ELEC 221 Lecture 10
Introducing the Fourier transform

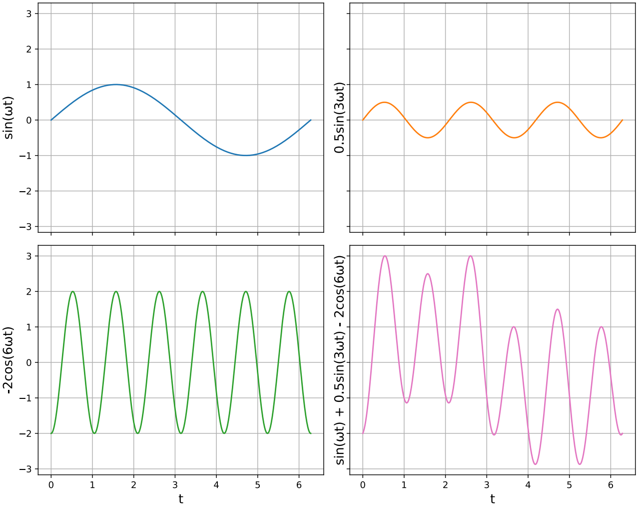
Thursday 10 October 2024

☆ more blank space on next one
☆ tutorial questions reflect exam contents better

- Midterm postmortem
- No tutorial on Monday (Thanksgiving holiday)
- Quiz 5 on Tuesday (based on today's material)

Last time (recap)

We've seen the Fourier series representation of **periodic** signals:



Last time (recap)

CT synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{c_k}_{\text{coefficients}} e^{jk\omega t}$$

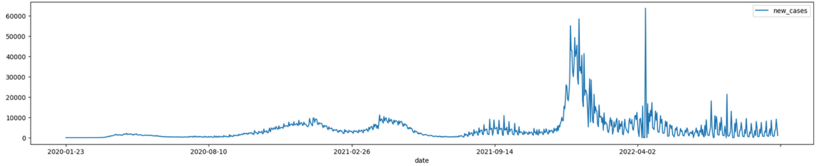
CT analysis equation:

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t} dt$$

A periodic signal is composed of complex exponential signals with *integer multiples of the fundamental frequency* only (harmonics).

Last time (recap)

In tutorial assignment 2, we saw signals that *weren't periodic*:



But, we were still doing *something* with Fourier analysis:

```
fourier_spectrum = np.fft.rfft(case_data)
```

Learning outcomes:

- Distinguish between the CT Fourier series and Fourier transform
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

The **Fourier transform** generalizes the Fourier series to **aperiodic signals**. It involves all possible frequencies.

not periodic

Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega t} dt$$

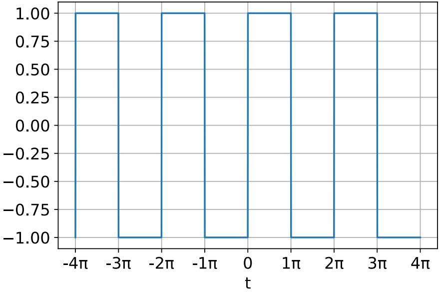
Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

How do we get here?

Towards the Fourier transform

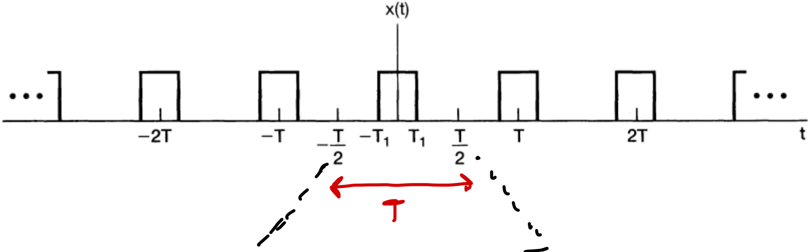
Previously, we looked at a 2π-periodic square wave:



We derived its Fourier series representation

$$x(t) = \sum_{k=1}^{\infty} \frac{4}{k\pi} \sin(kt) \quad \text{only odd } k$$

Let's generalize this. Consider the following square wave:



$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

Image credit: Oppenheim chapter 4.1

Towards the Fourier transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Let's compute its Fourier coefficients.

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega t}$$

Start with c_0:

$$C_0 = \frac{2T_1}{T}$$

$$C_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}$$

Towards the Fourier transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Now the c_k:

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$C_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\cdot\frac{1}{-jk\omega}\, e^{-jk\omega t}\Big|_{-T_1}^{T_1}$$

$$= \frac{1}{T}\cdot\frac{1}{-jk\omega}\left(e^{-jk\omega T_1} - e^{jk\omega T_1}\right) = \frac{1}{T}\cdot\frac{1}{\cancel{-j}k\omega}\cdot\left(-2\cancel{j}\sin(k\omega T_1)\right) = \frac{2\sin(k\omega T_1)}{k\omega T}$$

What does this function look like?

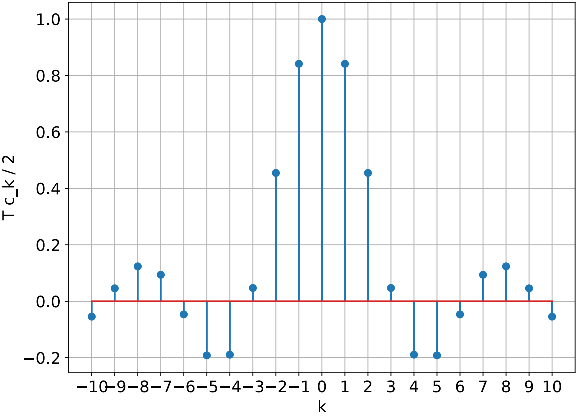
$$C_0 = \frac{2T_1}{T} \qquad C_k = \frac{2\sin(k\omega T_1)}{k\omega T}$$

Rearrange and express as a function of k:

$$f(k) = \frac{T C_k}{2} = \begin{cases} T_1 & k=0 \\ \dfrac{\sin(k\omega T_1)}{k\omega} & k \neq 0 \end{cases}$$

Let's plot this: set $T_1 = 1$, and $T = 2\pi$, so $\omega = \frac{2\pi}{T} = 1$.

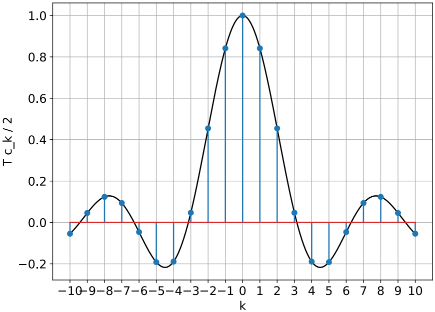
$$f(k) = \frac{T c_k}{2} = \begin{cases} T_1 & k=0 \\ \frac{\sin(k\omega T_1)}{k\omega} & k \neq 0 \end{cases}$$



with $T = 2\pi$

$\omega = 1$

$T_1 = 1$



These are **samples** of the function

$$f(k) = \begin{cases} T_1 & k=0 \\ \dfrac{\sin(k\omega T_1)}{k\omega} & k \neq 0 \end{cases}$$

at *integer values of k*.

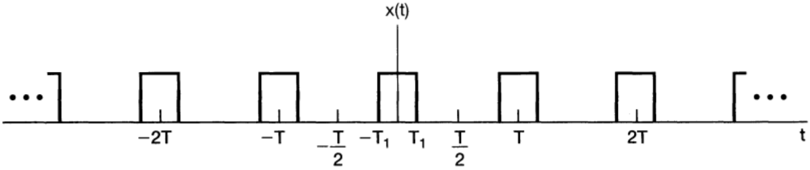
Let's consider this instead as a function of frequency, $\tilde{\omega} = k\omega$:

$$f(\tilde{\omega}) = \begin{cases} T_1 & \tilde{\omega} = 0 \\ \dfrac{\sin(\tilde{\omega} T_1)}{\tilde{\omega}} & \tilde{\omega} \neq 0 \end{cases}$$

The Fourier coefficients are samples of this function at *integer multiples* of fundamental frequency, $\tilde{\omega} = k\omega$, where $\omega = 2\pi/T$.

$$c_k = \frac{2}{T} f(\tilde{\omega}) = \frac{2}{T} f(k\omega)$$

Suppose T grows, but T_1 stays the same:



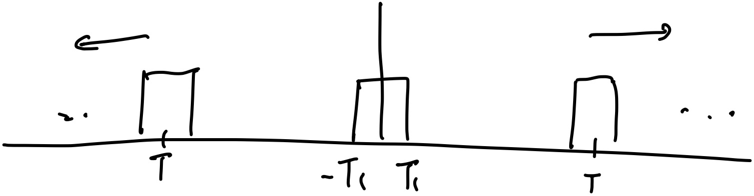


Image credit: Oppenheim chapter 4.1

Initially the spacing of samples is integer multiples of $\omega = 2\pi/T$.

ω gets smaller as T gets bigger

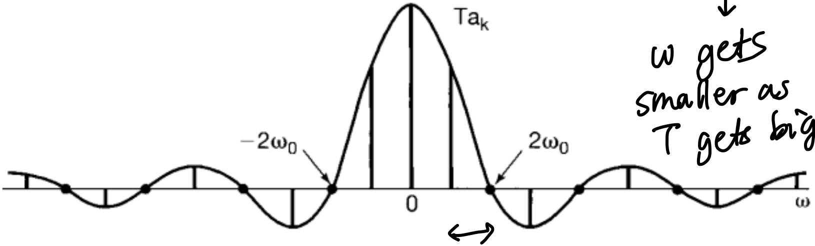


Image credit: Oppenheim chapter 4.1

As T grows, $\omega = 2\pi/T$ becomes smaller and smaller, so the integer multiples of it get closer and closer together.

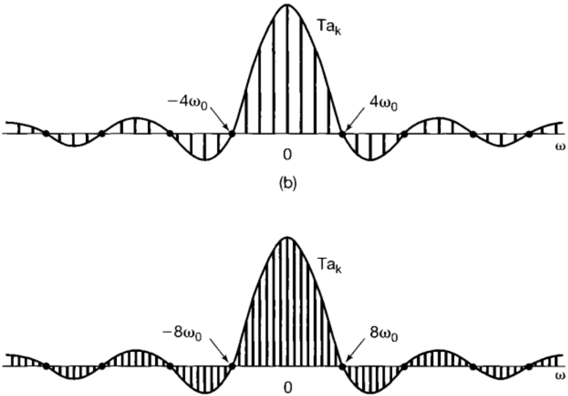


Image credit: Oppenheim chapter 4.1

Eventually, ω becomes so small that instead of

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

we may as well just consider the sum over integer multiples as a continuous integral over all possible ω:

$$x(t) \sim \int_{-\infty}^{\infty} c_\omega e^{j\omega t} d\omega$$

...but what does this have to do with non-periodic signals?

Given any aperiodic signal $x(t)$, we can always "pretend" it's periodic by constructing a **periodic extension**, $\tilde{x}(t)$ with period T.

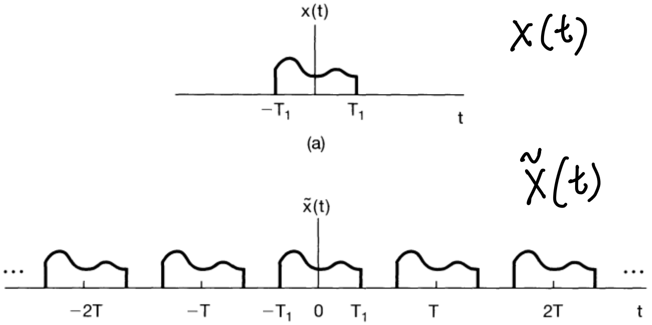


Image credit: Oppenheim chapter 4.1

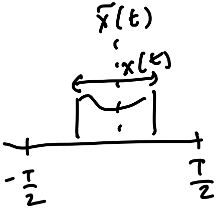
Motivation: Fourier transform

We can express $\tilde{x}(t)$ as a Fourier series (where $\omega = 2\pi/T$):

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$\boxed{c_k = \frac{1}{T} \int_T \tilde{x}(t) e^{-jk\omega t} \, dt}$$

What happens to the coefficients?



$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega t}\, dt}$$

Let's define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

so that

$$c_k = \frac{1}{T} X(jk\omega) \quad \Rightarrow \quad \text{Fourier coeffs of } \tilde{x}(t)$$

We can put this back in our Fourier series:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega) e^{jk\omega t} \qquad T = \frac{2\pi}{\omega}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \cdot \omega$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega$$

Consider what happens when $T \to \infty$...

1. $\tilde{x}(t)$ will look just like $x(t)$ for large enough T

$$\lim_{T \to \infty} \tilde{x}(t) = x(t)$$

2. ω will get smaller and smaller

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega) e^{jk\omega t} \omega \implies \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = x(t)$$

Inverse Fourier transform (synthesis equation):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Fourier transform (analysis equation, or Fourier *spectrum*):

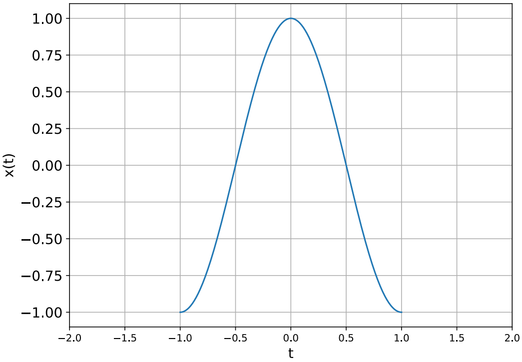
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Note: Sometimes the $1/2\pi$ prefactor appears on the spectrum, or sometimes both versions have $1/\sqrt{2\pi}$.

Example

Compute the Fourier spectrum of:

$$x(t) = \begin{cases} \cos(\pi t), & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$



$$x(t) = \begin{cases} \cos(\pi t), & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$

Start from the definition:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-1}^{1} \cos(\pi t) e^{-j\omega t} dt$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= \underbrace{\frac{1}{2} \int_{-1}^{1} e^{j(\pi-\omega)t} dt}_{\text{LHS}} + \underbrace{\frac{1}{2} \int_{-1}^{1} e^{-j(\pi+\omega)t} dt}_{\text{RHS}}$$

Example

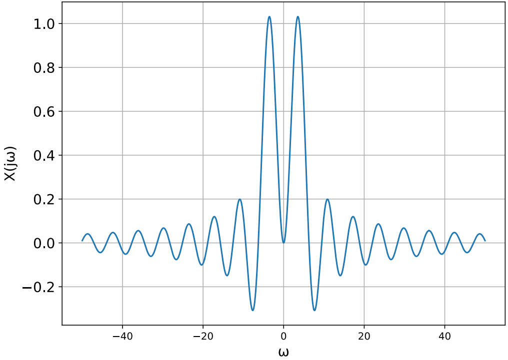
$$X(j\omega) = \frac{1}{2}\int_{-1}^{1} e^{j(\pi-\omega)t}dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\pi+\omega)t}dt$$

$$= \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega}$$

$$= \frac{\sin(\omega)}{\pi-\omega} - \frac{\sin(\omega)}{\pi+\omega}$$

Example

$$X(j\omega) = \frac{\sin(\omega)}{\pi - \omega} - \frac{\sin(\omega)}{\pi + \omega}$$



You've actually already seen the Fourier transform before...

Write a signal as a combination of shifted, weighted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Put this in an LTI system with impulse response $h(t)$:

$$x(t) \rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$x(t) \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

When the signal in question is a complex exponential,

$$x(t) = e^{j\omega t} \longrightarrow y(t) = \int_{-\infty}^{\infty} e^{j\omega\tau} h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} h(\tau)\,d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)d\tau$$

$$= x(t) \cdot H(j\omega)$$

The system function $H(j\omega)$, or frequency response

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)\, d\tau$$

is the **Fourier transform of the impulse response**!

We can use the inverse Fourier transform to obtain the impulse response from the frequency response:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} H(j\omega)\, d\omega$$

Learning outcomes:

- Distinguish between the CT Fourier series and Fourier transform
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

For next time

Content:

- Fourier transform for periodic signals
- Properties of the CT Fourier *transform*
- Time/frequency duality

Action items:

1. Quiz 5 Tuesday

Recommended reading:

- From today's class: Oppenheim 4.0-4.1
- Suggested problems: 4.1, 4.2a, 4.21abei, 4.22abde
- For next class: Oppenheim 4.2-4.4