ELEC 221 Lecture 07
CT and DT Fourier series

Thursday 26 September 2024

Announcements

- Assignment 2 due Saturday 5 Oct 23:59
- Tutorial Assignment 2 due Monday 23:59
- **No tutorial on Monday**
- Quiz 4 on Tuesday

Last time

Fourier synthesis equation:

Fourier analysis equation:

Last time

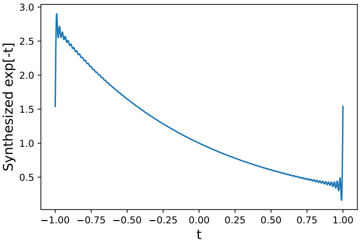
We evaluated the Fourier series coefficients of

$$x(t) = e^{-t}, \quad -1 \leq t < 1$$

We got an exact result

$$c_k = \frac{(-1)^k}{2(1 + jk\pi)} \left[e - e^{-1}\right]$$

But we saw some unusual behaviour when we tested it.



Today

Learning outcomes:

- Identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series
- Compute the fundamental period and frequency of a DT signal

Dirichlet conditions

Can we *always* express a signal as a Fourier series?

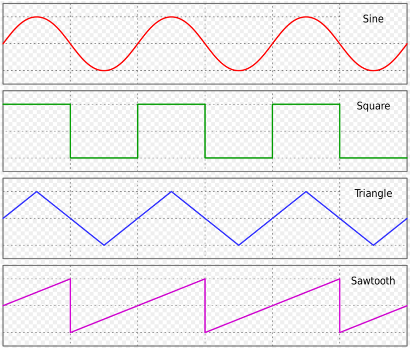


Image credit: *Sine, square, triangle, and sawtooth waveforms* (author: Omegatron)

`https://en.wikipedia.org/wiki/Triangle_wave#/media/File:Waveforms.svg` (CC BY-SA 3.0)

Dirichlet conditions

Given a **periodic function**, if over one period,

1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

then the Fourier series converges to

- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Examples that violate Dirichlet conditions

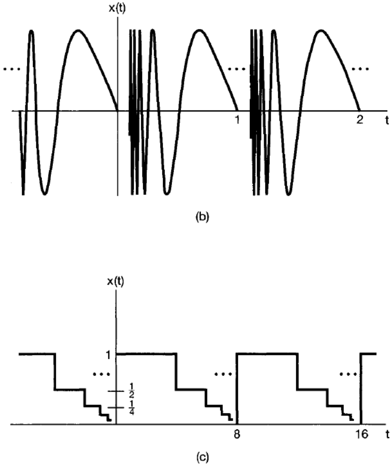
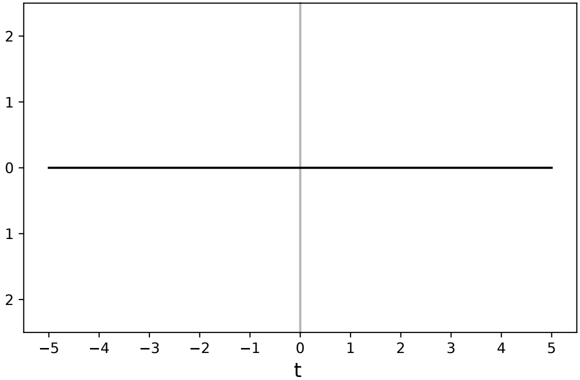


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Image: Oppenheim.

Example: the square wave

$$x(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi \end{cases}$$



Example: the square wave

Evaluate its Fourier coefficients. We can often take shortcuts based on the properties of a signal.

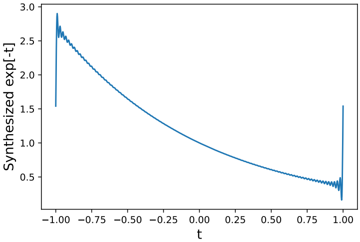
You can derive that for a 2π-periodic function, the coefficients have the following form:

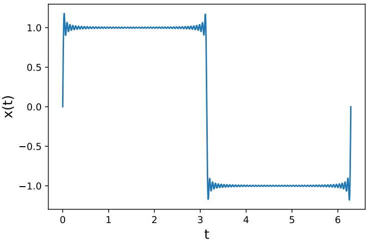
Example: the square wave

Since the function is symmetric, we can rewrite this:

Let's see how well this converges...

Gibb's phenomenon





The amount of ringing, or "overshoot", is about 9% of the jump of the discontinuity, no matter where we truncate.

Can derive from the *energy* of the error between the original and truncated signals (learn about energy / power of signals in A2).

Properties of Fourier series

In the square wave example, we leveraged some shortcuts to compute the Fourier coefficients.

They have other useful properties that help evaluate them.

Let's see what happens to the Fourier series when we apply:

- Superposition
- Time shift / scale / reversal
- Multiplication

Fourier coefficients combine linearly.

Suppose we have two signals $x(t), y(t)$ with period T,

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}, \\ y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t}, \end{aligned}$$

Then the signal $z(t) = Ax(t) + By(t)$ has the form

Properties of Fourier series

Time shift $x(t) \to x(t - t_0)$:

Time scale $x(t) \to x(\alpha t)$.

If original period was T, new period is T/α:

Multiplication leads to convolution of the coefficients:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}, \\ y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t}, \end{aligned}$$

Then the signal $z(t) = x(t)y(t)$ has the form

Properties of Fourier series

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Exercise

Go back to the square wave

$$x(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi \end{cases}$$

We obtained

$$x(t) \;\; = \;\; \sum_{k=1}^{\infty} b_k \sin(kt), \quad b_k = \begin{cases} 0, & k \text{ is even} \\ 4/k\pi, & k \text{ is odd} \end{cases}$$

What are the Fourier coefficients of the square wave

$$x(t) = \begin{cases} 1, & -\frac{\pi}{4} < t < \frac{\pi}{4}, \\ -1, & \frac{\pi}{4} < t < \frac{3\pi}{4} \end{cases}$$

Exercise

Step 1: express the b_k as the "original" coefficients c_k

Exercise

Step 2: apply the transformations

DT complex exponential signals

Recall our CT representation of complex exponential signals:

where α could be real or complex.

In DT, we write

where β can be real or complex.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is real.

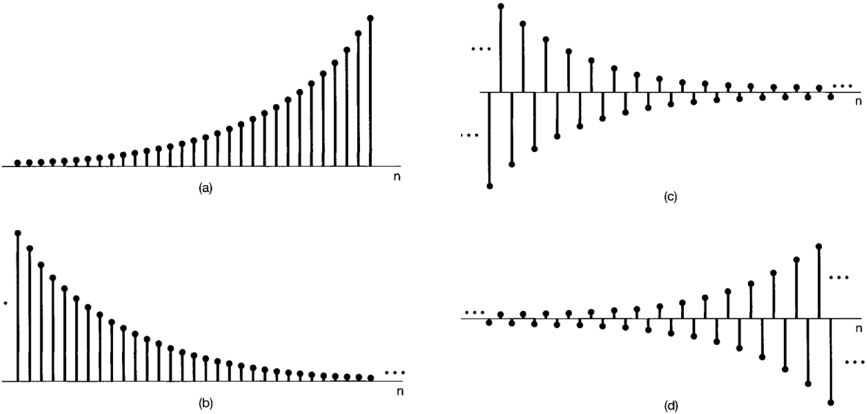


Image credit: Oppenheim chapter 1.3.

Case: $x[n] = C\beta^n$, where β is purely complex:

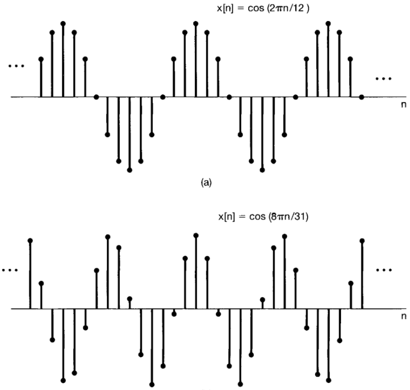


Image credit: Oppenheim chapter 1.3.

While these might look similar to their CT counterpoints, there is a **very important difference** relating to frequency.

In CT,

This is periodic with period $T = 2\pi/\omega$.

The bigger the frequency (ω) gets, the faster it oscillates!

Exercise: consider the DT signal

Does bigger ω always mean faster oscillation? If yes, why? If no, when does it stop getting faster?

For a DT signal with frequency ω, the signals with frequencies

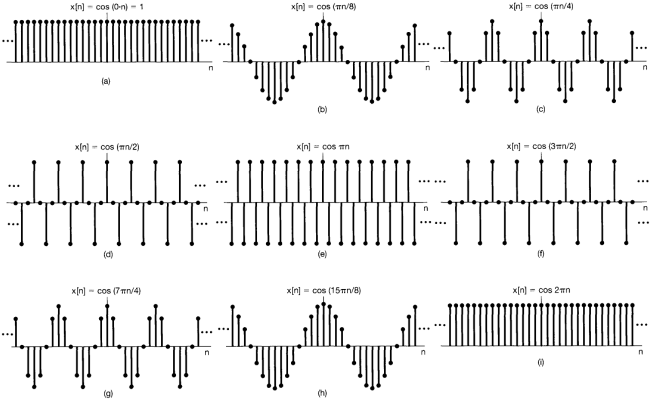


Image credit: Oppenheim chapter 1.3.

Exercise: What are the fundamental periods of

$$x(t) = \cos(3t), \quad \text{and} \quad x[n] = \cos(3n)$$

Frequency and period of DT complex exponential signals

Suppose the period is N:

This implies

must be rational for the signal to be periodic.

Frequency and period of DT complex exponential signals

Exercise: what is the fundamental period of

$$x[n] = \cos(5\pi n/6) + \sin(2\pi n/3)$$

Harmonics of DT complex exponential signals

What about harmonics?

In CT we had an infinite number of these. What about DT?

Note that, once we reach $k = N$, we get back to our original signal... so have only N distinct harmonics:

Consider a system with impulse response $h[n]$ and DT signal $x_m[n] = e^{jm\omega n}$. Use the convolution sum:

DT Fourier series

If we know how a system responds to complex exponential signals, we can learn its response to signals expressed in terms of them.

We need a Fourier series representation of DT signals:

How do we find the c_k?

Recap

Learning outcomes:

- Identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series
- Compute the fundamental period and frequency of a DT signal

Content:

- DT Fourier series coefficients
- Using the frequency response to design filter systems

Action items:

1. Tutorial Assignment due Monday at 23:59
2. Assignment 2 is due next Saturday at 23:59

Recommended reading:

- From today's class: Oppenheim 3.3-3.6
- Suggested problems: 3.2, 3.5, 3.8, 3.10-3.13, 3.14, 3.17, 3.22a,c, 3.23-3.26
- From today's class: Oppenheim 3.6-3.8