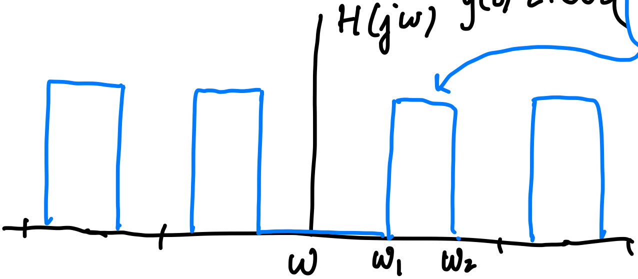
ELEC 221 Lecture 08
Introducing the Fourier transform

Tuesday 04 October 2022

- Assignment 3 due Friday
- Assignment 1 solutions available on PrairieLearn
- Assignment 4 available later this week (due after midterm)
- Quiz 4 today (beginning of class)

$y(t) = 2 + \cos\left(\frac{3\pi}{8}t - \frac{\pi}{3}\right)$



Today

Learning outcomes:

- Explain the concept of CT Fourier transform, and distinguish it from the CT Fourier series
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

Recap: Fourier series

So far, we have been working with the Fourier series representation of **periodic** CT and DT signals:

CT synthesis equation:

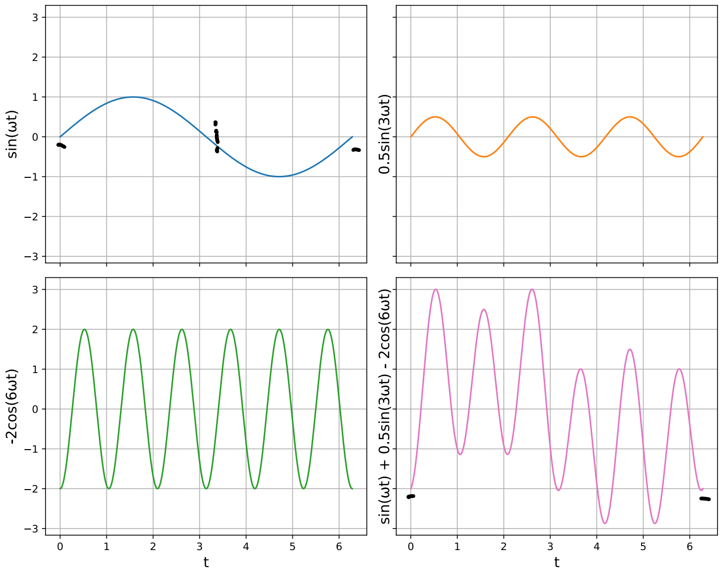
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

CT analysis equation:

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t} \, dt$$

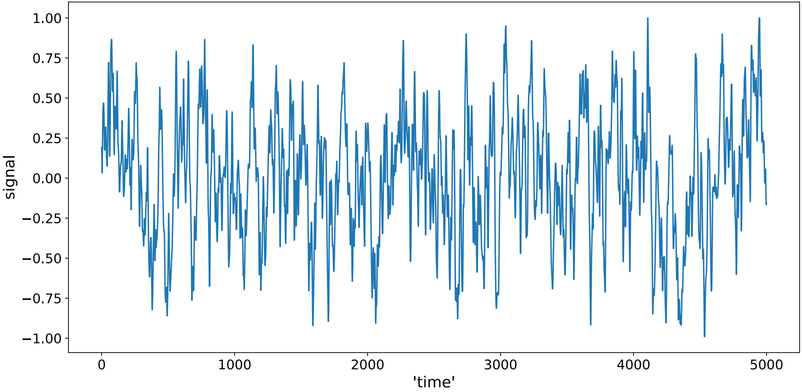
When the signal is periodic it can be represented using only the integer harmonics at the *same frequency* ω.

Recap: Fourier series



Towards the Fourier transform

On Thursday, we were working with audio signals:



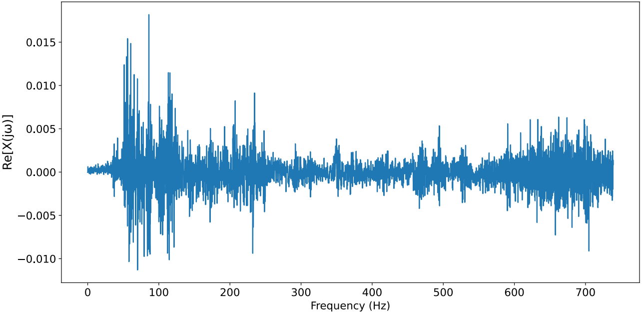
This is *not* a periodic signal.

Towards the Fourier transform

But, we were still doing *something* with Fourier analysis to it:

```
fourier_coefficients = np.fft.rfft(audio)

frequencies = np.fft.rfftfreq(
    len(audio), 1 / sample_rate
)
```



The **Fourier transform** extends our Fourier series methods to **aperiodic signals**. It involves a **spectrum** of different frequencies.

Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega t}\, dt$$

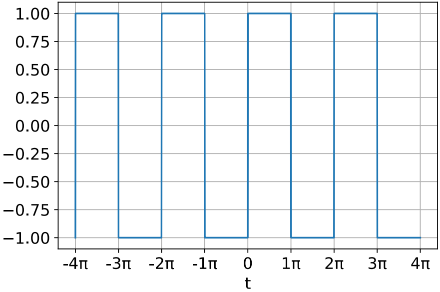
Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

How do we get here?

Towards the Fourier transform

Remember in lecture 4, we looked at a 2π-periodic square wave:

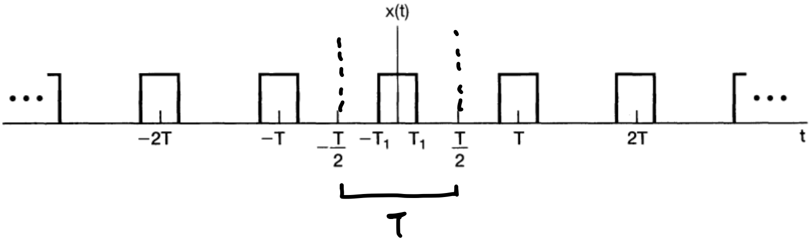


We derived its Fourier series representation

$$x(t) = \sum_{k=1}^{\infty} \frac{4}{k\pi} \sin(kt) \quad \text{only odd } k$$

Towards the Fourier transform

Let's generalize this a bit. Consider the following square wave:



$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

Image credit: Oppenheim chapter 4.1

Towards the Fourier transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

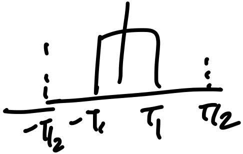
Let's compute its Fourier coefficients.

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t}$$

Start with c_0:

$$c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



Now the c_k:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega t}\, dt$$

$$= -\frac{1}{jk\omega T}\, e^{-jk\omega t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{jk\omega t}\boxed{\left(e^{-jk\omega T_1} - e^{jk\omega T_1}\right)}$$

$$= \frac{2\sin(k\omega T_1)}{k\omega T}$$

$$\sum_k c_k e^{jk\omega t}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2j}$$

Towards the Fourier transform

What does this function look like?

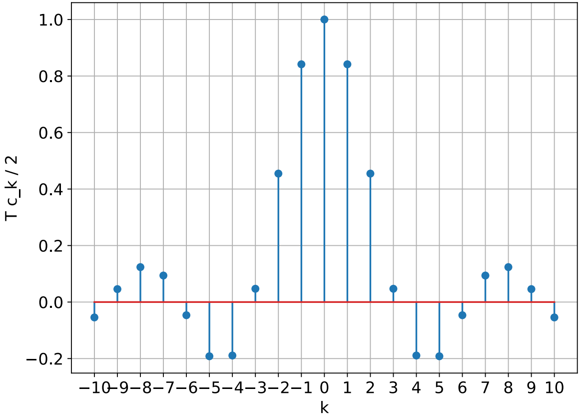
$$c_0 = \frac{2T_1}{T}, \quad c_k = \frac{2\sin(k\omega T_1)}{k\omega T}$$

Let's rearrange a bit:

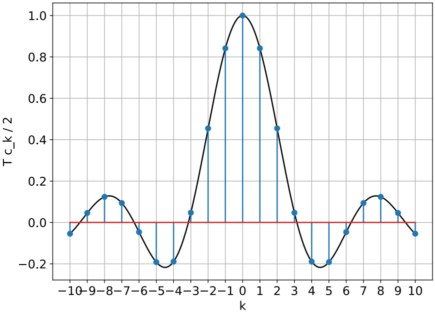
$$c_0 = \frac{2}{T}T_1 \quad c_k = \frac{2}{T}\underbrace{\frac{\sin(k\omega T_1)}{k\omega}}$$

Let's plot the "important part" for different values of k.

Towards the Fourier transform



(Set $T_1 = \omega = 1$ to plot)



These are **samples** of the function

$$f(k) = \begin{cases} 1 & k = 0 \\ \dfrac{\sin(k\omega T_1)}{k\omega} & k \neq 0 \end{cases}$$

at *integer values of k*.

$$f(k) = \begin{cases} 1, & k = 0 \\ \frac{\sin(k\omega T_1)}{k\omega}, & k \neq 0 \end{cases}$$

Let's consider this differently, i.e., as a function of $\tilde{\omega}$:

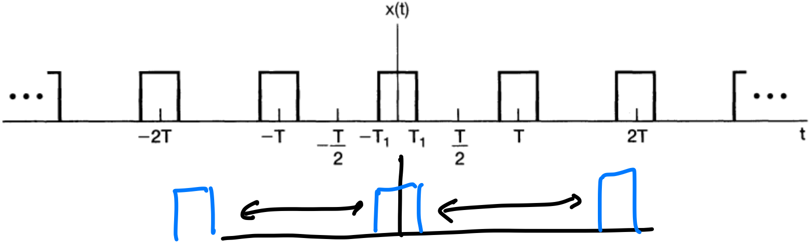
$$f(\tilde{\omega}) = \begin{cases} 1 & \tilde{\omega} = 0 \\ \frac{\sin(\tilde{\omega} T_1)}{\tilde{\omega}}, & \tilde{\omega} \neq 0 \end{cases}$$

The Fourier coefficients are samples of this function taken at $\underbrace{\textit{integer multiples } k\omega,}$ where $\omega = 2\pi/T$.

$$c_k = \frac{2}{T} f(k\omega)$$

$T = \frac{2\pi}{\omega}$

Suppose T grows (but T_1 stays the same)?



What happens to our samples from this function?

$$c_k \sim \frac{\sin(k\omega T_1)}{k\omega}$$

Image credit: Oppenheim chapter 4.1

Initially, we have some spacing of samples at integer values of $\omega = 2\pi/T$.

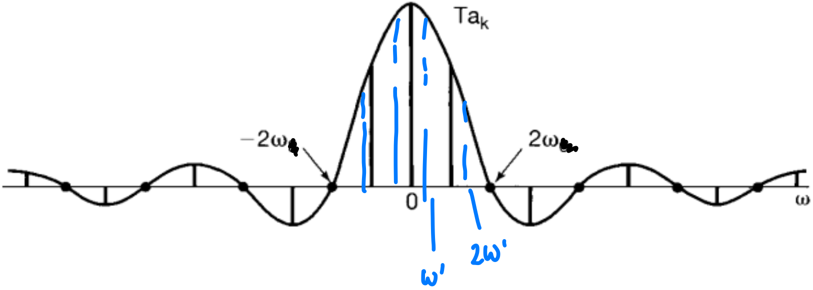


Image credit: Oppenheim chapter 4.1

As T grows, $\omega = 2\pi/T$ becomes smaller and smaller, so the integer multiples of it get closer and closer together.

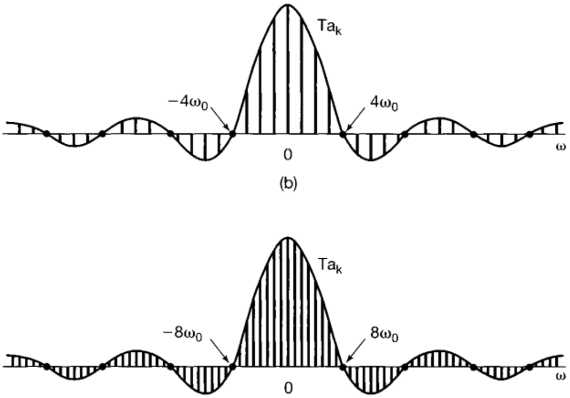


Image credit: Oppenheim chapter 4.1

Eventually, ω becomes so small that instead of

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

ω gets smaller as T increases

we may as well just consider the sum over integer multiples as a continuous integral over all possible ω:

$$x(t) \sim \int_{-\infty}^{\infty} c_k e^{j\omega t} d\omega$$

?

...but what does this have to do with non-periodic signals?

Towards the Fourier transform

Given any aperiodic signal $x(t)$, we can always "pretend" it's periodic by constructing a **periodic extension**, $\tilde{x}(t)$ with period T.

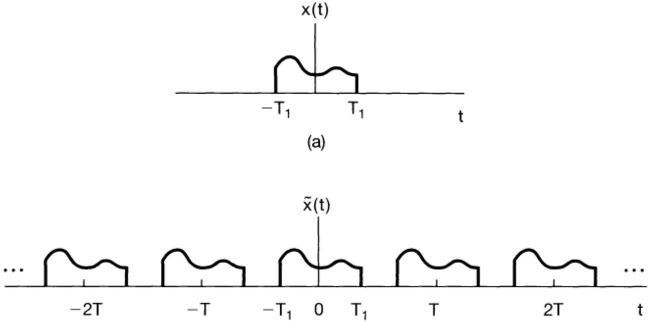


Image credit: Oppenheim chapter 4.1

$x(t) \longrightarrow \tilde{x}(t)$ periodic extension w/period T

Now that we made $\tilde{x}(t)$ look periodic, we can write it as a Fourier series (where $\omega = 2\pi/T$):

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$c_k = \frac{1}{T}\int_T \tilde{x}(t) e^{-jk\omega t}\, dt$$

$$c_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega t}dt$$

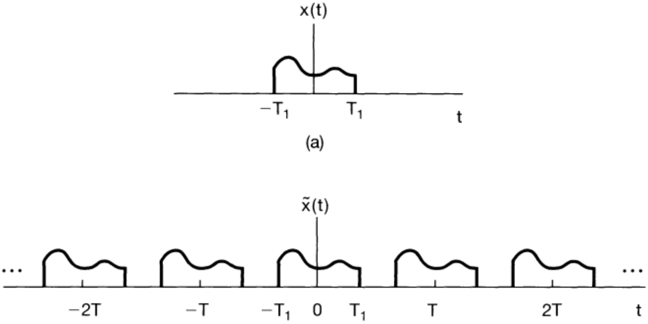
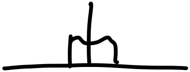
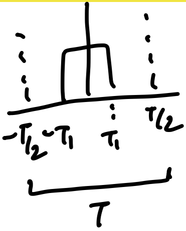


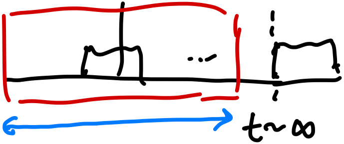
Image credit: Oppenheim chapter 4.1

What happens to the coefficients?





$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega t}\, dt$$



$t \sim \infty$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega t}\, dt}$$

Let's define

$$\underline{X(j\omega)} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

so that

$$c_k = \frac{1}{T} X(jk\omega)$$

Motivation: Fourier transform

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

We can put this back in our Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega) e^{jk\omega t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega$$

$T = \frac{2\pi}{\omega}$

Now consider what happens when $T \to \infty$...

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega)\, e^{jk\omega t}\, \omega$$

$x(t)$ $d\omega$

Two important things:

1. $\tilde{x}(t)$ will look just like $x(t)$ for large enough T
2. ω will get smaller and smaller

$$\lim_{T\to\infty} \tilde{x}(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}\,dw$$

$$\begin{aligned}\lim_{T\to\infty} \tilde{x}(t) = x(t) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega)e^{jk\omega t}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega\end{aligned}$$

This is the **Fourier transform**.

inverse

The Fourier transform

Inverse Fourier transform (synthesis equation):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

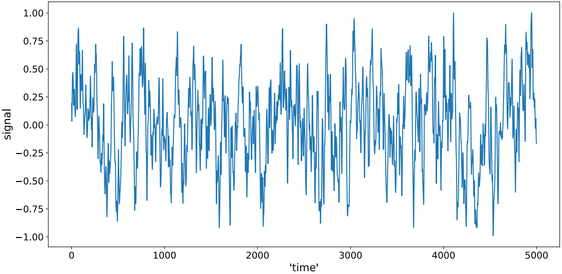
Fourier transform (analysis equation, or Fourier *spectrum*):

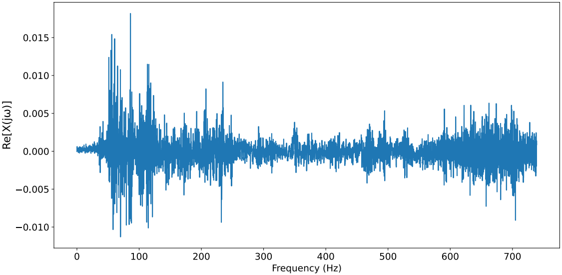
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Note: Sometimes the $1/2\pi$ prefactor appears on the spectrum, or sometimes both versions have $1/\sqrt{2\pi}$.

The Fourier transform

On Thursday, what we saw was a discretized version of this:

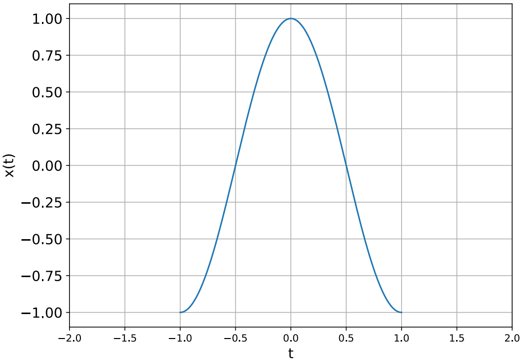




Example

Compute the Fourier spectrum of:

$$x(t) = \begin{cases} \cos(\pi t) & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$



$$x(t) = \begin{cases} \cos(\pi t), & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$

Start from the definition:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-1}^{1} \cos(\pi t) e^{-j\omega t} dt$$

$$\cos(\pi t) = \frac{e^{j\pi t} + e^{-j\pi t}}{2}$$

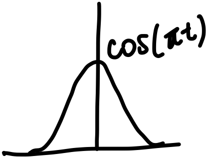
$$= \frac{1}{2}\int_{-1}^{1} e^{j(\pi-\omega)t} dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\pi+\omega)t} dt$$

Example

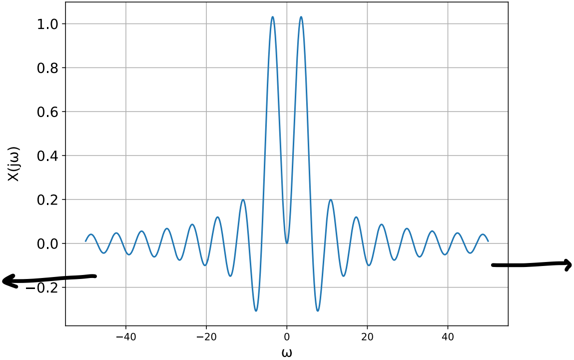
$$
\begin{aligned}
X(j\omega) &= \frac{1}{2}\int_{-1}^{1} e^{j(\pi-\omega)t}dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\pi+\omega)t}dt \\
&= \frac{1}{2}\frac{1}{j(\pi-\omega)}e^{j(\pi-\omega)t}\big|_{-1}^{1} + \frac{1}{2}\frac{-1}{j(\pi+\omega)}e^{-j(\pi+\omega)t}\big|_{-1}^{1} \\
&= \frac{1}{2j(\pi-\omega)}\left(e^{j(\pi-\omega)} - e^{-j(\pi-\omega)}\right) \\
&\quad - \frac{1}{2j(\pi+\omega)}\left(e^{-j(\pi+\omega)} - e^{j(\pi+\omega)}\right) \\
&= \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega}
\end{aligned}
$$

$$= \frac{\sin(\omega)}{\pi-\omega} - \frac{\sin(\omega)}{\pi+\omega}$$

Example



$$X(j\omega) = \frac{\sin(\omega)}{\pi - \omega} - \frac{\sin(\omega)}{\pi + \omega}$$



Fourier transform and impulse response

You've actually already (unknowingly) seen the Fourier transform when we discussed system functions and frequency response.

Recall the convolution integral representation of signals as a set of shifted, weighted impulses:

$X(j\omega)$

$H(j\omega)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

Put this in an LTI system with impulse response $h(t)$:

$$x(t) \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$x(t) \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

We found that, when the signal in question is a complex exponential, that

$$x(t) = e^{j\omega t} \to y(t) = \int_{-\infty}^{\infty} e^{j\omega \tau} h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} h(\tau)d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)d\tau$$

$$= x(t) \cdot H(j\omega)$$

Fourier transform and impulse response

The system function $H(j\omega)$, or frequency response

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau) \, d\tau$$

is the **Fourier transform of the impulse response**!

We can use the inverse Fourier transform to obtain the impulse response from the frequency response:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} H(j\omega) \, d\omega$$

The same thing works in discrete time:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{-j\omega n} h[n]$$

The impulse response can be obtained by computing the inverse discrete Fourier transform (recall we have only $\omega \in [0, 2\pi)$):

$$h[n] = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

We will cover the DTFT in detail next week / after the midterm; but this should help you solve some A3 problems.

Recap

Today's learning outcomes were:

- Explain the concept of CT Fourier transform, and distinguish it from the CT Fourier series
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

What topics did you find unclear today?

For next time

Content:

- Properties of the CT Fourier *transform*
- Convolution properties of the Fourier transform and time/frequency duality

Action items:

1. Assignment 3 is due Friday
2. Assignment 4 released later this week
3. Midterm 1 next Thursday

Recommended reading:

- From today's class: Oppenheim 4.0-4.1
- For next class: Oppenheim 4.2-4.4