ELEC 221 Lecture 21
The Laplace transform

Tuesday 26 November 2024

Announcements

- Quiz 9 today
- Tutorial assignment 5 due Monday 23:59
- First part of A5 released; due 8 Dec 23:59

Last time

We made frequency-division multiplexing more efficient with single-sideband modulation

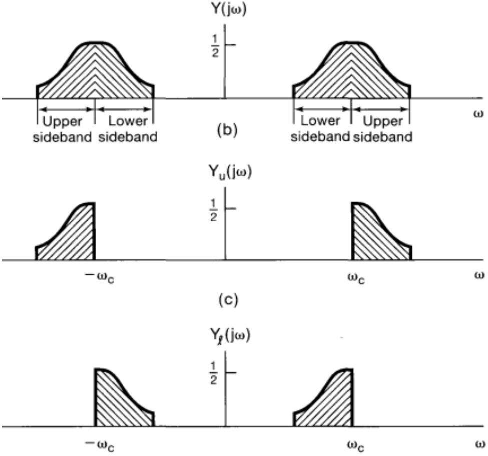


Image credit: Oppenheim 8.4

Last time

We saw how AM with a pulse-train carrier can be used for time-division multiplexing, and pulse-amplitude modulation.

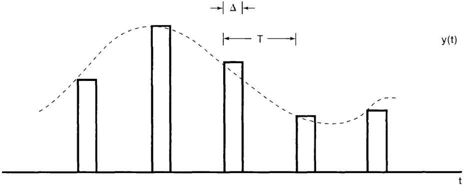
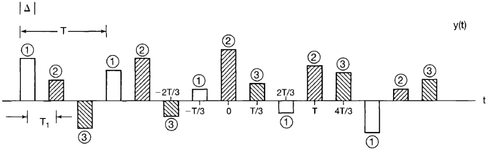


Figure 8.26 Transmitted waveform for a single PAM channel. The dotted curve represents the signal $x(t)$.

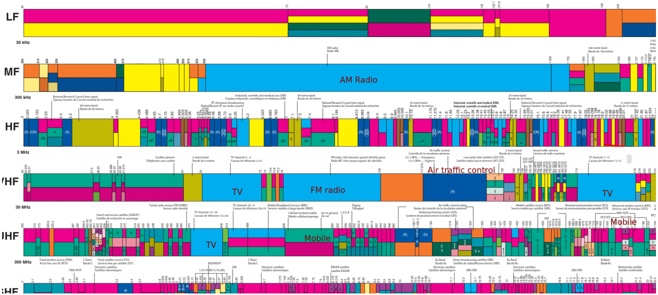


You will get to explore this more in A5.

Image credit: Oppenheim 8.6

Last time

We discussed how cell phones are radios and how the radio spectrum gets divided (and auctioned off).



https://ised-isde.canada.ca/site/spectrum-management-telecommunications/sites/default/files/attachments/2022/2018_Canadian_Radio_Spectrum_Chart.PDF

The course so far

Wayyyyy back in lecture 5:

LTI systems and complex exponential functions

To summarize:

$$x(t) = e^{st} \quad \rightarrow \quad y(t) = H(s) \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

17 / 45

Wayyyyy back in lecture 5:

The Fourier series

Let's consider a special set of signals[1]:

$$x(t) = e^{st} = e^{j\omega t}$$

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

[1] We will see the general case at the end of the course.

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

Input a signal into LTI system with impulse response $h(t)$:

If $s = j\omega$: **Fourier transform**

If $s = \sigma + j\omega$: (bilateral) **Laplace transform**

More generally,

We can relate the Laplace and Fourier transforms.

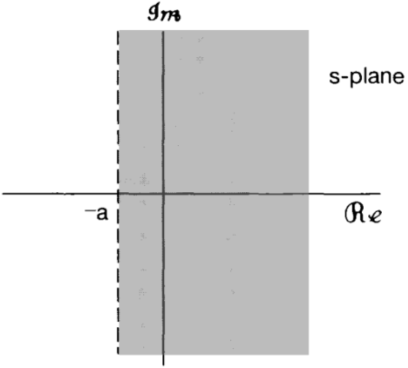
Example: Let $x(t) = e^{-at}u(t)$. What is $X(j\omega)$?

Recall: conditions on a?

Example: Let $x(t) = e^{-at}u(t)$. What is $X(s)$?

Conditions on a?

We must specify for which s the Laplace transform is valid.



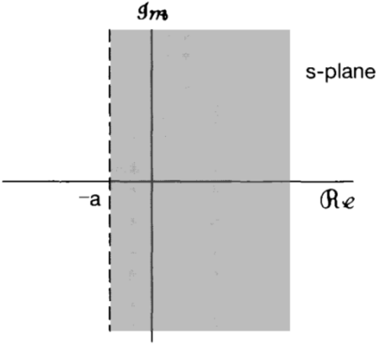
This is called the **region of convergence** (ROC).

Image credit: Oppenheim 9.1

Exercise: what is the Laplace transform and ROC of

$$x(t) = -e^{-at}u(-t)$$

Multiple signals can have the same algebraic Laplace transform, but different ROCs.



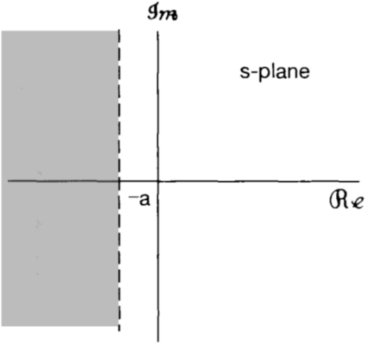


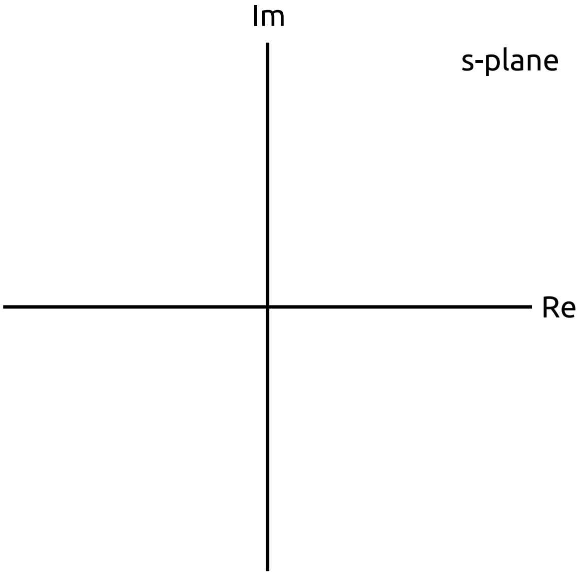
Image credit: Oppenheim 9.1

Exercise: what is the Laplace transform and ROC of

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

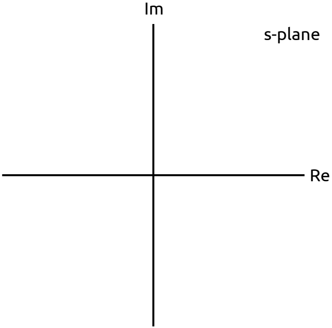
Hint: the Laplace transform is also linear!

Let's draw the ROC:



Pole-zero plots

$X(s)$ are often rational polynomials of s. Indicate roots on the s-plane using $\times$ for denominator (poles), $\circ$ for numerator (zeros):



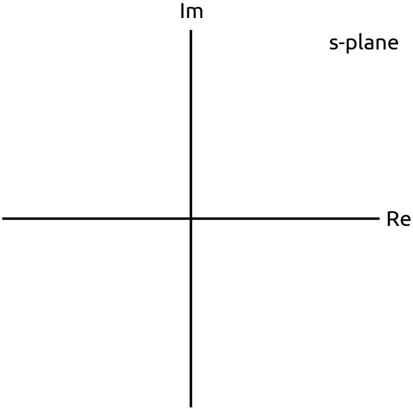
This is a **pole-zero plot**. (May also have poles/zeros at infinity if degree of polynomials is different)

Pole-zero plots

Exercise: compute the Laplace transform of

$$x(t) = -2e^{-3t}u(t) + 4e^{-4t}u(t)$$

and draw its pole-zero plot.



Regions of convergence

The ROC has many nice properties:

- if ROC doesn't contain $j\omega$ axis, FT does not converge
- ROC is strips parallel to $j\omega$ axis
- ROC of rational Laplace transform contains no poles

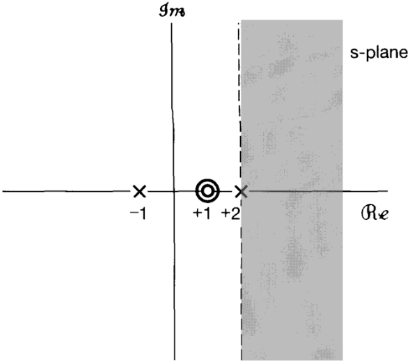


Image credit: Oppenheim 9.1

Regions of convergence

If $x(t)$ has finite duration and is absolutely integrable, the ROC is the entire s-plane.

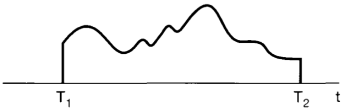
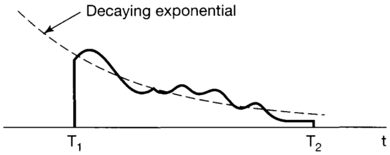


Figure 9.4 Finite-duration signal.



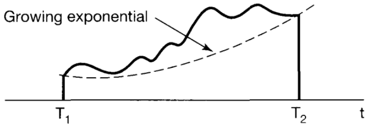
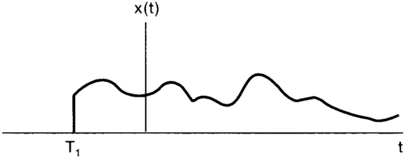
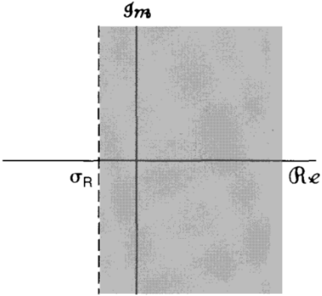


Image credit: Oppenheim 9.2

Right-sided signals

If $x(t)$ is right sided and $\text{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\text{Re}(s) > \sigma_0$ are also in the ROC.





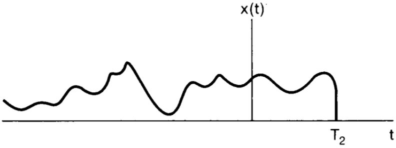
This ROC is called a **right-half plane**.

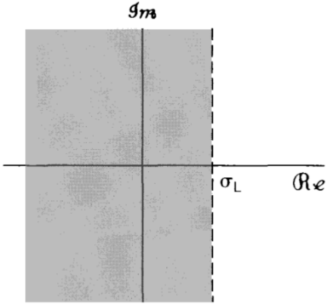
Intuition: if $\text{Re}(s) = \sigma_1 > \sigma_0$ the exponential in $x(t)e^{-\sigma t}$ decays even faster and will still converge.

Image credit: Oppenheim 9.2

Left-sided signals

If $x(t)$ is left sided and $\text{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\text{Re}(s) < \sigma_0$ are also in the ROC.





This ROC is called a **left-half plane**.

Image credit: Oppenheim 9.2

Two-sided signals

Any guesses?

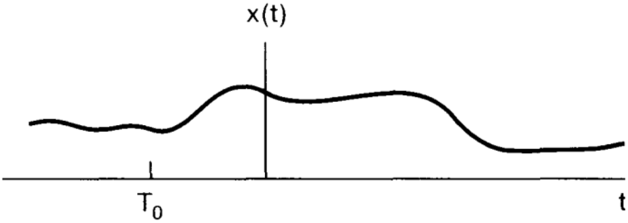
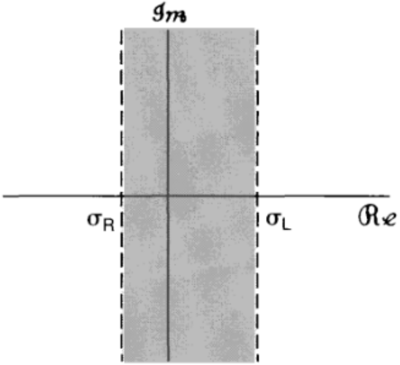


Image credit: Oppenheim 9.2

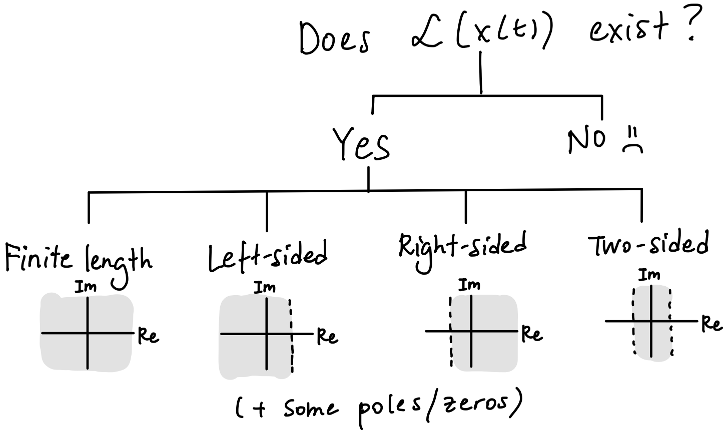
Two-sided signals



Only works if initial ROCs overlap - otherwise $X(s)$ doesn't exist!

Image credit: Oppenheim 9.2

For any signal $x(t)$...

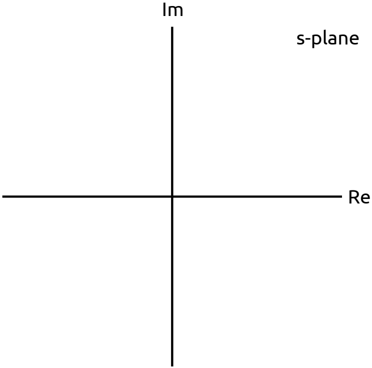


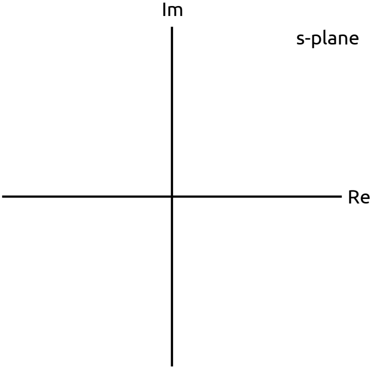
Regions of convergence

(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

$$\frac{s-1}{(s+2)(s+3)(s^2+s+1)}$$

Hint: draw pole-zero plot and identify possible ROCs.





Inverse Laplace transforms

From this, we can invert:

Make a change of variables $ds = jd\omega$:

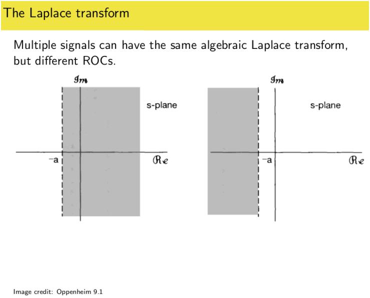
... we are not going to integrate this.

Inverse Laplace transforms

Suppose

where degree of denominator is higher than numerator.

To invert, we can use our handy identities, BUT the ROC matters.



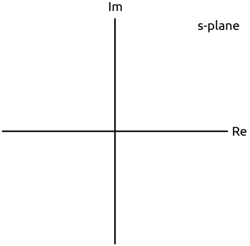
Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

Draw the s-plane:



For next time

Content:

- properties and system analysis with Laplace transform

Action items:

1. Thursday class back in person
2. Tutorial assignment 5 due Monday 23:59

Recommended reading:

- From this class: Oppenheim 9.0-9.3, 9.5 (skip 9.4)
- Suggested problems: Oppenheim 9.1-9.9, 9.21, 9.26
- For next class: 9.5-9.8 (skip 9.9)