ELEC 221 Lecture 23
The Laplace transform and feedback systems; introducing the z-transform

Tuesday 3 December 2024

Announcements

- Quiz 10 today
- Please fill out course evaluation survey if you have time after quiz
- Assignment 5 due Sunday at 23:59
- Exam info period office hours will be posted on Piazza/PrairieLearn this week

Last time

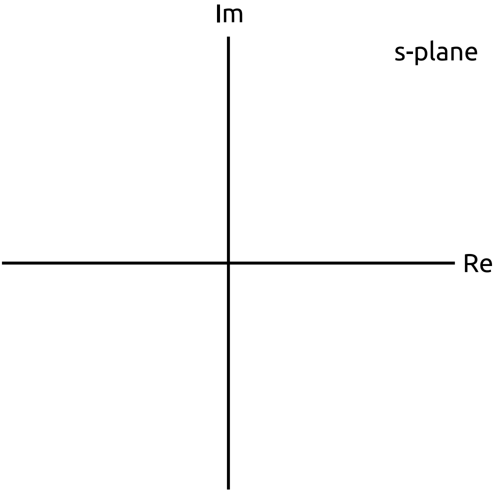
We explored various properties of the Laplace transform.

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R

Image credit: Oppenheim 9.5

We used the ROC to reason about the stability and causality of systems with rational Laplace transforms.



Today

Learning outcomes:

- compute the Laplace transform of systems described by constant-coefficient DEs
- compute system functions for feedback systems
- use the Laplace transform to design an inverse system
- define the z-transform and compute it and its ROC for basic signals

Systems described by constant-coefficient differential equations

Recall the situation with the Fourier transform:

Fourier transforms and systems described by differential equations

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

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Systems described by constant-coefficient differential equations

Same deal here. If system is described by the DE

then its system function is

Placement of zeros and poles is dictated by coefficients of $x(t)$ and $y(t)$ stuff respectively.

9.32. A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = (1/6)e^{2t}$ for all t.
2. The impulse response $h(t)$ satisfies the differential equation

$$\frac{dh(t)}{dt} + 2h(t) = (e^{-4t})u(t) + bu(t),$$

where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should *not* appear in the answer.

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9.34. Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

1. $H(1) = 0.2$.
2. When the input is $u(t)$, the output is absolutely integrable.
3. When the input is $tu(t)$, the output is not absolutely integrable.
4. The signal $d^2h(t)/dt^2 + 2\,dh(t)/dt + 2h(t)$ is of finite duration.
5. $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

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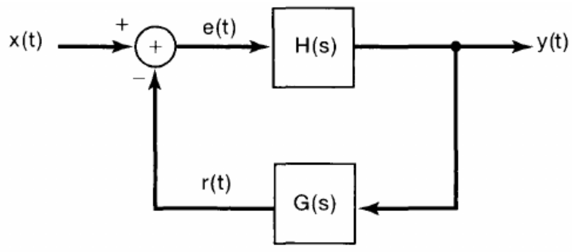
Feedback systems

An important application of Laplace transforms is the analysis of **feedback systems**.



Some important applications of feedback include:

- reducing sensitivity to disturbance and errors
- stabilizing unstable systems
- designing tracking systems

Image credit: Oppenheim 11.1

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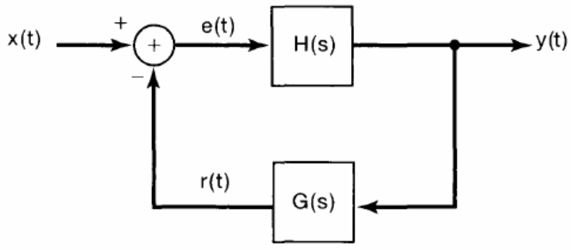


- $H(s)$ is the system function of the forward path
- $G(s)$ is the system function of the feedback path
- the combined function $Q(s)$ is the closed-loop system function

Let's compute $Q(s)$ in terms of $H(s)$ and $G(s)$.

Image credit: Oppenheim 11.1

Feedback systems

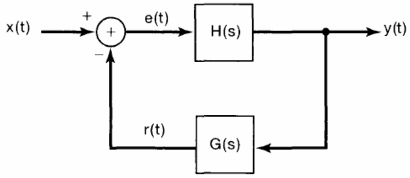
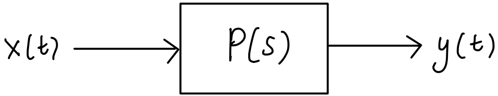


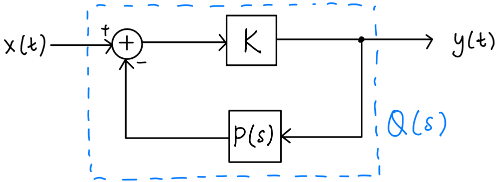
Image credit: Oppenheim 11.1

Application of feedback: constructing inverse systems

Suppose we have some LTI system



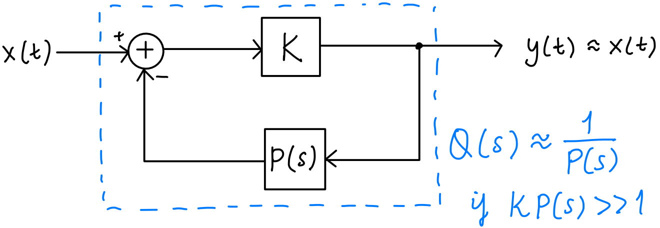
Let's use it as part of a larger system:



where the transfer function K is simply gain of strength K.

Exercise: What is $Q(s)$, and under what conditions can it act as the inverse of $P(s)$?

Solution: we can directly apply the expression for the closed-loop system function here

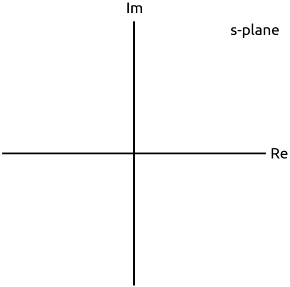


Application of feedback: stabilizing an unstable system

Consider a system described by the first order DE

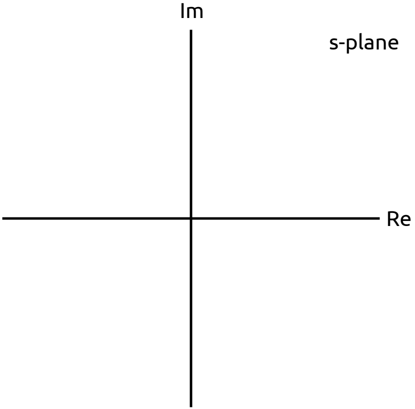
$$\frac{dy(t)}{dt} - ay(t) = bx(t)$$

Exercise: compute the system function and draw the ROC. Under what conditions is it stable?



Application of feedback: stabilizing an unstable system

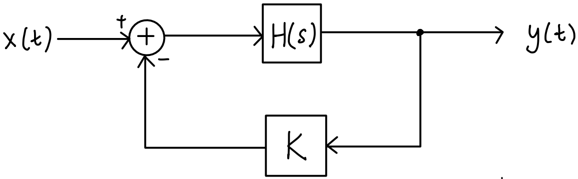
Suppose we have this setup ($a > 0$):

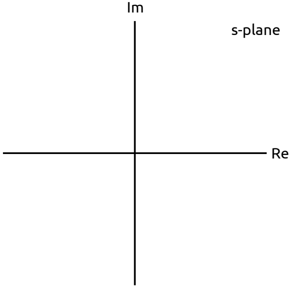


How can we make it stable?

Application of feedback: stabilizing an unstable system

Show that the following system will move the pole (under certain conditions on K):





Called a *proportional feedback system* since feeding back in a rescaled version of the output.

The z-transform

CT

Fourier series coefficients

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

Fourier transform (spectrum)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

DT

Fourier series coefficients

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi n}{N}}$$

Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Consider a DT complex exponential signal

If we put this in a system with impulse response $h[n]$, obtain

where

- $z = e^{j\omega}$: discrete-time Fourier transform
- $z = re^{j\omega}$: z-transform

The z-transform

For a general signal $x[n]$,

Just like in CT, this can be expressed with a DTFT involving $x[n]$:

Exercise: compute the z-transform of

$$x[n] = a^n u[n]$$

For what values of z does it converge?

Must be the case that , or .

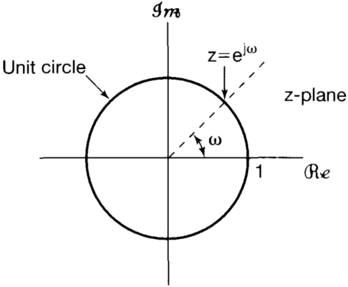
Exercise: compute the z-transform of

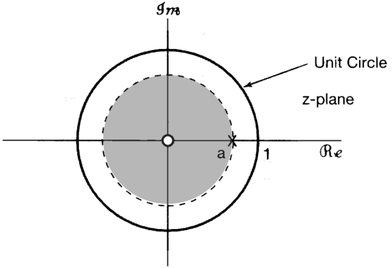
$$x[n] = -a^n u[-n-1]$$

For what values of z does it converge?

Must have , or . Then can write

For ROC of z-transform, we make pole-zero plots on the z-plane.





Unit circle $z = e^{j\omega}$ ($|z| = 1$) corresponds to the DTFT case (like the vertical axis $s = j\omega$ for CT).

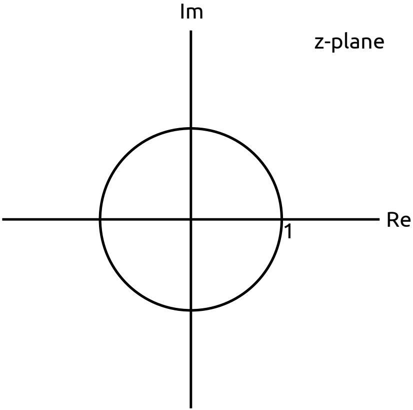
Image credit: Oppenheim 10.1

Regions of convergence

Exercise: compute the z-transform for

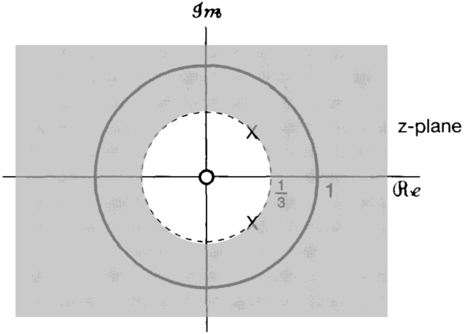
$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

and sketch the pole-zero plot of its ROC.



ROC of the z-transform has many properties:

- if ROC doesn't contain unit circle, DTFT doesn't converge
- it is a ring in the z-plane centred around origin (for $z = re^{j\omega}$, does not depend on ω, only r)
- it does not contain any poles



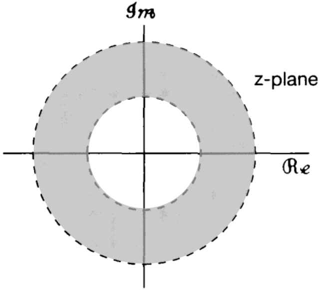


Image credit: Oppenheim 11.1, 11.2

If a signal $x[n]$ is of finite duration, its ROC is the entire z-plane *except possibly* $z = 0$ *and/or* $z = \infty$.

Exercise: compute the z-transform and ROC of

1. $z[n] = \delta[n]$
2. $z[n] = \delta[n-1]$
3. $z[n] = \delta[n+1]$

Solution:

Regions of convergence

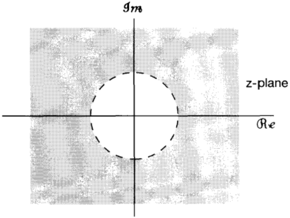
Right-sided signal: $X(z)$ has the form

$$X(z) = \sum_{n=N_1}^{\infty} x[n] z^{-n}$$

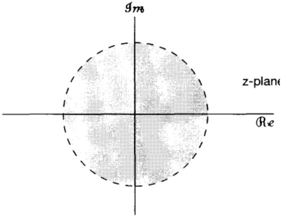
This may or may not include ∞ depending on the structure of the signal (in particular, if $N_1 < 0$, terms will become unbounded).

If $|z| = r_0$ is in the ROC for right-sided signal, then so are all *finite* z where $|z| > r_0$.

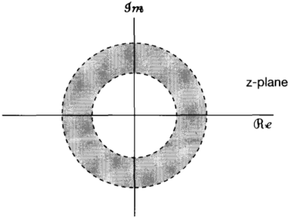
Similar argument for left-sided signals and the zero point.



Right-sided



Left-sided



Two-sided

Image credit: Oppenheim 10.2

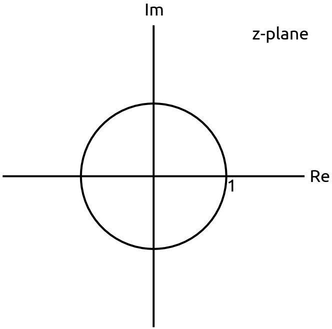
Regions of convergence

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

Draw the pole-zero plot and determine the possible ROCs.

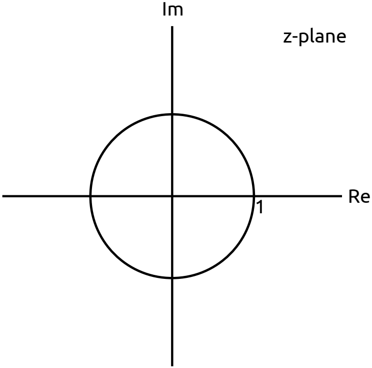
Hint: this function has 2 zeros; express it in a different way to find them.



When $X(z)$ is a rational function, we can compute the inverse using partial fractions. We still need the ROC to help us.

Exercise: compute the inverse z-transform of

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}, \quad |z| > 2$$



For next time

Content:

- more properties of z-transforms
- systems described by difference equations
- z-transforms and feedback system analysis

Action items:

1. Assignment 5 due Sunday 8 Dec at 23:59

Recommended reading:

- From this class: Oppenheim 9.7, 11.0-11.2, 10.1-10.3
- Suggested problems: 9.48, 11.1-11.4, 10.1-10.8, 10.21-10.23, 10.26
- For next class: 10.5-10.7, 11.2