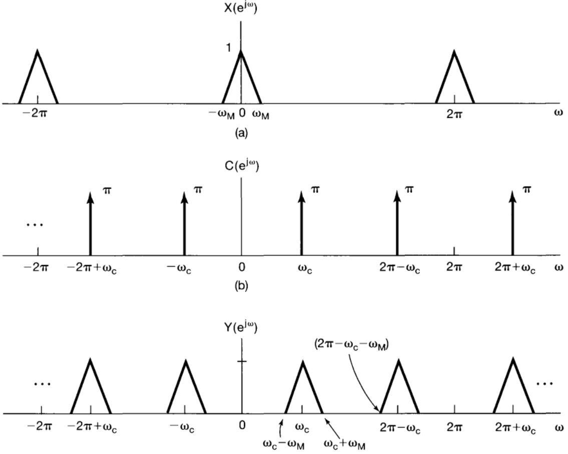
ELEC 221 Lecture 22
The Laplace transform

Thursday 24 November 2022

Announcements

- Midterm 2 available for pickup
- Quiz 10 Tuesday (last quiz)
- Assignment 6 (computational) due on Tuesday at 23:59
- Final assignment (pen and paper) released early next week

We did DT sinusoidal amplitude modulation.



We can still do frequency-division multiplexing, but we may need to upsample and shrink the spectra so things fit.

Image credit: Oppenheim 8.8

We briefly explored how frequency modulation works

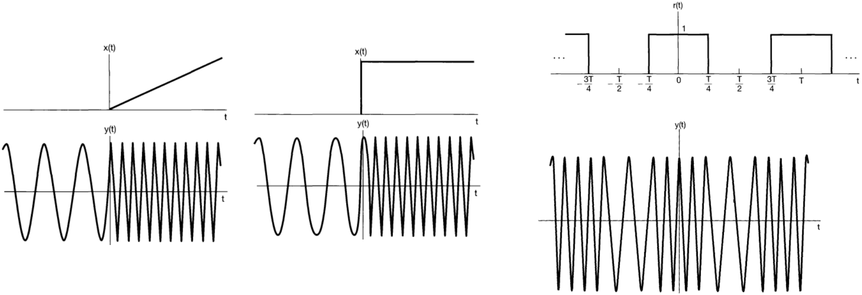
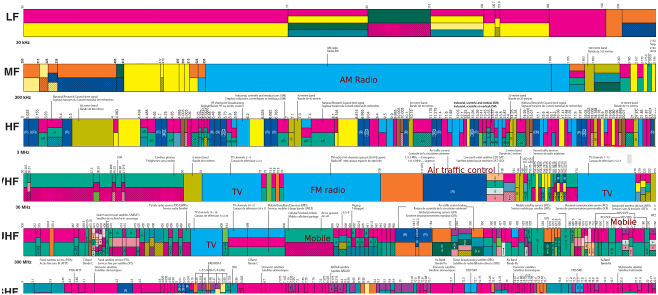


Image credit: Oppenheim 8.7

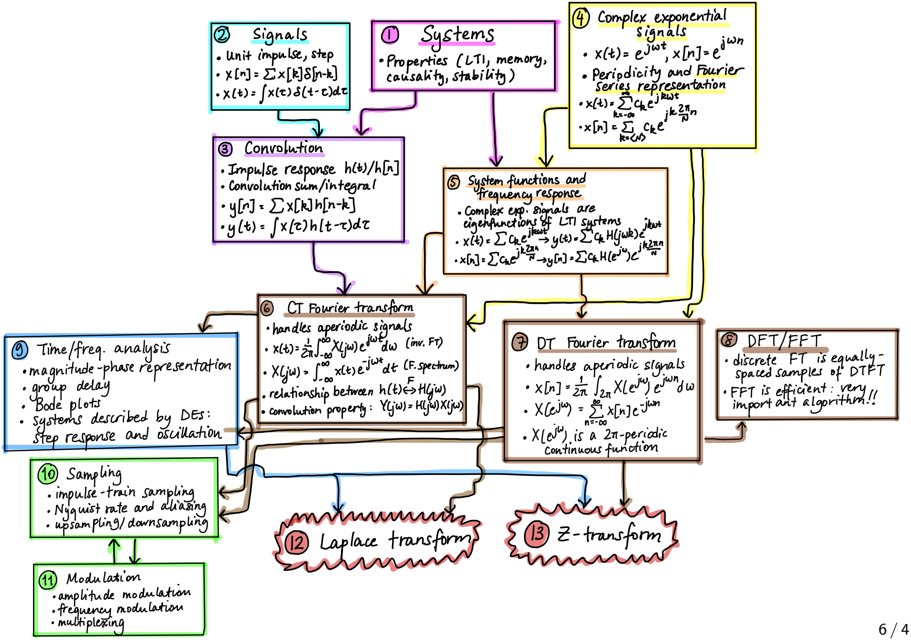
Last time

We discussed how cell phones are radios and how the radio spectrum gets divided (and auctioned off).



See the full graphic here: `https://www.ic.gc.ca/eic/site/smt-gst.nsf/vwapj/2018_Canadian_Radio_Spectrum_Chart.PDF/$FILE/2018_Canadian_Radio_Spectrum_Chart.PDF`

The course so far



Wayyyyy back in lecture 3:

LTI systems and complex exponential functions

Write $x(t) = e^{st}$. Then:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{st}e^{-s\tau}h(\tau)d\tau \\
&= e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau \\
&= e^{st}H(s)
\end{aligned}
$$

Wayyyyy back in lecture 3:

LTI systems and complex exponential functions

To summarize:

$$x(t) = e^{st} \quad \rightarrow \quad y(t) = H(s) \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

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Wayyyyy back in lecture 3:

...so what?

If all the $x_i(t)$ are complex exponential functions and

$$x(t) = e^{st} \quad \rightarrow \quad y(t) = H(s) \cdot e^{st}$$

then

$$x(t) = \sum_k c_k e^{s_k t} \quad \rightarrow \quad y(t) = \sum_k c_k H(s_k) e^{s_k t}$$

The response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

How can we express arbitrary signals as superpositions of complex exponentials?

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Wayyyyy back in lecture 3:

The Fourier series

Let's consider a special set of signals[1]:

$$x(t) = e^{st} = e^{j\omega t}$$

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

[1] We will see the general case at the end of the course.

Today

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

Consider a general complex exponential signal,

Input into LTI system with impulse response $h(t)$:

If $s = j\omega$: **Fourier transform**

If $s = \sigma + j\omega$: (bilateral) **Laplace transform**

More generally, the Laplace transform of an arbitrary signal is

We will write

We can relate the Laplace and Fourier transforms. Consider

for $s = \sigma + j\omega$.

Example: consider the signal $x(t) = e^{-at}u(t)$.

What is $X(j\omega)$?

Recall: conditions on a?

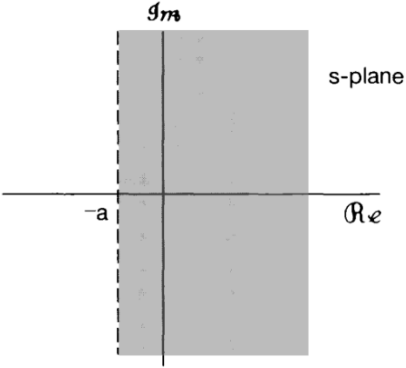
What is the Laplace transform?

What are the conditions here?

Take a closer look at $x(t) = e^{-at}u(t)$:

The Laplace transform exists in some regions where the Fourier transform does not!

We must specify for which s the Laplace transform is valid.



This is called the **region of convergence** (ROC).

Image credit: Oppenheim 9.1

Exercise: what is the Laplace transform of

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Solution:

Conditions are different though: need $\mathrm{Re}(s) < -a$.

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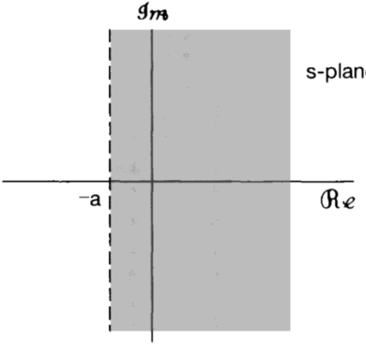
Conditions are different though: need $\mathrm{Re}(s) < -a$.

Multiple signals can have the same algebraic Laplace transform, but different ROCs.



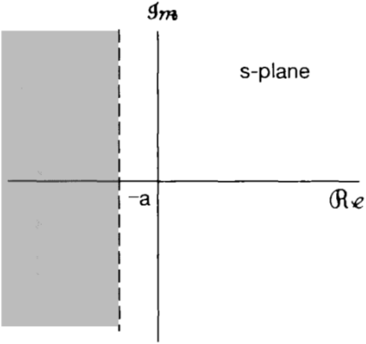


Image credit: Oppenheim 9.1

Exercise: what is the Laplace transform of

and what is its region of convergence?

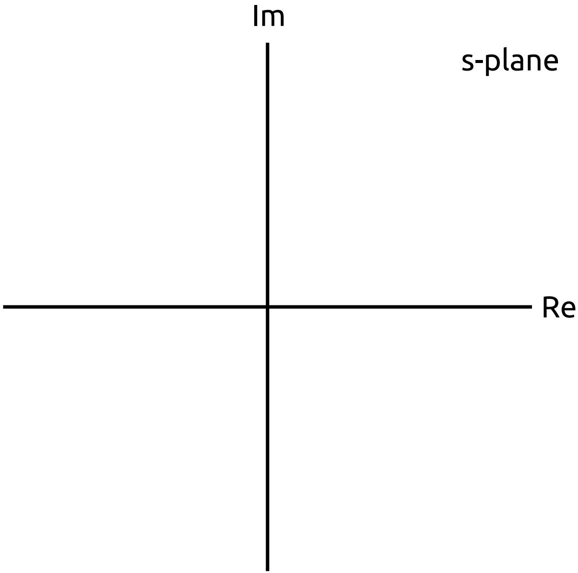
Hint: the Laplace transform is also linear!

Solution: by linearity,

To determine its ROC,

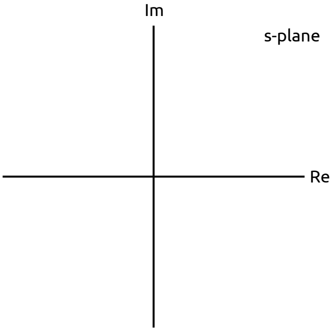
- first term tells us $\mathrm{Re}(s) > -1$
- second term tells us that $\mathrm{Re}(s) > -2$

Let's draw the ROC:



Pole-zero plots

Quite often the Laplace transforms will be rational polynomials of s. Generally the roots of these polynomials are indicated on the plots. Use $\times$ for denominator (poles), $\circ$ for numerator (zeros):



This is called a pole-zero plot. (May also have poles/zeros at infinity if degree of polynomials is different)

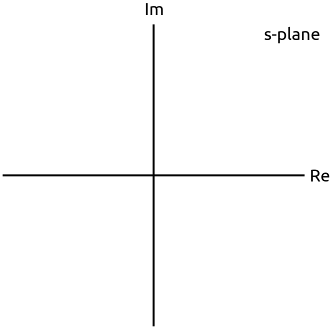
Exercise: compute the Laplace transform of

$$x(t) = -2e^{-3t}u(t) + 4e^{-4t}u(t)$$

and draw its pole-zero plot.

Pole-zero plots

Solution:



Regions of convergence

The ROC has many nice properties:

- if ROC doesn't contain $j\omega$ axis, FT does not converge
- ROC is strips parallel to $j\omega$ axis
- ROC of rational Laplace transform contains no poles

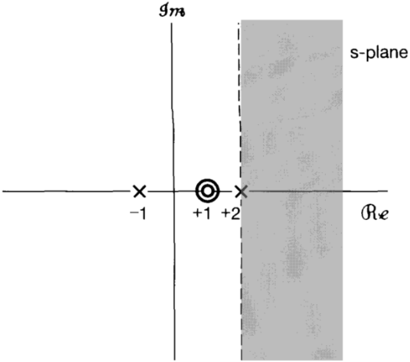


Image credit: Oppenheim 9.1

If $x(t)$ has finite duration and is absolutely integrable, the ROC is the entire s-plane.

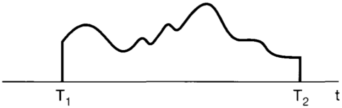
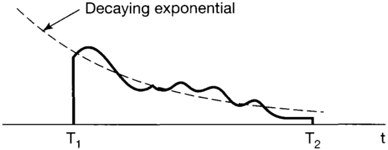


Figure 9.4 Finite-duration signal.



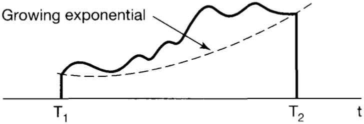
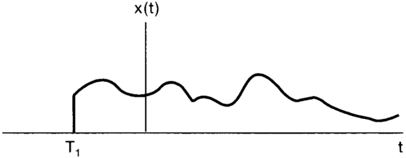
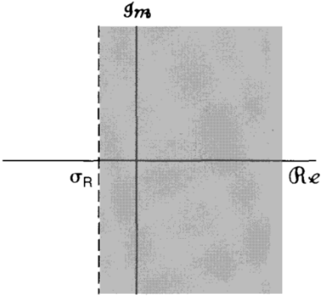


Image credit: Oppenheim 9.2

Right-sided signals

If $x(t)$ is right sided and $\mathrm{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\mathrm{Re}(s) > \sigma_0$ are also in the ROC.



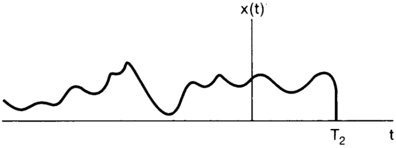


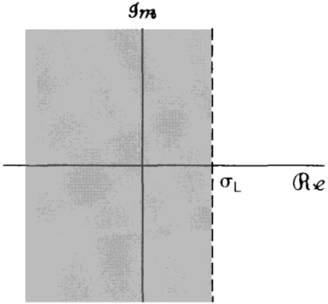
This ROC is called a **right-half plane**.

Intuition: if $\mathrm{Re}(s) = \sigma_1 > \sigma_0$ the exponential in $x(t)e^{-\sigma t}$ decays even faster and will still converge.

Image credit: Oppenheim 9.2

If $x(t)$ is left sided and $\text{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\text{Re}(s) < \sigma_0$ are also in the ROC.





This ROC is called a **left-half plane**.

Image credit: Oppenheim 9.2

Two-sided signals

Any guesses?

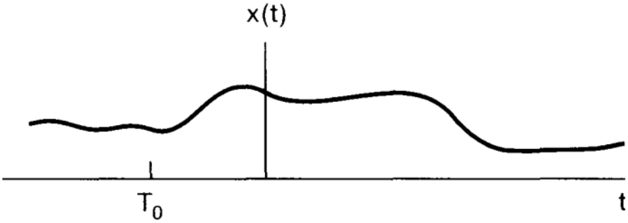
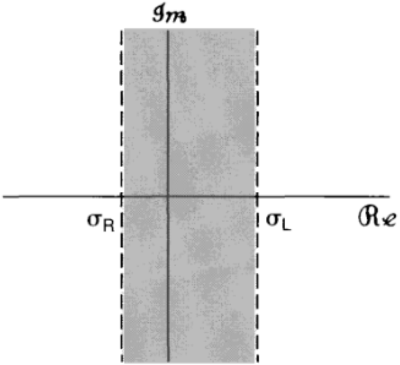


Image credit: Oppenheim 9.2

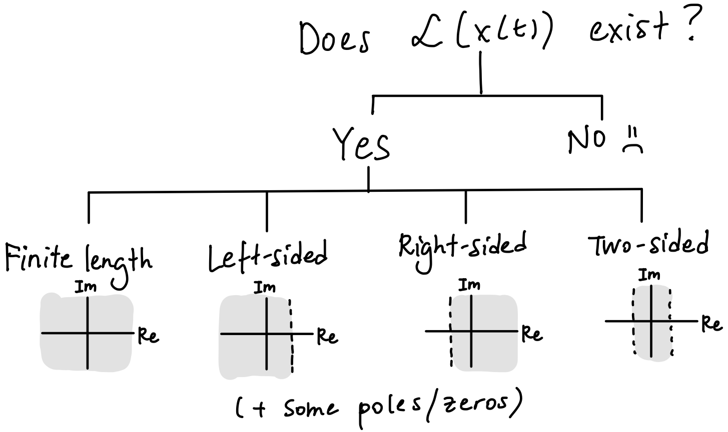
Two-sided signals



This is only the case if there was actually overlap - otherwise it doesn't exist!

Image credit: Oppenheim 9.2

For any signal $x(t)$...

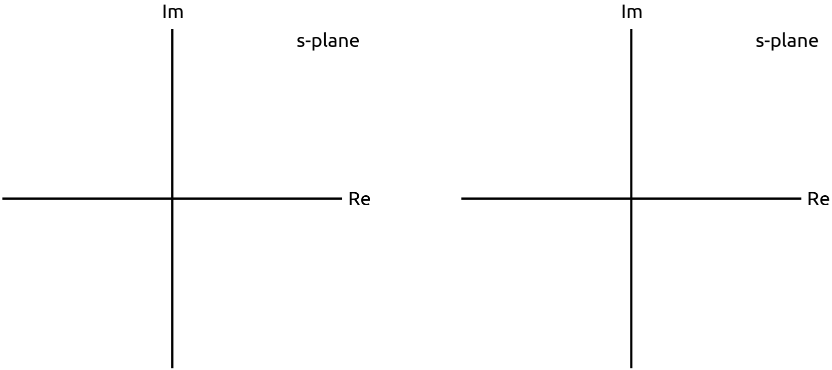


Regions of convergence

(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

$$\frac{s-1}{(s+2)(s+3)(s^2+s+1)}$$

in its ROC? (Hint: draw pole-zero plot and identify the regions)



From this, we can invert:

Make a change of variables $ds = jd\omega$:

... we are not going to integrate this.

Suppose the Laplace transform has the form

where degree of denominator is higher than numerator.

To invert this, we can use our handy identities, BUT the region of convergence does matter.

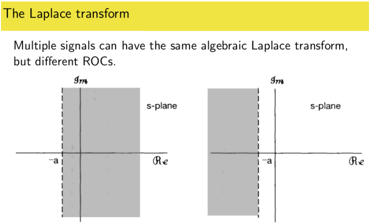


Image credit: Oppenheim 9.1

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

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Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
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Oppenheim practice problems: 9.1-9.9, 9.21, 9.26

Content:

- properties and system analysis with Laplace transform

Action items:

1. Quiz 10 on Tuesday
2. Assignment 6 due Tuesday at 23:59

Recommended reading:

- From this class: Oppenheim 9.0-9.3. 9.5 (skip 9.4)
- For next class: 9.5-9.8 (skip 9.9)