ELEC 221 Lecture 18
CT/DT conversion and sampling DT signals

Thursday 14 November 2024

Announcements

- Quiz 8 on Tuesday (L17 and L18)
- Assignment 4 due Saturday 23 Nov at 23:59 (do 4.2, 4.3, 4.4 after today; can try 4.5)
- Tutorial assignment 4 in Monday's tutorial (image processing)

We modeled **sampling** of CT signals as multiplication of a (band-limited) signal with a periodic impulse train:

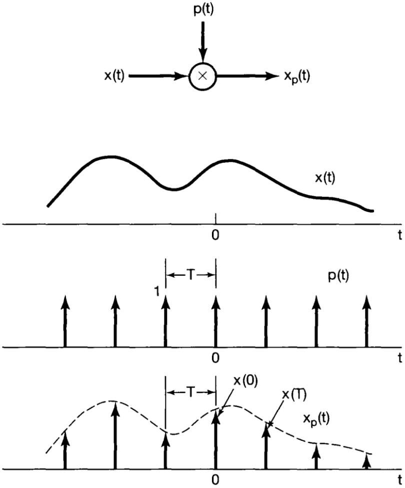


Image credit: Oppenheim 7.1

Last time

We went to the frequency domain to get a better understanding:

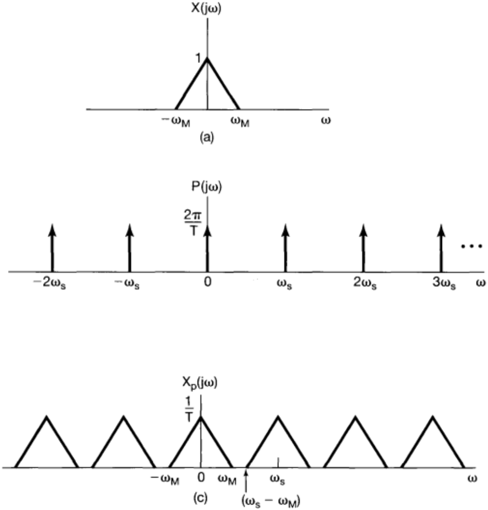


Image credit: Oppenheim 7.1

We recovered the original signal by applying a low pass filter

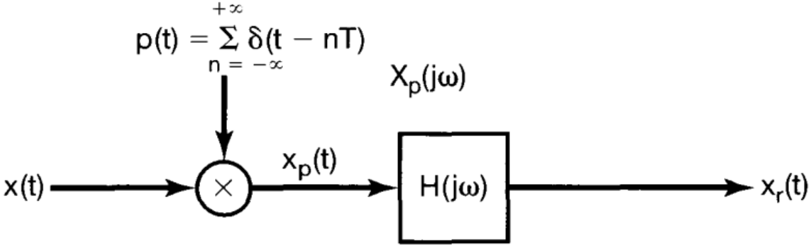


Image credit: Oppenheim 7.1

Last time

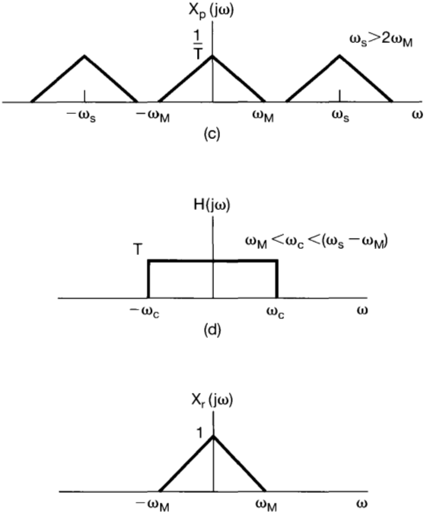


Image credit: Oppenheim 7.1

This only works if the sampling rate is higher than the **Nyquist rate**, i.e., $\omega_s > 2\omega_m$

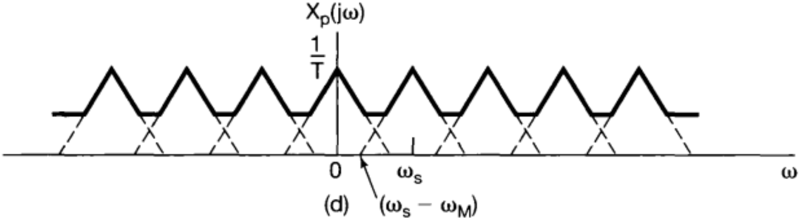
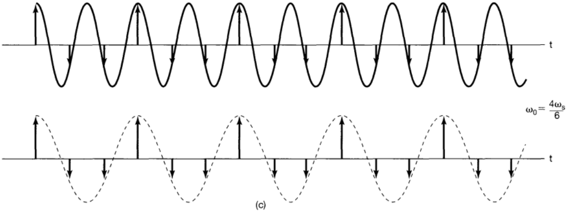


Image credit: Oppenheim 7.1

Last time

If the frequency isn't high enough, **aliasing** occurs.



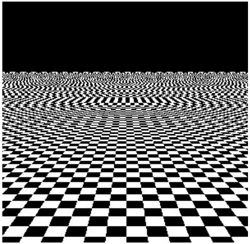


Image credit: Oppenheim 7.3, `https://textureingraphics.wordpress.com/what-is-texture-mapping/anti-aliasing-problem-and-mipmapping/`

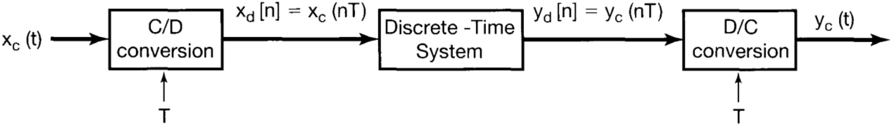
Today

Learning outcomes:

- describe the frequency domain effects of sampling from a discrete signal
- decimate and interpolate a DT signal
- determine how decimation and interpolation affect the spectrum of a DT signal

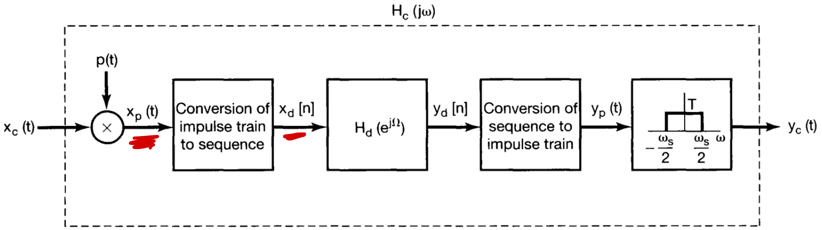
Converting between DT ↔ CT

Often convenient to process CT signals by first converting to DT, processing, then converting back.



What is the theory that makes this possible?

Image credit: Oppenheim 7.20



Let's explore what happens at the level of the spectra

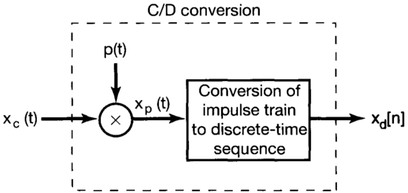
Note: we have *two frequencies*, one in CT, one in DT. Write:

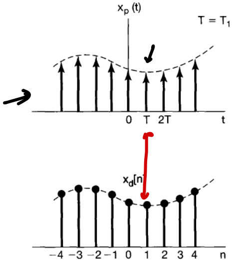
$$X_c(j\omega), \quad Y_c(j\omega)$$

$$X_d(e^{j\Omega}), \quad Y_d(e^{j\Omega})$$

Image credit: Oppenheim 7.24

First: how are $X_p(j\omega)$ and $X_d(e^{j\Omega})$ related?





$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT) \Rightarrow X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\,e^{-j\omega nT}$$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]\,e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)\,e^{-j\Omega n} = X_p\left(j\frac{\Omega}{T}\right)$$

Image credit: Oppenheim 7.21

Relate $X_d(e^{j\Omega})$ back to the original spectrum $X_c(j\omega)$

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega nT} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega}{T} - k\omega_s\right)\right)$$

$\omega_s = \frac{2\pi}{T}$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\,\frac{\Omega - 2\pi k}{T}\right)$$

Converting between DT $\leftrightarrow$ CT

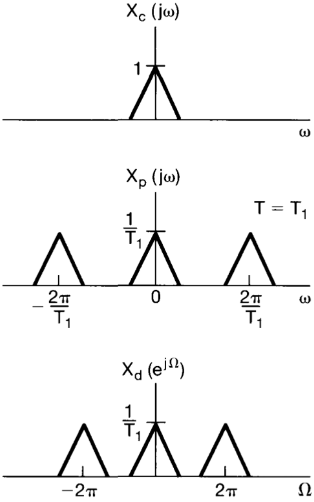
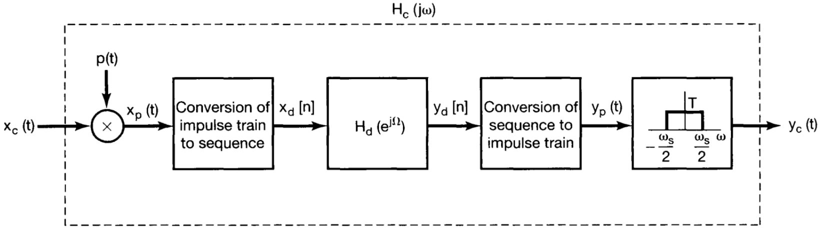


Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT



The converted signal $x_d[n]$ enters a DT system:

$$Y_d(e^{j\Omega}) = X_d(e^{j\Omega})\, H_d(e^{j\Omega})$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\,\frac{\Omega - 2\pi k}{T}\right) \cdot H_d(e^{j\Omega})$$

Image credit: Oppenheim 7.4

Converting between DT ↔ CT

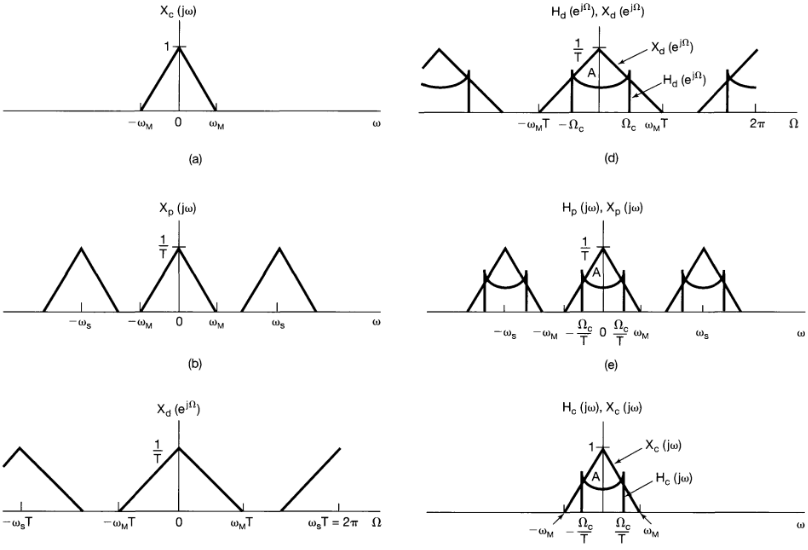
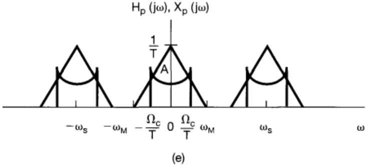
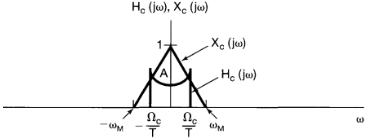


Image credit: Oppenheim 7.4

Sampling of DT signals



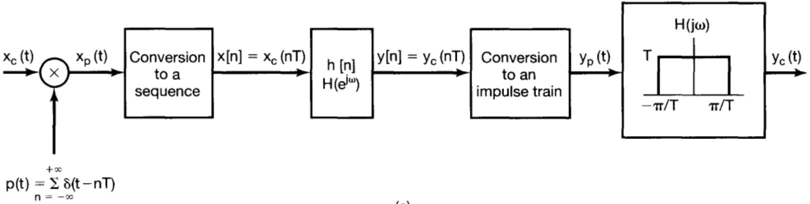


$$Y_c(j\omega) = H_c(j\omega)\, X_c(j\omega)$$

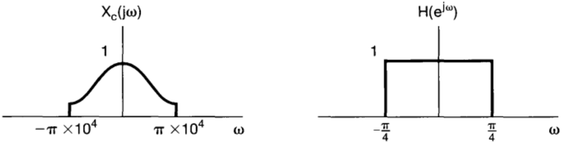
$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}) & |\omega| < \omega_s/2 \\ 0 & |\omega| > \omega_s/2 \end{cases}$$

Image credit: Oppenheim 7.4

Example



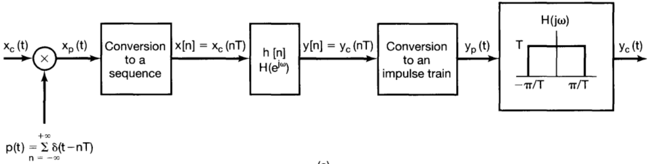
(a)



Sketch: $X_p(j\omega)$, $X_d(e^{j\omega})$, $Y_d(e^{j\omega})$, $Y_p(j\omega)$, $Y_c(j\omega)$ if $1/T = 20kHz$.

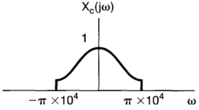
$\omega_s = \frac{2\pi}{T} = 4\pi \cdot 10^4$ Hz

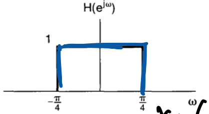
Image credit: Oppenheim Problem 7.29

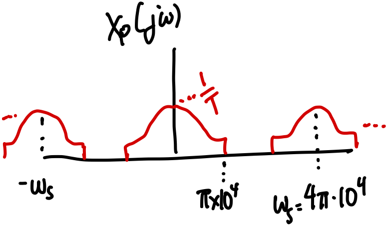


(a)

$\omega_s = 4\pi \cdot 10^4$







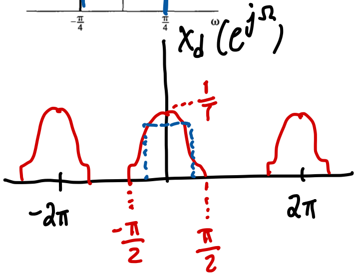
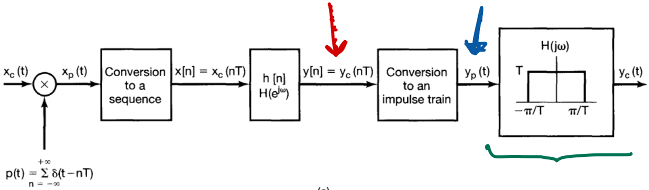
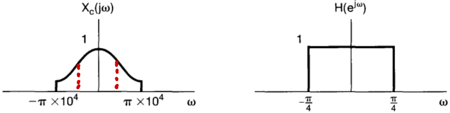


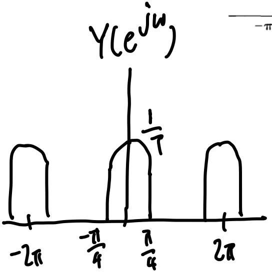
Image credit: Oppenheim Problem 7.29

Example



(a)





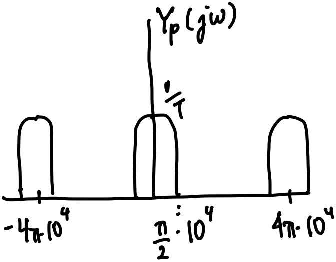
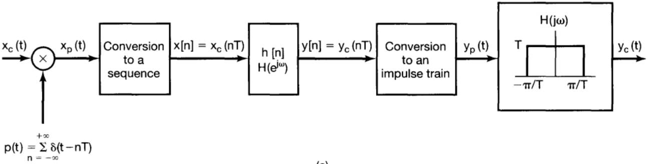
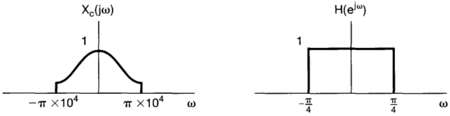


Image credit: Oppenheim Problem 7.29

Example



(a)



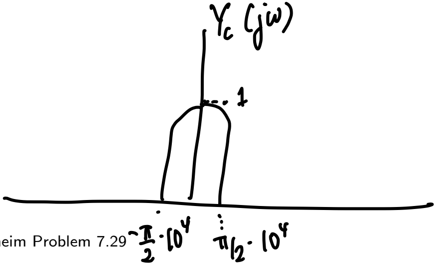


Image credit: Oppenheim Problem 7.29

Sampling of discrete-time signals

Sample with DT impulse train of period N:

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \qquad x_p[n] = \begin{cases} x[n], & n \text{ integer mult of } N \\ 0 & \text{otherwise} \end{cases}$$

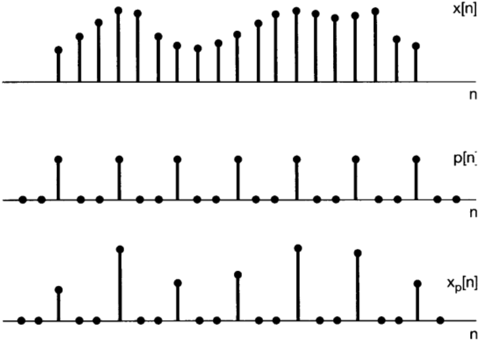


Image credit: Oppenheim 7.5

Take similar approach as we did in CT:

$$p[n] = \sum_{m=-\infty}^{\infty} \delta[n-mN] = \sum_{m=\langle N\rangle} c_m e^{jm\frac{2\pi n}{N}}$$

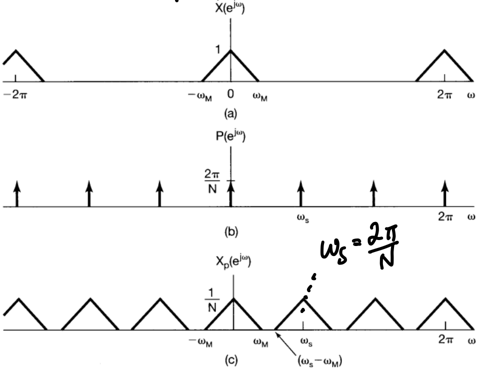
$$c_m = \frac{1}{N}$$

Exercise: derive $P(e^{j\omega})$

$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$x[n] \cdot p[n] \Rightarrow \frac{1}{2\pi} X(e^{j\omega}) * P(e^{j\omega})$$

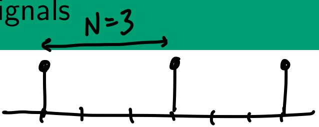
$$X_p(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) * P(e^{j\omega})$$
$$= \frac{1}{N} \sum_{k=0}^{N-1} X\left(e^{j(\omega - k\omega_s)}\right)$$



$$\omega_s = \frac{2\pi}{N}$$

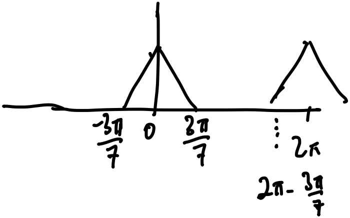
Image credit: Oppenheim 7.33

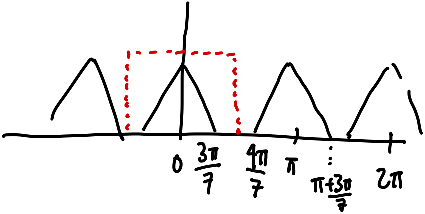
Sampling of discrete-time signals



Aliasing can happen in DT; some differences due to DT frequency range (π is the highest frequency).

Exercise: suppose $x[n]$ has $X(e^{j\omega})$ that is 0 for $3\pi/7 \leq |\omega| \leq \pi$. What is the largest sampling period N we can use without aliasing?





$$\omega_s = \frac{2\pi}{N} > \frac{6\pi}{7} \Rightarrow N_{max} = 2 \Rightarrow \omega_s = \pi$$

Sampling and then transmitting a DT signal in this way is inefficient

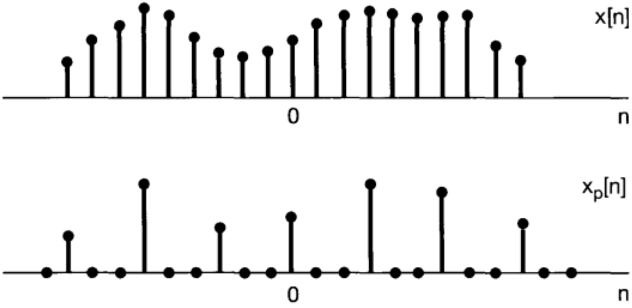
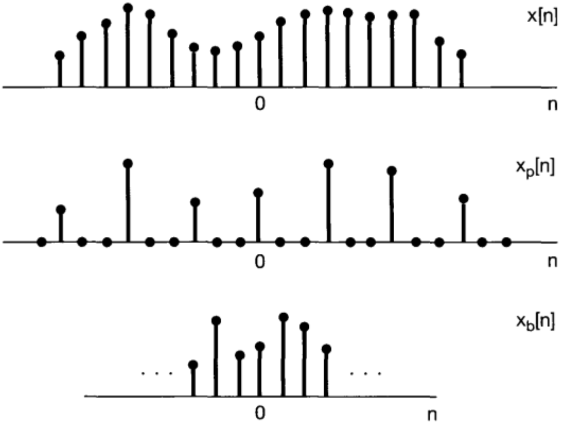


Image credit: Oppenheim 7.31

We can *compress* the representation:



$$x_b[n] = x[nN]$$

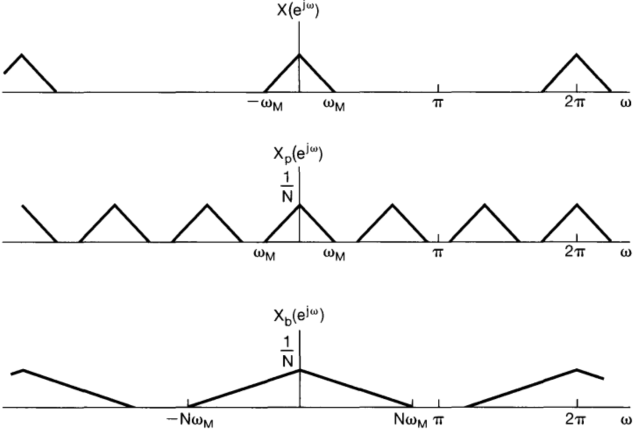
Image credit: Oppenheim 7.34

Decimation

Frequency domain effect:

$$X_b(e^{j\omega}) = X_p(e^{j\frac{\omega}{N}})$$

Decimation spreads out the spectrum



If original signal was CT, say that decimation has *downsampled* it.

Image credit: Oppenheim 7.35

Interpolation (upsampling)

Opposite of decimation: add $N-1$ points between.

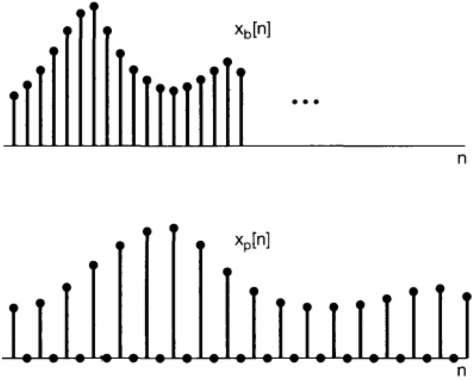


Image credit: Oppenheim 7.5

Interpolation (upsampling)

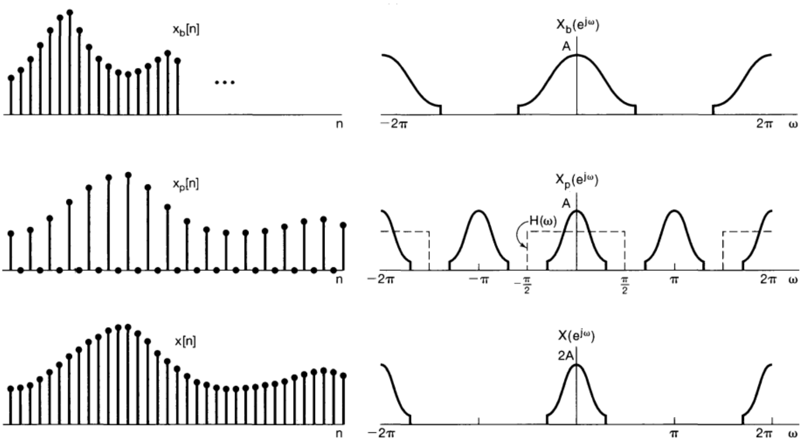


Image credit: Oppenheim 7.5

Example: down/upsampling

Oppenheim problem 7.19, ☆ try yourself!

7.19. Consider the system shown in Figure P7.19, with input $x[n]$ and the corresponding output $y[n]$. The zero-insertion system inserts two points with zero amplitude between each of the sequence values in $x[n]$. The decimation is defined by

$$y[n] = w[5n],$$

where $w[n]$ is the input sequence for the decimation system. If the input is of the form

$$x[n] = \frac{\sin \omega_1 n}{\pi n},$$

determine the output $y[n]$ for the following values of ω_1:

(a) $\omega_1 \le \frac{3\pi}{5}$

(b) $\omega_1 > \frac{3\pi}{5}$

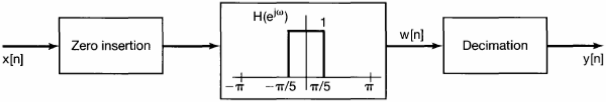


Figure P7.19

Example: down/upsampling

First case: $\omega_1 \leq \frac{3\pi}{5}$

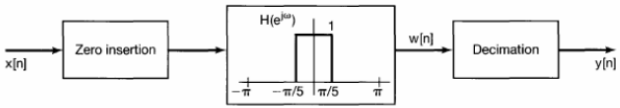


Figure P7.19

Second case: $\omega_1 > \frac{3\pi}{5}$

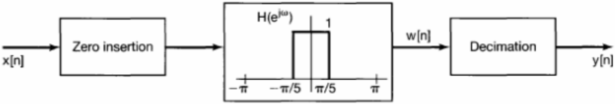


Figure P7.19

For next time

Content:

- moving into topic of modulation / communication systems

Action items:

1. Work on assignment 4
2. Prepare for quiz 8 on Tuesday (L17 and L18 material)
3. Tutorial Assignment 4 Monday

Recommended reading:

- From this class: Oppenheim 7.4-7.6
- Suggested problems: 7.17, 7.18, 7.20, 7.30, 7.32
- For next class: Oppenheim 8.0-8.4