ELEC 221 Lecture 03
DT impulse response and convolution sum; CT convolution integral

Thursday 12 September 2024

- Tutorial assignment 1 Monday 16 Sept 23:59
- Assignment 1 due Thursday 19 Sept 23:59
- Monday tutorial focus on practice problems - will post Piazza thread for requests
- New office hour Wed. 15:30-16:30 KALS 3065

every

Important:

- Quiz 2 on Tuesday
- Class next Thursday on Zoom (19th)
- Class next Tuesday also on Zoom if Air Canada goes on strike and cancels my flight (17th)

We defined **LTI (linear, time-invariant) systems**.

$$x_1(t) \to y_1(t)$$

Linearity:

$$a\,x_1(t) + b\,x_2(t) \to a\,y_1(t) + b\,y_2(t)$$

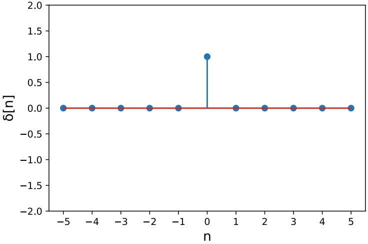
Time invariance:

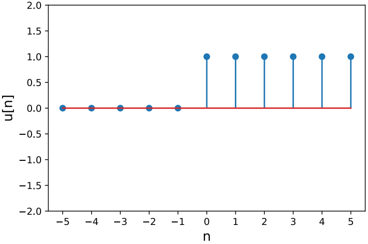
$$x(t) \to y(t)$$

$$x(t-t_0) \to y(t-t_0)$$

Last time

We defined the DT unit impulse and unit step





$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

... but then we got kind of stuck:

$$\delta[n] = u[n] - u[n-1] \qquad u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Today

First: clarify some points from last time

Learning outcomes:

- Define the convolution sum and use it to compute the output of a system
- Define the CT unit impulse and step functions
- Define the convolution integral and use it to compute the output of a system

In case we have time:

- Use the impulse response to determine whether a system is stable, causal, memoryless, or invertible

What space are we in?

Used to thinking in terms of mathematical functions:

$$y = f(x) \qquad \vec{y} = f(\vec{x})$$

output input

For us, the system plays the role of the function, and signals are the input and output:

$$y(t) = S\big(x(t)\big)$$

This is why linearity looks different.

A function $y = f(x)$ is linear if

$$f(ax) = af(x) = ay$$

$$f(x_1 + x_2) = f(x_1) + f(x_2) = y_1 + y_2$$

A system S that sends $x(t) \to y(t)$ is linear if

$$S\big(ax(t)\big) = aS\big(x(t)\big) = ay(t)$$

$$S\big(x_1(t) + x_2(t)\big) = S(x_1(t)) + S(x_2(t)) = y_1(t) + y_2(t)$$

"Linear" is also overloaded. Recall the system

$f(x) = ax + b$ degree 1

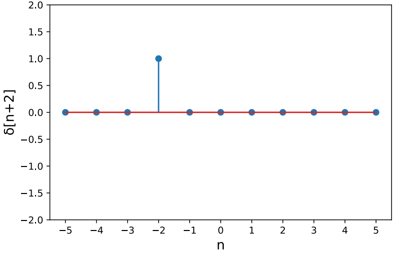
$$x(t) \to y(t) = x(t) + 1$$

is *not linear*, even though it looks like a linear equation.

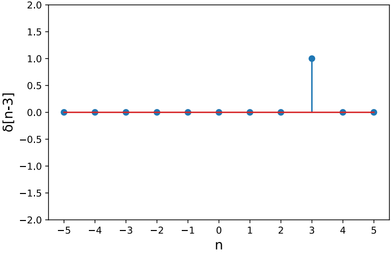
Weighted, shifted impulses

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$\delta[n+2]$

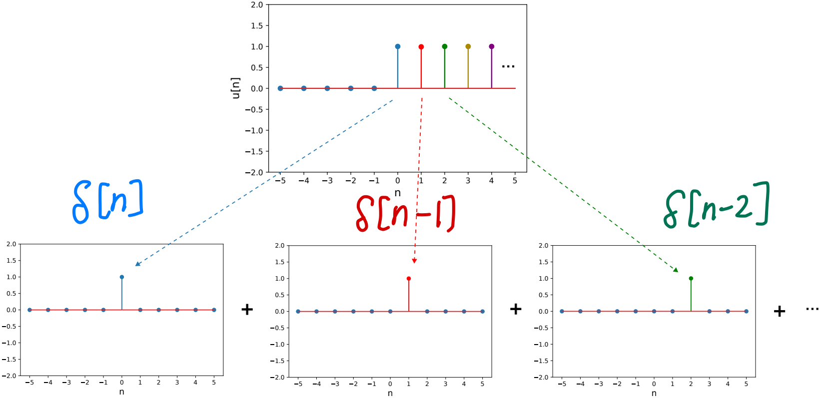


$\delta[n-3]$



Weighted, shifted impulses

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Weighted, shifted impulses

More generally, can write

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \dots$$

$$= \sum_{k=0}^{\infty} \delta[\underbrace{n-k}_{m}]$$

$$u[2] = \sum_{k=0}^{\infty} \delta[2-k]$$

only 0 contributes
$\Rightarrow k=2$

$\downarrow k=0$

Change variables ($m = n - k$)

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

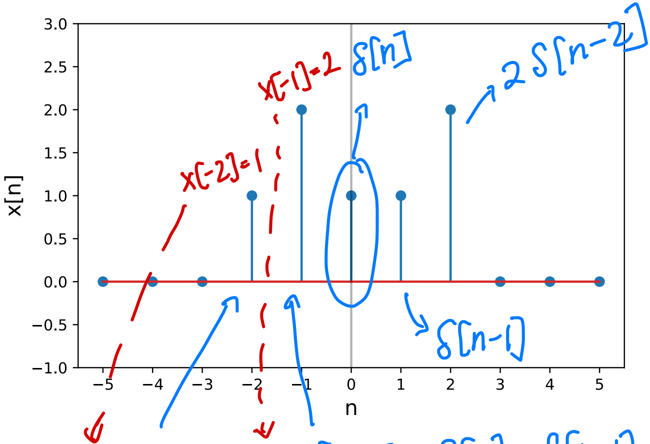
$$= \sum_{m=n}^{-\infty} \delta[m] \quad \text{(m decreases)}$$

$$= \sum_{m=-\infty}^{n} \delta[m]$$

$$u[2] = \sum_{m=-\infty}^{2} \delta[m]$$

$m=0$ contributes

Every point is a *weighted, shifted impulse*.



$$x[n] = \delta[n+2] + 2\,\delta[n+1] + \delta[n] + \delta[n-1] + 2\delta[n-2]$$

$$= x[-2]\cdot\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + \cdots$$

The unit impulse as a sampler

Any signal can be written as a **superposition of weighted impulses.**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Like a "deconstructed" version of the signal.

Multiplying by a shifted impulse "samples" the signal at that point:

$$x[n]\,\delta[n-k] = x[k] \cdot \underbrace{\delta[n-k]}_{}$$

How does an LTI system respond to a signal

$$x[n] = \sum_{k=-\infty}^{\infty} \overset{\text{coeff}}{x[k]}\, \delta[n-k]$$

into linear system: $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{\text{System}\big(\delta[n-k]\big)}_{h_k[n]}$$

Suppose it sends $\delta[n-k] \to h_k[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h_k[n]$$

$h_k[n]$ is called the **impulse response**.

Real-world example: nerve conduction study

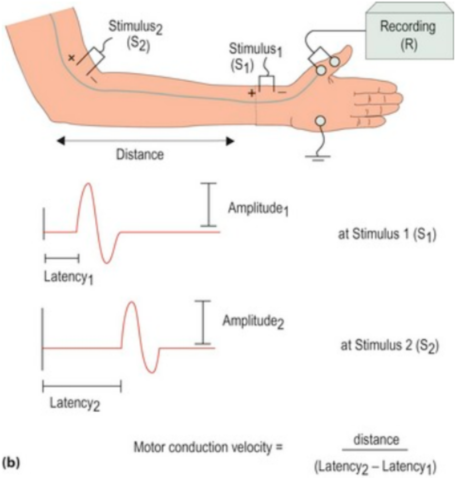


Image source: `https://neupsykey.com/nerve-conduction-studies-and-electromyography/`

The impulse response and time-invariance

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \underbrace{h_k[n]}_{\text{System}(\delta[n-k])}$$

What if the system is also time invariant?

$$x[n] \to y[n] \qquad x[n-k] \longrightarrow y[n-k]$$

Then

$$\delta[n] \to h[n]$$

$$\delta[n-k] \longrightarrow h[n-k]$$

If we know how a **linear** system responds to the unit impulse, we can learn how it responds to **any other signal**!

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

This is the **convolution sum**. We are "convolving" the sequences $x[n]$ and $h[n]$.

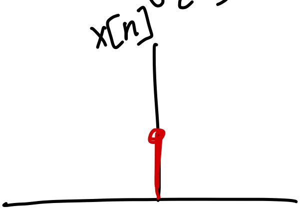
$$y[n] = x[n] * h[n]$$

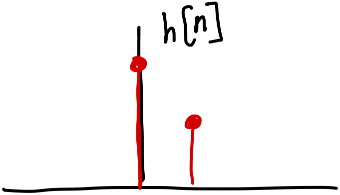
Consider an LTI system with input/output relationship

$$y[n] = 2x[n] + x[n-1]$$

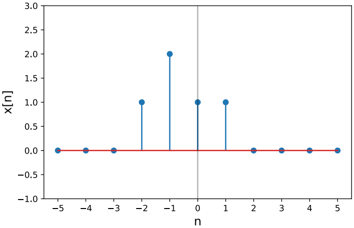
What is the impulse response of the system?

$$\delta[n] \longrightarrow h[n] = 2\cdot\delta[n] + \delta[n-1]$$

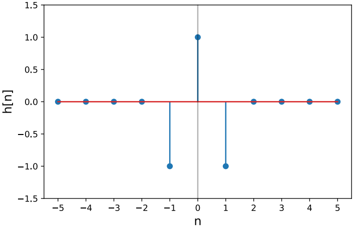




Consider the signal



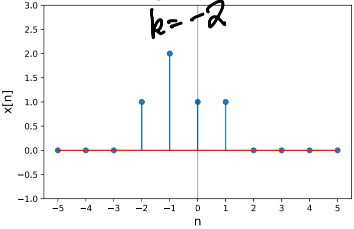
input to a system with impulse response



To learn the system output, we must consider the contribution of each weighted impulse response:

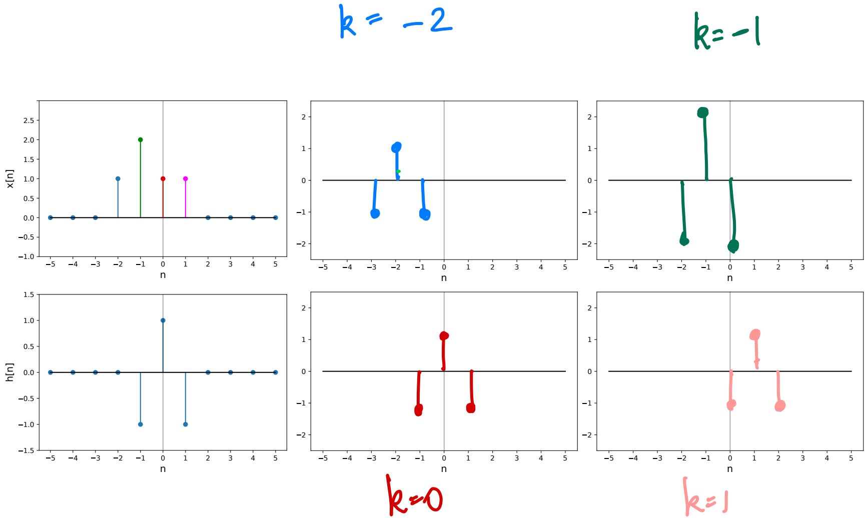
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

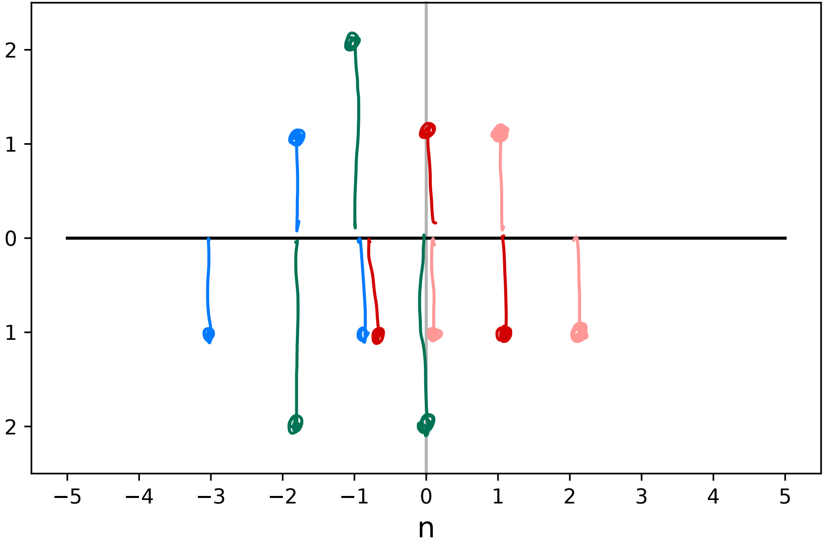
$$= \sum_{k=-2}^{1} x[k]\,h[n-k]$$



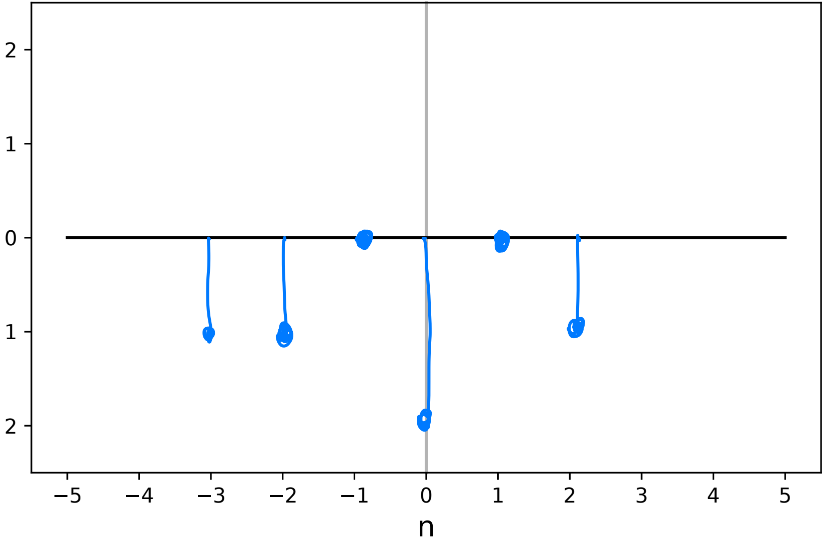
Only $x[k] \neq 0$ only for $k \in \{-2, -1, 0, 1\}$. So need to determine $x[k]h[n-k]$ for these cases, and sum them.

Example: convolution sum

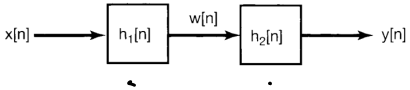



2
1
0
1
2
−5
−4
−3
−2
−1
0
1
2
3
4
5
n

Example: convolution sum



Properties of convolutions



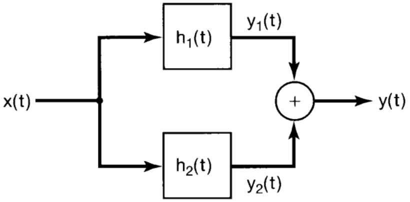


Image credits: Signals and Systems 2nd ed., Oppenheim

Convolution is:

- Associative:
  $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Commutative:
  $x[n] * h[n] = h[n] * x[n]$
- Distributive:
  $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

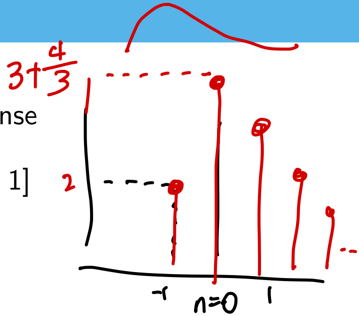
Example: convolution sum

Consider an LTI system with impulse response

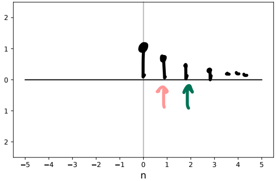
$$h[n] = 3\delta[n] + 2\delta[n+1]$$

What is output of the system if

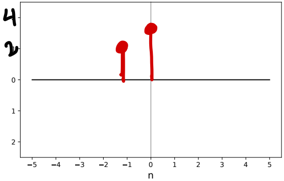
$$x[n] = \left(\frac{2}{3}\right)^n u[n]$$



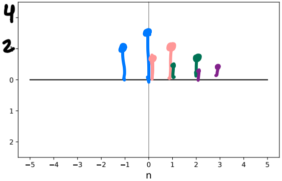
x[n]



h[n]



x[n] * h[n]



Example/exercise: convolution sum

What is output of the system

$$x[n] = \left(\frac{2}{3}\right)^n u[n], \quad h[n] = 3\delta[n] + 2\delta[n+1]$$

$$y[n] = x[n] * h[n] = x[n] * 3\delta[n] + x[n] * 2\delta[n+1]$$

$$= \underbrace{3\, x[n] * \delta[n]} + 2\, x[n] * \delta[n+1]$$

$$3\, x[n] * \delta[n] = 3 \sum_{k=-\infty}^{\infty} \left(\frac{2}{3}\right)^k u[k]\, \delta[n-k]$$

$$= 3 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \underbrace{\delta[n-k]} \rightarrow k=n \text{ survives}$$

$$= 3 \left(\frac{2}{3}\right)^n \delta[0] = 3 \left(\frac{2}{3}\right)^n \cdot u[n]$$

What is output of the system

$$x[n] = \left(\frac{2}{3}\right)^n u[n], \quad h[n] = 3\delta[n] + 2\delta[n+1]$$

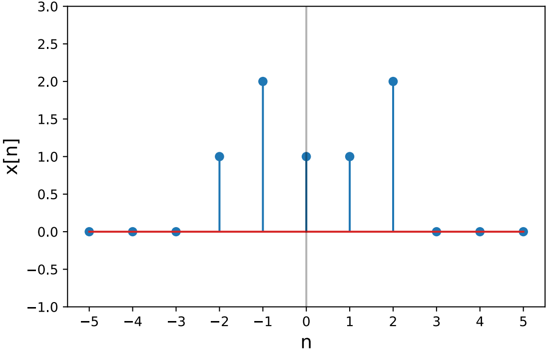
$2\, x[n] * \delta[n+1]$ only $k = n+1$ contributes

$= 2\left(\frac{2}{3}\right)^{n+1} u[n+1]$

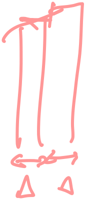
$\Rightarrow \; x[n] * h[n] = 3\left(\frac{2}{3}\right)^n u[n] + 2\left(\frac{2}{3}\right)^{n+1} u[n+1]$

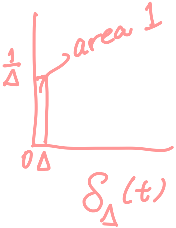
Shifted, weighted impulses

Return to this picture:



Shifted, weighted impulses





$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\cdot\Delta$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\underbrace{\delta(t-\tau)}_{}\,d\tau$$

The CT unit impulse

$\delta(t)$

$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

Recap

Today's learning outcomes were:

- Define the convolution sum and use it to compute the output of a system
- Define the CT unit impulse ~~and step functions~~
- ~~Define the convolution integral and use it to compute the output of a system~~

Content: CT unit impulse/step, convolution integral

- (Complex) exponential signals
- ~~The continuous time Fourier series representation~~

system properties from impulse response.

Action items:

1. Submit Tutorial Assignment 1 (Monday 23:59)
2. Work on Assignment 1 (can do most of Qs 2, 4, 5)
3. Quiz 2 Tuesday about this week's material

Recommended reading:

- From today's class: Oppenheim 1.4, 2.1-2.3
- practice problems: 2.1-2.12, 2.14-16, 2.21, 2.22, 2.28, 2.29
- For next class: Oppenheim 1.~~3, 3.0-3.3~~