ELEC221
Tutorial 5 – DTFT practice problems

1. Consider the discrete-time signal $x[n]$ with DTFT given by

$$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0)$$

Use the inverse DTFT to determine $x[n]$.

We can write $x[n]$ as

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega$$

However, because of the sifting property of the Dirac delta, $x[n]$ is zero for all values of ω, except when $\omega = \omega_0$. It follows then that $x[n] = e^{j\omega_0 n}$.

2. Consider the signal

$$y[n] = \sin(\omega_1 n)\, x[n]$$

Show that

$$Y(e^{j\omega}) = \frac{X(\omega - \omega_1) - X(\omega + \omega_1)}{2j}$$

Where $X(\omega) = DTFT(x[n])$

We start by decomposing $\sin(\omega_1 n)$ into complex exponentials

$$y[n] = \frac{e^{j\omega_1 n}x[n] - e^{-j\omega_1 n}x[n]}{2j}$$

Let $z[n] = e^{j\omega_1 n}x[n]$, so that

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_1 n}x[n]e^{-j\omega n} = e^{j\omega_1 n}X(e^{j\omega}) = X(\omega - \omega_1)$$

From here it is clear that the complex exponential is applying a frequency shift to the spectrum of $x[n]$. Because of the linearity property, the spectrum of $y[n]$ is expressed as\

$$Y(e^{j\omega}) = \frac{X(\omega - \omega_1) - X(\omega + \omega_1)}{2j}$$

3. Assume the following discrete-time signal

$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

And that S is an LTI system with frequency response $H(e^{j\omega}) = e^{-j\omega}$.

a) Find the DTFT of $x[n]$

b) Let $y[n] = S(x)$. Find the DTFT of $y[n]$.
c) Find $y[n] = S(x)$.

(a) Using the DTFT equation, we have that

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (\delta[n+1] + \delta[n] + \delta[n-1])e^{-j\omega n}$$

Again, because of the sifting property of δ, the sum only exists for certain values of n. So that the spectrum of $x[n]$ is given by

$$X(e^{j\omega}) = e^{-j\omega} + 1 + e^{j\omega} = 1 + 2\cos(\omega)$$

(b) The spectrum of $y[n]$ can be found by using $X(e^{j\omega})$ and $H(e^{j\omega})$ as follows

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = e^{-j\omega}(e^{-j\omega} + 1 + e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega}$$

(c) We need to apply the inverse DTFT to $Y(e^{j\omega})$, however, in order to avoid the integrals, we will exploit the parity between the DTFT and its inverse.

$$Y(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega}$$

By using linearity, we can rewrite the spectrum as $Y(e^{j\omega}) = Y_1 + Y_2 + Y_3$, where $Y_1 = 1$, $Y_2 = e^{-j\omega}$, and $Y_3 = e^{-2j\omega}$. We have already seen the transform pairs for all of these functions before, so that we can simply write

$$Y_1 = 1 = \delta[n]$$

$$Y_2 = e^{-j\omega} = \delta[n-1]$$

$$Y_3 = e^{-2j\omega} = \delta[n-2]$$

$$y[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

4. Suppose a discrete-time signal $x[n]$ has a DTFT given by

$$X(e^{j\omega}) = j\sin(K\omega)$$

for some positive integer K.

a) Is $x[n]$ real?
b) Find $x[n]$

a) Note that, by using $\sin(\theta) = -\sin(-\theta)$,

$$X(-\omega) = j\sin(-K\omega) = -j\sin(K\omega) = X^*(-\omega)$$

Thus, X is conjugate symmetric, which implies that $x[n]$ is real.

b) By decomposing the spectrum into complex exponentials, we get

$$X\left(e^{j\omega}\right) = (e^{jK\omega} - e^{-jK\omega})/2$$

Again, we can recognize the inverse DTFT of these terms from previous problems, so we can directly write

$$x[n] = (\delta[n+K] - \delta[n-K])/2$$

Extra practice problems

Find their DFTF

1. $x[n] = 3\delta[n-2] - \delta[n-3] + \delta[n-4]$
2. $x[n] = 7u[n-1] - 7u[n-9]$
3. $x[n] = \frac{\sin(0.25\pi n)}{9\pi n}$

Find their inverse DTFT

1. $Y\left(e^{j\omega}\right) = 2\pi$
2. $Y\left(e^{j\omega}\right) = 5e^{-j3\omega}$
3. $Y\left(e^{j\omega}\right) = 6\cos(3\omega)$