ELEC 221 Lecture 15
The discrete-time Fourier transform

Tuesday 29 October 2024

Announcements

- Quiz 7 today
- Assignment 3 due Saturday 23:59 (solutions posted after)
- Midterm 2 information posted on PrairieLearn

  → no calculators

Last time

$$a\frac{d^2y(t)}{dt^2} + y(t) = b\frac{dx(t)}{dt} + x(t)$$

We analyzed CT systems described by differential equations:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

$$H(j\omega) = \frac{-1 + 2j\omega - 3(j\omega)^2}{4 + 5j\omega + 3(j\omega)^2}$$

$$-x(t) + 2\frac{dx(t)}{dt} - 3\frac{d^2x(t)}{dt^2} = 4y(t) + 5\frac{dy(t)}{dt} + 3\frac{d^2y(t)}{dt^2}$$

For first-order systems

$$\tau \frac{dy(t)}{dt} + y(t) = x(t),$$

we determined

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

$$s(t) = (1 - e^{-t/\tau}) u(t)$$

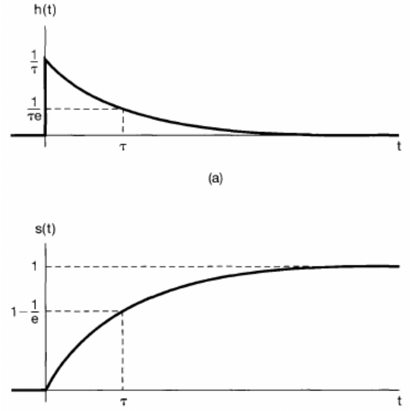


Image: Oppenheim Fig. 6.19

$$H(j\omega) = \frac{-1}{4+5j\omega+3(j\omega)^2}$$

$$= -\frac{1}{3\cdot\frac{4}{3}} \cdot \frac{1\cdot\frac{4}{3}}{\frac{4}{3}+\frac{5}{3}j\omega+(j\omega)^2}$$

$$= -\frac{1}{4}\cdot\frac{4/3}{4/3+\frac{5}{3}j\omega+(j\omega)^2}$$

$$\omega_n^2 = \frac{4}{3} \Rightarrow 2\zeta\omega_n = \frac{5}{3}$$

$$2\zeta\cdot\frac{2}{\sqrt{3}} = \frac{5}{3}$$

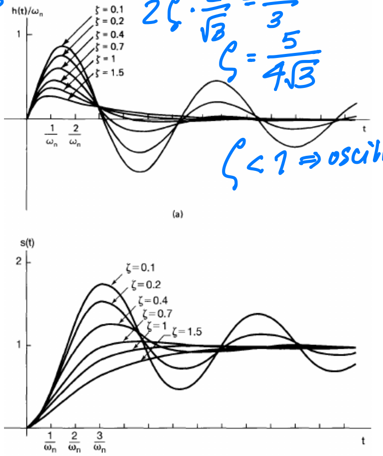
$$\zeta = \frac{5}{4\sqrt{3}}$$

For second-order systems,

$$\frac{d^2y(t)}{dt^2}+2\zeta\omega_n\frac{dy(t)}{dt}+\omega_n^2y(t)=\omega_n^2x(t)$$

the behaviour depends on ζ (zeta), the damping ratio.

generic: $$\frac{\omega_n^2}{\omega_n^2+2\zeta\omega_n(j\omega)+(j\omega)^2}$$



$\zeta < 1 \Rightarrow$ oscillates

Image: Oppenheim Fig. 6.22

Learning outcomes:

- Compute the DT Fourier transform of non-periodic DT signals
- State key properties of the DTFT
- Leverage convolution properties of DTFT to analyze the behaviour of LTI systems

Recap: CT Fourier series and transform

Fourier series pair:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t) e^{-j\omega k t} dt$$

Fourier transform pair:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

We can express a periodic DT signal (period N) as a discrete Fourier series.

DT synthesis equation:

$$x[n] = \sum_{k=\langle N\rangle} c_k e^{jk\cdot\frac{2\pi n}{N}}$$

DT analysis equation:

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

The DT Fourier transform

The discrete-time Fourier transform (DTFT) is the generalization of the discrete Fourier series to **aperiodic** signals.

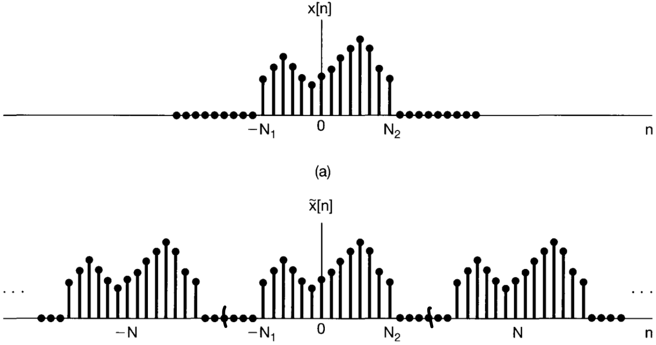


Image credit: Oppenheim chapter 5.1

Suppose $\tilde{x}[n]$ is a periodic extension of $x[n]$.

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} c_k e^{jk\frac{2\pi n}{N}} \qquad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\frac{2\pi n}{N}}$$

Set the bounds to consider where our signal actually is:

$$\tilde{x}[n] = \sum_{k=-N_1}^{N_2} c_k e^{jk\frac{2\pi n}{N}} \qquad c_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\frac{2\pi n}{N}}$$

What happens if we increase the period?

$$c_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\frac{2\pi n}{N}}$$

If $N \to \infty$, for any finite n, $\tilde{x}[n]$ looks just like $x[n]$:

$$c_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk\frac{2\pi n}{N}}$$

Since $x[n] = 0$ outside this range, we can extend the bounds:

$$c_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi n}{N}}$$

The DT Fourier transform

We have

$$c_k = \frac{1}{N} \underbrace{\sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi n}{N}}}_{X(e^{j\omega k})}$$

Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \omega = \frac{2\pi}{N}$$

$$c_k = \frac{1}{N} X(e^{jk\omega})$$

The DT Fourier transform

Substituting

$$c_k = \frac{1}{N} X(e^{jk\omega})$$

into the synthesis equation for $\tilde{x}[n]$ yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega}) \cdot e^{jk\omega n} \qquad \omega = \frac{2\pi}{N}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega}) e^{jk\omega n} \omega$$

What happens as $N \to \infty$?

$$\tilde{x}[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \Rightarrow \lim_{N \to \infty} \tilde{x}[n] = x[n]$$

Over what range should we integrate ω?

DT Fourier transform pair:

Inverse DTFT (synthesis equation)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

DTFT (analysis equation)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Example: DTFT of a square pulse

Compute the DTFT of the DT signal

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

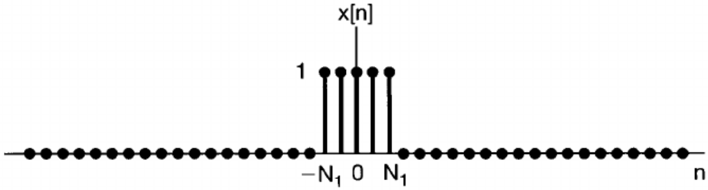
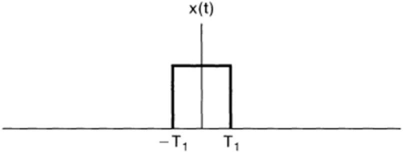


Image credit: Oppenheim chapter 5.1

Recall: FT of a CT square pulse

$$x(t) = \begin{cases} 1 & |t| \leq T_1, \\ 0 & |t| > T_1 \end{cases}$$



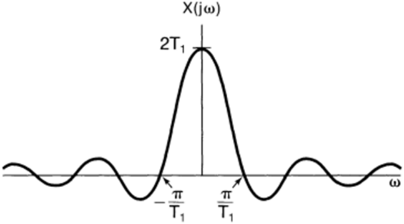


Image credit: Oppenheim chapter 4.1

Example: DTFT of a square pulse

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

Compute the DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\sum_{k=0}^{N} z^k = \frac{1-z^{N+1}}{1-z} \qquad = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

How do we evaluate this sum?

Change summation variable to $m = n + N_1$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} \Longrightarrow \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)} = e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

Use our handy identity:

$$\sum_{k=0}^{N} z^k = \frac{1-z^{N+1}}{1-z} \Longrightarrow X(e^{j\omega}) = e^{j\omega N_1} \frac{\left(1-e^{-j\omega(2N_1+1)}\right)}{1-e^{-j\omega}}$$

$$1-e^{-j\omega} = e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)$$

Do some reshuffling...

$$X(e^{j\omega}) = \frac{e^{j\omega N_1} - e^{-j\omega N_1 - j\omega}}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{e^{-j\omega/2}\left(e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$

why is it not 0 at 0?

consider L'Hopital's rule

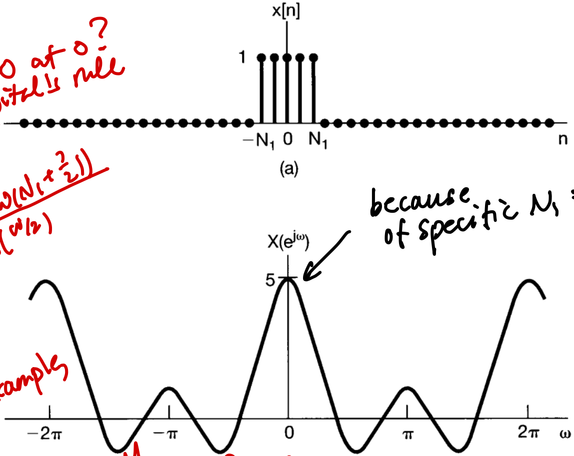
$$\lim_{\omega\to 0} \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$

$$= \lim_{\omega\to 0} \frac{(N_1+\frac{1}{2})\cos(\omega(N_1+\frac{1}{2}))}{\frac{1}{2}\cos(\omega/2)}$$

$$= \frac{N_1+\frac{1}{2}}{\frac{1}{2}}$$

$= 2N_1+1$ in example

since $N_1=2$ we get 5.



because of specific $N_1=2$

can also consider small angle approx, $\sin\theta\approx\theta$ for $\theta<<1$

Note that this function is **continuous** and **periodic**!

Image credit: Oppenheim chapter 5.1

Convergence criteria

In CT had Dirichlet criteria for both Fourier series and inverse Fourier transform. No conditions for DT Fourier series:

$$x[n] = \sum_{k=\langle N \rangle} c_k e^{jk\omega n} \qquad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega n}$$

What about the DT Fourier transform?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{OR} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

DTFT defines the relationship between impulse response and frequency response:

$$h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega})$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$h[n] = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

Convolution works the same way as in CT:

$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

Example: filters

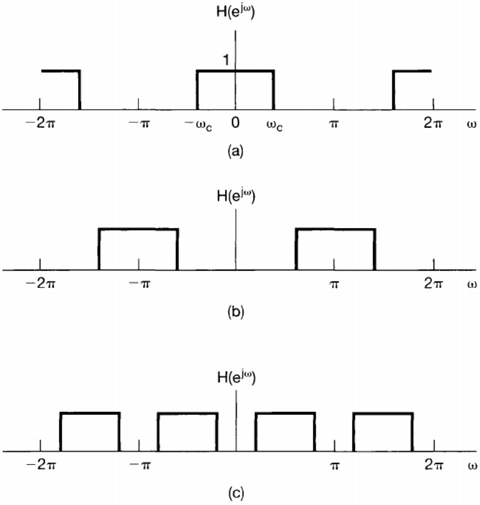


Image: Oppenheim 3.28

Example: filters

Determine the impulse response of an ideal DT low-pass filter,

$$H(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| < \pi \end{cases}$$

$$h[n] = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{\sin(\omega_c n)}{\pi n}$$

For an ideal DT high-pass filter,

$$H(e^{j\omega}) = \begin{cases} 1 & \pi - \omega_c \le |\omega| \le \pi + \omega_c \\ 0 & 0 \le |\omega| < \pi - \omega_c,\ \pi + \omega_c < |\omega| < 2\pi \end{cases}$$

$\Rightarrow$ try it yourself! answer will be similar to lowpass (hint: consider the pictures of the spectra, and the A3 question about multiplication property)

Example: convolution property

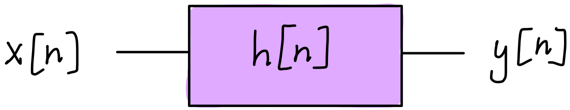
What is the DTFT of

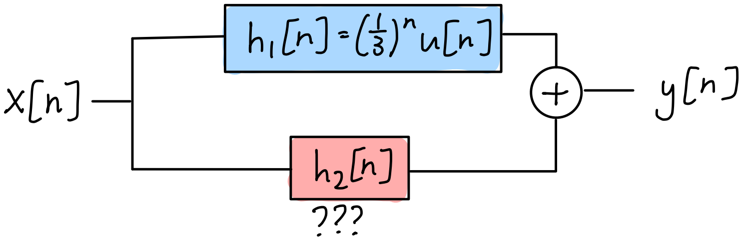
$$x[n] = a^n u[n], \quad |a| < 1$$

we will start here Thursday,
but you can already do this!

Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$





Using our identity:

For next time

Content:

- DTFT properties (linearity, time shift, etc.)
- DT systems based on difference equations

Action items:

1. Assignment 3 due Saturday 23:59 (solutions posted right after)

Recommended reading:

- From today's class: Oppenheim 5.1, 5.4
- Suggested problems: 5.1, 5.2, 5.5, 5.14, 5.21abcfj, 5.22a, 5.29
- For next class: Oppenheim 5.2, 5.3, 5.8, 6.6