ELEC 221 Lecture 12
The discrete-time Fourier transform

Tuesday 18 October 2022

Announcements

- No quiz today (quizzes resume next week)
- Assignment 4 will be made available this week Available
- Midterm grading underway

Midterm postmortem...

Last time

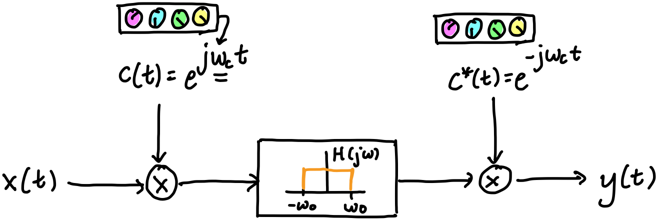
We saw the multiplication property of the CT Fourier transform:

$$y(t) = h(t) * x(t)$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

$$r(t) = s(t)\,p(t)$$

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)\, P(j(\omega-\theta))\, d\theta$$



Last time

We saw how the CT Fourier spectrum behaves under differentiation and integration:

$$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\leftrightarrow} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

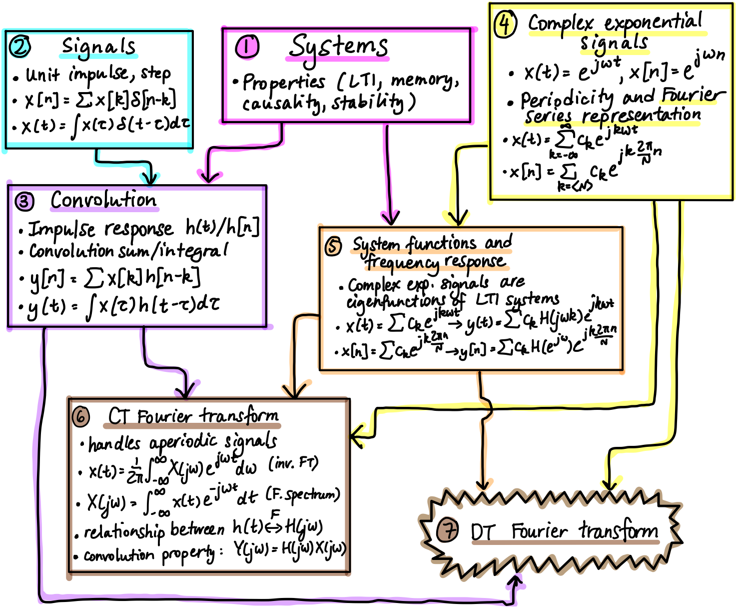
Last time

We leveraged differentiation/integration and the convolution property to compute impulse and frequency response for systems described by ODEs.

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

Where are we going?



DTFT

Learning outcomes:

- Compute the DT Fourier transform of non-periodic DT signals
- State key properties of the DTFT
- Leverage convolution properties of DTFT to analyze the behaviour of LTI systems

On Thursday and Tuesday:

- Learn how the fast Fourier transform algorithm works
- Hands-on with the NumPy FFT module: image processing

Fourier series pair:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega t}\, dt$$

Fourier transform pair:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

inv. Fourier trans.

When a DT signal is periodic (with period N) it can be represented using only the integer harmonics at the same frequency.

DT synthesis equation:

$$x[n] = \sum_{k=\langle N\rangle} c_k e^{jk\cdot\frac{2\pi n}{N}}$$

DT analysis equation:

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

The discrete-time Fourier transform (DTFT) is the generalization of the Fourier series representation to **aperiodic** signals.

We derive it just like we did in CT:

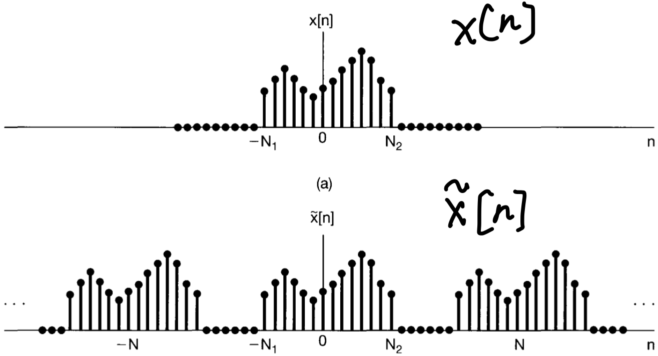


Image credit: Oppenheim chapter 5.1

Suppose $\tilde{x}[n]$ is a periodic extension of $x[n]$. We can write it as a DT Fourier series:

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} c_k e^{jk\frac{2\pi n}{N}}$$

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\frac{2\pi n}{N}}$$

We could just as well change the bounds of the sum to include where our signal actually is:

$$\tilde{x}[n] = \sum_{k=-N_1}^{N_2} c_k e^{jk\frac{2\pi n}{N}}$$

$$c_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\frac{2\pi n}{N}}$$

Now, what happens if we make the period larger and larger (i.e., increase the spacing?)

$$c_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} \tilde{x}[n]\, e^{-jk\frac{2\pi n}{N}}$$

If $N \to \infty$, for any finite n, our new signal $\tilde{x}[n]$ basically just looks like our old signal:

$$c_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

But since $x[n] = 0$ outside this range, we can change the bounds of the sum:

$$c_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

We have

$$c_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi n}{N}}$$

Let's define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi n}{N}} = \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega n}$$

Then

$$c_k = \frac{1}{N} X(e^{jk\omega})$$

Substituting

$$c_k = \frac{1}{N} X(e^{jk\omega})$$

back into the original synthesis equation for $\tilde{x}[n]$ yields

$$\omega = \frac{2\pi}{N}$$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega}) \cdot e^{jk\omega n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega}) e^{jk\omega n} \cdot \omega$$

$$\downarrow$$

$$[0, 2\pi)$$

Now what happens as $N \to \infty$?

The DT Fourier transform

As $N \to \infty$, $\omega \to 0$.

Consider what we are summing:

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{jk\omega}) e^{jk\omega n} \omega$$

This is going to be a sum of terms like $X(e^{jk\omega})e^{jk\omega n}\omega$ for very small ω. We can convert the sum to an integral:

$$\tilde{x}[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

Recall though that in this range, $\tilde{x}[n]$ is basically $x[n]$, and we only need to integrate from over 0 to 2π. The result is the **DT Fourier transform pair**.

Inverse DTFT (synthesis equation):

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

DTFT (analysis equation):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Example: DTFT of a square pulse

Let's compute the DTFT of the DT signal

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$

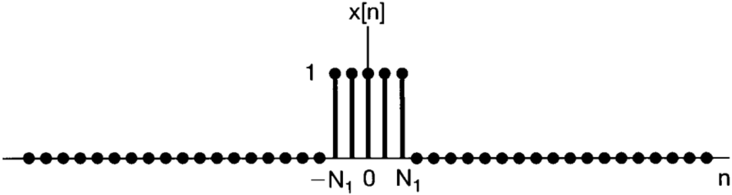


Image credit: Oppenheim chapter 5.1

Recall: FT of a CT square pulse

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

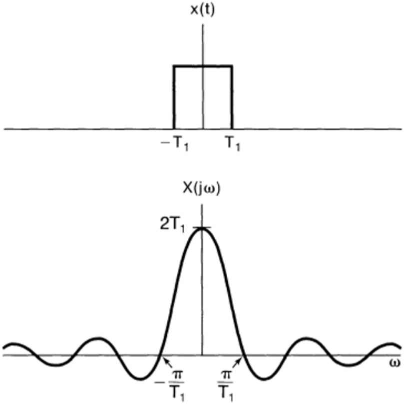


Image credit: Oppenheim chapter 4.1

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

Compute the DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

How do we evaluate this sum?

Change variable in the summation to $m = n + N_1$

$$X(e^{j\omega}) = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)} = \sum_{m=0}^{2N_1} e^{-j\omega m} e^{j\omega N_1}$$

$$= e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

Use our handy identity:

$$\sum_{k=0}^{N} z^k = \frac{1 - z^{N+1}}{1 - z}$$

$$\Rightarrow X(e^{j\omega}) = e^{j\omega N_1} \frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}}$$

Example: DTFT of a square pulse

$$e^{j\theta} = \cos\theta + j\sin\theta$$

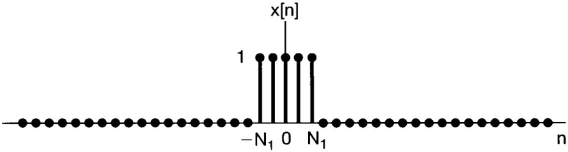
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

$$X(e^{j\omega}) = e^{j\omega N_1}\frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}}$$

Straightforward from here:

$$X(e^{j\omega}) = \cancel{e^{j\omega N_1}} \cdot \frac{\cancel{e^{-j\omega(N_1+1/2)}}\left(e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}\right)}{\cancel{e^{-j\omega/2}}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$



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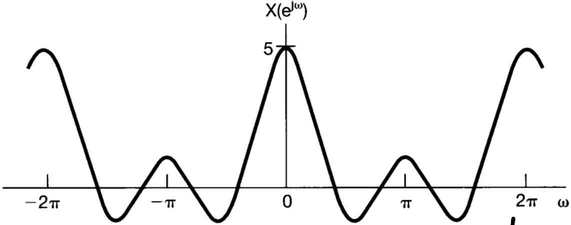
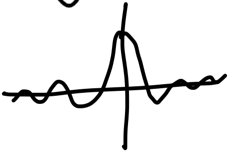




Note that this function is periodic!

Image credit: Oppenheim chapter 5.1

Convergence criteria

Recall in CT we had Dirichlet criteria for both Fourier series and inverse Fourier transform representations:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Convergence criteria

We didn't have this issue for the DT Fourier series:

$$x[n] = \sum_{k=\langle N \rangle} c_k e^{jk\omega n} \qquad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega n}$$

What about for the DT Fourier transform?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Convergence criteria

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The synthesis equation is fine; but the analysis equation has an infinite sum. One of the following must be satisfied:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Convolution works the same way as in CT:

$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

We also have the same relationship between impulse response and the frequency response:

$$h[n] \xleftrightarrow{F} H(e^{j\omega})$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Convolution

Convolution works the same way as in CT:

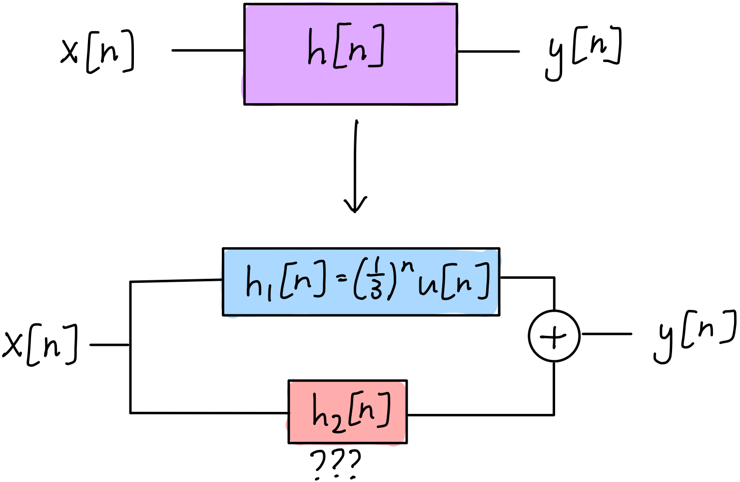
$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega})
\end{aligned}
$$

We also have the same relationship between impulse response and the frequency response:

$$
\begin{aligned}
h[n] &\overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}) \\
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\
h[n] &= \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{j\omega n}d\omega
\end{aligned}
$$

Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$



Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$

Hint:

$$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$$

Example: convolution property

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$

Hint: $(e^{-jw} - 3)(e^{-jw} - 4)$

$$a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}, \quad |a| < 1$$

$$b(e^{-jw} - 4) + a(e^{-jw} - 3) = -12 + 5e^{-jw}$$

$$a + b = 5$$

$$-3a - 4b = -12$$

$a = 8$ $\quad b = -3$

$$\frac{8}{e^{-jw} - 4} - \frac{3}{e^{-jw} - 3}$$

$8 \cdot u[n]$

Example: convolution property

$$
\begin{aligned}
H(e^{j\omega}) &= \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-2j\omega}} \\
&= \frac{-12+5e^{-j\omega}}{(3-e^{-j\omega})(4-e^{-j\omega})} \\
&= \frac{A}{(3-e^{-j\omega})}+\frac{B}{(4-e^{-j\omega})} \\
&= \frac{3}{(3-e^{-j\omega})}+\frac{-8}{(4-e^{-j\omega})} \\
&= \frac{1}{1-\frac{1}{3}e^{-j\omega}}+\frac{-2}{1-\frac{1}{4}e^{-j\omega}}
\end{aligned}
$$

Using our identity:

$$h[n] = h_1[n] + h_2[n] = \left(\frac{1}{3}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Many properties are the same as the CT analogs.

Linearity: If

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

then

$$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Many properties are the same as the CT analogs.

Time shift: If

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

then

$$x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$$

Frequency shift:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)})$$

Conjugation: If

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x^*[n] \overset{F}{\longleftrightarrow} X^*(e^{-j\omega})$$

If $x[n]$ is real,

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

Consequences for odd/even functions:

$$\text{Even}(x[n]) \overset{F}{\longleftrightarrow} \text{Re}(X(e^{j\omega}))$$
$$\text{Odd}(x[n]) \overset{F}{\longleftrightarrow} j\,\text{Im}(X(e^{j\omega}))$$

Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

Differentiation in frequency:

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jn \cdot x[n] e^{-j\omega n}$$

$$= -j \cdot \sum_{n=-\infty}^{\infty} n\, x[n] e^{-j\omega n}$$

$$n\, x[n] \overset{F}{\longleftrightarrow} j \frac{dX(e^{j\omega})}{d\omega}$$

Properties of the DT Fourier transform

Differencing:

$$x[n]-x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} (1-e^{-j\omega})X(e^{j\omega})$$

Accumulating:

$$\sum_{m=-\infty}^{n} x[m] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

Parseval's relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Here $|X(e^{j\omega})|^2$ is called the *energy-density spectrum*.

Today's learning outcomes were:

- Compute the DT Fourier transform of non-periodic DT signals
- State key properties of the DTFT
- Leverage convolution properties of DTFT to analyze the behaviour of LTI systems

What topics did you find unclear today?

Content:

- The discrete Fourier transform (DFT) and the Fast Fourier Transform (FFT) algorithm

Action items:

1. Keep an eye out for Assignment 4

Recommended reading:

- From today's class: Oppenheim 5.0-5.7
- For next class: Oppenheim extension problems 5.53-5.54