ELEC 221 Lecture 07
CT and DT Fourier series

Thursday 26 September 2024

Announcements

- Assignment 2 due Saturday 5 Oct 23:59
- Tutorial Assignment 2 due Monday 23:59
- **No tutorial on Monday**
- Quiz 4 on Tuesday

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$$

Fourier synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Fourier analysis equation:

$$c_k = \frac{1}{T} \int_T e^{-jk\omega t} x(t)\, dt$$

Last time

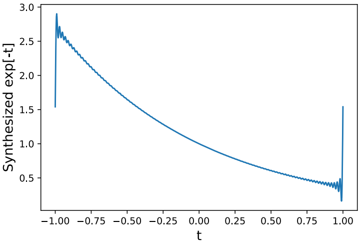
We evaluated the Fourier series coefficients of

$$x(t) = e^{-t}, \quad -1 \le t < 1$$

We got an exact result

$$c_k = \frac{(-1)^k}{2(1 + jk\pi)} \left[e - e^{-1} \right]$$

But we saw some unusual behaviour when we tested it.



Learning outcomes:

- Identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series
- Compute the fundamental period and frequency of a DT signal

Dirichlet conditions

Can we *always* express a signal as a Fourier series?

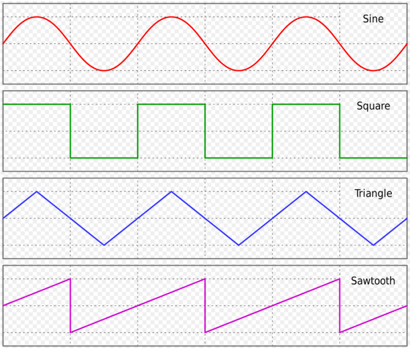
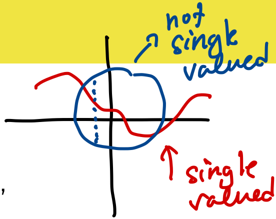


Image credit: *Sine, square, triangle, and sawtooth waveforms* (author: Omegatron)

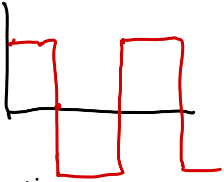
`https://en.wikipedia.org/wiki/Triangle_wave#/media/File:Waveforms.svg` (CC BY-SA 3.0)

Dirichlet conditions



Given a **periodic function**, if over one period,

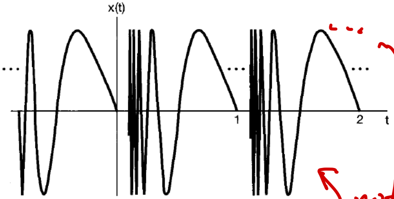
1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities



then the Fourier series converges to

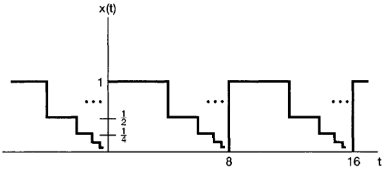
- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Examples that violate Dirichlet conditions



(b)

not okay



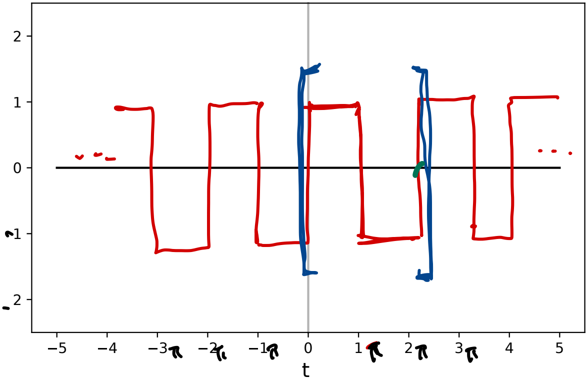
(c)

Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Image: Oppenheim.

Example: the square wave

$$x(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi \end{cases}$$



$T = 2\pi \rightarrow \omega = \frac{2\pi}{T} = 1$

Evaluate its Fourier coefficients. We can often take shortcuts based on the properties of a signal.

$c_{-k} = c_k^*$

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt}$

$= \cdots c_{-1} \ldots c_0 + c_1 e^{jt} + \ldots + c_k e^{jkt} + \ldots$

$= \cdots c_k^* e \ldots c_1 e^{jt} + \ldots$

$= c_0 + c_1 \ldots c_k e^{jkt} + c_k^* e^{-jkt}$

$= c_0 + (a_1 \ldots (a_2 - jb_2)e^{-2jt} + \ldots$

$= c_0 + a_1 \ldots e^{-2jt}) + jb_2(e^{2jt} - e^{-2jt}) + \ldots$

You can derive ... coefficients have the following form: ... $\sin(kt)$

$- 2b_2 \sin(2kt) \ldots$

$c_1 = a_1 + jb_1$

$\cos(kt) = \frac{e^{jkt} + e^{-jkt}}{2}$ $\sin(kt) = \frac{e^{jkt} - e^{-jkt}}{2j}$

When I did this in class, I thought I lost a sign, but it was actually correct. I rederived it more clearly on the next page anyways.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt} \quad (\omega = 1 \text{ because } T = 2\pi)$$

$$= \cdots + c_{-k} e^{-jkt} + \cdots + c_{-1} e^{-jt} + c_0 + c_1 e^{jt} + \cdots + c_k e^{jkt} + \cdots$$

Since $x(t)$ is real, $c_{-k} = c_k^*$

$$= \cdots + c_k^* e^{-jkt} + \cdots + c_1^* e^{-jt} + c_0 + c_1 e^{jt} + \cdots + c_k e^{jkt} + \cdots$$

Write $c_k = a_k + jb_k$, $c_k^* = a_k - jb_k$

$$= \cdots + (a_k - jb_k) e^{-jkt} + \cdots + (a_1 - jb_1) e^{-jt} + c_0 + (a_1 + jb_1) e^{jt}$$
$$+ \cdots + (a_k + jb_k) e^{jkt} + \cdots$$

Group the a_k and b_k

$$= c_0 + a_1 (e^{jt} + e^{-jt}) + jb_1 (e^{jt} - e^{-jt}) + \cdots + a_k (e^{jkt} + e^{-jkt})$$
$$+ jb_k (e^{jkt} - e^{-jkt}) + \cdots$$

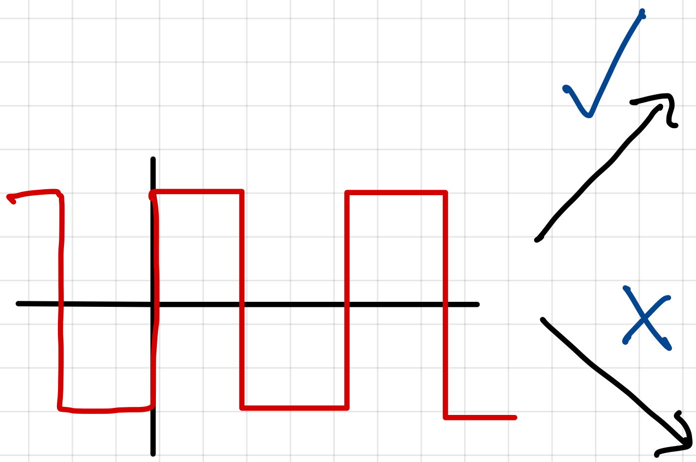
$$= c_0 + a_1 \cdot 2\cos(t) + jb_1 \cdot 2j\sin(t) + \cdots + a_k \cdot 2\cos(kt)$$
$$+ jb_k \cdot 2j\sin(kt) + \cdots$$

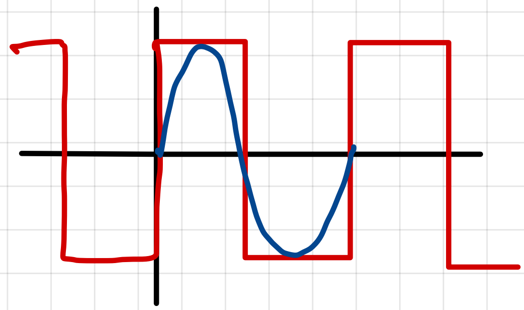
$$= c_0 + 2a_1 \cos(t) + \cdots + 2a_k \cos(kt) + \cdots$$
$$- 2b_1 \sin(t) - \cdots - 2b_k \sin(kt) - \cdots$$

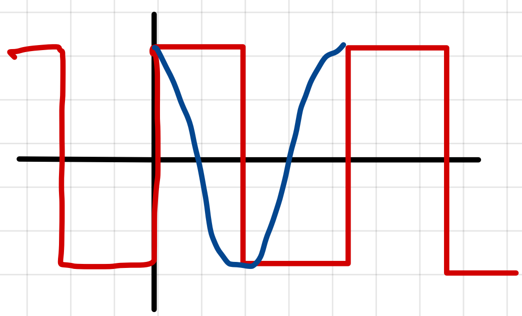
$$= c_0 + \sum_{k=1}^{\infty} 2a_k \cos(kt) - \sum_{k=1}^{\infty} 2b_k \sin(kt)$$

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2a_k \cos(kt) - \sum_{k=1}^{\infty} 2b_k \sin(kt)$$

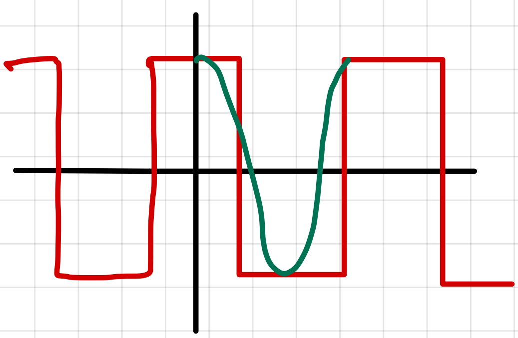
If we imagine our square wave, we can "fit" sin inside, but not cos. The square wave we are looking at is odd, like sin, while cos is even:







Note, though, that if we shift it, we would have cos:



but as we have defined it, we have sin only.

We also have a function that is symmetric around the x-axis, so $c_0 = 0$. We are thus left with

$$x(t) = -\sum_{k=1}^{\infty} 2b_k \sin(kt)$$

To determine the b_k, we can leverage the orthogonality of trig functions under integration:

$$\int_0^{2\pi} \sin(kt)\sin(mt)\,dt = \begin{cases} 0 & \text{if } k \neq m \\ \pi & \text{if } k = m \end{cases}$$

Multiply on both sides by $\sin(mt)$ and integrate over a period...

$$\int_0^{2\pi} \sin(mt)\,x(t)\,dt = -2\sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(mt)\sin(kt)\,dt$$

$$= -2b_m \cdot \pi$$

$$\Rightarrow \quad b_m = -\frac{1}{2\pi}\int_0^{2\pi} \sin(mt)\,x(t)\,dt$$

Our function is defined as

$$x(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & \pi \le t < 2\pi \end{cases}$$

So

$$b_m = -\frac{1}{2\pi}\int_0^{\pi} \sin(mt)\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} \sin(mt)\,dt$$

$$= -\frac{1}{2\pi}\cdot\left(-\frac{1}{m}\cos mt\right)\Big|_0^{\pi} + \frac{1}{2\pi}\cdot\left(-\frac{1}{m}\cos(mt)\right)\Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{m}\underbrace{\cos(m\pi)}_{(-1)^m} - \frac{1}{m}\right] - \frac{1}{2\pi}\left[\frac{1}{m}\underbrace{\cos(2\pi m)}_{1} - \frac{1}{m}\underbrace{\cos(m\pi)}_{(-1)^m}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^m}{m} - \frac{1}{m} - \frac{1}{m} + \frac{(-1)^m}{m}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2(-1)^m}{m} - \frac{2}{m}\right] = \frac{1}{m\pi}\left[(-1)^m - 1\right]$$

$$b_m = \frac{1}{m\pi}\left[(-1)^m - 1\right]$$

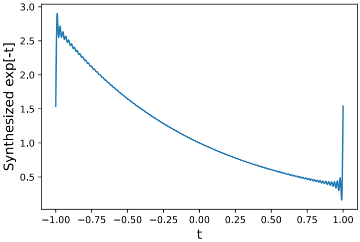
If m is even, $b_m = 0$.

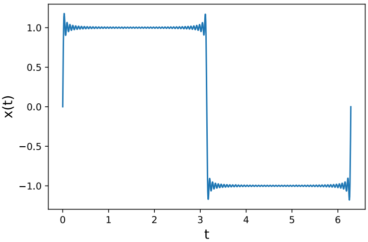
If m is odd, $b_m = \frac{-2}{m\pi}$

Thus, $x(t) = -2\sum_{k=1}^{\infty} b_k \sin(kt)$

$$= \sum_{\substack{k=1 \\ (k \text{ odd})}}^{\infty} \frac{4}{k\pi} \sin(kt)$$

Gibb's phenomenon





The amount of ringing, or "overshoot", is about 9% of the jump of the discontinuity, no matter where we truncate.

Can derive from the *energy* of the error between the original and truncated signals (learn about energy / power of signals in A2).

Properties of Fourier series

In the square wave example, we leveraged some shortcuts to compute the Fourier coefficients.

They have other useful properties that help evaluate them.

Let's see what happens to the Fourier series when we apply:

- Superposition
- Time shift / scale / reversal
- Multiplication

Fourier coefficients combine linearly.

Suppose we have two signals $x(t), y(t)$ with period T,

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}, \\
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t},
\end{aligned}
$$

Then the signal $z(t) = Ax(t) + By(t)$ has the form

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = Aa_k + Bb_k$$

$$x(t) = \sum c_k e^{jk\omega t}$$

Time shift $x(t) \to x(t - t_0)$:

$$x(t - t_0) = \sum_k c_k e^{jk\omega(t - t_0)}$$

$$= \sum c_k e^{jk\omega t} e^{-jk\omega t_0}$$

$$= \sum \underbrace{e^{-jk\omega t_0} c_k}_{} \cdot e^{jk\omega t}$$

$$c_k' = e^{-jk\omega t_0} c_k$$

$x(t) \to x(2t)$

Time scale $x(t) \to x(\alpha t)$.

$T \qquad \frac{T}{2}$

If original period was T, new period is T/α:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$T \to \frac{T}{\alpha}$$

$$\omega \to \omega\alpha$$

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega\alpha t}$$

$$a_{-2}e^{-2j\omega t} \cdot b_3 e^{3j\omega t} = a_2 b_3 e^{j\omega t}$$

Multiplication leads to convolution of the coefficients:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t},$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t},$$

Then the signal $z(t) = x(t)y(t)$ has the form

$$x(t)y(t) = \left[\cdots + a_{-k}e^{-jk\omega t} + \cdots + a_{-1}e^{-j\omega t} + a_0 + a_1 e^{j\omega t} + \cdots\right] \sum_{l=-\infty}^{\infty} b_l e^{jl\omega t}$$

$$= \left[+ a_{-2}b_3 + a_{-1}b_2 + \cdots\right] e^{j\omega t} + \left[a_{-2}b_4 + a_{-1}b_5 + \cdots\right] e^{2j\omega t} + \cdots$$

$$= \sum c_k e^{jk\omega t}, \quad c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

✓ convolution!

Properties of Fourier series

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Exercise

Go back to the square wave

$$x(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi \end{cases}$$

We obtained

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(kt), \quad b_k = \begin{cases} 0, & k \text{ is even} \\ 4/k\pi, & k \text{ is odd} \end{cases}$$

What are the Fourier coefficients of the square wave

$$x(t) = \begin{cases} 1, & -\frac{\pi}{4} < t < \frac{\pi}{4}, \\ -1, & \frac{\pi}{4} < t < \frac{3\pi}{4} \end{cases}$$

Exercise

Try it yourself!

Step 1: express the b_k as the "original" coefficients c_k

Exercise

Step 2: apply the transformations

Learning outcomes:

- Identify whether a signal can be expressed as a Fourier series
- Describe the Gibbs phenomenon
- State the key properties of Fourier series
- Compute the fundamental period and frequency of a DT signal

For next time

Content:

- DT Fourier series coefficients
- Using the frequency response to design filter systems

Action items:

1. Tutorial Assignment due Monday at 23:59
2. Assignment 2 is due next Saturday at 23:59

Recommended reading:

- From today's class: Oppenheim 3.3-3.6
- Suggested problems: 3.2, 3.5, 3.8, 3.10-3.13, 3.14, 3.17, 3.22a,c, 3.23-3.26
- From today's class: Oppenheim 3.6-3.8