ELEC 221 Lecture 03
DT impulse response and convolution sum; CT convolution integral

Thursday 12 September 2024

Announcements

- Tutorial assignment 1 Monday 16 Sept 23:59
- Assignment 1 due Thursday 19 Sept 23:59
- Monday tutorial focus on practice problems - will post Piazza thread for requests

Important:

- Quiz 2 on Tuesday
- Class next Thursday on Zoom
- Class next Tuesday also on Zoom if Air Canada goes on strike and cancels my flight

Last time

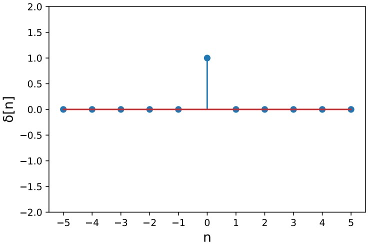
We defined **LTI (linear, time-invariant) systems**.

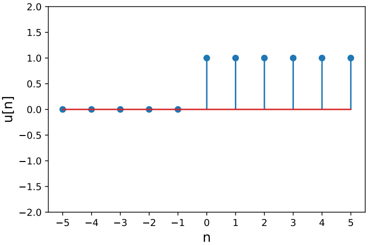
Linearity:

Time invariance:

Last time

We defined the DT unit impulse and unit step





... but then we got kind of stuck:

Today

First: clarify some points from last time

Learning outcomes:

- Define the convolution sum and use it to compute the output of a system
- Define the CT unit impulse and step functions
- Define the convolution integral and use it to compute the output of a system

In case we have time:

- Use the impulse response to determine whether a system is stable, causal, memoryless, or invertible

What space are we in?

Used to thinking in terms of mathematical functions:

For us, the system plays the role of the function, and signals are the input and output:

This is why linearity looks different.

What space are we in?

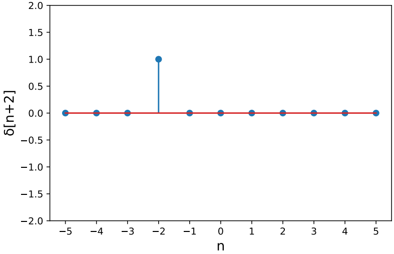
A function $y = f(x)$ is linear if

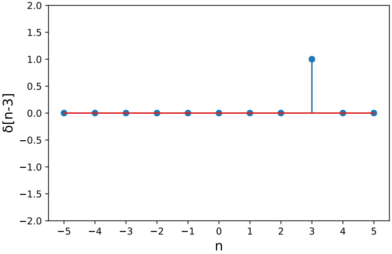
A system S that sends $x(t) \to y(t)$ is linear if

"Linear" is also overloaded. Recall the system

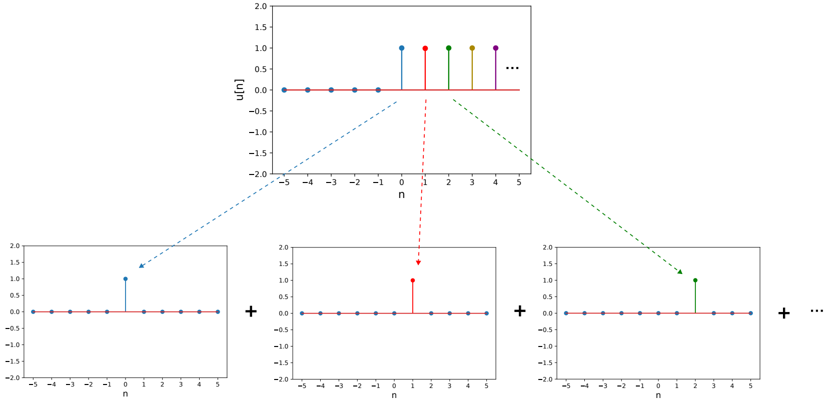
is *not linear*, even though it looks like a linear equation.

Weighted, shifted impulses





Weighted, shifted impulses



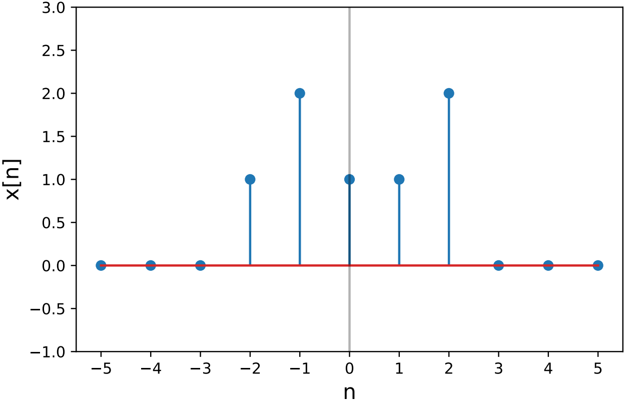
Weighted, shifted impulses

More generally, can write

Change variables ($m = n - k$)

The unit impulse as a sampler

Every point is a *weighted, shifted impulse*.



The unit impulse as a sampler

Any signal can be written as a **superposition of weighted impulses.**

Like a "deconstructed" version of the signal.

Multiplying by a shifted impulse "samples" the signal at that point:

The impulse response

How does an LTI system respond to a signal

Suppose it sends $\delta[n-k] \to h_k[n]$

$h_k[n]$ is called the **impulse response**.

Real-world example: nerve conduction study

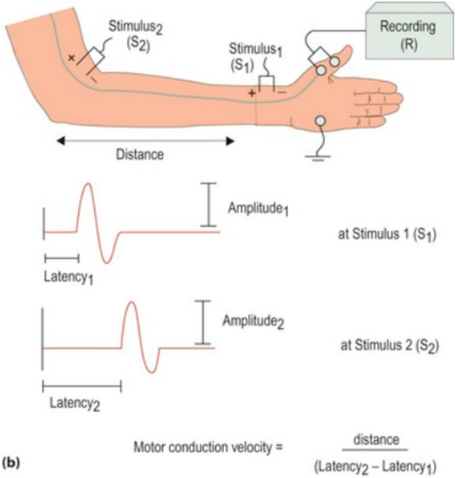


Image source: `https://neupsykey.com/nerve-conduction-studies-and-electromyography/`

What if the system is also time invariant?

Then

The convolution sum

If we know how a **linear** system responds to the unit impulse, we can learn how it responds to **any other signal**!

This is the **convolution sum**. We are "convolving" the sequences $x[n]$ and $h[n]$.

Exercise: impulse response

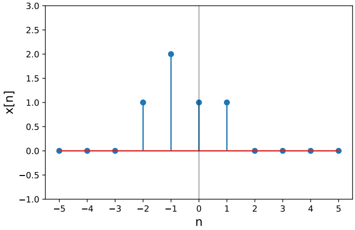
Consider an LTI system with input/output relationship

$$y[n] = 2x[n] + x[n-1]$$

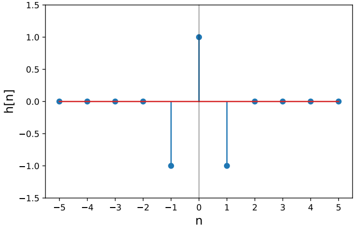
What is the impulse response of the system?

Example: convolution sum

Consider the signal

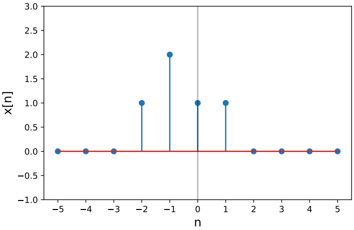


input to a system with impulse response



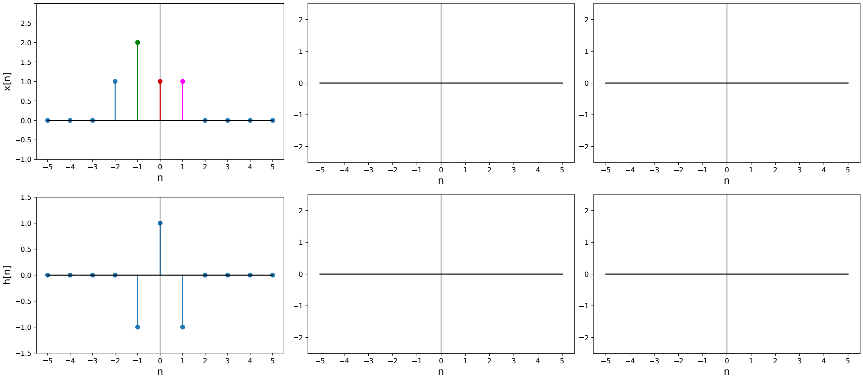
Example: convolution sum

To learn the system output, we must consider the contribution of each weighted impulse response:

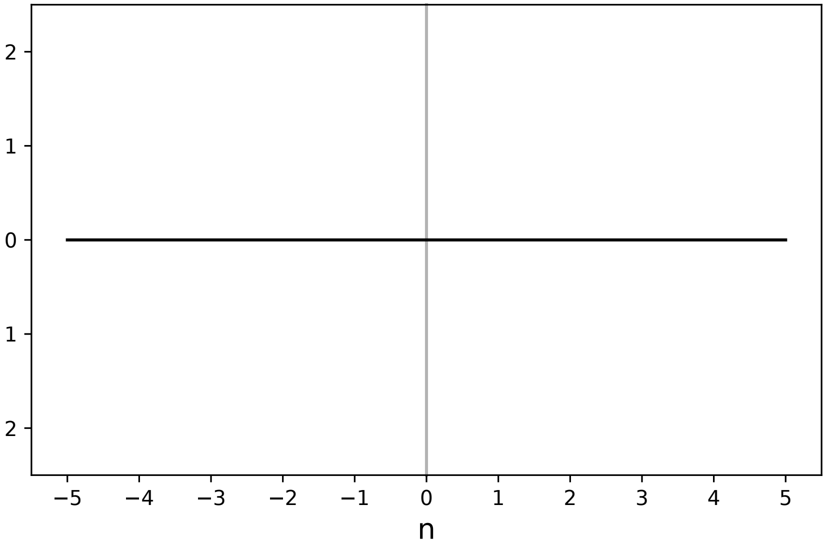


Only $x[k] \neq 0$ only for $k \in \{-2, -1, 0, 1\}$. So need to determine $x[k]h[n-k]$ for these cases, and sum them.

Example: convolution sum



Example: convolution sum



Properties of convolutions

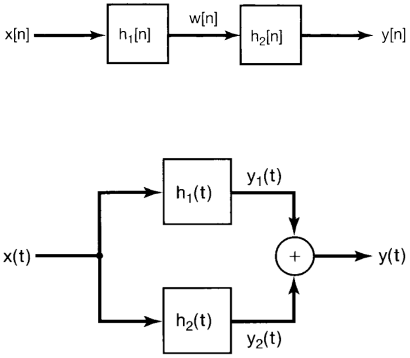


Image credits: Signals and Systems 2nd ed., Oppenheim

Convolution is:

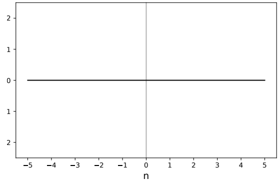
- Associative:
  $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Commutative:
  $x[n] * h[n] = h[n] * x[n]$
- Distributive:
  $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

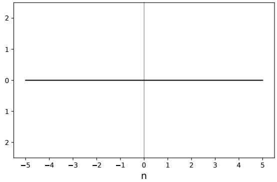
Consider an LTI system with impulse response

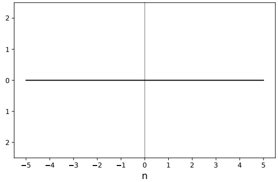
$$h[n] = 3\delta[n] + 2\delta[n+1]$$

What is output of the system if

$$x[n] = \left(\frac{2}{3}\right)^n u[n]$$







Example/exercise: convolution sum

What is output of the system

$$x[n] = \left(\frac{2}{3}\right)^n u[n], \quad h[n] = 3\delta[n] + 2\delta[n+1]$$

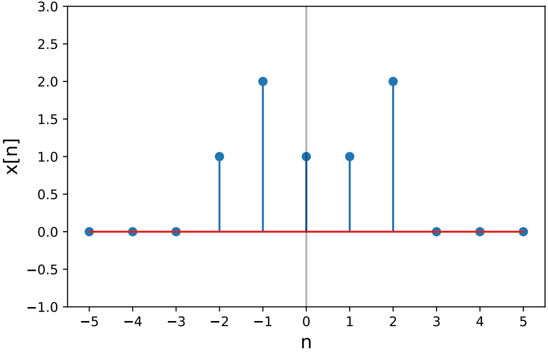
Example/exercise: convolution sum

What is output of the system

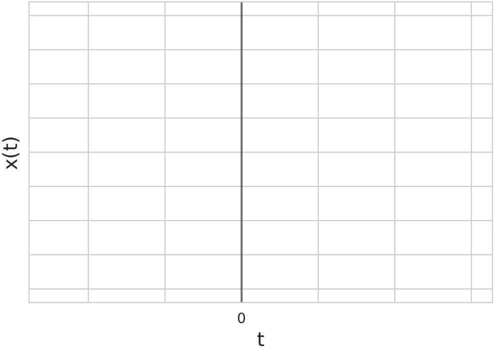
$$x[n] = \left(\frac{2}{3}\right)^n u[n], \quad h[n] = 3\delta[n] + 2\delta[n+1]$$

Shifted, weighted impulses

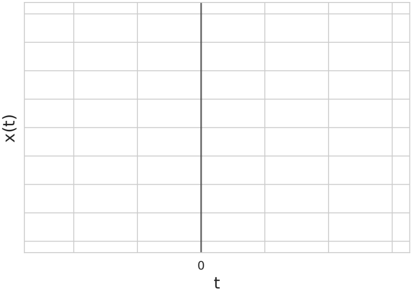
Return to this picture:



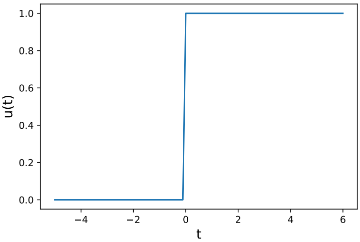
Shifted, weighted impulses



The CT unit impulse



The CT unit step



Just like in DT, the unit impulse and step are related:

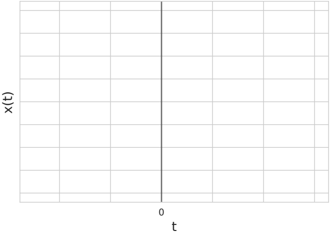
The CT analogue of convolution sum is the **convolution integral**.

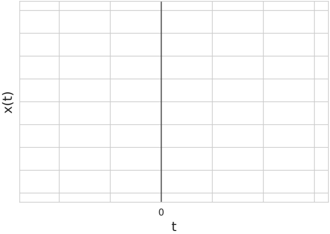
where $h(t)$ is the **CT impulse response**.

It has the same properties (commutative, associative, distributive).

(Oppenheim Ex. 2.6 Var.) Consider system with impulse response

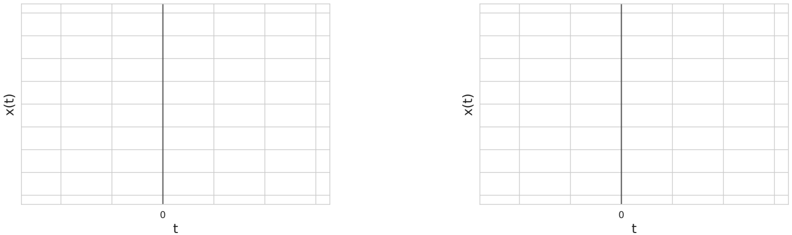
What is the output of the system for the input signal





Example: convolution

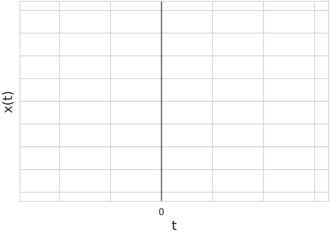
Example: convolution

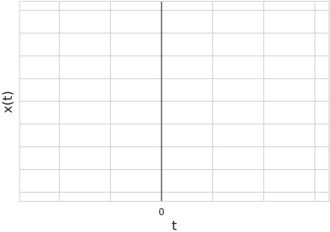


Exercise: convolution

(Oppenheim 2.8) Consider system with impulse response

What is the output of the system for the input signal

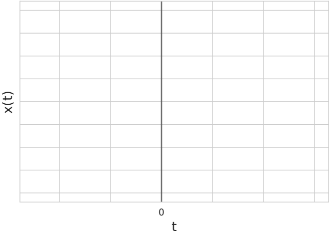


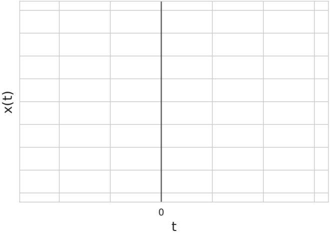


Exercise: convolution

Direct integration:

Visual intuition:





Impulse response and analysis of LTI systems

To reiterate: the convolution sum

and convolution integral

show that as long as we know how a system responds to a unit impulse, we can determine its response to any other signal.

Knowledge of the impulse response also allows us to determine key properties of systems such as causality, invertibility, and stability.

A system is memoryless if the output depends only on the input at the same time. This implies $h[n] = 0$ for $n \neq 0$, meaning

(And analogous for CT case)

Impulse response and invertibility

If a system is invertible, it has an inverse system.
Suppose impulse response of a system is $h(t)$. Then

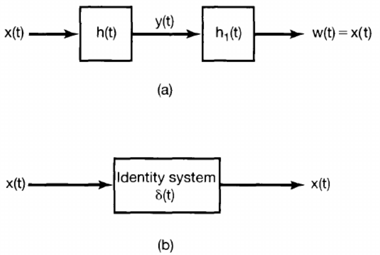


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

(And analagous for DT case. We will see this later in the course.)

Image: Oppenheim, Fig 2.26

Impulse response and stability

Suppose we have a signal $x(t)$ with a bounded input, $|x(t)| \leq B$.
If the system is stable, the output should be bounded.

(And analogous for DT case)

As long as

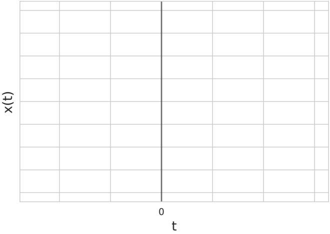
is bounded (i.e., $h(t)$ is absolutely integrable), then the system will be stable.

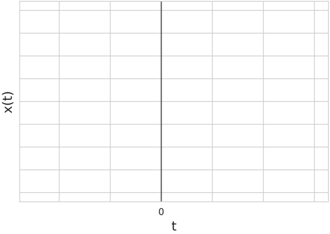
(And analogous for DT case)

Example/exercise: stability

Consider systems A and B with impulse responses

Are they stable?





Example/exercise: stability

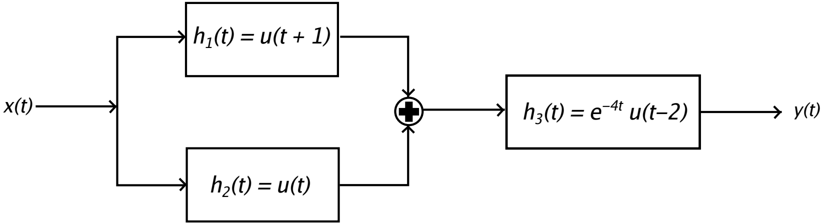
Impulse response and causality

Recall definition of causal signal and consider the convolution sum:

What properties does $h[n]$ need to have for system to be causal?

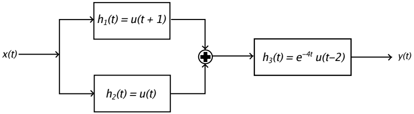
(Analogous holds for CT systems)

Consider the following combination of systems:

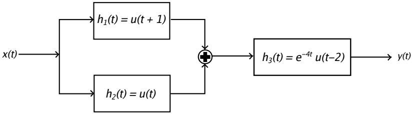


Is this system causal and/or stable?

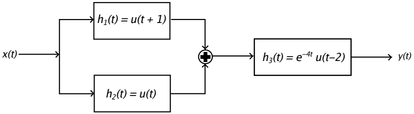
Final exercise



Final exercise



Final exercise



Recap

Today's learning outcomes were:

- Define the convolution sum and use it to compute the output of a system
- Define the CT unit impulse and step functions
- Define the convolution integral and use it to compute the output of a system

For next time

Content:

- (Complex) exponential signals
- The continuous-time Fourier series representation

Action items:

1. Submit Tutorial Assignment 1 (Monday 23:59)
2. Work on Assignment 1 (can do most of Qs 2, 4, 5)
3. Quiz 2 Tuesday about this week's material

Recommended reading:

- From today's class: Oppenheim 1.4, 2.1-2.3
- practice problems: 2.1-2.12, 2.14-16, 2.21, 2.22, 2.28, 2.29
- For next class: Oppenheim 1.3, 3.0-3.3