ELEC 221 Lecture 09
Properties of the CT the Fourier transform

Thursday 06 October 2022

Announcements

- Assignment 3 due tomorrow;
- Assignment 4 (computational) available after midterm
- Midterm 1 next Thursday

Midterm 1

What does it cover?

- Contents of lectures 1-9 (everything up to and incl. today)
- Pen-and-paper midterm; no Python, no programming
- All questions tie directly to the **learning outcomes** shared on the lecture slides

Practice problems:

- Review quizzes and assignment questions
- Oppenheim chapter problems (basic problems w/solutions, basic problems)
- Tutorial on Monday 17:30

Helpful for studying: Tables 3.1, 3.2, and 4.1

Midterm 1 provided formulas

$$\sum_{k=0}^{N} z^k = \frac{1 - z^{N+1}}{1 - z} \qquad \sum_{k=0}^{N-1} e^{\frac{2\pi jk}{N}} = 0$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega t}dt$$

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi}{N}n} \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \qquad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

Last time: the Fourier transform

We saw how we generalized from the CT Fourier series to the Fourier transform for aperiodic signals:

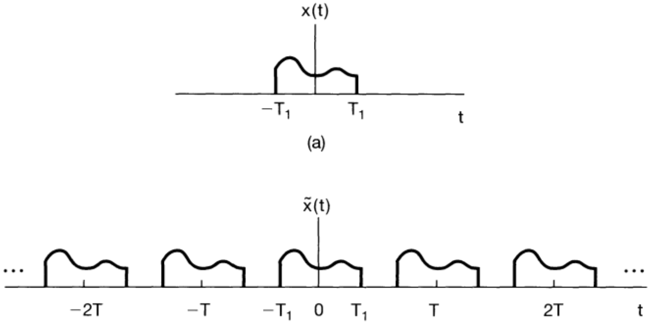


Image credit: Oppenheim chapter 4.1

We expressed the periodic extension of an aperiodic function as a Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

We computed its coefficients:

$$\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega t} dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega t} dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega t} dt \\
&= \frac{1}{T} X(jk\omega)
\end{aligned}$$

We put this back in our Fourier series:

$$\begin{aligned}\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega) e^{jk\omega t} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega\end{aligned}$$

and made some arguments as $T \to \infty$ $(\omega \to 0)$

$$\begin{aligned}\lim_{T\to\infty} \tilde{x}(t) = x(t) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega\end{aligned}$$

Last time: the Fourier transform

Inverse Fourier transform (synthesis equation):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier transform (analysis equation):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Last time: the Fourier transform

We found that the frequency response of a system is actually related to the impulse response by a Fourier transform:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} H(j\omega) d\omega$$

Today, we will build on this fact.

Today

Learning outcomes:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Leverage key properties of Fourier transform to simplify its computation
- Describe the duality between time and frequency domains
- Use convolution property to determine output of LTI systems

Dirichlet conditions for Fourier series

Back in lecture 4, we saw the Dirichlet conditions, which are sufficient for a **periodic** signal to be represented as a Fourier series.

If over **one period**, the function

1. is single-valued
2. is absolutely integrable ($\int_T |x(t)| dt < \infty$)
3. has a finite number of maxima and minima
4. has a finite number of discontinuities[1]

then the Fourier series converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

[1] 3/4: the signal has bounded variation over one period

Dirichlet conditions for Fourier transforms

There are similar sufficient criteria for Fourier transforms.

If the signal

1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)|dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

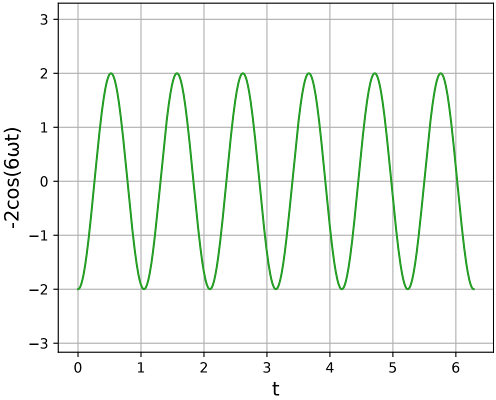
then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

Dirichlet conditions for Fourier transforms

Conclusion: absolutely integrable signals that are continuous or have a finite number of discontinuities have Fourier transforms.

...what about periodic signals?



Fourier transforms for periodic signals: a unified representation

Consider the following output of a Fourier transform:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

What signal does it correspond to?

Fourier transforms for periodic signals: a unified representation

Let's find it:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega \\
&= \int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega \\
&= e^{j\omega_0 t}
\end{aligned}
$$

Fourier transforms for periodic signals: a unified representation

That's good news - but that's just one complex exponential signal. What about when we have multiple harmonics?

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

Take the inverse Fourier transform of this...

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)e^{j\omega t}d\omega \\ &= \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} \delta(\omega - k\omega_0)e^{j\omega t}d\omega \\ &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \end{aligned}$$

Fourier transforms for periodic signals: a unified representation

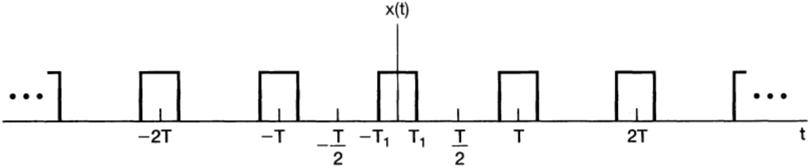
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

The Fourier transform of a periodic function is a train of impulses, positioned at the harmonically related frequencies.

The impulses have area $2\pi c_k$.

Fourier transforms for periodic signals: a unified representation

Remember our square wave from last time:



It had Fourier series coefficients

$$c_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

Image credit: Oppenheim chapter 4.1

Its Fourier *transform* will be

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \delta(\omega - k\omega_0)$$

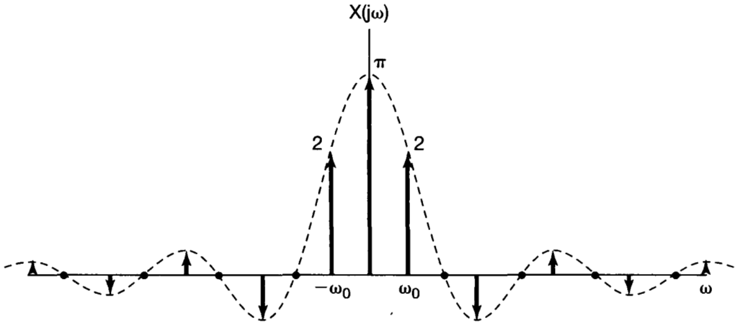


Image credit: Oppenheim chapter 4.2

Important properties of the Fourier transform

The Fourier transform has many useful properties that help with evaluating it for arbitrary functions.

Linearity. If

$$\begin{aligned} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\ y(t) &\overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) \end{aligned}$$

then

$$ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

Time shifting. If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Notice: $|X(j\omega)|$ does not change; we just add a linear phase shift.

Important properties of the Fourier transform

Conjugation. If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x^*(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

If $x(t)$ is purely real,

$$X(-j\omega) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(j\omega)$$

Important properties of the Fourier transform

You've already made use of this when we did audio processing:

```
# Gives the full spectrum
# Has redundant info is signal is real
np.fft.fft(signal)

# Gives only the positive part
np.fft.rfft(signal)
```

Behaviour under conjugation has other implications for even/odd portions of a real signal and its transform:

$$
\begin{aligned}
x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\
\mathrm{Even}(x(t)) &\overset{\mathcal{F}}{\longleftrightarrow} \mathrm{Re}(X(j\omega)) \\
\mathrm{Odd}(x(t)) &\overset{\mathcal{F}}{\longleftrightarrow} j \cdot \mathrm{Im}(X(j\omega))
\end{aligned}
$$

Important properties of the Fourier transform

Time scaling. If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Time reversal follows from this:

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(-j\omega\right)$$

You've all experienced the implications of this time scaling before!

Time/frequency duality of the FT

Let's consider a single square pulse:

$$x(t) = \begin{cases} 1 & |t| < T_1, \\ 0 & |t| > T_1 \end{cases}$$

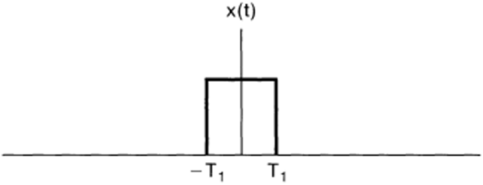


Image credit: Oppenheim chapter 4.1

Compute the Fourier transform:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt \\ &= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} \\ &= -\frac{1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right) \\ &= 2\frac{\sin(\omega T_1)}{\omega} \end{aligned}$$

Time/frequency duality of the FT

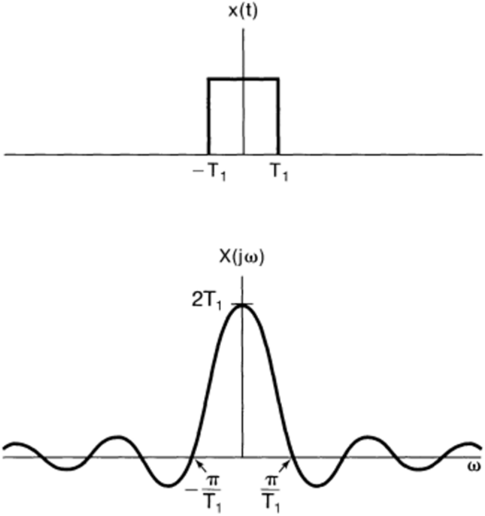


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Now let's consider a signal whose Fourier transform is

$$X(j\omega) = \begin{cases} 1 & |\omega| < W, \\ 0 & |\omega| > W \end{cases}$$

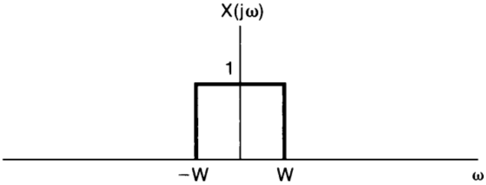


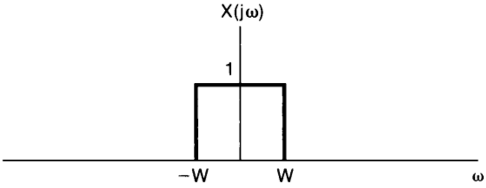
Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Compute the inverse Fourier transform:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega \\
&= \frac{1}{2\pi jt}e^{j\omega t}|_{-W}^{W} \\
&= \frac{1}{2\pi jt}\left(e^{jWt} - e^{-jWt}\right) \\
&= \frac{\sin(Wt)}{\pi t}
\end{aligned}
$$

Time/frequency duality of the FT



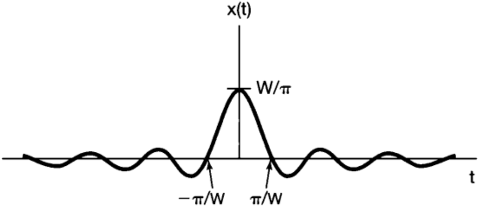


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

These are related...

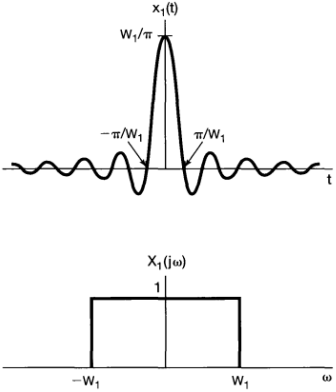
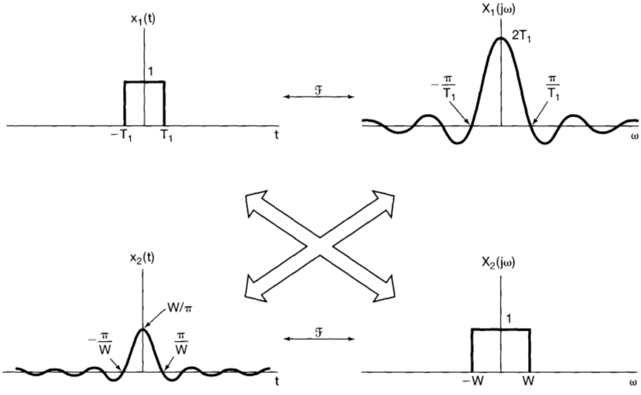


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Duality: for any transform pair ($x(t) \leftrightarrow X(j\omega)$), there is a *dual pair* with the time and frequency variables interchanged.



(We will explore this a bit more on Tuesday)

Image credit: Oppenheim chapter 4.3

Convolution and the Fourier transform

Recall the convolution integral representation: when a signal $x(t)$ is input into an LTI system with impulse response $h(t)$,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = h(t) * x(t)$$

Complex exponentials are eigenfunctions of LTI systems:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \\ y(t) &= h(t) * x(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \end{aligned}$$

where $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-jk\omega t} dt$

Recall how we arrived at the CT Fourier transform:

$$\begin{aligned} x(t) &= \lim_{\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \end{aligned}$$

What happens when we put $x(t)$, as expressed above, into an LTI system with impulse response $h(t)$?

What happens when we put $x(t)$, as expressed above, into an LTI system with impulse response $h(t)$?

$$
\begin{aligned}
x(t) &= \lim_{\omega\to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega \\
y(t) &= \lim_{\omega\to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) H(jk\omega) e^{jk\omega t} \omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega
\end{aligned}
$$

Convolution and the Fourier transform

We have **two** ways now to write a signal $y(t)$:

$$
\begin{aligned}
y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega \\
y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega
\end{aligned}
$$

This has an important implication:

$$
\begin{aligned}
y(t) &= h(t) * x(t) \\
Y(j\omega) &= H(j\omega)X(j\omega)
\end{aligned}
$$

Example: convolution

This can be helpful for evaluating the output of systems given $h(t)$ and $x(t)$ (or $h(t)$ given $y(t)$ and $x(t)$, etc.)

Example: what is $y(t)$ for an LTI system with the following input and impulse response?

$$
\begin{aligned}
x(t) &= e^{-t}u(t) \\
h(t) &= e^{t}u(-t)
\end{aligned}
$$

Example: convolution

Compute the Fourier transform of $x(t)$:

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-(j\omega+1)t}dt \\
&= \frac{-1}{1+j\omega}e^{-(j\omega+1)t}|_{0}^{\infty} \\
&= \frac{1}{1+j\omega}
\end{aligned}$$

Convenient general expression to remember:

$$\mathcal{F}(e^{-at}u(t)) = \frac{1}{a+j\omega}, \quad \text{Re}(a) > 0$$

Example: convolution

Compute the Fourier transform of $h(t) = e^t u(-t)$:

$$\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} e^t u(-t) e^{-j\omega t} dt \\
&= -\int_{\infty}^{-\infty} e^{-t} u(t) e^{j\omega t} dt \\
&= \int_{-\infty}^{\infty} e^{-t} u(t) e^{j\omega t} dt \\
&= \int_{0}^{\infty} e^{(j\omega - 1)t} dt \\
&= \frac{1}{1 - j\omega}
\end{aligned}$$

Example: convolution

Then,

$$
\begin{aligned}
Y(j\omega) &= H(j\omega)X(j\omega) \\
&= \frac{1}{1+j\omega}\frac{1}{1-j\omega}
\end{aligned}
$$

How to deal with this? Partial fractions:

$$
\begin{aligned}
Y(j\omega) &= \frac{A}{1+j\omega} + \frac{B}{1-j\omega} \\
&= \frac{1}{2}\frac{1}{1+j\omega} + \frac{1}{2}\frac{1}{1-j\omega}
\end{aligned}
$$

Example: convolution

Now we need to take the inverse Fourier transform:

$$Y(j\omega) = \frac{1}{2}\frac{1}{1+j\omega} + \frac{1}{2}\frac{1}{1-j\omega}$$

But we already know that

$$x(t) = e^{-t}u(t) \quad \leftrightarrow \quad X(j\omega) = \frac{1}{1+j\omega}$$

Similarly,

$$h(t) = e^{t}u(-t) \quad \leftrightarrow \quad H(j\omega) = \frac{1}{1-j\omega}$$

Example: convolution

$$Y(j\omega) = \frac{1}{2}\frac{1}{1+j\omega} + \frac{1}{2}\frac{1}{1-j\omega}$$

$$\begin{aligned}
y(t) &= \mathcal{F}^{-1}\left(\frac{1}{2}\frac{1}{1+j\omega}\right) + \mathcal{F}^{-1}\left(\frac{1}{2}\frac{1}{1-j\omega}\right) \\
&= \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t) \\
&= \frac{1}{2}e^{-|t|}
\end{aligned}$$

Recap

Today's learning outcomes were:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Leverage key properties of Fourier transform to simplify its computation
- Describe the duality between time and frequency domains
- Use convolution property to determine output of LTI systems

What topics did you find unclear today?

For next time

Content:

- Multiplication properties of the CT Fourier *transform*

Action items:

1. Assignment 3 is due tomorrow
2. Midterm 1 next Thursday

Recommended reading:

- From today's class: Oppenheim 4.2-4.4
- For next class: Oppenheim 4.5-4.7