ELECTED 221 Lecture 09
DT Fourier series; filters

Thursday 03 October 2024

Announcements

Q2.5 is updated;
please check PrairieLearn

- Assignment 2 due Saturday 23:59 (final question removed, deferred to A3)
- Midterm 1 on Tuesday (bring your student ID and writing implements)

We explored periodic DT complex exponential signals:

$$x[n] = e^{j\omega n} = e^{j\cdot\frac{2\pi}{N}n}$$

↳ fundamental period

We found that these signals behave differently than CT signals...

Difference 1: we only need to consider ω in the range $\underline{[0, 2\pi)}$.

$$x[n] = e^{j(\omega+2\pi)n}$$

$$= e^{j\omega n}\cdot\underbrace{e^{j2\pi n}}_{1}$$

$$= e^{j\omega n}$$

Difference 2: there are additional criteria for periodicity.

$$x[n+N] = e^{j\omega(n+N)}$$
$$= e^{j\omega n} \cdot \underbrace{e^{j\omega N}}_{1}$$

$$\omega N = 2\pi \cdot m$$
$\downarrow$
m is integer

Example: $x[n] = \sin(5\pi n/7)$ is periodic.

- In CT, period of $x(t) = \sin(5\pi t/7)$ is $T = \frac{2\pi}{\omega} = \frac{14}{5}$
- In DT, period of $x[n] = \sin(5\pi n/7)$ is $N = 14$

Example: $x[n] = \sin(5n/7)$ is NOT periodic in DT.

$$\frac{5\pi n}{7} = 2\pi \cdot m$$

$$\sum_{k=-\infty}^{\infty} e^{j\omega k t}$$

Difference 3: there are only finitely many harmonics.

$$x_0[n] = 1$$
$$x_1[n] = e^{j\cdot\frac{2\pi}{N}n} \qquad \Rightarrow x_k[n] = e^{j\frac{2\pi}{N}kn}$$
$$x_2[n] = e^{j\cdot 2\frac{2\pi}{N}n}$$
$$\vdots$$
$$x_{N-1}[n] = e^{j(N-1)\cdot\frac{2\pi}{N}n}$$
$$x_N[n] = e^{jN\frac{2\pi}{N}n} = e^{j\cdot 2\pi n} = 1$$
$$\vdots$$

Last time

We found DT complex exponential signals are also eigenfunctions of LTI systems.

$$x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega) \cdot e^{j\omega t}$$

$$x[n] = e^{jm\omega n}$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{jm\omega(n-k)}\, h[k] \\ &= e^{jm\omega n} \sum_{k=-\infty}^{\infty} e^{-jkm\omega}\, h[k] \\ &= x[n] \cdot H(e^{j\omega}) \end{aligned}$$

We need a Fourier series representation of DT signals:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\omega n}$$

Today

Learning outcomes:

- Evaluate Fourier series coefficients of DT signals
- Define filter systems through the frequency response
- ~~Distinguish between finite impulse response and infinite impulse response filters in DT~~

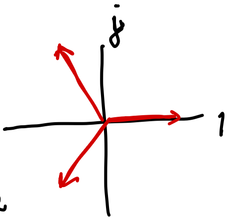
$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\omega n}$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$1 + e^{j\frac{2\pi}{5}n} + e^{j2\cdot\frac{2\pi}{5}n} + \cdots$$

Leverage the following identity about complex numbers:

$$\sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} = \begin{cases} N & \text{if } k=0,\pm N,\pm 2N\ldots \\ 0 & \text{otherwise} \end{cases}$$



We will multiply on both sides, and sum.

$N=2$: $e^{j\pi\cdot 0}=1$ $\quad e^{j\pi\cdot 1}\;-1$

$N=3$: $e^{j\frac{2\pi}{3}\cdot 0}$ $\quad e^{j\frac{2\pi}{3}}$ $\quad e^{j\frac{4\pi}{3}}$ $\quad e^{j\frac{2\pi}{N}k}$

DT Fourier coefficients

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\cdot\frac{2\pi}{N}n}$$

$$e^{-jm\frac{2\pi}{N}n} \quad x[n] = \sum_{k=0}^{N-1} c_k e^{-jm\frac{2\pi}{N}n} e^{jk\frac{2\pi}{N}n}$$

$$\sum_{n=0}^{N-1} e^{-jm\frac{2\pi}{N}n} \quad x[n] = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{-jm\frac{2\pi}{N}n} e^{jk\cdot\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{jn\frac{2\pi}{N}(k-m)}$$

$$= c_m \cdot N$$

$$c_m = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jm\frac{2\pi}{N}n}$$

$$x(t) = \sum c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega t}\,dt$$

DT Fourier coefficients

DT Fourier synthesis equation

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi}{N}n}$$

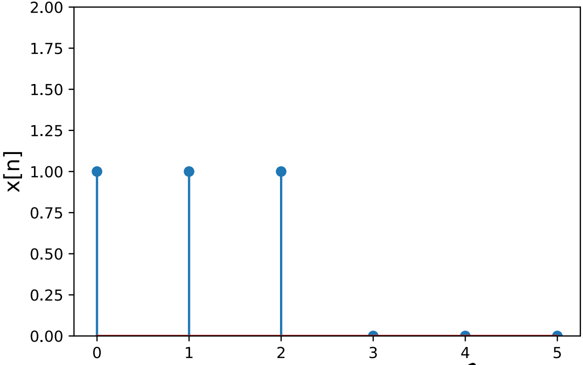
DT Fourier analysis equation

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$$

$N=5$

$$\begin{array}{ccccccccccccc}
c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \cdots \\
 & c_2^* & c_1^* & & & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & \cdots \\
 & & & & & & c_2^* & c_1^* & & & &
\end{array}$$

Compute the Fourier coefficients of this signal:



$N=6$

$$c_k = \frac{1}{6}\sum_{n=0}^{5} x[n] e^{-jk\frac{2\pi}{6}n} = \frac{1}{6}\sum_{n=0}^{5} x[n] e^{-\frac{jk\pi n}{3}}$$

Exercise: the DT square wave

$$c_k = \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-\frac{jk\pi n}{3}}$$

$$c_0 = \frac{1}{6}\sum_{n=0}^{5} x[n] = \frac{1}{6}\left[1+1+1+0+0+0\right] = \frac{1}{2}$$

$$c_1 = \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-\frac{j\pi n}{3}} = \frac{1}{6}\left[1 + e^{-j\frac{\pi}{3}} + e^{-j\cdot\frac{2\pi}{3}}\right] = \frac{1}{6}\left[1-\sqrt{3}j\right] \Rightarrow c_5$$

$$c_2 = \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi n}{3}} = \frac{1}{6}\left[1 + e^{-j\frac{2\pi}{3}} + e^{+\frac{2\pi j}{3}}\right] = 0$$

$$c_3 = \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-j\pi n} = \frac{1}{6}\left[1 + \underbrace{e^{-j\pi}}_{-1} + \underbrace{e^{-2j\pi}}_{+1}\right] = \frac{1}{6}$$

$$c_4 = 0$$

$$c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_2^* \quad c_1^*$$

$$c_5 = \frac{1}{6}\left[1+\sqrt{3}j\right] = c_1^*$$

Properties of DT Fourier coefficients

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Exercise: the DT square wave

Let's try the same thing as we did in CT:

- shift the signal left by 1
- speed it up by 2

try it yourself!

We've showed a couple important things so far.

Signals can be expressed in terms of weighted, shifted impulses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \qquad x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

If we know what an LTI system does to a unit impulse (the impulse response $h(t)$ or $h[n]$), we can learn what it does to any signal.

This was the convolution integral and sum:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Where do we go from here?

Complex exponential signals are eigenfunctions of LTI systems:

$$x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega) e^{j\omega t} \qquad H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(e^{j\omega}) e^{j\omega n} \qquad H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

$H(j\omega)$ in CT, and $H(e^{j\omega})$ in DT, are the **frequency response** of the system (more generally, system functions).

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jwkt} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} C_k H(jkw) e^{jwkt}$$

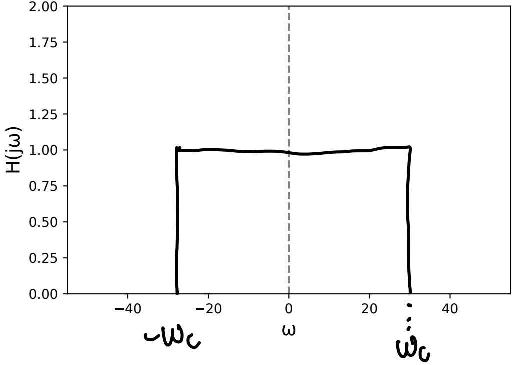
Through careful choice of $H(j\omega)$ or $H(e^{j\omega})$, we can change the behaviour of a system.

Example

What does a system with the following frequency response do?

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$y(t) = \sum_{k=-\infty}^{\infty} C_k H(jk\omega) e^{j\omega kt}$$



Filters

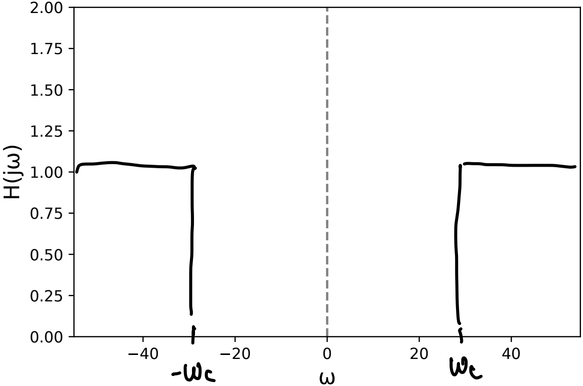
Filters are LTI systems that can be used to separate out, combine, or modify the components of a signal at specific frequencies.

Two key types:

- **Frequency-shaping**: change the amplitudes of parts of a signal at specified frequencies
- **Frequency-selective**: eliminate or attentuate parts of a signal at specified frequencies

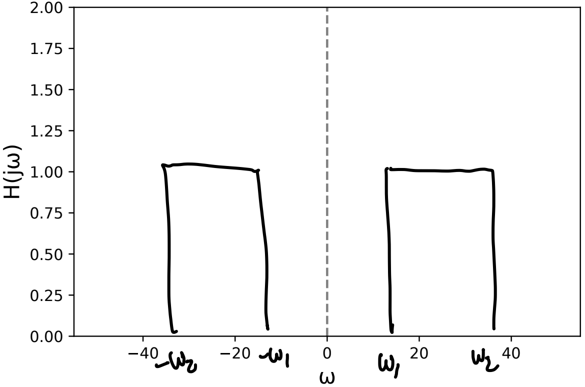
We can also consider an ideal **highpass filter**:

$$H(j\omega) = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & |\omega| \leq \omega_c \end{cases}$$



Or an ideal **bandpass filter**:

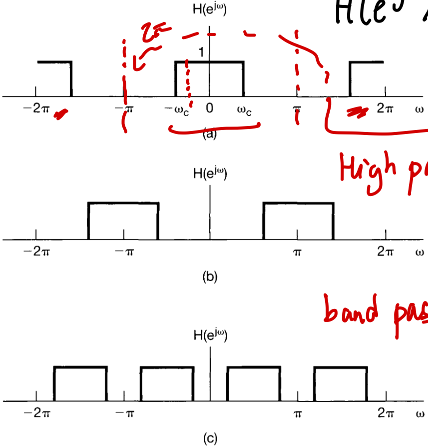
$$H(j\omega) = \begin{cases} 1 & \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{otherwise} \end{cases}$$



Recall that in DT, the frequency increases up until $\omega = \pi$, then decreases as it approaches 2π.

$$H(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| < \omega_c \\ 0 & \omega_c \le |\omega| < \pi \end{cases}$$

low pass



$\frac{3\pi}{2}$ ↕ $-\frac{\pi}{2}$

High pass

band pass

Image credit: Oppenheim chapter 3.9.

Recap

Today's learning outcomes were:

- Evaluate Fourier series coefficients of DT signals
- Define filter systems through the frequency response
- ~~Distinguish between finite impulse response and infinite impulse response filters in DT~~

For next time

Action items:

1. Assignment 2 due Saturday 23:59
2. Study for Midterm 1
3. Suggest tutorial topics on Piazza

Recommended reading:

- From today's class: Oppenheim 3.6-3.12
- Suggested problems: 3.2, 3.10-3.17, 3.27-3.31, 3.39