ELEC 221 Lecture 12
The discrete-time Fourier transform

Tuesday 18 October 2022

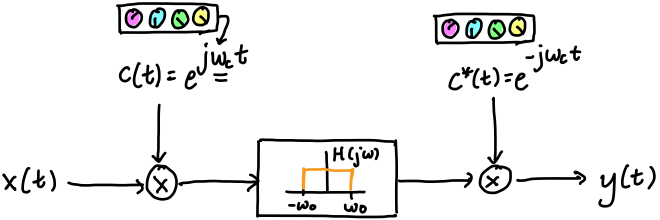
Announcements

- No quiz today (quizzes resume next week)
- Assignment 4 will be made available this week
- Midterm grading underway

Midterm postmortem...

Last time

We saw the multiplication property of the CT Fourier transform:



Last time

We saw how the CT Fourier spectrum behaves under differentiation and integration:

Last time

We leveraged differentiation/integration and the convolution property to compute impulse and frequency response for systems described by ODEs.

Where are we going?

① **Systems**

- Properties (LTI, memory, causality, stability)

② **Signals**

- Unit impulse, step
- $x[n] = \sum x[k]\delta[n-k]$
- $x(t) = \int x(\tau)\delta(t-\tau)d\tau$

③ **Convolution**

- Impulse response $h(t)/h[n]$
- Convolution sum/integral
- $y[n] = \sum x[k]h[n-k]$
- $y(t) = \int x(\tau)h(t-\tau)d\tau$

④ **Complex exponential signals**

- $x(t) = e^{j\omega t}$, $x[n] = e^{j\omega n}$
- Periodicity and Fourier series representation
- $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$
- $x[n] = \sum_{k=\langle N\rangle} c_k e^{jk\frac{2\pi}{N}n}$

⑤ **System functions and frequency response**

- Complex exp. signals are eigenfunctions of LTI systems
- $x(t) = \sum c_k e^{jk\omega t} \rightarrow y(t) = \sum c_k H(jk\omega) e^{jk\omega t}$
- $x[n] = \sum c_k e^{jk\frac{2\pi}{N}n} \rightarrow y[n] = \sum c_k H(e^{j\omega}) e^{jk\frac{2\pi n}{N}}$

⑥ **CT Fourier transform**

- handles aperiodic signals
- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ (inv. FT)
- $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ (F. spectrum)
- relationship between $h(t) \overset{F}{\leftrightarrow} H(j\omega)$
- convolution property: $Y(j\omega) = H(j\omega)X(j\omega)$

⑦ **DT Fourier transform**

Learning outcomes:

- Compute the DT Fourier transform of non-periodic DT signals
- State key properties of the DTFT
- Leverage convolution properties of DTFT to analyze the behaviour of LTI systems

On Thursday and Tuesday:

- Learn how the fast Fourier transform algorithm works
- Hands-on with the NumPy FFT module: image processing

Recap: CT Fourier series and transform

Fourier series pair:

Fourier transform pair:

When a DT signal is periodic (with period N) it can be represented using only the integer harmonics at the same frequency.

DT synthesis equation:

DT analysis equation:

The discrete-time Fourier transform (DTFT) is the generalization of the Fourier series representation to **aperiodic** signals.

We derive it just like we did in CT:

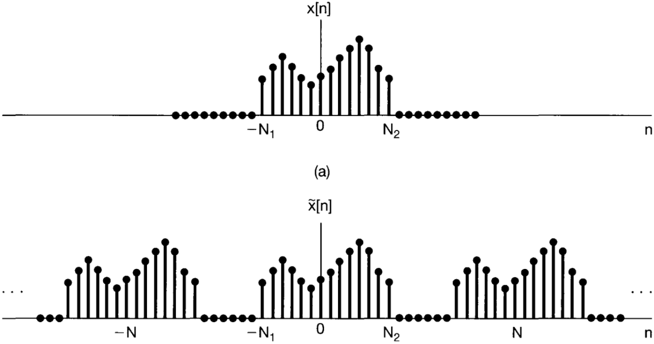


Image credit: Oppenheim chapter 5.1

Suppose $\tilde{x}[n]$ is a periodic extension of $x[n]$. We can write it as a DT Fourier series:

We could just as well change the bounds of the sum to include where our signal actually is:

Now, what happens if we make the period larger and larger (i.e., increase the spacing?)

If $N \to \infty$, for any finite n, our new signal $\tilde{x}[n]$ basically just looks like our old signal:

But since $x[n] = 0$ outside this range, we can change the bounds of the sum:

We have

Let's define

Then

Substituting

$$c_k = \frac{1}{N} X(e^{jk\omega})$$

back into the original synthesis equation for $\tilde{x}[n]$ yields

Now what happens as $N \to \infty$?

The DT Fourier transform

As $N \to \infty$, $\omega \to 0$.

Consider what we are summing:

This is going to be a sum of terms like $X(e^{jk\omega})e^{jk\omega n}\omega$ for very small ω. We can convert the sum to an integral:

Recall though that in this range, $\tilde{x}[n]$ is basically $x[n]$, and we only need to integrate from over 0 to 2π. The result is the **DT Fourier transform pair**.

Inverse DTFT (synthesis equation):

DTFT (analysis equation):

Example: DTFT of a square pulse

Let's compute the DTFT of the DT signal

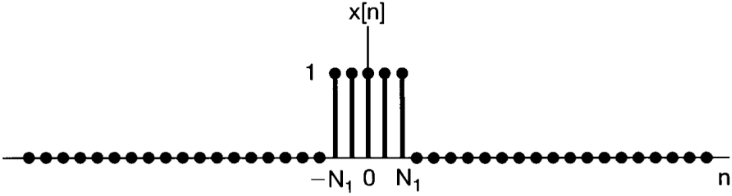


Image credit: Oppenheim chapter 5.1

Recall: FT of a CT square pulse

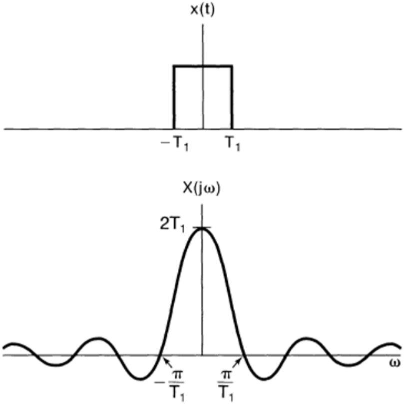


Image credit: Oppenheim chapter 4.1

Example: DTFT of a square pulse

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

Compute the DTFT:

How do we evaluate this sum?

Example: DTFT of a square pulse

Change variable in the summation to $m = n + N_1$

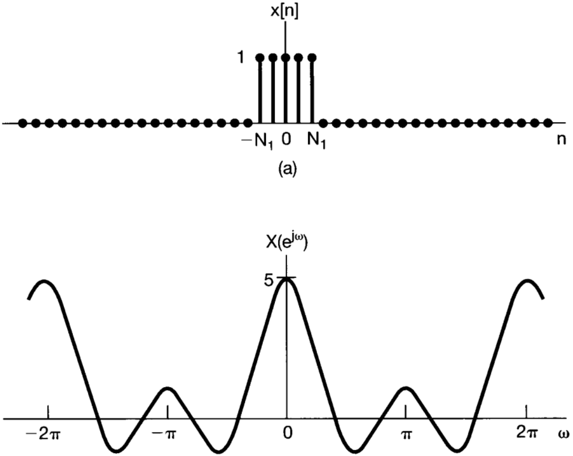
Use our handy identity:

Example: DTFT of a square pulse

$$X(e^{j\omega}) = e^{j\omega N_1} \frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}}$$

Straightforward from here:

Example: DTFT of a square pulse



Note that this function is periodic!

Image credit: Oppenheim chapter 5.1

Recall in CT we had Dirichlet criteria for both Fourier series and inverse Fourier transform representations:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \quad c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

We didn't have this issue for the DT Fourier series:

$$x[n] = \sum_{k=\langle N \rangle} c_k e^{jk\omega n} \qquad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega n}$$

What about for the DT Fourier transform?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The synthesis equation is fine; but the analysis equation has an infinite sum. One of the following must be satisfied:

Convolution

Convolution works the same way as in CT:

We also have the same relationship between impulse response and the frequency response:

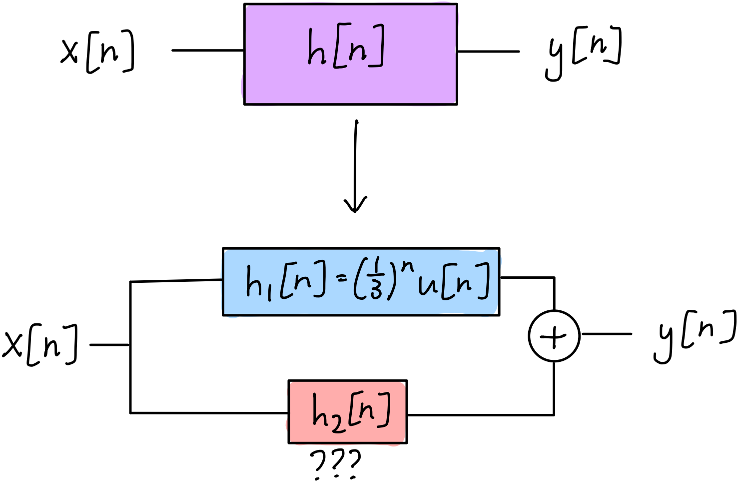
Convolution

Convolution works the same way as in CT:

We also have the same relationship between impulse response and the frequency response:

Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$



Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$

Hint:

Example: convolution property

Many properties are the same as the CT analogs.

Linearity: If

then

Properties of the DT Fourier transform

Many properties are the same as the CT analogs.

Time shift: If

then

Frequency shift:

Conjugation: If

then

If $x[n]$ is real,

Consequences for odd/even functions:

Periodicity:

Differentiation in frequency:

Differencing:

Accumulating:

Parseval's relation:

Here $|X(e^{j\omega})|^2$ is called the *energy-density spectrum*.

Today's learning outcomes were:

- Compute the DT Fourier transform of non-periodic DT signals
- State key properties of the DTFT
- Leverage convolution properties of DTFT to analyze the behaviour of LTI systems

What topics did you find unclear today?

For next time

Content:

- The discrete Fourier transform (DFT) and the Fast Fourier Transform (FFT) algorithm

Action items:

1. Keep an eye out for Assignment 4

Recommended reading:

- From today's class: Oppenheim 5.0-5.7
- For next class: Oppenheim extension problems 5.53-5.54