ELEC 221 Lecture 10
More properties of the CT Fourier transform

Tuesday 11 October 2022

Announcements

- Quiz 5 today
- Midterm 1 on Thursday
  - Closed book / closed notes; no calculators
  - Formula sheet provided (see last Thursday's lecture)
  - Please arrive on time and bring your ID
- Assignment 4 (computational) available after midterm
  - Statement of contributions worth 1 point from now on.
  - "No exceptions" means *no exceptions*

Last time: the Fourier transform

We saw the Dirichlet conditions for the Fourier transform.

If the signal

1. is single-valued
2. is absolutely integrable ($\int_{-\infty}^{\infty} |x(t)| dt < \infty$)
3. has a finite number of maxima and minima within any finite interval
4. has a finite number of finite discontinuities within any finite interval

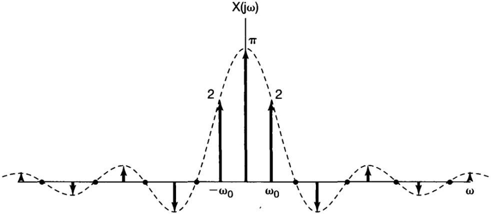
then the Fourier transform converges to

- $x(t)$ where it is continuous
- the average of the values on either side at a discontinuity

Last time: properties of the Fourier transform

We computed Fourier transforms of periodic signals.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$



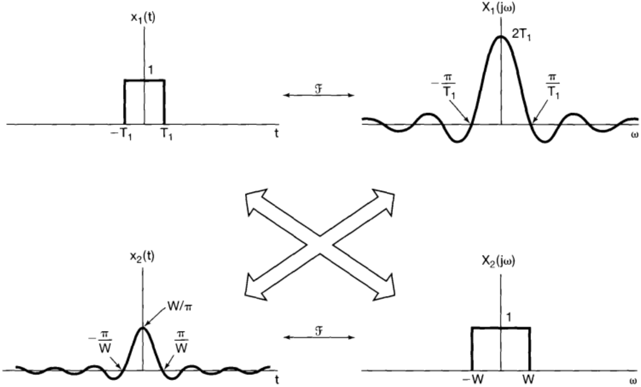
$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$$

Image credit: Oppenheim chapter 4.2

Last time: properties of the Fourier transform

We saw some important properties of the Fourier transform:

- Linearity
- Behaviour under time shift/scale/reverse/conjugation
- Time/frequency duality



Last time: properties of the Fourier transform

We explored how the **frequency response** of a system relates to its **impulse response** via a Fourier transform:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

We introduced the convolution property:

$$\begin{aligned} y(t) &= h(t) * x(t) \\ Y(j\omega) &= H(j\omega) X(j\omega) \end{aligned}$$

Last time: properties of the Fourier transform

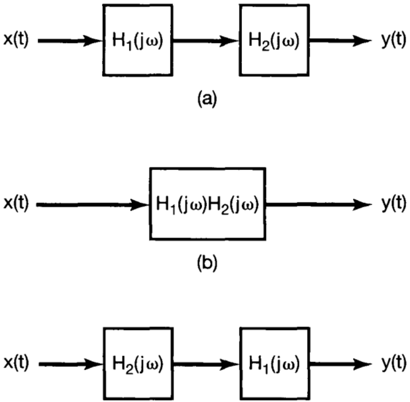


Image credit: Oppenheim chapter 4.4

Today

Learning outcomes:

- Describe the *multiplication property* of the Fourier transform
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

The multiplication property

We know that:

$$\begin{aligned} y(t) &= h(t) * x(t) \\ Y(j\omega) &= H(j\omega)X(j\omega) \end{aligned}$$

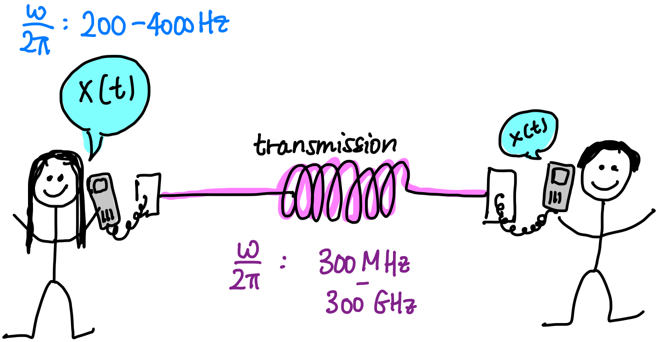
Something similar holds when we interchange time and frequency:

$$\begin{aligned} r(t) &= s(t)p(t) \\ R(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta \end{aligned}$$

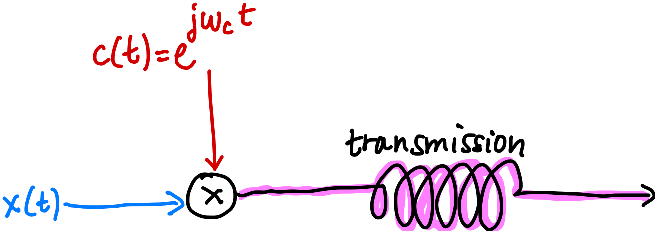
This is the **multiplication property**.

We are going to take a much closer look at this when we discuss communication systems and signal **modulation**.

For now, here is a taste:



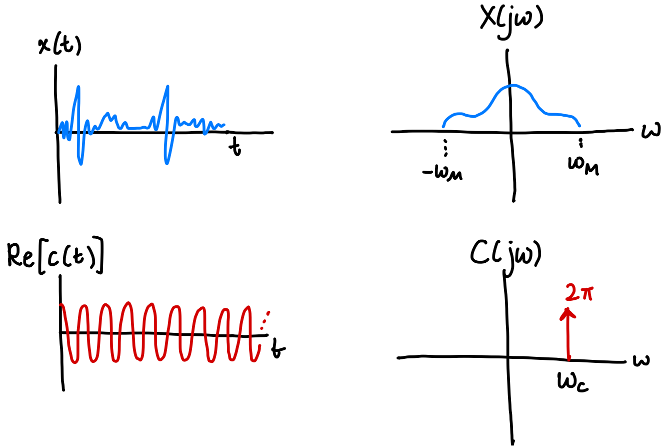
To shift our signal into the frequency range of transmission, we can multiply it by a **carrier signal** (amplitude modulation):



Is this doing what we think it is?

Example: the multiplication property

Consider the Fourier spectrum of both signals:



Example: the multiplication property

The multiplication property tells us

$$\begin{aligned} y(t) &= x(t)c(t) \\ Y(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)C(j(\omega-\theta))d\theta \end{aligned}$$

We have

$$\begin{aligned} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\ c(t) = e^{j\omega_c t} &\overset{\mathcal{F}}{\longleftrightarrow} C(j\omega) = 2\pi\delta(\omega-\omega_c) \end{aligned}$$

Example: the multiplication property

Let's convolve them:

$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)C(j(\omega-\theta))d\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)2\pi\delta((\omega-\omega_c)-\theta)d\theta \\
&= X(j(\omega-\omega_c))
\end{aligned}
$$

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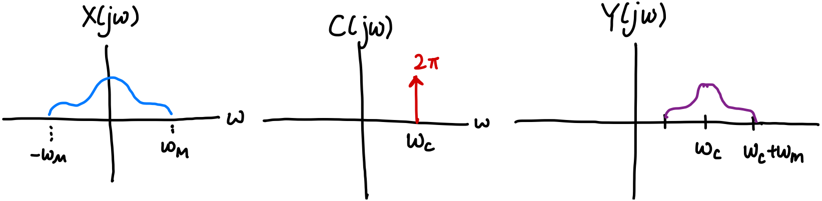
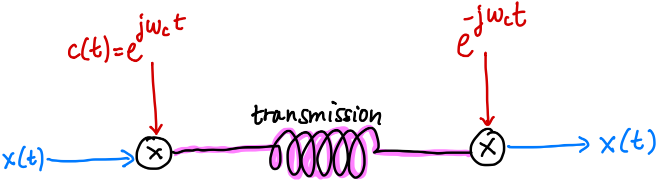
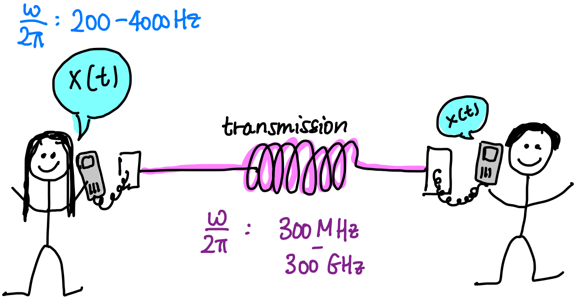
&= X(j(\omega-\omega_c))
\end{aligned}
$$

Multiplication with complex exponential carrier signal shifts the spectrum. We can move it into the desired frequency range.

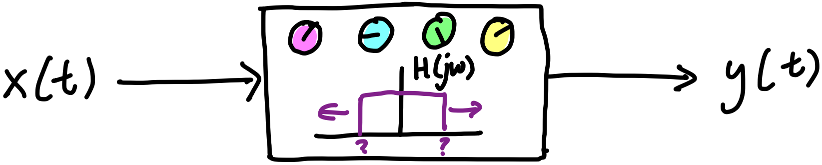


Example: the multiplication property

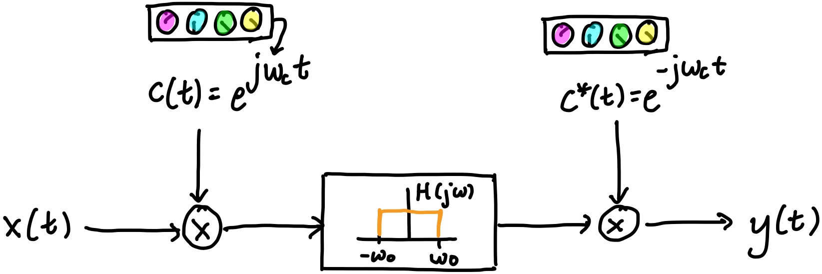




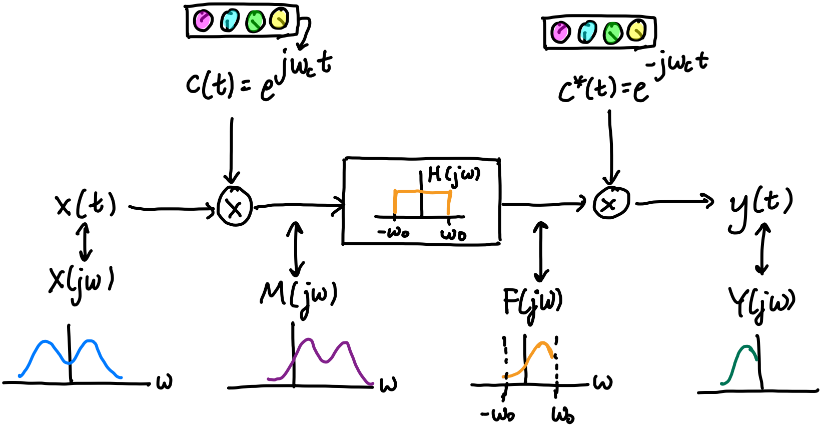
Example: frequency-selective filtering with variable centre frequency



Example: frequency-selective filtering with variable centre frequency



Example: frequency-selective filtering with variable centre frequency



Fourier transforms: differentiation

Consider the inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

What happens when we differentiate $x(t)$?

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)j\omega e^{j\omega t}d\omega$$

This means:

$$\begin{aligned} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\ \frac{dx(t)}{dt} &\overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \end{aligned}$$

What should happen here?

$$\begin{aligned} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\ \int_{-\infty}^{t} x(\tau)d\tau &\overset{\mathcal{F}}{\longleftrightarrow} ?? \end{aligned}$$

Good initial guess:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega)$$

More precisely:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

We can often take advantage of differentiation and integration properties to simplify computation of Fourier transforms and system outputs.

Example: Fourier transform properties and differentiation

Suppose

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

What is the Fourier transform of

$$z(t) = \frac{d^2}{dt^2} x(t-1)$$

Two properties to take advantage of here:

$$\begin{aligned} \frac{d}{dt} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \\ x(t-t_0) &\overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega) \end{aligned}$$

Example: Fourier transform properties and differentiation

$$z(t) = \frac{d^2}{dt^2}x(t-1)$$

First, consider: $p(t) = x(t-1)$:

$$\begin{aligned} p(t) &\overset{\mathcal{F}}{\longleftrightarrow} P(j\omega) \\ \frac{d^2}{dt^2}p(t) &\overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^2 P(j\omega) = -\omega^2 P(j\omega) \end{aligned}$$

But we know

$$p(t) = x(t-1) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega}X(j\omega) \quad \rightarrow \quad P(j\omega) = e^{-j\omega}X(j\omega)$$

So

$$\frac{d^2}{dt^2}x(t-1) \overset{\mathcal{F}}{\longleftrightarrow} -\omega^2 e^{-j\omega}X(j\omega)$$

Fourier transforms and systems described by differential equations

Back in lecture 6, we saw a system (RC circuit) described by a differential equation:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

We found its frequency response in the following way:

- Choosing input signal $v_s(t) = e^{j\omega t}$
- Since system is LTI, assuming output of the form $v_c(t) = H(j\omega)e^{j\omega t}$
- Plugging this into the ODE and solving for $H(j\omega)$

Nice properties of Fourier transforms give a much slicker method of computing frequency responses of such systems.

Consider a general system described by an ODE of arbitrary order:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

What is its frequency response $H(j\omega)$?

Fourier transforms and systems described by differential equations

First: take the Fourier transform of both sides.

$$\mathcal{F}\left(\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k}\right) = \mathcal{F}\left(\sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}\right)$$

Which property can we leverage next? Linearity

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

Fourier transforms and systems described by differential equations

$$\sum_{k=0}^{N} \alpha_k \mathcal{F}\left(\frac{d^k y(t)}{dt^k}\right) = \sum_{k=0}^{M} \beta_k \mathcal{F}\left(\frac{d^k x(t)}{dt^k}\right)$$

Now what? Differentiation

$$\sum_{k=0}^{N} \alpha_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} \beta_k (j\omega)^k X(j\omega)$$

We can simplify this even more:

$$Y(j\omega) \sum_{k=0}^{N} \alpha_k (j\omega)^k = X(j\omega) \sum_{k=0}^{M} \beta_k (j\omega)^k$$

$$Y(j\omega)\sum_{k=0}^{N}\alpha_k(j\omega)^k = X(j\omega)\sum_{k=0}^{M}\beta_k(j\omega)^k$$

Final property: Convolution

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M}\beta_k(j\omega)^k}{\sum_{k=0}^{N}\alpha_k(j\omega)^k}$$

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

Example: frequency response of systems described by ODEs

What are the **impulse response** and **frequency response** of the system described by

$$\frac{d^3y(t)}{dt^3} - 4\frac{dy(t)}{dt} = 3\frac{d^2x(t)}{dt^2} + x(t)$$

Start with frequency response:

$$\begin{aligned} H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} &= \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k} \\ &= \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega} \end{aligned}$$

Example: frequency response of systems described by ODEs

We can now leverage this to determine the impulse response:

$$H(j\omega) = \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega}$$

Use partial fractions:

$$\begin{aligned}
H(j\omega) &= \frac{3(j\omega)^2 + 1}{(j\omega)^3 - 4j\omega} \\
&= \frac{3(j\omega)^2 + 1}{j\omega((j\omega)^2 - 4)} \\
&= \frac{3(j\omega)^2 + 1}{j\omega(j\omega + 2)(j\omega - 2)} \\
&= \frac{A}{j\omega} + \frac{B}{j\omega + 2} + \frac{C}{j\omega - 2}
\end{aligned}$$

Details are left as an exercise:

$$H(j\omega) = \frac{-1/4}{j\omega} + \frac{13/8}{j\omega + 2} + \frac{13/8}{j\omega - 2}$$

To get the impulse response, we can take the inverse Fourier transform, and leverage linearity:

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega + 2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega - 2}\right)$$

Example: frequency response of systems described by ODEs

$$h(t) = -\frac{1}{4}\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega + 2}\right) + \frac{13}{8}\mathcal{F}^{-1}\left(\frac{1}{j\omega - 2}\right)$$

Last time, we learned a general expression for inverse Fourier transforms of this type:

$$\mathcal{F}(e^{-at}u(t)) = \frac{1}{a + j\omega}, \quad \mathrm{Re}(a) > 0$$

So we have:

$$h(t) = -\frac{1}{4}u(t) + \frac{13}{8}e^{-2t}u(t) + \frac{13}{8}e^{2t}u(t)$$

Today's learning outcomes were:

- Describe the *multiplication property* of the Fourier transform
- Describe the behaviour of the Fourier transform under differentiation and integration
- Use the convolution property to characterize LTI systems based on differential equations

What topics did you find unclear today?

For next time

Content (after the midterm):

- Discrete Fourier transform

Action items:

1. Midterm 1 on Thursday

Recommended reading:

- From today's class: Oppenheim 4.5-4.8
- For Tuesday's class: Oppenheim chapter 5.0-5.5