ELEC 221 Lecture 15
Time and frequency domain analysis I

Thursday 27 October 2022

- Midterms available for pickup after class (or at my office)
- Assignment 4 due on Saturday at 23:59
- (Bonus) Assignment 4.5 due on Saturday at 23:59
- Quiz 7 on Tuesday (will focus on today's content)

Previously

Complex exponential signals are eigenfunctions of LTI systems in both continuous time and discrete time.

If $x(t) = e^{st}$, for complex s

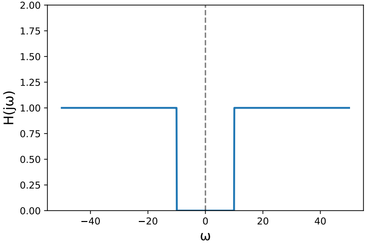
We have considered so far only $s = j\omega$

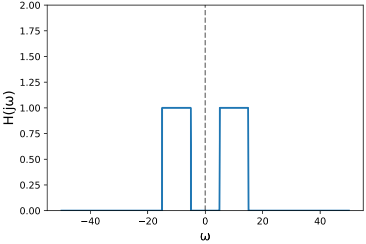
This is the **frequency response** of the system.

When we input a linear combination of signals,

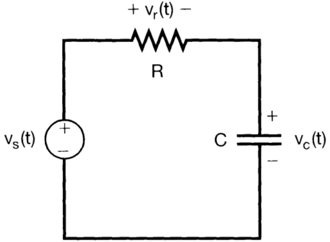
Previously

We have seen some simple frequency response of ideal filters:

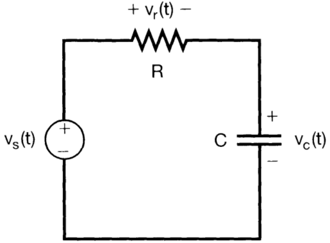




We have also seen more realistic ones.



If $v_s(t) = e^{j\omega t}$, then a solution is $v_c(t) = H(j\omega)e^{j\omega t}$ for some scaling $H(j\omega)$.

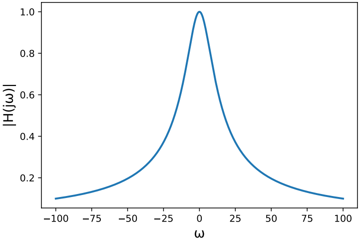


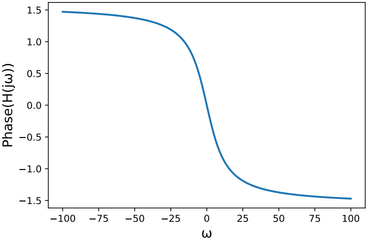
We found that

If we look at $H(j\omega)$ we can see that this is also a filter. The frequencies it attenuates depends on R and C.

Previously

In general $H(j\omega)$ has both a magnitude and a phase component.

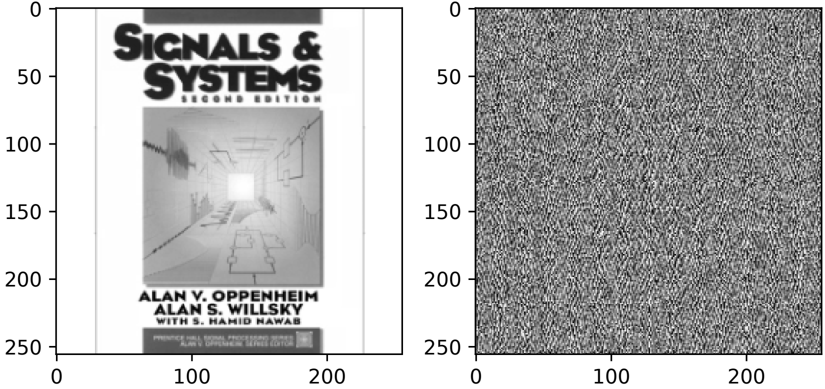




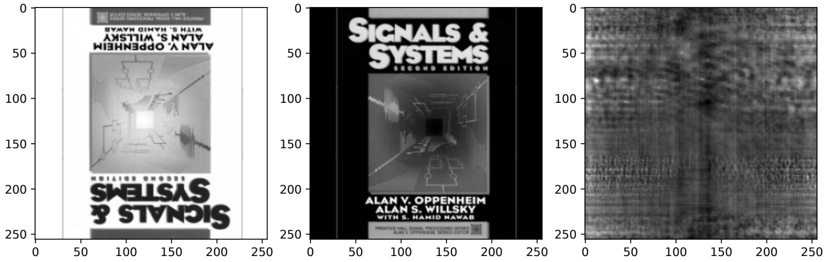
Increasing RC cuts off more frequencies, but there are design tradeoffs involved.

We haven't looked much at the phase response...

You hopefully learned from the hands-on that phase can be important!



Really important...



So we should probably consider this more in our analysis of systems.

Learning outcomes:

- express a frequency response in the magnitude-phase representation
- differentiate between linear and non-linear phase responses
- compute the group delay of a frequency response
- plot the frequency response using a Bode plot

The magnitude-phase representation

Since Fourier spectra are complex numbers, we can express them in terms of their magnitude and phase.

Recall the convolution property of the Fourier transform:

What happens to the output?

How does passing through the system with $H(j\omega)$ affect $|X(j\omega)|$ and $\sphericalangle X(j\omega)$?

Try it yourself. Given

$$
\begin{aligned}
X(j\omega) &= |X(j\omega)|e^{j\measuredangle X(j\omega)} \\
Y(j\omega) &= H(j\omega)X(j\omega)
\end{aligned}
$$

Determine

$$
\begin{aligned}
|Y(j\omega)| &= \\
\measuredangle Y(j\omega) &=
\end{aligned}
$$

We give these names:

- $|H(j\omega)|$ is the gain
- $\measuredangle H(j\omega)$ is the phase shift

Depending on what these are, the result can be either good, or bad (*distortion*).

Linear frequency response

If $|H(j\omega)| = 1$ everywhere, the system is called *all-pass* and is characterized by its phase response.

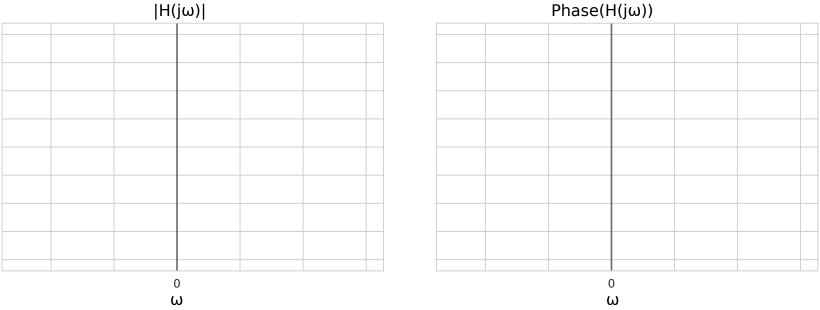
It is nicest when the phase shift is a *linear* function of the frequency.

Can you think of a system that causes a linear shift in phase?
(Hint: think back to properties of Fourier transform)

Time shift (or, a delay system):

Example: linear frequency response

Let's consider the ideal lowpass filter,



Example: linear frequency response

What is its impulse response? (inverse Fourier transform of frequency response)

Example: linear frequency response

Recall what this looks like graphically:

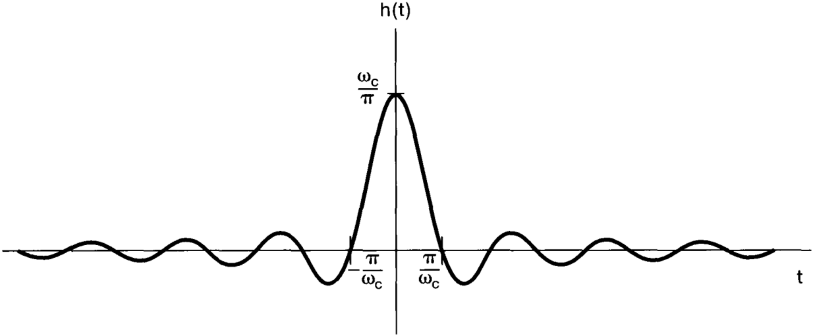
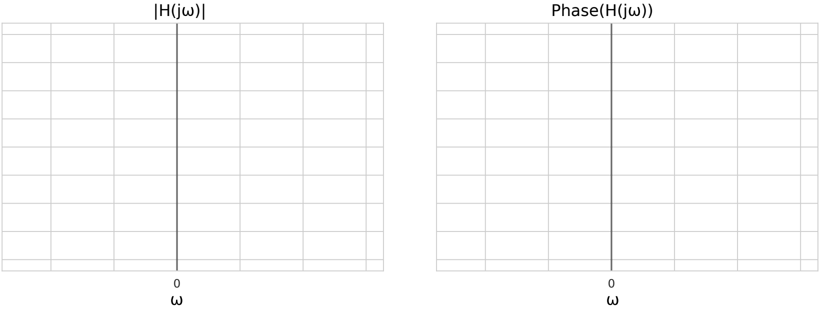


Image credit: Oppenheim 6.3

Example: linear frequency response

What happens if we add a linear phase?



Example: linear frequency response

The result is a shifted version of the original impulse response

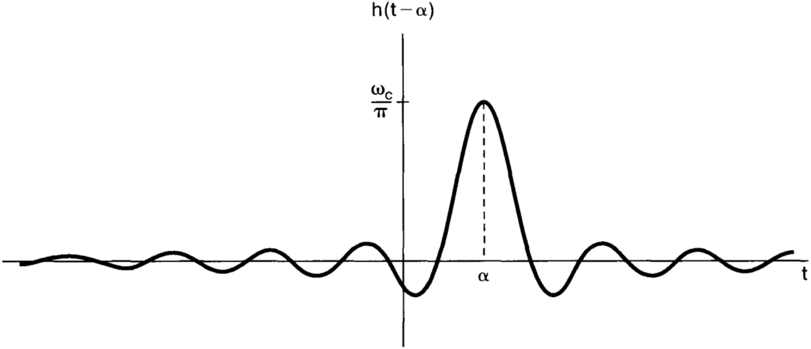


Image credit: Oppenheim 6.3

Exercise: linear frequency response

Consider the following frequency response:

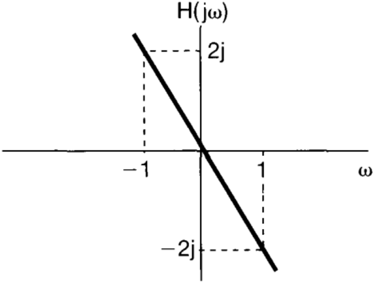


Figure P6.21

- What is $H(j\omega)$?
- Sketch $|H(j\omega)|$ and $\sphericalangle H(j\omega)$

Image credit: Oppenheim Problem 6.21

$$\begin{aligned} H(j\omega) &= \\ |H(j\omega)| &= \\ \sphericalangle H(j\omega) &= \end{aligned}$$

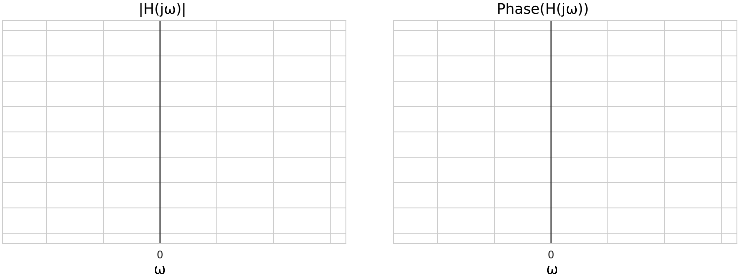


Image credit: Oppenheim Problem 6.21

Exercise: linear frequency response

Suppose a signal $x(t)$ with spectrum $X(j\omega) = \frac{1}{2+j\omega}$ is input into the system.

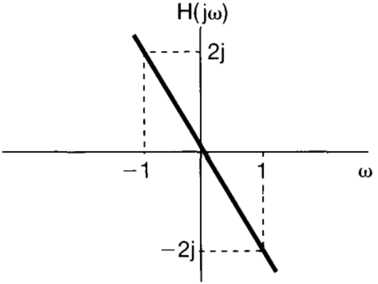


Figure P6.21

What is the output signal $y(t)$? (Hint: recall what happens to a Fourier spectrum of a function when you take its derivative)

Image credit: Oppenheim Problem 6.21

Exercise: linear frequency response

Group delay

Linear phase: same delay at all frequencies (shift the response).

Non-linear phase: different amount of delay at different frequencies

If we look at a small enough band of frequencies, we can make an approximation that it is...

Then:

$$Y(j\omega) \approx X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha}$$

The parameter α represents an effective common delay of the frequencies in this small band.

It is called the **group delay**:

Non-linear phase and group delay has a lot of real-world implications.

Exercise: group delay

Consider a filter with frequency response

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- What are $|H(j\omega)|$ and $\sphericalangle H(j\omega)$
- What is the group delay?

Exercise: group delay

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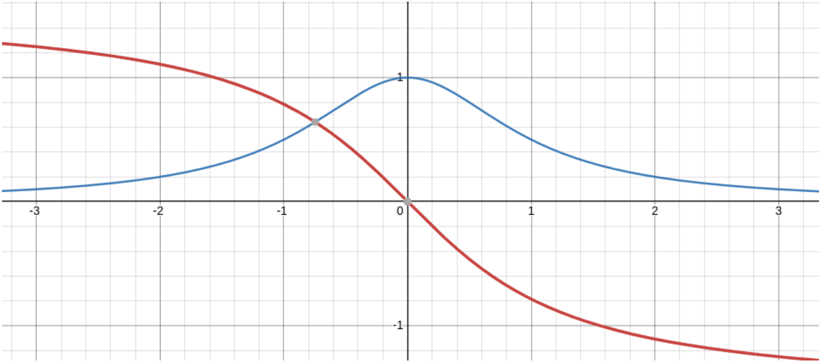
Exercise: group delay



Bode plots

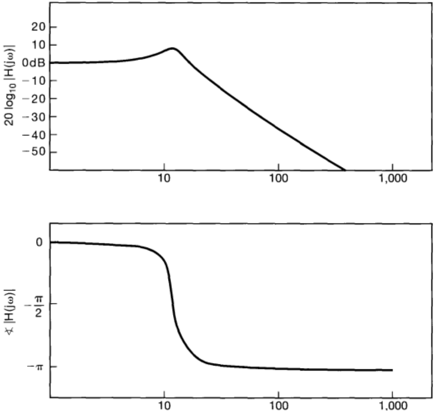
Recall:

Magnitude is multiplicative and phase is additive... would be nicer if both were additive.

Rather than making plots of $|H(j\omega)|$ and $\sphericalangle H(j\omega)$, it is common to make plots of $20\log_{10}|H(j\omega)|$ and $\sphericalangle H(j\omega)$ against $\log_{10}\omega$.

Bode plots

These are called *Bode plots*:



We will see more of these on Tuesday.

Image credit: Oppenheim 6.2

Ideal filter step response

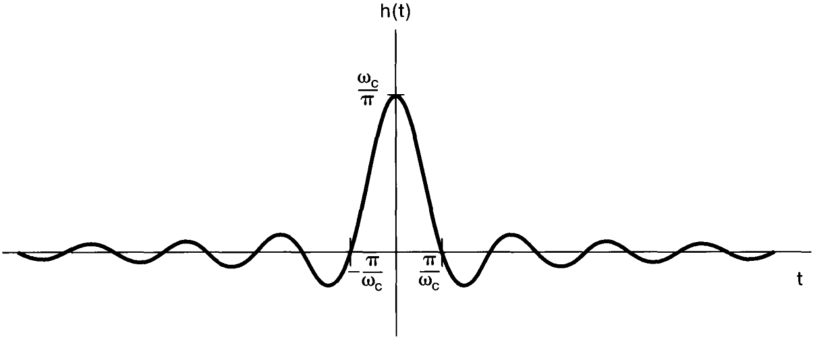


Image credit: Oppenheim 6.3

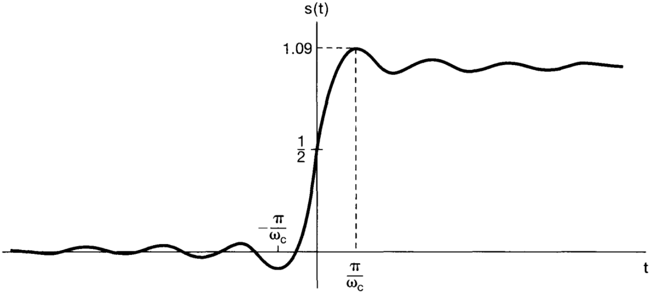
Ideal filter step response

It is also important to consider *step response* of filters.

Recall that

By linearity, if we put this in a system, the result is

Ideal filter step response



By changing the design of the filters, we can limit the amount of ringing. More on Tuesday!

Today

Learning outcomes:

- express a frequency response in the magnitude-phase representation
- differentiate between linear and non-linear phase responses
- compute the group delay of a frequency response
- plot the frequency response using a Bode plot

Oppenheim practice problems:

- (DT) 6.2, 6.4, 6.37, 6.39 (choose a couple)
- (CT) 6.21a-c, 6.23, 6.27, 6.42

For next time

Content:

- Properties of non-ideal filters
- Filters described by first/second-order difference equations

Action items:

1. Quiz 7 Tuesday
2. Assignment 4 due Saturday 23:59
3. Bonus activity due Saturday 23:59

Recommended reading:

- For next class: Oppenheim 6.4-6.7