Midterm 2 Practice Problems

NOTE: Also look at the material already available from Tutorial 5 and Tutorial 6

This document includes some of the examples and exercises from Tutorial 7.

1. Consider the following impulse response of a discrete-time, LTI system

$$h[n] = \delta[n] + 2\delta[n-1]$$

   a. What is the $H(e^{j\omega})$ of the system?
   b. What is its magnitude and phase representation?
   c. What would be the output if the input is $x[n] = \cos(\pi n/2 + \pi/6) + \sin(\pi n + \pi/3)$?

a) The frequency response of the system can be found by calculating the DTFT of the impulse response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = \sum_{k=-\infty}^{\infty} \left(\delta[k] + 2\delta[k-1]\right)e^{-j\omega k}$$

By the sifting property, this sum is non-zero only for $k = 0,1$, thus the frequency response is

$$H(e^{j\omega}) = 1 + 2e^{-j\omega}$$

b) The magnitude can be calculated from $|a + jb| = \sqrt{a^2 + b^2}$, so we can convert the complex exponential to its sine and cosine form to separate the real and imaginary parts. The following identity will be useful $\cos^2\theta + \sin^2\theta = 1$.

$$|H(e^{j\omega})| = |1 + 2\cos(\omega) - 2j\sin(\omega)| = \sqrt{(1+2\cos(\omega))^2 + (2\sin(\omega))^2} = \sqrt{5 + 4cos(\omega)}$$

The phase can be obtained from $\angle(a + jb) = \tan^{-1}(b/a)$:

$$\angle H(e^{j\omega}) = \angle(1 + 2\cos(\omega) - 2j\sin(\omega)) = -\tan^{-1}(2\sin(\omega)/(1 + 2\cos(\omega))$$

c) The input $x[n]$ has two sinusoids at different frequencies. We need to determine how the system will affect these functions

For $\omega = \pi/2$

$$|H(\pi/2)| = \sqrt{5 + 4cos(\pi/2)} = \sqrt{5}$$

$$\angle H(\pi/2) = -\tan^{-1}(2\sin(\pi/2)/(1 + 2\cos(\pi/2)) = -\tan^{-1}(2) \approx -1.107$$

For $\omega = \pi$

$$|H(\pi)| = \sqrt{5+4cos(\pi)} = 1$$

$$\angle H(\pi) = -\tan^{-1}(2\sin(\pi)/(1+2\cos(\pi)) = 0$$

So, the output of the system will be

$$y[n] = \sqrt{5}\cos(\pi n/2+\pi/6-1.107)+\sin(\pi n+\pi/3)$$

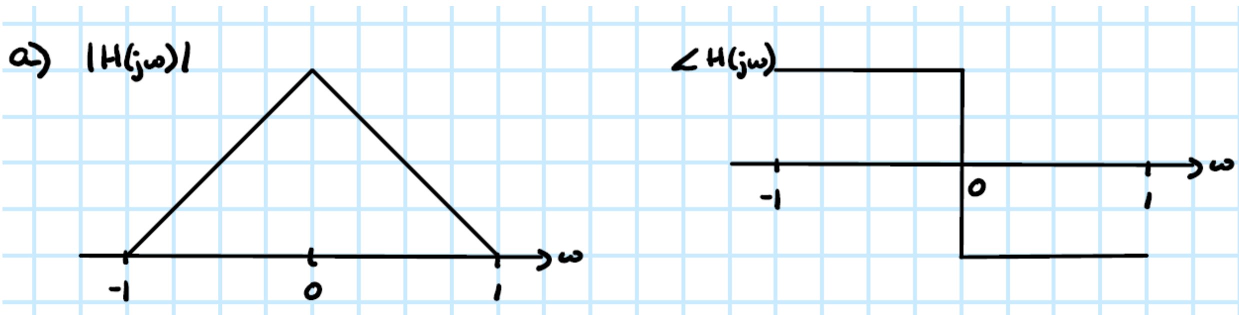
2. Consider a system with the frequency response $H(j\omega)$ described below

$$|H(j\omega)=(\omega+1)u(\omega+1)-2(\omega)u(\omega)+(\omega-1)u(\omega-1)$$

$$\angle H(j\omega)=\frac{\pi}{2}u(-\omega)-\frac{\pi}{2}u(\omega),\ \text{where } \angle H(0)=0$$

a) Sketch the magnitude and phase of $H(j\omega)$
b) For the input $x(t)=1+\cos(t/2)$, sketch the magnitude and phase of $X(j\omega)$
c) Find the output $y(t)$.

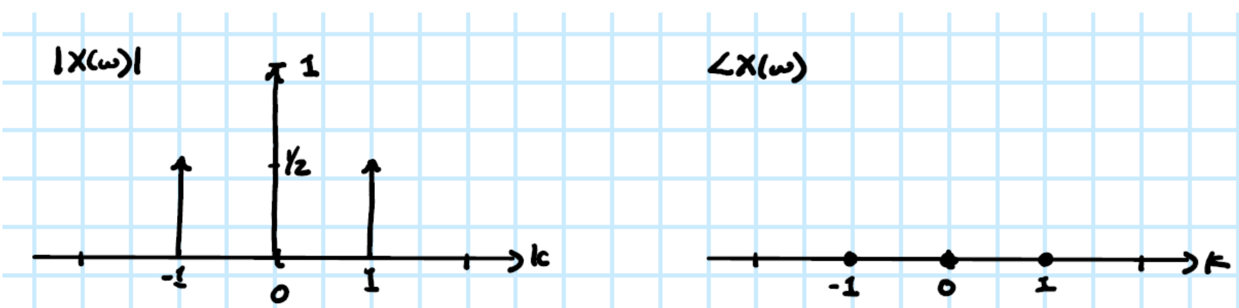
a)



b) Since $x(t)$ is a periodic signal, we can know its frequency content by finding its Fourier coefficients

$$x(t)=1+\frac{1}{2}\left(e^{jt/2}+e^{-jt/2}\right)$$

From there we can identify the coefficients to be $C_0 = 1$, and $C_1 = C_1 = 1/2$.



c) To obtain the output $y(t)$ we only need to calculate the magnitude and phase of the frequency response at the frequencies in $x(t)$, which are $\omega = 0, 0.5\ rad/s$

$$y(t)=|H(0)|1+|H(0.5)|\cos(t/2+\angle H(0.5))=1+\frac{1}{2}cos(t/2-pi/2)$$

3. The frequency response of a filter is

$$H(s)=\frac{\sqrt{60}s}{s^2+2s\sqrt{14}+15}$$

Where $s = j\omega$

a) Find the magnitude and phase (in degrees) of the frequency response at the frequencies$\omega = 0, 10,$ and 100.
b) Roughly sketch its frequency response. What type of filter is this?
c) If a signal generator that produces a biased sinusoid $x(t) = B + Acos(\omega t)$ is connected to this filter, find the frequency, or frequencies, ω_0 at which the filter's output is $y(t) = Acos(\omega_0 t + \theta)$, i.e., the amplitude of the sinusoid remain the same.

a) First, we need to find the magnitude and phase expressions for $H(e^{j\omega})$. The following my be useful

$$|A(j\omega)|=\frac{|B(j\omega)|}{|C(j\omega)|}$$
$$\angle A(j\omega)=\angle B(j\omega)-\angle C(j\omega)$$

$$H(j\omega)=\frac{\sqrt{60}\,j\omega}{(j\omega)^2+(2\sqrt{14})\,j\omega+15}=\frac{j\omega\sqrt{60}}{(15-\omega^2)+j\omega 2\sqrt{14}}$$

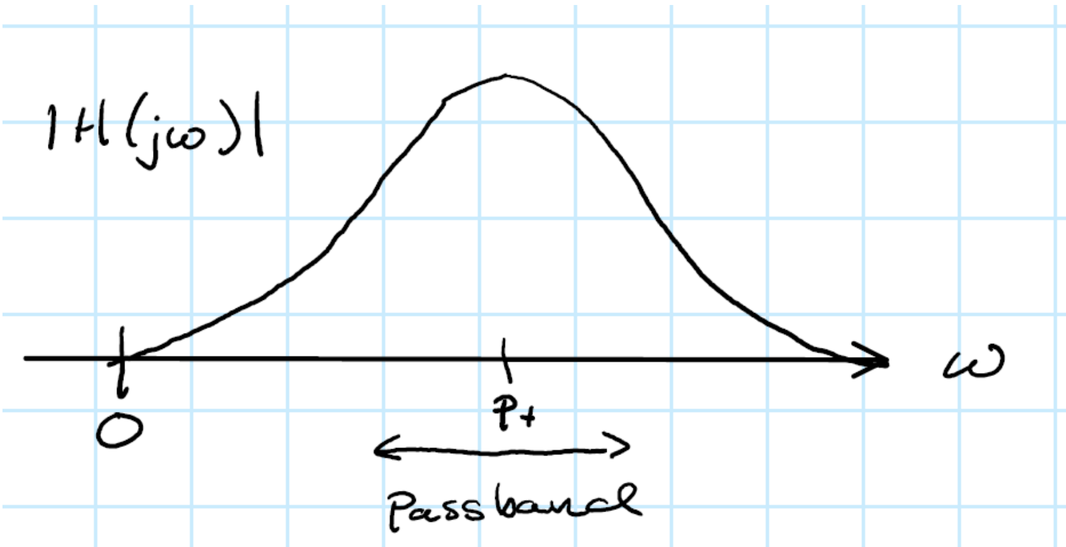
$$|H(e^{j\omega})|=\frac{\omega\sqrt{60}}{\sqrt{(15-\omega^2)^2+56\omega^2}}$$
$$\angle H(e^{j\omega})=\tan^{-1}(\omega\sqrt{60}/0)-\tan^{-1}(\omega 2\sqrt{14}/(15-\omega^2))=90-\tan^{-1}(\omega 2\sqrt{14}/(15-\omega^2))$$

Now we can calculate the values for magnitude and phase for the requested frequencies. For $\omega = 0$, $|H(0)| = 0$, meaning that it blocks the DC component of a signal, thus phase is irrelevant.

For $\omega = 10$, $|H(10)| = 0.684$, $\angle H(10) = 131.360°$.

For $\omega = 100$, $|H(100)| = 0.077$, $\angle H(100) = 94.286°$

b) By the way it attenuates frequencies low and high frequencies, it can be deduced that this is a passband filter. The frequency response of this filter would like something like below.



c) Given the input signal $x(t) = B + Acos(\omega t)$, we know that the output signal will be

$$y(t) = |H(0)|B + |H(\omega_0)|\cos(\omega_0 t + \angle H(\omega_0))$$

We are interested on finding the frequencies at which $|H(\omega_0)| = 1$

$$|H(\omega_0)| = \frac{\omega_0\sqrt{60}}{\sqrt{(15-\omega_0{}^2)^2 + 56\omega_0{}^2}} = 1$$

If we elevate to the power of 2 both sides, we can eliminate the square roots and rearrange the terms as follows

$$60\omega_0{}^2 = (15-\omega_0{}^2)^2 + 56\omega_0{}^2$$
$$60\omega_0^2 = 225 - 30\omega_0^2 + \omega_0^4 + 56\omega_0^2$$
$$0 = \omega_0^4 - 34\omega_0^2 + 225$$
$$0 = (\omega_0^2 - 9)(\omega_0^2 - 25)$$

We need to find the values of ω_0 for which the terms above are zero. These are $\omega_0 = 3$, and $\omega_0 = 5$.

4. The frequency response of a causal, continuous-time LTI system is given by

$$H(j\omega)=\frac{j\omega+5}{28-\omega^2+11j\omega}$$

a) Find the differential equation that describes this system.
b) Find the impulse response $h(t)$ of the system.
c) Find the expression for the damping ratio ζ of the system.

a) We know

$$\sum_{k=0}^{N}\alpha_k\frac{d^k y(t)}{dt^k}=\sum_{k=0}^{M}\beta_k\frac{d^k x(t)}{dt^k}$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{\sum_{k=0}^{M}\beta_k(j\omega)^k}{\sum_{k=0}^{N}\alpha_k(j\omega)^k}$$

It follows that the frequency response can be rearranged as

$$Y(j\omega)((j\omega)^2+11j\omega+28)=X(j\omega)(j\omega+5)$$

$$\frac{d^2y(t)}{dt^2}+11\frac{dy(t)}{dx}+28y(t)=\frac{dx(t)}{dt}+5x(t)$$

b) Let $s=j\omega$. We can then rearrange the frequency response as

$$H(j\omega)=\frac{s+5}{s^2+11s+28}=\frac{s+5}{(s+4)(s+7)}$$

We can simplify this by using partial fractions

$$\frac{s+5}{(s+4)(s+7)}=\frac{A}{(s+4)}+\frac{B}{(s+7)}$$

$$s+5=A(s+4)+B(s+7)$$

We make zero one of the terms. First, let $s=-4$

$$-4+5=A(-4+7)$$

$$1=3A$$

$$1/3=A$$

Let $s=-7$

$$
\begin{aligned}
-7+5 &= B(-7+4) \\
-2 &= -3B \\
2/3 &= B
\end{aligned}
$$

We can then rearrange the transfer function as below

$$H(j\omega)=\frac{1}{3}\frac{1}{4+j\omega}+\frac{2}{3}\frac{1}{7+j\omega}$$

We know from previous exercises (or by looking at a FT table) that this form of the Fourier transform corresponds to the following in the time domain

$$h(t)=\left(\frac{1}{3}e^{-4t}+\frac{2}{3}e^{-7t}\right)u(t)$$

c) We know that the typical transfer function for a second order system is the following

$$H(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}$$

In our transfer function we need to make equal the corresponding terms for ω_n^2. We can accomplish that by finding a multiplying factor so that $5a = 28$. That factor is 5.6. Thus

$$H(j\omega)=\frac{\frac{5.6}{5.6}\frac{j\omega+5}{1}}{(j\omega)^2+11j\omega+28}=\frac{1}{5.6}\frac{5.6j\omega+28}{\left((j\omega)^2+11j\omega+28\right)}$$

Now it is clear that $\omega_n^2 = 28$, and we can calculate the damping ration as

$$
\begin{aligned}
2\zeta\omega_n &= 11 \\
\zeta &= \frac{11}{2\omega_n} = 1.039
\end{aligned}
$$

Which corresponds to an overdamped system.

5. Consider a continuous-time LTI system for which the response to the input

$$x(t) = \left(e^{-8t} + e^{-3t}\right)u(t)$$

In given by

$$y(t) = \left(6e^{-8t} - 6e^{-5t}\right)u(t)$$

a) Find the frequency response of the system.
b) Determine the system's impulse response.
c) Find the differential equation that relates the output and input of this system
d) What is the damping ratio of the system?

a) First, we apply FT to both the input and output using the FT table

$$X(j\omega) = \frac{1}{j\omega+8} + \frac{1}{j\omega+3}$$
$$X(j\omega) = \frac{2j\omega+11}{(j\omega+8)(j\omega+3)}$$

And

$$Y(j\omega) = \frac{6}{(j\omega+8)} - \frac{6}{(j\omega+5)}$$
$$Y(j\omega) = \frac{-18}{(j\omega+8)(j\omega+5)}$$

The frequency response is given by

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{-18}{(j\omega+8)(j\omega+5)} \frac{(j\omega+8)(j\omega+3)}{2j\omega+11} = -18\frac{j\omega+3}{(2j\omega+11)(j\omega+5)}$$

b) We need to use partial fraction expansion. Let $s = j\omega$.

$$-18(s+3) = A(2s+11) + B(s+5)$$

If we let $s = -5$, we obtain $A = 36$. If we set $s = -11/2$, we obtain $B = -90$. So that

$$H(j\omega) = \frac{-90}{2j\omega+11} + \frac{36}{j\omega+5} = \frac{-45}{j\omega+11/2} + \frac{36}{j\omega+5}$$

From the FT tables we know that

$$h(t) = \left(-45e^{-\frac{11}{2}t} + 36e^{-5t}\right)u(t)$$

c) We can rearrange the frequency response as

$$H(j\omega)=\frac{-18(j\omega+3)}{(2j\omega+11)(j\omega+5)}=\frac{-18(j\omega+3)}{2(j\omega)^2+21j\omega+55}$$

It follows that this corresponds to the following differential equation

$$2(j\omega)^2Y(j\omega)+21j\omega Y(j\omega)+55Y(j\omega)=-18\left(j\omega X(j\omega)+3X(j\omega)\right)$$
$$2\frac{d^2y(t)}{dt^2}+21\frac{dy(t)}{dx}+55y(t)=-18\frac{dx(t)}{dt}-54x(t)$$

d) Very similar to the previous problem. We know that the typical transfer function for a second order system is the following

$$H(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}$$

And our transfer function/frequency response is the following

$$H(j\omega)=\frac{-18j\omega-54}{2(j\omega)^2+21j\omega+55}$$

First, we need to leave the term $(j\omega)^2$ by itself

$$H(j\omega)=\frac{1}{2}\frac{-18j\omega-54}{\left((j\omega)^2+10.5j\omega+27.5\right)}$$

In our transfer function we need to make equal the corresponding terms for ω_n^2. For simplicity, it is better if we manipulate the terms at the top. We can find a multiplying factor so that $54a=27.5$, however. For simplicity, let $a=55/108$.

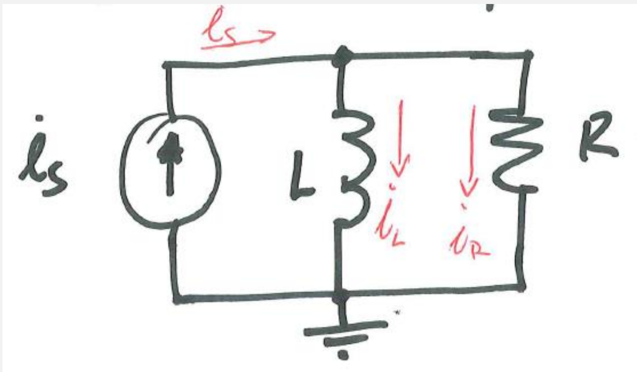
$$H(j\omega)=\frac{1}{2a}\frac{-aj\omega-27.5}{\left((j\omega)^2+10.5j\omega+27.5\right)}$$

Now it is clear that $\omega_n^2=27.5$, and we can calculate the damping ration as

$$2\zeta\omega_n=10.5$$
$$\zeta=\frac{10.5}{2\omega_n}=1.001$$

Which is very close to being a critically damped system.

6. Consider an LTI system that is implemented as an RL circuit shown in the figure below. The input signal $x(t)$ is generated by the current source, and the output $y(t)$ is measured as the current through the inductor.



a) Find the differential equation that describes the system.
b) Find the frequency response of the system.
c) Calculate the impulse response and the step response of the system.

a) The voltage across the inductor is defined as

$$V_L(t) = L\frac{dy(t)}{dt}$$

And the current through the resistor is defined as

$$i_R(t) = \frac{V_L(t)}{R} = \frac{L}{R}\frac{dy(t)}{dt}$$

Notice that the voltage is the same for the inductor and resistor.

From the diagram we can define the following sum of currents

$$x(t) = i_L(t) + i_R(t)$$
$$x(t) = y(t) + \frac{L}{R}\frac{dy(t)}{dt}$$

Which is the differential equation that relates input and output.

b) Using the FT for differential equations we have

$$X(j\omega) = Y(j\omega) + \frac{L}{R}j\omega Y(j\omega)$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1+\frac{L}{R}j\omega}$$

c) We can use the FT tables to apply the inverse transform of $H(j\omega)$ after some rearranging of the terms

$$H(j\omega) = \frac{L}{R}\frac{1}{R/_{L} + j\omega}$$

$$h(t) = \frac{L}{R}e^{-Rt/_{L}}u(t)$$

Since it is a causal system, its impulse response is defined only for $t \geq 0$, hence the unit step.

The step response is found as follows

$$s(t) = \int_{-\infty}^{\tau} h(\tau)d\tau = \frac{R}{L}\int_{-\infty}^{\tau} e^{-R\tau/_{L}}u(\tau)d\tau = \frac{R}{L}\int_{0}^{\tau} e^{-R\tau/_{L}}d\tau = \frac{R/L}{-R/L}e^{-R\tau/_{L}}\Big|_{0}^{t}$$

$$s(t) = (1 - e^{-Rt/_{L}})u(t)$$

7. A CT signal $x(t)$ is a superposition of real sinusoids (cosines) of frequencies $f_1 = 300Hz$, $f_2 = 400Hz$, $f_3 = 1.3kHz$, $f_4 = 3.6kHz$, and $f_5 = 4.3kHz$. All the sinusoids have amplitude 1.
   a) Compute the Fourier transform of x(t).
   b) Assume that $x(t)$ undergoes impulse train sampling to generate $x_s(t) = \sum x(nT_s)\delta(t - nT_s)$ with $1/T_s = 9kHz$ and that $x_s(t)$ goes through an ideal low-pass filter with cut-off frequency $f_c = 4.5kHz$. What are the frequencies present in the output of the low-pass filter?
   c) Repeat part (b) but this time assume $1/T_s = 2kHz$ and $f_c = 900Hz$.

a) Since the signal $x(t)$ is a superposition of cosines, it can be represented as a superposition of complex exponentials of amplitude 0.5 (recall that $cos\theta = 0.5(e^{j\theta} + e^{-j\theta})$). Also, we know the FT pair for a complex exponential, i.e., $\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$. Thus, the FT of $x(t)$ is

$$X(j\omega) = \pi \sum_{\omega_0 \in S} \delta(\omega - \omega_0)$$

Where $S = \{\pm 300, \pm 400, \pm 1{,}300, \pm 3{,}600, \pm 4{,}300\} \ni \omega_0$.

b) The highest frequency present in $x(t)$ is $f_5 = 4.3kHz$, thus the Nyquist rate is given by $2(4.3kHz) = 8.6kHz$. The sampling frequency of $f_s = 9kHz$ is larger than the Nyquist rate, so no aliasing should occur. Also, the filter's cut-off frequency is $4.5kHz = f_s/2$, which meets the requirements of the sampling theorem. This means that all the frequencies present in $x(t)$ will be also present in $x_s(t)$.

c) This is clearly a case of undersampling. We know that the frequency content of $X_s(j\omega)$ for this case contains copies of $X(j\omega)$ shifted by f_s, so that the frequencies present in $X_s(j\omega)$ are $\omega_0 \in S + 2{,}000k$, for $k \in \mathbb{Z}$, i.e., $\{\pm 300 + 2000k, \pm 400 + 2000k, \pm 1{,}300 + 2000k, \pm 3{,}600 + 2000k, \pm 4{,}300 + 2000k\}$.

Also, notice that the reconstructed signal $x_r(t)$ will have its frequencies higher than $f_s/2$ be seen as $f_s - f$. The low-pass filter will let through all the frequencies in the interval $[-900Hz, 900Hz]$, which means that the frequencies present at the output will be $\pm 300Hz$, $\pm 400Hz$, and $\pm 700Hz$, this last one resulting from undersampling $f_3 = 1{,}300Hz$.

8. Consider the signal $x(t) = \cos(2t) + \sin(3t)$,
   a) Find the largest period T_s such that $x[n] = x(nT_s)$ satisfies the Nyquist sampling criterion and is periodic with period $N_0 = 15$.
   b) Find the ideal reconstruction filter equation $H(j\omega)$ based on the sampling period T_s from (a).

a) First, we need to determine the Nyquist rate for $x(t)$ based on the highest frequency present,

$$\omega_N = 2\omega_{max} = 2(3) = 6$$
$$T_s < \frac{2\pi}{\omega_N} = \frac{\pi}{3}$$

We can now look for a sampling period that satisfies $N = 15$

$$T_s = 2\pi\frac{k}{15} < \frac{\pi}{3} = \frac{5\pi}{15}$$

Which we find that this condition exists for $k = 1,2$, but since the question asks for the largest period possible, thus

$$T_s = 2\pi\frac{2}{15}$$

The resulting sampled signal is

$$x[n] = \cos\left(2\pi\frac{4}{15}n\right) + \sin\left(2\pi\frac{6}{15}n\right)$$

b) The ideal reconstruction filter is an ideal low-pass filter defined as

$$H_R(j\omega) = T_s\left(u(\omega + \omega_s/2) - u(\omega - \omega_s/2)\right)$$

By substituting $T_s = 4\pi/15$

$$H_R(j\omega) = \frac{4\pi}{15}\left(u(\omega + 15/4) - u(\omega - 15/4)\right)$$