ELEC 221 Lecture 22
The Laplace transform

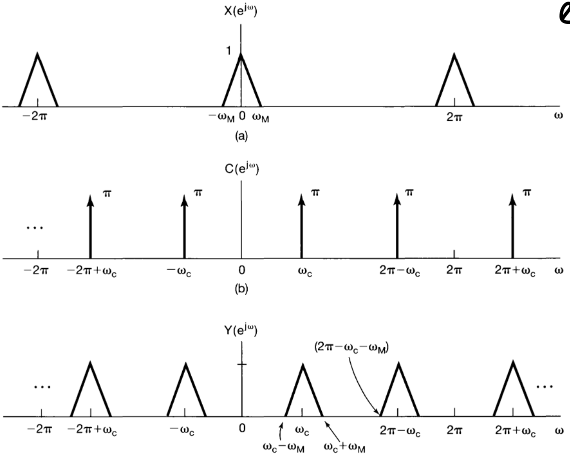
Thursday 24 November 2022

- Midterm 2 available for pickup
- Quiz 10 Tuesday (last quiz) – today's lecture
- Assignment 6 (computational) due on Tuesday at 23:59
- Final assignment (pen and paper) released early next week

Last time

We did DT sinusoidal amplitude modulation.

★ know for exam

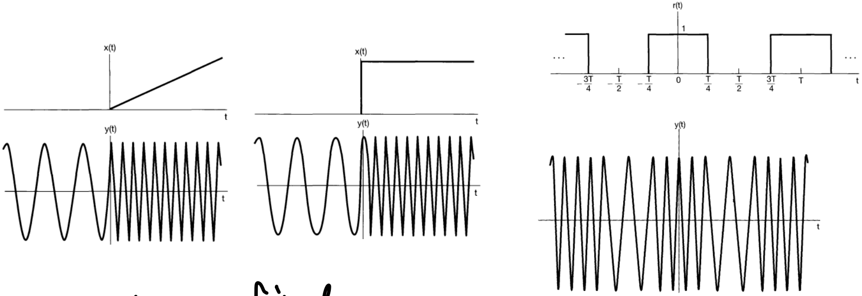


We can still do frequency-division multiplexing, but we may need to upsample and shrink the spectra so things fit.

Image credit: Oppenheim 8.8

Last time

We briefly explored how frequency modulation works

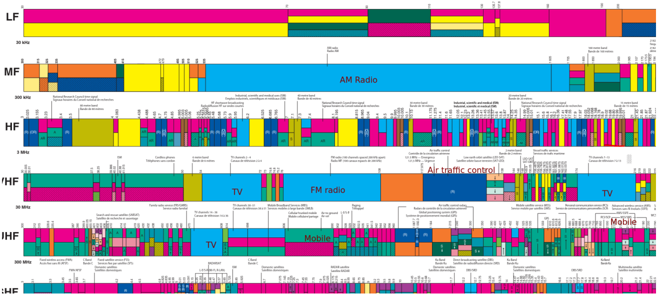


not on final

Image credit: Oppenheim 8.7

Last time

We discussed how cell phones are radios and how the radio spectrum gets divided (and auctioned off).



not on exam

See the full graphic here: `https://www.ic.gc.ca/eic/site/smt-gst.nsf/vwapj/2018_Canadian_Radio_Spectrum_Chart.PDF/$FILE/2018_Canadian_Radio_Spectrum_Chart.PDF`

The course so far

① Systems
- Properties (LTI, memory, causality, stability)

② Signals
- Unit impulse, step
- $x[n] = \sum x[k]\delta[n-k]$
- $x(t) = \int x(\tau)\delta(t-\tau)d\tau$

③ Convolution
- Impulse response $h(t)/h[n]$
- Convolution sum/integral
- $y[n] = \sum x[k]h[n-k]$
- $y(t) = \int x(\tau)h(t-\tau)d\tau$

④ Complex exponential signals
- $x(t) = e^{j\omega t}$, $x[n] = e^{j\omega n}$
- Periodicity and Fourier series representation
- $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$
- $x[n] = \sum_{k=\langle N\rangle} c_k e^{jk\frac{2\pi}{N}n}$

⑤ System functions and frequency response
- Complex exp. signals are eigenfunctions of LTI systems
- $x(t) = \sum c_k e^{jk\omega t} \rightarrow y(t) = \sum c_k H(jk\omega)e^{jk\omega t}$
- $x[n] = \sum c_k e^{jk\frac{2\pi}{N}n} \rightarrow y[n] = \sum c_k H(e^{j\omega})e^{jk\frac{2\pi}{N}n}$

⑥ CT Fourier transform
- handles aperiodic signals
- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ (inv. FT)
- $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ (F. spectrum)
- relationship between $h(t) \overset{F}{\leftrightarrow} H(j\omega)$
- convolution property: $Y(j\omega) = H(j\omega)X(j\omega)$

⑦ DT Fourier transform
- handles aperiodic signals
- $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
- $X(e^{j\omega})$ is a 2π-periodic continuous function

⑧ DFT/FFT
- discrete FT is equally-spaced samples of DTFT
- FFT is efficient: very important algorithm!!

⑨ Time/freq. analysis
- magnitude-phase representation
- group delay
- Bode plots
- Systems described by DEs: step response and oscillation

⑩ Sampling
- impulse-train sampling
- Nyquist rate and aliasing
- upsampling/downsampling

⑪ Modulation
- amplitude modulation
- frequency modulation
- multiplexing

⑫ Laplace transform

⑬ Z-transform

Wayyyyy back in lecture 3:

LTI systems and complex exponential functions

Write $x(t) = e^{st}$. Then:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{st}e^{-s\tau}h(\tau)d\tau \\ &= e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau \\ &= e^{st}H(s) \end{aligned}$$

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The course so far

Wayyyyy back in lecture 3:

LTI systems and complex exponential functions

To summarize:

$$x(t) = e^{st} \quad \rightarrow \quad y(t) = H(s) \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

Wayyyyy back in lecture 3:

...so what?

If all the $x_i(t)$ are complex exponential functions and

$$x(t) = e^{st} \quad \rightarrow \quad y(t) = H(s) \cdot e^{st}$$

then

$$x(t) = \sum_k c_k e^{s_k t} \quad \rightarrow \quad y(t) = \sum_k c_k H(s_k) e^{s_k t}$$

The response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

How can we express arbitrary signals as superpositions of complex exponentials?

Wayyyyy back in lecture 3:

The Fourier series

Let's consider a special set of signals[1]:

$$x(t) = e^{st} = e^{j\omega t}$$

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

[1] We will see the general case at the end of the course.

Today

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

Consider a general complex exponential signal,

$$x(t) = e^{st}$$

Input into LTI system with impulse response $h(t)$:

$$y(t) = h(t) * x(t) = H(s)\, x(t)$$

If $s = j\omega$: **Fourier transform**

$$H(jw) = \int_{-\infty}^{\infty} e^{-j\omega t} h(t)\, dt$$

If $s = \sigma + j\omega$: (bilateral) **Laplace transform**

$$H(s) = \int_{-\infty}^{\infty} e^{-st} h(t)\, dt$$

More generally, the Laplace transform of an arbitrary signal is

$$x(t) \to \quad X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt$$

We will write

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

We can relate the Laplace and Fourier transforms. Consider

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\,dt$$

for $s = \sigma + j\omega$.

$$X(s) = \int_{-\infty}^{\infty} e^{-(\sigma + j\omega)t} x(t)\,dt$$

$$= \int_{-\infty}^{\infty} \left[e^{-\sigma t} x(t)\right] e^{-j\omega t}\,dt$$

$$= \mathcal{F}\left(e^{-\sigma t} x(t)\right)$$

The Laplace transform

Example: consider the signal $x(t) = e^{-at}u(t)$.

What is $X(j\omega)$?

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt$$
$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$
$$= \frac{1}{j\omega + a}$$

Recall: conditions on a? $\mathrm{Re}(a) > 0$

What is the Laplace transform?

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$= \int_0^{\infty} e^{-at} e^{-(\sigma + j\omega)t} dt$$

$$= \int_0^{\infty} e^{-(a+\sigma)t} e^{-j\omega t} dt$$

$$= \frac{1}{(\sigma + a) + j\omega}$$

$$= \frac{1}{s+a}$$

What are the conditions here?

$$\sigma + a > 0$$

$$\mathrm{Re}(s) > -a$$

Take a closer look at $x(t) = e^{-at}u(t)$:

$$F(x(t)) = \frac{1}{a+j\omega} \qquad a>0$$

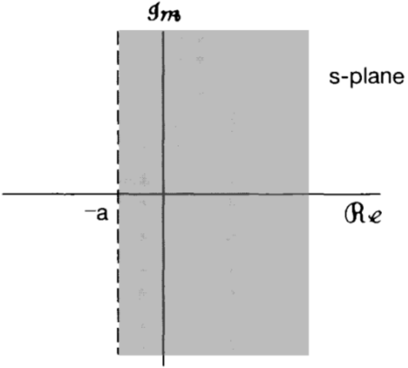
$$\mathcal{L}(x(t)) = \frac{1}{a+s} \qquad \mathrm{Re}(s) > -a$$

$$\mathcal{F}(x(t)) = X(j\omega) = \frac{1}{a+j\omega}, \quad a>0$$

$$\mathcal{L}(x(t)) = X(s) = \frac{1}{a+s}, \quad \mathrm{Re}(s) > -a$$

The Laplace transform exists in some regions where the Fourier transform does not!

We must specify for which s the Laplace transform is valid.



This is called the **region of convergence** (ROC).

Image credit: Oppenheim 9.1

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt$$

Exercise: what is the Laplace transform of

$$x(t) = -e^{-at} u(-t)$$

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$$x(t) = -e^{-at}u(-t)$$

Solution:

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt$$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}\,dt$$

$$= -\int_{-\infty}^{0} e^{-(a+s)t}\,dt$$

$$= \frac{1}{a+s}$$

Conditions are different though: need $\text{Re}(s) < -a$.

Multiple signals can have the same algebraic Laplace transform, but different ROCs.

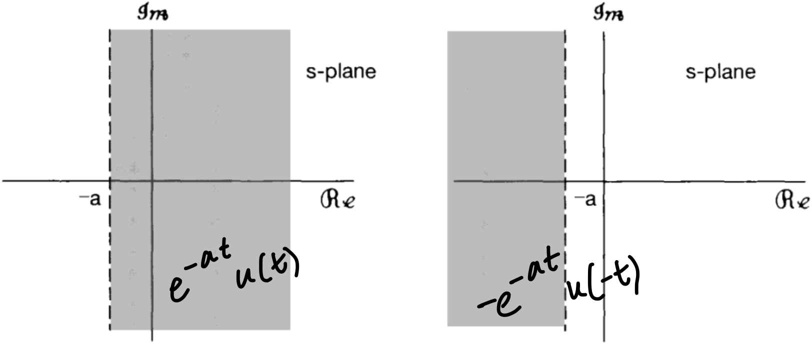


Image credit: Oppenheim 9.1

Exercise: what is the Laplace transform of

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

and what is its region of convergence?

Hint: the Laplace transform is also linear!

The Laplace transform

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

Solution: by linearity,

$$X(s) = 3\mathcal{L}(e^{-2t}u(t)) - 2\mathcal{L}(e^{-t}u(t))$$

$$= 3 \cdot \frac{1}{s+2} - 2 \cdot \frac{1}{s+1}$$

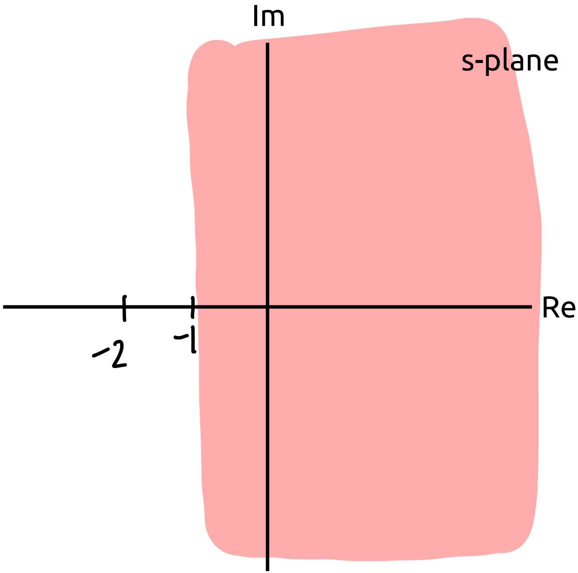
$$\mathrm{Re}(s) > -1$$

$$\begin{aligned} X(s) &= 3\mathcal{L}(e^{-2t}u(t)) - 2\mathcal{L}(e^{-t}u(t)) \\ &= \frac{3}{s+2} - \frac{2}{s+1} \\ &= \frac{s-1}{s^2+3s+2} \end{aligned}$$

To determine its ROC,

- first term tells us $\mathrm{Re}(s) > -1$
- second term tells us that $\mathrm{Re}(s) > -2$

Let's draw the ROC:



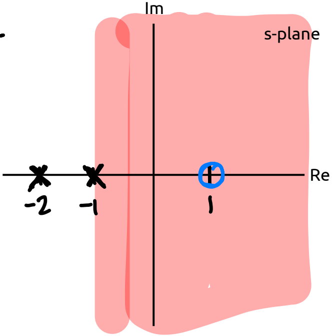
Pole-zero plots

Quite often the Laplace transforms will be rational polynomials of s. Generally the roots of these polynomials are indicated on the plots. Use $\times$ for denominator (poles), $\circ$ for numerator (zeros):

$$3 \cdot \frac{1}{s+2} - 2 \cdot \frac{1}{s+1}$$

$$= \frac{3s+3-2s-4}{(s+2)(s+1)}$$

$$= \frac{s-1}{s^2+3s+2}$$



This is called a pole-zero plot. (May also have poles/zeros at infinity if degree of polynomials is different)

Pole-zero plots

Exercise: compute the Laplace transform of

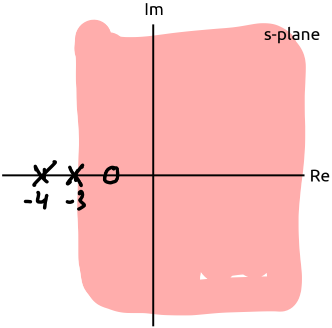
$$x(t) = -2e^{-3t}u(t) + 4e^{-4t}u(t)$$

and draw its pole-zero plot.

Solution:

$$X(s) = -2\,\frac{1}{s+3} + 4\,\frac{1}{s+4}$$

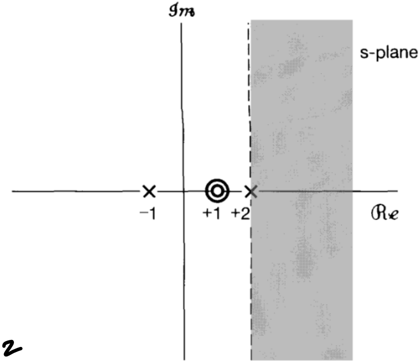
$$= \frac{2(s+2)}{(s+3)(s+4)}$$



Regions of convergence

The ROC has many nice properties:

- if ROC doesn't contain $j\omega$ axis, FT does not converge
- ROC is strips parallel to $j\omega$ axis
- ROC of rational Laplace transform contains no poles



$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)} \quad \mathrm{Re}(s) > 2$$

Image credit: Oppenheim 9.1

Regions of convergence

If $x(t)$ has finite duration and is absolutely integrable, the ROC is the entire s-plane.

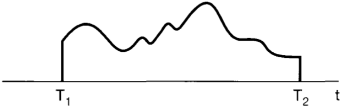
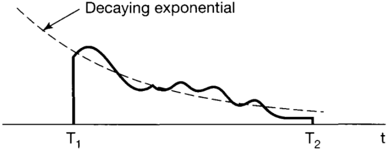


Figure 9.4 Finite-duration signal.



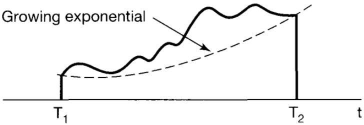
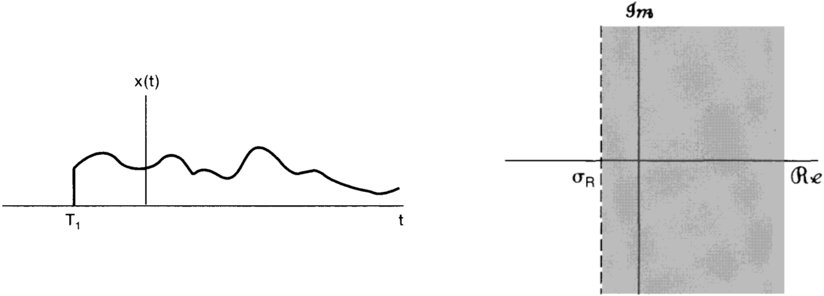


Image credit: Oppenheim 9.2

If $x(t)$ is right sided and $\mathrm{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\mathrm{Re}(s) > \sigma_0$ are also in the ROC.



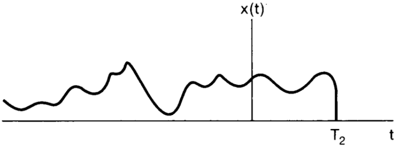
This ROC is called a **right-half plane**.

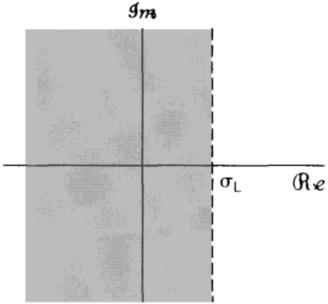
Intuition: if $\mathrm{Re}(s) = \sigma_1 > \sigma_0$ the exponential in $x(t)e^{-\sigma t}$ decays even faster and will still converge.

Image credit: Oppenheim 9.2

Left-sided signals

If $x(t)$ is left sided and $\text{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\text{Re}(s) < \sigma_0$ are also in the ROC.

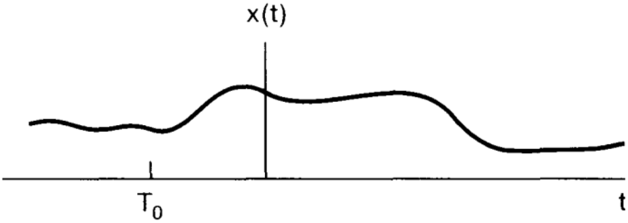




This ROC is called a **left-half plane**.

Image credit: Oppenheim 9.2

Any guesses?



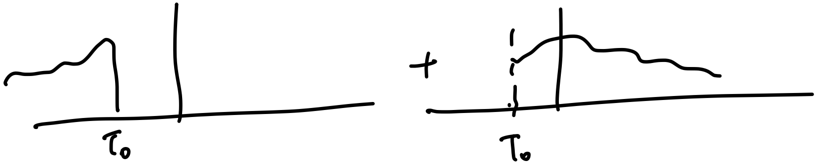
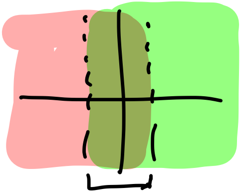
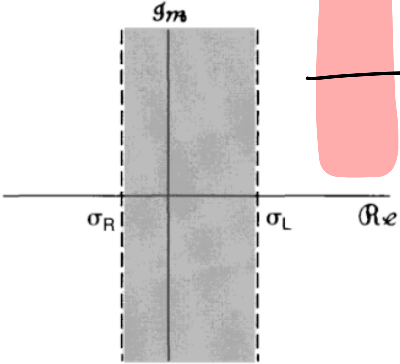
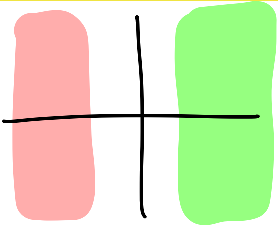


Image credit: Oppenheim 9.2

Two-sided signals



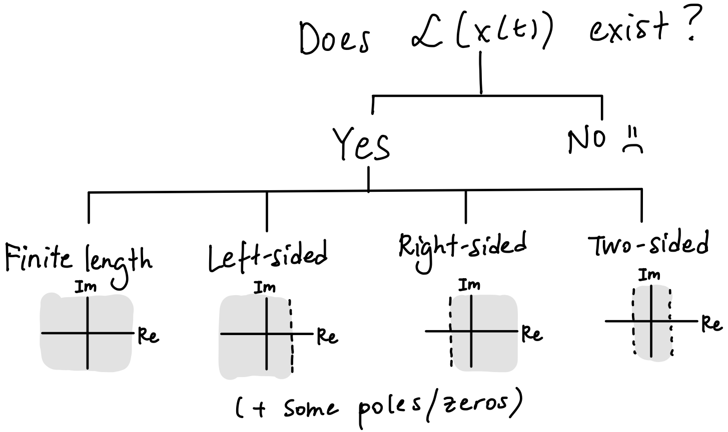




This is only the case if there was actually overlap - otherwise it doesn't exist!

Image credit: Oppenheim 9.2

For any signal $x(t)$...

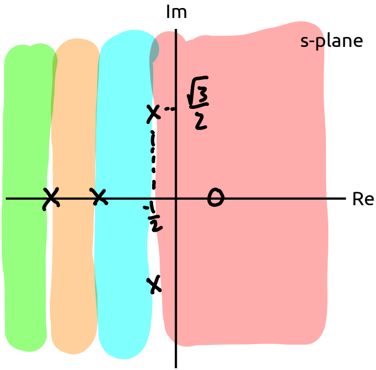


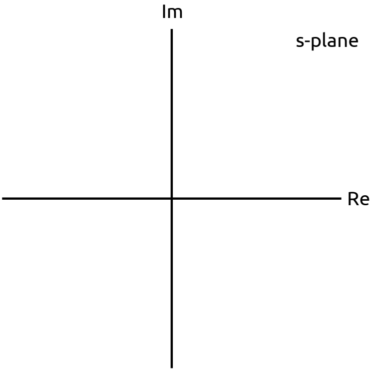
Regions of convergence

(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

$$\frac{s-1}{(s+2)(s+3)(s^2+s+1)}$$

in its ROC? (Hint: draw pole-zero plot and identify the regions)





$$X(s) = X(\sigma + j\omega) = \mathcal{F}(x(t)e^{-\sigma t}) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

From this, we can invert:

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}(X(\sigma + j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\underbrace{\sigma + j\omega}_{s})e^{\overbrace{(\sigma + j\omega)}^{s}t}\,d\omega$$

Make a change of variables $ds = jd\omega$:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

... we are not going to integrate this.

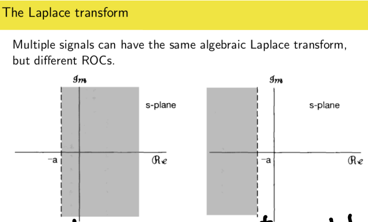
Inverse Laplace transforms

Suppose the Laplace transform has the form

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$$

where degree of denominator is higher than numerator.

To invert this, we can use our handy identities, BUT the region of convergence does matter.



$e^{-at}u(t)$ $\qquad$ $-e^{-at}u(-t)$

Image credit: Oppenheim 9.1

Inverse Laplace transforms

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

Inverse Laplace transforms

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

Start by using partial fractions:

$$X(s) = \frac{s+2}{(s+3)(s+4)}$$

$$= -1\frac{1}{s+3} + 2\cdot\frac{1}{s+4}$$

Taking inverse LT our options are:

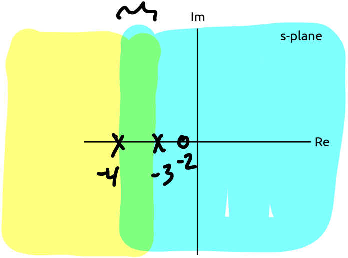
$$\frac{1}{s+3} \to e^{-3t}u(t) \quad \text{or} \quad -e^{-3t}u(-t)$$

$$\frac{1}{s+4} \to e^{-4t}u(t) \quad \text{or} \quad -e^{-4t}u(-t)$$

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

Draw the s-plane:



The signal must be two-sided, so:

$$X(s) = -1\frac{1}{s+3} + 2\cdot\frac{1}{s+4} \rightarrow x(t) = e^{-3t}u(-t) + 2e^{-4t}u(t)$$

Today

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

Oppenheim practice problems: 9.1-9.9, 9.21, 9.26

For next time

Content:

- properties and system analysis with Laplace transform

Action items:

1. Quiz 10 on Tuesday
2. Assignment 6 due Tuesday at 23:59

Recommended reading:

- From this class: Oppenheim 9.0-9.3. 9.5 (skip 9.4)
- For next class: 9.5-9.8 (skip 9.9)