ELEC 221 Lecture 18
CT $\leftrightarrow$ DT signals and sampling

Tuesday 8 November 2022

- Quiz 8 today
- Assignment 5 available due 11:59 Friday Nov. 11 (**no extensions**; solutions to be posted immediately after for studying)

Midterm 2 on Monday 14 Nov 17:30 (tutorial session).
Two-stage exam:

- Individual portion 60 minutes (85%)
- Group portion 40 minutes (15%, similar questions)
- If grade on group portion is lower than individual, your individual grade will count for 100%

We modeled **sampling** of CT signals as multiplication of a (band-limited) signal with a periodic impulse train:

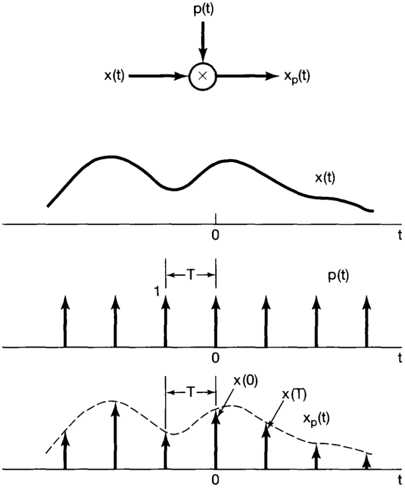


Image credit: Oppenheim 7.1

Last time

We went to the frequency domain to get a better understanding:

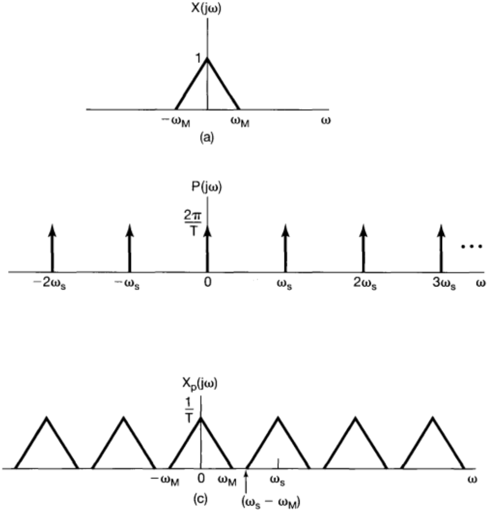


Image credit: Oppenheim 7.1

Last time

We are able to recover our original signal from our samples by applying a low pass filter...

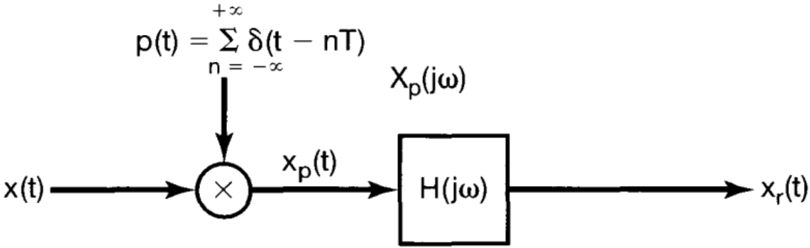


Image credit: Oppenheim 7.1

Last time

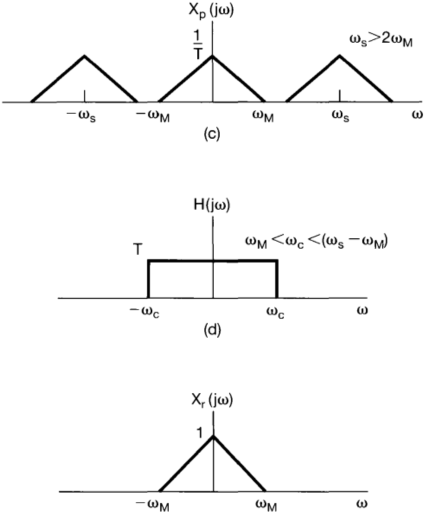


Image credit: Oppenheim 7.1

...but only if the sampling rate is higher than the **Nyquist rate**, i.e., at least twice as high as the highest frequency in the signal.

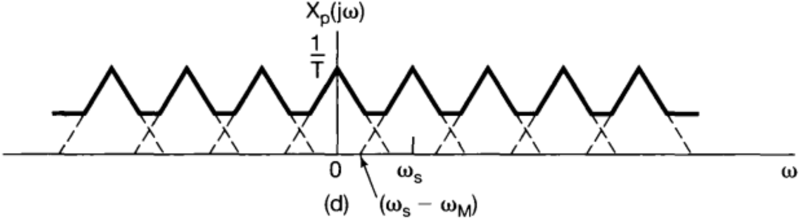
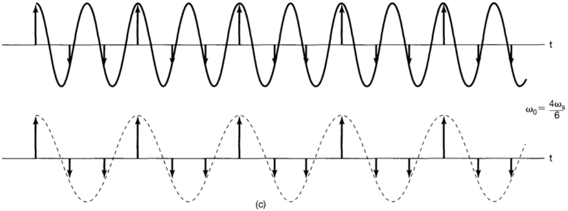


Image credit: Oppenheim 7.1

Last time

If the frequency isn't high enough, **aliasing** occurs.



(c)

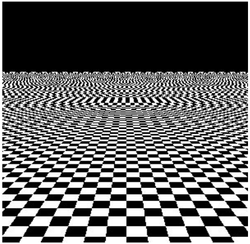


Image credit: Oppenheim 7.3, `https://textureingraphics.wordpress.com/what-is-texture-mapping/anti-aliasing-problem-and-mipmapping/`

If the frequency *is* high enough, we can use various methods of interpolation to recover our original signal.

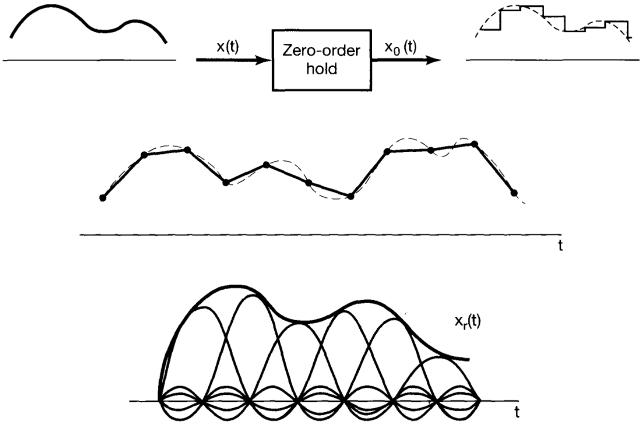


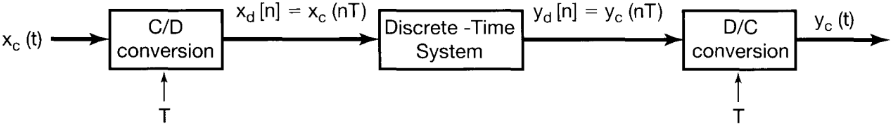
Image credit: Oppenheim 7.2

Today

Learning outcomes:

- describe the frequency domain effects of sampling from a discrete signal
- decimate and interpolate a DT signal
- determinate how decimation and interpolation affect the spectrum of a signal

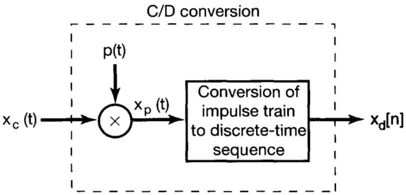
Often convenient to process CT signals by first converting to DT, processing, then converting back.



What is the theory that makes this possible?

Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT



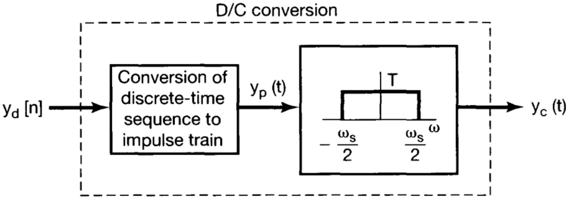
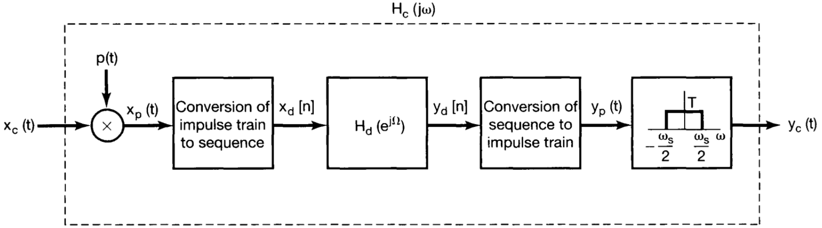


Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT



Let's explore what happens at the level of the spectra again.

Note: we have *two frequencies*, one in CT, one in DT. Write:

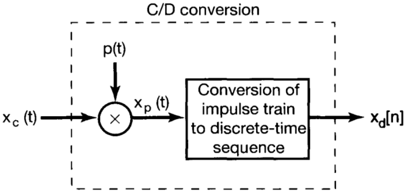
$$X(j\omega), \quad Y(j\omega)$$

$$X(e^{j\Omega}), \quad Y(e^{j\Omega})$$

Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT

First: how are $X_p(j\omega)$ and $X_d(e^{j\Omega})$ related?



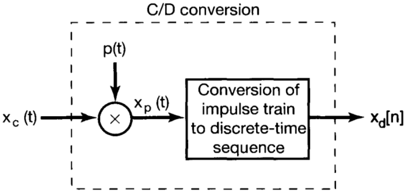
Last time we found

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \rightarrow X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT}$$

Image credit: Oppenheim 7.4

First: how are $X_p(j\omega)$ and $X_d(e^{j\Omega})$ related?

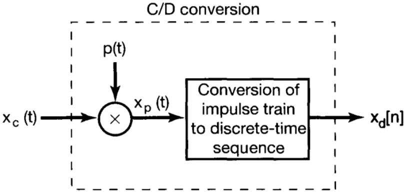


$$
\begin{aligned}
X_p(j\omega) &= \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT} \\
X_d(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} = X_p(j\Omega/T)
\end{aligned}
$$

Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT

First: how are $X_p(j\omega)$ and $X_d(e^{j\Omega})$ related?



$$\begin{aligned} X_d(e^{j\Omega}) &= X_p(j\Omega/T) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega/T - k\omega_s)) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T) \end{aligned}$$

Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)$$

The DT spectrum is also copies of the spectrum of $x_c(t)$, but

- the frequency is rescaled: $\Omega = \omega T$
- they are periodic over the interval $[0, 2\pi)$

Converting between DT $\leftrightarrow$ CT

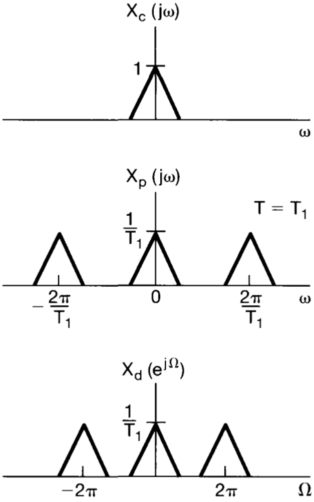


Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT

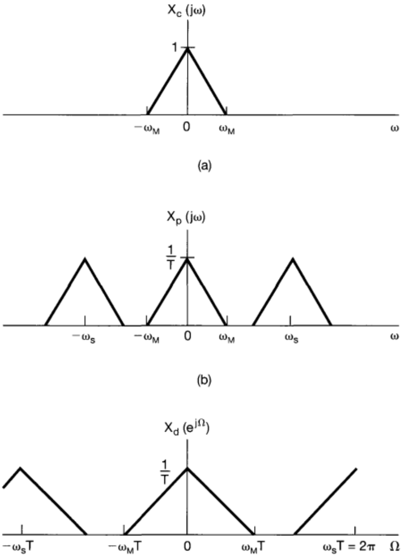
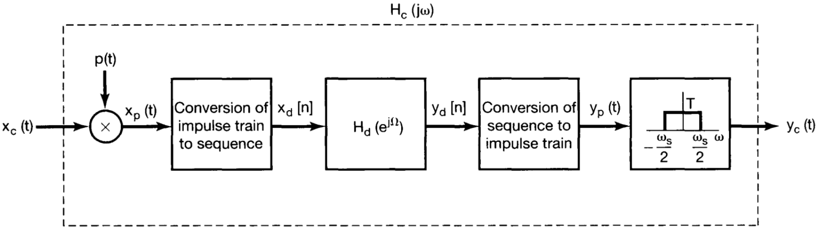


Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT



The converted signal $x_d[n]$ now goes through some DT system:

$$
\begin{aligned}
Y_d(e^{j\Omega}) &= H_d(e^{j\Omega})X_d(e^{j\Omega}) \\
&= H_d(e^{j\Omega})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)
\end{aligned}
$$

Image credit: Oppenheim 7.4

Converting between DT $\leftrightarrow$ CT

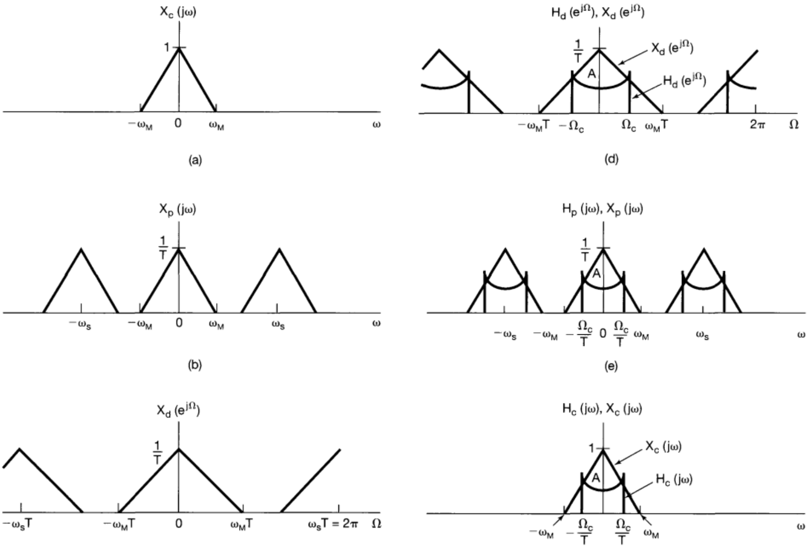
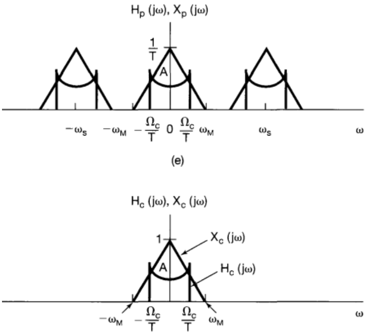


Image credit: Oppenheim 7.4



Still end up with the correct output,

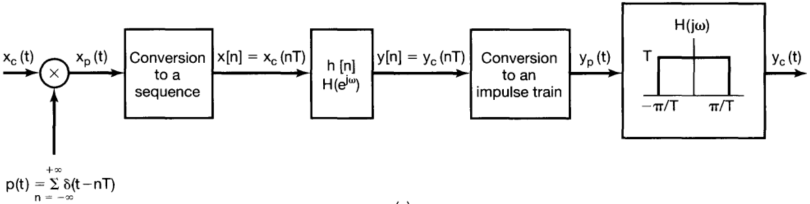
$$Y_c(j\omega) = H_c(j\omega)X_c(j\omega)$$

where

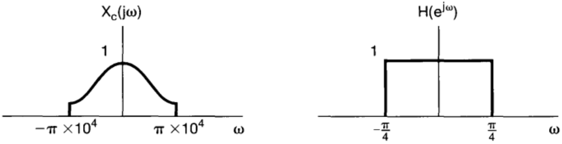
$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2, \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

Image credit: Oppenheim 7.4

Example



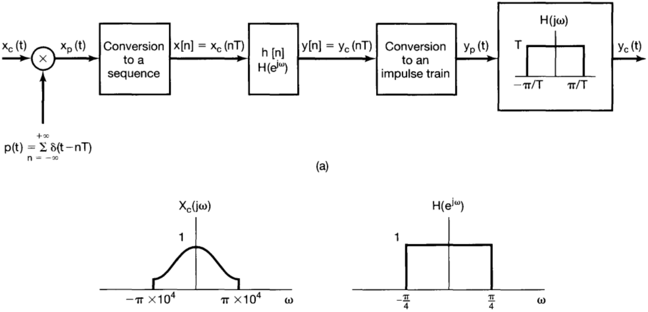
(a)



Sketch: $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, $Y_c(j\omega)$ if $1/T = 20kHz$.

Image credit: Oppenheim Problem 7.29

Example



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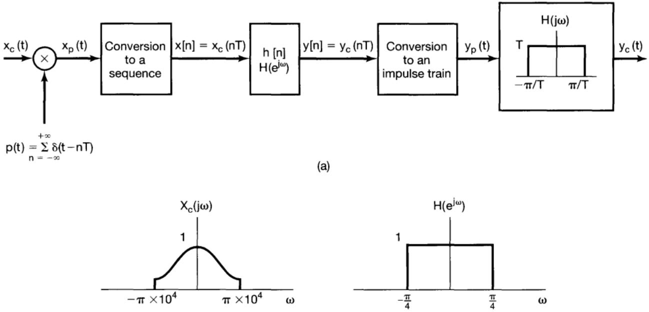
Example



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Example

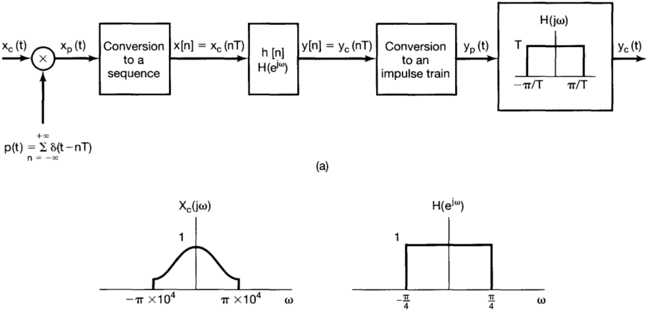


Image credit: Oppenheim Problem 7.29

Suppose we sample with DT impulse train of period N:

$$x_p[n] = \begin{cases} x[n], & n \text{ integer multiple of } N \\ 0, & \text{otherwise} \end{cases}$$

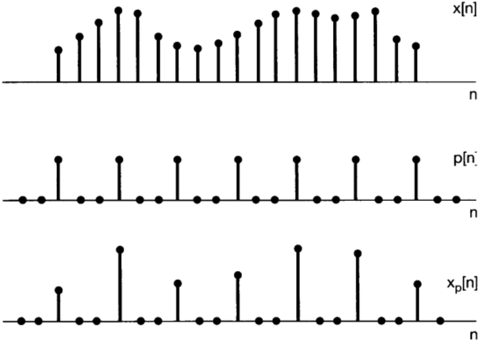


Image credit: Oppenheim 7.5

Sampling of discrete-time signals

Same thing happens to the spectrum:

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$

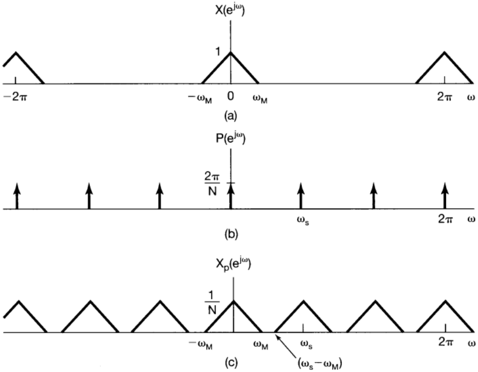


Image credit: Oppenheim 7.5

Sampling of discrete-time signals

Aliasing can happen in DT as well but some differences due to DT frequency range (ω is the highest frequency).

Exercise: suppose $x[n]$ has $X(e^{j\omega})$ that is 0 for $3\pi/7 \leq |\omega| \leq \pi$. What is the largest sampling period N we can use?

Aliasing can happen in DT as well but some differences due to DT frequency range (ω is the highest frequency).

Exercise: suppose $x[n]$ has $X(e^{j\omega})$ that is 0 for $3\pi/7 \leq |\omega| \leq \pi$. What is the largest sampling period N we can use?

Solution: set $\omega_s = 2\pi/N$ at least 2x highest frequency.

$$\frac{2\pi}{N} \geq \frac{6\pi}{7} \;\rightarrow\; N \leq 7/3 \;\rightarrow\; N_{max} = 2, \omega_2 = \;\; \pi$$

Sampling DT signals in this way is inefficient:

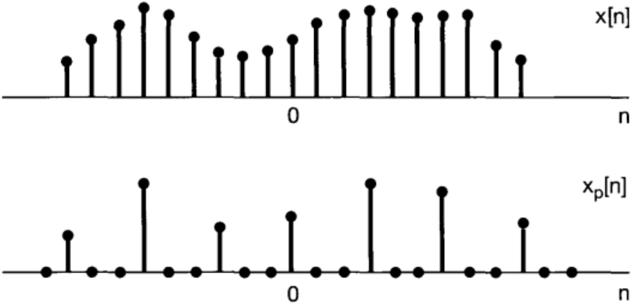
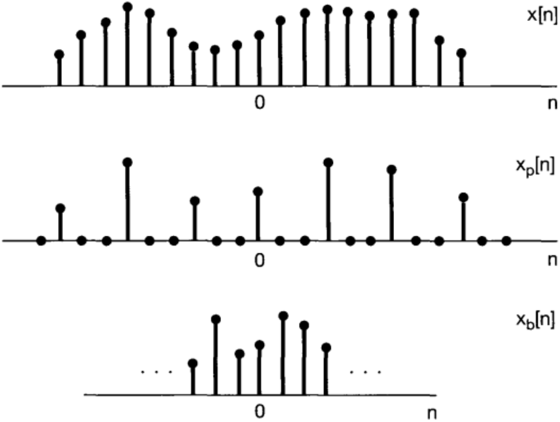


Image credit: Oppenheim 7.5

This is a much nicer way:



$$x_b = x[nN]$$

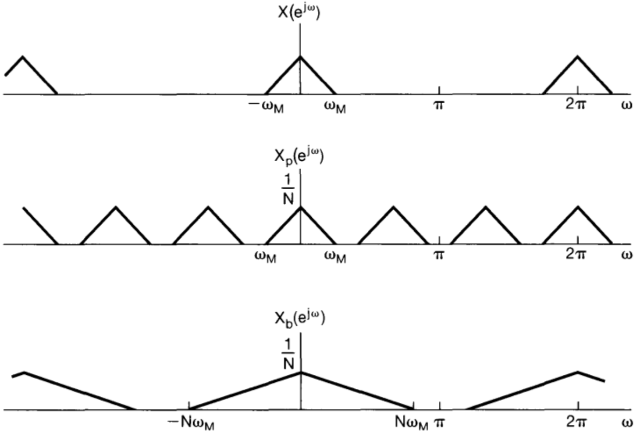
Image credit: Oppenheim 7.5

Decimation

Frequency domain effect:

$$
\begin{aligned}
X_b(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} x_b[k]e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} x_p[kN]e^{-j\omega k} \\
&= \sum_{n} x_p[n]e^{-j\omega n/N}, \quad n \text{ int. mults. of } N \\
&= \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N} \\
&= X_p(e^{j\omega/N})
\end{aligned}
$$

Decimation spreads out the spectrum.



If original signal was CT, say that decimation has *downsampled* it.

Image credit: Oppenheim 7.5

Opposite of decimation: add $N-1$ points between.

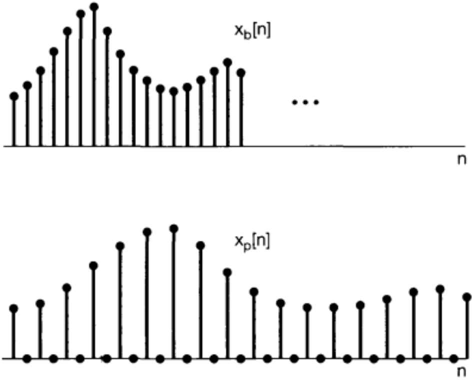


Image credit: Oppenheim 7.5

Interpolation (upsampling)

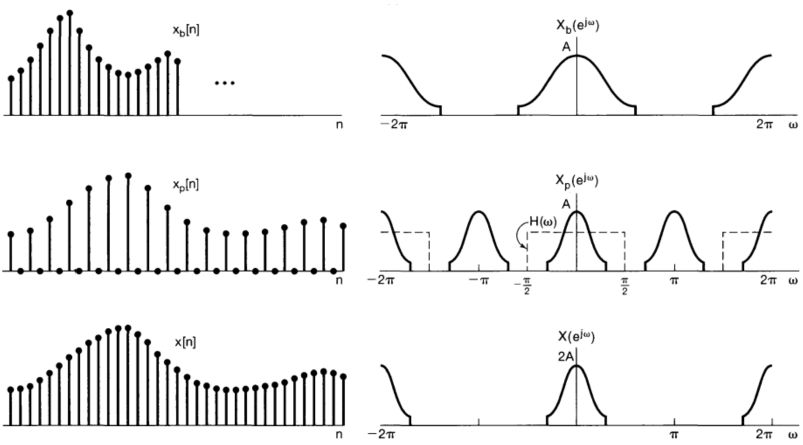


Image credit: Oppenheim 7.5

Example: down/upsampling

Today

Learning outcomes:

- describe the frequency domain effects of sampling from a discrete signal
- decimate and interpolate a DT signal
- determinate how decimation and interpolation affect the spectrum of a signal

Oppenheim practice problems: 7.17, 7.18, 7.20, 7.30, 7.32

For next time

Content:

- hands-on lecture on Tuesday 15
- moving into topic of modulation / communication systems

Action items:

1. Assignment 5 due 11:59pm Friday 11 Nov
2. Midterm 2 Monday 14 Nov during tutorial

Recommended reading:

- From this class: Oppenheim 7.4-7.6