ELEC 221 Lecture 14
Analysis of CT systems based on first- and second-order differential equations

Thursday 24 October 2024

Announcements

- Tutorial assignment 3 due Monday 23:59
- Assignment 3 available, due Saturday 2 Nov 23:59
- Monday's tutorial will focus on problem solving
- MT2 information available next week

We expressed systems in the magnitude phase representation

For a signal $x(t)$ and system with frequency response $H(j\omega)$,

$|H(j\omega)|$ is the gain and $\sphericalangle H(j\omega)$ is the phase shift. We plotted these separately.

Last time

We derived the behaviour of the Fourier transform under differentiation and integration:

From these, we determined

Last time

Finally, we combined many properties to write an expression for the frequency responses of systems described by differential equations, What are the **impulse response** and **frequency response** of the system described by

$$\frac{d^3y(t)}{dt^3} - 4\frac{dy(t)}{dt} = 3\frac{d^2x(t)}{dt^2} + x(t)$$

We computed frequency response and began using partial fractions:

$$\begin{aligned} H(j\omega) &= \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega} \\ &= \frac{3(j\omega)^2 + 1}{j\omega((j\omega)^2 - 4)} \\ &= \frac{3(j\omega)^2 + 1}{j\omega(j\omega + 2)(j\omega - 2)} \\ &= \frac{A}{j\omega} + \frac{B}{j\omega + 2} + \frac{C}{j\omega - 2} \end{aligned}$$

Last time: exercise

Details are left as an exercise:

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

Last time: exercise

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega + 2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega - 2}\right)$$

Check Table 4.2 - two expressions to leverage:

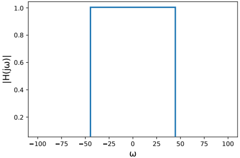
So we have:

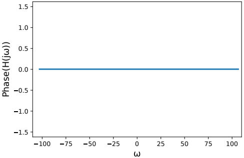
Today

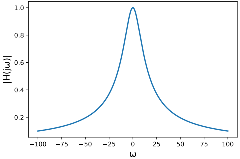
Learning outcomes:

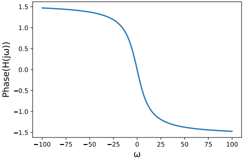
- define and compute the unit step response of a system
- characterize the oscillatory behaviour of CT systems described by second-order differential equations
- read a Bode plot

We've seen two versions of lowpass filters:









What's the difference?

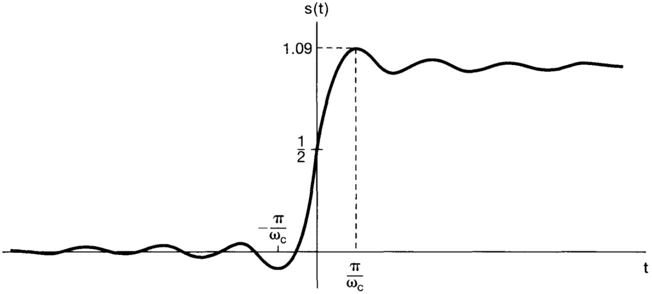
Lowpass filters

Look at their impulse responses:

Step response

It is also important to consider the *step response* of filters:

Ideal filter step response



An ideal filter leads to **ringing** in the step response.

Image credit: Oppenheim 6.3

Ideal filter step response

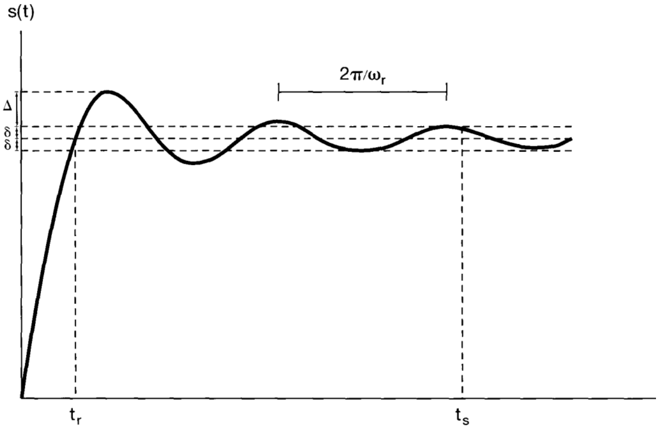


Figure 6.17 Step response of a continuous-time lowpass filter, indicating the rise time t_r, overshoot Δ, ringing frequency ω_r, and settling time t_s—i.e., the time at which the step response settles to within $\pm\delta$ of its final value.

Image credit: Oppenheim 6.4

There are **tradeoffs** in filter design. Compromises in the frequency domain can lead to nicer behaviour in the time domain.

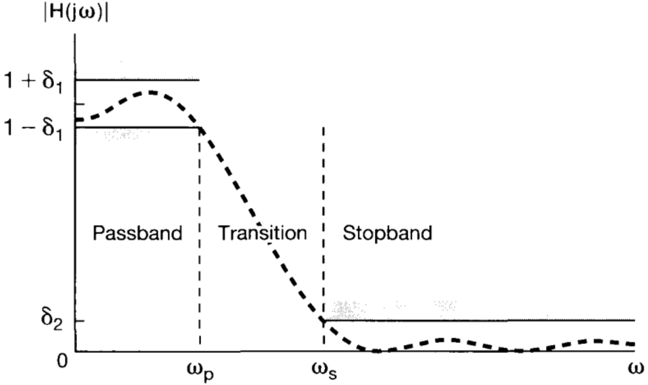
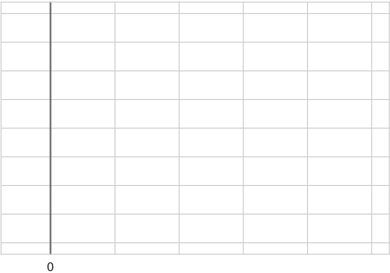


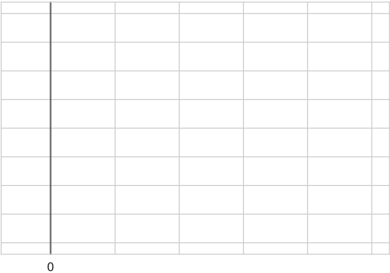
Image credit: Oppenheim 6.4

Exercise: What is the step response of a system described by a first-order ODE?

The impulse and step response of the system are

T is the **time constant** of the system.





Second-order systems

Consider a system described by the ODE

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Exercise: what is the frequency response?

Let's explore this in a little more detail and compute the impulse and step response of this system.

where

Three cases to consider:

- $\zeta = 1$
- $\zeta > 1$
- $\zeta < 1$

Second-order systems

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}, \quad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case: $\zeta = 1$.

Use handy table of Fourier transform pairs to find

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}, \quad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case: $\zeta \neq 1$. Do a partial fraction expansion:

Use table of Fourier transform pairs to find

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(e^{c_+ t} - e^{c_- t}\right) u(t), \qquad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Exercise: derive the step response,

$$s(t) = \left[1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{c_+ t}}{c_+} - \frac{e^{c_- t}}{c_-}\right)\right] u(t)$$

The form of the exponential depends whether $\zeta > 1$ or $\zeta < 1$

- $\zeta < 1$: $c_\pm$ are imaginary; complex exponentials, so the response will oscillate!
- $\zeta > 1$: $c_\pm$ real and negative; decaying exponentials

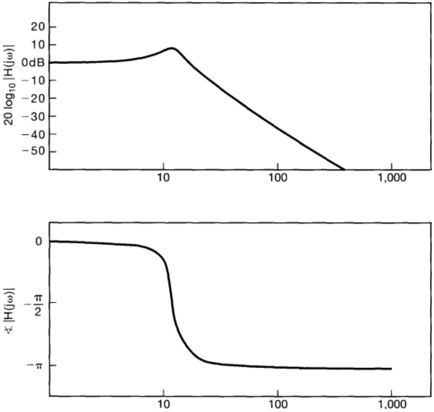
Let's go plot these.

Recall:

Magnitude is multiplicative and phase is additive... would be nicer if both were additive.

Rather than plotting $|H(j\omega)|$ and $\sphericalangle H(j\omega)$, it is common to plot $20 \log_{10} |H(j\omega)|$ and $\sphericalangle H(j\omega)$ against $\log_{10} \omega$.

These are called *Bode plots*:



The logarithmic scale also allows us to view the response over a much wider range of frequencies.

Image credit: Oppenheim 6.2

Bode plots: first-order systems

$$T\frac{dy(t)}{dt} + y(t) = x(t), \qquad H(j\omega) = \frac{1}{1 + j\omega T}$$

Let's view these in the magnitude-phase representation:

From this, we find

We have

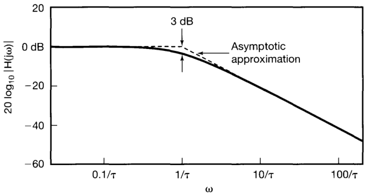
To make our Bode plot, compute

If $\omega << 1/T$,

If $\omega >> 1/T$, $\omega T >> 1$

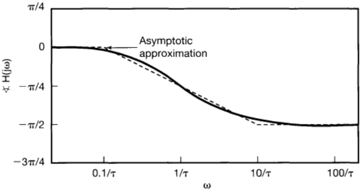
If $\omega = 1/T$,

Bode plots: first-order systems



Can make similar approximations to recover plot of phase

$$H(j\omega) = \tan^{-1}(-\omega T)$$



Let's replot the second-order system functions as Bode plots.

Image credit: Oppenheim 6.5

For next time

Content:

- Discrete-time Fourier transform

Action items:

1. Tutorial Assignment 3 due Monday 23:59
2. Assignment 3 due next Saturday 23:59

Recommended reading:

- From today's class: Oppenheim 6.1-6.5, 6.7
- Suggested problems: 6.1, 6.3a, 6.5, 6.9, 6.15, 6.21-6.23
- For next class: Oppenheim 5.0-5.7