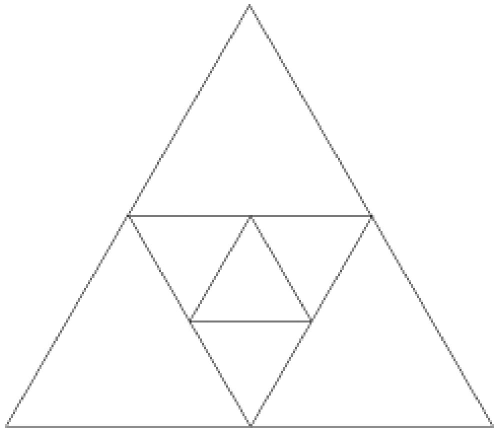
ELEC 221
Tutorial 4

Monday 17 October 2022

Fourier **Series** *vs* Fourier **Transform**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega t}dt$$

- Signal must be periodic
- Frequency content is harmonically related
- Keywords: series, periodic, coefficients



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Works in aperiodic signals
- Frequency content is an spectrum
- Keywords: transform, spectrum, frequency response

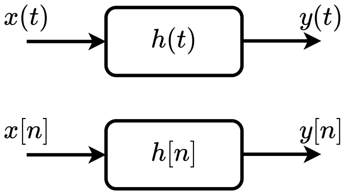
LTI System

- CT:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- DT:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



CT Fourier Transform

Inverse Fourier transform (synthesis equation):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier transform (analysis equation):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\begin{aligned} y(t) &= h(t) * x(t) \\ Y(j\omega) &= H(j\omega) X(j\omega) \end{aligned}$$

Time shifting:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Frequency shifting:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x(t) e^{j\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

Conjugation:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

If $x(t)$ is purely real,

$$X(-j\omega) \overset{\mathcal{F}}{\longleftrightarrow} X^*(j\omega)$$

Properties of Fourier Transform

Time scaling:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Time reversal follows from this:

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X\left(-j\omega\right)$$

Example

1. What is $y(t)$ for an LTI system with the following input and impulse response?

$$
\begin{aligned}
x(t) &= e^{-t}u(t) \\
h(t) &= e^{t}u(-t)
\end{aligned}
$$

2. What about $x_1(t) = x(t-2)$?

Example

Time Domain Solution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} e^{-\tau} \underbrace{u(\tau)}_{\tau \geqslant 0}\, e^{(t-\tau)} \underbrace{u(-(t-\tau))}_{\tau \geqslant t}\, d\tau = \int_{\max(0,t)}^{\infty} e^{t-2\tau}\, d\tau$$

Example

$$y(t) = -\frac{1}{2} e^{t-2\tau} \Big|_{\max(0,t)}^{\infty} = \frac{1}{2} e^{t-2\max(0,t)}$$

$$\left.\begin{aligned} t \geq 0 &\longrightarrow y(t) = \frac{1}{2} e^{-t} \\ t < 0 &\longrightarrow y(t) = \frac{1}{2} e^{t} \end{aligned}\right\} \longrightarrow y(t) = \frac{1}{2} e^{-|t|}$$

Example

Frequency Domain Solution

$$y(t) = x(t) * h(t) \longrightarrow F\{y(t)\} = F\{x(t) * h(t)\}$$

$$Y(j\omega) = X(j\omega) \times H(j\omega)$$

$$F\{e^{-at}u(t)\} = \int_{-\infty}^{+\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \frac{-1}{a+j\omega}\Big|_{0}^{\infty}$$

$$F\{e^{-at}u(t)\} \overset{Re(a)>0}{=\!=\!=} \frac{1}{a+j\omega}$$

Example

$$X(j\omega) = \mathcal{F}\left\{ e^{-t} u(t) \right\} = \frac{1}{1 + j\omega}$$

$$H(j\omega) = \mathcal{F}\left\{ e^{t} u(-t) \right\} = \frac{1}{1 - j\omega}$$

time reversal Property

$$Y(j\omega) = X(j\omega)\, H(j\omega) = \frac{1}{1+j\omega} \times \frac{1}{1-j\omega} = \frac{1}{1+\omega^2}$$

$$y(t) = \mathcal{F}^{-1}\left\{ Y(j\omega) \right\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(j\omega)\, e^{j\omega t}\, d\omega \longrightarrow ☹$$

Example

$$Y(j\omega) = \frac{A}{1+j\omega} + \frac{B}{1-j\omega} = \frac{(A+B) + (B-A)j\omega}{(1+j\omega)(1-j\omega)} = \frac{1}{(1+j\omega)(1-j\omega)}$$

$$\longrightarrow \begin{cases} A+B=1 \\ B-A=0 \end{cases} \longrightarrow A = B = \frac{1}{2}$$

$$Y(j\omega) = \frac{1}{2} \times \frac{1}{1+j\omega} + \frac{1}{2} \times \frac{1}{1-j\omega}$$

$$y(t) = F^{-1}\left\{Y(j\omega)\right\} = \frac{1}{2}F^{-1}\left\{\frac{1}{1+j\omega}\right\} + \frac{1}{2}F^{-1}\left\{\frac{1}{1-j\omega}\right\} = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

$$\left.\begin{array}{l} t \geq 0 \longrightarrow y(t) = \frac{1}{2}e^{-t} \\ t < 0 \longrightarrow y(t) = \frac{1}{2}e^{t} \end{array}\right\} \longrightarrow y(t) = \frac{1}{2}e^{-|t|}$$

Example

*

$$x_1(t) = x(t-2) \longrightarrow y_1(t) = ?$$

Approach 1

$$\text{LTI system} \longrightarrow y_1(t) = y(t-2) = \frac{1}{2} e^{-|t-2|}$$

Approach 2

$$x_1(t) = x(t-2) \longrightarrow X_1(j\omega) = X(j\omega)\, e^{-j\omega 2} = \frac{e^{-2j\omega}}{1+j\omega}$$

Example

$$Y_1(j\omega) = X_1(j\omega)\,H(j\omega) = \frac{e^{-2j\omega}}{1+j\omega} \times \frac{1}{1-j\omega} = \frac{e^{-2j\omega}}{(1+j\omega)(1-j\omega)}$$

$$Y_1(j\omega) = \frac{(A+B) + (B-A)j\omega}{(1+j\omega)(1-j\omega)} = \frac{e^{-2j\omega}}{(1+j\omega)(1-j\omega)}$$

$$\left.\begin{aligned} B - A &= 0 \\ A + B &= e^{-2j\omega} \end{aligned}\right\} \longrightarrow A = B = \frac{1}{2} e^{-2j\omega}$$

$$Y_1(j\omega) = \frac{1}{2} \times \frac{e^{-2j\omega}}{1+j\omega} + \frac{1}{2} \times \frac{e^{-2j\omega}}{1-j\omega}$$

Example

$$Re(a) > 0 \longrightarrow F\left\{ e^{-at} u(t) \right\} = \frac{1}{a + jw}$$

$$F\left\{ e^{-a(t-t_0)} u(t-t_0) \right\} = \frac{e^{-jwt_0}}{a + jw}$$

$$\longrightarrow y(t) = \frac{1}{2} e^{-(t-2)} u(t-2) + \frac{1}{2} e^{(t-2)} u(-(t-2)) = \frac{1}{2} e^{-|t-2|}$$

Example

Example

Example

Example

Example

Example

Example

Example