ELEC 221 Lecture 16
DT systems based on difference equations

Thursday 31 October 2024

Announcements

- Assignment 3 due Saturday 23:59; solutions posted shortly after deadline
- Midterm 2 Monday 4 Nov during tutorial time
- No class on Tuesday 5 Nov

We derived the **discrete-time Fourier transform** (DTFT)

Inverse DTFT (synthesis equation)

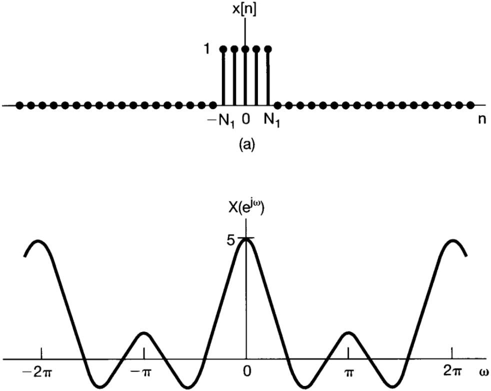
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

DTFT (analysis equation)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Last time

We computed the DTFT for a square pulse



The DTFT is **continuous**, and 2π-**periodic**!

Image credit: Oppenheim chapter 5.1

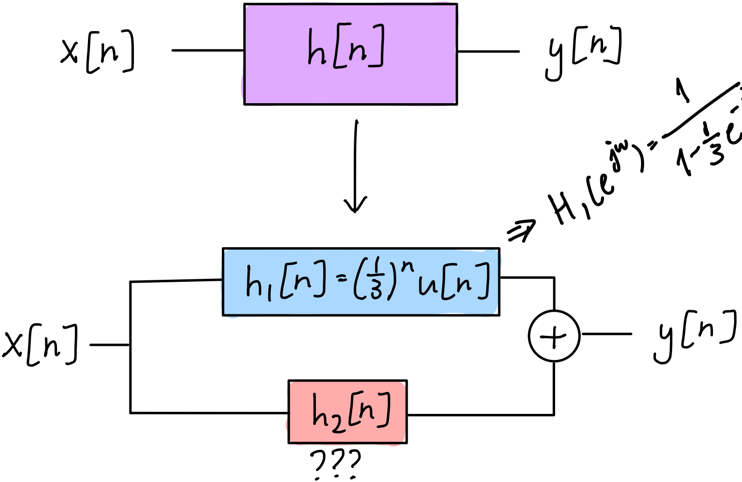
Today

Learning outcomes:

- Leverage key properties of the DTFT to simplify its computation
- Use the convolution property of the DTFT to analyze the behaviour of LTI systems
- Construct and analyze DT systems based on difference equations

Example

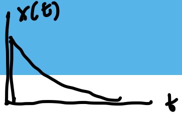
$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$

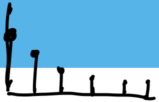


$$\Rightarrow H_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

Example

What is the DTFT of





$$x[n] = a^n u[n], \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

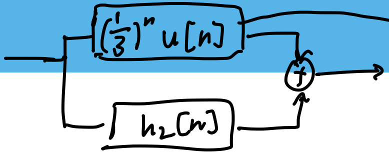
$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad |z| < 1$$

Linearity: If

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$
$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

then

$$a\,x_1[n] + b\,x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega})$$



$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-2j\omega}}$$

$$= \frac{-12+5e^{-j\omega}}{(3-e^{-j\omega})(4-e^{-j\omega})}$$

$$= \frac{A}{3-e^{-j\omega}} + \frac{B}{4-e^{-j\omega}} = \frac{4A - Ae^{-j\omega} + 3B - Be^{-j\omega}}{(3-e^{-j\omega})(4-e^{-j\omega})}$$

$$4A+3B = -12$$
$$-A-B = 5$$
$$\Rightarrow A=3 \quad B=-8$$

$$= \frac{3}{3-e^{-j\omega}} + \frac{-8}{4-e^{-j\omega}}$$

$$= \frac{1}{1-\frac{1}{3}e^{-j\omega}} + \quad ???$$

$$H_2(e^{j\omega}) = -2\,\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$
$$\Downarrow$$
$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

$$h[n] = \left(\frac{1}{3}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Time shift: If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

Frequency shift:

$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

Example: time and frequency shift

$\delta(t) \xleftrightarrow{\mathcal{F}} 1$

What is the DTFT of

$$x[n] = \delta[n] + 2\delta[n-1] + e^{3jn}\delta[n-2]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

$$\delta[n] \xleftrightarrow{\mathcal{F}} 1$$

$$2\delta[n-1] \xleftrightarrow{\mathcal{F}} 2 \cdot e^{-j\omega}$$

$$e^{3jn}\delta[n-2] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega-3)}) = e^{-2j(\omega-3)} = e^{6j} \cdot e^{-2j\omega}$$

$$\updownarrow$$

$$e^{-2j\omega}$$

Conjugation: If

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

then

$$x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

If $x[n]$ is real,

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

Consequences for odd/even functions:

$$\mathrm{Even}(x[n]) \xleftrightarrow{\mathcal{F}} \mathrm{Re}(X(e^{j\omega}))$$

$$\mathrm{Odd}(x[n]) \xleftrightarrow{\mathcal{F}} j \cdot \mathrm{Im}(X(e^{j\omega}))$$

Time expansion:

$$x_k[n] = \begin{cases} x[n/k] & n \text{ a multiple of } k \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$x_k[n] \longleftrightarrow X(e^{jk\omega})$$

Properties of the DT Fourier transform

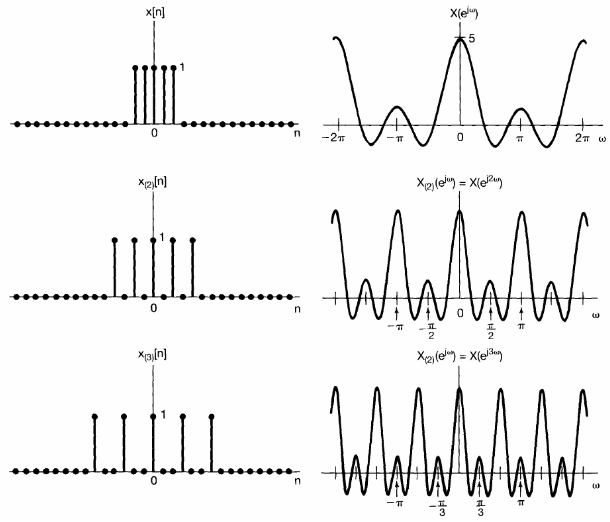


Image credit: Oppenheim Figure 5.14

Differentiation in frequency:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} x[n]\,(-jn)\, e^{-j\omega n}$$

$$n\,x[n] \xleftrightarrow{\mathcal{F}} j\,\frac{dX(e^{j\omega})}{d\omega}$$

$$\frac{dX(e^{j\omega})}{d\omega} = -j \sum_{n=-\infty}^{\infty} n\,x[n]\, e^{-j\omega n}$$

$$j\,\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} \big(n\,x[n]\big)\, e^{-j\omega n}$$

Differencing:

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) - e^{-j\omega} X(e^{j\omega})$$

$$= (1 - e^{-j\omega})\, X(e^{j\omega})$$

Example: differentiation in frequency

What is the DTFT of

$$x[n] = n\overbrace{\delta[n+3]}^{e^{3j\omega}} \quad X(e^{j\omega})$$

$$X'(e^{j\omega}) = j\frac{d\,e^{3j\omega}}{d\omega}$$

$$= -3e^{3j\omega}$$

$$x[n] = n\,a^n u[n]$$

Example: difference equations

What is the frequency response of a system defined by a linear, constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

linearity: $$\left[\sum_{k=0}^{N} a_k e^{-j\omega k}\right] Y(e^{j\omega}) = \left[\sum_{k=0}^{M} b_k e^{-j\omega k}\right] X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

Exercise: compute frequency response of a system characterized by

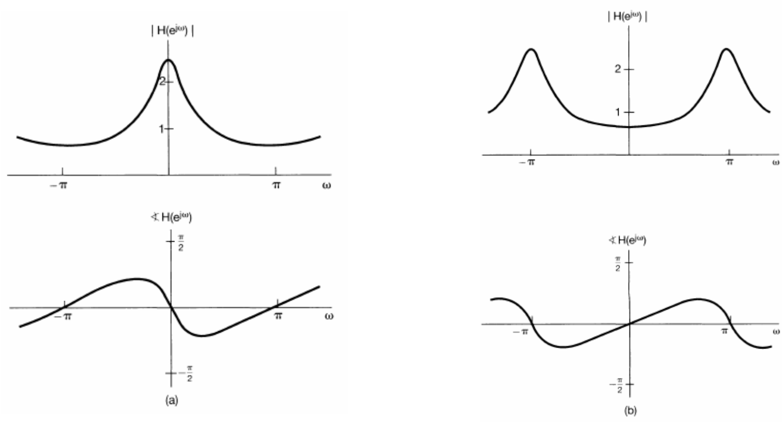
$$y[n] - ay[n-1] = x[n]$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$\Downarrow$$

$$h[n] = a^n u[n]$$

"First-order recursive DT filters"



$0 < a < 1 \ (a = 0.6)$

$-1 < a < 0 \ (a = -0.6)$

Image credit: Oppenheim Figure 3.34

Exercise: compute frequency response of a system characterized by

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

$$H(e^{j\omega}) = \frac{1}{3}\left(e^{j\omega} + 1 + e^{-j\omega}\right)$$

$$= \frac{1}{3}\left(1 + 2\cos(\omega)\right)$$

Exercise: compute frequency response of a system characterized by

$$y[n] = \frac{1}{5}(x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2])$$

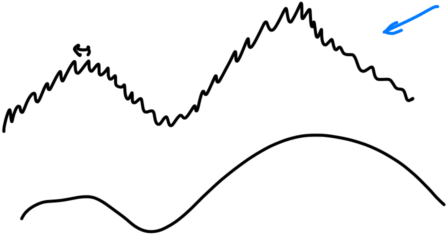
$$H(e^{j\omega}) = \frac{1}{5}\left(e^{2j\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}\right)$$

$$= \frac{1}{5}\left(1 + 2\cos\omega + 2\cos(2\omega)\right)$$

moving average

Exercise: compute frequency response of a system characterized by

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$



small N and M; lots of variation kept

large N and M; smoother curve because higher freqs. are attenuated more

see pictures

"Non-recursive DT filters"

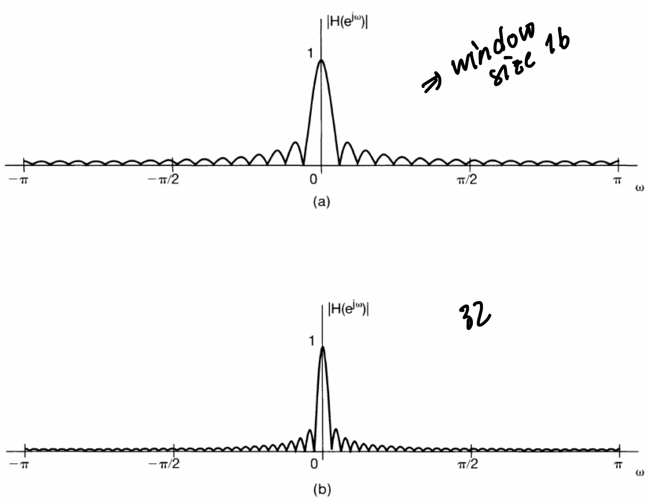


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Image credit: Oppenheim Figure 3.36

Content:

- The sampling theorem
- Basics of interpolation
- The Nyquist rate and aliasing

Action items:

1. Assignment 3
2. Midterm 2

Recommended reading:

- From this class: Oppenheim 5.2-5.9
- Suggested problems: 5.4b, 5.6, 5.8, 5.19, 5.22bcdfgh, 5.25, 5.29, 5.31, 5.33-5.36
- For next class (7 Nov): Oppenheim 7.1-7.3