ELEC 221 Lecture 25
The z-transform

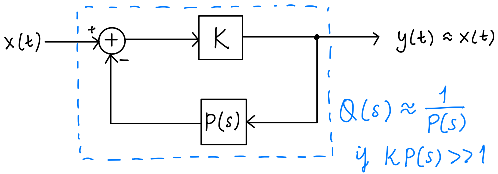
Tuesday 6 December 2022

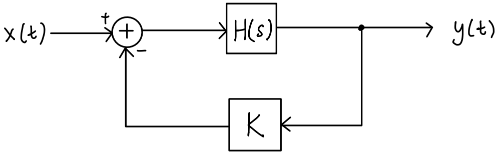
- Last class!
- Please come pick up your midterms
- Assignment 7 due ~~tonight~~ Thursday at 23:59 (hard deadline, no extensions) - please submit on **Canvas** if file too big for PL
- Details for final exam to be posted on Piazza when available
- My office hours the same until exam (Friday 9:00-10:00, open-door after 13:00)

Last time

We saw how knowledge of the Laplace transform can help us:

- analyze feedback systems
- stabilize unstable systems
- find inverse systems

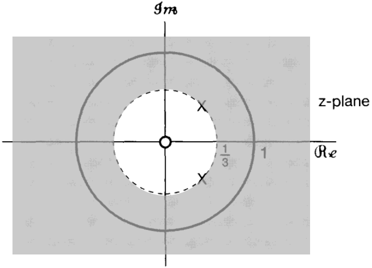




We introduced the DT counterpart, the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

We represented its region of convergence on the z-plane



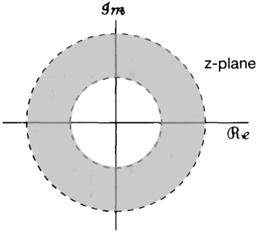


Image credit: Oppenheim 11.1, 11.2

Today

Learning outcomes:

- use the z-transform to determine whether a system is causal or stable
- apply the z-transform to systems described by difference equations
- analyze simple feedback systems with the z-transform

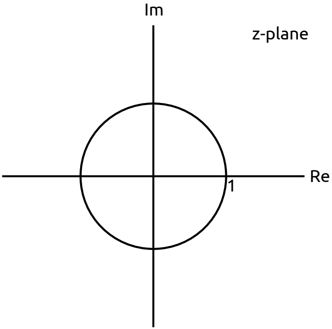
Regions of convergence

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

Draw the pole-zero plot and determine the possible ROCs.

Hint: this function has 2 zeros; express it in a different way to find them.

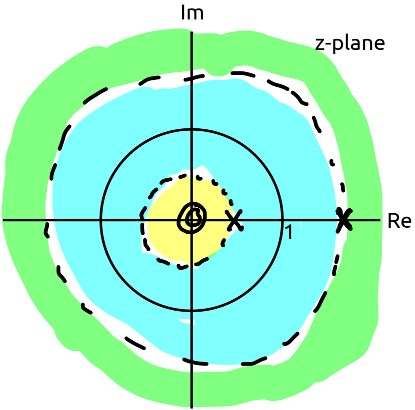


Regions of convergence

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

Solution:



$$X(z) = \frac{1}{z^{-2}(z-\frac{1}{3})(z-2)}$$

$$= \frac{z^2}{(z-\frac{1}{3})(z-2)}$$

When the z-transform can be expressed as a rational function, we can compute the inverse using partial fractions. We still need the ROC to help us.

$-\frac{1}{5}\cdot\left(\frac{1}{3}\right)^n u[n]$

Exercise: compute the inverse z-transform of $+\frac{6}{5}(2^n)u[n]$

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

if ROC is specified to be $|z| > 2$.

$\alpha^n u[n] \longleftrightarrow \frac{1}{1-\alpha z^{-1}}$ $|z| > \alpha$ right-sided

$-\alpha^n u[-n-1] \longleftrightarrow \frac{1}{1-\alpha z^{-1}}$ $|z| < \alpha$ left-sided

Exercise: compute the inverse z-transform of

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

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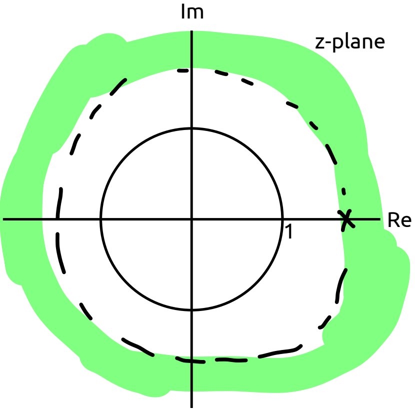


Use partial fractions:

$$X(z) = \frac{A}{1-\frac{1}{3}z^{-1}} + \frac{B}{1-2z^{-1}}$$

$$= \frac{-1/5}{1-\frac{1}{3}z^{-1}} + \frac{6/5}{1-2z^{-1}}$$

From ROC, signal is right-sided:

$$x[n] = -\frac{1}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{6}{5}(2)^n u[n]$$

Inverse z-transforms

Take a closer look at the structure of $X(z)$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

This is a *power series in z*. If we can do the expansion, we can recover $x[n]$ from the coefficients.

Exercise 1: what is the inverse z-transform of

$$X(z) = 3z^2 - 1 + 2z^{-3}, \quad 0 < |z| < \infty$$

Solution: $x[n] = 3 \cdot \delta[n+2] - \delta[n] + 2\delta[n-3]$

Inverse z-transforms

Particularly helpful for non-linear cases.

Exercise 2 (Oppenheim 10.63a): what is the inverse z-transform of

$$X(z) = \log(1-2z), \quad |z| < \frac{1}{2}$$

Hint:

$$\log(1-w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \quad |w| < 1$$

Solution:

$n = -i$ $\quad$ $x[n]$

$$X(z) = \log(1-2z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i} = -\sum_{n=-\infty}^{-1} \left(\frac{2^{-n}}{-n}\right) z^{-n}$$

$$x[n] = \begin{cases} \dfrac{2^{-n}}{n} & n \leq -1 \\ 0 & n > -1 \end{cases} = \frac{2^{-n}}{n} u[-n-1]$$

Properties of the z-transform

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \quad \text{w/ROC } R_1$$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z) \quad \text{w/ROC } R_2$$

Linearity:

$$a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z) \quad \text{w/ROC contain } R_1 \cap R_2$$

Example: $a^n u[n]$, $a^n u[n-1]$ both have ROC of $|z| > |a|$. What is the ROC of z-transform of

$$a^n u[n] - a^n u[n-1]$$

Solution: $\delta[n]$ w/ROC all of z-plane

$x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$

$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$ w/ROC R

Time shift:

$x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} X(z)$ w/ROC R maybe add/delete 0 or ∞

Time reversal:

$$x[-n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{1}{z}\right) \quad \text{w/ROC } 1/R$$

Time expansion (zero-insertion of $k-1$ zeros):

$$x_{(k)}[n] \xleftrightarrow{\mathcal{Z}} X(z^k) \quad \text{w/ROC } R^{1/k}$$

The z-transform and causality

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \quad \text{w/ROC } R$$

Scaling in z:

$$z_0^n x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right) \quad \text{w/ROC } |z_0| R$$

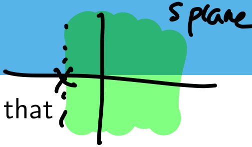
Conjugation:

$$x^*[n] \longleftrightarrow X^*(z^*) \quad \text{w/ROC } R$$

If $x[n]$ is real, the poles and zeros come in *conjugate pairs*.

The z-transform and causality

$z = \underbrace{r}e^{j\omega}$



The convolution property of the z-transform tells us that

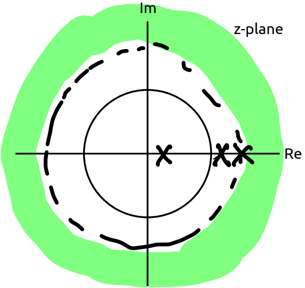
$X(s) \quad s = \sigma + j\omega$

$$Y(z) = H(z)X(z)$$

Remember how we previously tested causality: $h[n] = 0$ for all $n < 0$ (it is right-sided).

A DT LTI system with rational z-transform is causal if:

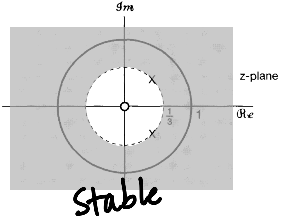
- the ROC is the exterior of a circle outside the outermost pole (including infinity)
- with $H(z)$ expressed in polynomials of z, order of numerator does not exceed order of the denominator

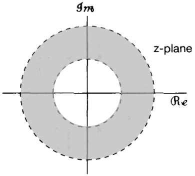


Previously, to compute stability, we checked if the impulse response was absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

This was also a condition required for the DTFT to exist.





An LTI system is stable if ROC includes the unit circle $|z| = 1$.

Image credit: Oppenheim 11.1, 11.2

If $x[n] = 0$ for all $n < 0$, then

$$\lim_{z\to\infty} X(z) = x[0]$$

How? Look again at expression for $X(z)$:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1] \cdot \frac{1}{z} + \cdots$$

Consequences:

- if $x[n]$ is causal, $\lim_{z\to\infty} X(z)$ is finite
- if $X(z)$ is a ratio of polynomials, order of numerator cannot be greater than order of denominator (cannot have more finite zeros than finite poles)

10.17. Suppose we are given the following five facts about a particular LTI system S with impulse response $h[n]$ and z-transform $H(z)$:

1. $h[n]$ is real.
2. $h[n]$ is right sided.
3. $\lim_{z \to \infty} H(z) = 1$.
4. $H(z)$ has two zeros.
5. $H(z)$ has one of its poles at a nonreal location on the circle defined by $|z| = 3/4$.

Answer the following two questions:

(a) Is S causal? **(b)** Is S stable?

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causal

does it go both ways??

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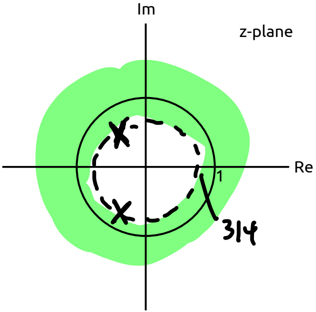


Consider LTI system described by a DT difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Using properties of the DTFT (convolution, time shift, linearity):

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Using analogous properties of the z-transform,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Exercise (Oppenheim 10.36): consider LTI system described by a DT difference equation

$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$$

Suppose the system is stable; what is its impulse response? $h[n]$

$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$$

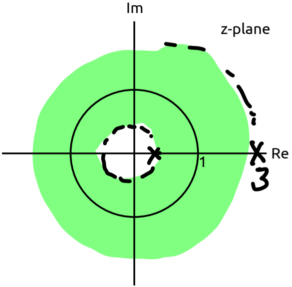
Solution:

$$H(z) = \frac{1}{z - 10/3 + z^{-1}}$$

$$= \frac{1}{z\left(1 - \frac{10}{3}z^{-1} + z^{-2}\right)}$$

$$= \frac{z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 3z^{-1}\right)}$$

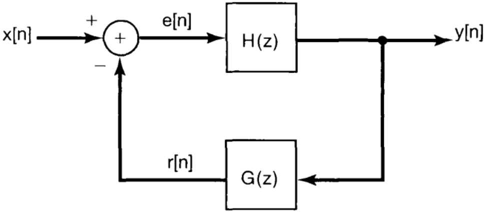
$$= \frac{-3/8}{1 - \frac{1}{3}z^{-1}} + \frac{3/8}{1 - 3z^{-1}}$$



$$h[n] = -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{8}(3^n)\, u[-n-1]$$

Feedback systems

The *z*-transform can help us with the analysis of feedback systems (using them for stabilization, etc.) like we did in CT with the Laplace transform.

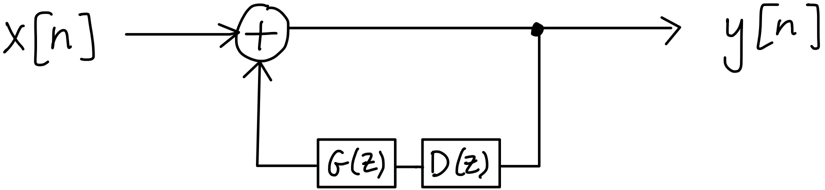


The closed-loop system function has the same form:

$$Q(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1 + G(z)H(z)}$$

Image credit: Oppenheim 11.1

One type of system with this structure is called the **comb filter**



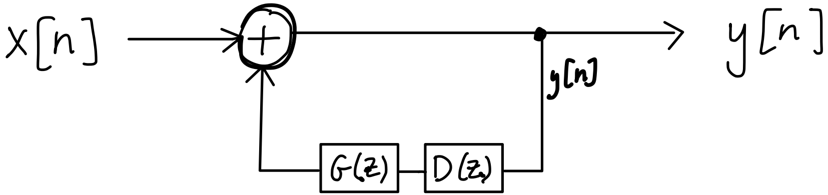
Suppose:

- $D(z)$ is a system that causes a delay of K steps
- $G(z)$ is a system with gain g

Exercise:

- what is the difference equation that describes the entire system?
- what is the closed-loop system function? (hint: you can compute it in two ways!)

Example: comb filters



Difference equation: $y[n] - g\,y[n-K] = x[n]$

$$y[n] = x[n] + gy[n-K]$$

System function: $Q(z) = \dfrac{1}{1-g\cdot z^{-K}} = \dfrac{z^K}{z^K-g}$

$$Q(z) = \frac{1}{1-gz^{-K}} = \frac{z^K}{z^K-g}$$

$$Q(z) = \frac{1}{1 - gz^{-K}} = \frac{z^K}{z^K - g}$$

$$Q(z) = \frac{z^K}{z^K - g}$$

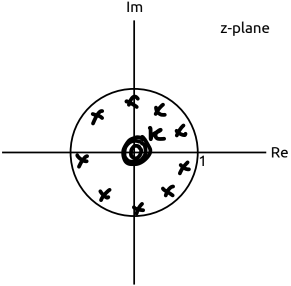
$\downarrow$

$e^{j\omega}$

What are the poles and zeros?

$z^K = g$

$z = g^{1/K}$



Why is it called the comb filter? Let's look at its frequency response (take $z = e^{j\omega}$).

Example: Karplus-Strong

Another example of this is the Karplus-Strong algorithm!

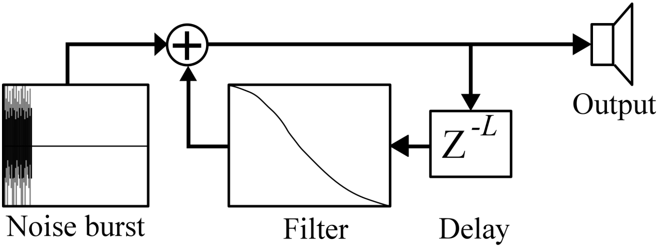
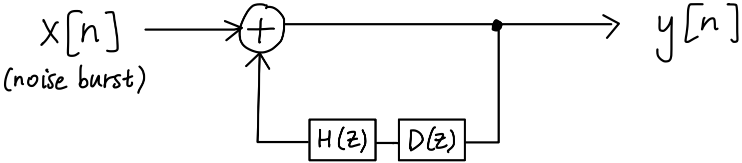


Image credit: https://commons.wikimedia.org/wiki/File:Karplus-strong-schematic.svg Author: PoroCYon

Example: Karplus-Strong

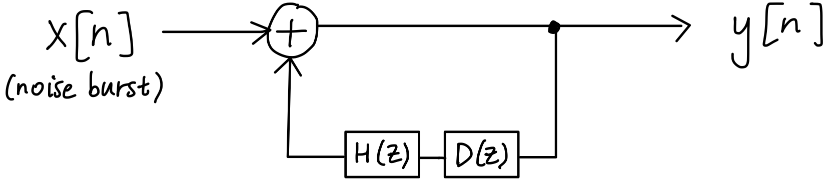


Suppose:

- $D(z)$ is a system that causes a delay of K steps
- $H(z)$ is a lowpass filter described by DE
  $y[n] = \frac{1}{2}(x[n] + x[n-1])$

Exercise:

- what is the difference equation that describes the entire system?
- what is the closed-loop system function?



Difference equation:

$$y[n] = x[n] + \frac{1}{2}\left(y[n-K] + y[n-K-1]\right)$$

System function:

$$Q(z) = \frac{1}{1 - \frac{1}{2}z^{-K} - \frac{1}{2}z^{-K-1}}$$

Learning outcomes:

- use the z-transform to determine whether a system is causal or stable
- apply the z-transform to systems described by difference equations
- analyze simple feedback systems with the z-transform

Oppenheim practice problems: 10.13-10.16, 10.25-10.27, 10.31, 10.33-10.35, 11.1

For next time

Action items:

1. Assignment 7 due tonight at 23:59 (submit on Canvas)

Recommended reading: 10.5-10.7, 11.2