

ELEC 221 Lecture 22
**The Laplace transform: properties and
system analysis**

Thursday 28 November 2024

Announcements

- Midterm 2 available for pickup at my office (some remaining MT1 as well)
- Quiz 10 Tuesday (last quiz)
- Final tutorial on Monday (problem solving - post suggestions on Piazza @226)
- Tutorial Assignment 5 due Monday 23:59
- Assignment 5 due Sunday 8 December 23:59
- Final exam details available late next week

Last time

We introduced the Laplace transform,

where $s = \sigma + j\omega$.

If $s = j\omega$, reduces to **Fourier transform**

$X(s)$ can exist in regions that $X(j\omega)$ does not (allows us to analyze more kinds of systems), but still doesn't exist everywhere.

Last time

We introduced the s -plane and pole-zero plots. We used them to plot the region of convergence (ROC) of $X(s)$.

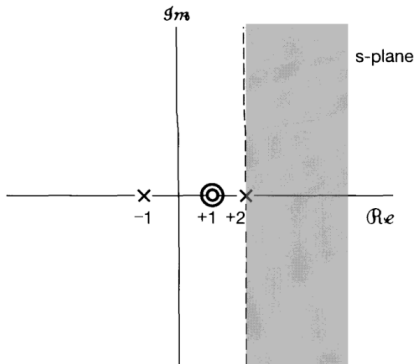
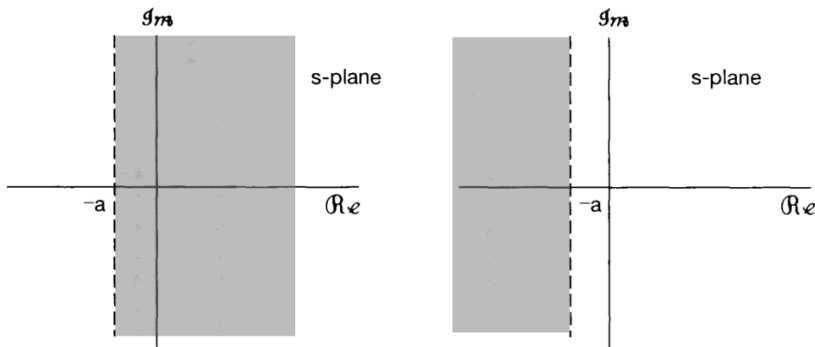


Image credit: Oppenheim 9.1

Last time

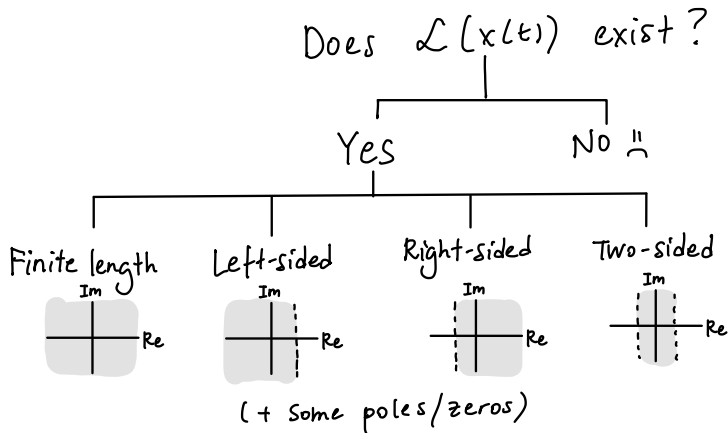
The ROC is essential for computing inverse Laplace transforms.



Both ROC associated to algebraic expression $X(s) = \frac{1}{s+a}$, but came from different signals.

Last time

We distinguished between types of signals and their ROCs.



Learning outcomes:

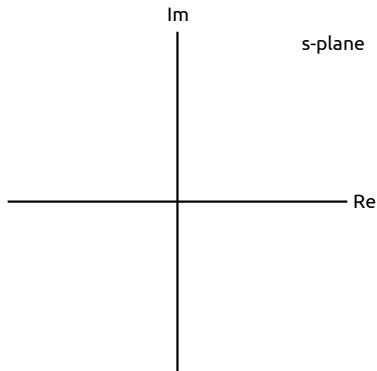
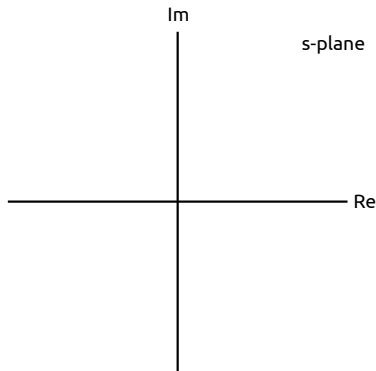
- apply key properties of the Laplace transform to its computation
- use the Laplace transform to determine whether a system is causal or stable
- compute the Laplace transform of systems described by constant-coefficient DEs

Regions of convergence

(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

$$\frac{s - 1}{(s + 2)(s + 3)(s^2 + s + 1)}$$

Hint: draw pole-zero plot and identify possible ROCs.



Inverse Laplace transforms

From this, we can invert:

Make a change of variables $ds = j d\omega$:

... we are not going to integrate this.

Inverse Laplace transforms

Suppose

where degree of denominator is higher than numerator.

To invert, we can use our handy identities, BUT the ROC matters.

The Laplace transform

Multiple signals can have the same algebraic Laplace transform, but different ROCs.

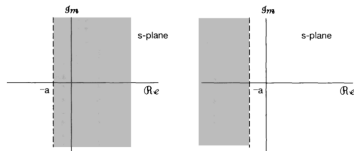


Image credit: Oppenheim 9.1

Inverse Laplace transforms

Exercise: what is the inverse Laplace transform of

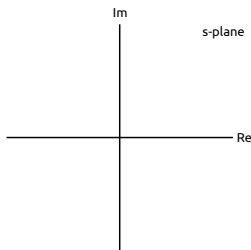
$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \operatorname{Re}(s) < -3$$

Inverse Laplace transforms

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s + 2}{s^2 + 7s + 12}, \quad -4 < \operatorname{Re}(s) < -3$$

Draw the s -plane:



Properties of the Laplace transform

We've made use of many nice properties of the Fourier transform:

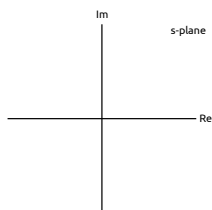
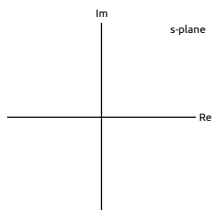
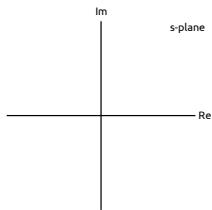
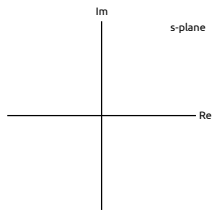
- linearity
- time shift/scale
- differentiation
- conjugation
- convolution

All have analogs with Laplace transform, but factor in the ROC.

Linearity

Example: $x(t) = e^{-b|t|}$.

Properties of the Laplace transform

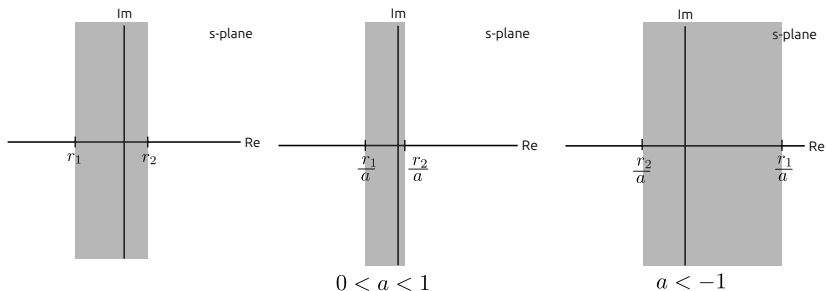


Properties of the Laplace transform

Time shifting.

Time scaling.

Time reversal.



Properties of the Laplace transform

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s}{s^2 + 9}, \quad \operatorname{Re}(s) < 0$$

Hint:

$$\cos(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_0^2}, \quad \operatorname{Re}(s) > 0$$

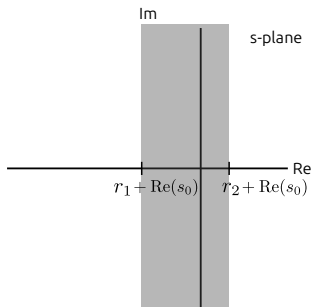
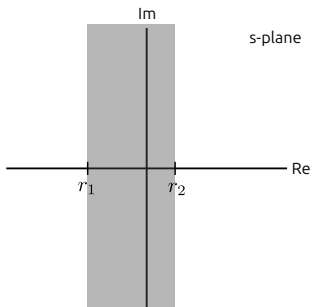
Properties of the Laplace transform

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!} u(t)$	$\frac{1}{s^n}$	$\Re\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!} u(-t)$	$\frac{1}{s^n}$	$\Re\{s\} < 0$
6	$e^{-\alpha t} u(t)$	$\frac{1}{s + \alpha}$	$\Re\{s\} > -\alpha$
7	$-e^{-\alpha t} u(-t)$	$\frac{1}{s + \alpha}$	$\Re\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t)$	$\frac{1}{(s + \alpha)^n}$	$\Re\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(-t)$	$\frac{1}{(s + \alpha)^n}$	$\Re\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos \omega_0 t] u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\Re\{s\} > 0$
12	$[\sin \omega_0 t] u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\Re\{s\} > 0$
13	$[e^{-\alpha t} \cos \omega_0 t] u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$	$\Re\{s\} > -\alpha$
14	$[e^{-\alpha t} \sin \omega_0 t] u(t)$	$\frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$	$\Re\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n \delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re\{s\} > 0$

Properties of the Laplace transform

s shifting



Differentiation in time.

Differentiation in s .

Properties of the Laplace transform

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Properties of the Laplace transform

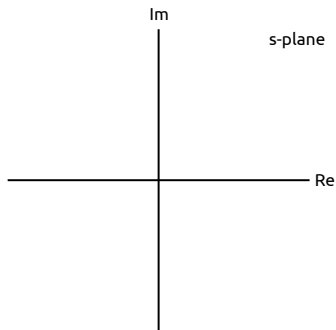
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Properties of the Laplace transform

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Conjugation.

Initial/final-value theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or singularities at the origin,

Furthermore if $x(t)$ has finite limit as $t \rightarrow \infty$,

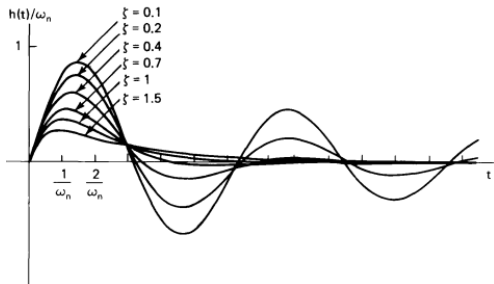
Convolution.

Recall the convolution property:

The ROC of the system (transfer) function can tell us a lot about a system, including systems whose Fourier transforms don't exist.

$H(s)$ and causality

Recall that a system is causal if $h(t) = 0$ for $t < 0$.

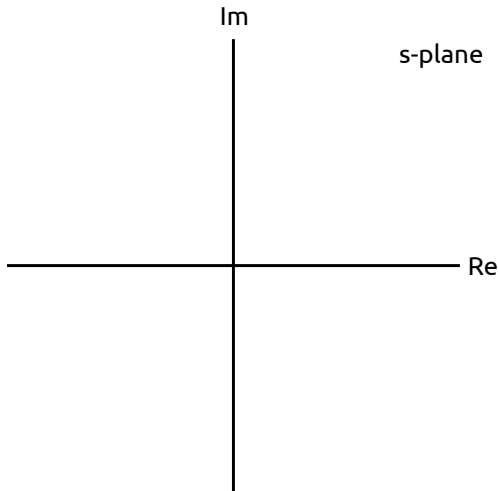


Means $h(t)$ is right-sided, so its ROC is a right-half plane.

Image credit: Oppenheim 6.5

$H(s)$ and causality

Note that the converse is not necessarily true! But if $H(s)$ is rational, the ROC is the right-half plane to right of right-most pole.



Our original criteria for stability in terms of impulse response was if

Related to Dirichlet conditions: if a signal is absolutely integrable, its **Fourier transform** converges.

An LTI system with **rational** $H(s)$ is stable iff its ROC includes the entire $j\omega$ axis ($\text{Re}(s) = 0$), and there aren't more zeros than poles.

9.28. Consider an LTI system for which the system function $H(s)$ has the pole-zero pattern shown in Figure P9.28.

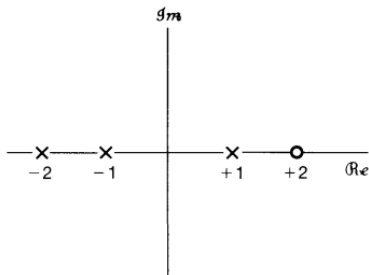


Figure P9.28

- (a) Indicate all possible ROCs that can be associated with this pole-zero pattern.
- (b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

Systems described by constant-coefficient differential equations

Recall the situation with the Fourier transform:

Fourier transforms and systems described by differential equations

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^M \beta_k (j\omega)^k}{\sum_{k=0}^N \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection!** (and vice versa)

Systems described by constant-coefficient differential equations

Same deal here. If system is described by the DE

then its system function is

Placement of zeros and poles is dictated by solutions of $x(t)$ and $y(t)$ stuff respectively.

(Oppenheim 9.31) Consider a CT LTI system described by the DE

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

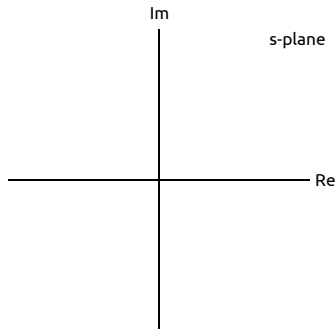
- Determine $H(s)$ as a ratio of polynomials in s and sketch the pole-zero plot.
- Determine $h(t)$ for each of the following cases:
 1. The system is stable
 2. The system is causal
 3. The system is neither causal nor stable

Systems described by constant-coefficient differential equations

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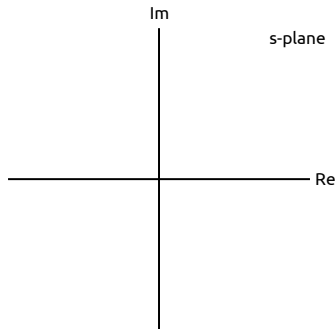
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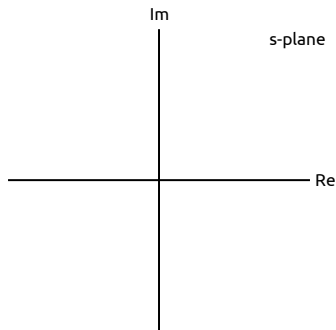
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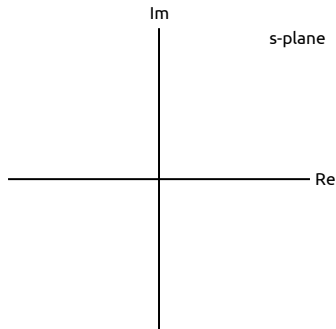
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For next time

Content:

- the Laplace transform and feedback systems
- introducing the z-transform

Action items:

1. Suggest problems for tutorial
2. Tutorial assignment 5 due Monday 23:59
3. Assignment 5 due 8 Dec 23:59

Recommended reading:

- From this class: Oppenheim 9.5-9.7
- Suggested problems: 9.13-9.16, 9.21, 9.22, 9.26, 9.29, 9.32, 9.33
- For next class: 9.7, 11.0-11.2, 10.1-10.3