Practice problems

1. Consider the discrete-time signal $x[t]$ where

$$x[n] = 1 + \cos(4\pi n/9)$$

(a) Fin the period N
(b) What is the corresponding fundamental frequency f_0 and ω_0?
(c) What are the coefficients for its Fourier series expansion?

2. Consider the continuous-time signal $x(t)$ where

$$x(t) = 1 + \cos(\pi t) + \cos(2\pi t)$$

Suppose that x is the input to an LTI system with frequency response given by

$$H(j\omega) = \begin{cases} e^{j\omega}, & |\omega| < 4\ rad/s \\ 0, & otherwise \end{cases}$$

What will be the output of the system?

3. Suppose that the continuous time signal $x(t)$ is periodic with period T. Let the fundamental frequency be $\omega_0 = 2\pi/T$. Suppose that the Fourier series coefficients for this signal are known constants $C_0, C_1, C_2, \cdots$. Give the Fourier series coefficients $C'_0, C'_1, C'_2, \cdots$ for each of the following signals:

(a) $ax(t)$, where a is a real valued constant
(b) $x(t - t_0)$, where t_0 is a constant
(c) $S(x)$, where S is an LTI system with frequency response $H(j\omega)$ given by

$$H(j\omega) = \begin{cases} 1, & \omega = 0 \\ 0, & otherwise \end{cases}$$

(d) Let $y(t)$ be another periodic signal with period T. Suppose $y(t)$ has Fourier series coefficients $C''_0, C''_1, C''_2, \cdots$. Give Fourier series coefficients of $x(t) + y(t)$.

From Lee-Varaiya, 2011.

4. Consider a continuous-time periodic signal $x(t)$ with fundamental frequency $\omega_0 = 1\, rad/s$. Suppose that the Fourier series coefficients are

$$C_k = \begin{cases} 1, & k = 0, 1, 2 \\ 0, & otherwise \end{cases}$$

(a) Given the continuous-time LTI system $Filter$, with a frequency response

$$H(j\omega) = \cos{(\pi\omega/2)}$$

find $y(t) = Filter(x)$.

(b) What is the fundamental frequency in rad/s for $y(t)$ calculated in (a)?

5. Suppose that the frequency response $H(e^{j\omega})$ of a discrete-time LTI system $Filter$ is given by:

$$H(e^{j\omega}) = |\omega|$$

where ω has units of rad/s. What is the output $y[t]$ of the system $Filter$ for each of the following inputs $x[t]$:

(a) $x[n] = \cos{(\pi n/2)}$
(b) $x[n] = 5$
(c) $x[n] = \begin{cases} +1, & n\ is\ even \\ -1, & n\ is\ odd \end{cases}$

6. Consider a continuous-time LTI system with impulse response given by

$$h(t) = \delta(t-1) + \delta(t-2)$$

where δ is the Dirac delta function.

(a) Find a simple equation relating the input $x(t)$ and output $y(t)$ of this system
(b) Find the frequency response of this system

7. Suppose the following difference equation relates the input $x[t]$ and output $y[t]$ of a discrete-time, causal LTI system S,

$$y[n] + \alpha y[n-1] = x[n] + x[n-1]$$

for some constant α.

(a) Find the impulse response $h[n]$
(b) Find the frequency response $H(e^{j\omega})$
(c) Find a sinusoidal input with non-zero amplitude such that the output is zero

8. Each of the statements below refers to a discrete-time system S with input $x[n]$ and output $y[n]$. Determine whether the statement is true or false.

(a) Suppose you know that if $x[n]$ is a sinusoid then $y[n]$ is a sinusoid. Then you can conclude that S is LTI.

(b) Suppose you know that S is LT, and that if $x[n] = cos(\pi n/2)$, then $y[n] = 2cos(\pi n/2)$. Then you have enough information to determine the frequency response.

(c) Suppose you know that S is LTI, and that if $x[n] = \delta[n]$, then $y[n] = (0.9)^n u[n]$ then you have enough information to determine the frequency response.

(d) Suppose you know that S is causal, and that input $x[n] = \delta[n]$ produces output $y[n] = \delta[n] + \delta[n-1]$, and input $x'[n] = \delta[n-2]$ produces output $y'[n] = 2\delta[n-2] + \delta[n-3]$. Then you can conclude that S is not LTI.

(e) Suppose you know that S is causal, and that if $x[n] = \delta[n] + \delta[n-2]$ then $y[n] = \delta[n] + \delta[n-1] + 2\delta[n-2] + \delta[n-3]$. Then you can conclude that S is not LTI.

9. Consider an LTI discrete-time system $Filter$ with impulse response

$$h[n] = \delta[n] + \delta[n-2]$$

where δ is the Kronecker delta function.

(a) Sketch $h[n]$

(b) Find the output when the input is the unit step function $u[n]$

(c) Find the output when the input is a ramp

$$r[n] = \begin{cases} n, & for\ n \geq 0 \\ 0, & otherwise \end{cases}$$

(d) Suppose the input signal $x[n]$ is such that

$$x[n] = \cos(\omega n)$$

where $\omega = \pi/2$. Give a simple expression for $y[n] = Filter[x]$.

(e) What is the frequency response $H(e^{j\omega})$ of the system $Filter$?

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$$x[n] = 1 + \cos(4\pi n/9)$$

(a) $N_0 = ?$

By definition $\omega = \frac{2\pi}{N}$

$$\omega = \frac{4}{9}\pi = \frac{2\pi}{N}$$

$$N = \frac{9}{4\cancel{\pi}} \cdot 2\cancel{\pi} = \frac{9}{2}$$

Needs to be an integer but if we wait for 2 periods

$$2\left(\frac{9}{2}\right) = 9$$

$$\therefore N_0 = 9$$

(b) ω_0, f_0

$$\omega_0 = \frac{2}{9}\pi \text{ rad/s}$$

$$\omega_0 = 2\pi f_0 \longrightarrow f_0 = 1/9 \text{ Hz}$$

(c) $C_k = ?$

Remember

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$k=1$

$$x[n] = \underbrace{1}_{C_0} + \underbrace{\frac{1}{2}}_{C_1}\left(e^{j\overbrace{\tfrac{4}{9}\pi}^{\omega} n} + e^{-j\overbrace{\tfrac{4}{9}\pi}^{\omega} n}\right)$$

$$C_0 = 1, \quad C_1 = 1/2$$

②

$$x(t) = 1 + \cos(\pi t) + \cos(2\pi t)$$

$$H(j\omega) = \begin{cases} e^{j\omega}, & |\omega| < 4 \text{ rad/s} \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = ?$$

- Find Fourier coefficients for $x(t)$

$$x(t) = 1 + \frac{1}{2}(e^{j\pi t} + e^{-j\pi t}) + \frac{1}{2}(e^{2j\pi t} + e^{-2j\pi t})$$

- Multiply by frequency response $H(j\omega)$ to find $y(t)$

$$y(t) = H(0)\,1 + \frac{1}{2}H(\pi)(e^{j\pi t} + e^{-j\pi t}) + \frac{1}{2}H(2\pi)(e^{2j\pi t} + e^{-2j\pi t})$$

$$H(0) = e^{j0} = 1$$

Euler's identity

$$e^{j\pi} + 1 = 0$$

$$H(\pi) = e^{j\pi} = -1$$

$H(2\pi) = 0$, since $|2\pi| > 4$ rad/s

$$\therefore\ y(t) = 1 - \frac{1}{2}(e^{j\pi t} + e^{-j\pi t}) = \underline{1 - \cos(\pi t)}$$

(3) - $x(t)$, signal with period T, $\omega_0 = 2\pi/T$

- Fourier coefficients $C_0, C_1, C_2, \cdots, C_K$

(a) $a\,x(t)$

- Linearity property

$$C_K' = a\,C_K$$

(b) $x(t-t_0)$

- Time shifting property

$$C_K' = e^{-jK\omega t_0} C_K$$

(c) Frequency response

$$H(j\omega) = \begin{cases} 1, & \omega = 0 \\ 0, & \text{otherwise} \end{cases}$$

- This system preserves only the DC component of the input signal, eliminating anything else

$$\therefore C_0' = C_0, \quad C_K = 0, \text{ for } K \neq 0$$

(d) - Linearity property

$$C_K' = C_K + C_K''$$

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$x(t)$, with $\omega_0 = 1$ rad/s, a periodic signal

$$C_K = \begin{cases} 1, & K = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find output $y(t)$ if frequency response is

$$H(j\omega) = \cos(\pi\omega/2)$$

- We can reconstruct $x(t)$ from its C_K and complex exponentials

$$x(t) = 1 + \underbrace{(e^{jt} + e^{-jt}) + (e^{2jt} + e^{-2jt})}_{\text{grouped by k value}}$$

- Since system is LTI, output is given by

$$y(t) = H(0) + H(1)e^{jt} + H(-1)e^{-jt} + H(2)e^{2jt} + H(-2)e^{-2jt}$$

- Evaluating $H(j\omega) = \cos(\pi\omega/2)$

$$H(0) = 1,\ H(1) = H(-1) = 0,\ H(2) = H(-2) = -1$$

$$\therefore y(t) = 1 - (e^{2jt} + e^{-2jt}) = 1 - 2\cos(2t)$$

(b) The fundamental frequency for $y(t)$ is

$$\omega_0' = 2 \text{ rad/s}$$

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Frequency response

$$H(e^{j\omega}) = |\omega|$$

- LTI system
- In discrete time.

(a) $x[n] = \cos(\pi n/2)$

- Convert to complex exponentials

$$x[n] = \frac{1}{2}\left(e^{j\pi n/2} + e^{-j\pi n/2}\right)$$

- Multiply by $H(e^{j\omega})$ to obtain $y[n]$

$$y[n] = \frac{1}{2} H(\pi/2)\, e^{j\pi n/2} + \frac{1}{2} H(-\pi/2)\, e^{-j\pi n/2}$$

$$H(\pi/2) = H(-\pi/2) = \pi/2$$

$$y[n] = \frac{\pi}{2} \cdot \frac{1}{2}\left(e^{j\pi n/2} + e^{-j\pi n/2}\right)$$

$$\therefore y[n] = \frac{\pi}{2} \cos(\pi n/2)$$

(b) $x[n] = 5 \longrightarrow x[n] = 5e^{j0n}$

$$\therefore y[n] = 5H(0)e^{j0n} = 0$$

(c) - Using Euler's identity

Euler's identity $e^{j\pi} = -1$

$$x[n] = \begin{cases} +1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases} = e^{j\pi n}$$

$$y[n] = H(\pi)e^{j\pi n} = \pi e^{j\pi n}$$

$$\therefore y[n] = \begin{cases} \pi, & \text{if } n \text{ is even} \\ -\pi, & \text{if } n \text{ is odd} \end{cases}$$

⑥ Impulse response

$$h(t) = \delta(t-1) + \delta(t-2)$$

- CT and LTI

(a) Relate output $y(t)$ and input $x(t)$

- Use convolution

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(\delta(\tau-1) + \delta(\tau-2)\right) x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau-1)\, x(t-\tau)\, d\tau + \int_{-\infty}^{\infty} \delta(\tau-2)\, x(t-\tau)\, d\tau$$

- Using the sifting property

$$\underline{y(t) = x(t-1) + x(t-2)}$$

(b) Find $H(j\omega)$, the frequency response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\delta(t-1) + \delta(t-2)\right) e^{-j\omega t}\, dt \leftarrow \text{Use sifting property again}$$

$$\underline{H(j\omega) = e^{-j\omega} + e^{-2j\omega}}$$

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Consider the system

$$y[n] + \alpha\, y[n-1] = x[n] + x[n-1]$$

- It is DT, LTI, and causal

- Reorganize to

$$y[n] = x[n] + x[n-1] - \alpha\, y[n-1]$$

(a) Find $h[n]$, the impulse response

- Since it is causal, $h(n) = 0$ for $n < 0$
- Assuming the input is an impulse, it follows that

$$h[n] = \delta[n] + \delta[n-1] - \alpha\, h[n-1]$$

- Evaluating $h[n]$

$$
\begin{array}{lcccccc}
 & & \delta[n] & + & \delta[n-1] & - & \alpha h[n-1] \\
h[0] & = & 1 & + & 0 & + & 0 \\
h[1] & = & 0 & + & 1 & - & \alpha \\
h[2] & = & 0 & + & 0 & & -\alpha(1-\alpha) \\
h[3] & = & 0 & + & 0 & & +\alpha^2(1-\alpha) \\
h[4] & = & 0 & + & 0 & & -\alpha^3(1-\alpha) \\
\cdots & & & & & & \\
h[n] & = & 0 & + & 0 & & +(-\alpha)^{n-1} + (-\alpha)^n
\end{array}
$$

$$\therefore\ \underline{h[n] = (-\alpha)^{n-1} u[n-1] + (-\alpha)^n u[n]}$$

(b) Find $H(e^{j\omega})$

- Assume the input to be a complex exponential

$$x[n] = e^{j\omega n}$$

then

$$y[n] = H(e^{j\omega})\, e^{j\omega n}$$

it follows that

$$H(e^{j\omega})\, e^{j\omega n} + \alpha H(e^{j\omega})\, e^{j\omega(n-1)} = e^{j\omega n} + e^{j\omega(n-1)}$$

$$H(e^{j\omega})\left(1+\alpha e^{-j\omega}\right) = 1 + e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1+\alpha e^{-j\omega}}$$

(c) $y[n] = 0$ if $H(e^{j\omega}) = 0$

- for this to be the case,

$$e^{-j\omega} = -1$$

which happens if $\omega = \pi$ (Euler's identity)

$\therefore$ $x[n] = \cos(\pi n)$ will make $y[n]$ have zero amplitude.

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(a) False. The output frequency may not be the same as the input frequency.

(b) False. You only know the response to one frequency.

(c) True. The frequency response is the discrete time Fourier transform of the impulse response.

(d) True. If the system were LTI, the response to the delayed impulse would be the delayed impulse response.

(e) False. The system might be LTI with impulse response given by

$$h[n] = \delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6] + \cdots.$$

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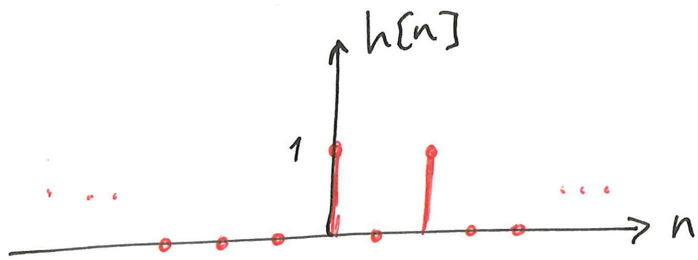
LTI, DT Filter with impulse response

$$h[n] = \delta[n] + \delta[n-2]$$

(a)



(b) Input $x[n] = u[n]$

- Use convolution and sifting property

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$\Rightarrow y[n] = x[n] + x[n-2]$$

- when $x[n] = u[n]$,

$$y[n] = u[n] + u[n-2] = \begin{cases} 0, & \text{if } n < 0 \\ 1, & \text{if } 0 \le n \le 1 \\ 2, & \text{if } n \ge 2 \end{cases}$$

(c) Input $x[n] = r[n]$, the ramp function

- Using the result of the convolution sum in (b), for $x[n] = r[n]$

$$y[n] = r[n] + r[n-2] = \begin{cases} 0, & \text{if } n < 0 \\ n, & \text{if } 0 \le n \le 1 \\ 2n-2, & \text{if } n \ge 2 \end{cases}$$

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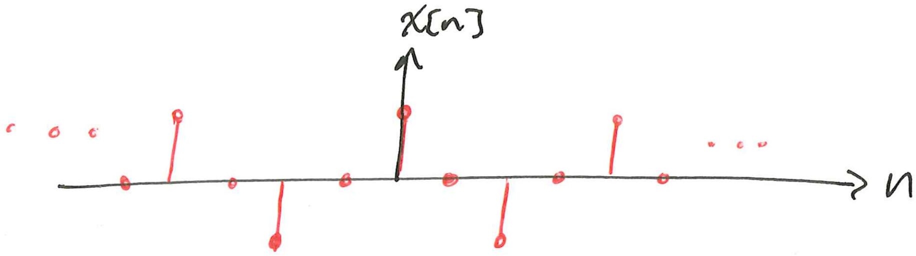
(d) Input $x[n] = \cos(\omega n)$, for $\omega = \pi/2$

- Using the convolution result from (b)

$$y[n] = x[n] + x[n-2]$$

and by plotting $x[n]$



by inspection, it is clear that, since the output is the sum of two samples separated by one, the result will always be zero.

$$\therefore y[n] = 0$$

(e) Frequency response $H(e^{j\omega})$

- Assume input is a complex exponential

$$x[n] = e^{j\omega n}$$

$$y[n] = H(e^{j\omega})\, e^{j\omega n}$$

and using the convolution result from (b)

$$H(e^{j\omega})\, e^{j\omega n} = e^{j\omega n} + e^{j\omega(n-2)}$$

$$H(e^{j\omega}) = 1 + e^{-2j\omega}$$