthogonality** relations w.r.t. integration.

Example: let's compute c_m.

Multiply on both sides by the conjugate of the basis function:

Evaluating Fourier coefficients

$$e^{-jm\omega t}x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} e^{-jm\omega t}$$

Integrate over a period:

Evaluating Fourier coefficients

Evaluate the integral

Case 1: $k = m$

Case 2: $k \neq m$

Evaluating Fourier coefficients

From here:

$$\frac{1}{T}\int_0^T e^{-jm\omega t}x(t)dt = \sum_{k=-\infty}^{\infty} c_k \cdot \frac{1}{T}\int_0^T e^{j(k-m)\omega t}dt$$

Only the $k = m$ term survives so

Evaluating Fourier coefficients

Since we integrate over one period, write

Fourier coefficients tell us *how much* each harmonic contributes to the total signal.

Note that c_0 is a constant offset:

(Similar techniques can be used to derive a_k and b_k for the sin and cos representation. Try it!)

Recap: key expressions

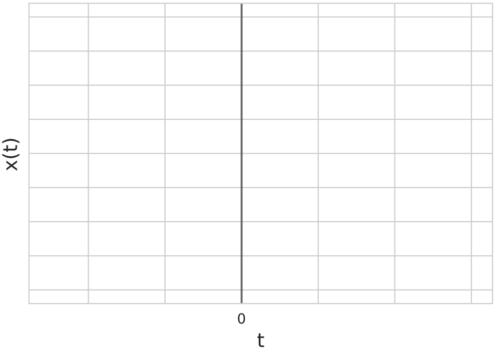
Fourier synthesis equation:

Fourier analysis equation:

Exercise

What is the Fourier series of

Start with a plot, and determine T and ω.



Exercise

Exercise

Recap

Today's learning outcomes were:

- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems
- Define and compute the system function (frequency response) of an LTI system
- Express a periodic CT signal as a Fourier series, and compute its Fourier coefficients

For next time

Content:

- Dirichlet conditions and the Gibbs phenomenon
- Properties of Fourier series and Fourier coefficients

Action items:

1. Assignment 1 is due tonight
2. Tuesday quiz on this week's material

Recommended reading:

- From today's class: Oppenheim 1.3, 3.0-3.3
- Suggested problems: 1.6, 1.8, 1.10, 3.1, 3.3, 3.4, 3.13, 3.17, 3.22a,c
- For next class: Oppenheim 3.4-3.5