ELEC 221 Lecture 17
The sampling theorem

Thursday 3 November 2022

- Midterms available for pickup at my office
- Assignment 5 available; due 11:59 Friday Nov. 11 (**no extensions**; solutions to be posted immediately after for studying)

Monday 14 Midterm

Important: on Zoom for the next week.

- Nov. 8 class
- Office hours this Friday and next Friday
- Still available by appointment

Links will be distributed on Canvas.

We introduced the **step response** of filters.

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

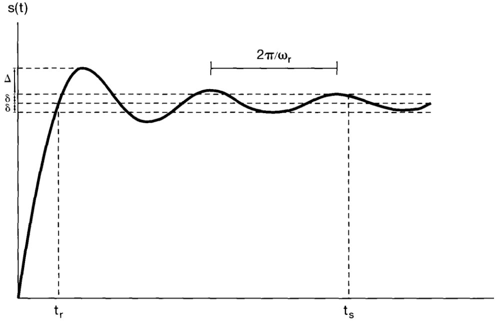


Figure 6.17 Step response of a continuous-time lowpass filter, indicating the rise time t_r, overshoot Δ, ringing frequency ω_r, and settling time t_s—i.e., the time at which the step response settles to within $\pm\delta$ of its final value.

Image credit: Oppenheim 6.4

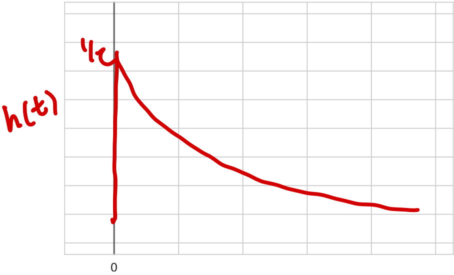
Last time

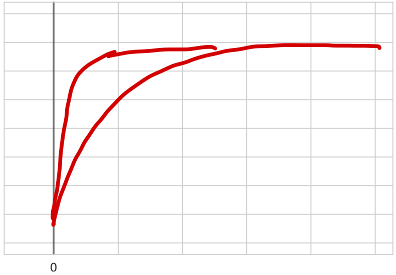
$$\tau \frac{dy(t)}{dt} + y(t) = x(t), \qquad H(j\omega) = \frac{1}{1 + j\omega\tau}$$

The impulse and step response of the system are

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \qquad s(t) = \left(1 - e^{-t/\tau}\right) u(t)$$

τ is the **time constant** of the system.





Last time

We drew some simple Bode plots.

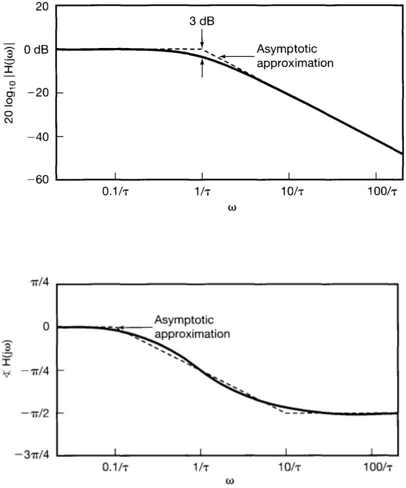
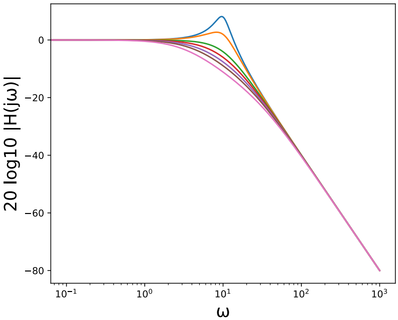


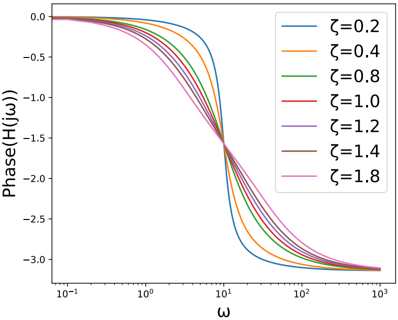
Image credit: Oppenheim 6.5

Last time

We looked at systems described by second-order ODEs.

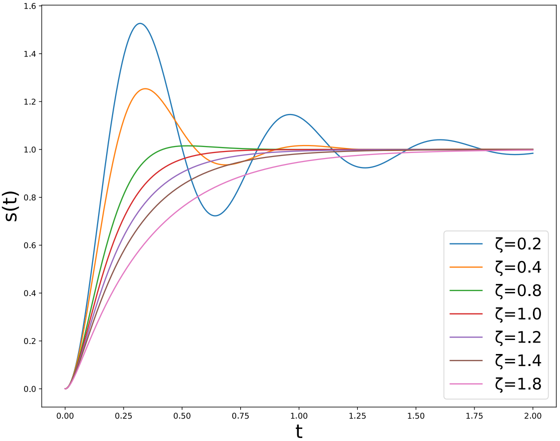
$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$



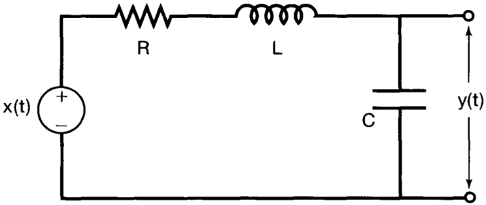


Last time

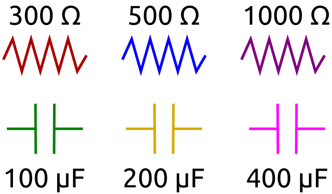
ζ is the damping ratio (can be under, over, or critically damped).



Last time



Suppose $L = 6\text{H}$. We have a box of capacitors and resistors:



What is the best choice to ensure step response doesn't oscillate?

Image credit: Oppenheim P6.19.

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

Solution: compute the frequency response

$$H(j\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + 1}$$

$$= \frac{1}{(j\omega/(1/\sqrt{LC}))^2 + 2(R/2)\sqrt{C/L}\,\frac{j\omega}{1/\sqrt{LC}} + 1}$$

Find that $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$

If $\zeta = (R/2)\sqrt{C/L}$, and $L = 6\text{H}$, we want

$$\frac{R}{2}\sqrt{\frac{C}{L}} \geq 1$$

$$\frac{R}{2}\sqrt{\frac{C}{L}} \geq 1$$

$$R^2 C \geq 4L = 24$$

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Best choice is $R = 500\Omega$, and $C = 100\mu F$ ($R^2 C = 25$)

$R = 500\Omega$ $C = 100\mu F$ ($R^2 C = 25$)

Today

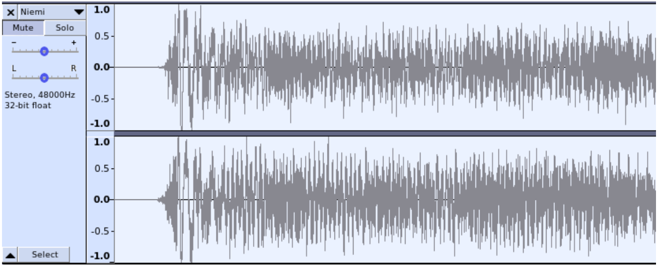
- state the sampling theorem
- define the Nyquist sampling rate and determine if a sampling rate is sufficient to reconstruct a signal
- construct systems of filters to interpolate a signal from its samples
- describe the phenomenon of aliasing

Lecture 04 Demos

```
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import Audio
```

Demo 1: fun with square waves

```
tone = 65             # A frequency in Hz
duration = 2           # The length of the audio signal (in seconds)
sample_rate = 48000    # The number of samples per second to take
t_range = np.linspace(0, duration, sample_rate * duration) # Range of time
```



Sampling

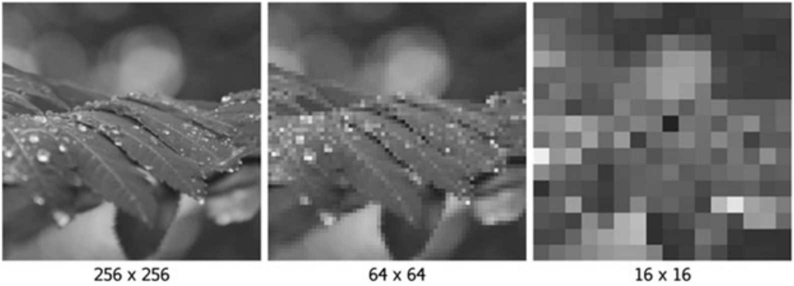


Image credit: `https://what-when-how.com/introduction-to-video-and-image-processing/image-acquisition-introduction-to-video-and-image-processing-part-2/`

Sampling

History of frame rate in film:
https://www.youtube.com/watch?v=mjYjFEp9Yx0

Image credit: https://www.mediacollege.com/video/frame-rate/img/frame-rates.jpg

We saw that the discrete Fourier transform was a set of equally-spaced samples of the discrete-time Fourier transform.

The discrete Fourier transform

What if we sample this signal at particular values of $k\omega = k2\pi/N$?

$$
\begin{aligned}
X(e^{jk2\pi/N}) &= \sum_{n=0}^{N-1} x[n]e^{-jk2\pi n/N} = N\tilde{X}[k] \\
\frac{1}{N}X(e^{jk2\pi/N}) &= \tilde{X}[k]
\end{aligned}
$$

The discrete Fourier transform is a set of equally spaced samples of the full discrete-time Fourier transform.

Key point 1: Any signal $x[n]$ can be uniquely specified by a finite set of samples from its DTFT (i.e., its DFT).

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The unit impulse as a sampler

Multiplying the signal by a shifted impulse picks out the value of the signal at that point:

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k]$$

This allows us to write any signal as a **superposition of weighted impulses.**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$$

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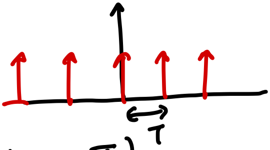
In continuous time:

$$x(t)\cdot\delta(t-t_0) = x(t_0)\cdot\delta(t-t_0)$$

What if we have more than one?



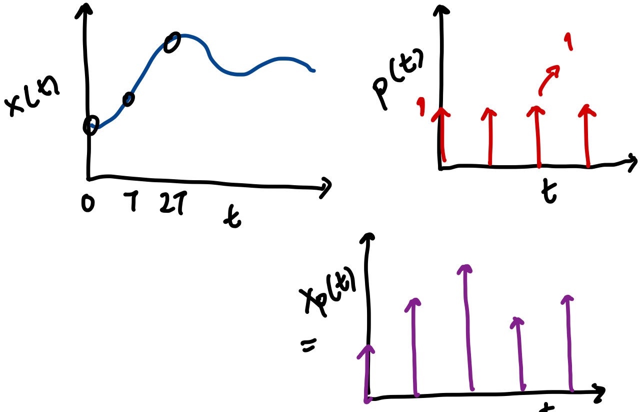
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

where

$$\omega_s = \frac{2\pi}{T}$$

What does the following signal look like?

$$x_p(t) = x(t) \cdot p(t)$$



Impulse train sampling

$$x(t) \qquad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The combined signal is

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

This is all time domain; what happens in the frequency domain?

By the multiplication property,

$$x_p(t) = x(t)\,p(t) \leftrightarrow X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\,P(j(\omega-\theta))\,d\theta$$

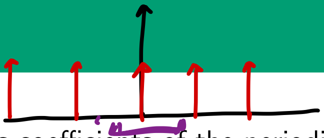
But what is $P(j\omega)$? We haven't evaluated this yet...

We have a periodic impulse train. Recall what Fourier transforms of periodic signals looked like:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \longleftrightarrow x(t) = e^{j\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \longleftrightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}$$

Impulse train sampling



We need to find the Fourier series coefficients of the periodic impulse train.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega t}\, dt$$

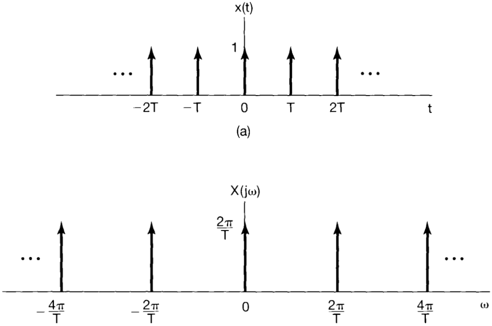
$$= \frac{1}{T}$$

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega t}$$

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \cdot \delta(\omega - k\omega_s)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T}$$



Impulse train sampling

$$X(j\omega)$$

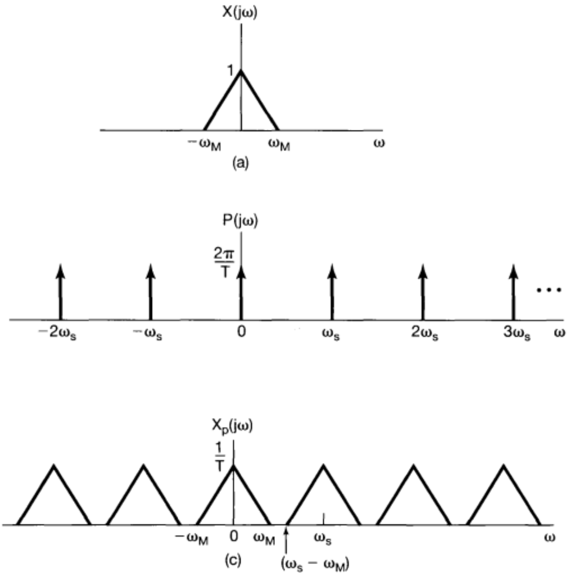
$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) \cdot P(j(\omega-\theta))\, d\theta$$

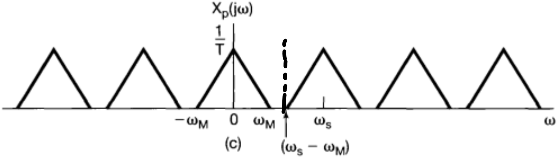
$$= \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} X(j\theta) \cdot \frac{\cancel{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s - \theta) \cdot d\theta$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Impulse train sampling



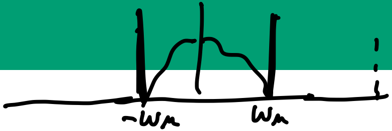
Suppose we have sampled...



How do we recover our original signal from this spectrum?

Image credit: Oppenheim 7.1

The sampling theorem

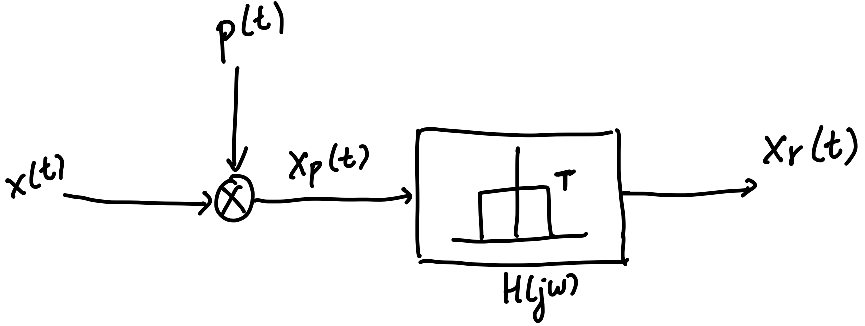


"Let $x(t)$ be a **band-limited** signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M \qquad \omega_s = \frac{2\pi}{T}$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$."

Let's show this graphically:



The Nyquist rate

The sampling frequency is key:

- $\omega_s = 2\omega_M$ is referred to as the **Nyquist rate**
- $\omega_M = \omega_s/2$ is referred to as the **Nyquist frequency**

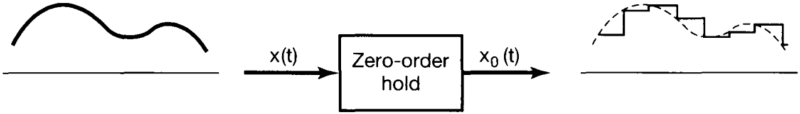
Exercise: suppose we perform impulse-train sampling with period $T = 10^{-4}$. If a signal $x(t)$ has $X(j\omega) = 0$ for $|\omega| > 15000\pi$, can we reconstruct it exactly from the samples?

$$\omega_s \quad > \quad 30000\pi$$

$$T = 10^{-4} \quad \rightarrow \quad \omega \approx 62800 < 30000\pi$$

Sampling in practice: zero-order hold

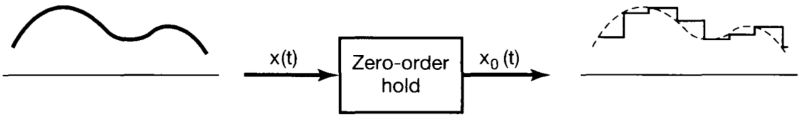
In reality we cannot generate perfect narrow, large-amplitude impulses. Instead:



Can we still reconstruct our signal?

Image credit: Oppenheim 7.1

Sampling in practice: zero-order hold



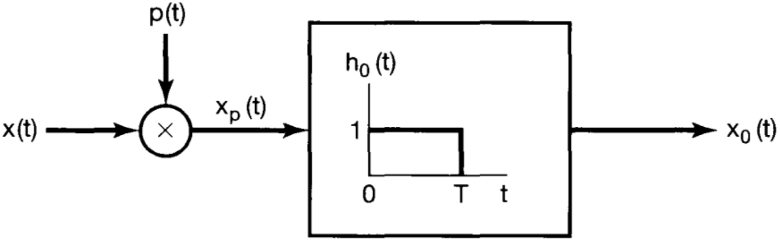
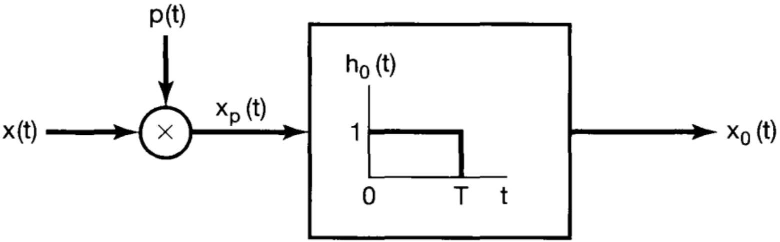
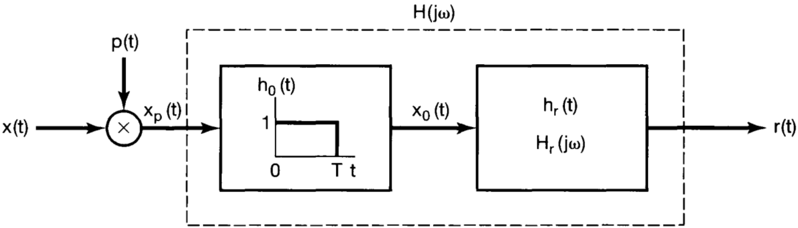


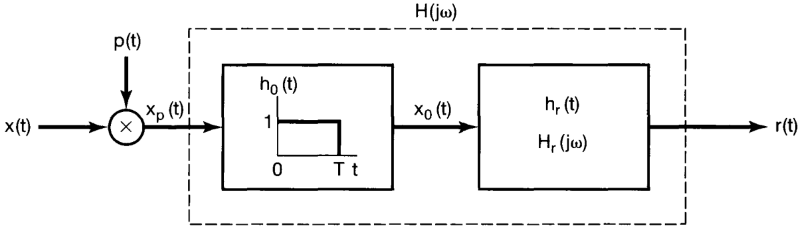
Image credit: Oppenheim 7.1

Sampling in practice: zero-order hold





Sampling in practice: zero-order hold

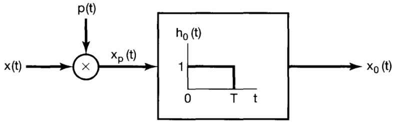


To obtain $r(t) = x(t)$, need $H_r(j\omega)H_0(j\omega) = H(j\omega)$ for ideal lowpass filter.

But what is $H_0(j\omega)$?

Image credit: Oppenheim 7.1

Sampling in practice: zero-order hold



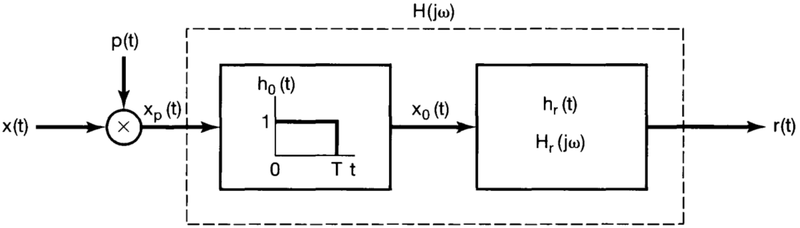
Square pulse between $-T_1$ and T_1:

$$X(j\omega) = 2\frac{\sin(\omega T_1)}{\omega}$$

Use properties of the Fourier transform to obtain

$$H_0(j\omega) = e^{-j\omega T/2}\left(2\frac{\sin(\omega T/2)}{\omega}\right)$$

Image credit: Oppenheim 7.1



If $H_r(j\omega)H_0(j\omega) = H(j\omega)$ (ideal lowpass filter) and

$$H_0(j\omega) = e^{-j\omega T/2}\left(2\frac{\sin(\omega T/2)}{\omega}\right)$$

then we need

$$H_r(j\omega) = \frac{e^{j\omega T/2}}{2\frac{\sin(\omega T/2)}{\omega}}H(j\omega)$$

Sampling in practice: zero-order hold

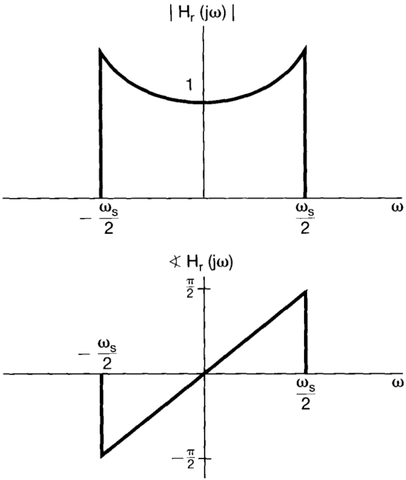
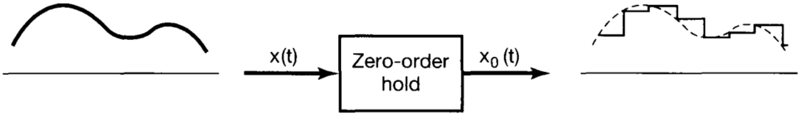


Image credit: Oppenheim 7.1

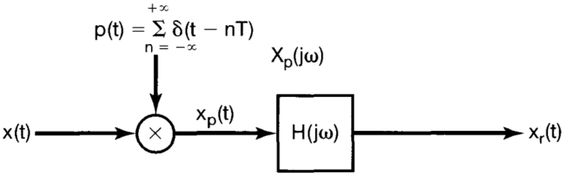
In some cases, the ZOH actually provides a good enough interpolation:



But we can do a lot better using, e.g., linear (first-order hold) or higher-order polynomial reconstruction methods.

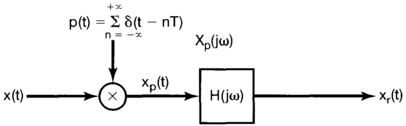
Image credit: Oppenheim 7.1

Band-limited interpolation



$$
\begin{aligned}
x_p(t) &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \\
x_r(t) &= x_p(t) * h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)h(t - nT)
\end{aligned}
$$

Image credit: Oppenheim 7.2



For lowpass filter with cutoff ω_c and gain T,

$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}$$

Then

$$\begin{aligned} x_r(t) &= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT) \frac{\omega_c T \sin(\omega_c (t - nT))}{\pi \omega_c (t - nT)} \end{aligned}$$

Image credit: Oppenheim 7.2

Sometimes zero- or first-order are good enough; increasing the order will improve interpolation at the cost of complexity.

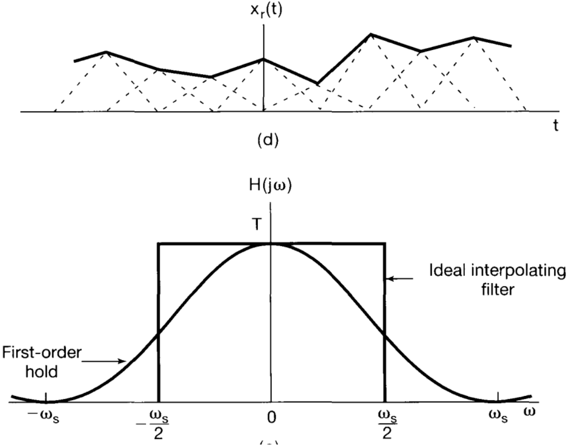
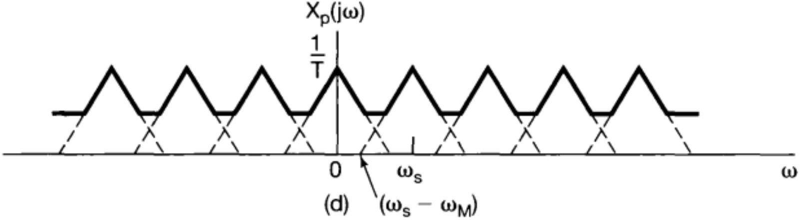


Image credit: Oppenheim 7.2

Aliasing

What happens when you don't sample at a high enough rate?



Aliasing

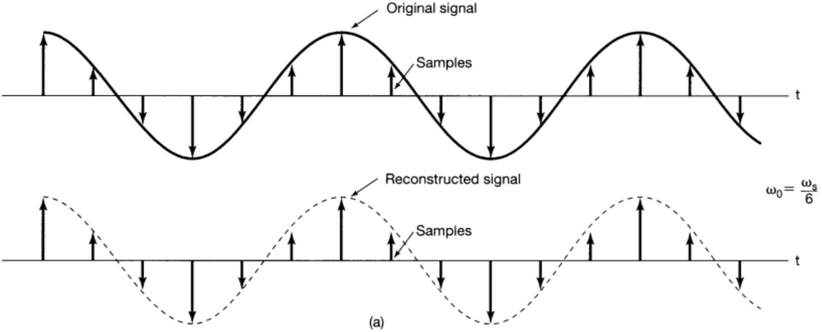


Image credit: Oppenheim 7.3

Aliasing

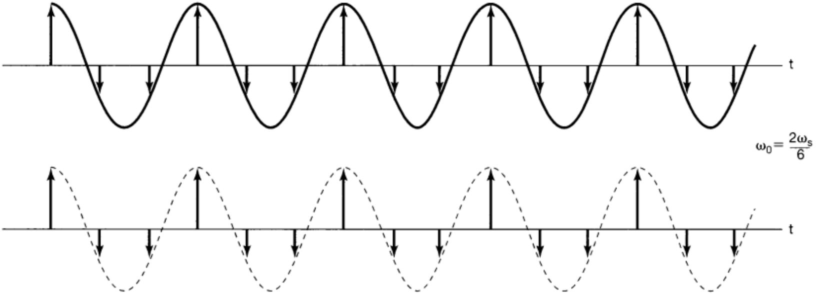


Image credit: Oppenheim 7.3

Aliasing

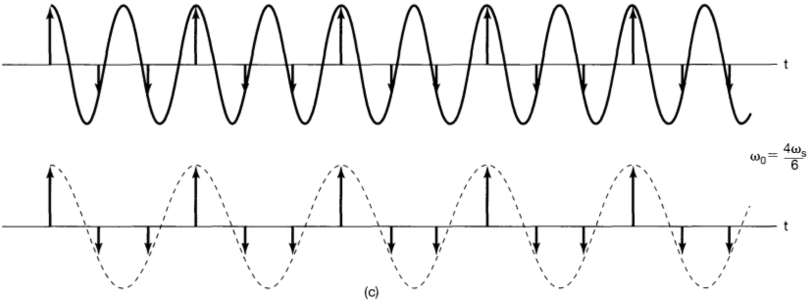


Image credit: Oppenheim 7.3

Aliasing

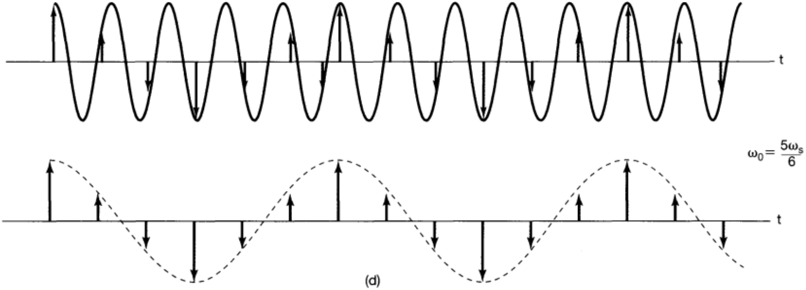
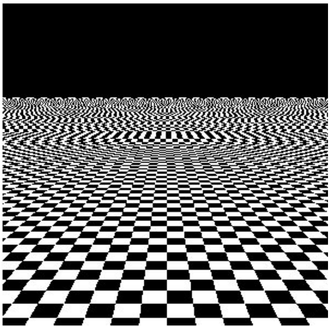


Image credit: Oppenheim 7.3

See Oppenheim Fig. 7.15

Real-world examples



Fun on your own: read up about Moiré patterns, and various **anti-aliasing** techniques that are used in music/images/games!

Image credit: `https://textureingraphics.wordpress.com/what-is-texture-mapping/anti-aliasing-problem-and-mipmapping/`

Today

Learning outcomes:

- state the sampling theorem
- define the Nyquist sampling rate and determine if a sampling rate is sufficient to reconstruct a signal
- construct systems of filters to interpolate a signal from its samples
- describe the phenomenon of aliasing

Oppenheim practice problems: 7.1-7.7, 7.21, 7.25

For next time

Content:

- DT processing of CT signals
- Sampling in discrete time
- Decimation/interpolation

Action items:

1. Assignment 5 due 11:59pm Friday 11 Nov
2. Midterm 2 Monday 14 Nov during tutorial

Recommended reading:

- From this class: Oppenheim 7.1-7.3
- For next class: Oppenheim 7.4-7.6