ELEC 221 Lecture 24
Feedback systems

Thursday 5 December 2024

Announcements

- Last class!
- Please come pick up your midterms
- Will post final exam info (incl. practice final) on PrairieLearn
- Assignment 5 due Sunday at 23:59

The Laplace transform, with info about input/output relationships, can help characterize systems described by differential equations.

$$\sum_{k=0}^{N} a_k \frac{d y^k(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d x^k(t)}{dt^k}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

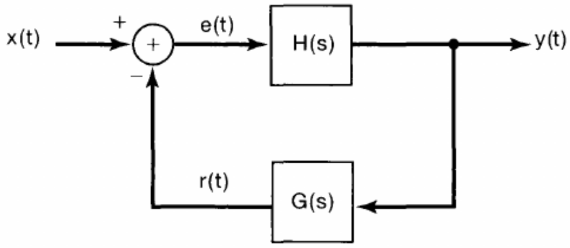
Today

Learning outcomes:

- compute system functions for feedback systems
- use the Laplace transform and feedback systems to design inverse systems and stabilize unstable systems
- identify the z-transform

Feedback systems

An important application of Laplace transforms is the analysis of **feedback systems**.

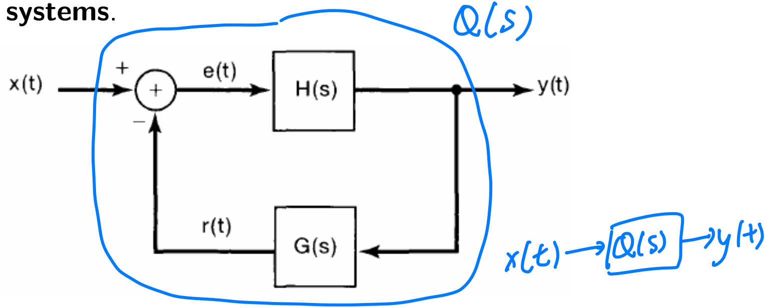


Some important applications of feedback include:

- reducing sensitivity to disturbance and errors
- stabilizing unstable systems
- designing tracking systems

Image credit: Oppenheim 11.1

An important application of Laplace transforms is the analysis of **feedback systems**.

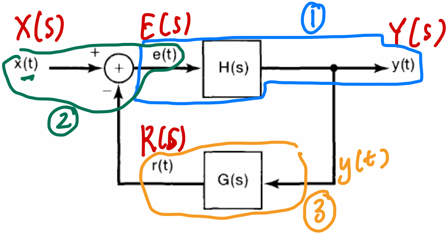


- $H(s)$ is the system function of the forward path
- $G(s)$ is the system function of the feedback path
- the combined function $Q(s)$ is the closed-loop system function

Let's compute $Q(s)$ in terms of $H(s)$ and $G(s)$.

Image credit: Oppenheim 11.1

Feedback systems



$$X(s)\,Q(s) = Y(s)$$

$$Q(s) = \frac{Y(s)}{X(s)}$$

1 $\quad E(s)\,H(s) = Y(s)$

2 $\quad e(t) = x(t) - r(t) \qquad E(s) = X(s) - R(s) \Rightarrow X(s) = R(s) + E(s)$

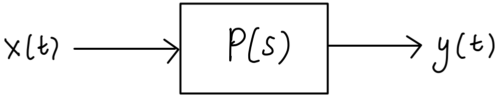
3 $\quad Y(s)\,G(s) = R(s)$

$$Q(s) = \frac{H(s)\,E(s)}{E(s) + R(s)} = \frac{H(s)\,E(s)}{E(s) + Y(s)\,G(s)} = \frac{H(s)\,\cancel{E(s)}}{\cancel{E(s)} + \cancel{E(s)}\,H(s)\,G(s)}$$

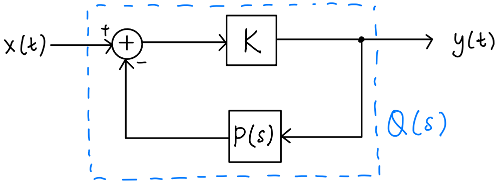
$$\boxed{Q(s) = \frac{H(s)}{1 + G(s)H(s)}}$$

Image credit: Oppenheim 11.1

Suppose we have some LTI system



Let's use it as part of a larger system:



where the transfer function K is simply gain of strength K.

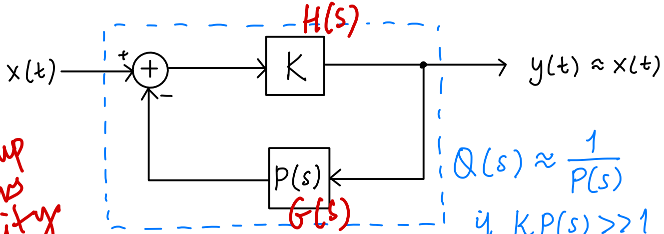
Exercise: What is $Q(s)$, and under what conditions can it act as the inverse of $P(s)$?

constant gain

Solution: we can directly apply the expression for the closed-loop system function here

$$Q(s) = \frac{H(s)}{1+G(s)H(s)} = \frac{K}{1+KP(s)} \approx \frac{K}{KP(s)} \sim \frac{1}{P(s)}$$

for large K



$Q(s) \approx \frac{1}{P(s)}$

if $KP(s) >> 1$

* look up what happens w/ causality

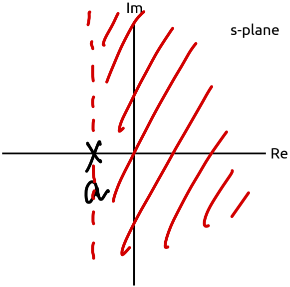
↳ when P(s) is a delay system, inverse is not causal. could not find explanation in text; colleague also wasn't sure, but agreed not physical. at any rate, the inverse here is approximate (for large enough K).

Application of feedback: stabilizing an unstable system

Consider a causal system described by the first order DE

$$\frac{dy(t)}{dt} - ay(t) = bx(t)$$

Exercise: compute the system function and draw the ROC. Under what conditions is it stable?



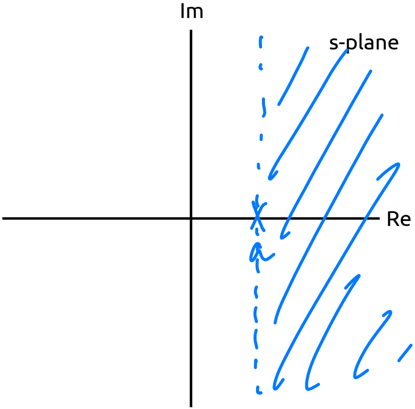
$$H(s) = \frac{b}{s-a}$$

pole at a

Stable if $a < 0$

Application of feedback: stabilizing an unstable system

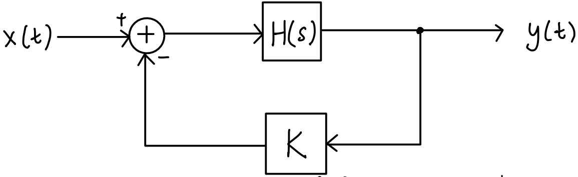
Suppose we have this setup ($a > 0$):



How can we make it stable?

Application of feedback: stabilizing an unstable system

Show that the following system will move the pole (under certain conditions on K):

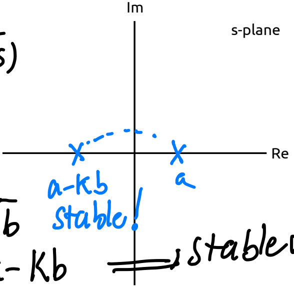


$$Q(s) = \frac{H(s)}{1 + H(s)G(s)} = \frac{H(s)}{1 + K \cdot H(s)}$$

$$H(s) = \frac{b}{s-a}$$

$$Q(s) = \frac{\frac{b}{s-a}}{1 + K \cdot \frac{b}{s-a}} = \frac{b}{(s-a)\left[1 + K\frac{b}{s-a}\right]} = \frac{b}{s-a+Kb}$$

pole: $a - Kb$ $\Longrightarrow$ stable if $K > \frac{a}{b}$ $\rightarrow a - Kb < 0$

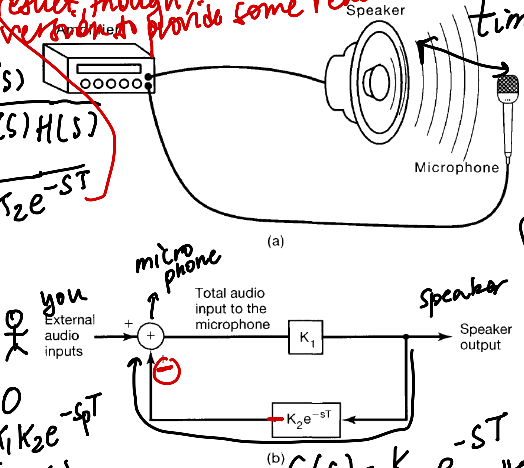


Called a *proportional feedback system* since feeding back in a rescaled version of the output.

Real-world example: audio feedback

technically need to use Nyquist stability criterion: s is complex, we treated as real.
(gives same result, though): consider as a simplified version to provide some real-world intuition.

$$Q(s) = \frac{H(s)}{1+G(s)H(s)} = \frac{K_1}{1-K_1K_2e^{-sT}}$$



delay of time T

$$s_p = \frac{\log(K_1K_2)}{T}$$

want negative

$$\boxed{0 < K_1K_2 < 1 \text{ for stability}}$$

$$1 - K_1K_2e^{-s_pT} = 0$$

$$1 = K_1K_2e^{-s_pT}$$

$$\log(e^{s_pT}) = \log(K_1K_2)$$

$$s_pT = \log(K_1K_2)$$

$$\Rightarrow s_p = \frac{\log(K_1K_2)}{T}$$

$$G(s) = K_2e^{-sT}$$

delay

attenuation would depend on distance

big K_2: less attenuation

Image credit: Oppenheim, Fig. 11.8

$$1 = K_1 K_2 e^{-s_p T} = K_1 K_2 e^{-\sigma_p T} e^{-j\omega T}$$

$$e^{\sigma_p T} e^{j\omega T} = K_1 K_2$$

The z-transform

CT

Fourier series coefficients

$$c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Fourier transform (spectrum)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

DT

Fourier series coefficients

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Consider a DT complex exponential signal

$$x[n] = \underbrace{e^{j\omega}}_{z}{}^{n} = z^n$$

If we put this in a system with impulse response $h[n]$, obtain

$$y[n] = h[n] * x[n] = \underbrace{H(e^{j\omega})}_{\Downarrow \; H(z)} x[n]$$

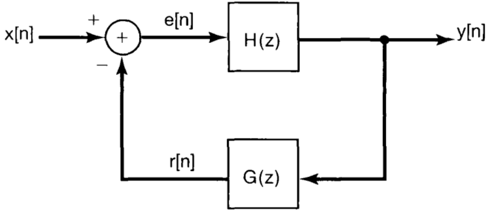
where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- $z = e^{j\omega}$: discrete-time Fourier transform
- $z = re^{j\omega}$: z-transform

DT feedback systems

The z-transform can help us analyze feedback systems (using them for stabilization, etc.), just like Laplace transform in CT.

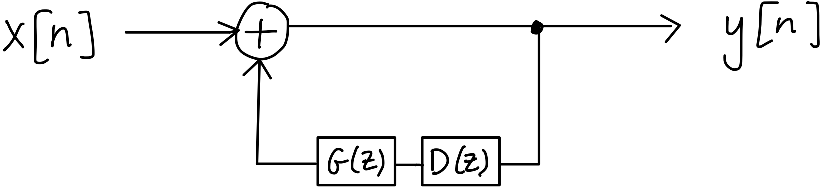


The closed-loop system function has the same form:

$$Q(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1 + H(z)G(z)}$$

Image credit: Oppenheim 11.1

One type of system with this structure is called the **comb filter**



- $D(z)$ is a system that causes a delay of K steps
- $G(z)$ is a system with gain g

Difference equation: $y[n] = x[n] + g\,y[n-K]$

System function:

$$Q(z) = \frac{1}{1 - gz^{-K}} = \frac{z^K}{z^K - g}$$

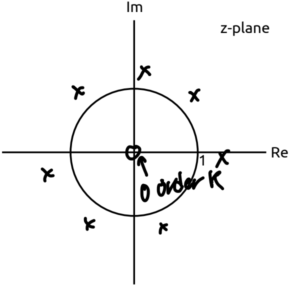
Example: comb filters

$$Q(z) = \frac{z^K}{z^K - g}$$

What are the poles and zeros?

poles where $z^K = g$

$z = re^{j\omega}$



Why is it called the comb filter? Let's look at its frequency response (take $z = e^{j\omega}$).

Another example of this is the Karplus-Strong algorithm!

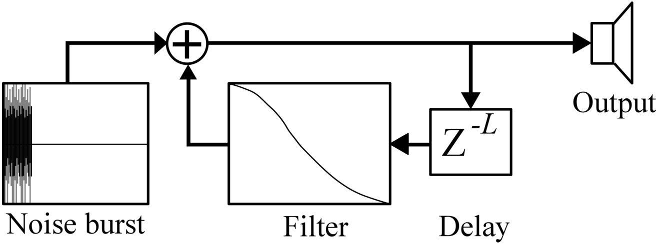
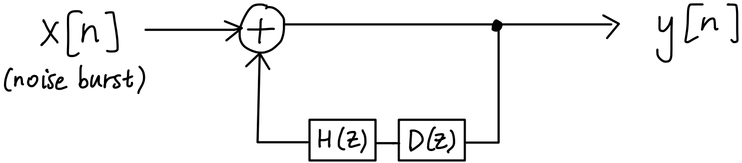


Image credit: https://commons.wikimedia.org/wiki/File:Karplus-strong-schematic.svg Author: PoroCYon CC BY-SA 3.0

Example: Karplus-Strong



- $D(z)$ is a system that causes a delay of K steps
- $H(z)$ is a lowpass filter described by DE
  $y[n] = \frac{1}{2}(x[n] + x[n-1])$

Difference equation: $y[n] = x[n] + \frac{1}{2}\left(y[n-K] + y[n-1-K]\right)$

System function:

$$Q(z) = \frac{1}{1 - \frac{1}{2}z^{-K} - \frac{1}{2}z^{-K-1}}$$

For next time

Action items:

1. Assignment 5 due Sunday at 23:59

Recommended reading:

- From this class: Oppenheim 11.1-11.2
- Suggested problems: 11.2-11.4