ELEC 221 Lecture 13
Differentiation and integration properties; systems based on differential equations

Tuesday 22 October 2024

Announcements

- Quiz 6 today
- TA3 due Monday 23:59; A3 available soon
- Exam time announced: Sunday 15 December, 7pm

We saw some important properties of the Fourier transform:

- Linearity
- Behaviour under time shift/scale/reverse/conjugation

The most important was for **convolution**:

This made it easier to analyze LTI systems!
In assignment 3, you will explore the related relationship for **multiplication**:

Today

Learning outcomes:

- Express a Fourier transform using the magnitude-phase representation
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

The magnitude-phase representation

Since Fourier spectra are complex we can express them in terms of their magnitude and phase.

Recall the convolution property:

Exercise: How does a system $H(j\omega)$ affect $|X(j\omega)|$ and $\sphericalangle X(j\omega)$?

Example: Lowpass filters in practice

The magnitude-phase representation can help us both visualize and characterize the behaviour of systems.

Example: a resistor combined with a capacitor creates an LTI system that behaves as a lowpass filter.

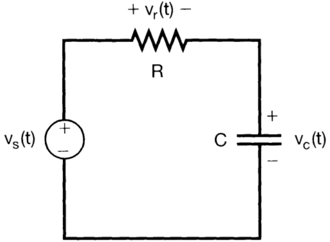
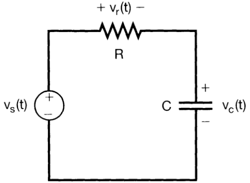


Image credit: Oppenheim chapter 3.10.

Example: lowpass filters in practice

What is the voltage across the capacitor if $v_s = e^{j\omega t}$?



Derive two expressions for current, using resistor and capacitor:

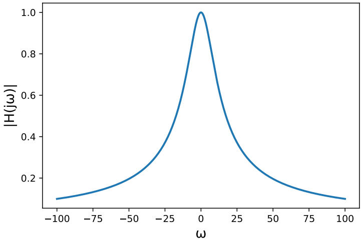
Image credit: Oppenheim chapter 3.10.

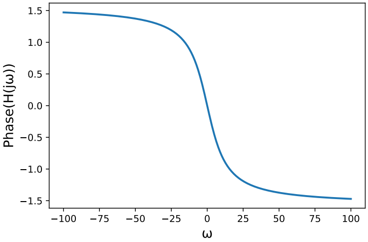
Example: lowpass filters in practice

Put these together to form a differential equation:

Example: lowpass filters in practice

Results in the following frequency response (setting $RC = 0.1$):





Adjusting RC controls the frequency response; increasing RC cuts off more frequencies.

Consider the inverse Fourier transform:

What happens when we differentiate $x(t)$?

This means:

What should happen here?

Good initial guess:

More precisely:

Exercise: what are the Fourier transforms of the unit impulse and unit step?

Example: Fourier transform properties and differentiation

We can take advantage of differentiation and integration properties to simplify computations.

Suppose

What is the Fourier transform of

Two properties to take advantage of here:

Example: Fourier transform properties and differentiation

$$z(t) = \frac{d^2}{dt^2} x(t-1)$$

First, consider: $p(t) = x(t-1)$:

Fourier transforms and systems described by differential equations

The RC circuit from earlier was based on a differential equation:

We found its frequency response by:

- Choosing input signal
- Since system is LTI, assuming output of the form
- Plugging this into the ODE and solving for

There is a better way to do this!

Consider a general system described by an ODE of arbitrary order:

What is its frequency response $H(j\omega)$?

Fourier transforms and systems described by differential equations

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

Fourier transforms and systems described by differential equations

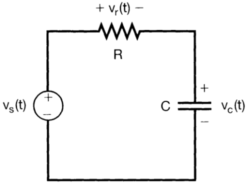
$$Y(j\omega)\sum_{k=0}^{N}\alpha_k(j\omega)^k = X(j\omega)\sum_{k=0}^{M}\beta_k(j\omega)^k$$

Final property:

This representation allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

Exercise: frequency response of systems described by ODEs

What are the **impulse response** and **frequency response** of our RC circuit filter?



Exercise: frequency response of systems described by ODEs

What are the **impulse response** and **frequency response** of the system described by

Start with frequency response:

Example: frequency response of systems described by ODEs

We can now leverage this to determine the impulse response:

$$H(j\omega) = \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega}$$

Use partial fractions:

Example: frequency response of systems described by ODEs

Details are left as an exercise:

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

Example: frequency response of systems described by ODEs

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega + 2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega - 2}\right)$$

Check Table 4.2 - two expressions to leverage:

So we have:

For next time

Content:

- Systems described by first- and second-order ODEs
- Step response
- Bode plots

Action items:

1. Tutorial assignment 3

Recommended reading:

- For today's class: Oppenheim 4.3, 4.6-4.7, 6.1-6.2
- Suggested problems: 4.5, 4.8, 4.22, 4.25, 4.29, 4.33-4.36
- For next class: Oppenheim 4.7, 6.3-6.5