ELEC 221 Lecture 10
More properties of the CT Fourier transform

Tuesday 11 October 2022

Announcements

- Quiz 5 today
- Midterm 1 on Thursday
  - Closed book / closed notes; no calculators
  - Formula sheet provided (see last Thursday's lecture)
  - Please arrive on time and bring your ID
- Assignment 4 (computational) available after midterm
  - Statement of contributions worth 1 point from now on.
  - "No exceptions" means *no exceptions*

Last time: the Fourier transform

We saw the Dirichlet conditions for the Fourier transform.

If the signal

1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

Last time: properties of the Fourier transform

We computed Fourier transforms of periodic signals.

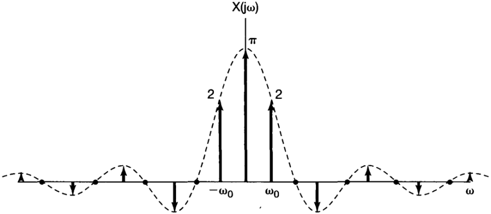
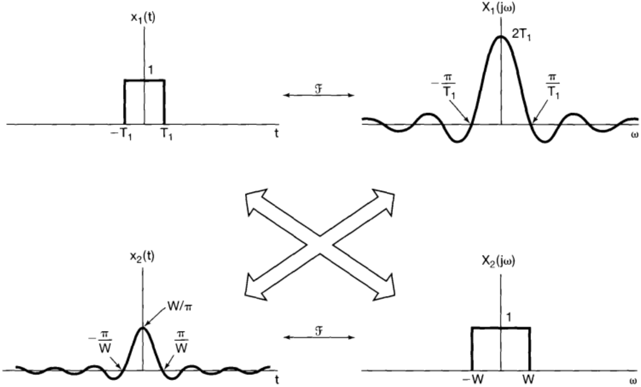


Image credit: Oppenheim chapter 4.2

Last time: properties of the Fourier transform

We saw some important properties of the Fourier transform:

- Linearity
- Behaviour under time shift/scale/reverse/conjugation
- Time/frequency duality



Last time: properties of the Fourier transform

We explored how the **frequency response** of a system relates to its **impulse response** via a Fourier transform:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

We introduced the convolution property:

Last time: properties of the Fourier transform

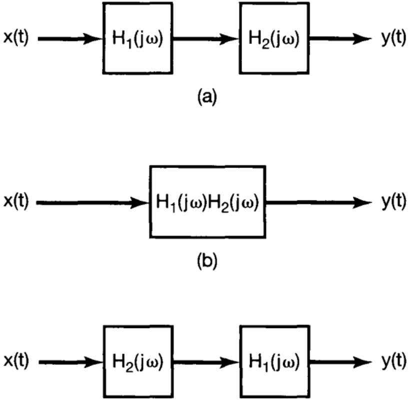


Image credit: Oppenheim chapter 4.4

Today

Learning outcomes:

- Describe the *multiplication property* of the Fourier transform
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

The multiplication property

We know that:

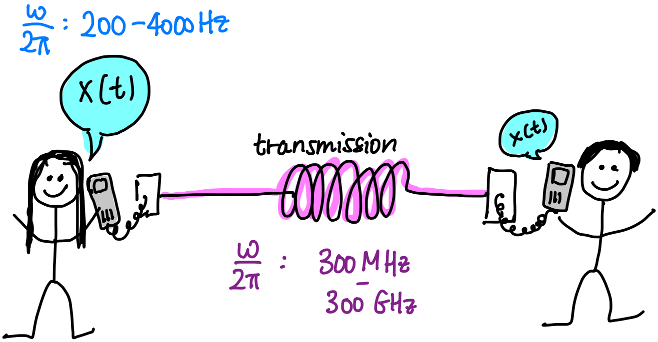
Something similar holds when we interchange time and frequency:

This is the **multiplication property**.

Example: the multiplication property

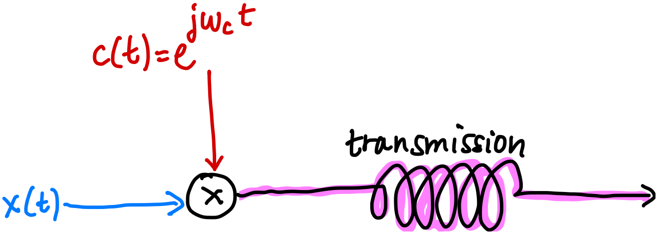
We are going to take a much closer look at this when we discuss communication systems and signal **modulation**.

For now, here is a taste:



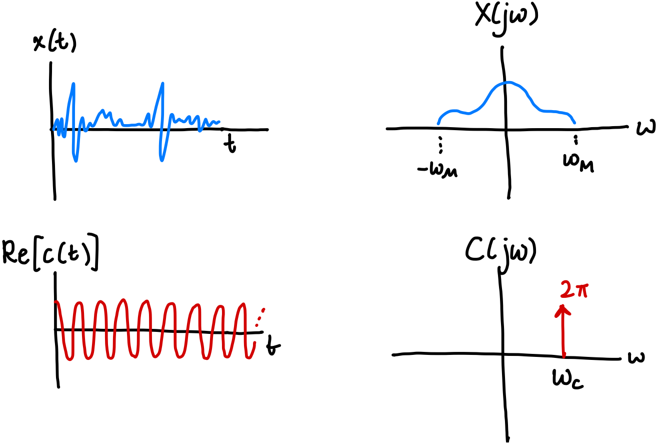
Example: the multiplication property

To shift our signal into the frequency range of transmission, we can multiply it by a **carrier signal** (amplitude modulation):



Is this doing what we think it is?

Consider the Fourier spectrum of both signals:



Example: the multiplication property

The multiplication property tells us

We have

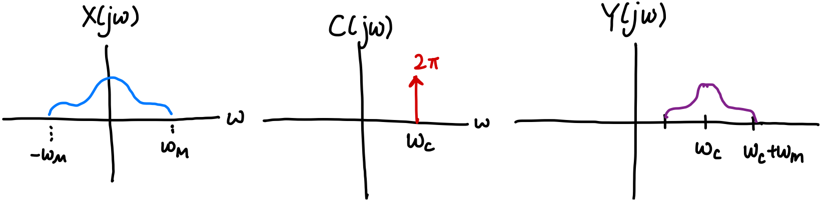
Example: the multiplication property

Let's convolve them:

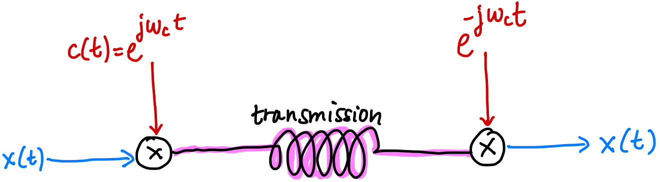
Example: the multiplication property

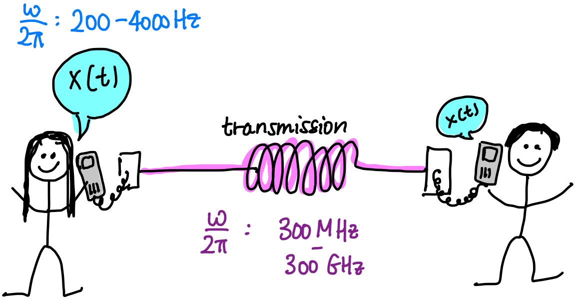
Let's convolve them:

Multiplication with complex exponential carrier signal shifts the spectrum. We can move it into the desired frequency range.

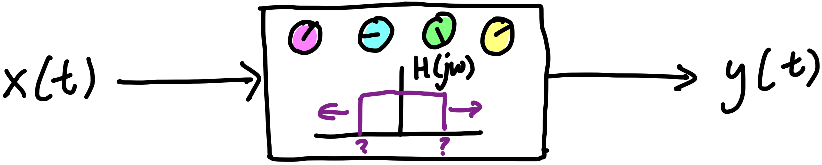


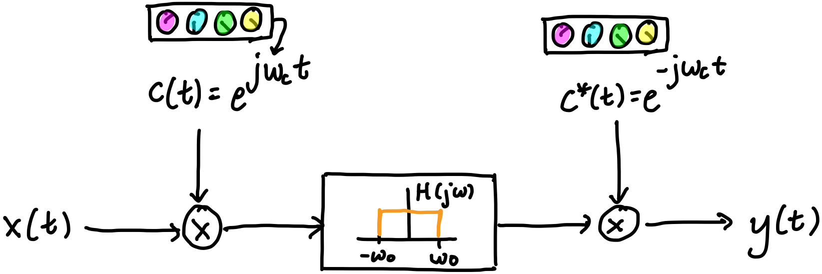
Example: the multiplication property

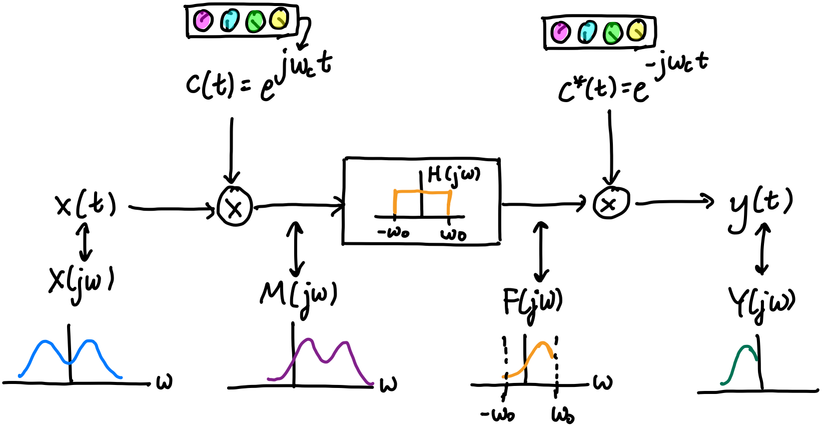




Example: frequency-selective filtering with variable centre frequency




$c(t) = e^{j\omega_c t}$
$c^*(t) = e^{-j\omega_c t}$
$x(t)$
$H(j\omega)$
$-\omega_0$
ω_0
$y(t)$



c(t) = e^{jω_c t}
c*(t) = e^{-jω_c t}
x(t)
X(jω)
M(jω)
H(jω)
-ω_0
ω_0
F(jω)
y(t)
Y(jω)
ω

Consider the inverse Fourier transform:

What happens when we differentiate $x(t)$?

This means:

What should happen here?

Good initial guess:

More precisely:

We can often take advantage of differentiation and integration properties to simplify computation of Fourier transforms and system outputs.

Example: Fourier transform properties and differentiation

Suppose

What is the Fourier transform of

Two properties to take advantage of here:

Example: Fourier transform properties and differentiation

$$z(t) = \frac{d^2}{dt^2} x(t-1)$$

First, consider: $p(t) = x(t-1)$:

But we know

So

Back in lecture 6, we saw a system (RC circuit) described by a differential equation:

We found its frequency response in the following way:

- Choosing input signal
- Since system is LTI, assuming output of the form
- Plugging this into the ODE and solving for

Nice properties of Fourier transforms give a much slicker method of computing frequency responses of such systems.

Consider a general system described by an ODE of arbitrary order:

What is its frequency response $H(j\omega)$?

First: take the Fourier transform of both sides.

Which property can we leverage next?

Fourier transforms and systems described by differential equations

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

Now what?

We can simplify this even more:

$$Y(j\omega)\sum_{k=0}^{N}\alpha_k(j\omega)^k = X(j\omega)\sum_{k=0}^{M}\beta_k(j\omega)^k$$

Final property:

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

Example: frequency response of systems described by ODEs

What are the **impulse response** and **frequency response** of the system described by

Start with frequency response:

Example: frequency response of systems described by ODEs

We can now leverage this to determine the impulse response:

$$H(j\omega) = \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega}$$

Use partial fractions:

Example: frequency response of systems described by ODEs

Details are left as an exercise:

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

Example: frequency response of systems described by ODEs

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega + 2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega - 2}\right)$$

Last time, we learned a general expression for inverse Fourier transforms of this type:

So we have:

Recap

Today's learning outcomes were:

- Describe the *multiplication property* of the Fourier transform
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

What topics did you find unclear today?

For next time

Content (after the midterm):

- Discrete Fourier transform

Action items:

1. Midterm 1 on Thursday

Recommended reading:

- From today's class: Oppenheim 4.5-4.8
- For Tuesday's class: Oppenheim chapter 5.0-5.5