ELEC 221 Lecture 16
Time and frequency domain analysis II

Tuesday 1 November 2022

Announcements

- Midterms available for pickup at my office
- Quiz 7 today
- Assignment 5 released soon (last one before midterm 2)

Important:

- Nov. 8 class on Zoom
- Office hours this Friday and next Friday on Zoom (same time)

Links will be distributed on Canvas.

Last time

We formalized the magnitude-phase representation of spectra:

$$H(j\omega) = |H(j\omega)|e^{\sphericalangle H(j\omega)}$$

where

- $|H(j\omega)|$ is the gain
- $\sphericalangle H(j\omega)$ is the phase shift

We used these to analyze how systems affect phase:

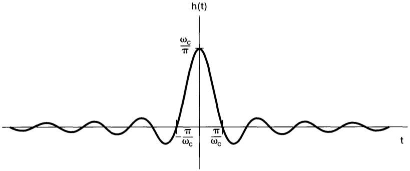
$$\begin{aligned} |Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\ \sphericalangle Y(j\omega) &= \sphericalangle H(j\omega) + \sphericalangle X(j\omega) \end{aligned}$$

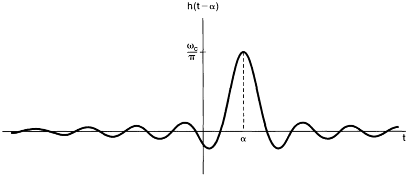
Last time

We saw how linear shifts in phase affect a system's behaviour:

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \qquad\qquad H(j\omega) = \begin{cases} e^{-\alpha\omega}, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$$h(t) = \frac{\sin(\omega_c t)}{\pi t} \qquad\qquad h(t) = \frac{\sin(\omega_c (t-\alpha))}{\pi (t-\alpha)}$$





Last time

We analyzed non-linear shifts by making an approximation that they are linear for small bands of frequencies:

$$\sphericalangle H(j\omega) \approx -\phi - \alpha\omega$$

We extended this to the idea of group delay:

$$\tau(\omega) = -\frac{d}{d\omega}(\sphericalangle H(j\omega))$$

Illustrative example (Oppenheim Ex. 6.1): group delay

Suppose we have some system whose frequency response is

$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega),$$

$$H_i(j\omega) = \frac{1 + \left(j\omega/\omega_i\right)^2 - 2j\zeta_i\left(\omega/\omega_i\right)}{1 + \left(j\omega/\omega_i\right)^2 + 2j\zeta_i\left(\omega/\omega_i\right)},$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec and } \zeta_1 = 0.066, \\ \omega_2 = 943 \text{ rad/sec and } \zeta_2 = 0.033, \\ \omega_3 = 1888 \text{ rad/sec and } \zeta_3 = 0.058. \end{cases}$$

Actual frequencies: $f_1 \approx 50$ Hz, $f_2 \approx 150$ Hz, $f_3 = 300$ Hz.

Image credit: Oppenheim 6.2

Illustrative example (Oppenheim Ex. 6.1): group delay

Can find that $|H(j\omega)| = 1$, and the phase component is

$$\sphericalangle H(j\omega) = -2\sum_{i=1}^{3} \tan^{-1}\left[\frac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right]$$

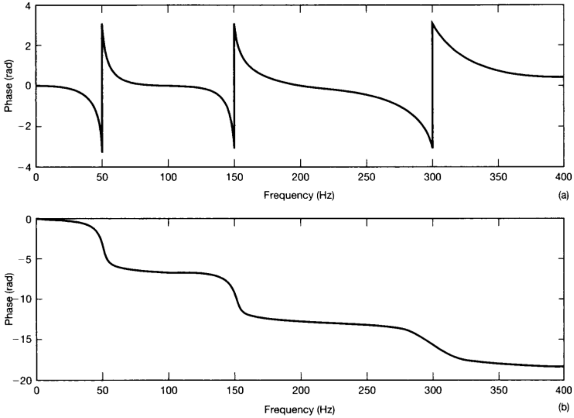


Image credit: Oppenheim 6.2

Illustrative example (Oppenheim Ex. 6.1): group delay

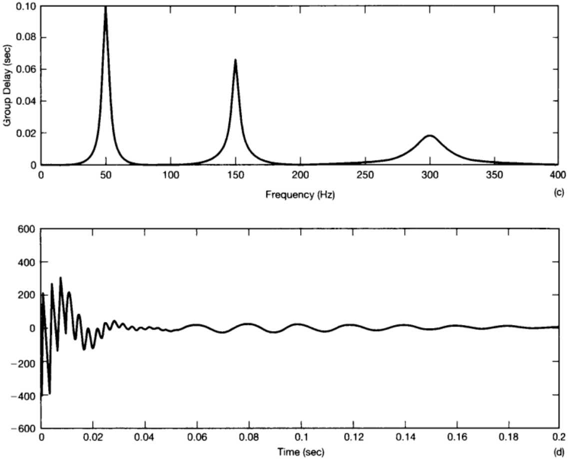


Image credit: Oppenheim 6.2

Learning outcomes:

- define and compute the unit step response of a system
- plot frequency response using a Bode plot
- characterize the oscillatory behaviour of CT systems described by second-order differential equations

We will continue to work in CT: you will get practice problems and assignment problems about the DT case (it is very similar).

Ideal filter step response

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

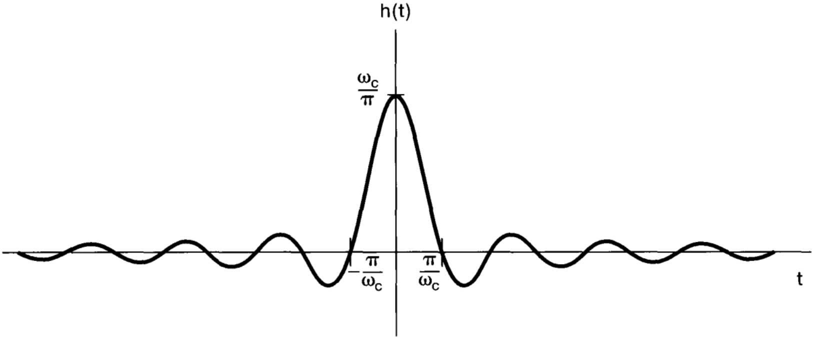


Image credit: Oppenheim 6.3

Ideal filter step response

It is also important to consider *step response* of filters.

Recall that

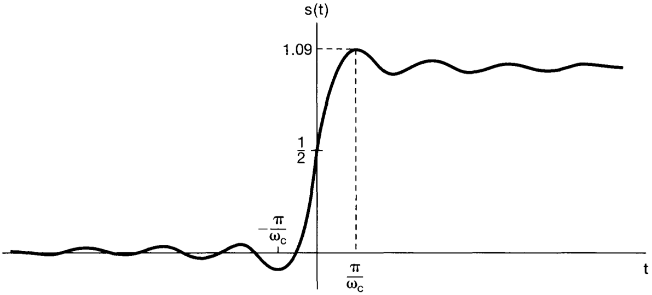
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

By linearity, if we put this in a system, the result is

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

Ideal filter step response

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$



An ideal filter leads to **ringing** in the step response.

Image credit: Oppenheim 6.3

Ideal filter step response

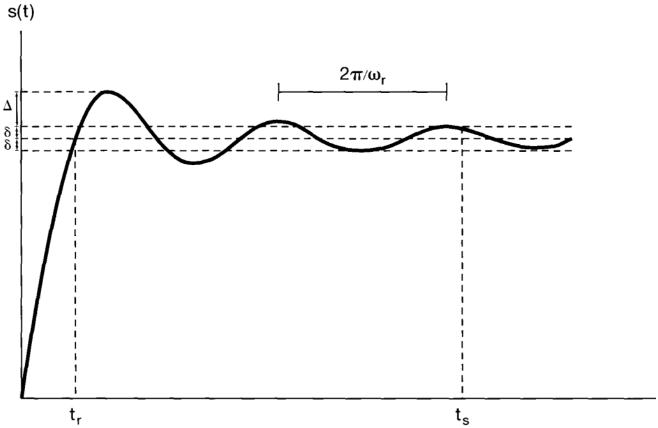


Figure 6.17 Step response of a continuous-time lowpass filter, indicating the rise time t_r, overshoot Δ, ringing frequency ω_r, and settling time t_s—i.e., the time at which the step response settles to within $\pm\delta$ of its final value.

Image credit: Oppenheim 6.4

There are **tradeoffs** in filter design. Compromises in the frequency domain can lead to nicer behaviour in the time domain.

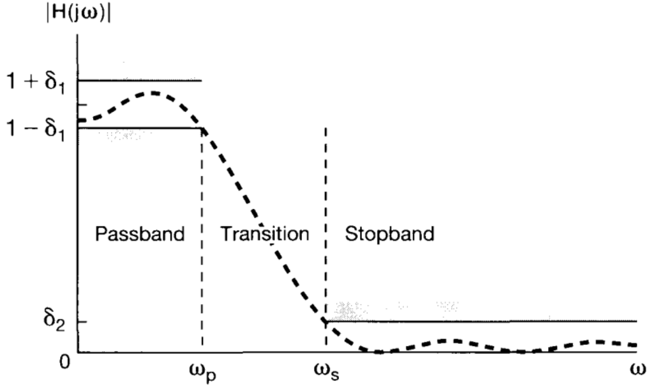


Image credit: Oppenheim 6.4

First-order systems

LTI system described by a first-order ODE:

$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

Exercise: what is the frequency response $H(j\omega)$?

Solution: recall the handy formula we derived from the convolution property. Given an arbitrary-order ODE,

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

the frequency response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

So for our system,

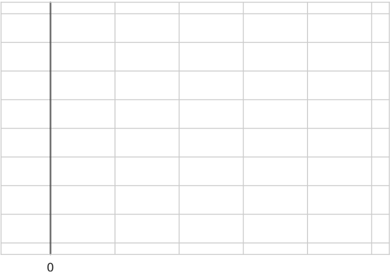
$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

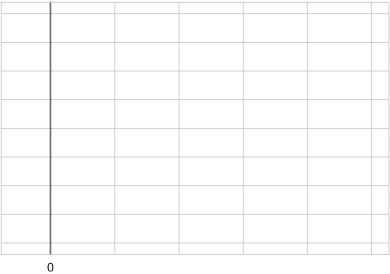
$$\tau \frac{dy(t)}{dt} + y(t) = x(t), \qquad H(j\omega) = \frac{1}{1 + j\omega\tau}$$

The impulse and step response of the system are

$$h(t) = \frac{1}{\tau} e^{-t/T} u(t), \qquad s(t) = (1 - e^{-t\tau})u(t)$$

τ is the **time constant** of the system.





$$\tau \frac{dy(t)}{dt} + y(t) = x(t), \qquad H(j\omega) = \frac{1}{1 + j\omega\tau}$$

Let's view these in the magnitude-phase representation:

$$H(j\omega) = \frac{1}{1 + j\omega\tau} \cdot \frac{1 - j\omega\tau}{1 - j\omega\tau} = \frac{1}{(\omega\tau)^2 + 1} - j\frac{\omega\tau}{(\omega\tau)^2 + 1}$$

From this, we find

$$\begin{aligned} |H(j\omega)| &= \frac{1}{\sqrt{(\omega\tau)^2 + 1}} \\ \sphericalangle H(j\omega) &= \tan^{-1}(-\omega\tau) \end{aligned}$$

Let's plot these in a new way...

Bode plots

Recall:

$$\begin{aligned} |Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\ \measuredangle Y(j\omega) &= \measuredangle H(j\omega) + \measuredangle X(j\omega) \end{aligned}$$

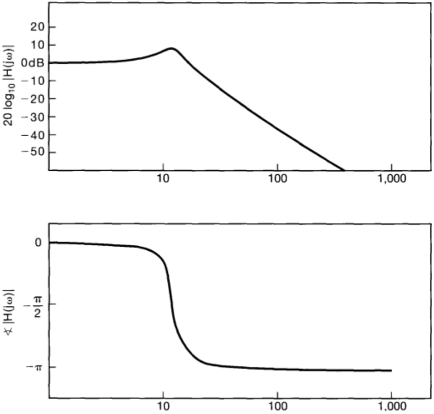
Magnitude is multiplicative and phase is additive... would be nicer if both were additive.

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

Rather than making plots of $|H(j\omega)|$ and $\measuredangle H(j\omega)$, it is common to make plots of $20\log_{10}|H(j\omega)|$ and $\measuredangle H(j\omega)$ against $\log_{10}\omega$.

Bode plots

These are called *Bode plots*:



The logarithmic scale also allows us to view the response over a much wider range of frequencies.

Image credit: Oppenheim 6.2

First-order systems

We have

$$
\begin{aligned}
|H(j\omega)| &= \frac{1}{\sqrt{(\omega\tau)^2 + 1}} \\
\measuredangle H(j\omega) &= \tan^{-1}(-\omega\tau)
\end{aligned}
$$

To make our Bode plot, compute

$$
\begin{aligned}
20\log_{10}|H(j\omega)| &= 20\log_{10}\left(\frac{1}{\sqrt{(\omega\tau)^2 + 1}}\right) \\
&= -20\log_{10}\left((\omega\tau)^2 + 1\right)^{1/2} \\
&= -10\log_{10}((\omega\tau)^2 + 1)
\end{aligned}
$$

$$20\log_{10}|H(j\omega)| = -10\log_{10}((\omega\tau)^2 + 1)$$

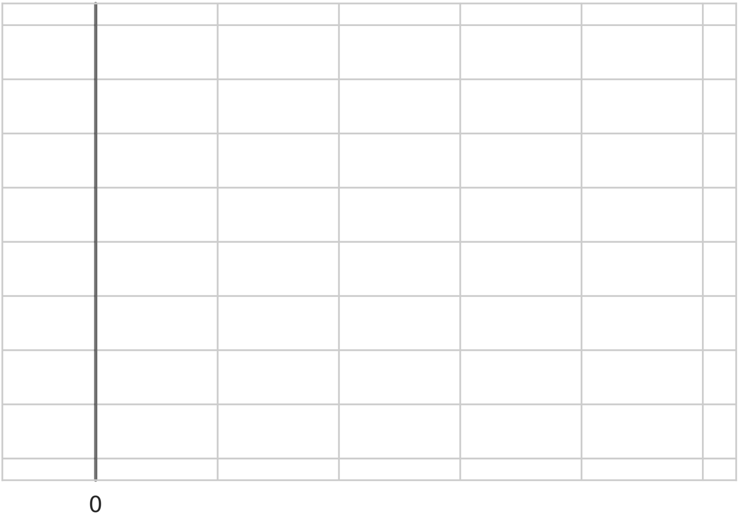
If $\omega << 1/\tau$,

$$20\log_{10}|H(j\omega)| \approx 0$$

If $\omega >> 1/\tau$, $\omega\tau >> 1$

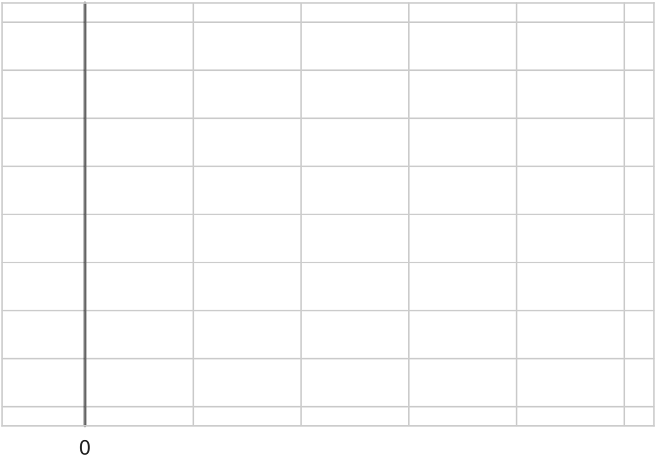
$$\begin{aligned}
20\log_{10}|H(j\omega)| &\approx -10\log_{10}((\omega\tau)^2) \\
&= -20\log_{10}(\omega\tau)) \\
&= -20\log_{10}(\omega) - 20\log_{10}(\tau)
\end{aligned}$$

First-order systems



First-order systems

But that is just an approximation. In reality,



The case of $\omega = 1/\tau$ has a special name.

First-order systems

$$20\log_{10}|H(j\omega)| = -10\log_{10}((\omega\tau)^2+1)$$

If $\omega = 1/\tau$,

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= -10\log_{10}(((1/\tau)\tau)^2+1) \\ &= -10\log_{10}(2) \\ &\approx -3 \end{aligned}$$

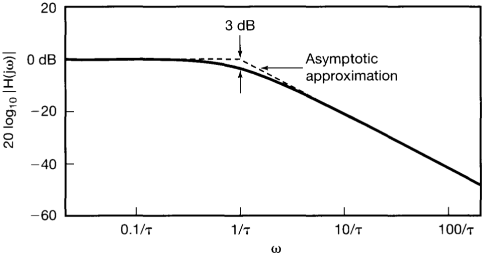


Image credit: Oppenheim 6.5

Can make similar approximations to recover plot of the phase

$$H(j\omega) = \tan^{-1}(-\omega\tau)$$

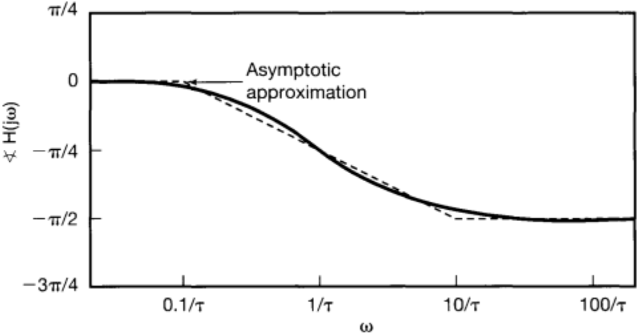


Image credit: Oppenheim 6.5

Consider a system described by the ODE

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Exercise: what is the frequency response?

Let's explore this in a little more detail and compute the impulse and step response of this system.

$$
\begin{aligned}
H(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \\
&= \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}
\end{aligned}
$$

where

$$c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Three cases to consider:

- $\zeta = 1$
- $\zeta > 1$
- $\zeta < 1$

Second-order systems

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}, \quad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case: $\zeta = 1$.

$$\begin{aligned} c_\pm &= -\zeta\omega_n = -\omega_n \\ H(j\omega) &= \frac{\omega_n^2}{(j\omega + \omega_n)^2} \end{aligned}$$

Use handy table of Fourier transform pairs to find

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$$

Second-order systems

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}, \quad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case: $\zeta \neq 1$. Do a partial fraction expansion:

$$\begin{aligned} H(j\omega) &= \frac{A}{j\omega - c_+} + \frac{B}{j\omega - c_-} \\ &= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\frac{1}{j\omega - c_+} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\frac{1}{j\omega - c_-} \end{aligned}$$

Use handy table of Fourier transform pairs to find

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(e^{c_+ t} - e^{c_- t}\right)u(t)$$

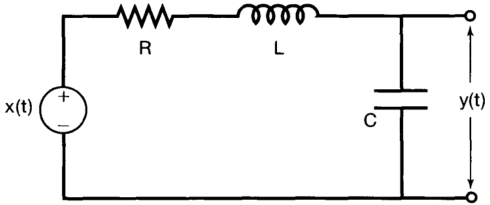
$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(e^{c_+ t} - e^{c_- t} \right) u(t), \qquad c_\pm = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

The form of the exponential depends on whether $\zeta > 1$ or $\zeta < 1$

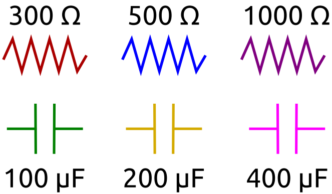
- $\zeta < 1$: $c_\pm$ are imaginary; complex exponentials, so the response will oscillate!
- $\zeta > 1$: $c_\pm$ real and negative; decaying exponentials

Let's go plot these.

Example: RLC circuit



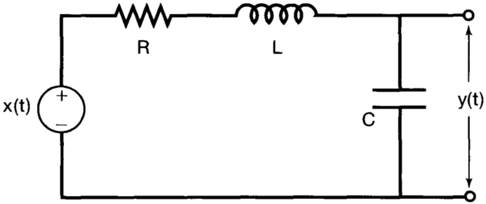
Suppose $L = 6$ H. We have a box of capacitors and resistors:



What is the best choice to ensure step response doesn't oscillate?

Image credit: Oppenheim P6.19.

Example: RLC circuit



First, we need to set up the ODE for the system.

$$\begin{aligned} V_{TOTAL} &= V_R(t) + V_I(t) + V_L(t) \\ x(t) &= LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) \end{aligned}$$

Image credit: Oppenheim P6.19.

Example: RLC circuit

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

Example: RLC circuit

Learning outcomes:

- define and compute the unit step response of a system
- plot frequency response using a Bode plot
- characterize the oscillatory behaviour of CT systems described by second-order differential equations

Oppenheim practice problems:

- (DT) 6.35, 6.36, 6.41, 6.42, 6.65
- (CT) 6.15, 6.28 (choose a couple), 6.32, 6.33, 6.53

For next time

Content:

- The sampling theorem
- Basics of interpolation
- The Nyquist rate and aliasing

Action items:

1. Work through Oppenheim section 6.5-6.7
2. Assignment 5 coming soon
3. Midterm 2 in two weeks

Recommended reading:

- From this class: Oppenheim 6.4-6.8
- For next class: Oppenheim 7.1-7.3