ELEC 221 Lecture 09
Properties of the CT the Fourier transform

Thursday 06 October 2022

Announcements

- Assignment 3 due tomorrow;
- Assignment 4 (computational) available after midterm
- Midterm 1 next Thursday

Midterm 1

What does it cover?

- Contents of lectures 1-9 (everything up to and incl. today)
- Pen-and-paper midterm; no Python, no programming
- All questions tie directly to the **learning outcomes** shared on the lecture slides

Practice problems:

- Review quizzes and assignment questions
- Oppenheim chapter problems (basic problems w/solutions, basic problems)
- Tutorial on Monday 17:30

Helpful for studying: Tables 3.1, 3.2, and 4.1

Midterm 1 provided formulas

$$\sum_{k=0}^{N} z^k = \frac{1 - z^{N+1}}{1 - z} \qquad \sum_{k=0}^{N-1} e^{\frac{2\pi jk}{N}} = 0$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega t}dt$$

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi}{N}n} \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \qquad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

Last time: the Fourier transform

We saw how we generalized from the CT Fourier series to the Fourier transform for aperiodic signals:

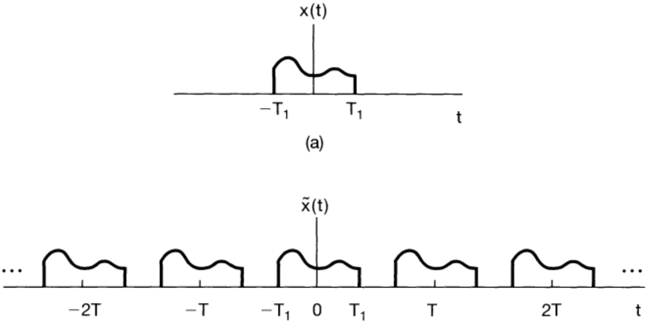


Image credit: Oppenheim chapter 4.1

Last time: the Fourier transform

We expressed the periodic extension of an aperiodic function as a Fourier series:

We computed its coefficients:

Last time: the Fourier transform

We put this back in our Fourier series:

and made some arguments as $T \to \infty$ ($\omega \to 0$)

Inverse Fourier transform (synthesis equation):

Fourier transform (analysis equation):

Last time: the Fourier transform

We found that the frequency response of a system is actually related to the impulse response by a Fourier transform:

Today, we will build on this fact.

Learning outcomes:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Leverage key properties of Fourier transform to simplify its computation
- Describe the duality between time and frequency domains
- Use convolution property to determine output of LTI systems

Dirichlet conditions for Fourier series

Back in lecture 4, we saw the Dirichlet conditions, which are sufficient for a **periodic** signal to be represented as a Fourier series.

If over **one period**, the function

1. is single-valued
2. is absolutely integrable ($\int_T |x(t)| dt < \infty$)
3. has a finite number of maxima and minima
4. has a finite number of discontinuities[1]

then the Fourier series converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

[1] 3/4: the signal has bounded variation over one period

Dirichlet conditions for Fourier transforms

There are similar sufficient criteria for Fourier transforms.

If the signal

1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

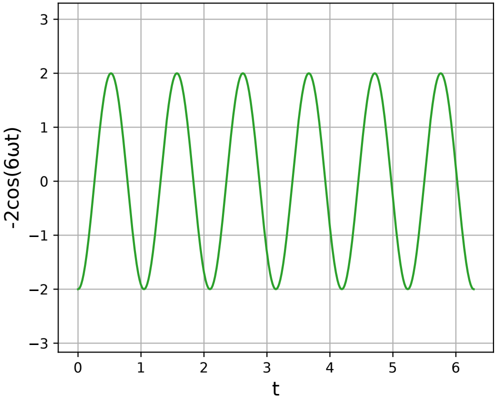
then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

Dirichlet conditions for Fourier transforms

Conclusion: absolutely integrable signals that are continuous or have a finite number of discontinuities have Fourier transforms.

...what about periodic signals?



Consider the following output of a Fourier transform:

What signal does it correspond to?

Let's find it:

That's good news - but that's just one complex exponential signal. What about when we have multiple harmonics?

Take the inverse Fourier transform of this...

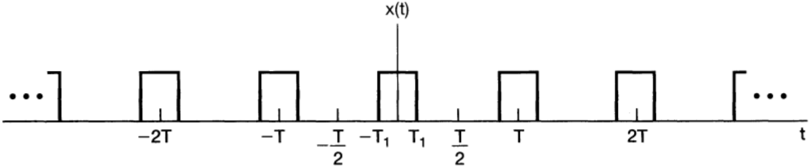
Fourier transforms for periodic signals: a unified representation

The Fourier transform of a periodic function is a train of impulses, positioned at the harmonically related frequencies.

The impulses have area $2\pi c_k$.

Fourier transforms for periodic signals: a unified representation

Remember our square wave from last time:



It had Fourier series coefficients

Image credit: Oppenheim chapter 4.1

Its Fourier *transform* will be

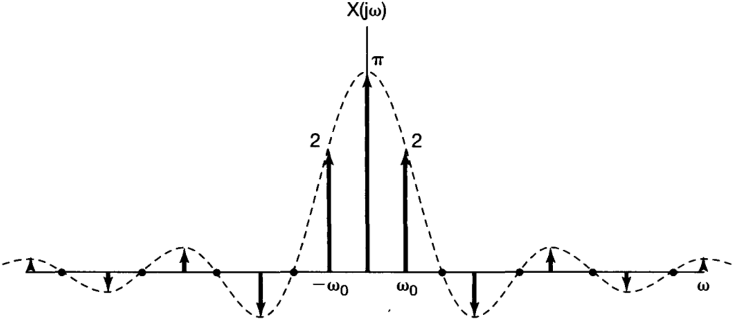


Image credit: Oppenheim chapter 4.2

Important properties of the Fourier transform

The Fourier transform has many useful properties that help with evaluating it for arbitrary functions.

Linearity. If

then

Time shifting. If

then

Notice: $|X(j\omega)|$ does not change; we just add a linear phase shift.

Important properties of the Fourier transform

Conjugation. If

then

If $x(t)$ is purely real,

Important properties of the Fourier transform

You've already made use of this when we did audio processing:

```
# Gives the full spectrum
# Has redundant info is signal is real
np.fft.fft(signal)

# Gives only the positive part
np.fft.rfft(signal)
```

Behaviour under conjugation has other implications for even/odd portions of a real signal and its transform:

Important properties of the Fourier transform

Time scaling. If

then

Time reversal follows from this:

You've all experienced the implications of this time scaling before!

Let's consider a single square pulse:

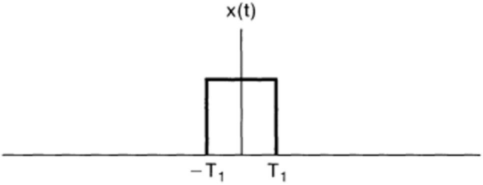


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Compute the Fourier transform:

Time/frequency duality of the FT

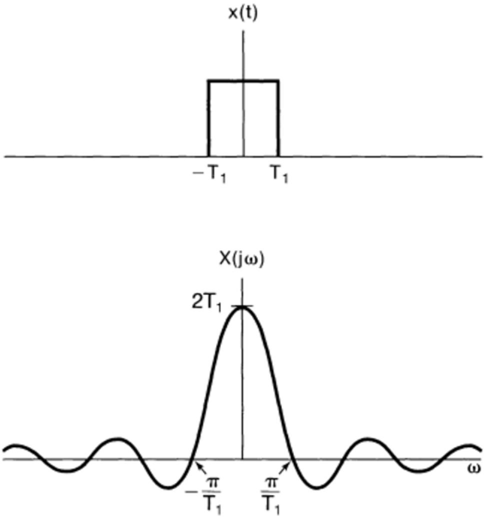


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Now let's consider a signal whose Fourier transform is

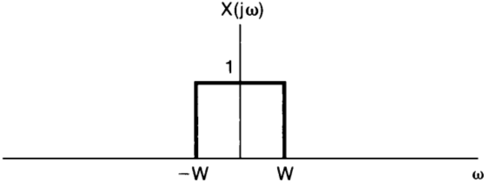


Image credit: Oppenheim chapter 4.1

Compute the inverse Fourier transform:

Time/frequency duality of the FT

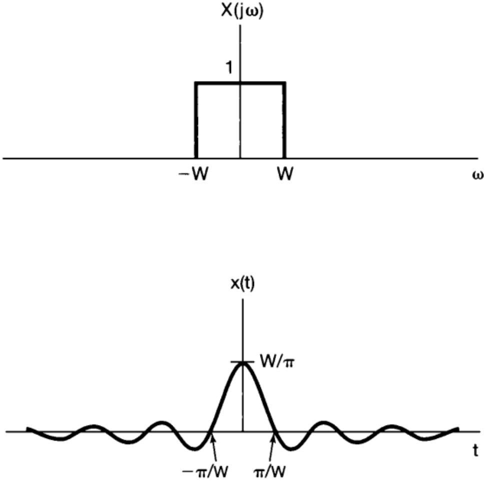


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

These are related...

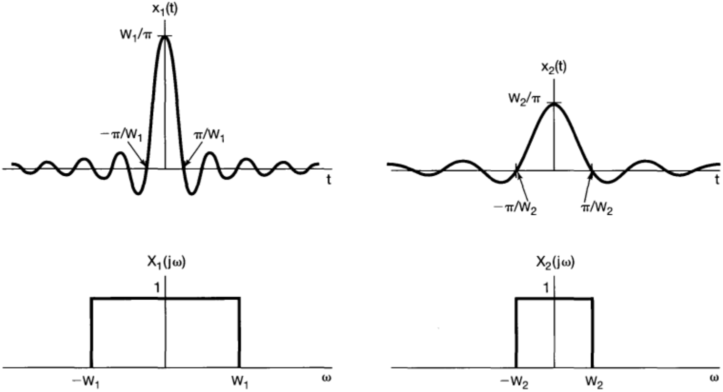
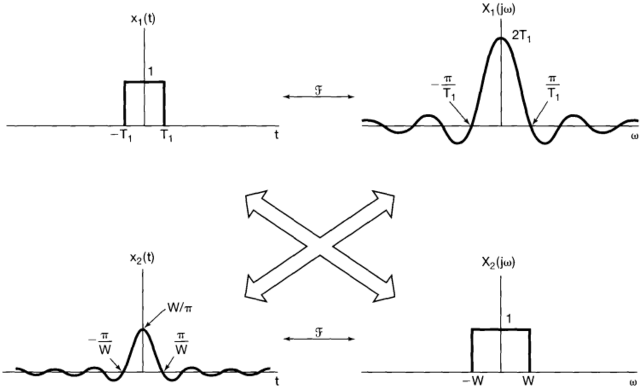


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Duality: for any transform pair $(x(t) \leftrightarrow X(j\omega))$, there is a *dual pair* with the time and frequency variables interchanged.



(We will explore this a bit more on Tuesday)

Image credit: Oppenheim chapter 4.3

Convolution and the Fourier transform

Recall the convolution integral representation: when a signal $x(t)$ is input into an LTI system with impulse response $h(t)$,

Complex exponentials are eigenfunctions of LTI systems:

where

Convolution and the Fourier transform

Recall how we arrived at the CT Fourier transform:

What happens when we put $x(t)$, as expressed above, into an LTI system with impulse response $h(t)$?

Convolution and the Fourier transform

What happens when we put $x(t)$, as expressed above, into an LTI system with impulse response $h(t)$?

We have **two** ways now to write a signal $y(t)$:

This has an important implication:

Example: convolution

This can be helpful for evaluating the output of systems given $h(t)$ and $x(t)$ (or $h(t)$ given $y(t)$ and $x(t)$, etc.)

Example: what is $y(t)$ for an LTI system with the following input and impulse response?

Example: convolution

Compute the Fourier transform of $x(t)$:

Convenient general expression to remember:

Example: convolution

Compute the Fourier transform of $h(t) = e^t u(-t)$:

Then,

How to deal with this? Partial fractions:

Example: convolution

Now we need to take the inverse Fourier transform:

But we already know that

Similarly,

Example: convolution

$$Y(j\omega) = \frac{1}{2}\frac{1}{1+j\omega} + \frac{1}{2}\frac{1}{1-j\omega}$$

Recap

Today's learning outcomes were:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Leverage key properties of Fourier transform to simplify its computation
- Describe the duality between time and frequency domains
- Use convolution property to determine output of LTI systems

What topics did you find unclear today?

For next time

Content:

- Multiplication properties of the CT Fourier *transform*

Action items:

1. Assignment 3 is due tomorrow
2. Midterm 1 next Thursday

Recommended reading:

- From today's class: Oppenheim 4.2-4.4
- For next class: Oppenheim 4.5-4.7