ELEC 221 Lecture 12
The CT Fourier transform properties: convolution and multiplication

Thursday 17 October 2024

Announcements

- Quiz 6 Tuesday
- Please prepare a 4-5 second excerpt of your favourite song (as a `.wav` file) for Monday's tutorial assignment

Last time

We saw the Dirichlet conditions for the Fourier transform.

this condition is correct; it is for aperiodic signals.
for periodic signals, must allow δ in the transform.

If the signal

1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

We computed Fourier transforms of periodic signals.

$$x(t) = e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \cdot \delta(\omega - k\omega_0)$$

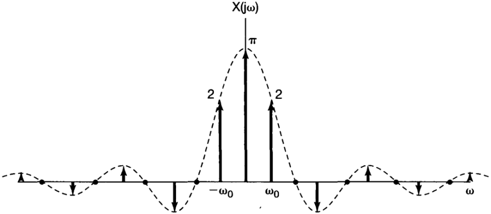


Image credit: Oppenheim chapter 4.2

Last time

Duality: for any transform pair $(x(t) \leftrightarrow X(j\omega))$, there is a *dual pair* with the time and frequency variables interchanged.

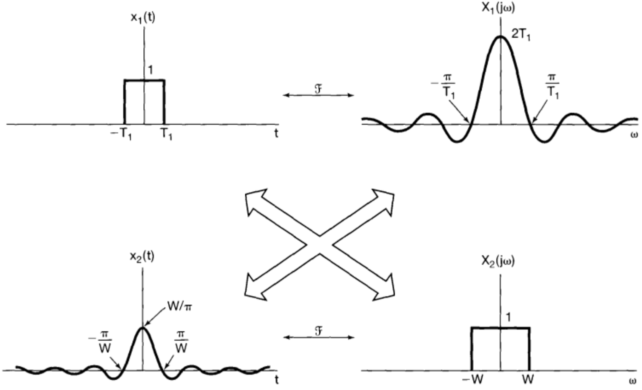


Image credit: Oppenheim chapter 4.3

We will make a big step towards answering the question
"Why are we even doing this?"

Learning outcomes:

- Leverage key properties of Fourier transform to simplify its computation
- Apply the *convolution property* of the Fourier transform to characterize LTI system behaviour
- Describe the *multiplication property* of the Fourier transform and provide applications of its use

Clarification

Last class, I wrote

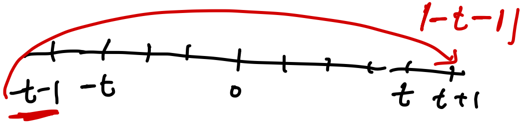
$$e^{-|t|} = e^{-t} u(t) + e^{t} u(-t)$$

and you asked about $u(0)$.

There are different conventions:

- we are treating it as undefined
- sometimes it's defined as 1
- sometimes it's defined as 0
- sometimes it's defined as $1/2$ ("half-maximum convention")

For how we're using it (in integrals), it doesn't matter.



What is the Fourier transform of $x(t) = e^{-2|t-1|}$?

$$x(t) = \begin{cases} e^{-2(t-1)} & t > 1 \\ e^{-2(-t+1)} & t < 1 \end{cases}$$

$$= e^{-2(t-1)} u(t-1) + e^{-2(-t+1)} u(-t+1)$$

Linearity.

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

$$a\,x(t) + b\,y(t) \overset{\mathcal{F}}{\longleftrightarrow} a\,X(j\omega) + b\,Y(j\omega)$$

Our example:

$$\mathcal{F}\left[e^{-2(t-1)} u(t-1) + e^{-2(-t+1)} u(-t+1) \right]$$

$$= \mathcal{F}\left(e^{-2(t-1)} u(t-1)\right) + \mathcal{F}\left(e^{-2(-t+1)} u(-t+1)\right)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Time shifting. If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

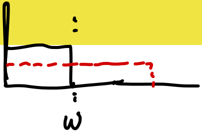
then

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Notice: $|X(j\omega)|$ does not change; we just add a linear phase shift.

Our example: time shift $e^{-2t} u(t) \longrightarrow e^{-2(t-1)} u(t-1)$

$$e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} \Longrightarrow e^{-2(t-1)} u(t-1) \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{\frac{e^{-j\omega}}{2+j\omega}}_{?}$$



Time scaling. If

$$x(t) \overset{F}{\longleftrightarrow} X(jw)$$

then

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{jw}{a}\right)$$

positive a: $w \to a$

Time reversal follows from this:

$$x(-t) \longleftrightarrow X(-jw)$$

Our example: we have

$$\underbrace{e^{-2(t-1)}u(t-1)}_{z(t)} + e^{-2(-t+1)}u(-t+1) = z(t) + z(-t+2)$$

reverse time, + time shift

this is correct: in class, I forgot I skipped a step in my notes. see left.

$$\mathcal{F}\left(e^{-2|t-1|}\right) = \frac{e^{-j\omega}}{2+j\omega} + e^{-2j\omega}\cdot\frac{e^{j\omega}}{2-j\omega}$$

$$= \frac{e^{-j\omega}}{2+j\omega} + \frac{e^{-j\omega}}{2-j\omega}$$

$$= \frac{4e^{-j\omega}}{4+\omega^2}$$

First, shift:

$$z(t) \rightarrow z(t+2) = z(t-\overbrace{(-2)}^{t_0=-2})$$

$$X(j\omega) \rightarrow e^{2j\omega}X(j\omega)$$

Then reverse:

$$e^{2j\omega}X(j\omega) \rightarrow e^{-2j\omega}X(-j\omega)$$

Conjugation. If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

If $x(t)$ is purely real,

$$X(-j\omega) = X^*(j\omega)$$

Implications for even/odd parts of a (real) signal:

$$x(t) = \mathrm{Even}(x(t)) + \mathrm{Odd}(x(t))$$

$$\mathcal{F}(x(t)) = \mathcal{F}(\mathrm{Even}(x(t))) + \mathcal{F}(\mathrm{Odd}(x(t)))$$

$$\mathrm{Even}(x(t)) = \frac{1}{2}(x(t) + x(-t))$$

$$\mathcal{F}(\mathrm{Even}(x(t))) = \frac{1}{2}(X(j\omega) + X(-j\omega)) = \mathrm{Re}(X(j\omega))$$

$$\mathrm{Odd}(x(t)) = \frac{1}{2}(x(t) - x(-t))$$

$$\mathcal{F}(\mathrm{Odd}(x(t))) = \frac{1}{2}(X(j\omega) - X(-j\omega)) = j \cdot \mathrm{Im}(X(j\omega))$$

Recall complex exponentials are eigenfunctions of LTI systems. If we input signal $x(t)$ into LTI system with impulse response $h(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} c_k \underbrace{H(jk\omega)}_{} e^{jk\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

where

This came from the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau = x(t) \cdot H(j\omega)$$

$x(t) = e^{j\omega t}$

Let's express $x(t)$ using the inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

and put this into the convolution integral...

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-\tau)}\,d\omega\right]h(\tau)\,d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-\tau)}\,d\omega\, h(\tau)\,d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,\underline{e^{-j\omega\tau}}\,d\omega\cdot \underline{h(\tau)}\,\underline{d\tau} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left[\int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)\,d\tau\right]e^{j\omega t}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)H(j\omega)e^{j\omega t}\,d\omega
\end{aligned}$$

We have **two** ways to write $y(t)$:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(X(j\omega)\, H(j\omega)\right) e^{j\omega t}\, d\omega$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, d\omega$$

This has an important implication:

$$y(t) = h(t) * x(t)$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

This can be helpful for evaluating the output of systems given $h(t)$ and $x(t)$ (or $h(t)$ given $y(t)$ and $x(t)$, etc.)

Example: suppose a signal $x(t) = \frac{\sin(\omega_0 t)}{\pi t}$ is input into a lowpass filter with frequency response

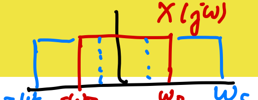
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases}$$

Method 1: inverse FT $H(j\omega)$ to get $h(t)$, then convolve.

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega = \frac{\sin(\omega_c t)}{\pi t}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} \frac{\sin(\omega_0 \tau)}{\pi t} \frac{\sin(\omega_c (t-\tau))}{\pi (t-\tau)}\, d\tau$$

Example: convolution



Method 2: compute $X(j\omega)$ then use convolution property.

We just computed $h(t)$ and found

$$h(t) = \frac{\sin(\omega_c t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases}$$

$$x(t) = \frac{\sin(\omega_0 t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| \geq \omega_0 \end{cases}$$

$$Y(j\omega) = H(j\omega)\,X(j\omega) = \begin{cases} 1 & |\omega| < \omega_s \\ 0 & |\omega| \geq \omega_s \end{cases} \qquad \omega_s = \min(\omega_0, \omega_c)$$

$$y(t) = \begin{cases} \dfrac{\sin(\omega_c t)}{\pi t} & \omega_c < \omega_0 \\[2ex] \dfrac{\sin(\omega_0 t)}{\pi t} & \omega_0 < \omega_c \end{cases}$$

Exercise: convolution

Consider an LTI system that sends

$$x(t) = e^{-t} u(t) \qquad y(t) = \frac{1}{2} e^{-|t|}$$

What is its impulse response?

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

???

$$H(j\omega) = \frac{1}{2}\left[1 + \frac{j\omega + 1}{-j\omega + 1}\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad X(j\omega) = \frac{1}{1+j\omega}$$

$$y(t) = \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{t} u(-t) = \frac{1}{2} \frac{1}{1+j\omega} + \frac{1}{2} \frac{1}{1-j\omega}$$

time reversal

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2}\left[1 + \frac{1+j\omega}{1-j\omega}\right]$$

$$= \frac{1}{2}\left[\frac{1-j\omega + 1 + j\omega}{1-j\omega}\right]$$

$$= \frac{1}{2}\,\frac{2}{1-j\omega}$$

$$= \frac{1}{1-j\omega}$$

$$= H(j\omega) \qquad \Rightarrow h(t) = e^{t}\,u(-t)$$

Today's learning outcomes were:

- Leverage key properties of Fourier transform to simplify its computation
- Apply the *convolution property* of the Fourier transform to characterize LTI system behaviour
- Describe the *multiplication property* of the Fourier transform and provide applications of its use

For next time

Content:

- Behaviour of the Fourier transform under differentiation and integration
- LTI systems based on differential equations

Action items:

1. Tutorial Assignment 3 on Monday - bring music!

Recommended reading:

- From today's class: Oppenheim 4.4-4.6
- Suggested problems: 4.4, 4.6, 4.9, 4.12, 4.15, 4.17, 4.19, 4.26, 4.32
- For Tuesday's class: Oppenheim chapter 4.7, 6.1-6.2