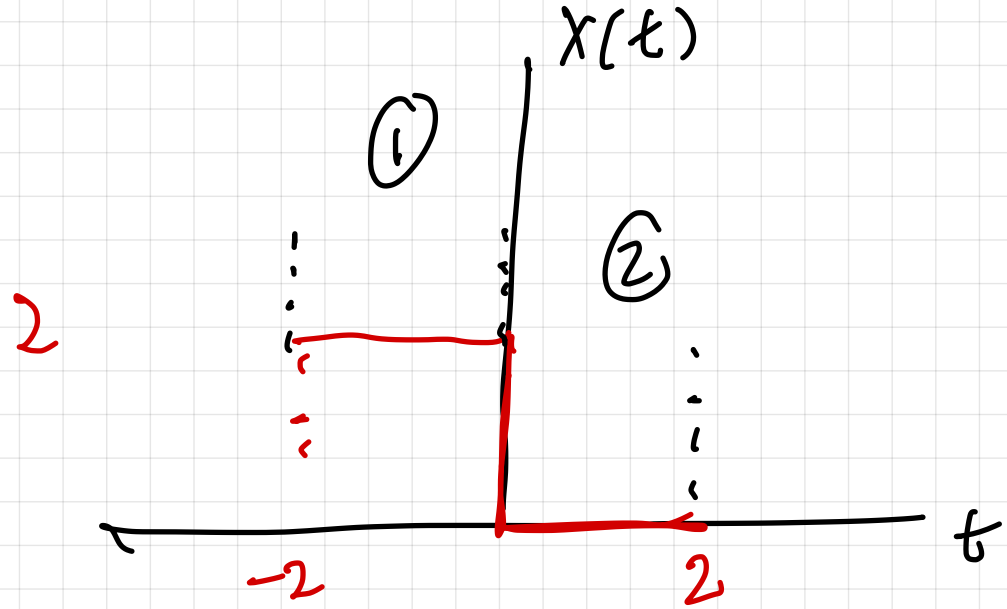
ELEC 221 Lecture 11
Properties of the CT Fourier Transform

Tuesday 15 October 2024

- Quiz 5 today (based on Lecture 10 material)
- TAs are very busy with grading!
- Expect A3 and TA3 next week



$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$= \int_{-2}^{0} \underbrace{2e^{-jwt}}\, dt + \int_{0}^{2} \cancel{0 \cdot e^{-jwt}}\, dt$$

$$= 2\int_{-2}^{0} e^{-jwt}\, dt$$

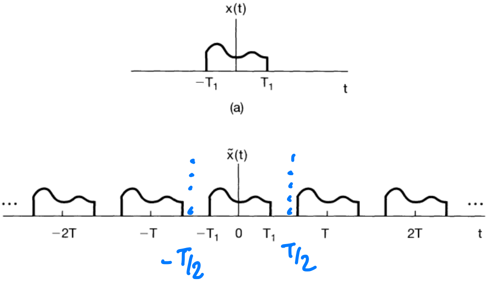
$$= 2 \cdot -\frac{1}{jw}\, e^{-jwt}\Big|_{-2}^{0}$$

$$= -\frac{2}{jw}\left[1 - e^{2jw}\right] \qquad \frac{\pi}{2}$$

$$= -\frac{2}{j\cdot\pi/2}\left[1 - e^{\cancel{2}j\cdot\frac{\pi}{\cancel{2}}}\right] = -\frac{4}{j\pi}\, 2 = \frac{-8}{j\pi}$$

Last time

We generalized the CT Fourier series (for periodic signals) to the Fourier transform (for aperiodic signals):



We expressed the periodic extension of an aperiodic function as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Image credit: Oppenheim chapter 4.1

We computed its Fourier coefficients:

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega t}\, dt}_{X(jk\omega)}$$

$$= \frac{1}{T} X(jk\omega)$$

We put this into the Fourier series and let $T_\infty \to \infty$ ($\omega \to 0$):

$$\tilde{x}(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\omega)\, e^{jk\omega t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega)\, e^{jk\omega t}\cdot\omega$$

$$\lim_{T\to\infty} \tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = x(t)$$

Last time

Inverse Fourier transform (synthesis equation):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw) e^{jwt} dw$$

Fourier transform (analysis equation):

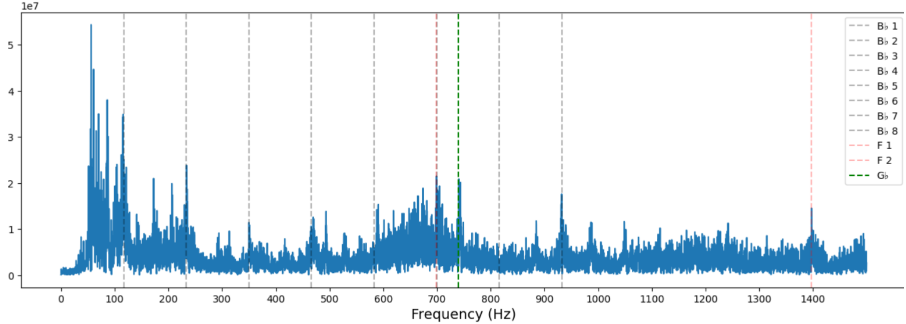
$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

Frequency response: $H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$

Preview

The Fourier spectrum contains a lot of important and useful information about signals!

```
fourier_spectrum = np.fft.rfft(audio, norm="forward")
frequencies =  np.fft.rfftfreq(len(audio), 1 / sample_rate)
```



You will experience this directly in Tutorial Assignment 3.

Today

Learning outcomes:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Describe the duality between time and frequency domains
- Leverage key properties of Fourier transform to simplify its computation

Dirichlet conditions for Fourier transforms

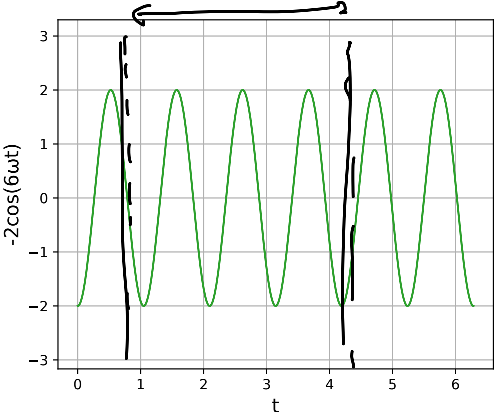
If a signal

1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

The “**within any finite interval**” takes care of periodic signals:



Exercise 1

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

What signal does the following Fourier transform belong to?

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = e^{j\omega_0 t}$$

$$\frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega - \omega_0) e^{j\omega t} d\omega = \underbrace{e^{j\omega_0 t}}_{\text{periodic!}}$$

number of C_k? $\underline{\underline{1}}$

Exercise 2

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

What signal does the following Fourier transform belong to?

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

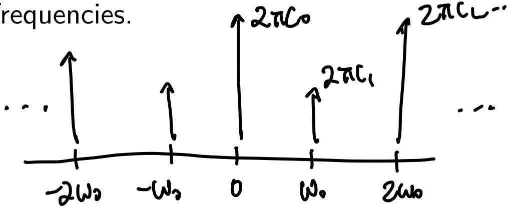
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Fourier transforms for periodic signals: a unified representation

The Fourier transform of a periodic function is an **impulse train**.

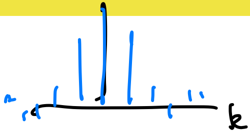
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \, \delta(\omega - k\omega_0)$$

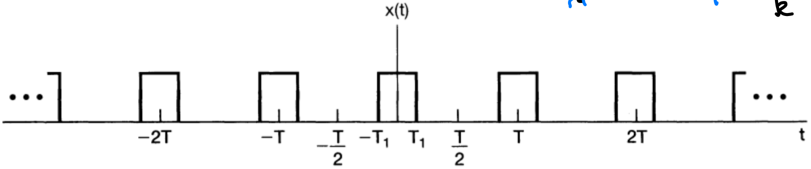
The impulses have area $2\pi c_k$ and are positioned at the harmonically related frequencies.



Fourier transforms for periodic signals: a unified representation

Remember our square wave from last time:





It had Fourier series coefficients

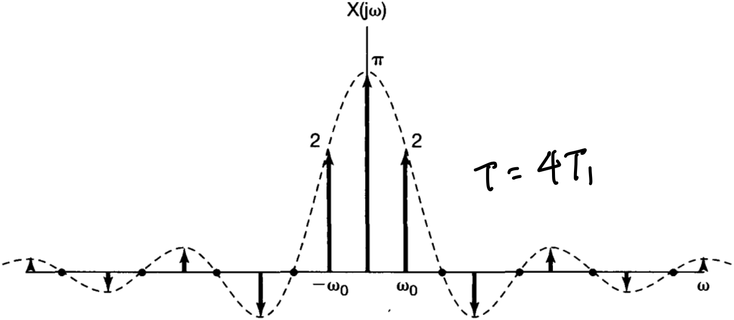
$$c_0 = \frac{2T_1}{T} \qquad c_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$\omega_0 = \frac{2\pi}{T}$$

Image credit: Oppenheim chapter 4.1

Its Fourier *transform* will be

$$X(j\omega) = 2\pi c_0 \cdot \delta(\omega) + \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty}{}' \, 2\pi \cdot \left(\frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \right) \delta(\omega - \omega_0 k)$$



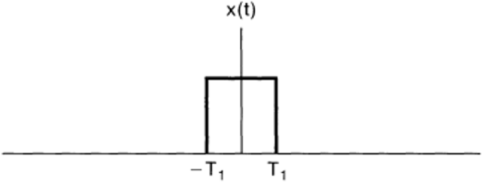
$T = 4T_1$

Image credit: Oppenheim chapter 4.2

Time/frequency duality of the FT

Let's consider a single square pulse:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$



Its Fourier spectrum is

$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt = \int_{-T_1}^{T_1} e^{-jwt}\, dt = \frac{2\sin(wT_1)}{w}$$

Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

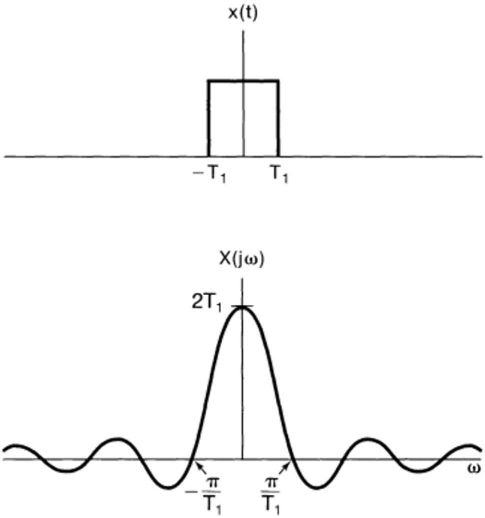
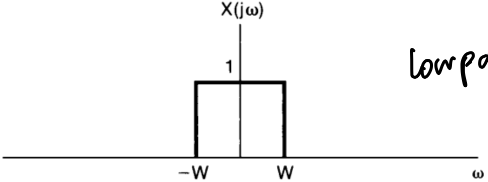


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Now let's consider a signal whose Fourier transform is

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$



lowpass filter

Image credit: Oppenheim chapter 4.1

Compute the inverse Fourier transform:

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi j t} \left[e^{j\omega t} \right]_{-W}^{W} = \frac{1}{2\pi j t} \left[e^{jWt} - e^{-jWt} \right]$$

$$= \frac{\sin(Wt)}{\pi t}$$

Time/frequency duality of the FT

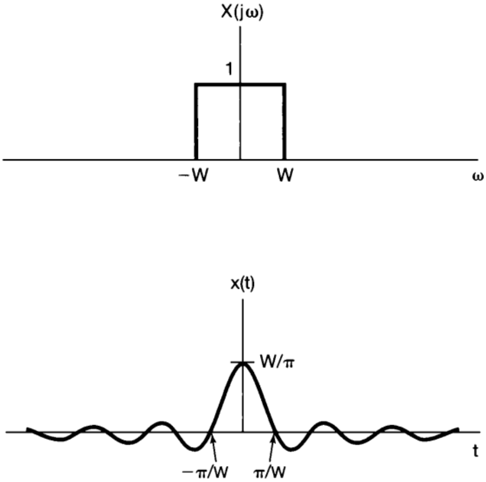


Image credit: Oppenheim chapter 4.1

Time/frequency duality of the FT

Duality: for any transform pair $(x(t) \leftrightarrow X(j\omega))$, there is a *dual pair* with the time and frequency variables interchanged.

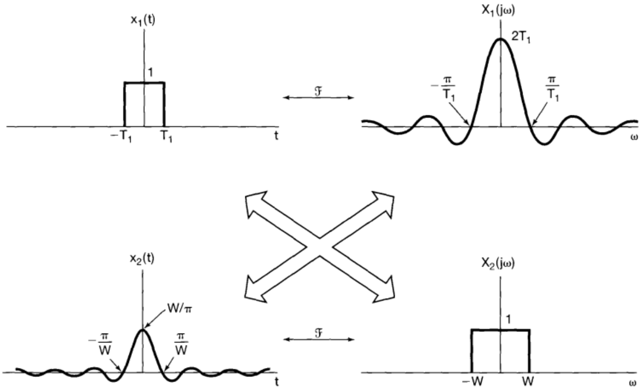


Image credit: Oppenheim chapter 4.3

Exercise

What is the Fourier transform of $x(t) = \underline{\underline{e^{-at}u(t)}}$ $(a > 0)$?

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{-a-j\omega}\, e^{-at}\, e^{-j\omega t}\Big|_{0}^{\infty}$$

$$= \frac{1}{-a-j\omega}\,(0-1)$$

$$= \frac{1}{a+j\omega}$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$$

$(\mathrm{Re}(a)>0)$

$$e^{-|t|} \longrightarrow \begin{cases} e^{-t} & t>0 \\ e^{t} & t<0 \end{cases} = e^{-t}u(t) + e^{t}\cdot u(-t)$$

$$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

What is the Fourier transform of $x(t) = e^{-2|t-1|}$?

$$x(t) = \begin{cases} e^{-2(t-1)} & t > 1 \\ e^{-2(-t+1)} & t < 1 \end{cases}$$

$$= e^{-2(t-1)}u(t-1) + e^{-2(-t+1)}u(-t+1)$$

$$e^{-at}u(t)$$

Recap

Learning outcomes:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Describe the duality between time and frequency domains
- Leverage key properties of Fourier transform to simplify its computation

For next time

Content:

- Convolution and multiplication property of the Fourier transform
- More Fourier transform pairs
- Differentiation and integration of FT

Recommended reading:

- From today's class: Oppenheim 4.2-4.3
- Suggested problems: 4.2-4.4, 4.6, 4.9, 4.21bcdgh, 4.27
- For next class: Oppenheim 4.4-4.6