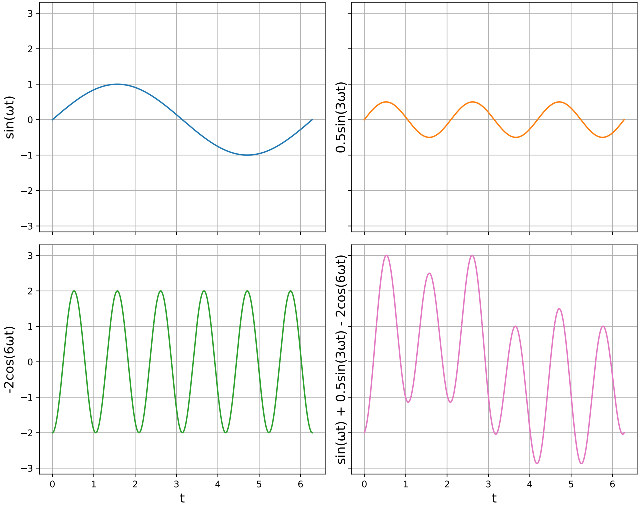
ELEC 221 Lecture 10
Introducing the Fourier transform

Thursday 10 October 2024

- Midterm postmortem
- No tutorial on Monday (Thanksgiving holiday)
- Quiz 5 on Tuesday (based on today's material)

Last time (recap)

We've seen the Fourier series representation of **periodic** signals:



Last time (recap)

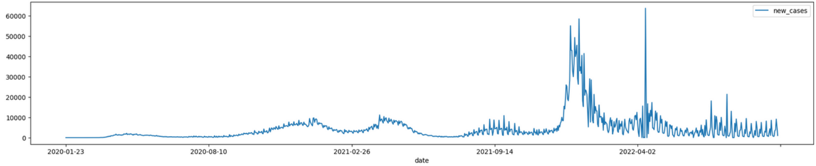
CT synthesis equation:

CT analysis equation:

A periodic signal is composed of complex exponential signals with *integer multiples of the fundamental frequency* only (harmonics).

Last time (recap)

In tutorial assignment 2, we saw signals that *weren't periodic*:



But, we were still doing *something* with Fourier analysis:

```
fourier_spectrum = np.fft.rfft(case_data)
```

Today

Learning outcomes:

- Distinguish between the CT Fourier series and Fourier transform
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

The **Fourier transform** generalizes the Fourier series to **aperiodic signals**. It involves all possible frequencies.

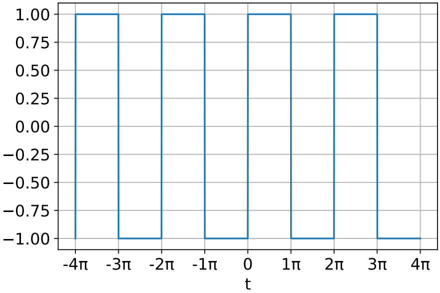
Fourier series:

Fourier transform:

How do we get here?

Towards the Fourier transform

Previously, we looked at a 2π-periodic square wave:



We derived its Fourier series representation

Let's generalize this. Consider the following square wave:

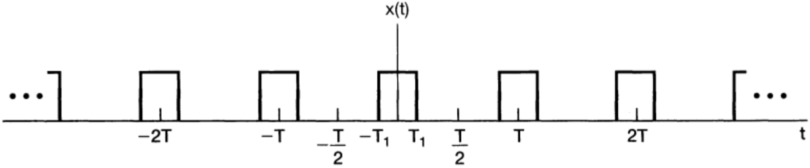


Image credit: Oppenheim chapter 4.1

Towards the Fourier transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Let's compute its Fourier coefficients.

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t}$$

Start with c_0:

Towards the Fourier transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

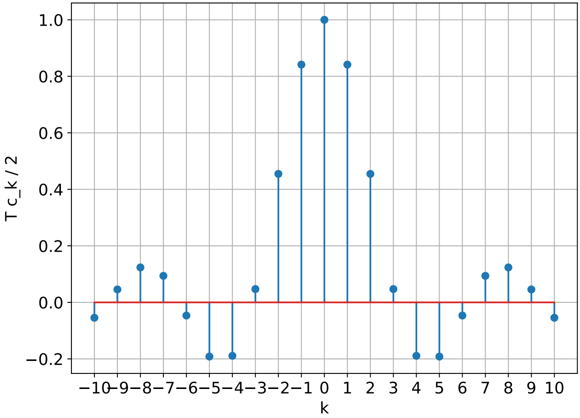
Now the c_k:

What does this function look like?

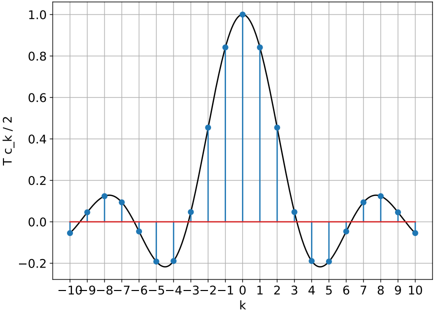
Rearrange and express as a function of k:

Let's plot this: set $T_1 = 1$, and $T = 2\pi$, so $\omega = \frac{2\pi}{T} = 1$.

Towards the Fourier transform



Towards the Fourier transform



These are **samples** of the function

at *integer values of* k.

Towards the Fourier transform

Let's consider this instead as a function of frequency, $\tilde{\omega} = k\omega$:

The Fourier coefficients are samples of this function at *integer multiples* of fundamental frequency, $\tilde{\omega} = k\omega$, where $\omega = 2\pi/T$.

Suppose T grows, but T_1 stays the same:

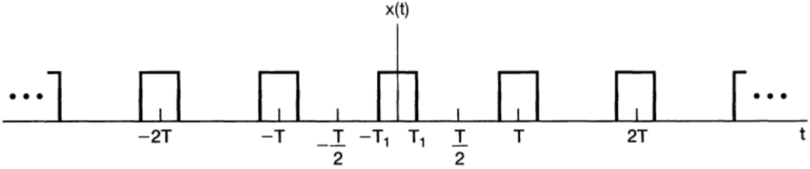


Image credit: Oppenheim chapter 4.1

Initially the spacing of samples is integer multiples of $\omega = 2\pi/T$.

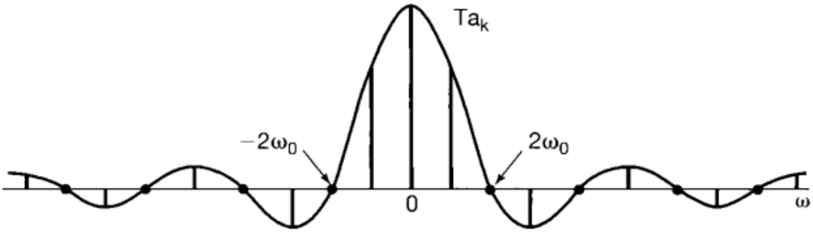


Image credit: Oppenheim chapter 4.1

As T grows, $\omega = 2\pi/T$ becomes smaller and smaller, so the integer multiples of it get closer and closer together.

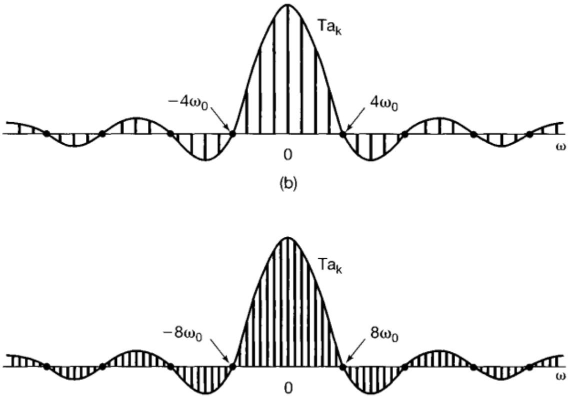


Image credit: Oppenheim chapter 4.1

Towards the Fourier transform

Eventually, ω becomes so small that instead of

we may as well just consider the sum over integer multiples as a continuous integral over all possible ω:

...but what does this have to do with non-periodic signals?

Given any aperiodic signal $x(t)$, we can always "pretend" it's periodic by constructing a **periodic extension**, $\tilde{x}(t)$ with period T.

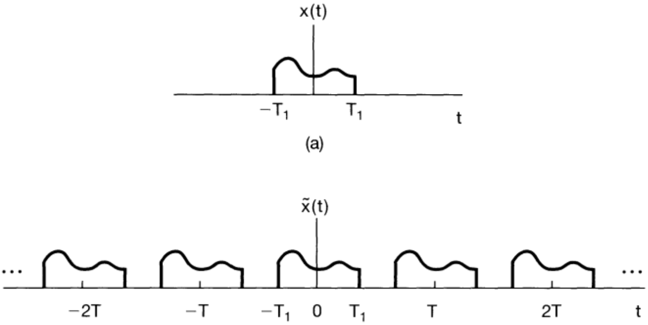


Image credit: Oppenheim chapter 4.1

Motivation: Fourier transform

We can express $\tilde{x}(t)$ as a Fourier series (where $\omega = 2\pi/T$):

Motivation: Fourier transform

What happens to the coefficients?

Let's define

so that

We can put this back in our Fourier series:

Motivation: Fourier transform

Consider what happens when $T \to \infty$...

1. $\tilde{x}(t)$ will look just like $x(t)$ for large enough T

2. ω will get smaller and smaller

Inverse Fourier transform (synthesis equation):

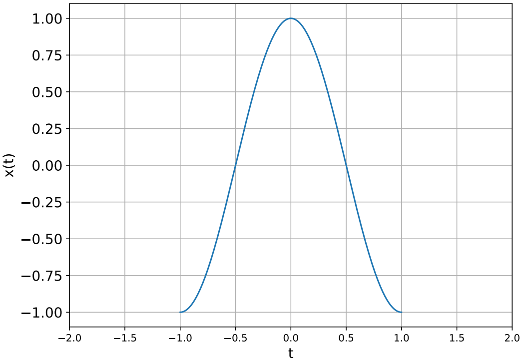
Fourier transform (analysis equation, or Fourier *spectrum*):

Note: Sometimes the $1/2\pi$ prefactor appears on the spectrum, or sometimes both versions have $1/\sqrt{2\pi}$.

Example

Compute the Fourier spectrum of:

$$x(t) = \begin{cases} \cos(\pi t), & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$



Example

$$x(t) = \begin{cases} \cos(\pi t), & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$

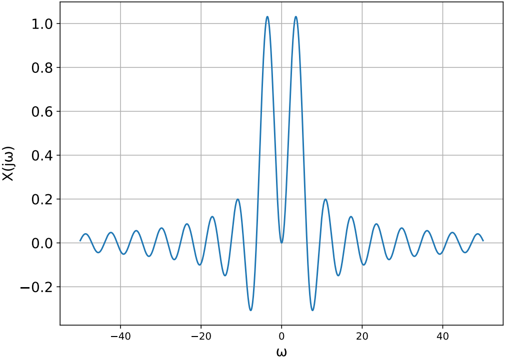
Start from the definition:

Example

$$X(j\omega) \qquad \frac{1}{2}\int_{-1}^{1} e^{j(\pi-\omega)t}dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\pi+\omega)t}dt$$

Example

$$X(j\omega) = \frac{\sin(\omega)}{\pi - \omega} - \frac{\sin(\omega)}{\pi + \omega}$$



You've actually already seen the Fourier transform before...

Write a signal as a combination of shifted, weighted impulses:

Put this in an LTI system with impulse response $h(t)$:

Fourier transform and impulse response

$$x(t) \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

When the signal in question is a complex exponential,

Fourier transform and impulse response

The system function $H(j\omega)$, or frequency response

is the **Fourier transform of the impulse response**!

We can use the inverse Fourier transform to obtain the impulse response from the frequency response:

Learning outcomes:

- Distinguish between the CT Fourier series and Fourier transform
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For next time

Content:

- Fourier transform for periodic signals
- Properties of the CT Fourier *transform*
- Time/frequency duality

Action items:

1. Quiz 5 Tuesday

Recommended reading:

- From today's class: Oppenheim 4.0-4.1
- Suggested problems: 4.1, 4.2a, 4.21abei, 4.22abde
- For next class: Oppenheim 4.2-4.4