ELEC 221 Lecture 14
Analysis of CT systems based on first- and second-order differential equations

Thursday 24 October 2024

- Tutorial assignment 3 due Monday 23:59
- Assignment 3 available, due Saturday 2 Nov 23:59
- Monday's tutorial will focus on problem solving
- MT2 information available next week

Erratum from last class:

$$z(t) = x(t)y(t) \Rightarrow Z(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

Last time

We expressed systems in the magnitude phase representation

$$X(j\omega) = |X(j\omega)| e^{j \sphericalangle X(j\omega)}$$

For a signal $x(t)$ and system with frequency response $H(j\omega)$,

$$|Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|$$

$$\sphericalangle Y(j\omega) = \sphericalangle H(j\omega) + \sphericalangle X(j\omega)$$

$|H(j\omega)|$ is the gain and $\sphericalangle H(j\omega)$ is the phase shift. We plotted these separately.

We derived the behaviour of the Fourier transform under differentiation and integration:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

From these, we determined

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \Delta(j\omega) = 1$$

$$u(t) \overset{\mathcal{F}}{\longleftrightarrow} U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Last time

Finally, we combined many properties to write an expression for the frequency responses of systems described by differential equations,

$$\sum_{k=0}^{N} \alpha_k \frac{dy^k(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{dx^k(t)}{dt^k}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

Last time: exercise

What are the **impulse response** and **frequency response** of the system described by

$$\frac{d^3y(t)}{dt^3} - 4\frac{dy(t)}{dt} = 3\frac{d^2x(t)}{dt^2} + x(t)$$

We computed frequency response and began using partial fractions:

$$\begin{aligned} H(j\omega) &= \frac{3(j\omega)^2+1}{(j\omega)^3 - 4j\omega} \\ &= \frac{3(j\omega)^2+1}{j\omega((j\omega)^2-4)} \\ &= \frac{3(j\omega)^2+1}{j\omega(j\omega+2)(j\omega-2)} \\ &= \frac{A}{j\omega} + \frac{B}{j\omega+2} + \frac{C}{j\omega-2} \end{aligned}$$

Details are left as an exercise:

$$H(j\omega) = \frac{-1/4}{j\omega} + \frac{13/8}{j\omega+2} + \frac{13/8}{j\omega-2}$$

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega+2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega-2}\right)$$

Last time: exercise

$x(t) = 1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$

time reversed

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega+2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega-2}\right)$$

$$-\frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{2-j\omega}\right)$$

Check Table 4.2 - two expressions to leverage:

$$\mathcal{F}\left[e^{-at}u(t)\right] = \frac{1}{a+j\omega} \quad Re(a) > 0$$

$$\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) = \mathcal{F}^{-1}\left(\underbrace{\frac{1}{j\omega} + \pi\delta(\omega)}\, - \pi\delta(\omega)\right) = \underline{u(t) - \frac{1}{2}}$$

So we have:

$$h(t) = -\frac{1}{4}u(t) + \frac{1}{8} + \frac{13}{8}e^{-2t}u(t) - \frac{13}{8}e^{2t}u(-t)$$

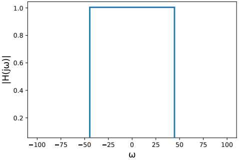
Learning outcomes:

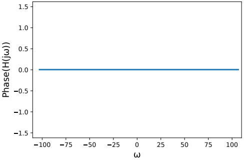
- define and compute the unit step response of a system
- characterize the oscillatory behaviour of CT systems described by second-order differential equations
- read a Bode plot

Lowpass filters

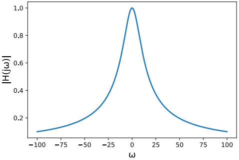
We've seen two versions of lowpass filters:

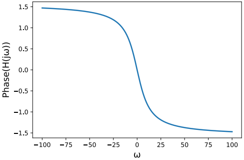
$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$





$$H(j\omega) = \frac{1}{1 + j\omega RC}$$



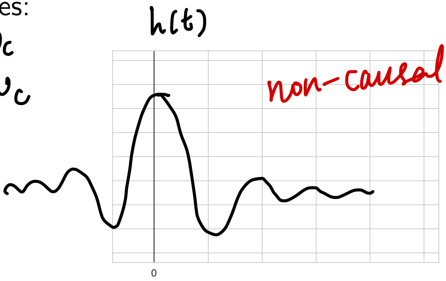


What's the difference?

Look at their impulse responses:

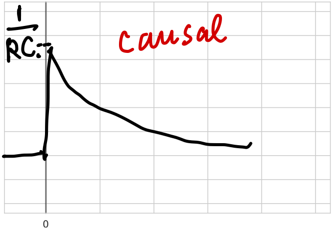
$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}$$



$$H(j\omega) = \frac{1}{1+j\omega RC} = \frac{1}{RC\left(\frac{1}{RC} + j\omega\right)}$$

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



Step response

It is also important to consider the *step response* of filters:

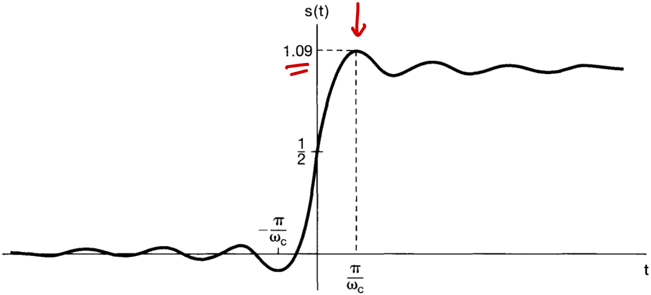
$$s(t) = h(t) * u(t) \qquad\qquad t-\tau > 0$$

$$= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} h(\tau)\, d\tau$$

Ideal filter step response

$$h(t) = \frac{\sin(\omega_c t)}{\pi t} \qquad s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$



An ideal filter leads to **ringing** in the step response.

Image credit: Oppenheim 6.3

Ideal filter step response

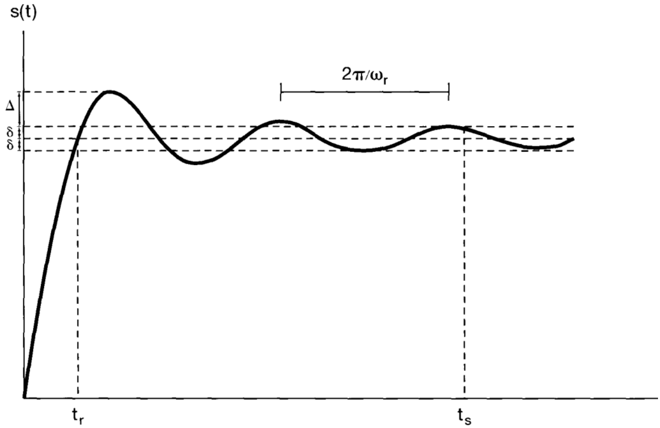


Figure 6.17 Step response of a continuous-time lowpass filter, indicating the rise time t_r, overshoot Δ, ringing frequency ω_r, and settling time t_s—i.e., the time at which the step response settles to within $\pm\delta$ of its final value.

Image credit: Oppenheim 6.4

There are **tradeoffs** in filter design. Compromises in the frequency domain can lead to nicer behaviour in the time domain.

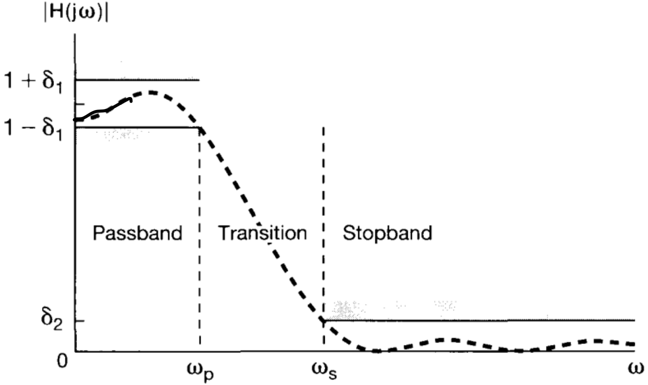


Image credit: Oppenheim 6.4

$$S(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

Exercise: What is the step response of a system described by a first-order ODE?

$$T\,\frac{dy(t)}{dt} + y(t) = x(t)$$

$$H(j\omega) = \frac{1}{1 + j\omega T}$$

$$h(t) = \frac{1}{T} e^{-t/T} u(t)$$

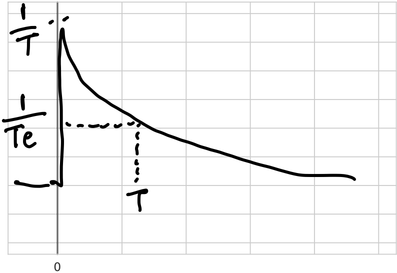
$$s(t) = \left(1 - e^{-t/T}\right) u(t)$$

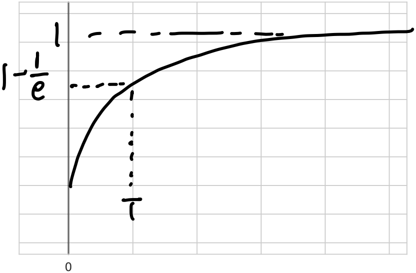
$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \frac{1}{T}\int_{-\infty}^{t} e^{-\tau/T} u(\tau)\, d\tau = \frac{1}{T}\int_{0}^{t} e^{-\tau/T} d\tau = \left(1 - e^{-t/T}\right) u(t)$$

The impulse and step response of the system are

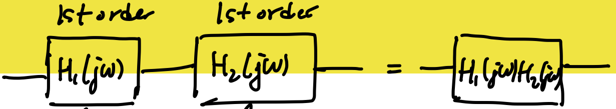
$$h(t) = \frac{1}{T} e^{-t/T} u(t) \qquad s(t) = \left(1 - e^{-t/T}\right) u(t)$$

T is the **time constant** of the system.





Second-order systems



$\frac{1}{a_1 + j\omega}$ $\frac{1}{a_2 + j\omega}$ $\frac{1}{a_1 a_2 + (a_1 + a_2)j\omega + (j\omega)^2}$

Consider a system described by the ODE

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Exercise: what is the frequency response?

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

Let's explore this in a little more detail and compute the impulse and step response of this system.

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}$$

where

$$c_{\pm} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Three cases to consider:

- $\zeta = 1$
- $\zeta > 1$
- $\zeta < 1$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}, \quad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case: $\zeta = 1$.

$$c_\pm = -\zeta \omega_n = -\omega_n$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

Use handy table of Fourier transform pairs to find

$$h(t) = \omega_n^2 \, t \, e^{-\omega_n t} u(t)$$

$$s(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}) u(t)$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}, \quad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case: $\zeta \neq 1$. Do a partial fraction expansion:

$$H(j\omega) = \frac{A}{j\omega - c_+} + \frac{B}{j\omega - c_-}$$

$$= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[\frac{1}{j\omega - c_+} - \frac{1}{j\omega - c_-}\right]$$

Use table of Fourier transform pairs to find

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(e^{c_+ t} - e^{c_- t}\right)u(t)$$

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(e^{c_+ t} - e^{c_- t}\right) u(t), \qquad c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Exercise: derive the step response,

$$s(t) = \left[1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{c_+ t}}{c_+} - \frac{e^{c_- t}}{c_-}\right)\right] u(t)$$

The form of the exponential depends whether $\zeta > 1$ or $\zeta < 1$

- $\zeta < 1$: $c_\pm$ are imaginary; complex exponentials, so the response will oscillate!
- $\zeta > 1$: $c_\pm$ real and negative; decaying exponentials

Let's go plot these.

Bode plots

Recall:

$$|Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|$$
$$\measuredangle Y(j\omega) = \measuredangle H(j\omega) + \measuredangle X(j\omega)$$

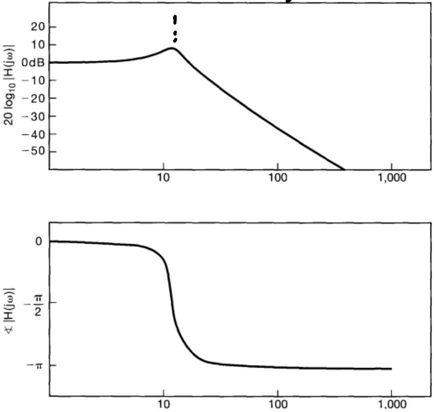
Magnitude is multiplicative and phase is additive... would be nicer if both were additive.

$$\log_{10}|Y(j\omega)| = \log_{10}|H(j\omega)| + \log_{10}|X(j\omega)|$$

Rather than plotting $|H(j\omega)|$ and $\measuredangle H(j\omega)$, it is common to plot $20\log_{10}|H(j\omega)|$ and $\measuredangle H(j\omega)$ against $\log_{10}\omega$.

Bode plots

These are called *Bode plots*: $Q = \frac{1}{2\zeta}$



The logarithmic scale also allows us to view the response over a much wider range of frequencies.

Image credit: Oppenheim 6.2

Bode plots: first-order systems

☆ We didn't cover this in lecture; just here for reference.

$$T\frac{dy(t)}{dt} + y(t) = x(t), \qquad H(j\omega) = \frac{1}{1 + j\omega T}$$

Let's view these in the magnitude-phase representation:

$$H(j\omega) = \frac{1}{1 + j\omega T} \cdot \frac{1 - j\omega T}{1 - j\omega T} = \frac{1}{(\omega T)^2 + 1} - j\frac{\omega T}{(\omega T)^2 + 1}$$

From this, we find

$$\begin{aligned} |H(j\omega)| &= \frac{1}{\sqrt{(\omega T)^2 + 1}} \\ \sphericalangle H(j\omega) &= \tan^{-1}(-\omega T) \end{aligned}$$

We have

$$\begin{aligned} |H(j\omega)| &= \frac{1}{\sqrt{(\omega T)^2 + 1}} \\ \measuredangle H(j\omega) &= \tan^{-1}(-\omega T) \end{aligned}$$

To make our Bode plot, compute

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= 20\log_{10}\left(\frac{1}{\sqrt{(\omega T)^2+1}}\right) \\ &= -20\log_{10}\left((\omega T)^2+1\right)^{1/2} \\ &= -10\log_{10}((\omega T)^2+1) \end{aligned}$$

Bode plots: first-order systems

$$20\log_{10}|H(j\omega)| = -10\log_{10}((\omega T)^2 + 1)$$

If $\omega << 1/T$,

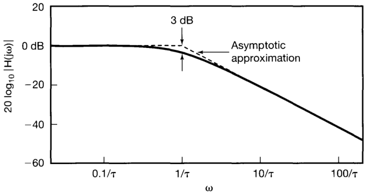
$$20\log_{10}|H(j\omega)| \approx 0$$

If $\omega >> 1/T$, $\omega T >> 1$

$$\begin{aligned} 20\log_{10}|H(j\omega)| &\approx -10\log_{10}((\omega T)^2) \\ &= -20\log_{10}(\omega T)) \\ &= -20\log_{10}(\omega) - 20\log_{10}(T) \end{aligned}$$

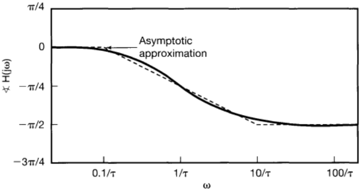
If $\omega = 1/T$,

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= -10\log_{10}(((1/T)T)^2 + 1) \\ &= -10\log_{10}(2) \approx -3 \end{aligned}$$



Can make similar approximations to recover plot of phase

$$H(j\omega) = \tan^{-1}(-\omega T)$$



Let's replot the second-order system functions as Bode plots.

Image credit: Oppenheim 6.5

For next time

Content:

- Discrete-time Fourier transform

Action items:

1. Tutorial Assignment 3 due Monday 23:59
2. Assignment 3 due next Saturday 23:59

Recommended reading:

- From today's class: Oppenheim 6.1-6.5, 6.7
- Suggested problems: 6.1, 6.3a, 6.5, 6.9, 6.15, 6.21-6.23
- For next class: Oppenheim 5.0-5.7