ELEC 221 Lecture 06
CT Fourier series coefficients and properties

Tuesday 24 September 2024

Announcements

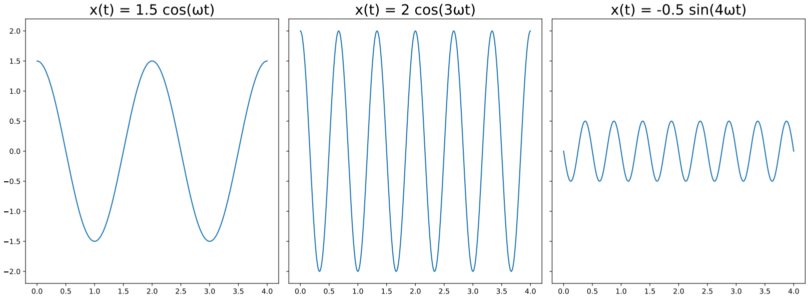
- Quiz 3 today
- Assignment 2 available at 12pm Tuesday, due 5 Oct 23:59
- Tutorial Assignment 2 due Monday 30 Sept 23:59
- No tutorial next Monday

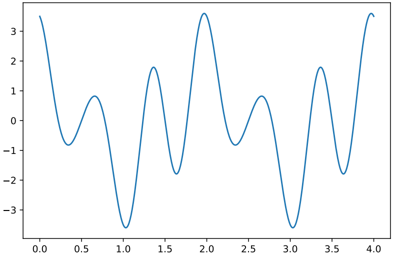
Last time

Complex exponential signals are eigenfunctions of LTI systems:

$H(s)$ is the **system function**.

Last time





Last time

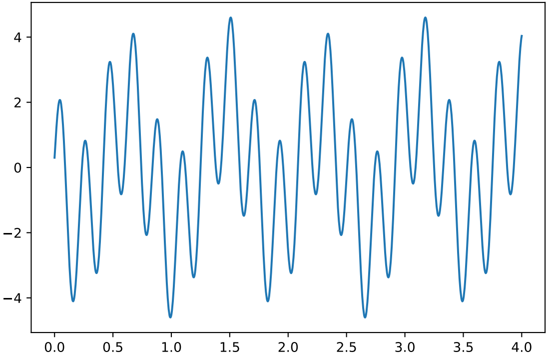
When we restricted to complex values only, i.e.,

the system function is called the **frequency response**,

If a superposition of such signals is input into an LTI system, each signal is rescaled by the frequency response:

Last time

What if we are given a signal like this:



Can we determine the c_k?

Learning outcomes:

- Compute the Fourier series coefficients of a CT periodic signal
- State the Dirichlet conditions and identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series

Evaluating Fourier coefficients

Given a signal $x(t)$, let's compute c_k.

The $e^{jk\omega t}$ are **basis functions** and have **orthogonality** relations.

Let $\phi_k(t) = e^{jk\omega t}$. Let's integrate over a period:

where * indicates the *complex conjugate*.

Evaluating Fourier coefficients

Exercise: evaluate the integral

$$\frac{1}{T}\int_0^T \phi_k(t)\phi_m^*(t)dt = \frac{1}{T}\int_0^T e^{jk\omega t}e^{-jm\omega t}dt$$

Case 1: $k = m$

Case 2: $k \neq m$

Evaluating Fourier coefficients

Exercise: express $x(t)$ as a Fourier series and evaluate the integral

$$\frac{1}{T}\int_0^T \phi_m^*(t)x(t)dt$$

Fourier coefficients tell us *how much* each harmonic contributes.

Note that c_0 is a constant offset:

(Similar techniques can be used to derive a_k and b_k for the sin and cos representation. Try it!)

Recap: key expressions

Fourier synthesis equation:

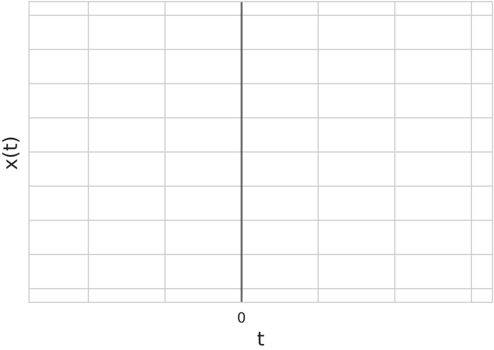
Fourier analysis equation:

Exercise

What is the Fourier series of

$$x(t) = e^{-t}, \quad -1 \leq t < 1$$

Start with a plot, and determine T and ω.



Exercise

Exercise

Dirichlet conditions

Can we *always* express a signal as a Fourier series?

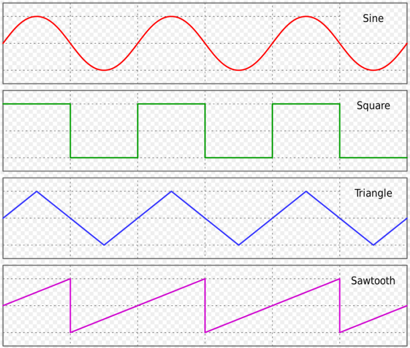


Image credit: *Sine, square, triangle, and sawtooth waveforms* (author: Omegatron)

`https://en.wikipedia.org/wiki/Triangle_wave#/media/File:Waveforms.svg` (CC BY-SA 3.0)

Dirichlet conditions

Given a **periodic function**, if over one period,

1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

then the Fourier series converges to

- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Examples that violate Dirichlet conditions

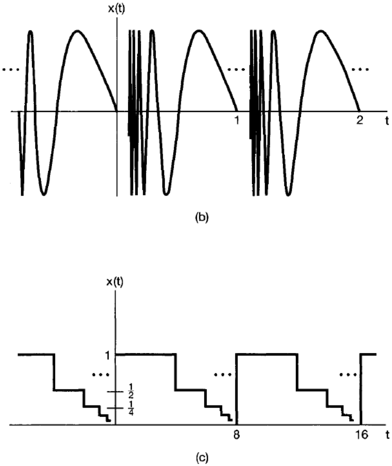


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Image: Oppenheim.

Fourier series converge

Consider what happens if we truncate the Fourier series:

Look at approximation error:

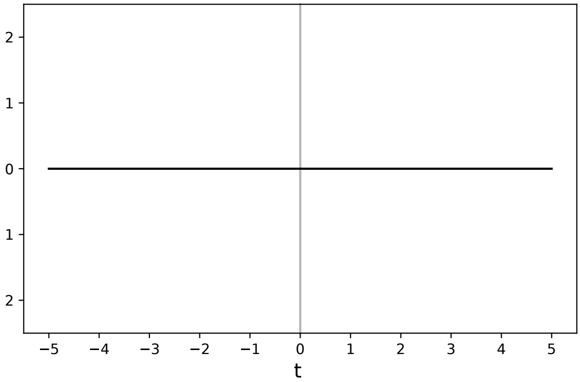
How much error is there over a given period?

This should go to 0 as $N \to \infty$.

Example: the square wave

Consider a square wave signal with period 2π:

$$x(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi \end{cases}$$



Example: the square wave

Evaluate its Fourier coefficients. We can often take shortcuts based on the properties of a signal.

You can derive that for a 2π-periodic function, the coefficients have the following form:

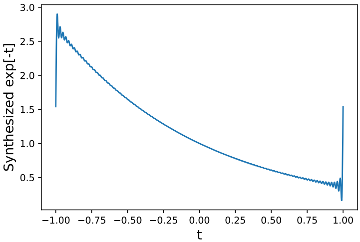
Example: the square wave

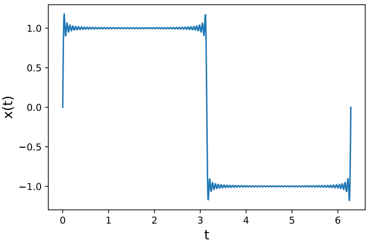
Since the function is symmetric, we can rewrite this:

Let's see how well this converges...

Live code: Fourier series

This is called the Gibb's phenomenon.





The amount of ringing, or "overshoot", is about 9% of the jump of the discontinuity, no matter the size of N.

Can derive from the *energy* of the error between the original and truncated signals (learn about energy / power of signals in A2).

Properties of Fourier series

Fourier coefficients have some really useful properties that help us evaluate them.

What happens when we apply the following transformations to the Fourier series representations of our signals?

- Superposition
- Time shift / scale / reversal
- Multiplication

Fourier coefficients combine linearly.

Suppose we have two signals $x(t), y(t)$ with period T,

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}, \\
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t},
\end{aligned}
$$

Then the signal $z(t) = Ax(t) + By(t)$ has the form

Time shift $x(t) \to x(t - t_0)$:

Properties of Fourier series

Time scale $x(t) \to x(\alpha t)$.

If original period was T, new period is T/α:

Multiplication leads to convolution of the coefficients:

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}, \\
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t},
\end{aligned}
$$

Then the signal $z(t) = x(t)y(t)$ has the form

Properties of Fourier series

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Recap

Learning outcomes:

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- State the Dirichlet conditions and identify whether a signal can be expressed as a Fourier series
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For next time

Content:

- Discrete-time Fourier series coefficients and properties

Action items:

1. Tutorial Assignment 2 due Monday
2. Assignment 2 due on 5 October (do Q3 and Q5)

Recommended reading:

- From today's class: Oppenheim 3.0-3.5
- Suggested problems: 3.4, 3.5, 3.8, 3.13, 3.17, 3.22a,c, 3.23-3.26
- For next class: Oppenheim 3.6-3.7