ELEC 221 Lecture 15
Time and frequency domain analysis I

Thursday 27 October 2022

Announcements

- Midterms available for pickup after class (or at my office)
- Assignment 4 due on Saturday at 23:59
- (Bonus) Assignment 4.5 due on Saturday at 23:59
- Quiz 7 on Tuesday (will focus on today's content)

Previously

Complex exponential signals are eigenfunctions of LTI systems in both continuous time and discrete time.

If $x(t) = e^{st}$, for complex s

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{st}e^{-s\tau}h(\tau)d\tau \\
&= e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau \\
&= e^{st}H(s)
\end{aligned}$$

Previously

We have considered so far only $s = j\omega$

$$H(s) \rightarrow H(j\omega)$$

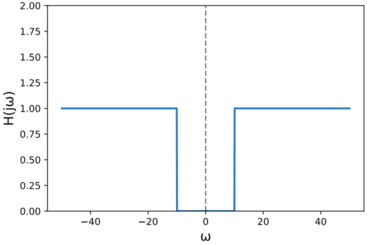
This is the **frequency response** of the system.

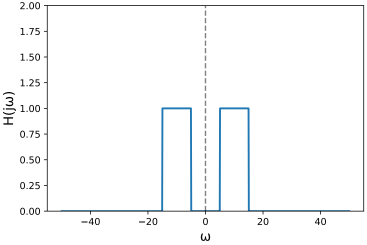
When we input a linear combination of signals,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \quad \rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$$

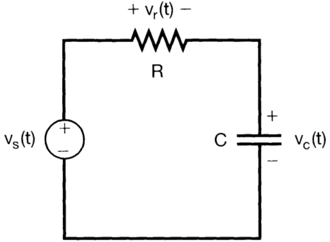
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \rightarrow \quad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(j\omega) e^{j\omega t}$$

We have seen some simple frequency response of ideal filters:





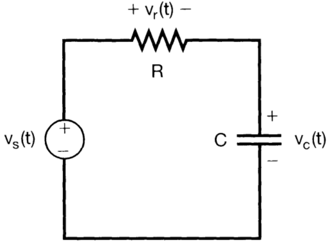
We have also seen more realistic ones.



$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

If $v_s(t) = e^{j\omega t}$, then a solution is $v_c(t) = H(j\omega)e^{j\omega t}$ for some scaling $H(j\omega)$.

Previously



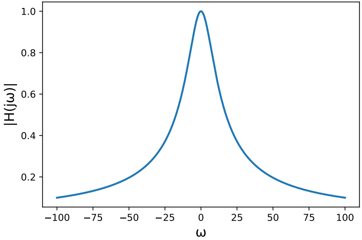
We found that

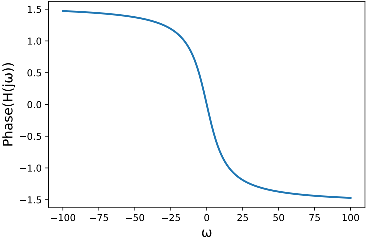
$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

If we look at $H(j\omega)$ we can see that this is also a filter. The frequencies it attenuates depends on R and C.

Previously

In general $H(j\omega)$ has both a magnitude and a phase component.

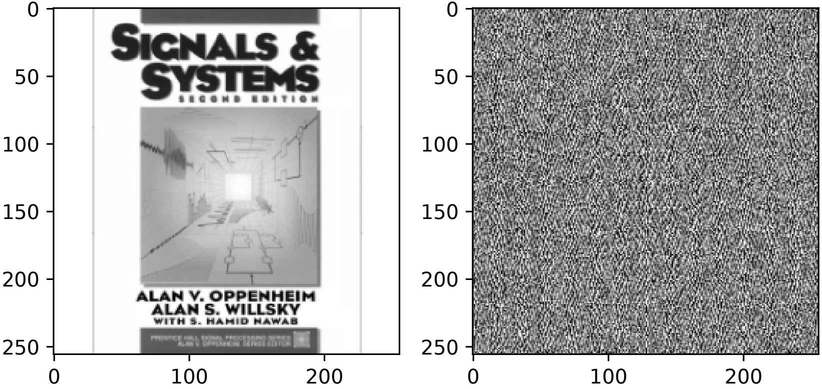




Increasing RC cuts off more frequencies, but there are design tradeoffs involved.

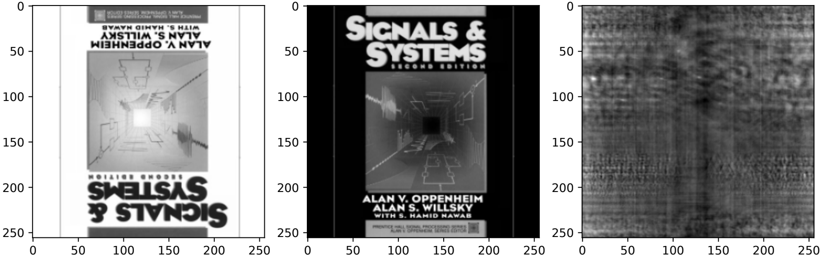
We haven't looked much at the phase response...

You hopefully learned from the hands-on that phase can be important!



Previously

Really important...



So we should probably consider this more in our analysis of systems.

Today

Learning outcomes:

- express a frequency response in the magnitude-phase representation
- differentiate between linear and non-linear phase responses
- compute the group delay of a frequency response
- plot the frequency response using a Bode plot

The magnitude-phase representation

Since Fourier spectra are complex numbers, we can express them in terms of their magnitude and phase.

$$\begin{aligned} X(j\omega) &= |X(j\omega)|e^{j\sphericalangle X(j\omega)} \\ X(e^{j\omega}) &= |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})} \end{aligned}$$

Recall the convolution property of the Fourier transform:

$$
\begin{aligned}
Y(j\omega) &= H(j\omega)X(j\omega) \\
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega})
\end{aligned}
$$

What happens to the output?

How does passing through the system with $H(j\omega)$ affect $|X(j\omega)|$ and $\sphericalangle X(j\omega)$?

Try it yourself. Given

$$\begin{aligned} X(j\omega) &= |X(j\omega)|e^{j\sphericalangle X(j\omega)} \\ Y(j\omega) &= H(j\omega)X(j\omega) \end{aligned}$$

Determine

$$\begin{aligned} |Y(j\omega)| &= \\ \sphericalangle Y(j\omega) &= \end{aligned}$$

$$\begin{aligned} |Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\ \sphericalangle Y(j\omega) &= \sphericalangle H(j\omega) + \sphericalangle X(j\omega) \end{aligned}$$

We give these names:

- $|H(j\omega)|$ is the gain
- $\sphericalangle H(j\omega)$ is the phase shift

Depending on what these are, the result can be either good, or bad (*distortion*).

Linear frequency response

If $|H(j\omega)| = 1$ everywhere, the system is called *all-pass* and is characterized by its phase response.

It is nicest when the phase shift is a *linear* function of the frequency.

$$\sphericalangle H(j\omega) = \omega t + b$$

Can you think of a system that causes a linear shift in phase? (Hint: think back to properties of Fourier transform)

Time shift (or, a delay system):

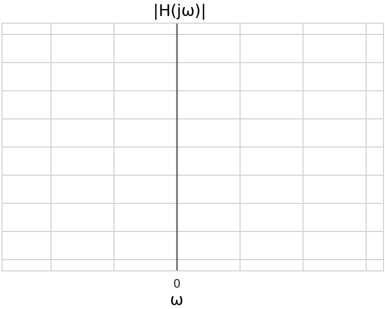
$$y(t) = x(t - t_0)$$

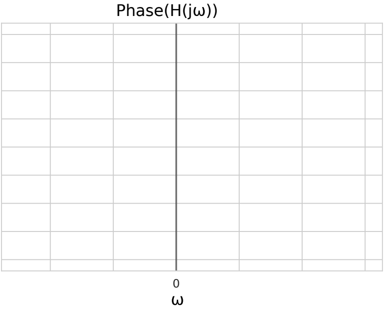
$$Y(j\omega) = e^{-j\omega t_0} X(j\omega)$$

$$\begin{aligned} |Y(j\omega)| &= |X(j\omega)| \\ \sphericalangle Y(j\omega) &= -\omega t_0 + \sphericalangle X(j\omega) \end{aligned}$$

Let's consider the ideal lowpass filter,

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$





What is its impulse response? (inverse Fourier transform of frequency response)

$$
\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\frac{1}{jt}e^{j\omega t}\Big|_{-\omega_c}^{\omega_c} \\
&= \frac{1}{2\pi}\frac{1}{jt}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right) \\
&= \frac{1}{2\pi}\frac{1}{jt}2j\sin(\omega_c t) \\
&= \frac{\sin(\omega_c t)}{\pi t}
\end{aligned}
$$

Example: linear frequency response

Recall what this looks like graphically:

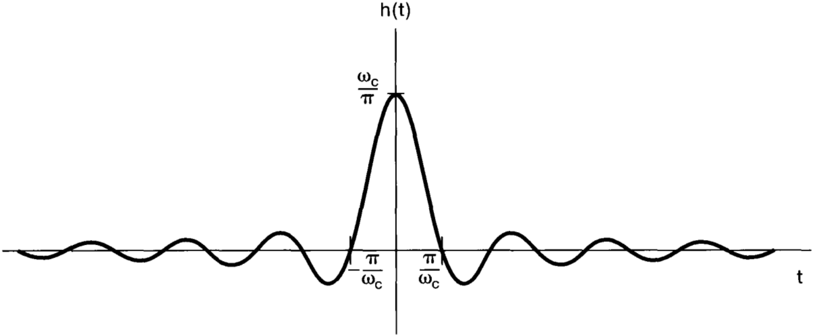
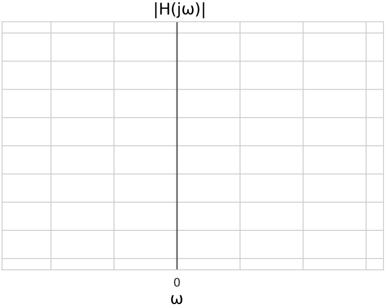
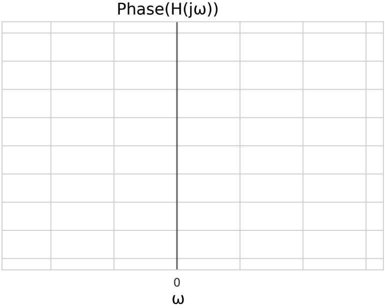


Image credit: Oppenheim 6.3

What happens if we add a linear phase?

$$H(j\omega) = \begin{cases} e^{-\alpha\omega}, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$





Example: linear frequency response

$$
\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\frac{1}{j(t-\alpha)}e^{j\omega(t-\alpha)}\Big|_{-\omega_c}^{\omega_c} \\
&= \frac{1}{2\pi}\frac{1}{j(t-\alpha)}\left(e^{j\omega_c(t-\alpha)} - e^{-j\omega_c(t-\alpha)}\right) \\
&= \frac{1}{2\pi}\frac{1}{j(t-\alpha)}2j\sin(\omega_c(t-\alpha)) \\
&= \frac{\sin(\omega_c(t-\alpha))}{\pi(t-\alpha)}
\end{aligned}
$$

Example: linear frequency response

The result is a shifted version of the original impulse response

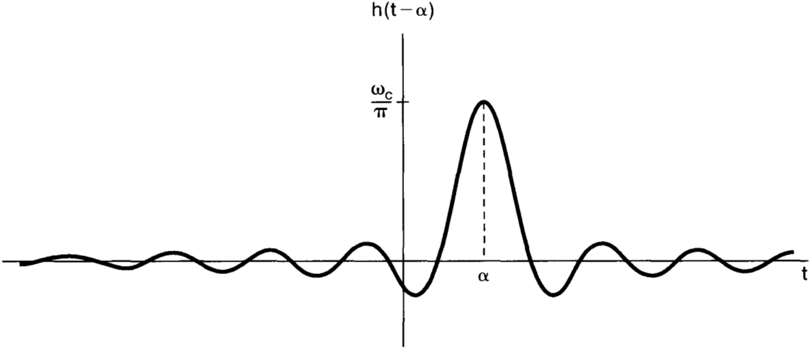


Image credit: Oppenheim 6.3

Consider the following frequency response:

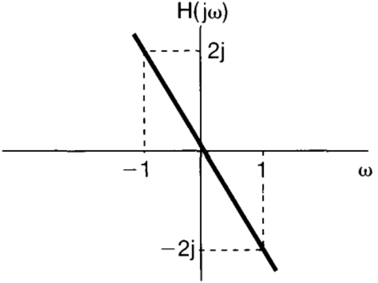


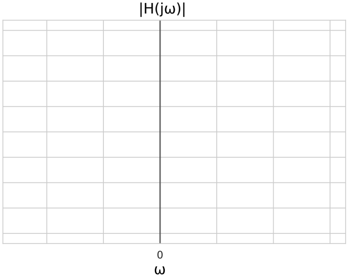
Figure P6.21

- What is $H(j\omega)$?
- Sketch $|H(j\omega)|$ and $\sphericalangle H(j\omega)$

Image credit: Oppenheim Problem 6.21

Exercise: linear frequency response

$$
\begin{aligned}
H(j\omega) &= \\
|H(j\omega)| &= \\
\sphericalangle H(j\omega) &=
\end{aligned}
$$



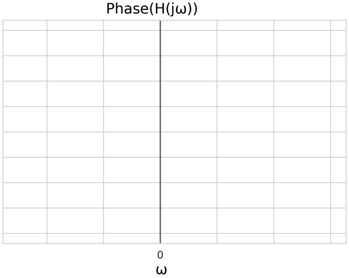


Image credit: Oppenheim Problem 6.21

Exercise: linear frequency response

Suppose a signal $x(t)$ with spectrum $X(j\omega) = \frac{1}{2+j\omega}$ is input into the system.

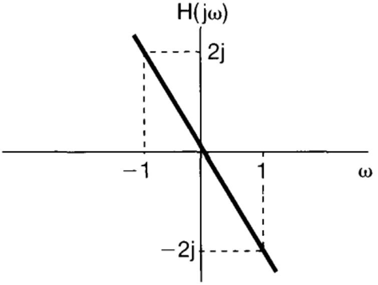


Figure P6.21

What is the output signal $y(t)$? (Hint: recall what happens to a Fourier spectrum of a function when you take its derivative)

Image credit: Oppenheim Problem 6.21

Exercise: linear frequency response

Group delay

Linear phase: same delay at all frequencies (shift the response).

Non-linear phase: different amount of delay at different frequencies

If we look at a small enough band of frequencies, we can make an approximation that it is...

$$\sphericalangle H(j\omega) \approx -\phi - \omega\alpha$$

Then:

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &\approx X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha} \end{aligned}$$

$$Y(j\omega) \approx X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha}$$

The parameter α represents an effective common delay of the frequencies in this small band.

It is called the **group delay**:

$$\begin{aligned} \sphericalangle H(j\omega) &\approx -\phi - \omega\alpha \\ -\frac{d}{d\omega}(\sphericalangle H(j\omega)) &= \alpha = \tau(\omega) \end{aligned}$$

Non-linear phase and group delay has a lot of real-world implications.

Exercise: group delay

Consider a filter with frequency response

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- What are $|H(j\omega)|$ and $\sphericalangle H(j\omega)$
- What is the group delay?

Exercise: group delay

$$H(j\omega) = \frac{1}{1 + j\omega}$$

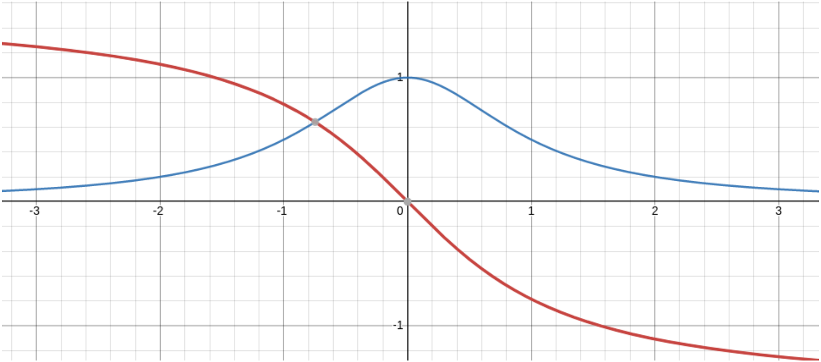
- What are $|H(j\omega)|$ and $\sphericalangle H(j\omega)$
- What is the group delay?

Exercise: group delay

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- What are $|H(j\omega)|$ and $\sphericalangle H(j\omega)$
- What is the group delay?

$$\measuredangle H(j\omega) = \tan^{-1}(-\omega), \quad \tau(\omega) = \frac{1}{1+\omega^2}$$



Recall:

$$\begin{aligned} |Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\ \sphericalangle Y(j\omega) &= \sphericalangle H(j\omega) + \sphericalangle X(j\omega) \end{aligned}$$

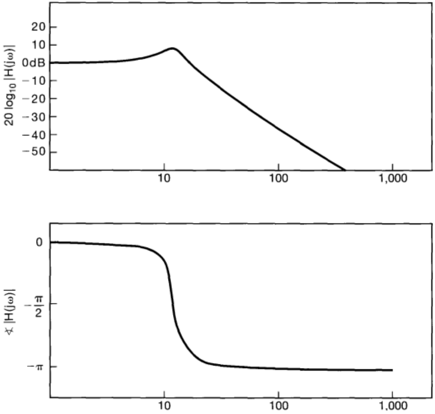
Magnitude is multiplicative and phase is additive... would be nicer if both were additive.

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

Rather than making plots of $|H(j\omega)|$ and $\sphericalangle H(j\omega)$, it is common to make plots of $20\log_{10}|H(j\omega)|$ and $\sphericalangle H(j\omega)$ against $\log_{10}\omega$.

Bode plots

These are called *Bode plots*:



We will see more of these on Tuesday.

Image credit: Oppenheim 6.2

Ideal filter step response

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

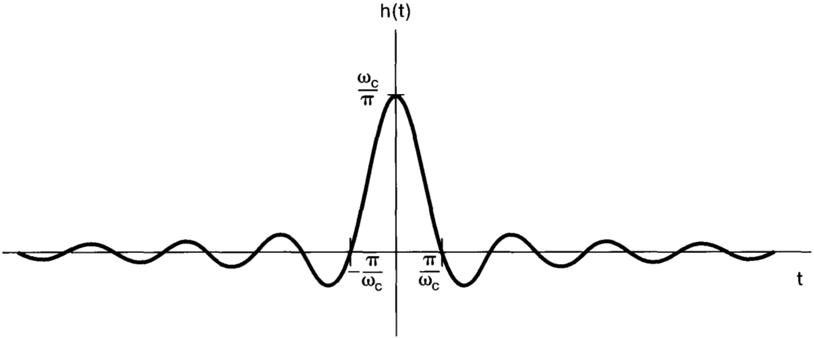


Image credit: Oppenheim 6.3

Ideal filter step response

It is also important to consider *step response* of filters.

Recall that

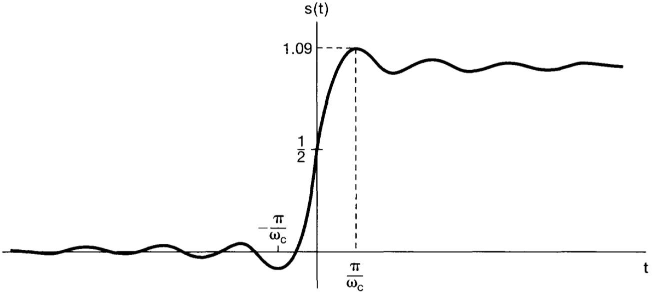
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

By linearity, if we put this in a system, the result is

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

Ideal filter step response

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$



By changing the design of the filters, we can limit the amount of ringing. More on Tuesday!

Today

Learning outcomes:

- express a frequency response in the magnitude-phase representation
- differentiate between linear and non-linear phase responses
- compute the group delay of a frequency response
- plot the frequency response using a Bode plot

Oppenheim practice problems:

- (DT) 6.2, 6.4, 6.37, 6.39 (choose a couple)
- (CT) 6.21a-c, 6.23, 6.27, 6.42

For next time

Content:

- Properties of non-ideal filters
- Filters described by first/second-order difference equations

Action items:

1. Quiz 7 Tuesday
2. Assignment 4 due Saturday 23:59
3. Bonus activity due Saturday 23:59

Recommended reading:

- For next class: Oppenheim 6.4-6.7