1 Discrete Time Step Response

We can calculate s[n] directly from h[n]

$$s[n] = \sum_{k=-\infty}^{\infty} u[n-k]\,h[k] = \sum_{k=-\infty}^{n} h[k]$$

We can obtain h[n] back from s[n]

$$s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k]$$

$$= h[n]$$

Just as the h[n] completely describes the input-output characteristics of a system, it follows that s[n] does the same

Ex.

$$h[n] = 0.6^n u[n]$$

$$s[n] = \sum_{k=-\infty}^{n} 0.6^k u[n] = \sum_{k=0}^{n} 0.6^k$$

$$\rightarrow \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

Using a geometric series

$$\therefore s[n] = \frac{1-0.6^{n+1}}{1-0.6}$$

System is causal, so don't forget to add u[n]

$$\boxed{s[n] = 2.5(1-0.6^{n+1})\,u[n]}$$

Getting h[n] from s[n]

$$h[n] = s[n] - s[n-1]$$

$$= 2.5(1-0.6^{n+1})u[n]$$

$$-2.5(1-0.6^{n})u[n-1]$$

$\rightarrow$ for $n=0$

$$h[0] = 2.5[1-0.6] = \underline{1}$$

$\rightarrow$ for $n \geq 1$

$$h[n] = 2.5(1-0.6^{n+1}-1+0.6^{n})$$

$$= 2.5(-0.6^{n+1}+0.6^{n})$$

$$= 2.5(1-0.6)0.6^{n}$$

$$\therefore \underline{h[n] = 0.6^{n}u[n]} \,//$$

Ex. 2

DT, LTI system

$$h[n] = \delta[n] + 2\delta[n-1]$$

$$H(e^{j\omega}) = ?$$

DTFT

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} \left(\delta[k] + 2\delta[k-1]\right) e^{-j\omega k}$$

$$\underline{H(e^{j\omega}) = 1 + 2e^{-j\omega}} \; //$$

$$\cos^2\omega + \sin^2\omega = 1$$

$$e^{j\omega} = \cos\omega + j\sin\omega$$

$$|a + jb| = \sqrt{a^2 + b^2}$$

$$|H(e^{j\omega})| = |1 + 2e^{-j\omega}|$$

$$|H(e^{j\omega})| = |1 + 2\cos\omega - 2j\sin\omega|$$

$$|H(e^{j\omega})| = \sqrt{(1 + 2\cos\omega)^2 + (2\sin\omega)^2}$$

$$\underline{|H(e^{j\omega})| = \sqrt{5 + 4\cos\omega}} \; //$$

$\angle H(e^{j\omega})$ $\qquad\qquad$ $\angle(a+jb) = \tan^{-1}(b/a)$

$\angle H(e^{j\omega}) = \angle(1+2\cos\omega - 2j\sin\omega)$

$\angle H(e^{j\omega}) = \tan^{-1}\left[-2\sin\omega / (1+2\cos\omega)\right]$

$\angle H(e^{j\omega}) = -\tan^{-1}\left[2\sin\omega / (1+2\cos\omega)\right]$

$x[n] = \cos(\pi n/2 + \pi/6) + \sin(\pi n + \pi/3)$

$\omega = \pi/2$ $\qquad\qquad$ $\omega = \pi$

$\rightarrow$ for $\omega = \pi/2$

$|H(\pi/2)| = \sqrt{5+4\cos(\pi/2)} = \sqrt{5}$

$\angle H(\pi/2) = -\tan^{-1}\underbrace{\left[\frac{2\sin(\pi/2)}{1+2\cos\pi/2}\right]}_{2}$

$\angle H(\pi/2) \approx -1.107$

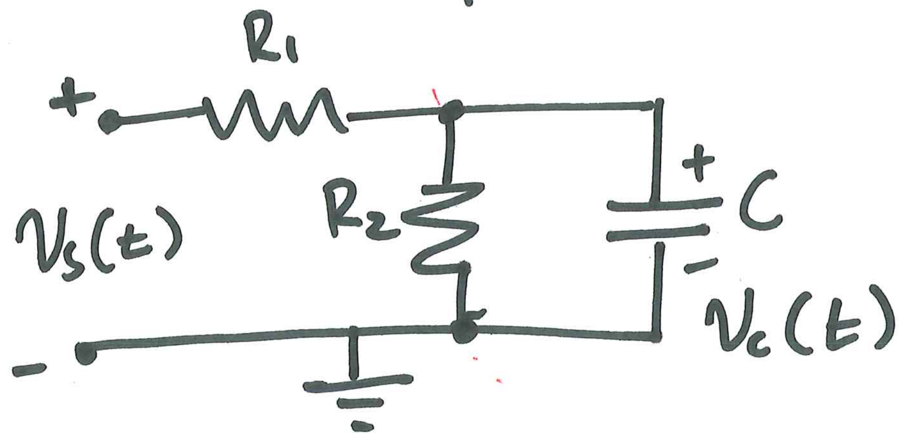
$\rightarrow$ for $\omega = \pi$

$|H(\pi)| = \sqrt{5+4\cos\pi} = \underline{1}$

$\angle H(\pi) = -\tan^{-1}\left[\frac{2\sin\pi}{1+2\cos\pi}\right] = \underline{\emptyset}$

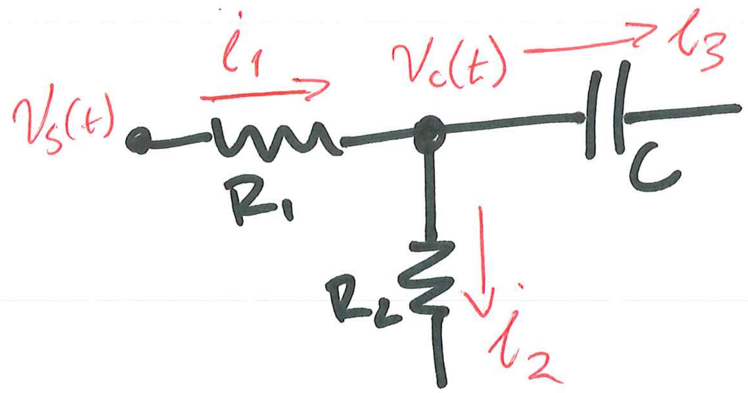
$$y[n] = \sqrt{5}\,\cos(\pi n/2 + \pi/6 - 1.107)$$
$$+ \sin(\pi \text{[illegible]} + \pi/3)$$

6 Circuit Example 1



- Input is voltage source $v_s(t)$
- Output is voltage capacitor $v_c(t)$

$\rightarrow H(j\omega)$, $h(t)$, $s(t)$



$x(t) = v_s(t)$

$y(t) = v_c(t)$

$$i_1 = i_2 + i_3$$

$$\frac{v_s(t) - v_c(t)}{R_1} = \frac{v_c(t)}{R_2} + C\frac{dv_c(t)}{dt}$$

$$\frac{v_s(t)}{R_1} = v_c(t)\underbrace{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}_{\rho} + C\frac{dv_c(t)}{dt}$$

Using ρ to represent that term

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k} = \frac{V_c(j\omega)}{V_s(j\omega)}$$

$$H(j\omega) = \frac{\frac{1}{R_1}}{\rho + Cj\omega} = \frac{1}{R_1 \rho + R_1 C j\omega}$$

This is another way of using the differentiation rule above

$$Y(j\omega) \sum_{k=0}^{N} \alpha_k (j\omega)^k = X(j\omega) \sum_{k=0}^{M} \beta_k (j\omega)^k$$

$$\frac{V_s(j\omega)}{R_1} = V_c(j\omega)\rho + Cj\omega V_c(j\omega)$$

$$H(j\omega) = \frac{V_c(j\omega)}{V_s(j\omega)} = \frac{1}{R_1 \rho + R_1 C j\omega}$$

$$H(j\omega) = \frac{1}{\rho/c + j\omega} \cdot \frac{1}{R_1 C}$$

For convenience when we get transform back to time domain, it is better to keep the term jω by itself

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

Time Domain | Frequency Domain

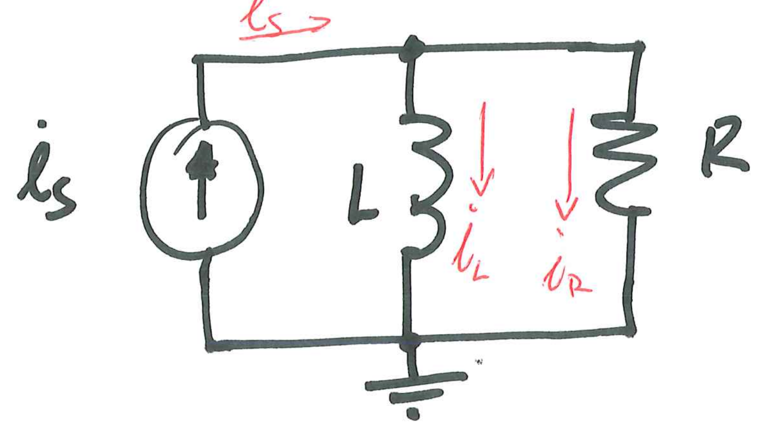
$$\therefore h[t] = \frac{1}{R_1 C} e^{-\rho t/c} u(t)$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t} \frac{1}{R_1 C} e^{-\rho\tau/c} u(\tau)\,d\tau$$

$$= \frac{1}{R_1 C}\int_0^t e^{-\rho\tau/c}\,d\tau = \frac{1}{-R_1\rho} e^{-\rho\tau/c}\Big|_0^t$$

$$s(t) = \frac{1}{R_1\rho}\left(1 - e^{-\rho t/c}\right)$$

9 Circuit example #2



$x(t) = i_s(t)$

$y(t) = i_L(t)$

$H(j\omega)$, $h(t)$, $s(t)$

$$V_L(t) = L \frac{d i_L(t)}{dt}$$

$$i_R(t) = \frac{V_L(t)}{R} = \frac{L}{R} \frac{d i_L(t)}{dt}$$

$$i_s(s) = i_L(t) + i_R(t)$$

$$H(j\omega) = ?$$

$$I_s(j\omega) = I_L(j\omega) + \frac{L}{R} j\omega I_L(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{I_L(j\omega)}{I_s(j\omega)} = \frac{1}{1 + \frac{L}{R} j\omega}$$

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$$H(j\omega) = \frac{R}{L} \cdot \frac{1}{R/L + j\omega}$$

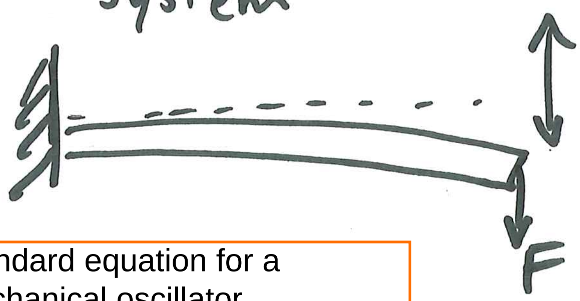
$$h(t) = \frac{R}{L} e^{-Rt/L} u(t)$$

$$s(t) = \frac{R}{L} \int_0^t e^{-R\tau/L} d\tau = \frac{R/L}{-R/L} e^{-R\tau/L} \Big|_0^t$$

$$s(t) = \left(-e^{-Rt/L} + 1\right) u(t)$$

11 Second order System

Second order systems that represent a mechanical system can also be of interest from a signals and systems perspective



Standard equation for a mechanical oscillator

$$F = kx$$

Hooke's law for springs

$$m\ddot{y} + \lambda \dot{y} + ky = F$$

We want that equation to resemble this one

$$\rightarrow \ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 X$$

$$\ddot{y} + \frac{\lambda}{m}\dot{y} + \frac{k}{m}y = \frac{kX}{m}$$

Start by substituting Hooke's law and making sure that y'' is by itself

$$\omega_n^2 = \frac{k}{m} \rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

$$2\zeta\omega_n$$

Notice that now y and x are being multiplied by the same coeff.

$$\ddot{y} + \frac{\lambda}{m}\dot{y} + \omega_n^2 y = \omega_n^2 X$$

$$m\sqrt{\frac{k}{m}} = \sqrt{\frac{km^2}{m}}$$

λ is the mechanical dampening coefficient of the system

$$\frac{\lambda}{m} = 2\zeta\omega_n$$

But we are interested in the damping ratio ζ

$$\zeta = \frac{\lambda}{2m\omega_n} = \frac{\lambda}{2\sqrt{mk}}$$

$$(j\omega)^2 Y(j\omega) + \frac{\lambda}{m} j\omega\, Y(j\omega) + \omega_n^2 Y(j\omega)$$

$$= X(j\omega)\, \omega_n^2$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + \frac{\lambda}{m} j\omega + \omega_n^2}$$