ELEC 221 Lecture 22
The Laplace transform: properties and system analysis

Thursday 28 November 2024

Announcements

- Midterm 2 available for pickup at my office (some remaining MT1 as well)
- Quiz 10 Tuesday (last quiz)
- Final tutorial on Monday (problem solving - post suggestions on Piazza @226)
- Tutorial Assignment 5 due Monday 23:59
- Assignment 5 due Sunday 8 December 23:59 → 2 more Qs.
- Final exam details available late next week

Last time

We introduced the Laplace transform,

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\,dt$$

where $s = \sigma + j\omega$.

$$= \mathcal{F}\left[e^{-\sigma t} x(t) \right]$$

If $s = j\omega$, reduces to **Fourier transform**

$\sigma = 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\,dt$$

$X(s)$ can exist in regions that $X(j\omega)$ does not (allows us to analyze more kinds of systems), but still doesn't exist everywhere.

Last time

We introduced the s-plane and pole-zero plots. We used them to plot the region of convergence (ROC) of $X(s)$.

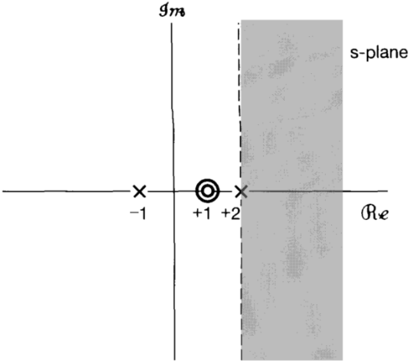
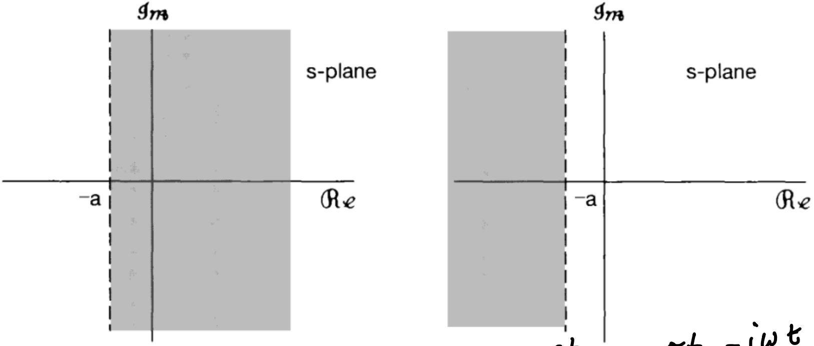


Image credit: Oppenheim 9.1

Last time

The ROC is essential for computing inverse Laplace transforms.



$$e^{-st} = e^{-\sigma t} e^{-j\omega t}$$

Both ROC associated to algebraic expression $X(s) = \frac{1}{s+a}$, but came from different signals.

Image credit: Oppenheim 9.1

We distinguished between types of signals and their ROCs.

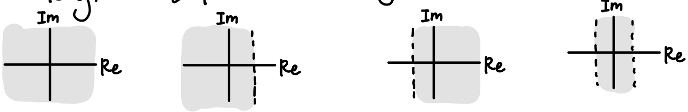
Does $\mathcal{L}(x(t))$ exist?

Yes

No :(

exists for all t

Finite length | Left-sided | Right-sided | Two-sided



(+ some poles/zeros)

Learning outcomes:

- apply key properties of the Laplace transform to its computation
- use the Laplace transform to determine whether a system is causal or stable
- compute the Laplace transform of systems described by constant-coefficient DEs

Regions of convergence

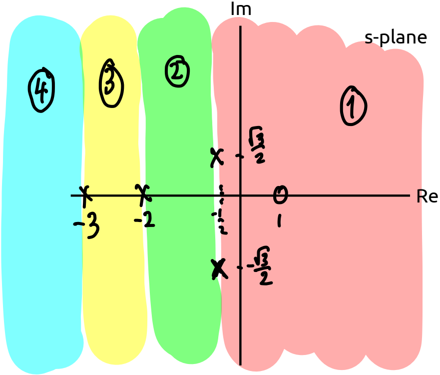
(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

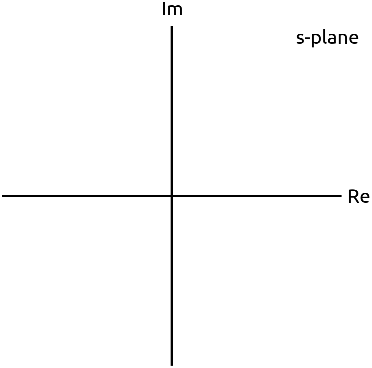
roots:

$$s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$

$$X(s) = \frac{s-1}{(s+2)(s+3)(s^2+s+1)}$$

Hint: draw pole-zero plot and identify possible ROCs.





$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} e^{-\sigma t} e^{-j\omega t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left[e^{-\sigma t} x(t) \right] e^{-j\omega t}\, dt$$

From this, we can invert:

$$\Rightarrow \quad x(t)\, e^{-\sigma t} = \mathcal{F}^{-1}\left[X(\sigma + j\omega) \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega) t}\, d\omega$$

Make a change of variables $ds = jd\omega$:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}\, ds$$

... we are not going to integrate this.

Inverse Laplace transforms

real coefficients

Suppose

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s+a_i}$$

$$\frac{1}{s+a} \leftarrow \begin{cases} e^{-at}u(t) \\ e^{-at}u(-t) \end{cases}$$

where degree of denominator is higher than numerator.
(for the expanded expression)

To invert, we can use our handy identities, BUT the ROC matters.

$$\frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \dots$$

$$\frac{s^2+3s}{s^5-3s^4+\dots}$$

degree is less

The Laplace transform

Multiple signals can have the same algebraic Laplace transform, but different ROCs.

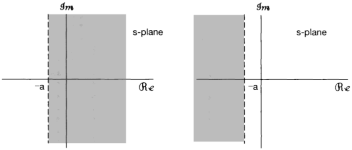
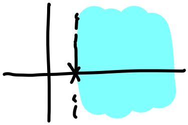


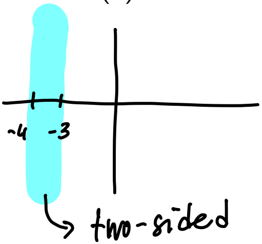
Image credit: Oppenheim 9.1

Exercise: what is the inverse Laplace transform of

right-sided signal



$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$



$$X(s) = \frac{-1}{s+3} + \frac{2}{s+4}$$

$$(-1)\cdot\frac{1}{s+3} \rightarrow e^{-3t}u(t) \quad -e^{-3t}u(-t).$$

right-sided   left-sided

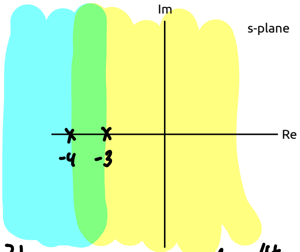
$$2\cdot\frac{1}{s+4} \rightarrow e^{-4t}u(t) \quad -e^{-4t}u(-t).$$

Inverse Laplace transforms

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

Draw the s-plane:



$$x(t) = -1 \cdot \left(-e^{-3t} u(-t)\right) + 2\left(e^{-4t} u(t)\right)$$ ✓

Properties of the Laplace transform

We've made use of many nice properties of the Fourier transform:

- linearity
- time shift/scale
- differentiation
- conjugation
- convolution

All have analogs with Laplace transform, but factor in the ROC.

Linearity

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad \text{w/ ROC } R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \quad \text{w/ ROC } R_2$$

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s)$$

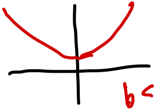
w/ ROC containing $R_1 \cap R_2$

Example: $x(t) = e^{-b|t|}$.

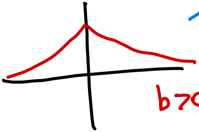
$$x(t) = e^{-bt} u(t) + e^{bt} u(-t)$$

r.s. l.s.

guess: $b>0$ only some region will exist.



$b<0$

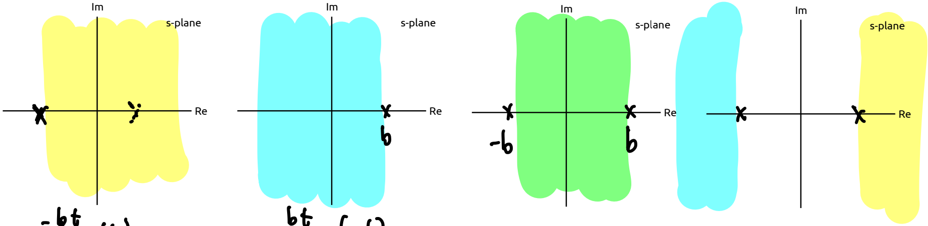


$b>0$

Properties of the Laplace transform

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} \qquad Re(s) > -b$$

$$e^{bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b} \qquad Re(s) < b$$



$e^{-bt}u(t)$ $\quad$ $e^{bt}u(-t)$

$b > 0$ $\quad$ $b > 0$ $\quad$ $b > 0$ $\quad$ $b \leq 0$

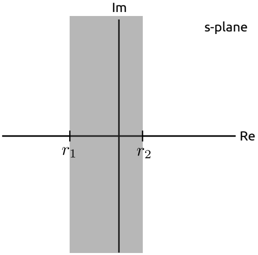
Properties of the Laplace transform

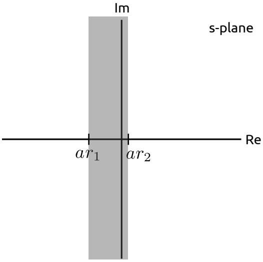
$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{w/ ROC } R$$

Time shifting. $x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s) \quad \text{w/ROC } R$

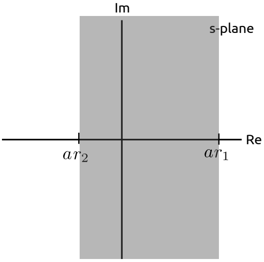
Time scaling. $x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{w/ROC } aR$

Time reversal. $x(-t) \xleftrightarrow{\mathcal{L}} X(-s) \quad \text{w/ROC } -R$





$0 < a < 1$



$a < -1$

★ We did not cover slides 17-24 in the lecture; please review on your own time.

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s}{s^2+9}, \quad \text{Re}(s) < 0$$

Hint:

$$\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_0^2}, \quad \text{Re}(s) > 0$$

The hint tells us $\cos(3t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+9}, \quad \text{Re}(s) > 0$ but the ROC is wrong. Time reversal will change the ROC.

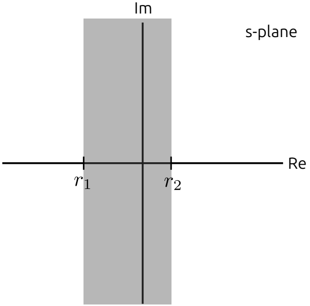
$$\begin{aligned}
x(-t) &\overset{\mathcal{L}}{\longleftrightarrow} X(-s), \quad \text{w/ ROC} - R \\
\cos(-3t)u(-t) &\overset{\mathcal{L}}{\longleftrightarrow} \frac{-s}{s^2+9}, \quad \text{Re}(s) < 0 \\
\cos(3t)u(-t) &\overset{\mathcal{L}}{\longleftrightarrow} -\frac{s}{s^2+9}, \quad \text{Re}(s) < 0
\end{aligned}$$

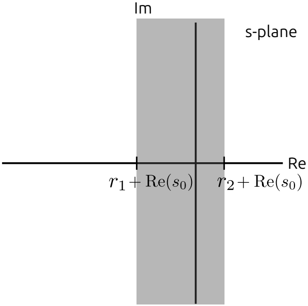
$$x(t) = -\cos(3t)u(-t)$$

Properties of the Laplace transform

s **shifting**

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0) \quad \text{w/ROC } R + \mathrm{Re}(s_0)$$





Note that will moves the poles/zeros. Consider $e^{j\omega_0 t} x(t)$,

$X(s)$ w/pole/zero at $a \to X(s - s_0)$ has pole/zero at $a + j\omega_0$

Properties of the Laplace transform

Differentiation in time.

$$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \quad \text{w/ROC } R$$

$$\frac{dx(t)}{dt} \quad \overset{\mathcal{L}}{\leftrightarrow} \quad sX(s) \quad \text{w/ROC containing } R$$

Differentiation in s.

$$x(t) \quad \overset{\mathcal{L}}{\leftrightarrow} \quad X(s) \quad \text{w/ROC } R$$

$$-tx(t) \quad \overset{\mathcal{L}}{\leftrightarrow} \quad \frac{dX(s)}{ds} \quad \text{w/ROC } R$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Solution: We have $t\cdot$ something, so use **differentiation in s**.
$x(t) = te^{-2|t|} = tz(t)$.

$$z(t) \overset{\mathcal{L}}{\longleftrightarrow} Z(s) \quad \text{w/ROC } R$$

$$x(t) = tz(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{dZ(s)}{ds} \quad \text{w/ROC } R$$

Next, compute the Laplace transform of $z(t) = e^{-2|t|}$.

$$\begin{aligned} z(t) &= \begin{cases} e^{-2t} & t > 0 \\ e^{2t} & t < 0 \end{cases} \\ &= e^{-2t}u(t) + e^{2t}u(-t) \end{aligned}$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

From earlier,

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}(s) > -2 \qquad e^{2t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{1}{s-2}, \quad \mathrm{Re}(s) < 2$$

$$Z(s) = \frac{1}{s+2} - \frac{1}{s-2}, \quad -2 < \mathrm{Re}(s) < 2$$

To get $X(s)$...

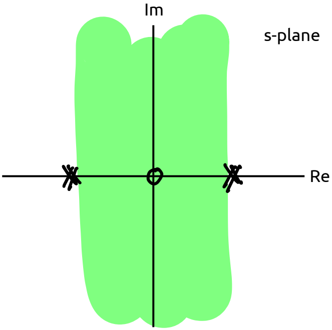
$$\begin{aligned} X(s) &= -\frac{dZ(s)}{ds}, \quad -2 < \mathrm{Re}(s) < 2 \\ &= \frac{1}{(s+2)^2} - \frac{1}{(s-2)^2} \\ &= \frac{-8s}{(s+2)^2(s-2)^2} \end{aligned}$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

Let's make a pole-zero plot:

$$X(s) = \frac{-8s}{(s+2)^2(s-2)^2}, \quad -2 < \text{Re}(s) < 2$$



Conjugation.

$$
\begin{aligned}
x(t) &\overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{w/ROC } R \\
x^*(t) &\overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*) \quad \text{w/ROC } R
\end{aligned}
$$

Initial/final-value theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or singularities at the origin,

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

Furthermore if $x(t)$ has finite limit as $t \to \infty$,

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Properties of the Laplace transform

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$

Image credit: Oppenheim 9.6

Convolution.

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \text{ w/ROC } R_1$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \text{ w/ROC } R_2$$

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s) \text{ w/ROC containing } R_1 \cap R_2$$

Recall the convolution property:

$$h(t) \xleftrightarrow{\mathcal{L}} H(s) \qquad Y(s) = H(s)X(s)$$

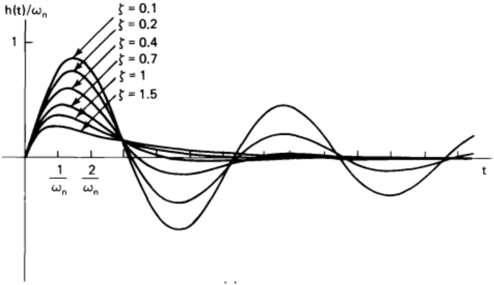
The ROC of the system (transfer) function can tell us a lot about a system, including systems whose Fourier transforms don't exist.

$H(s)$ and causality

$y[n] = x[n-1] \to$ causal

$y[n] = x[n+1] \to$ not causal

Recall that a system is causal if $h(t) = 0$ for $t < 0$.

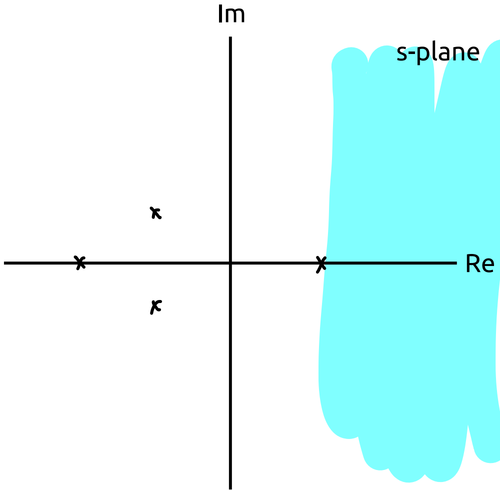


Means $h(t)$ is right-sided, so its ROC is a right-half plane.

Image credit: Oppenheim 6.5

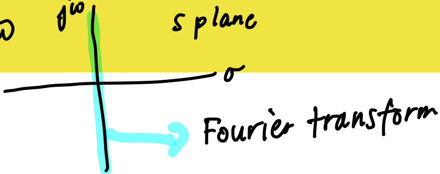
$H(s)$ and causality

Note that the converse is not necessarily true! But if $H(s)$ is rational, the ROC is the right-half plane to right of right-most pole.



$H(s)$ and stability

$s = \sigma + j\omega$



Our original criteria for stability in terms of impulse response was if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

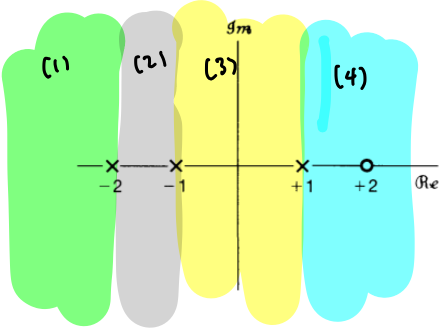
Related to Dirichlet conditions: if a signal is absolutely integrable, its **Fourier transform** converges.

An LTI system with **rational** $H(s)$ is stable iff its ROC includes the entire $j\omega$ axis ($\mathrm{Re}(s) = 0$), and there aren't more zeros than poles.

$$\sum \frac{A_i}{s + a_i} \qquad \cancel{\frac{s^5 + s^4 - s^3}{s^2 + 1}}$$

$H(s)$ and causality / stability

9.28. Consider an LTI system for which the system function $H(s)$ has the pole-zero pattern shown in Figure P9.28.



(1) not stable, not causal (anticausal)

(2) neither

(3) not causal; stable

(4) not stable; causal

Figure P9.28

(a) Indicate all possible ROCs that can be associated with this pole-zero pattern.

(b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

Image credit: Oppenheim 9.28

For next time

Content:

- the Laplace transform and feedback systems
- introducing the z-transform

Action items:

1. Suggest problems for tutorial
2. Tutorial assignment 5 due Monday 23:59
3. Assignment 5 due 8 Dec 23:59

Recommended reading:

- From this class: Oppenheim 9.5-9.7
- Suggested problems: 9.13-9.16, 9.21, 9.22, 9.26, 9.29, 9.32, 9.33
- For next class: 9.7, 11.0-11.2, 10.1-10.3