# ELEC 221 Lecture 11
# Properties of the CT Fourier Transform

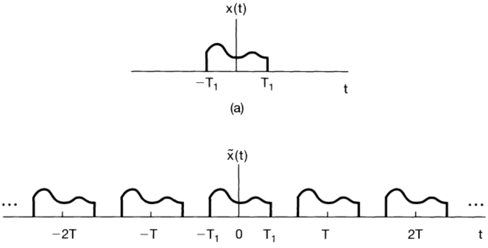
Tuesday 15 October 2024

# Announcements

- Quiz 5 today (based on Lecture 10 material)
- TAs are very busy with grading!
- Expect A3 and TA3 next week

# Last time

We generalized the CT Fourier series (for periodic signals) to the Fourier transform (for aperiodic signals):




We expressed the periodic extension of an aperiodic function as

Image credit: Oppenheim chapter 4.1

# Last time

We computed its Fourier coefficients:

We put this into the Fourier series and let $T \to \infty$ $(\omega \to 0)$:

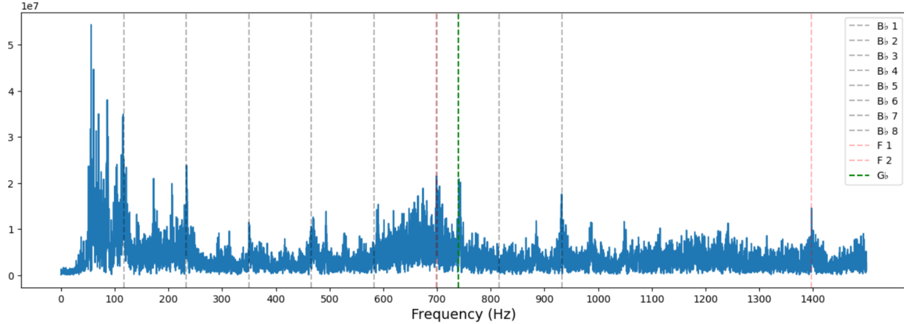
# Last time

Inverse Fourier transform (synthesis equation):

Fourier transform (analysis equation):

The Fourier spectrum contains a lot of important and useful information about signals!

```
fourier_spectrum = np.fft.rfft(audio, norm="forward")
frequencies =  np.fft.rfftfreq(len(audio), 1 / sample_rate)
```




You will experience this directly in Tutorial Assignment 3.

# Today

Learning outcomes:

- State sufficient criteria for a signal to have a Fourier transform
- Compute the Fourier transform of a periodic signal
- Describe the duality between time and frequency domains
- Leverage key properties of Fourier transform to simplify its computation

# Dirichlet conditions for Fourier transforms

If a signal

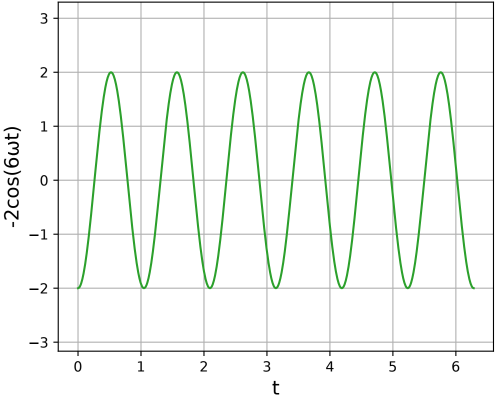
1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

# Dirichlet conditions for Fourier transforms

The "**within any finite interval**" takes care of periodic signals:



# Exercise 1

What signal does the following Fourier transform belong to?

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

# Exercise 2

What signal does the following Fourier transform belong to?

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

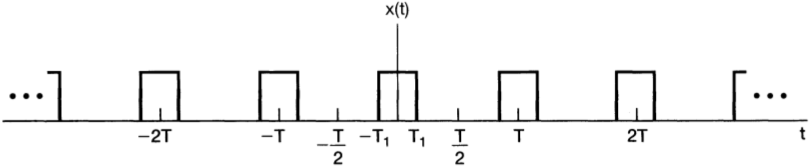
# Fourier transforms for periodic signals: a unified representation

The Fourier transform of a periodic function is an **impulse train**.

The impulses have area $2\pi c_k$ and are positioned at the harmonically related frequencies.

# Fourier transforms for periodic signals: a unified representation

Remember our square wave from last time:




It had Fourier series coefficients

Image credit: Oppenheim chapter 4.1

Its Fourier *transform* will be

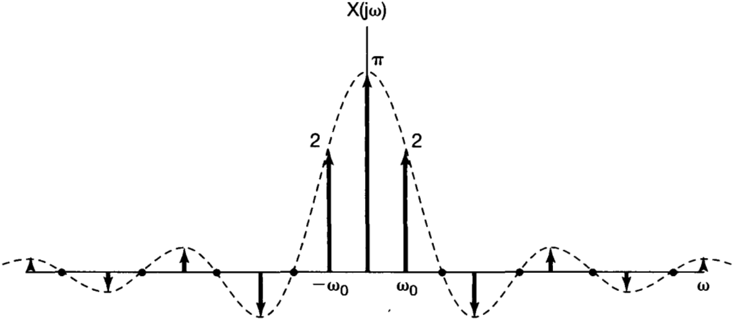
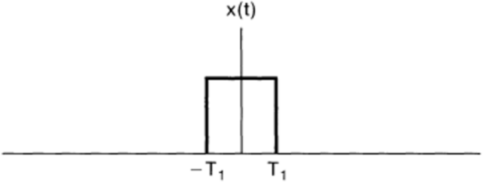



Image credit: Oppenheim chapter 4.2

# Time/frequency duality of the FT

Let's consider a single square pulse:




Its Fourier spectrum is

Image credit: Oppenheim chapter 4.1

# Time/frequency duality of the FT

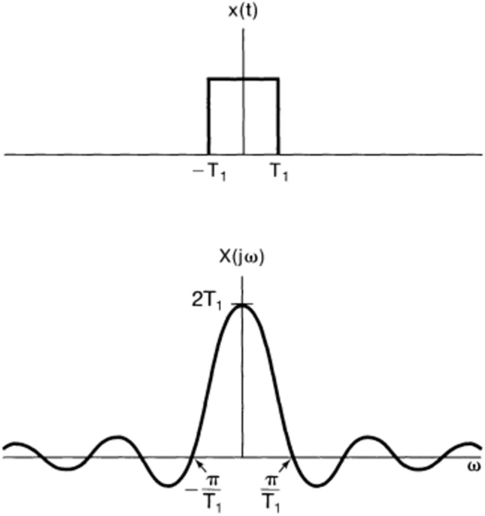



Image credit: Oppenheim chapter 4.1

Now let's consider a signal whose Fourier transform is

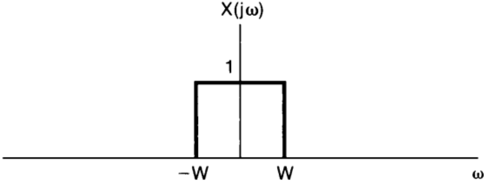



Image credit: Oppenheim chapter 4.1

# Time/frequency duality of the FT

Compute the inverse Fourier transform:

# Time/frequency duality of the FT

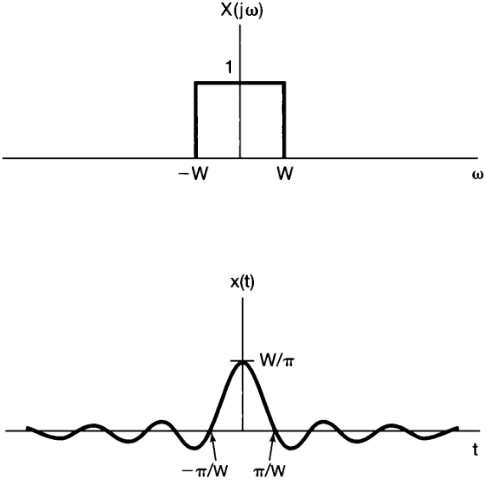



Image credit: Oppenheim chapter 4.1

# Time/frequency duality of the FT

Duality: for any transform pair $(x(t) \leftrightarrow X(j\omega))$, there is a *dual pair* with the time and frequency variables interchanged.

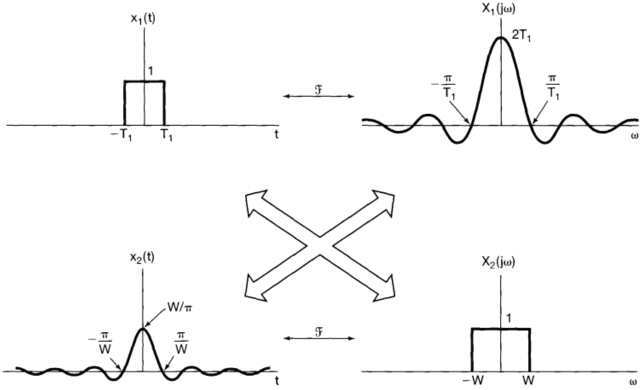



Image credit: Oppenheim chapter 4.3

# Exercise

What is the Fourier transform of $x(t) = e^{-at}u(t)$ $(a > 0)$?

# Example: Fourier transform properties

What is the Fourier transform of $x(t) = e^{-2|t-1|}$?

The Fourier transform has many useful properties that help with evaluating it for arbitrary functions.

**Linearity**.

Our example:

**Time shifting**. If

then

Notice: $|X(j\omega)|$ does not change; we just add a linear phase shift.

Our example:

**Time scaling**. If

then

**Time reversal** follows from this:

# Important properties of the Fourier transform

Our example: we have

**Conjugation**. If

then

If $x(t)$ is purely real,

Implications for even/odd parts of a signal:

# Important properties of the Fourier transform

Recall we can write functions in terms of their odd/even parts:

Learning outcomes:

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## For next time

Content:

- Convolution and multiplication property of the Fourier transform
- More Fourier transform pairs
- Differentiation and integration of FT

Recommended reading:

- From today's class: Oppenheim 4.2-4.3
- Suggested problems: 4.2-4.4, 4.6, 4.9, 4.21bcdgh, 4.27
- For next class: Oppenheim 4.4-4.6