ELEC 221 Lecture 08
DT Fourier series

Tuesday 01 October 2024

Announcements

- Quiz 4 today
- Assignment 2 due Saturday 23:59 (solutions posted immediately after deadline)
- Midterm details (+past midterm) available on PrairieLearn

Midterm 1 details

See details in "Practice" section on PrairieLearn:

- List of learning outcomes available; covers up to end of L9 (less emphasis on L8-L9);
- Formula sheet provided; no calculators (they aren't needed)
- Should understand *how* and *why* things are done in A1/A2 questions (midterm questions are less involved)
- Practice w/textbook questions and 2022 midterm (ignore content about Fourier transform)

Midterm 1 preparation

Office hours:

- Tuesday 12:30-1:30pm KAIS 3047 (TA)
- Wednesday 3:30-4:30pm KAIS 3065 (TA)
- Thursday 5:00-6:00pm KAIS 3047 (TA)
- Friday 2:30-3:30pm KAIS 3043 (prof; also by appointment)

Monday 7 Oct tutorial:

- problem solving with TAs
- can request focus on specific topics in advance

Assignment feedback from TAs

- You need to show your scratch work for full marks. Just the final graph/expression is not enough
- "Evaluating the convolution" means finding an expression, not just computing the value of the convolution for a few points.
- Explain why you are doing something/what you are doing. This way an arithmetic mistakes can be awarded partial marks
- Read the entire question. Often they ask for multiple things or reflection/commentary on your work.

Fourier synthesis equation:

Fourier analysis equation:

Last time

Dirichlet conditions: given a periodic function, if over one period it

1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

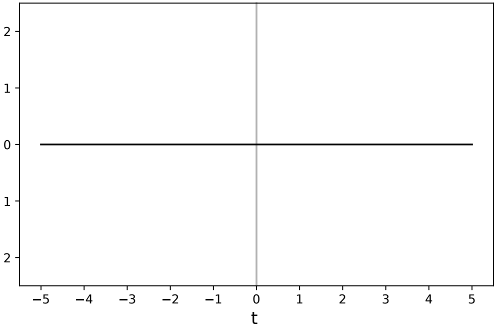
then the Fourier series converges to

- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Last time

We evaluated the Fourier series coefficients of a square wave:

$$x(t) = \begin{cases} 1, & 0 \leq t < \pi, \\ -1, & \pi \leq t < 2\pi \end{cases}$$



$$x(t) \;=\; \sum_{k=1}^{\infty} b_k \sin(kt), \quad b_k = \begin{cases} 0, & k \text{ is even} \\ 4/k\pi, & k \text{ is odd} \end{cases}$$

Last time

We determined how Fourier series coefficients transform.

Superposition of two signals with same ω:

Time shift

Time scale

Multiplication leads to convolution:

Today

Learning outcomes:

- Determine Fourier coefficients of a signal after transformation
- Compute the fundamental period and frequency of DT signals
- Evaluate Fourier series coefficients of DT signals

Exercise

Go back to the square wave

$$x(t) = \begin{cases} 1, & 0 \le t < \pi, \\ -1, & \pi \le t < 2\pi \end{cases}$$

We obtained

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(kt), \quad b_k = \begin{cases} 0, & k \text{ is even} \\ 4/k\pi, & k \text{ is odd} \end{cases}$$

What are the Fourier coefficients of the square wave

$$x(t) = \begin{cases} 1, & -\frac{\pi}{4} \le t < \frac{\pi}{4}, \\ -1, & \frac{\pi}{4} \le t < \frac{3\pi}{4} \end{cases}$$

Exercise

Step 1: express the b_k as the “original” coefficients c_k

Exercise

Step 2: apply the transformations

DT complex exponential signals

Recall our CT representation of complex exponential signals:

where α could be real or complex.

In DT, we write

where β can be real or complex.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is real.

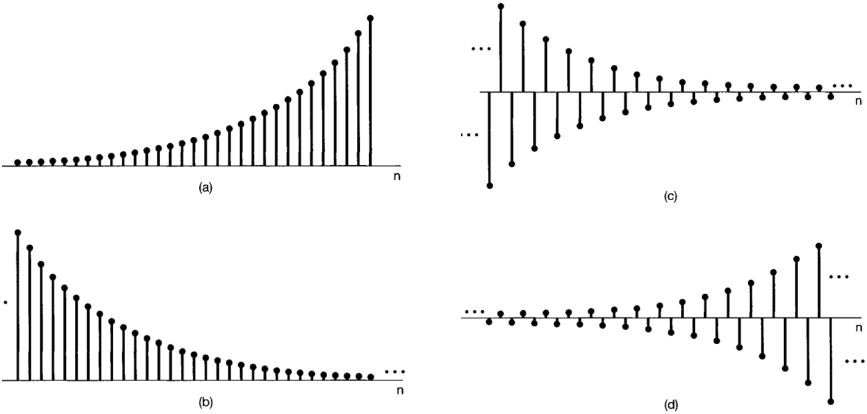


Image credit: Oppenheim chapter 1.3.

Case: $x[n] = C\beta^n$, where β is purely complex:

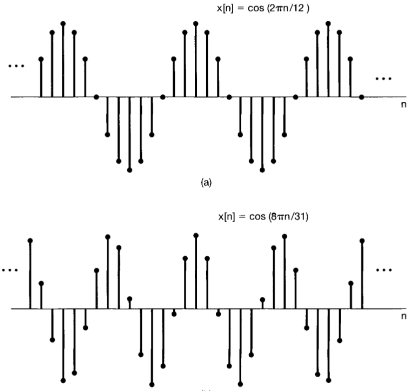


Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

While these might look similar to their CT counterpoints, there is a **very important difference** relating to frequency.

In CT,

This is periodic with period

The bigger the frequency (ω) gets, the faster it oscillates!

Exercise: consider the DT signal

Does bigger ω always mean faster oscillation? If yes, why? If no, when does it stop getting faster?

For a DT signal with frequency ω, the signals with frequencies

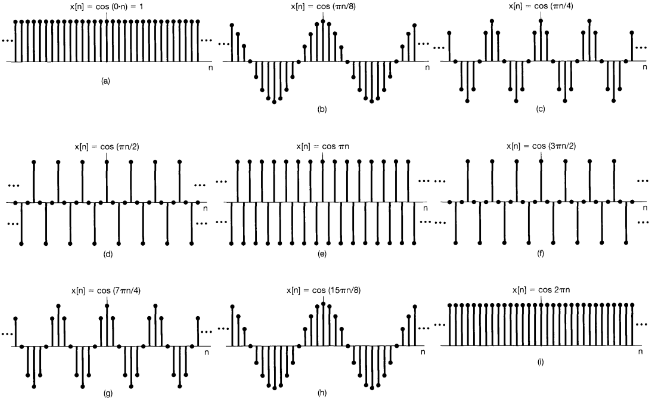


Image credit: Oppenheim chapter 1.3.

Exercise: What are the fundamental periods of

$$x(t) = \cos(3t), \quad \text{and} \quad x[n] = \cos(3n)$$

Frequency and period of DT complex exponential signals

Suppose the period is N:

This implies

must be rational for the signal to be periodic.

Exercise: what is the fundamental period of

$$x[n] = \cos(5\pi n/6) + \sin(2\pi n/3)$$

What about harmonics?

In CT we had an infinite number of these. What about DT?

Consider a system with impulse response $h[n]$ and DT signal $x_m[n] = e^{jm\omega n}$. Use the convolution sum:

DT Fourier series

If we know how a system responds to complex exponential signals, we can learn its response to signals expressed in terms of them.

We need a Fourier series representation of DT signals:

How do we find the c_k?

DT Fourier coefficients

Leverage the following identity about complex numbers:

We will multiply on both sides, and sum.

DT Fourier coefficients

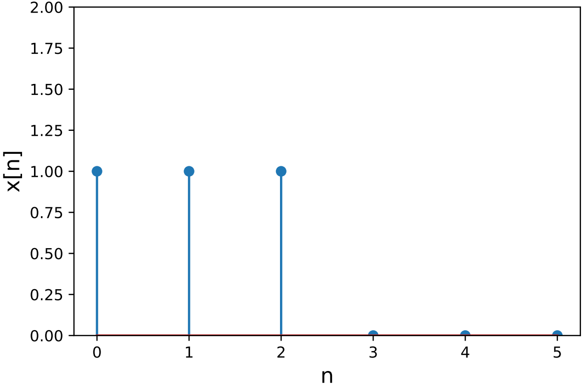
DT Fourier coefficients

DT Fourier synthesis equation

DT Fourier analysis equation

Exercise: the DT square wave

Compute the Fourier coefficients of this signal:



Exercise: the DT square wave

Properties of DT Fourier coefficients

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Where do we go from here?

We've showed a couple important things so far.

Signals can be expressed in terms of weighted, shifted impulses.

Where do we go from here?

If we know what an LTI system does to a unit impulse (the impulse response $h(t)$ or $h[n]$), we can learn what it does to any signal.

This was the convolution integral and sum:

Where do we go from here?

Complex exponential signals are eigenfunctions of LTI systems:

$H(j\omega)$ in CT, and $H(e^{j\omega})$ in DT, are the **frequency response** of the system (more generally, system functions).

Next class, we will see that the frequency response leads to a useful and intuitive description of a special type of system: filters.

Today's learning outcomes were:

- Determine Fourier coefficients of a signal after transformation
- Compute the fundamental period and frequency of DT signals
- Evaluate Fourier series coefficients of DT signals

For next time

Content:

- Using the frequency response to design filter systems

Action items:

1. Assignment 2 due Saturday 23:59

Recommended reading:

- From today's class: Oppenheim 3.5-3.7
- Suggested problems: 3.2, 3.10-3.12, 3.14, 3.17, 3.23-3.26, 3.28, 3.30, 3.31
- From today's class: Oppenheim 3.8-3.12