Magnitude and phase

Extra notes for Tutorial 6 on Lecture 15

Calculating magnitude

Using as example the frequency response we saw on the tutorial today

$$H(j\omega) = \frac{a - j\omega}{a + j\omega}$$

There are two ways that $|H(j\omega)|$ can be calculated. The first one involves separating the real and imaginary parts by multiplying both top and bottom by the complex conjugate of the denominator

$$H(j\omega) = \frac{(a - j\omega)}{(a + j\omega)}\frac{(a - j\omega)}{(a - j\omega)} = \frac{a^2 - 2aj\omega + j^2\omega^2}{a^2 - j^2\omega^2} = \frac{a^2 - \omega^2}{a^2 + \omega^2} + j\frac{2a\omega}{a^2 + \omega^2}$$

Then the magnitude follows

$$|H(j\omega)| = \sqrt{\frac{(a^2 - \omega^2)^2}{(a^2 + \omega^2)^2} + j\frac{4a^2\omega^2}{(a^2 + \omega^2)^2}} = \sqrt{\frac{a^4 - 2a^2\omega^2 + \omega^4 + 4a^2\omega^2}{a^4 + 2a^2\omega^2 + \omega^4}} = 1$$

Alternatively, if we remember the relationship between the magnitude of the frequency response and the input and output

$$|H(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|} = \frac{\sqrt{\Re\{Y(j\omega)\}^2 + \Im\{Y(j\omega)\}^2}}{\sqrt{\Re\{X(j\omega)\}^2 + \Im\{X(j\omega)\}^2}}$$

It follows that

$$|H(j\omega)| = \frac{\sqrt{a^2 + \omega^2}}{\sqrt{a^2 + \omega^2}} = 1$$

Calculating phase

Phase can be calculated as

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\Im\{H(j\omega)\}}{\Re\{H(j\omega)\}}\right)$$

However, in this case it results in a rather complicated equation.

Alternatively, we can represent the top and bottom terms of $H(j\omega)$ using the complex polar notation. Let $\alpha = a - j\omega$, and $\beta = a + j\omega$, so that

$$H(j\omega) = \frac{\alpha}{\beta} = \frac{|\alpha|e^{j\theta_1}}{|\beta|e^{j\theta_2}} = \frac{|\alpha|}{|\beta|}e^{j(\theta_1 - \theta_2)} = |H(j\omega)|e^{j\angle H(j\omega)}$$

Thus, the phase becomes

$$\angle H(j\omega) = \theta_1 - \theta_2 = \tan^{-1}\left(\frac{\Im\{\alpha\}}{\Re\{\alpha\}}\right) - \tan^{-1}\left(\frac{\Im\{\beta\}}{\Re\{\beta\}}\right)$$

Which in this case is

$$\angle H(j\omega) = \tan^{-1}\left(\frac{-\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{a}\right) = -2\tan^{-1}\left(\frac{\omega}{a}\right)$$

Group delay

Now that we know the phase of $H(j\omega)$, we can calculate the group delay, which is defined as

$$\tau(\omega) = \frac{-d}{d\omega}\big(\angle H(j\omega)\big) = \alpha$$

So that

$$\tau(\omega) = \frac{-d}{d\omega}\left(-2\tan^{-1}\left(\frac{\omega}{a}\right)\right) = \frac{-2a^2}{a^2 + \omega^2}$$