ELEC 221 Lecture 10
More properties of the CT Fourier transform

Tuesday 11 October 2022

Announcements

no quiz next week

- Quiz 5 today
- Midterm 1 on Thursday
  - Closed book / closed notes; no calculators
  - Formula sheet provided (see last Thursday's lecture)
  - Please arrive on time and bring your ID
- Assignment 4 (computational) available after midterm
  - Statement of contributions worth 1 point from now on.
  - "No exceptions" means *no exceptions*

Last time: the Fourier transform

We saw the Dirichlet conditions for the Fourier transform.

If the signal

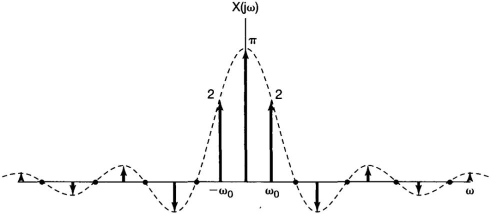
1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

We computed Fourier transforms of periodic signals.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k (\omega - k\omega_0)$$



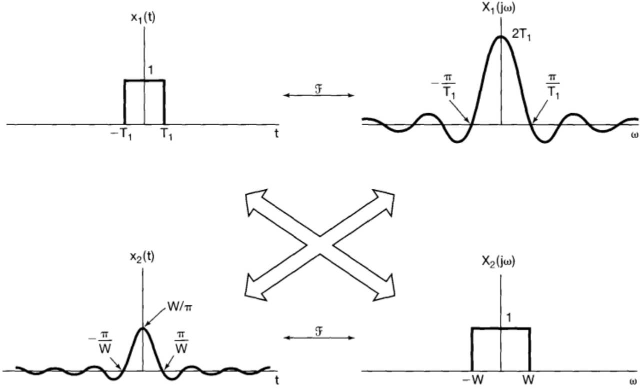
$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$

Image credit: Oppenheim chapter 4.2

Last time: properties of the Fourier transform

We saw some important properties of the Fourier transform:

- Linearity
- Behaviour under time shift/scale/reverse/conjugation
- Time/frequency duality



$H(e^{j\omega})$: discrete-time analog

We explored how the **frequency response** of a system relates to its **impulse response** via a Fourier transform:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

We introduced the convolution property:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Last time: properties of the Fourier transform

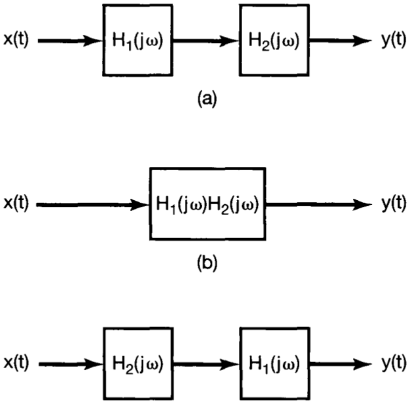
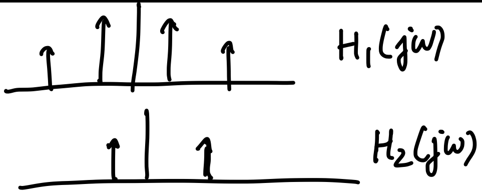


Image credit: Oppenheim chapter 4.4



Learning outcomes:

- Describe the *multiplication property* of the Fourier transform
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

We know that:

$$y(t) = h(t) * x(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Something similar holds when we interchange time and frequency:

$$r(t) = s(t)p(t)$$

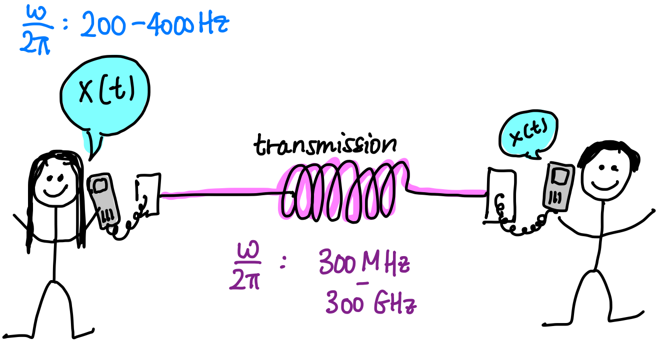
$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))\,d\theta$$

This is the **multiplication property**.

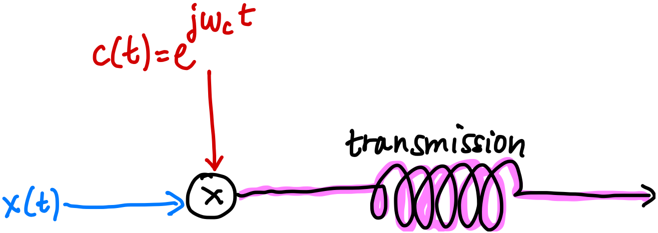
Example: the multiplication property

We are going to take a much closer look at this when we discuss communication systems and signal **modulation**.

For now, here is a taste:



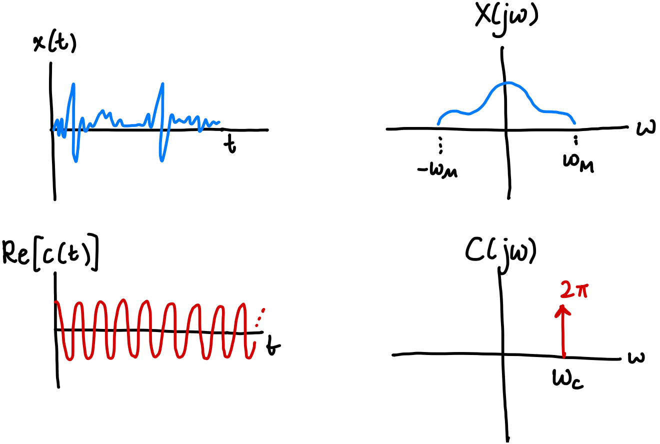
To shift our signal into the frequency range of transmission, we can multiply it by a **carrier signal** (amplitude modulation):



Is this doing what we think it is?

Example: the multiplication property

Consider the Fourier spectrum of both signals:



The multiplication property tells us

$$y(t) = x(t)c(t)$$

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)C(j(\omega-\theta))\,d\theta$$

We have

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$c(t) = e^{j\omega_c t} \xleftrightarrow{\mathcal{F}} C(j\omega) = 2\pi\delta(\omega-\omega_c)$$

Let's convolve them:

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)C(j(\omega-\theta))\,d\theta$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} X(j\theta)\cdot \cancel{2\pi}\,\delta\big(\underbrace{(\omega-\omega_c)}-\underbrace{\theta}\big)\,d\theta$$

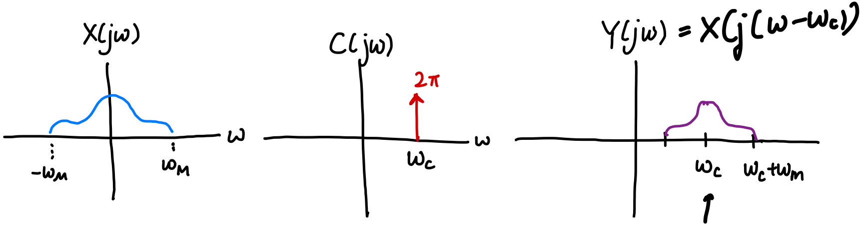
$$= X(j(\omega-\omega_c))$$

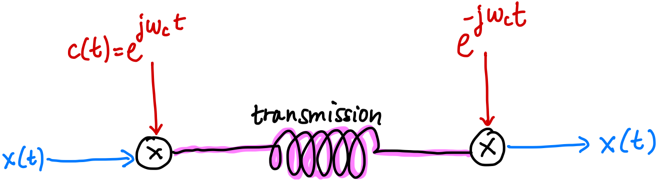
Example: the multiplication property

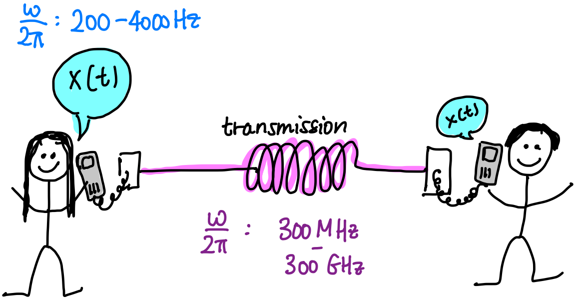
Let's convolve them:

$$\begin{aligned} Y(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)C(j(\omega-\theta))d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)2\pi\delta((\omega-\omega_c)-\theta)d\theta \\ &= X(j(\omega-\omega_c)) \end{aligned}$$

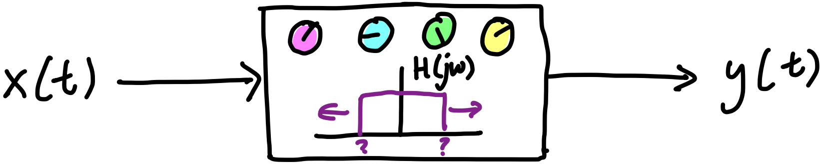
Multiplication with complex exponential carrier signal shifts the spectrum. We can move it into the desired frequency range.

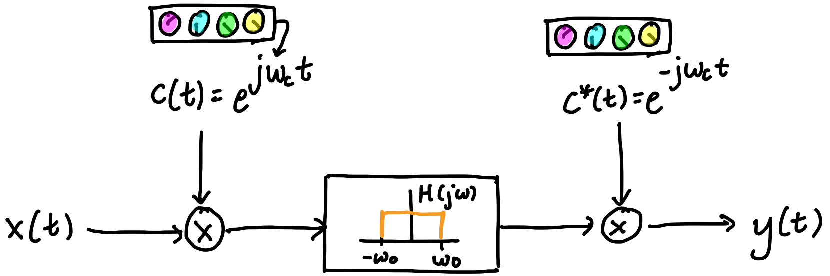



$c(t)=e^{j\omega_c t}$
$e^{-j\omega_c t}$
transmission
$x(t)$
$x(t)$

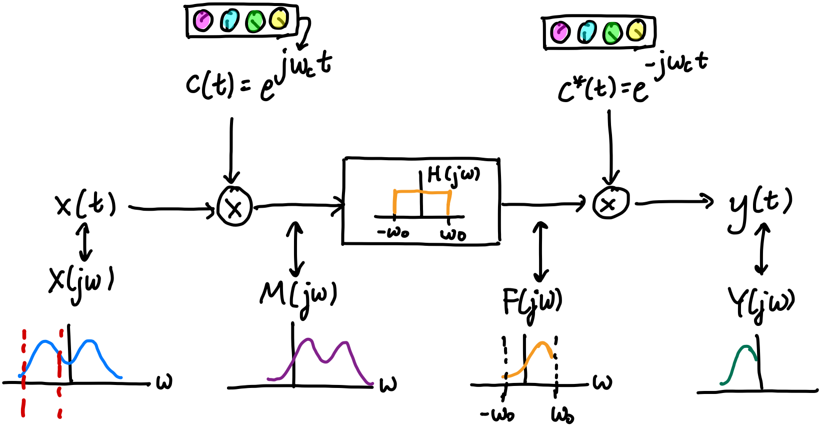

$\frac{\omega}{2\pi}$: 200-4000 Hz
x(t)
x(t)
transmission
$\frac{\omega}{2\pi}$: 300 MHz - 300 GHz

Example: frequency-selective filtering with variable centre frequency




$c(t) = e^{j\omega_c t}$
$c^*(t) = e^{-j\omega_c t}$
$x(t)$
$H(j\omega)$
$-\omega_0$
ω_0
$y(t)$

Example: frequency-selective filtering with variable centre frequency



Consider the inverse Fourier transform:

$$\underbrace{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

What happens when we differentiate $x(t)$?

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\underline{e^{j\omega t}}\, d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

This means:

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} \underline{j\omega}\, X(j\omega)$$

What should happen here?

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} ??$$

Good initial guess:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega)$$

More precisely:

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

We can often take advantage of differentiation and integration properties to simplify computation of Fourier transforms and system outputs.

Example: Fourier transform properties and differentiation

Suppose

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

What is the Fourier transform of

$$z(t) = \frac{d^2}{dt^2} x(t-1)$$

timeshift

d/dt

Two properties to take advantage of here:

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(j\omega)$$

$$x(t-t_0) \overset{F}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

$$z(t) = \frac{d^2}{dt^2}\underbrace{x(t-1)}$$

First, consider: $p(t) = x(t-1)$:

$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(j\omega)$$

$$\frac{d^2}{dt^2}p(t) \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^2 P(j\omega) = -\omega^2 P(j\omega)$$

But we know

$$p(t) = x(t-1) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega}X(j\omega) = \boxed{P(j\omega)}$$

So

$$\frac{d^2}{dt^2}x(t-1) \overset{\mathcal{F}}{\longleftrightarrow} -\omega^2 e^{-j\omega}X(j\omega)$$

Back in lecture 6, we saw a system (RC circuit) described by a differential equation:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

We found its frequency response in the following way:

- Choosing input signal $v_s(t) = e^{j\omega t}$
- Since system is LTI, assuming output of the form $v_c(t) = H(j\omega) \cdot e^{j\omega t}$
- Plugging this into the ODE and solving for $H(j\omega)$

Fourier transforms and systems described by differential equations

Nice properties of Fourier transforms give a much slicker method of computing frequency responses of such systems.

Consider a general system described by an ODE of arbitrary order:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

What is its frequency response $H(j\omega)$?

$$Y(j\omega) = \underline{H(j\omega)}\, X(j\omega)$$

First: take the Fourier transform of both sides.

$$\mathcal{F}\left(\sum_{k=0}^{N}\alpha_k \frac{d^k y(t)}{dt^k}\right) = \mathcal{F}\left(\sum_{k=0}^{M}\beta_k \frac{d^k x(t)}{dt^k}\right)$$

Which property can we leverage next? Linearity

$$\sum_{k=0}^{N}\alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M}\beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

$$\frac{d^k x(t)}{dt^k} \overset{\mathcal{F}}{\leftrightarrow} (j\omega)^k \cdot X(j\omega)$$

Now what? Differentiation

$$\sum_{k=0}^{N} \alpha_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} \beta_k (j\omega)^k X(j\omega)$$

We can simplify this even more:

$H(j\omega)$

$$Y(j\omega) \sum_{k=0}^{N} \alpha_k (j\omega)^k = X(j\omega) \sum_{k=0}^{M} \beta_k (j\omega)^k$$

$$X(j\omega)H(j\omega) = Y(j\omega)$$

$$Y(j\omega)\boxed{\sum_{k=0}^{N} \alpha_k (j\omega)^k} = X(j\omega)\boxed{\sum_{k=0}^{M} \beta_k (j\omega)^k}$$

Final property: Convolution

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

Example: frequency response of systems described by ODEs

What are the **impulse response** and **frequency response** of the system described by

α β

$$\frac{d^3y(t)}{dt^3} - 4\frac{dy(t)}{dt} = 3\frac{d^2x(t)}{dt^2} + x(t)$$

Start with frequency response:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

$$= \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega}$$

We can now leverage this to determine the impulse response:

$$H(j\omega) = \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega}$$

Use partial fractions:

$$\begin{aligned} H(j\omega) &= \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega} \\ &= \frac{3(j\omega)^2 + 1}{(j\omega)((j\omega)^2 - 4)} \\ &= \frac{3(j\omega)^2 + 1}{j\omega(j\omega + 2)(j\omega - 2)} \\ &= \frac{A}{j\omega} + \frac{B}{j\omega + 2} + \frac{C}{j\omega - 2} \end{aligned}$$

Details are left as an exercise:

$$H(j\omega) = \frac{-1/4}{j\omega} + \frac{13/8}{j\omega+2} + \frac{13/8}{j\omega-2}$$

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega+2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega-2}\right)$$

Example: frequency response of systems described by ODEs

can show that $F(u(t)) = \frac{1}{j\omega}$

can show $F(-e^{2t}u(-t)) = \frac{1}{j\omega - 2}$

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega+2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega-2}\right)$$

only

Last time, we learned a general expression for inverse Fourier transforms of this type:

$$\mathcal{F}(e^{-at}u(t)) = \frac{1}{a+j\omega} \qquad \mathrm{Re}(a) > 0$$

So we have:

$$h(t) = -\frac{1}{4}u(t) + \frac{13}{8}e^{-2t}u(t) + \frac{13}{8}\left(-e^{2t}u(-t)\right)$$

Recap

Today's learning outcomes were:

- Describe the *multiplication property* of the Fourier transform
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

What topics did you find unclear today?

For next time

Content (after the midterm):

- Discrete Fourier transform

Action items:

1. Midterm 1 on Thursday

Recommended reading:

- From today's class: Oppenheim 4.5-4.8
- For Tuesday's class: Oppenheim chapter 5.0-5.5