ELEC 221 Lecture 25
The z-transform

Tuesday 6 December 2022

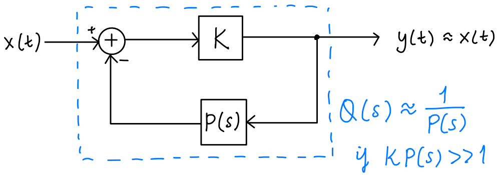
Announcements

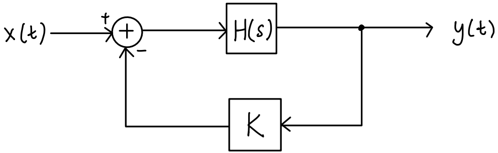
- Last class!
- Please come pick up your midterms
- Assignment 7 due tonight at 23:59 (hard deadline, no extensions)
- Details for final exam to be posted on Piazza when available

Last time

We saw how knowledge of the Laplace transform can help us:

- analyze feedback systems
- stabilize unstable systems
- find inverse systems





Last time

We introduced the DT counterpart, the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

We represented its region of convergence on the z-plane

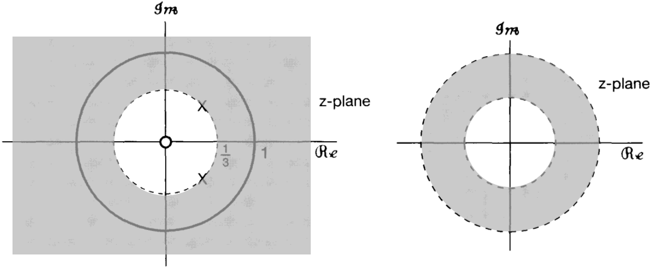


Image credit: Oppenheim 11.1, 11.2

Today

Learning outcomes:

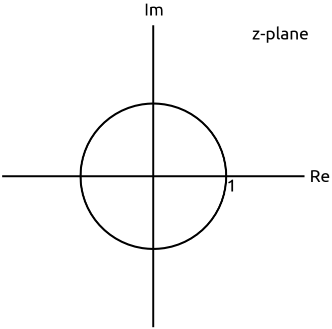
- use the z-transform to determine whether a system is causal or stable
- apply the z-transform to systems described by difference equations
- analyze simple feedback systems with the z-transform

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

Draw the pole-zero plot and determine the possible ROCs.

Hint: this function has 2 zeros; express it in a different way to find them.

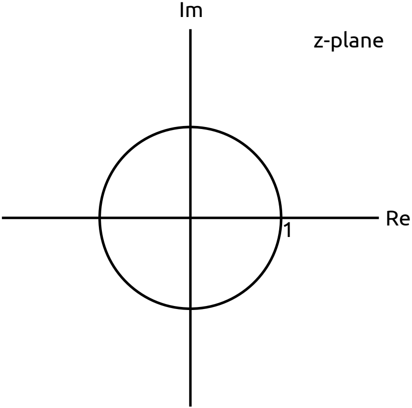


Regions of convergence

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

Solution:



$$\begin{aligned} X(z) &= \frac{1}{z^{-2}(z-\frac{1}{3})(z-2)} \\ &= \frac{z^2}{(z-\frac{1}{3})(z-2)} \end{aligned}$$

When the z-transform can be expressed as a rational function, we can compute the inverse using partial fractions. We still need the ROC to help us.

Exercise: compute the inverse z-transform of

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

if ROC is specified to be $|z| > 2$.

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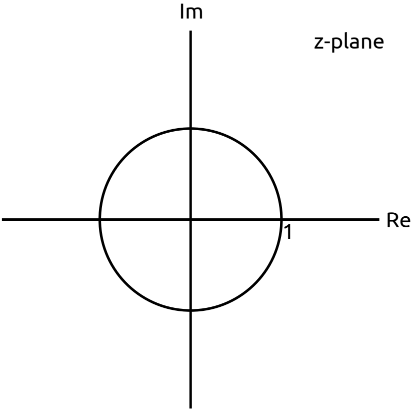


Use partial fractions:

$$\begin{aligned} X(z) &= \frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B}{1 - 2z^{-1}} \\ &= \frac{-1/5}{1 - \frac{1}{3}z^{-1}} + \frac{6/5}{1 - 2z^{-1}} \end{aligned}$$

From ROC, signal is right-sided:

$$x[n] = -\frac{1}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{6}{5}2^n u[n]$$

Take a closer look at the structure of $X(z)$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

This is a *power series in z*. If we can do the expansion, we can recover $x[n]$ from the coefficients.

Exercise 1: what is the inverse z-transform of

$$X(z) = 3z^2 - 1 + 2z^{-3}, \quad 0 < |z| < \infty$$

Solution:

$$x[n] = 3\delta[n+2] - \delta[n] + 2\delta[n-3]$$

Particularly helpful for non-linear cases.

Exercise 2 (Oppenheim 10.63a): what is the inverse z-transform of

$$X(z) = \log(1 - 2z), \quad |z| < \frac{1}{2}$$

Hint:

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \ |w| < 1$$

Solution:

$$\begin{aligned} X(z) &= \log(1 - 2z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i} = -\sum_{n=-\infty}^{-1} \frac{2^{-n}}{-n} z^{-n} \\ x[n] &= \begin{cases} \frac{2^{-n}}{n} & n \leq -1 \\ 0 & n > -1 \end{cases} = \frac{2^{-n}}{n} u[-n-1] \end{aligned}$$

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \quad \text{w/ROC } R_1$$
$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z) \quad \text{w/ROC } R_2$$

Linearity:

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z) \quad \text{w/ROC containing } R_1 \cap R_2$$

Example: $a^n u[n]$, $a^n u[n-1]$ both have ROC of $|z| > |a|$. What is the ROC of z-transform of

$$a^n u[n] - a^n u[n-1]$$

Solution: $\delta[n]$ w/ROC entire z-plane.

Properties of the z-transform

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \quad \text{w/ROC } R$$

Time shift:

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z) \quad \text{w/ROC } R \text{ and maybe add/delete } 0 \text{ or } \infty$$

Time reversal:

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right) \quad \text{w/ROC } 1/R$$

Time expansion (zero-insertion of $k-1$ zeros):

$$x_{(k)}[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z^k) \quad \text{w/ROC } R^{1/k}$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \quad \text{w/ROC } R$$

Scaling in z:

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{z_0}\right) \quad \text{w/ROC } |z_0|R$$

Conjugation:

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*) \quad \text{w/ROC } R$$

If $x[n]$ is real, the poles and zeros come in *conjugate pairs*.

The z-transform and causality

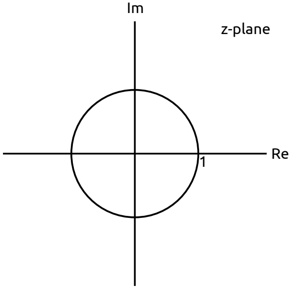
The convolution property of the z-transform tells us that

$$Y(z) = H(z)X(z)$$

Remember how we previously tested causality: $h[n] = 0$ for all $n < 0$ (it is right-sided).

A DT LTI system with rational z-transform is causal if:

- the ROC is the exterior of a circle outside the outermost pole (including infinity)
- with $H(z)$ expressed in polynomials of z, order of numerator does not exceed order of the denominator

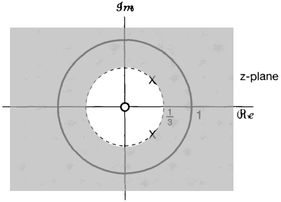


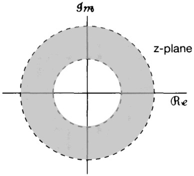
The z-transform and stability

Previously, to compute stability, we checked if the impulse response was absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]|z^{-n} < \infty$$

This was also a condition required for the DTFT to exist.





An LTI system is stable if ROC includes the unit circle $|z| = 1$.

Image credit: Oppenheim 11.1, 11.2

The initial value theorem

If $x[n] = 0$ for all $n < 0$, then

$$\lim_{z\to\infty} X(z) = x[0]$$

How? Look again at expression for $X(z)$:

$$\sum_{n=0}^{\infty} x[n] z^{-n}$$

Consequences:

- if $x[n]$ is causal, $\lim_{z\to\infty} X(z)$ is finite
- if $X(z)$ is a ratio of polynomials, order of numerator cannot be greater than order of denominator (cannot have more finite zeros than finite poles)

10.17. Suppose we are given the following five facts about a particular LTI system S with impulse response $h[n]$ and z-transform $H(z)$:

1. $h[n]$ is real.
2. $h[n]$ is right sided.
3. $\lim_{z \to \infty} H(z) = 1$.
4. $H(z)$ has two zeros.
5. $H(z)$ has one of its poles at a nonreal location on the circle defined by $|z| = 3/4$.

Answer the following two questions:

(a) Is S causal? **(b)** Is S stable?

Leveraging z-transform properties

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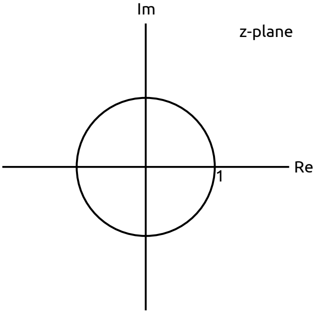


Systems described by difference equations

Consider LTI system described by a DT difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Using properties of the DTFT (convolution, time shift, linearity):

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Using analogous properties of the z-transform,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Exercise (Oppenheim 10.36): consider LTI system described by a DT difference equation

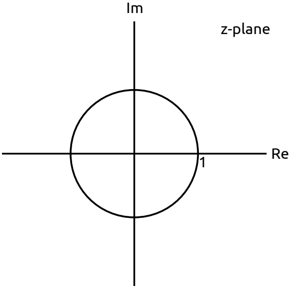
$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$$

Suppose the system is stable; what is its impulse response?

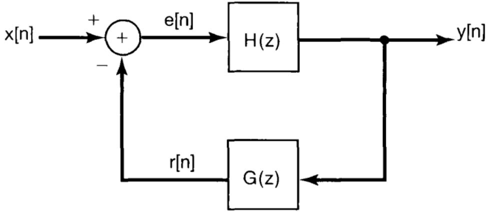
$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$$

Solution:

$$\begin{aligned}
H(z) &= \frac{1}{z - 10/3 + z^{-1}} \\
&= \frac{z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}} \\
&= \frac{z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - 3z^{-1})} \\
&= \frac{-3/8}{1 - \frac{1}{3}z^{-1}} + \frac{3/8}{1 - 3z^{-1}} \\
h[n] &= -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{8}3^n u[-n-1]
\end{aligned}$$



The z-transform can help us with the analysis of feedback systems (using them for stabilization, etc.) like we did in CT with the Laplace transform.

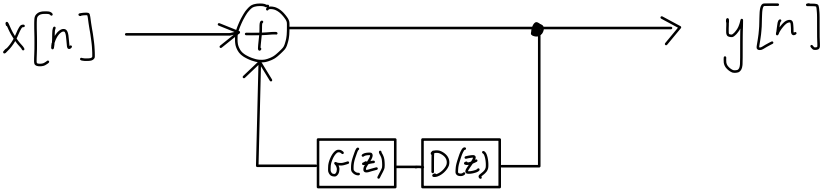


The closed-loop system function has the same form:

$$Q(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1 + G(z)H(z)}$$

Image credit: Oppenheim 11.1

One type of system with this structure is called the **comb filter**

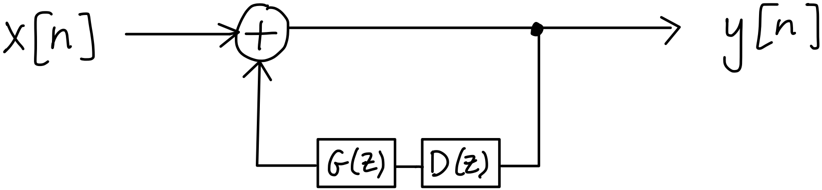


Suppose:

- $D(z)$ is a system that causes a delay of K steps
- $G(z)$ is a system with gain g

Exercise:

- what is the difference equation that describes the entire system?
- what is the closed-loop system function? (hint: you can compute it in two ways!)



Difference equation:

$$y[n] = x[n] + gy[n-K]$$

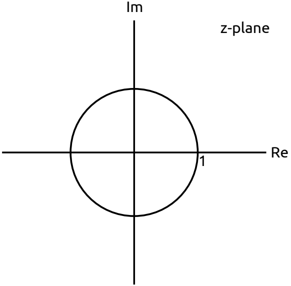
System function:

$$Q(z) = \frac{1}{1 - gz^{-K}} = \frac{z^K}{z^K - g}$$

Example: comb filters

$$Q(z) = \frac{1}{1 - gz^{-K}} = \frac{z^K}{z^K - g}$$

What are the poles and zeros?



Why is it called the comb filter? Let's look at its frequency response (take $z = e^{j\omega}$).

Example: Karplus-Strong

Another example of this is the Karplus-Strong algorithm!

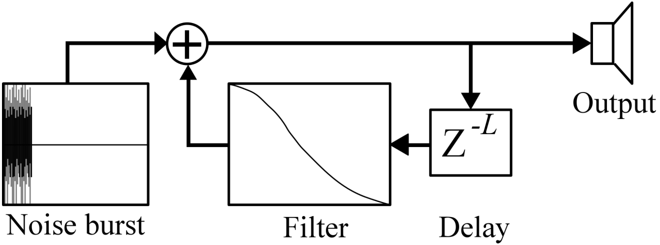
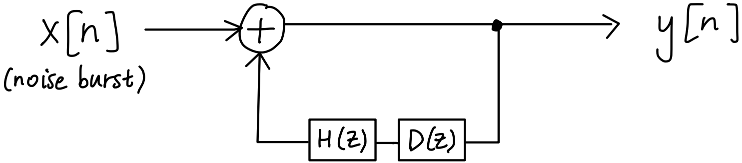


Image credit: https://commons.wikimedia.org/wiki/File:Karplus-strong-schematic.svg Author: PoroCYon CC BY-SA 3.0



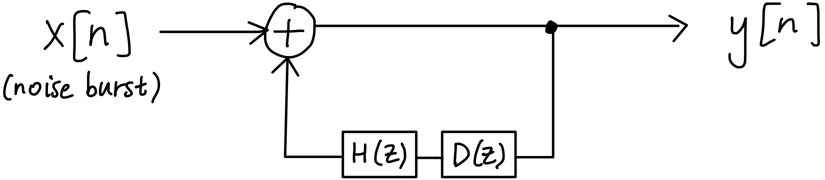
Suppose:

- $D(z)$ is a system that causes a delay of K steps
- $H(z)$ is a lowpass filter described by DE
  $y[n] = \frac{1}{2}(x[n] + x[n-1])$

Exercise:

- what is the difference equation that describes the entire system?
- what is the closed-loop system function?

Example: Karplus-Strong



Difference equation:

$$y[n] = x[n] + \frac{1}{2}(y[n-K] + y[n-K-1])$$

System function:

$$Q(z) = \frac{1}{1 - \frac{1}{2}z^{-K} - \frac{1}{2}z^{-K-1}} = \frac{z^{K+1}}{z^{K+1} - \frac{1}{2}z - \frac{1}{2}}$$

Today

Learning outcomes:

- use the z-transform to determine whether a system is causal or stable
- apply the z-transform to systems described by difference equations
- analyze simple feedback systems with the z-transform

Oppenheim practice problems: 10.13-10.16, 10.25-10.27, 10.31, 10.33-10.35, 11.1

For next time

Action items:

1. Assignment 7 due tonight at 23:59

Recommended reading: 10.5-10.7, 11.2