ELEC 221 Lecture 15
Time and frequency domain analysis I

Thursday 27 October 2022

in hallway (next class has a midterm)

after 4:30pm (meeting at 3:30)

- Midterms available for pickup after class (or at my office)
- Assignment 4 due on Saturday at 23:59
- (Bonus) Assignment 4.5 due on Saturday at 23:59
- Quiz 7 on Tuesday (will focus on today's content)

Complex exponential signals are eigenfunctions of LTI systems in both continuous time and discrete time.

If $x(t) = e^{st}$, for complex s

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} e^{st}e^{-s\tau}h(\tau)\,d\tau \\
&= e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)\,d\tau \\
&= e^{st}\cdot H(s)
\end{aligned}
$$

We have considered so far only $s = j\omega$

$$H(s) \to H(j\omega)$$

This is the **frequency response** of the system.

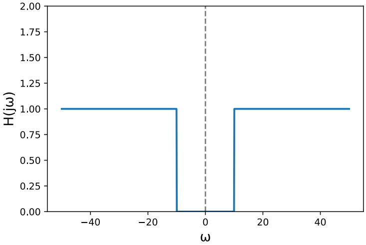
When we input a linear combination of signals,

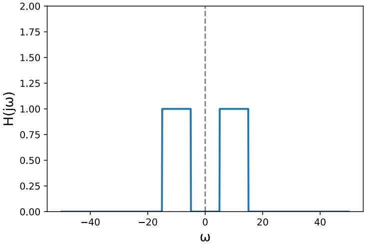
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \mapsto y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega) c_k e^{jk\omega t}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega \mapsto y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) X(j\omega) e^{j\omega t}$$

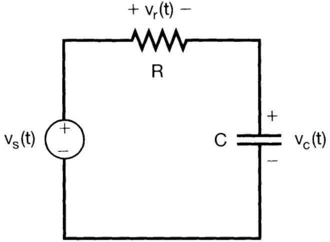
Previously

We have seen some simple frequency response of ideal filters:





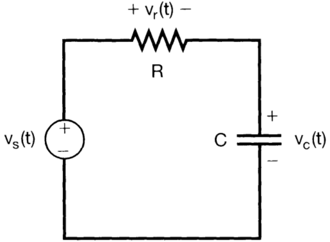
We have also seen more realistic ones.



$$RC\frac{dV_c(t)}{dt}+V_c(t)=V_s(t)$$

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)$$

If $v_s(t)=e^{j\omega t}$, then a solution is $v_c(t)=H(j\omega)e^{j\omega t}$ for some scaling $H(j\omega)$.



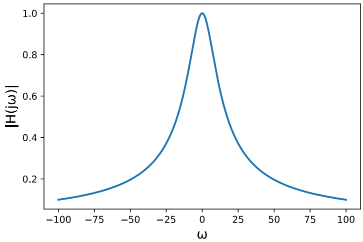
We found that

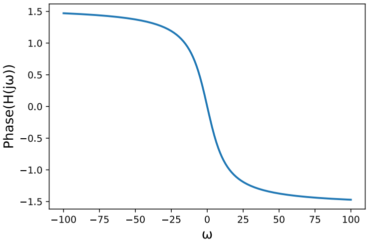
$$H(jw) = \frac{1}{1 + jwRC}$$

If we look at $H(j\omega)$ we can see that this is also a filter. The frequencies it attenuates depends on R and C.

Previously

In general $H(j\omega)$ has both a magnitude and a phase component.

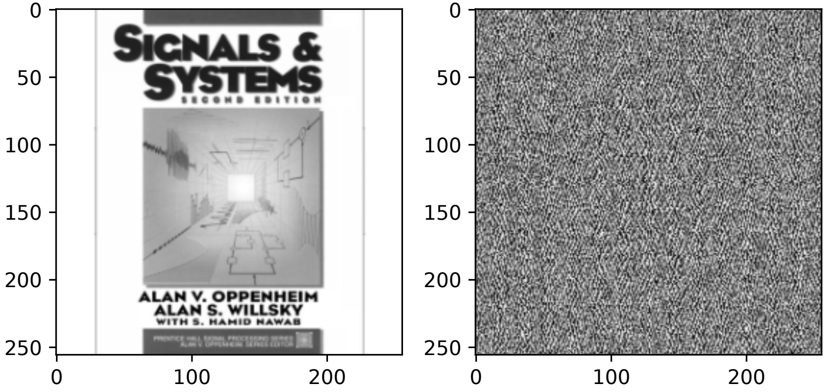




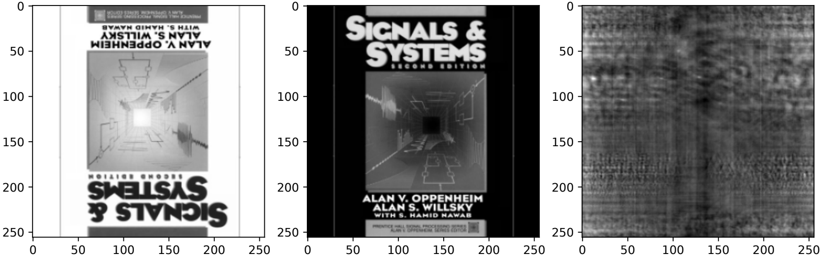
Increasing RC cuts off more frequencies, but there are design tradeoffs involved.

We haven't looked much at the phase response...

You hopefully learned from the hands-on that phase can be important!



Really important...



So we should probably consider this more in our analysis of systems.

Today

Learning outcomes:

- express a frequency response in the magnitude-phase representation
- differentiate between linear and non-linear phase responses
- compute the group delay of a frequency response
- plot the frequency response using a Bode plot

$\measuredangle X(j\omega)$

Since Fourier spectra are complex numbers, we can express them in terms of their magnitude and phase.

$$X(j\omega) = |X(j\omega)| \cdot e^{j\measuredangle X(j\omega)}$$

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\measuredangle X(e^{j\omega})}$$

Frequency response of LTI systems

Recall the convolution property of the Fourier transform:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

What happens to the output?

How does passing through the system with $H(j\omega)$ affect $|X(j\omega)|$ and $\sphericalangle X(j\omega)$?

Try it yourself. Given

$$
\begin{aligned}
X(j\omega) &= |X(j\omega)|e^{j\sphericalangle X(j\omega)} \\
Y(j\omega) &= H(j\omega)X(j\omega)
\end{aligned}
$$

$|H(j\omega)|e^{j\sphericalangle H(j\omega)}$

Determine

$$
\begin{aligned}
|Y(j\omega)| &= \\
\sphericalangle Y(j\omega) &=
\end{aligned}
$$

$$|Y(j\omega)| = |H(j\omega)| \cdot |X(j\omega)|$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\measuredangle Y(j\omega) = \measuredangle H(j\omega) + \measuredangle X(j\omega)$$

$$\sphericalangle Y(j\omega) = \sphericalangle H(j\omega) + \sphericalangle X(j\omega)$$

We give these names:

- $|H(j\omega)|$ is the gain
- $\sphericalangle H(j\omega)$ is the phase shift

Depending on what these are, the result can be either good, or bad (*distortion*).

If $|H(j\omega)| = 1$ everywhere, the system is called *all-pass* and is characterized by its phase response.

It is nicest when the phase shift is a *linear* function of the frequency.

linear equation

$$\measuredangle H(j\omega) = \omega t + b$$

$$H(j\omega) = |H(j\omega)| \cdot e^{j(\omega t + b)}$$

Can you think of a system that causes a linear shift in phase?
(Hint: think back to properties of Fourier transform)

time shift

Time shift (or, a delay system):

$$y(t) = x(t - t_0)$$

$$Y(j\omega) = \underbrace{e^{-j\omega t_0}} X(j\omega)$$

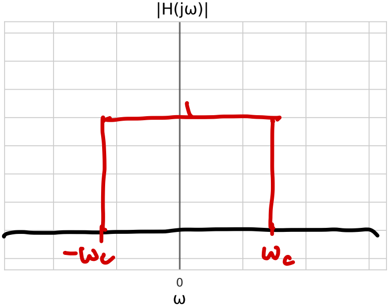
$$|Y(j\omega)| = |X(j\omega)|$$

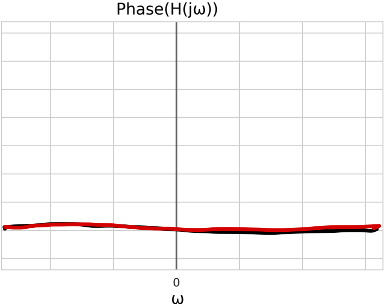
$$\measuredangle Y(j\omega) = -\omega t_0 + \measuredangle X(j\omega)$$

Example: linear frequency response

Let's consider the ideal lowpass filter,

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$





What is its impulse response? (inverse Fourier transform of frequency response)

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\,e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\underbrace{\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)}_{2j\sin(\omega_c t)}$$

$$= \frac{\sin(\omega_c t)}{\pi t}$$

Example: linear frequency response

Recall what this looks like graphically:

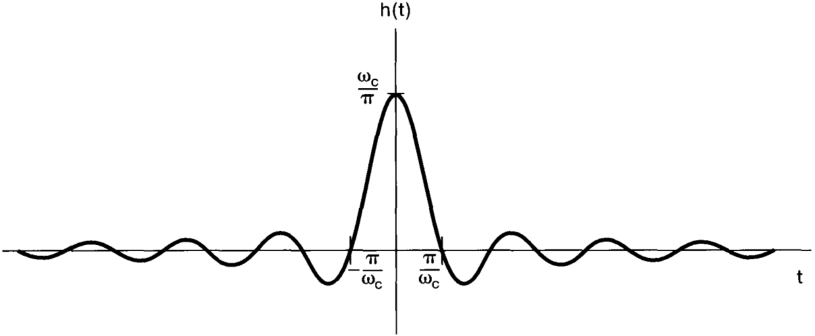
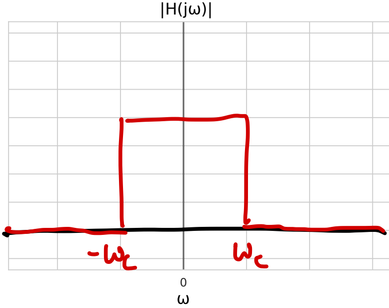


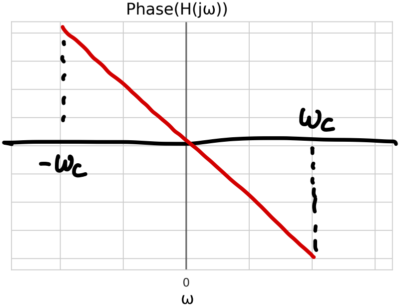
Image credit: Oppenheim 6.3

Example: linear frequency response

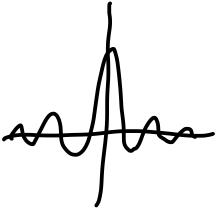
What happens if we add a linear phase?

$$H(j\omega) = \begin{cases} e^{-j\alpha\omega} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$





Example: linear frequency response



$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega} \cdot e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{1}{j(t-\alpha)}\, e^{j\omega(t-\alpha)} \Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\cancel{2}\pi} \cdot \frac{1}{\cancel{j}(t-\alpha)} \cdot \cancel{2j} \sin(\omega_c(t-\alpha))$$

$$= \frac{\sin(\omega_c(t-\alpha))}{\pi(t-\alpha)}$$

$$\frac{\sin(\omega_c t)}{\pi t}$$

Example: linear frequency response

The result is a shifted version of the original impulse response

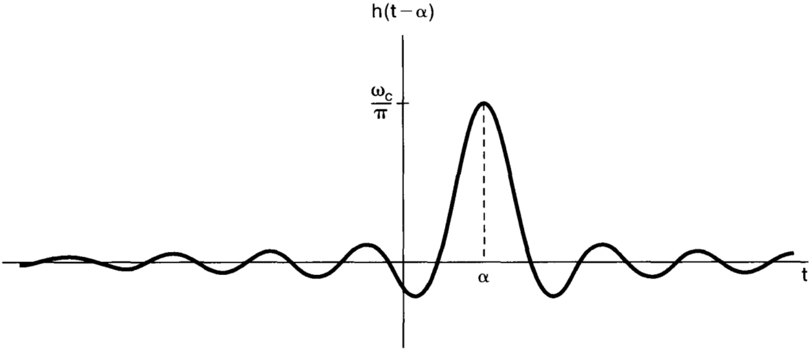


Image credit: Oppenheim 6.3

Exercise: linear frequency response

Consider the following frequency response:

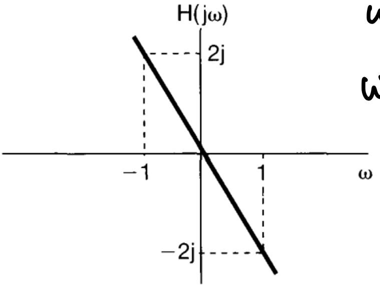


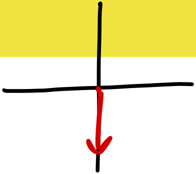
Figure P6.21

$\omega = -1 \Rightarrow H(j\omega) = 2j$

$\omega = 1 \Rightarrow H(j\omega) = -2j$

- What is $H(j\omega)$?
- Sketch $|H(j\omega)|$ and $\measuredangle H(j\omega)$

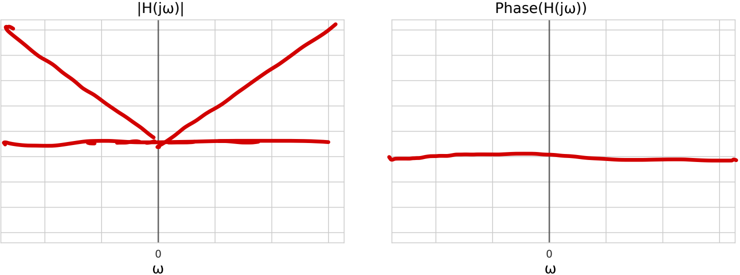
Image credit: Oppenheim Problem 6.21



$$H(j\omega) = -2j\omega$$

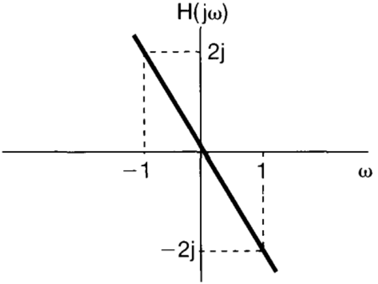
$$|H(j\omega)| = |2\omega|$$

$$\measuredangle H(j\omega) = -\frac{\pi}{2}$$



Exercise: linear frequency response

Suppose a signal $x(t)$ with spectrum $X(j\omega) = \frac{1}{2+j\omega}$ is input into the system.



$H(j\omega) = -2j\omega$

Figure P6.21

What is the output signal $y(t)$? (Hint: recall what happens to a Fourier spectrum of a function when you take its derivative)

Image credit: Oppenheim Problem 6.21

From the convolution property,

$$Y(j\omega) = H(j\omega)X(j\omega) = -2(j\omega)X(j\omega)$$

This means that

$$y(t) = -2\frac{dx(t)}{dt}$$

Our handy identity tells us that

$$X(j\omega) = \frac{1}{2+j\omega} \rightarrow x(t) = e^{-2t}u(t)$$

So,

$$\begin{aligned} y(t) &= -2\frac{d}{dt}\left(e^{-2t}u(t)\right) \\ &= -2\left(-2e^{-2t}u(t) + e^{-2t}\delta(t)\right) \\ &= 4e^{-2t}u(t) - 2\delta(t) \end{aligned}$$

Group delay

Linear phase: same delay at all frequencies (shift the response).

Non-linear phase: different amount of delay at different frequencies

If we look at a small enough band of frequencies, we can make an approximation that it is...

$$\measuredangle H(j\omega) \approx -\phi - \omega\alpha$$

Then:

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$
$$\approx X(j\omega) \cdot |H(j\omega)| \cdot e^{-j\phi} e^{-j\omega\alpha}$$

$$Y(j\omega) \approx X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha}$$

The parameter α represents an effective common delay of the frequencies in this small band.

It is called the **group delay**:

$$\sphericalangle H(j\omega) \approx -\phi - \omega\alpha$$

$$-\frac{d}{d\omega}\left(\sphericalangle H(j\omega)\right) = \tau(\omega)$$

$$\sphericalangle H(j\omega) \approx -\phi - \omega\alpha$$

$$\tau(\omega) = -\frac{d}{d\omega}\left(\sphericalangle H(j\omega)\right) = \alpha$$

Non-linear phase and group delay has a lot of real-world implications.

Exercise: group delay

Consider a filter with frequency response

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- What are $|H(j\omega)|$ and $\measuredangle H(j\omega)$
- What is the group delay?

$$\tau(\omega) = \frac{-d}{d\omega}\left(\measuredangle H(j\omega)\right)$$

Exercise: group delay

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- What are $|H(j\omega)|$ and $\measuredangle H(j\omega)$
- What is the group delay?

First, write

$$H(j\omega) = \frac{1}{(1+j\omega)}\frac{(1-j\omega)}{(1-j\omega)} = \frac{1-j\omega}{1+\omega^2} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

From this, we find

$$|H(j\omega)| = \sqrt{\left(\frac{1}{1+\omega^2}\right)^2 + \left(\frac{\omega}{1+\omega^2}\right)^2} = \sqrt{\frac{1+\omega^2}{(1+\omega^2)^2}} = \frac{1}{\sqrt{1+\omega^2}}$$

$$\measuredangle H(j\omega) = \tan^{-1}(-\omega)$$

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- What are $|H(j\omega)|$ and $\measuredangle H(j\omega)$

$$\measuredangle H(j\omega) = \tan^{-1}(-\omega)$$

- What is the group delay?

$$\tau(\omega) = -\frac{d}{d\omega}\left(\tan^{-1}(-\omega)\right)$$

$$= -\frac{1}{1+(-\omega)^2}(-1)$$

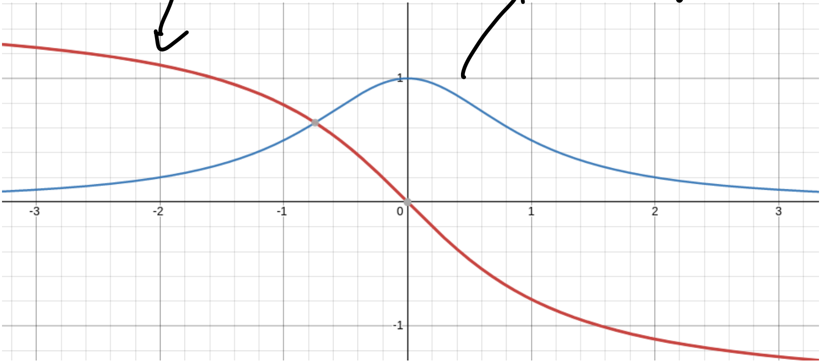
$$= \frac{1}{1+\omega^2}$$

$$\tau(\omega) = -\frac{d}{d\omega}\left(\tan^{-1}(-\omega)\right) = -\frac{1}{1+(-\omega)^2}\cdot(-1) = \frac{1}{1+\omega^2}$$

Exercise: group delay

$\angle H(j\omega) = \tan^{-1}(-\omega)$

$\tau(\omega) = \frac{1}{1+\omega^2}$



Recall:

★ We will start here next time.

$$
\begin{aligned}
|Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\
\measuredangle Y(j\omega) &= \measuredangle H(j\omega) + \measuredangle X(j\omega)
\end{aligned}
$$

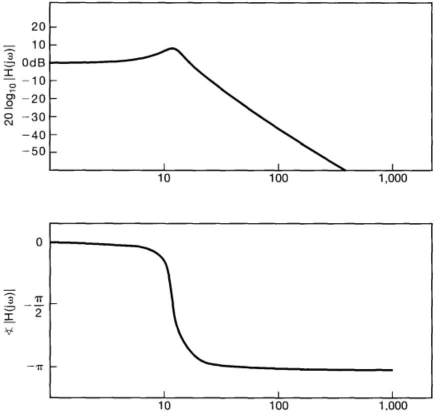
Magnitude is multiplicative and phase is additive... would be nicer if both were additive.

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

Rather than making plots of $|H(j\omega)|$ and $\measuredangle H(j\omega)$, it is common to make plots of $20\log_{10}|H(j\omega)|$ and $\measuredangle H(j\omega)$ against $\log_{10}\omega$.

Bode plots

These are called *Bode plots*:



We will see more of these on Tuesday.

Image credit: Oppenheim 6.2

Ideal filter step response

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

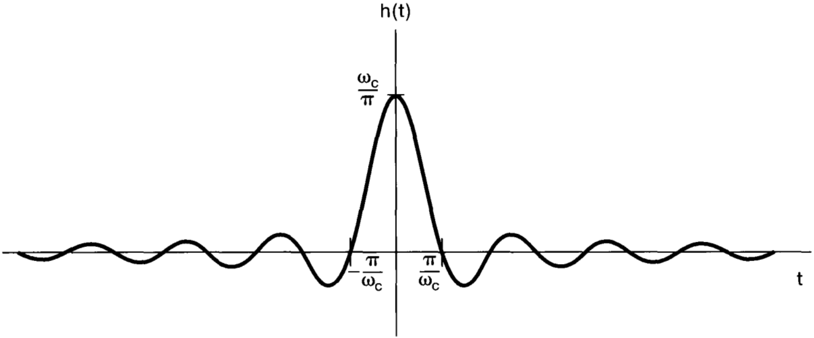


Image credit: Oppenheim 6.3

Ideal filter step response

It is also important to consider *step response* of filters.

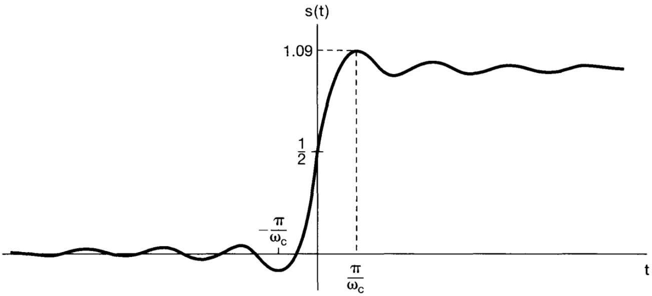
Recall that

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

By linearity, if we put this in a system, the result is

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$



By changing the design of the filters, we can limit the amount of ringing. More on Tuesday!

Today

Learning outcomes:

- express a frequency response in the magnitude-phase representation
- differentiate between linear and non-linear phase responses
- compute the group delay of a frequency response
- plot the frequency response using a Bode plot

next time

Oppenheim practice problems:

- (DT) 6.2, 6.4, 6.37, 6.39 (choose a couple)
- (CT) 6.21a-c, 6.23, 6.27, 6.42

For next time

Content:

- Properties of non-ideal filters
- Filters described by first/second-order difference equations

Action items:

1. Quiz 7 Tuesday
2. Assignment 4 due Saturday 23:59
3. Bonus activity due Saturday 23:59

Recommended reading:

- For next class: Oppenheim 6.4-6.7