ELEC 221 Lecture 23
The Laplace transform: properties and system analysis

Tuesday 29 November 2022

Announcements

- Midterm 2 available for pickup (some remaining MT1 as well)
- Quiz 10 today (last quiz)
- Assignment 6 (computational) due Thursday at 23:59
- Assignment 7 released soon; will be short and due Tuesday Dec. 6 at 23:59 (hard deadline, no extensions)

We introduced the Laplace transform of a signal

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt$$

for $s = \sigma + j\omega$ an arbitrary complex number.

If $s = j\omega$, this reduces to the **Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

Last time

We introduced the s-plane and made pole-zero plots of the region of convergence of Laplace transforms.

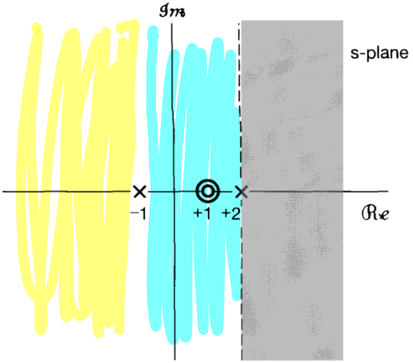
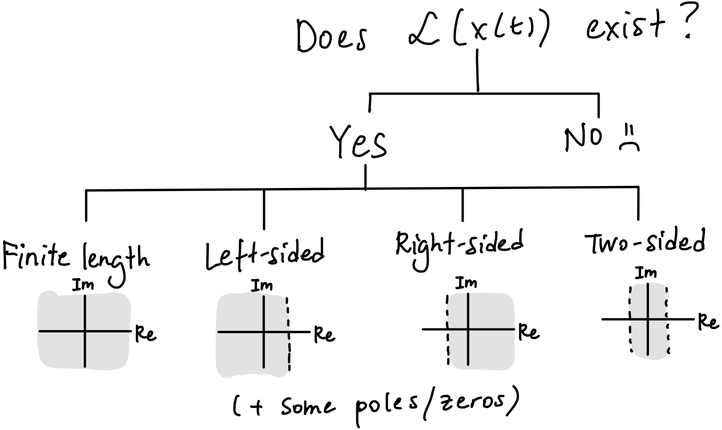
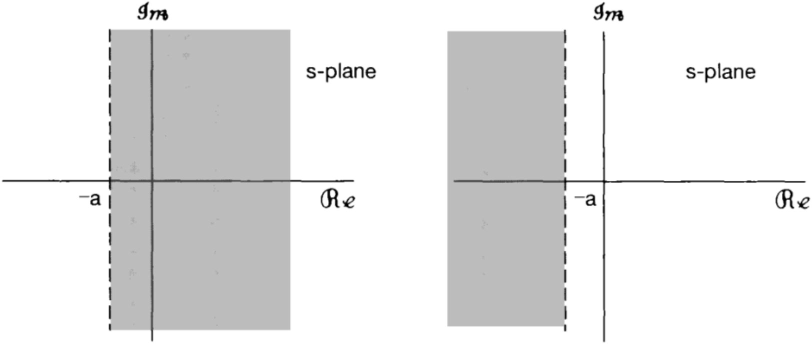


Image credit: Oppenheim 9.1

We distinguished between types of signals and their ROCs.



We saw that the region of convergence is very important in computing inverse Laplace transforms.



Both ROC associated to algebraic expression $X(s) = \frac{1}{s+a}$, but came from different signals.

Image credit: Oppenheim 9.1

Learning outcomes:

- apply key properties of the Laplace transform to its computation
- use the Laplace transform to determine whether a system is causal or stable
- compute the Laplace transform of systems described by constant-coefficient DEs

Properties of the Laplace transform

We've made use of many nice properties of the Fourier transform:

- linearity
- time shift/scale
- differentiation
- conjugation
- convolution

All of these have analogs with the Laplace transform as well!

But we must factor in the ROC.

Linearity. If

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad \text{w/ ROC } R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \quad \text{w/ ROC } R_2$$

then

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s) \quad \text{w/ ROC } R_1 \cap R_2$$

(Combined ROC may actually be larger than original ones!)

Time shifting. If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{w/ROC } R$$

then

$$x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s) \quad \text{w/ROC } R$$

s **shifting**. If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{w/ROC } R$$

then

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0) \quad \text{w/ROC } R + \mathrm{Re}(s_0)$$

Time scaling. If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{w/ROC } R$$

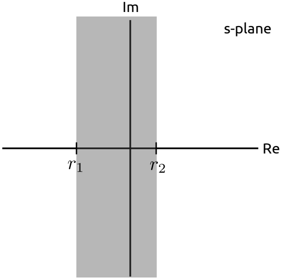
then

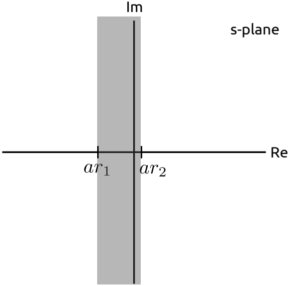
$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{w/ROC } aR$$

Time reversal.

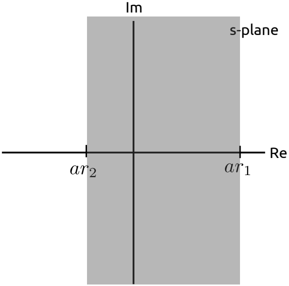
$$x(-t) \xleftrightarrow{\mathcal{L}} X(-s) \quad \text{w/ROC } -R$$

Properties of the Laplace transform





$0 < a < 1$



$a < -1$

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s}{s^2 + 9}, \quad \text{Re}(s) < 0$$

Hint:

$$\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + \omega_0^2}, \quad \text{Re}(s) > 0$$

$$(-)\cos(3t)\, u(-t)$$

Properties of the Laplace transform

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s}{s^2 + 9}, \quad \text{Re}(s) < 0$$

Hint:

$$\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + \omega_0^2}, \quad \text{Re}(s) > 0$$

The hint tells us that

$$\cos(3t)u(t) \leftrightarrow \frac{s}{s^2 + 9} \quad \text{Re}(s) > 0$$

but the ROC is wrong.

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s}{s^2+9}, \quad \mathrm{Re}(s) < 0$$

Time reversal will change the ROC.

$$x(-t) \leftrightarrow X(-s) \quad \text{w/ ROC } -R$$

$$\cos(-3t)u(-t) \leftrightarrow \frac{-s}{s^2+9} \quad \mathrm{Re}(s) < 0$$

$$\cos(3t)u(-t) \leftrightarrow -\frac{s}{s^2+9} \quad \mathrm{Re}(s) < 0$$

Thus,

$$x(t) = -\cos(3t)u(-t)$$

Properties of the Laplace transform

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Image credit: Oppenheim 9.6

Conjugation. If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{w/ROC } R$$

then

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*) \quad \text{w/ROC } R$$

Convolution. If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \quad \text{w/ROC } R_1$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \quad \text{w/ROC } R_2$$

then

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s) \quad \text{w/ROC containing } R_1 \cap R_2$$

Differentiation in time. If

$$x(t) \leftrightarrow X(s) \quad \text{w/ROC } R$$

then

$$\frac{dx(t)}{dt} \leftrightarrow sX(s) \quad \text{w/ROC contains } R$$

Differentiation in s**.** If

$$x(t) \leftrightarrow X(s) \quad \text{w/ROC } R$$

then

$$-t\,x(t) \leftrightarrow \frac{dX(s)}{ds} \quad \text{w/ROC } R$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

$$\frac{1}{(s+2)^2} - \frac{1}{(s-2)^2} \qquad -2 < Re(s) < 2$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Solution: We have $t\cdot$ something, so use **differentiation in s**.

Let $x(t) = te^{-2|t|} = tz(t)$.

$$z(t) \longleftrightarrow Z(s) \quad \text{w/ROC } R$$

$$z(t) \longleftrightarrow Z(s) \quad \text{w/ROC } R$$

$$x(t) = tz(t) \stackrel{\mathcal{L}}{\longleftrightarrow} -\frac{dZ(s)}{ds} \quad \text{w/ROC } R$$

$$x(t) = tz(t) \longleftrightarrow -\frac{dZ(s)}{ds} \quad \text{w/Roc } R$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Next, compute the Laplace transform of $z(t) = e^{-2|t|}$.

$$z(t) = \begin{cases} e^{-2t} & t > 0 \\ e^{2t} & t < 0 \end{cases}$$

$$= e^{-2t}u(t) + e^{2t}u(-t)$$

Evaluate the transforms of each term:

$$e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}(s) > -2$$

$$e^{2t}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} -\frac{1}{s-2}, \quad \mathrm{Re}(s) < 2$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Put these together:

$$Z(s) = \frac{1}{s+2} - \frac{1}{s-2} \qquad -2 < \mathrm{Re}(s) < 2$$

To get $X(s)$...

$$X(s) = -\frac{dZ(s)}{ds} \qquad -2 < \mathrm{Re}(s) < 2$$

$$= -\frac{d}{ds}\left[(s+2)^{-1} - (s-2)^{-1}\right]$$

$$= -\left[-(s+2)^{-2} + (s-2)^{-2}\right]$$

$$= \frac{1}{(s+2)^2} - \frac{1}{(s-2)^2} = \frac{(s-2)^2 - (s+2)^2}{(s+2)^2(s-2)^2}$$

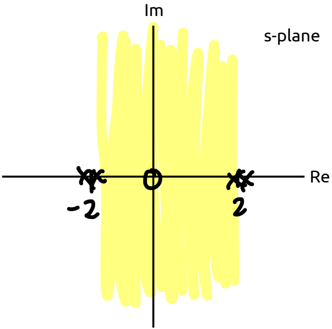
$$= \frac{s^2 - 4s + 4 - s^2 - 4s - 4}{(s+2)^2(s-2)^2} = \frac{-8s}{(s+2)^2(s-2)^2}$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Let's make a pole-zero plot:

$$X(s) = \frac{-8s}{(s+2)^2(s-2)^2} \qquad -2 < \mathrm{Re}(s) < 2$$



While computing Laplace transforms for their own sake is fun, we actually want to use them for something use: analysis and characterization of LTI systems.

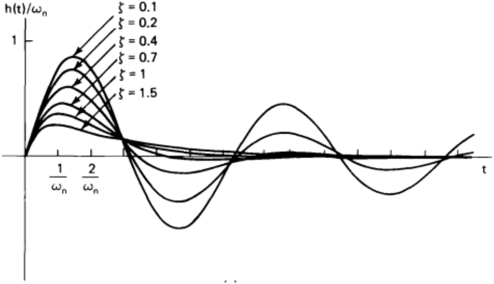
Recall the convolution property:

$$Y(s) = H(s)\, X(s)$$

The ROC of the system function (transfer function) can tell us a lot about a system!

$H(s)$ and causality

Recall that a system is causal only if its impulse response $h(t) = 0$ for $t < 0$ (see Piazza post @161).

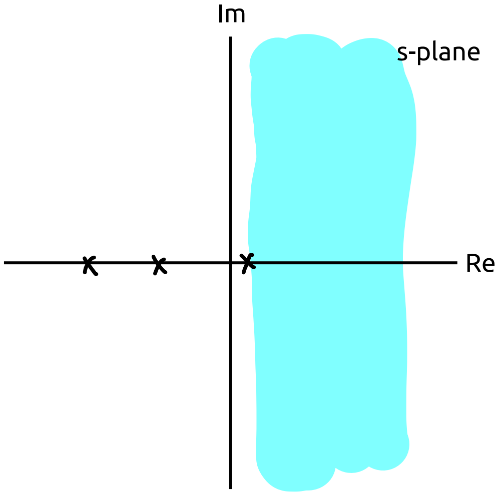


Means $h(t)$ is right-sided, so its ROC is a right-half plane.

Image credit: Oppenheim 6.5

$H(s)$ and causality

Note that the converse is not necessarily true! But if $H(s)$ is rational, the ROC is the right-half plane to right of right-most pole.



$H(s)$ and stability

Our original criteria for stability in terms of impulse response was if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

then the system is stable.

Related also to the Dirichlet conditions: if a signal is absolutely integrable, its **Fourier transform** converges.

An LTI system with rational $H(s)$ is stable if and only if the ROC of its system function includes the entire $j\omega$ axis ($\mathrm{Re}(s) = 0$), and there are not more zeros than poles.

$H(s)$ and causality / stability

9.28. Consider an LTI system for which the system function $H(s)$ has the pole-zero pattern shown in Figure P9.28.

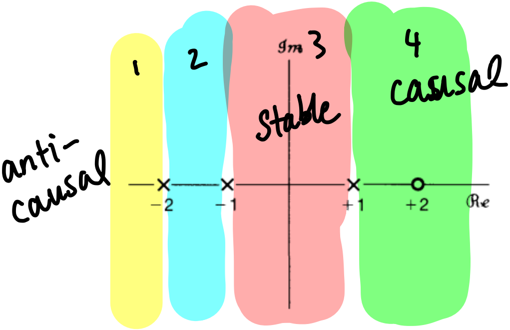


Figure P9.28

(a) Indicate all possible ROCs that can be associated with this pole-zero pattern.
(b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

Image credit: Oppenheim 9.28

$H(s)$ and causality / stability

9.28. Consider an LTI system for which the system function $H(s)$ has the pole-zero pattern shown in Figure P9.28.

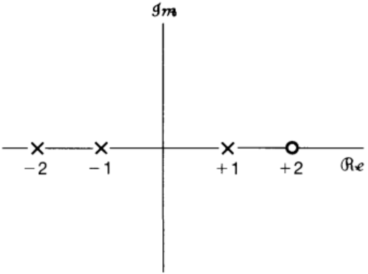


Figure P9.28

(a) Indicate all possible ROCs that can be associated with this pole-zero pattern.

(b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

Image credit: Oppenheim 9.28

Systems described by constant-coefficient differential equations

Recall the situation with the Fourier transform (lecture 10):

Fourier transforms and systems described by differential equations

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

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Same deal here. If system is described by the DE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

then its system function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Placement of zeros and poles is dictated by solutions of $x(t)$ and $y(t)$ stuff respectively.

Systems described by constant-coefficient differential equations

(Oppenheim 9.31) Consider a CT LTI system described by the DE

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

- Determine $H(s)$ as a ratio of polynomials in s and sketch the pole-zero plot.
- Determine $h(t)$ for each of the following cases:
    1. The system is stable
    2. The system is causal
    3. The system is neither causal nor stable

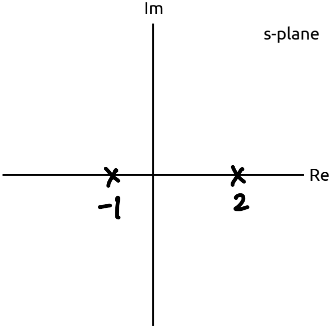
(Oppenheim 9.31) Consider a CT LTI system described by the DE

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

- Determine $H(s)$ as a ratio of polynomials in s and sketch the pole-zero plot.

$$H(s) = \frac{1}{s^2 - s - 2}$$

$$= \frac{1/3}{s-2} - \frac{1/3}{s+1}$$



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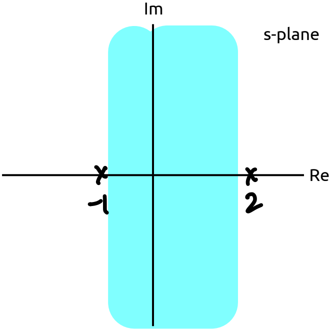
- Determine $h(t)$ for each of the following cases:
  1. The system is stable
  2. The system is causal
  3. The system is neither causal nor stable

ROC : $-1 < Re(s) < 2$

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$$



Systems described by constant-coefficient differential equations

(Oppenheim 9.31) Consider a CT LTI system described by the DE

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

- Determine $h(t)$ for each of the following cases:
  1. The system is stable
  2. The system is causal
  3. The system is neither causal nor stable

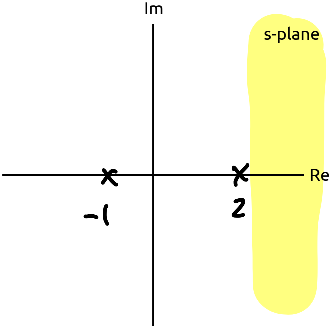
$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

ROC $Re(s) > 2$

ROC $\quad \mathrm{Re}(s) > 2 \quad$ (RHP)

$h(t) = \frac{1}{3} e^{2t} u(t) - \frac{1}{3} e^{-t} u(t)$

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)$$



Systems described by constant-coefficient differential equations

(Oppenheim 9.31) Consider a CT LTI system described by the DE

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

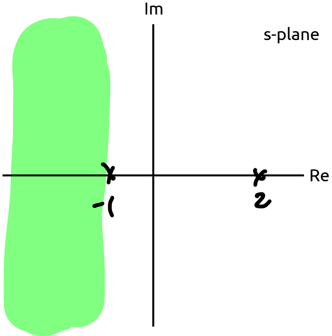
- Determine $h(t)$ for each of the following cases:
  1. The system is stable
  2. The system is causal
  3. The system is neither causal nor stable

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

ROC $\quad \mathrm{Re}(s) < -1 \quad$ (LHP)

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$



Today

Learning outcomes:

- apply key properties of the Laplace transform to its computation
- use the Laplace transform to determine whether a system is causal or stable
- compute the Laplace transform of systems described by constant-coefficient DEs

Oppenheim practice problems: 9.13-9.16, 9.21, 9.22, 9.26, 9.29, 9.32, 9.33

For next time

Content:

- the Laplace transform and feedback systems
- introducing the z-transform

Action items:

1. Assignment 6 due Thursday at 23:59
2. Assignment 7 released soon

Recommended reading:

- From this class: Oppenheim 9.5-9.7
- For next class: 9.7, 11.0-11.2, 10.1-10.3