ELEC 221 Lecture 17
The sampling theorem

Thursday 7 November 2024

☆ Highlight changes from group portion

- Assignment 4 to be released soon (focus on chapters 7/8)
- No class Tuesday (reading break)
- No prof office hours this Friday

☆ make <u>new</u> PL assignment w/ MC questions

Recap

Continuous time:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t) e^{-j\omega k t} dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} d\omega$$

Discrete time:

$$x[n] = \sum_{k=\langle N \rangle} c_k e^{jk\cdot\frac{2\pi n}{N}} \qquad c_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jk\cdot\frac{2\pi}{N}}$$

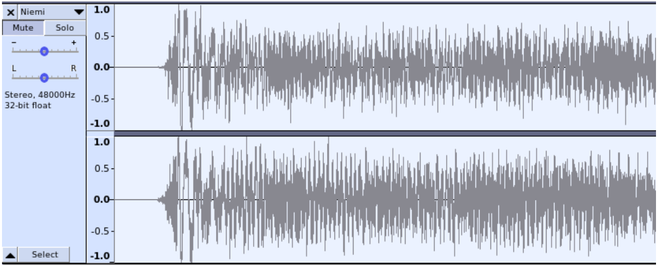
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Lecture 04 Demos

```
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import Audio
```

Demo 1: fun with square waves

```
tone = 65             # A frequency in Hz
duration = 2           # The length of the audio signal (in seconds)
sample_rate = 48000   # The number of samples per second to take
t_range = np.linspace(0, duration, sample_rate * duration) # Range of time
```



Motivation

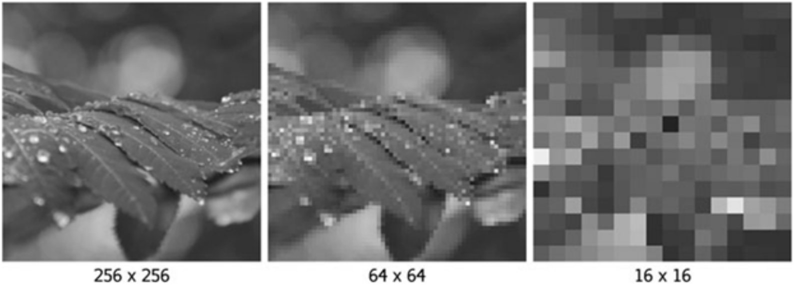


Image credit: https://what-when-how.com/introduction-to-video-and-image-processing/image-acquisition-introduction-to-video-and-image-processing-part-2/

`https://youtu.be/B8EMI3_0TO0?t=9`

History of frame rate in film:
`https://www.youtube.com/watch?v=mjYjFEp9Yx0`

Image credit: `https://www.mediacollege.com/video/frame-rate/img/frame-rates.jpg`

Today

Core question: under what conditions can we recover a continuous time signal using only information from its samples?

Learning outcomes:

- state the sampling theorem
- define the Nyquist sampling rate and determine if a sampling rate is sufficient to reconstruct a signal from its samples
- describe the phenomenon of aliasing

Impulse train sampling

The unit impulse as a sampler

Multiplying the signal by a shifted impulse picks out the value of the signal at that point:

$$x[n]\cdot\delta[n-k] = x[k]\cdot\delta[n-k]$$

This allows us to write any signal as a **superposition of weighted impulses.**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]$$

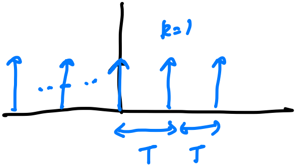
Impulse train sampling

In continuous time:

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

What if we have more than one?

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$



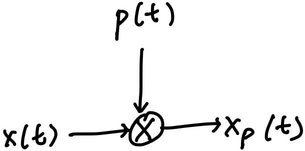
T= sampling period

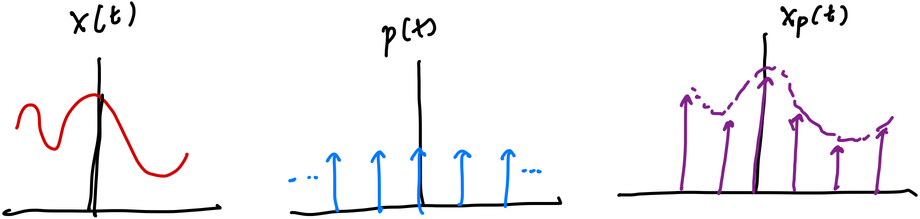
where $w_s = \frac{2\pi}{T}$ is the sampling frequency.

Impulse train sampling

What does the following signal look like?

$$x_p(t) = x(t)\,p(t)$$





The combined signal in the time domain is

$$x_p(t) = x(t)\,p(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)$$

What happens in the *frequency domain*?

$$x_p(t) = x(t)\,p(t) \longleftrightarrow X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\cdot P(j(\omega-\theta))\,d\theta$$

We have a periodic impulse train. Recall what Fourier transforms of periodic signals looked like:

$$X(j\omega) = 2\pi\delta(\omega-\omega_0) \longleftrightarrow x(t) = e^{j\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \longleftrightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Impulse train sampling

We need to find the Fourier series coefficients of the periodic impulse train.

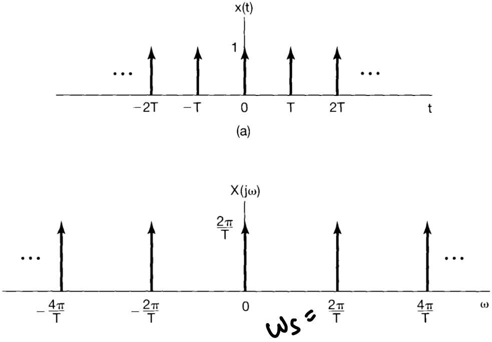
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

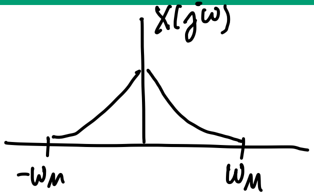
$$a_m = \frac{1}{T} \int_{-T/2}^{T/2} p(t)\, e^{-jm\omega t}\, dt$$

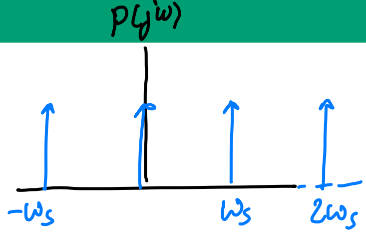
$$= \frac{1}{T}$$

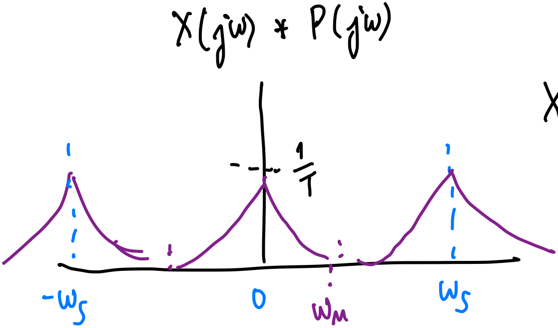
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$



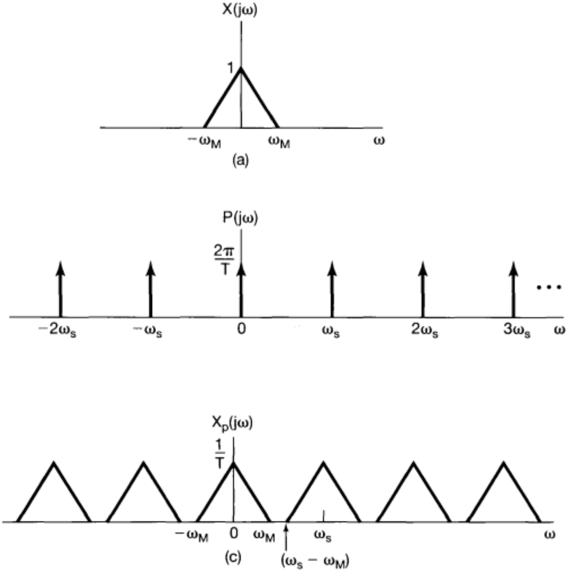




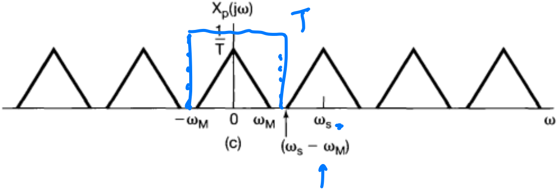


$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Impulse train sampling



Suppose we have sampled...



How do we recover our original signal from this spectrum?

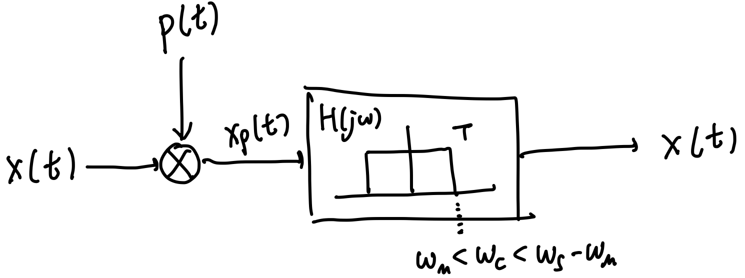
Image credit: Oppenheim 7.1

"Let $x(t)$ be a **band-limited** signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M \qquad \omega_s = \frac{2\pi}{T}$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$."

Let's show this graphically:



The sampling theorem

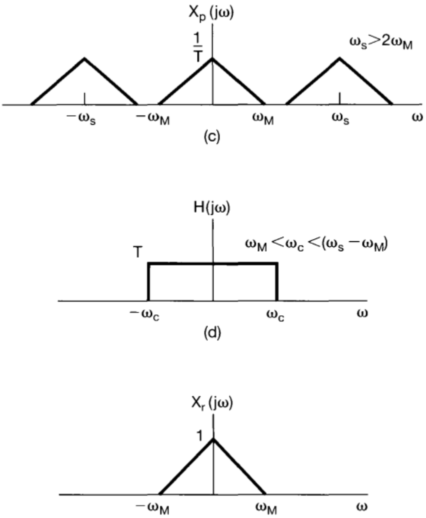


Image credit: Oppenheim 7.1

The sampling frequency is key:

- $\omega_s = 2\omega_M$ is referred to as the **Nyquist rate**
- $\omega_M = \omega_s/2$ is referred to as the **Nyquist frequency**

Exercise: suppose we perform impulse-train sampling with period $T = 10^{-4}$. If a signal $x(t)$ has $X(j\omega) = 0$ for $|\omega| > 15000\pi$, can we reconstruct it exactly from the samples?

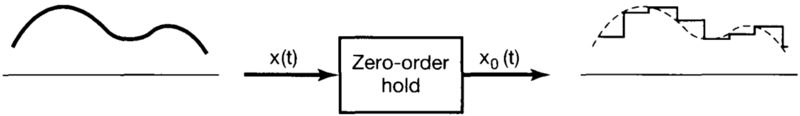
$$\rightarrow \quad \omega = \frac{2\pi}{10^{-4}} \approx 62800$$

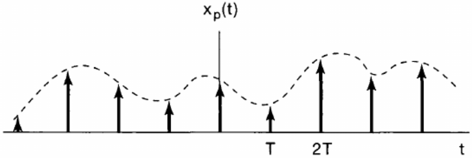
$$\omega_s > 30000\pi \checkmark$$

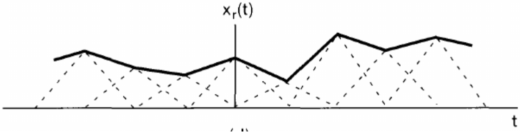
No!

Interpolation

In reality we cannot generate a perfect, ideal impulse train. But, we can still interpolate (you will explore this in A4)







What happens when you don't sample at a high enough rate?

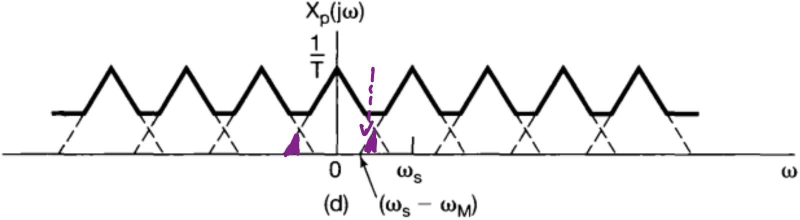


Image credit: Oppenheim 7.1

Aliasing

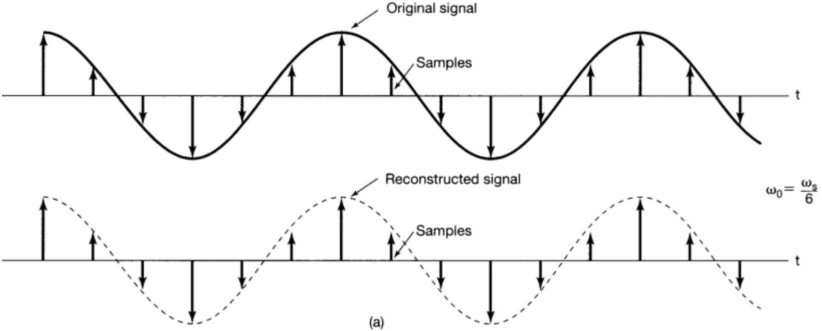


Image credit: Oppenheim 7.3

Aliasing

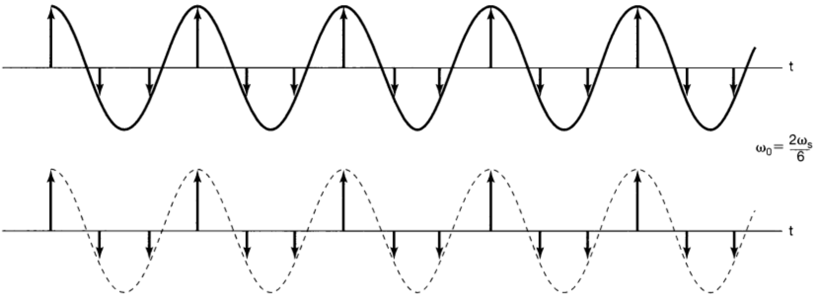


Image credit: Oppenheim 7.3

Aliasing

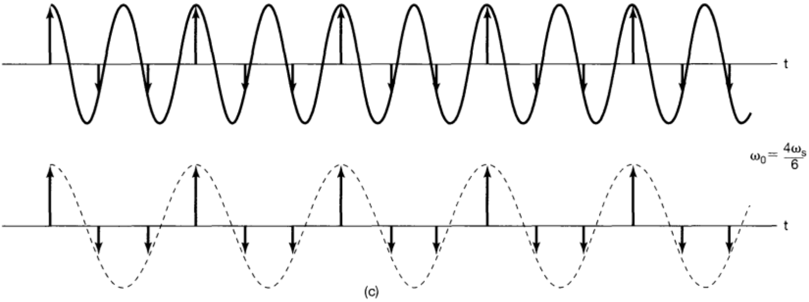


Image credit: Oppenheim 7.3

Aliasing

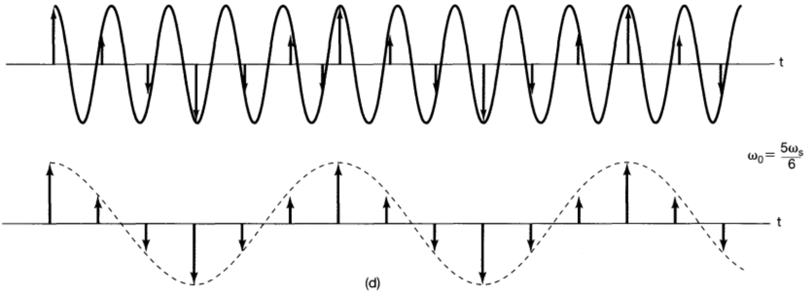


Image credit: Oppenheim 7.3

Simulation

`https://visualize-it.github.io/stroboscopic_effect/simulation.html`

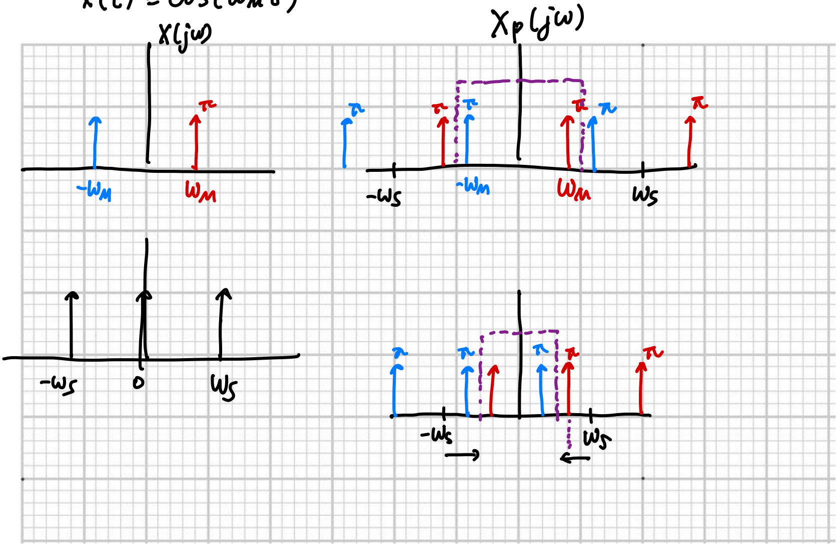
Two aspects to consider here:

- Why does the interpreted frequency *decrease* as the true frequency increases?
- Why does it it look like it goes *backwards*?

We can understand both by looking at the spectra.

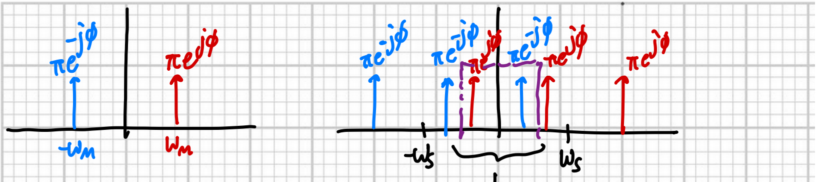
Frequency misattribution

$x(t) = \cos(\omega_M t)$



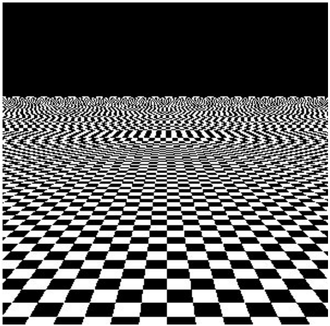
Backwards-ness

$x(t) = \cos(\omega_m t + \phi)$



$x'(t) = \cos(\omega t - \phi)$

Real-world examples



Fun on your own: read up about Moiré patterns, and various **anti-aliasing** techniques that are used in music/images/games!

Image credit: `https://textureingraphics.wordpress.com/what-is-texture-mapping/anti-aliasing-problem-and-mipmapping/`

For next time

Content:

- DT processing of CT signals
- Sampling in discrete time
- Decimation/interpolation

Action items:

1. Watch for A4 → first 2 questions are available

Recommended reading:

- From this class: Oppenheim 7.0-7.3
- Suggested problems: 7.1-7.6, 7.21, 7.25
- For next class: Oppenheim 7.4-7.6