ELEC 221 Lecture 13
Differentiation and integration properties; systems based on differential equations

Tuesday 22 October 2024

- Quiz 6 today
- TA3 due Monday 23:59; A3 available soon
- Exam time announced: Sunday 15 December, 7pm

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Last time

We saw some important properties of the Fourier transform:

- Linearity
- Behaviour under time shift/scale/reverse/conjugation

The most important was for **convolution**:

$$y(t) = h(t) * x(t) \qquad Y(j\omega) = H(j\omega)X(j\omega)$$

This made it easier to analyze LTI systems!
In assignment 3, you will explore the related relationship for **multiplication**:

$$z(t) = y(t)\,x(t) \qquad Z(j\omega) = \frac{1}{2\pi} Y(j\omega) * X(j\omega)$$

Today

Learning outcomes:

- Express a Fourier transform using the magnitude-phase representation
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

The magnitude-phase representation $z = a + bj \quad z = re^{j\theta}$

Since Fourier spectra are complex we can express them in terms of their magnitude and phase.

$$X(j\omega) = |X(j\omega)|\, e^{j \sphericalangle X(j\omega)}$$

Recall the convolution property:

$$H(j\omega) = |H(j\omega)| \cdot e^{j \sphericalangle H(j\omega)}$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

Exercise: How does a system $H(j\omega)$ affect $|X(j\omega)|$ and $\sphericalangle X(j\omega)$?

$$|Y(j\omega)| = |X(j\omega)| \cdot |H(j\omega)| \rightarrow \text{gain}$$

$$\sphericalangle Y(j\omega) = \sphericalangle X(j\omega) + \underbrace{\sphericalangle H(j\omega)}_{\text{phase shift}}$$

Example: Lowpass filters in practice

The magnitude-phase representation can help us both visualize and characterize the behaviour of systems.

Example: a resistor combined with a capacitor creates an LTI system that behaves as a lowpass filter.

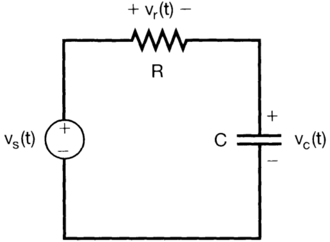
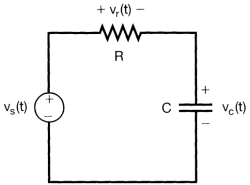


Image credit: Oppenheim chapter 3.10.

Example: lowpass filters in practice

What is the voltage across the capacitor if $v_s = e^{j\omega t}$?



Derive two expressions for current, using resistor and capacitor:

$$i(t) = \frac{V_s(t) - V_c(t)}{R}$$

$$i(t) = C \cdot \frac{dv_c(t)}{dt}$$

Image credit: Oppenheim chapter 3.10.

 $RC \cdot \frac{dy(t)}{dt} + y(t) = x(t)$

Put these together to form a differential equation:

$$C \frac{dv_c(t)}{dt} = \frac{v_s(t) - v_c(t)}{R}$$

$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t) = e^{j\omega t}$$

$$RC \cdot j\omega \cdot H(j\omega) \cdot e^{j\omega t} + H(j\omega) \cdot e^{j\omega t} = e^{j\omega t}$$

$$RC j\omega \cdot H(j\omega) + H(j\omega) = 1$$

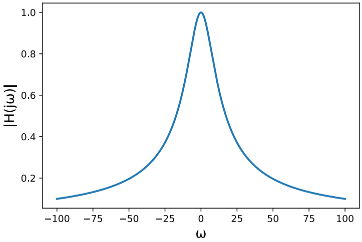
$$[RCj\omega + 1] H(j\omega) = 1$$

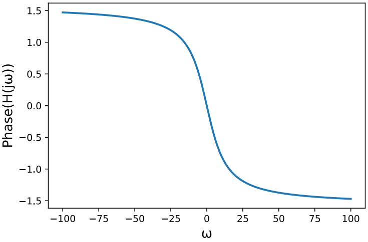
$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$|H(j\omega)|$

$\measuredangle H(j\omega)$

Results in the following frequency response (setting $RC = 0.1$):





Adjusting RC controls the frequency response; increasing RC cuts off more frequencies.

Consider the inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

What happens when we differentiate $x(t)$?

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \underbrace{j\omega e^{j\omega t}} d\omega$$

This means:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

What should happen here?

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{F}} ??$$

Good initial guess:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega)$$

More precisely:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\, \delta(\omega)$$

Exercise: what are the Fourier transforms of the unit impulse and unit step?

$$\delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

$$\Delta(j\omega) = \int_{-\infty}^{\infty} \delta(\tau) e^{-j\omega\tau} d\tau = 1$$

$$U(j\omega) = \frac{1}{j\omega}\Delta(j\omega) + \pi \cdot \Delta(0)\,\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Example: Fourier transform properties and differentiation

We can take advantage of differentiation and integration properties to simplify computations.

Suppose

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

What is the Fourier transform of

$$z(t) = \frac{d^2}{dt^2} x(t-1)$$

Two properties to take advantage of here:

Example: Fourier transform properties and differentiation

$$z(t) = \frac{d^2}{dt^2} x(t-1)$$

First, consider: $p(t) = x(t-1)$:

Try it yourself!

The RC circuit from earlier was based on a differential equation:

$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

We found its frequency response by:

- Choosing input signal $e^{j\omega t}$
- Since system is LTI, assuming output of the form $H(j\omega)e^{j\omega t}$
- Plugging this into the ODE and solving for $H(j\omega)$

There is a better way to do this!

Consider a general system described by an ODE of arbitrary order:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

What is its frequency response $H(j\omega)$?

$$\mathcal{F}\left(\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k}\right) = \mathcal{F}\left(\sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}\right)$$

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

$$\sum_{k=0}^{N} \alpha_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} \beta_k (j\omega)^k X(j\omega)$$

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

$$\sum_{k=0}^{N} \alpha_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} \beta_k (j\omega)^k X(j\omega)$$

$$Y(j\omega)\left[\sum_{k=0}^{N} \alpha_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} \beta_k (j\omega)^k\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$= \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k} \qquad \frac{X \text{ coeffs}}{Y \text{ coeffs}}$$

$$Y(j\omega)\sum_{k=0}^{N}\alpha_k(j\omega)^k = X(j\omega)\sum_{k=0}^{M}\beta_k(j\omega)^k$$

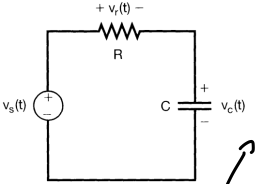
Final property:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M}\beta_k(j\omega)^k}{\sum_{k=0}^{N}\alpha_k(j\omega)^k}$$

This representation allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

Exercise: frequency response of systems described by ODEs

What are the **impulse response** and **frequency response** of our RC circuit filter?



$$\frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

$y(t)$ $x(t)$

$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

What are the **impulse response** and **frequency response** of the system described by

$$\frac{d^3y(t)}{dt^3} - 4\frac{dy(t)}{dt} = 3\frac{d^2x(t)}{dt^2} + x(t)$$

Start with frequency response:

$$H(jw) = \frac{3(jw)^2 + 1}{(jw)^3 - 4jw}$$

We can now leverage this to determine the impulse response:

$$H(j\omega) = \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega}$$

Use partial fractions:

$$H(jw) = \frac{3(jw)^2 + 1}{(jw)\left((jw)^2 - 4\right)}$$

$$= \frac{3(jw)^2 + 1}{jw(jw-2)(jw+2)}$$

$$= \frac{A}{jw} + \frac{B}{jw-2} + \frac{C}{jw+2} \Rightarrow \text{get } h(t)$$

Details are left as an exercise:

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

Example: frequency response of systems described by ODEs

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega+2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega-2}\right)$$

Check Table 4.2 - two expressions to leverage:

Try yourself - I'll share the solution on Thursday!

So we have:

For next time

Content:

- Systems described by first- and second-order ODEs
- Step response
- Bode plots

Action items:

1. Tutorial assignment 3

Recommended reading:

- For today's class: Oppenheim 4.3, 4.6-4.7, 6.1-6.2
- Suggested problems: 4.5, 4.8, 4.22, 4.25, 4.29, 4.33-4.36
- For next class: Oppenheim 4.7, 6.3-6.5