ELEC 221 Lecture 15
The discrete-time Fourier transform

Tuesday 29 October 2024

Announcements

- Quiz 7 today
- Assignment 3 due Saturday 23:59 (solutions posted after)
- Midterm 2 information posted on PrairieLearn

Last time

We analyzed CT systems described by differential equations:

Last time

For first-order systems

$$T\frac{dy(t)}{dt} + y(t) = x(t),$$

we determined

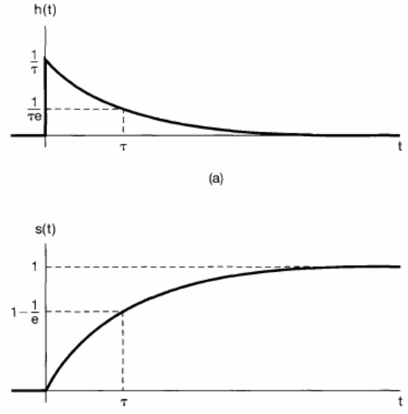


Image: Oppenheim Fig. 6.19

For second-order systems,

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

the behaviour depends on ζ (zeta), the damping ratio.

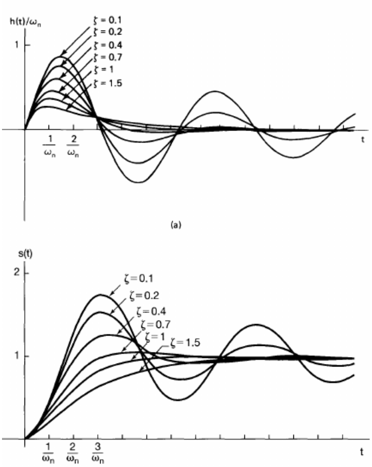


Image: Oppenheim Fig. 6.22

Today

Learning outcomes:

- Compute the DT Fourier transform of non-periodic DT signals
- State key properties of the DTFT
- Leverage convolution properties of DTFT to analyze the behaviour of LTI systems

Recap: CT Fourier series and transform

Fourier series pair:

Fourier transform pair:

Recap: DT Fourier series

We can express a periodic DT signal (period N) as a discrete Fourier series.

DT synthesis equation:

DT analysis equation:

The discrete-time Fourier transform (DTFT) is the generalization of the discrete Fourier series to **aperiodic** signals.

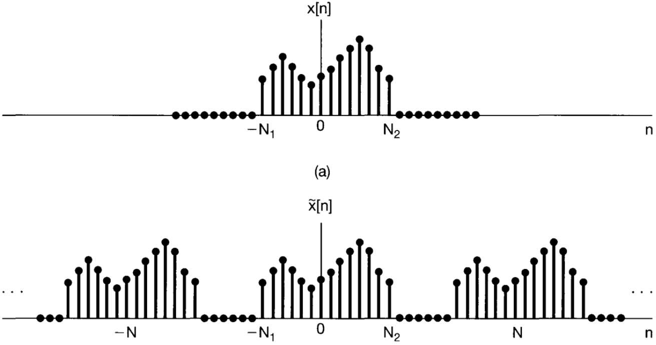


Image credit: Oppenheim chapter 5.1

Suppose $\tilde{x}[n]$ is a periodic extension of $x[n]$.

Set the bounds to consider where our signal actually is:

What happens if we increase the period?

$$c_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\frac{2\pi n}{N}}$$

If $N \to \infty$, for any finite n, $\tilde{x}[n]$ looks just like $x[n]$:

Since $x[n] = 0$ outside this range, we can extend the bounds:

The DT Fourier transform

We have

Define

Substituting

$$c_k = \frac{1}{N} X(e^{jk\omega})$$

into the synthesis equation for $\tilde{x}[n]$ yields

What happens as $N \to \infty$?

Over what range should we integrate ω?

DT Fourier transform pair:

Inverse DTFT (synthesis equation)

DTFT (analysis equation)

Example: DTFT of a square pulse

Compute the DTFT of the DT signal

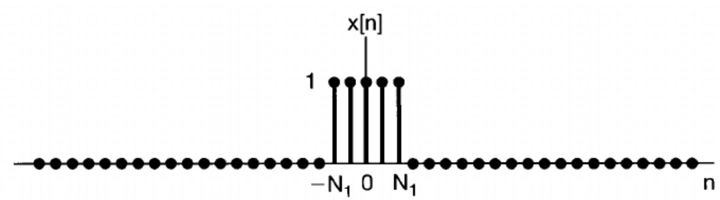


Image credit: Oppenheim chapter 5.1

Recall: FT of a CT square pulse

$$x(t) = \begin{cases} 1 & |t| < T_1, \\ 0 & |t| > T_1 \end{cases}$$

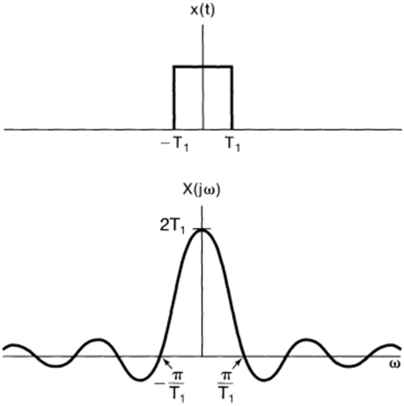


Image credit: Oppenheim chapter 4.1

Example: DTFT of a square pulse

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

Compute the DTFT:

How do we evaluate this sum?

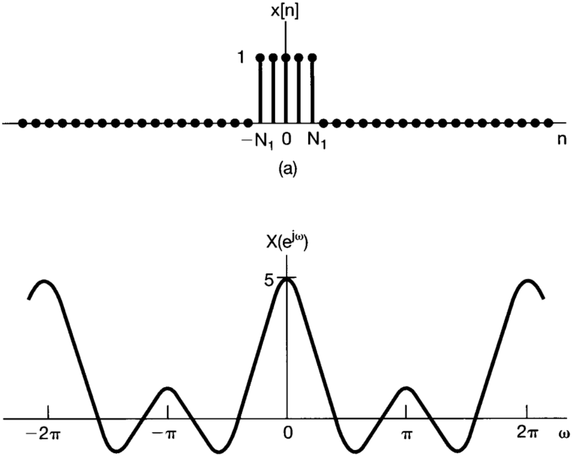
Example: DTFT of a square pulse

Change summation variable to $m = n + N_1$

Use our handy identity:

Do some reshuffling...

Example: DTFT of a square pulse



Note that this function is **continuous** and **periodic**!

Image credit: Oppenheim chapter 5.1

In CT had Dirichlet criteria for both Fourier series and inverse Fourier transform. No conditions for DT Fourier series:

$$x[n] = \sum_{k=\langle N \rangle} c_k e^{jk\omega n} \qquad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega n}$$

What about the DT Fourier transform?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Convolution

DTFT defines the relationship between impulse response and frequency response:

Convolution works the same way as in CT:

Example: filters

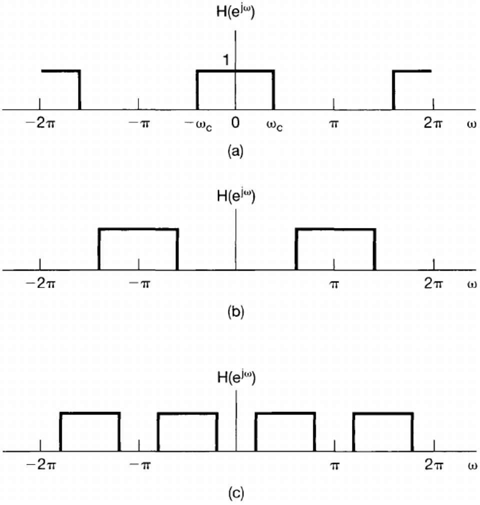


Image: Oppenheim 3.28

Example: filters

Determine the impulse response of an ideal DT low-pass filter,

$$H(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| < \pi \end{cases}$$

Example: filters

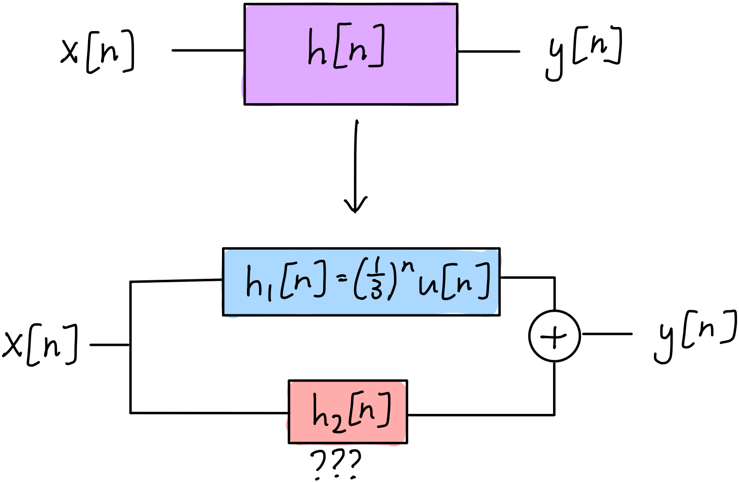
For an ideal DT high-pass filter,

Example: convolution property

What is the DTFT of

Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$



Example: convolution property

Using our identity:

For next time

Content:

- DTFT properties (linearity, time shift, etc.)
- DT systems based on difference equations

Action items:

1. Assignment 3 due Saturday 23:59 (solutions posted right after)

Recommended reading:

- From today's class: Oppenheim 5.1, 5.4
- Suggested problems: 5.1, 5.2, 5.5, 5.14, 5.21abcfj, 5.22a, 5.29
- For next class: Oppenheim 5.2, 5.3, 5.8, 6.6