

ELEC 221 Lecture 22

The Laplace transform

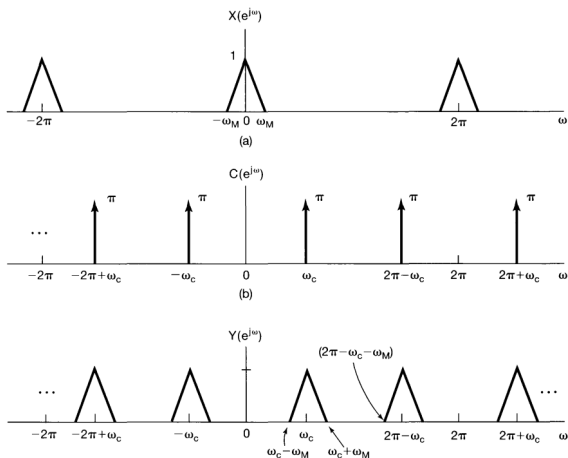
Thursday 24 November 2022

Announcements

- Midterm 2 available for pickup
- Quiz 10 Tuesday (last quiz)
- Assignment 6 (computational) due on Tuesday at 23:59
- Final assignment (pen and paper) released early next week

Last time

We did DT sinusoidal amplitude modulation.



We can still do frequency-division multiplexing, but we may need to upsample and shrink the spectra so things fit.

Last time

We briefly explored how frequency modulation works

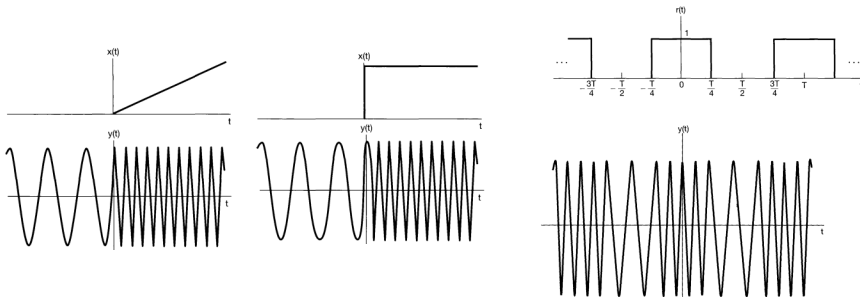
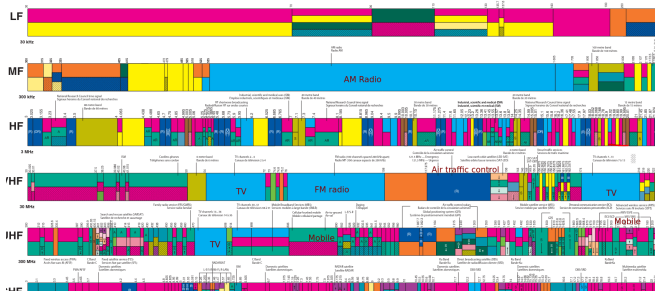


Image credit: Oppenheim 8.7

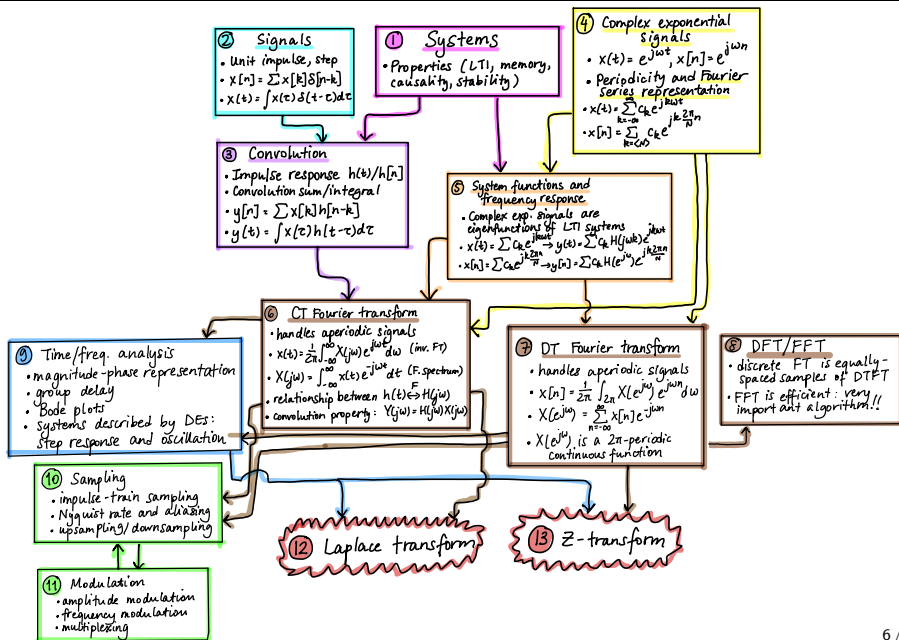
Last time

We discussed how cell phones are radios and how the radio spectrum gets divided (and auctioned off).



See the full graphic here: [https://www.ic.gc.ca/eic/site/smt-gst.nsf/vwapj/2018_Canadian_Radio_Spectrum_Chart.PDF/\\$FILE/2018_Canadian_Radio_Spectrum_Chart.PDF](https://www.ic.gc.ca/eic/site/smt-gst.nsf/vwapj/2018_Canadian_Radio_Spectrum_Chart.PDF/$FILE/2018_Canadian_Radio_Spectrum_Chart.PDF)

The course so far



The course so far

Wayyyyyy back in lecture 3:

LTI systems and complex exponential functions

Write $x(t) = e^{st}$. Then:

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\&= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau \\&= \int_{-\infty}^{\infty} e^{st}e^{-s\tau}h(\tau)d\tau \\&= e^{st} \int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau \\&= e^{st}H(s)\end{aligned}$$

The course so far

Wayyyyy back in lecture 3:

LTI systems and complex exponential functions

To summarize:

$$x(t) = e^{st} \rightarrow y(t) = H(s) \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

The course so far

Wayyyyy back in lecture 3:

...so what?

If all the $x_i(t)$ are complex exponential functions and

$$x(t) = e^{st} \rightarrow y(t) = H(s) \cdot e^{st}$$

then

$$x(t) = \sum_k c_k e^{s_k t} \rightarrow y(t) = \sum_k c_k H(s_k) e^{s_k t}$$

The response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

How can we express arbitrary signals as superpositions of complex exponentials?

The course so far

Wayyyyy back in lecture 3:

The Fourier series

Let's consider a special set of signals¹:

$$x(t) = e^{st} = e^{j\omega t}$$

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

¹We will see the general case at the end of the course.

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

The Laplace transform

Consider a general complex exponential signal,

$$x(t) = e^{st}$$

Input into LTI system with impulse response $h(t)$:

$$y(t) = h(t) * x(t) = H(s)x(t)$$

If $s = j\omega$: **Fourier transform**

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} h(t) dt$$

If $s = \sigma + j\omega$: (bilateral) **Laplace transform**

$$H(s) = \int_{-\infty}^{\infty} e^{-st} h(t) dt$$

The Laplace transform

More generally, the Laplace transform of an arbitrary signal is

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t) dt$$

We will write

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

The Laplace transform

We can relate the Laplace and Fourier transforms. Consider

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t) dt$$

for $s = \sigma + j\omega$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-(\sigma+j\omega)t} x(t) dt \\ &= \int_{-\infty}^{\infty} (e^{-\sigma t} x(t)) e^{-j\omega t} dt \\ &= \mathcal{F}(e^{-\sigma t} x(t)) \end{aligned}$$

The Laplace transform

Example: consider the signal $x(t) = e^{-at}u(t)$.

What is $X(j\omega)$?

$$\begin{aligned}X(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\&= \int_0^{\infty} e^{-at}e^{-j\omega t}dt \\&= \frac{1}{j\omega + a}\end{aligned}$$

Recall: conditions on a ?

The Laplace transform

What is the Laplace transform?

$$\begin{aligned}X(s) &= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt \\&= \int_0^{\infty} e^{-(a+\sigma)t} e^{-j\omega t} dt \\&= \frac{1}{(\sigma + a) + j\omega} \\&= \frac{1}{a + s}\end{aligned}$$

What are the conditions here?

$$\sigma + a > 0$$

$$\operatorname{Re}(s) > -a$$

The Laplace transform

Take a closer look at $x(t) = e^{-at}u(t)$:

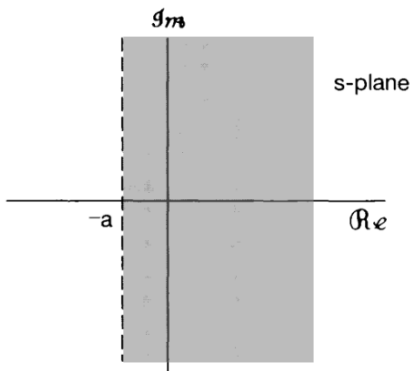
$$\mathcal{F}(x(t)) = X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0$$

$$\mathcal{L}(x(t)) = X(s) = \frac{1}{a + s}, \quad \text{Re}(s) > -a$$

The Laplace transform exists in some regions where the Fourier transform does not!

The Laplace transform

We must specify for which s the Laplace transform is valid.



This is called the **region of convergence** (ROC).

The Laplace transform

Exercise: what is the Laplace transform of

$$x(t) = -e^{-at}u(-t)$$

The Laplace transform

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$$x(t) = -e^{-at}u(-t)$$

Solution:

$$\begin{aligned}X(s) &= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt \\&= -\int_{-\infty}^0 e^{-(a+s)t}dt \\&= \frac{1}{a+s}\end{aligned}$$

Conditions are different though: need $\operatorname{Re}(s) < -a$.

The Laplace transform

Exercise: what is the Laplace transform of

$$x(t) = -e^{-at}u(-t)$$

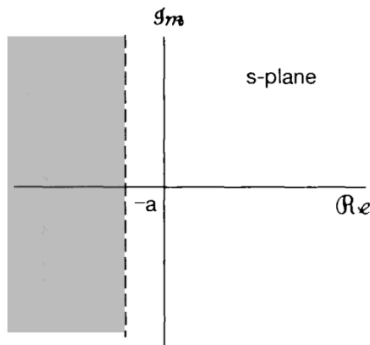
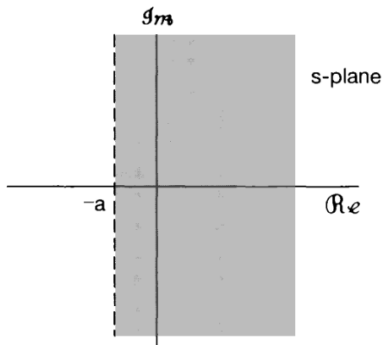
Solution:

$$\begin{aligned}X(s) &= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt \\&= -\int_{-\infty}^0 e^{-(a+s)t}dt \\&= \frac{1}{a+s}\end{aligned}$$

Conditions are different though: need $\operatorname{Re}(s) < -a$.

The Laplace transform

Multiple signals can have the same algebraic Laplace transform, but different ROCs.



The Laplace transform

Exercise: what is the Laplace transform of

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

and what is its region of convergence?

Hint: the Laplace transform is also linear!

The Laplace transform

Solution: by linearity,

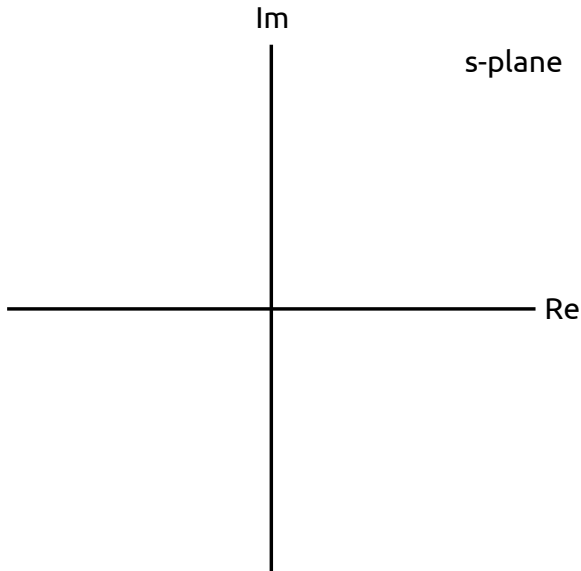
$$\begin{aligned}X(s) &= 3\mathcal{L}(e^{-2t}u(t)) - 2\mathcal{L}(e^{-t}u(t)) \\&= \frac{3}{s+2} - \frac{2}{s+1} \\&= \frac{s-1}{s^2+3s+2}\end{aligned}$$

To determine its ROC,

- first term tells us $\text{Re}(s) > -1$
- second term tells us that $\text{Re}(s) > -2$

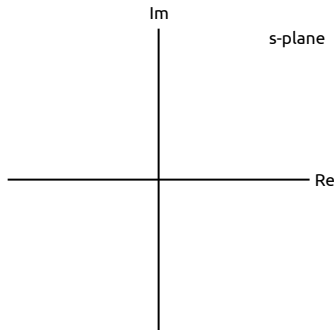
The Laplace transform

Let's draw the ROC:



Pole-zero plots

Quite often the Laplace transforms will be rational polynomials of s . Generally the roots of these polynomials are indicated on the plots. Use \times for denominator (poles), \circ for numerator (zeros):



This is called a pole-zero plot. (May also have poles/zeros at infinity if degree of polynomials is different)

Exercise: compute the Laplace transform of

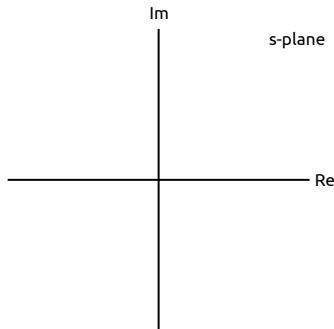
$$x(t) = -2e^{-3t}u(t) + 4e^{-4t}u(t)$$

and draw its pole-zero plot.

Pole-zero plots

Solution:

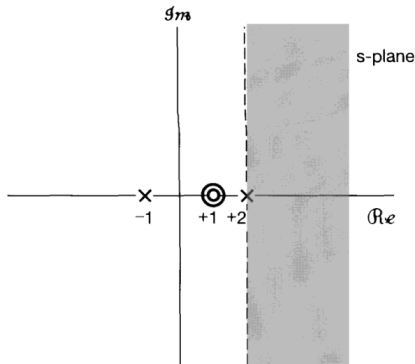
$$\begin{aligned}X(s) &= -2\frac{1}{s+3} + 4\frac{1}{s+4} \\&= \frac{-2s-8+4s+12}{(s+3)(s+4)} \\&= \frac{2(s+2)}{s^2+7s+12}\end{aligned}$$



Regions of convergence

The ROC has many nice properties:

- if ROC doesn't contain $j\omega$ axis, FT does not converge
- ROC is strips parallel to $j\omega$ axis
- ROC of rational Laplace transform contains no poles



$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}, \quad \Re(s) > 2$$

Regions of convergence

If $x(t)$ has finite duration and is absolutely integrable, the ROC is the entire s -plane.

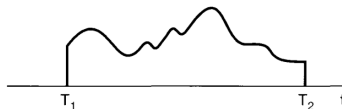


Figure 9.4 Finite-duration signal.

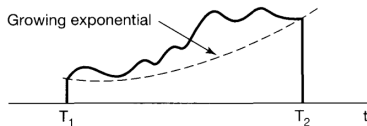
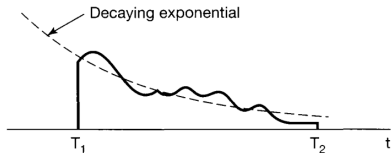
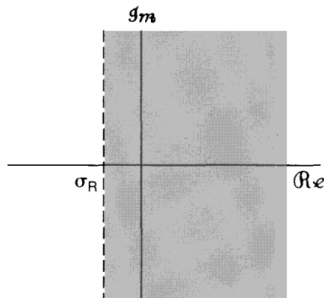
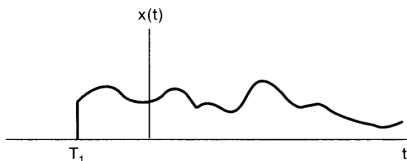


Image credit: Oppenheim 9.2

Right-sided signals

If $x(t)$ is right sided and $\text{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\text{Re}(s) > \sigma_0$ are also in the ROC.

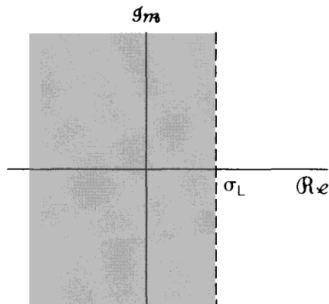
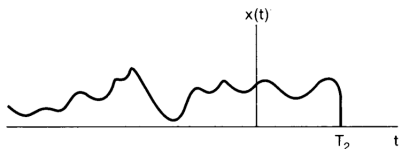


This ROC is called a **right-half plane**.

Intuition: if $\text{Re}(s) = \sigma_1 > \sigma_0$ the exponential in $x(t)e^{-\sigma t}$ decays even faster and will still converge.

Left-sided signals

If $x(t)$ is left sided and $\text{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\text{Re}(s) < \sigma_0$ are also in the ROC.



This ROC is called a **left-half plane**.

Image credit: Oppenheim 9.2

Two-sided signals

Any guesses?

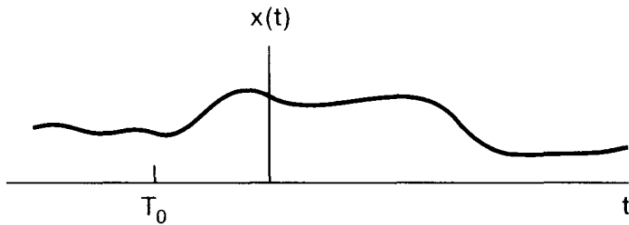
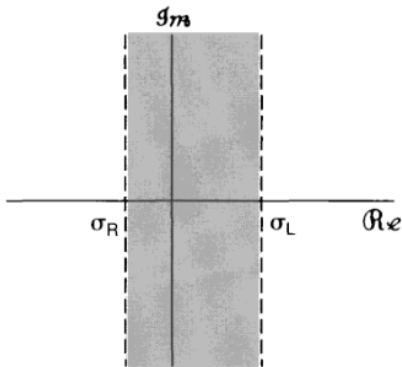


Image credit: Oppenheim 9.2

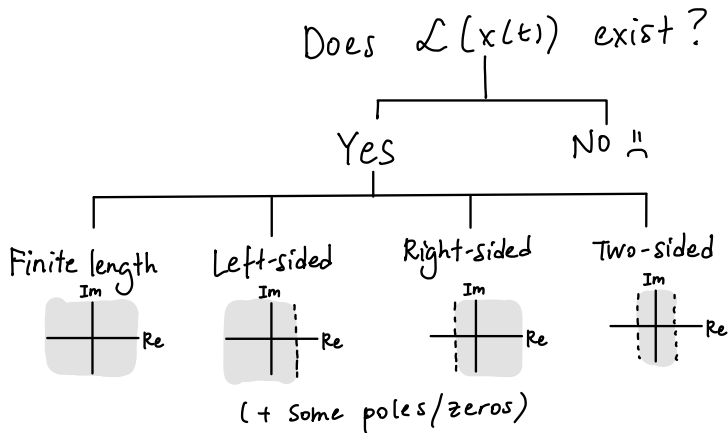
Two-sided signals



This is only the case if there was actually overlap - otherwise it doesn't exist!

Regions of convergence

For any signal $x(t)$...

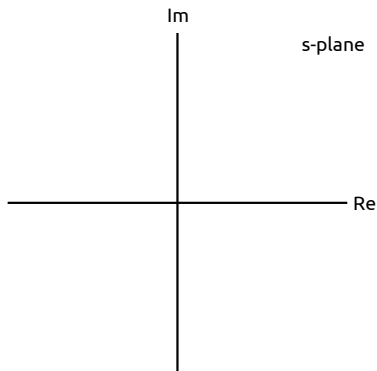
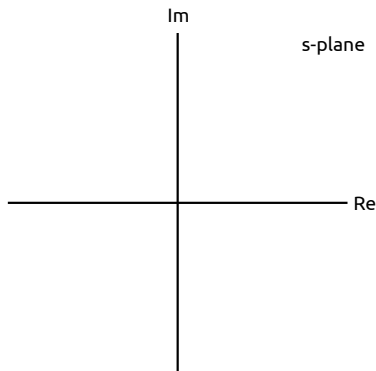


Regions of convergence

(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

$$\frac{s - 1}{(s + 2)(s + 3)(s^2 + s + 1)}$$

in its ROC? (Hint: draw pole-zero plot and identify the regions)



Inverse Laplace transforms

$$X(s) = X(\sigma + j\omega) = \mathcal{F}(x(t)e^{-\sigma t}) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

From this, we can invert:

$$\begin{aligned}x(t)e^{-\sigma t} &= \mathcal{F}^{-1}(X(\sigma + j\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \\x(t) &= \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega\end{aligned}$$

Make a change of variables $ds = jd\omega$:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

... we are not going to integrate this.

Inverse Laplace transforms

Suppose the Laplace transform has the form

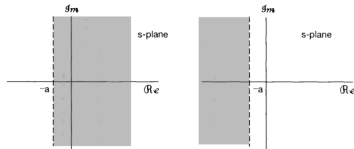
$$X(s) = \sum_{i=1}^m \frac{A_i}{s + a_i}$$

where degree of denominator is higher than numerator.

To invert this, we can use our handy identities, BUT the region of convergence does matter.

The Laplace transform

Multiple signals can have the same algebraic Laplace transform, but different ROCs.



Inverse Laplace transforms

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \operatorname{Re}(s) < -3$$

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

Oppenheim practice problems: 9.1-9.9, 9.21, 9.26

For next time

Content:

- properties and system analysis with Laplace transform

Action items:

1. Quiz 10 on Tuesday
2. Assignment 6 due Tuesday at 23:59

Recommended reading:

- From this class: Oppenheim 9.0-9.3. 9.5 (skip 9.4)
- For next class: 9.5-9.8 (skip 9.9)