ELEC 221 Lecture 06
CT Fourier series coefficients and properties

Tuesday 24 September 2024

- Quiz 3 today
- Assignment 2 available at 12pm Tuesday, due 5 Oct 23:59
- Tutorial Assignment 2 due Monday 30 Sept 23:59
- No tutorial next Monday

$h(t) \rightarrow H(j\omega)$

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$

$y(t) = \sum_{k=-\infty}^{\infty} c_k \cdot H(jk\omega) e^{jk\omega t}$

$\frac{e^{j\omega t} + e^{-j\omega t}}{2}$ $\quad$ $\frac{e^{3j\omega t} - e^{-3j\omega t}}{2j}$

$x(t) = \underset{k=0}{3} + \underset{k=1,\ k=-1}{3\cos(\omega t)} + \underset{k=3,\ k=-3}{3\sin(3\omega t)}$

$c_0 = 3 \quad c_1 = \frac{3}{2} \quad c_{-1} = \frac{3}{2} \quad c_3 = \frac{3}{2j} = -\frac{3}{2}j \quad c_{-3} = \frac{3}{2}j$

$$x(t) = \underset{k=0}{3} + \underset{k=1,\ k=-1}{3\cos(wt)} + \underset{k=3,\ k=-3}{3\sin(3wt)}$$

$$c_0 = 3 \quad c_1 = \frac{3}{2} \quad c_{-1} = \frac{3}{2} \quad c_3 = \frac{3}{2j} = -\frac{3}{2}j \quad c_{-3} = \frac{3}{2}j$$

$$x(t) = \sum c_k e^{jkwt} \rightarrow y(t) = \sum c_k H(jkw) \cdot e^{jkwt}$$

$$c_0 \longrightarrow H(j \cdot 0w)\, c_0$$

$$c_1 \longrightarrow H(j \cdot 1w)\, c_1$$

$$\vdots$$

$$H(jw) = \frac{1}{1+w} \qquad H(jkw) = \frac{1}{1+kw}$$

$$c_0 \longrightarrow c_0$$

$$c_1 \longrightarrow H(jw) = \frac{1}{1+w} = \frac{1}{2} \ (w=1)$$

$$c_1' = \frac{3}{4}$$

$$c_3 : \ H(3jw) = \frac{1}{1+3w} = \frac{1}{4} \ (w=1) \Rightarrow c_3' = -\frac{3}{8}j$$

Complex exponential signals are eigenfunctions of LTI systems:

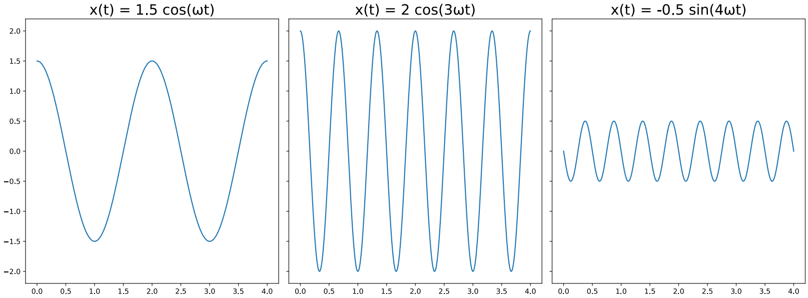
$$x(t) = e^{st} \qquad y(t) = H(s) \cdot e^{st} = H(s) \cdot x(t)$$

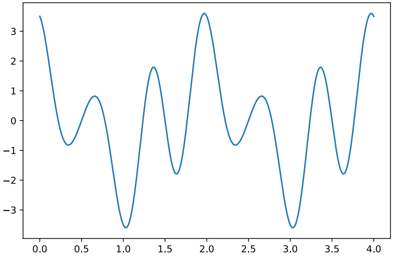
$H(s)$ is the **system function**.

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

consider only $s = j\omega$ (complex)

Last time





When we restricted to complex values only, i.e.,

$$x(t) = e^{j\omega t}$$

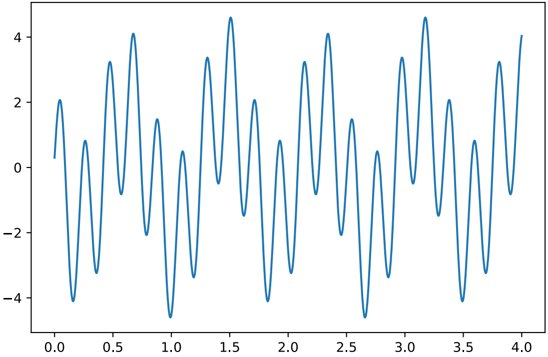
the system function is called the **frequency response**,

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)\, d\tau$$

If a superposition of such signals is input into an LTI system, each signal is rescaled by the frequency response:

$$x(t) = \sum_k c_k e^{j\omega k t} \longrightarrow y(t) = \sum_k c_k H(jk\omega) e^{j\omega k t}$$

What if we are given a signal like this:



Can we determine the c_k?

Today

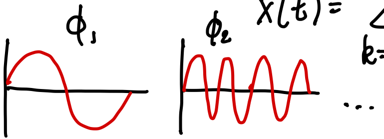
Learning outcomes:

- Compute the Fourier series coefficients of a CT periodic signal
- State the Dirichlet conditions and identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series

Evaluating Fourier coefficients

Given a signal $x(t)$, let's compute c_k.

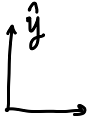
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$



The $e^{jk\omega t}$ are **basis functions** and have **orthogonality** relations.

Let $\phi_k(t) = e^{jk\omega t}$. Let's integrate over a period:

$$\underbrace{\frac{1}{T}\int_0^T \phi_k(t)\,\phi_m^*(t)\,dt}$$



$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where * indicates the *complex conjugate*.

analogous process

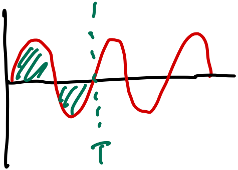
$$\vec{v}_1 \cdot \vec{v}_2 = (1 \;\; 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Evaluating Fourier coefficients

Exercise: evaluate the integral

$$\frac{1}{T}\int_0^T \phi_k(t)\phi_m^*(t)dt = \frac{1}{T}\int_0^T \underline{e^{jk\omega t}}\,\underline{e^{-jm\omega t}}\,dt$$

Case 1: $k = m$ (my) left class half



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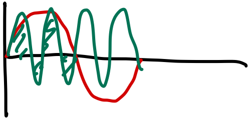
$$\frac{1}{T}\int_0^T 1\,dt = \frac{1}{T}\cdot T = 1$$

Case 2: $k \neq m$ (my) right class half

0

$$\frac{1}{T}\int_0^T e^{j(k-m)\omega t}dt = \frac{1}{T}\int_0^T \left[\cos\left((k-m)\omega t\right) + j\sin\left((k-m)\omega t\right)\right]dt$$



$$= 0$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Exercise: express $x(t)$ as a Fourier series and evaluate the integral

$$\frac{1}{T}\int_0^T \phi_m^*(t)x(t)dt \quad = c_m$$

$$= \frac{1}{T}\int_0^T \phi_m^*(t) \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k \cdot \underbrace{\frac{1}{T}\int_0^T \phi_m^*(t) e^{jk\omega t} \, dt}$$

$$\phi_m^*(t) = e^{-jm\omega t}$$

$$= c_m$$

Evaluating Fourier coefficients

$$c_m = \frac{1}{T}\int_T e^{-jm\omega t} x(t)\,dt$$

Fourier coefficients tell us *how much* each harmonic contributes.

Note that c_0 is a constant offset:

$$c_0 = \frac{1}{T}\int_T x(t)\,dt$$

(Similar techniques can be used to derive a_k and b_k for the sin and cos representation. Try it!)

★ IMPORTANT!!!

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$$

Fourier synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Fourier analysis equation:

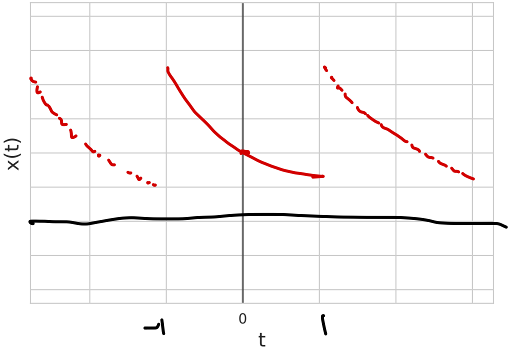
$$c_k = \frac{1}{T} \int_T e^{-jk\omega t} x(t)\, dt$$

Exercise

What is the Fourier series of

$$x(t) = e^{-t}, \quad -1 \leq t < 1$$

Start with a plot, and determine T and ω.



$$c_k = \frac{1}{T}\int_T e^{-jk\omega t} x(t)\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-jk\omega t} e^{-t}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t(jk\omega+1)}\,dt$$

$$= \frac{1}{2}\left(\frac{-1}{jk\omega+1}\right) e^{-t(jk\omega+1)}\Big|_{-1}^{1}$$

$$\vdots$$

$$= \frac{1}{2(1+jk\omega)}\left[e^{(1+jk\omega)} - e^{-(1+jk\omega)}\right]$$

$$\omega = \pi$$

$$\vdots$$

$$= \frac{1}{2(1+jk\omega)} \left[e^{(1+jk\omega)} - e^{-(1+jk\omega)} \right]$$

$$\omega = \pi$$

$$= \frac{1}{2(1+jk\pi)} \left[e^{1+jk\pi} - e^{-(1+jk\pi)} \right]$$

$$= \frac{1}{2(1+jk\pi)} \left[e \cdot \underbrace{e^{jk\pi}}_{(-1)^k} - e^{-1} \underbrace{e^{-jk\pi}}_{(-1)^k} \right]$$

$$= \frac{(-1)^k}{2(1+jk\pi)} \left[e - e^{-1} \right]$$

$$e^{jk\pi}$$

$$\downarrow$$

$k=0 \quad 1$

$k=1 \quad -1$

$k=2 \quad 1$

$$\vdots$$

$$(-1)^k$$

Dirichlet conditions

Can we *always* express a signal as a Fourier series?

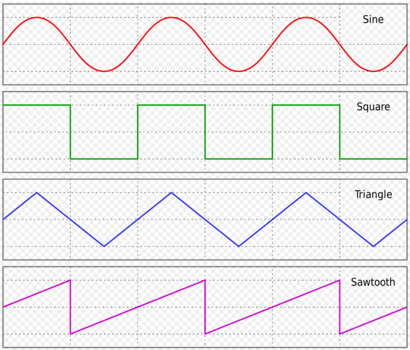


Image credit: *Sine, square, triangle, and sawtooth waveforms* (author: Omegatron)

`https://en.wikipedia.org/wiki/Triangle_wave#/media/File:Waveforms.svg` (CC BY-SA 3.0)

Recap

Learning outcomes:

- Compute the Fourier series coefficients of a CT periodic signal
- State the Dirichlet conditions and identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series

For next time

Content:

- Discrete-time Fourier series coefficients and properties

Action items:

1. Tutorial Assignment 2 due Monday
2. Assignment 2 due on 5 October (do Q3 and Q5)

Recommended reading:

- From today's class: Oppenheim 3.0-3.5
- Suggested problems: 3.4, 3.5, 3.8, 3.13, 3.17, 3.22a,c, 3.23-3.26
- For next class: Oppenheim 3.6-3.7