ELEC 221 Lecture 23
The Laplace transform and feedback systems; introducing the z-transform

Tuesday 3 December 2024

Announcements

- Quiz 10 today
- Please fill out course evaluation survey if you have time after quiz
- Assignment 5 due Sunday at 23:59
- Exam info period office hours will be posted on Piazza/PrairieLearn this week

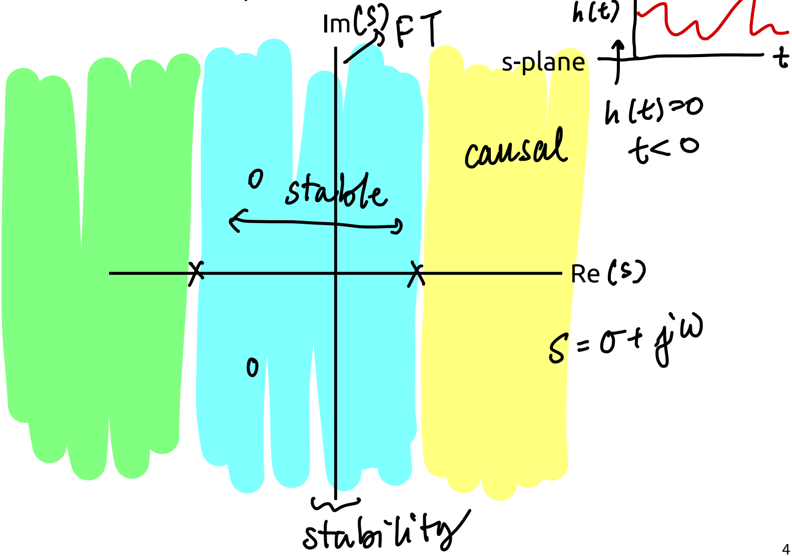
We explored various properties of the Laplace transform.

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R

Image credit: Oppenheim 9.5

We used the ROC to reason about the stability and causality of systems with rational Laplace transforms.



Learning outcomes:

- compute the Laplace transform of systems described by constant-coefficient DEs
- compute system functions for feedback systems
- use the Laplace transform to design an inverse system
- define the z-transform and compute it and its ROC for basic signals

Systems described by constant-coefficient differential equations

Recall the situation with the Fourier transform:

Fourier transforms and systems described by differential equations

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

$$\frac{d^3 y(t)}{dt^3} + 2\frac{dy(t)}{dt} + y(t) = 2\frac{dx(t)}{dt} + x(t)$$

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Systems described by constant-coefficient differential equations

$b_k (j\omega)^k$

Same deal here. If system is described by the DE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ w/ROC R

then its system function is

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{Y(s)}{X(s)}$$

$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$ w/ROC containing R

Placement of zeros and poles is dictated by coefficients of $x(t)$ and $y(t)$ stuff respectively.

9.32. A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = (1/6)e^{2t}$ for all t.
2. The impulse response $h(t)$ satisfies the differential equation

$x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega)e^{j\omega t}$

$$\frac{dh(t)}{dt} + 2h(t) = (e^{-4t})u(t) + bu(t),$$

$e^{st} \rightarrow H(s)e^{st}$ still eigenfunctions!

where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should *not* appear in the answer.

$x(t) = e^{2t} \longrightarrow \boxed{H(s)} \longrightarrow y(t) = \frac{1}{6}e^{2t} \Rightarrow H(s)?$

$s=2$

$H(2) = \frac{1}{6}$

Left / Right

$$\mathcal{L}\left(\frac{dh(t)}{dt} + 2h(t)\right) = \mathcal{L}\left(e^{-4t}u(t) + b\cdot u(t)\right)$$

$$sH(s) + 2H(s) = \frac{1}{s+4} + \frac{b}{s} \Rightarrow H(s)(s+2) = \frac{1}{s+4} + \frac{b}{s} = \frac{s+b(s+4)}{s(s+4)}$$

$$H(s) = \frac{s+b(s+4)}{(s+2)s(s+4)}$$

→ not stable!

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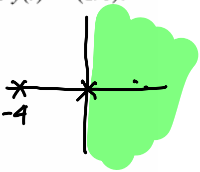

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Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should *not* appear in the answer.

$$H(s) = \frac{s + b(s+4)}{(s+2)s(s+4)} \qquad H(2) = \frac{1}{6}$$

$$H(2) = \frac{2+6b}{4\cdot 2\cdot 6}$$

$$\frac{1}{6} = \frac{2+6b}{48} \Rightarrow b = 1$$

$$H(s) = \frac{s+s+4}{s(s+2)(s+4)} = \frac{2\cancel{(s+2)}}{s\cancel{(s+2)}(s+4)} = \frac{2}{s(s+4)}$$

③ $tu(t) \longleftrightarrow \frac{1}{s^2}$ $\quad H(s) = \frac{s \cdot \text{stuff}}{\text{stuff}}$ $\Rightarrow$ $H(s)$ has one zero at 0

9.34. Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

#3 [1. $H(1) = 0.2$.

#1 [2. When the input is $u(t)$, the output is absolutely integrable.
3. When the input is $tu(t)$, the output is not absolutely integrable.

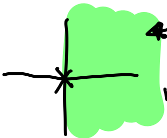
#2 [4. The signal $d^2h(t)/dt^2 + 2\,dh(t)/dt + 2h(t)$ is of finite duration.
5. $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

② $u(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$ absolutely integrable

$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$ $\quad \mathrm{Re}(s) > 0$ $\qquad e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$ $\quad \mathrm{Re}(s) > -a$

↳ pole at 0



not stable

$Y(s) = H(s)X(s) \Rightarrow H(0) = 0$

$H(s) = \frac{s \cdot \text{stuff}}{\text{stuff}}$

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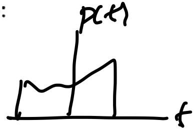

④ $p(t) = \dfrac{d^2h(t)}{dt^2} + \dfrac{2\,dh(t)}{dt} + 2h(t)$

$P(s) = s^2H(s) + 2sH(s) + 2H(s)$

$\Rightarrow H(s) = \dfrac{P(s)}{s^2+2s+2} \longrightarrow P(s) = \prod_i (s-a_i) = s\cdot(s-a_1)(s-a_2)\cdots$

$= \dfrac{\underline{s}\cdot(s-a_i)(s-a_j)\cdots}{s^2+2s+2}$

⑤ deg (numerator) < deg (denominator) $\nearrow$ $H(s) = \dfrac{c\cdot s}{s^2+2s+2}$

① $c = 1$

Systems described by constant-coefficient differential equations

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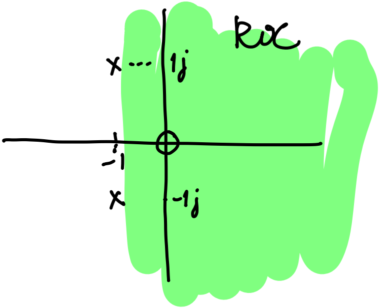
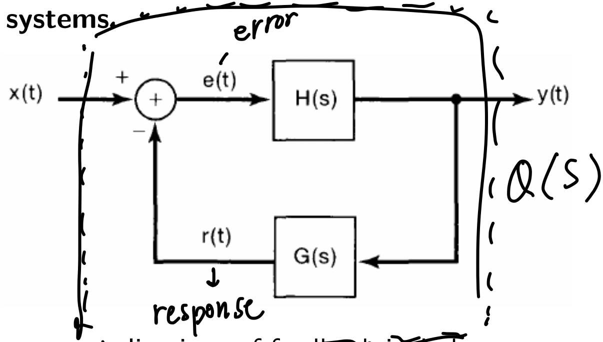
Determine $H(s)$ and its region of convergence.

$$H(s) = \frac{s}{s^2+2s+2}$$



Feedback systems

An important application of Laplace transforms is the analysis of **feedback systems**.



Some important applications of feedback include:

- reducing sensitivity to disturbance and errors
- stabilizing unstable systems
- designing tracking systems

Image credit: Oppenheim 11.1

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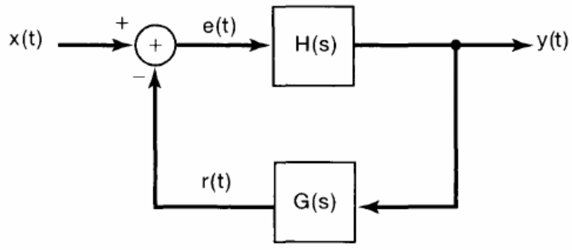
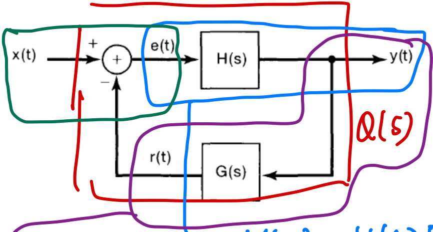


- $H(s)$ is the system function of the forward path
- $G(s)$ is the system function of the feedback path
- the combined function $Q(s)$ is the closed-loop system function

Let's compute $Q(s)$ in terms of $H(s)$ and $G(s)$.

Image credit: Oppenheim 11.1

 ☆ We will revisit on Thursday.



$$Y(s) = Q(s)\,X(s)$$

① $$Q(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{H(s)E(s)}{X(s)}$$

$$= \frac{H(s)\left[X(s) - R(s)\right]}{X(s)}$$

$\Rightarrow Y(s) = H(s)E(s)$

$$e(t) = x(t) - r(t)$$

$$E(s) = X(s) - R(s)$$

$\Rightarrow R(s) = G(s)Y(s)$

$$\Rightarrow Q(s) = \frac{H(s)}{1 + G(s)H(s)}$$

For next time

Content:

we will continue with CT feedback systems; I will show a bit of z transform if time.

- ~~more properties of z-transforms~~
- ~~systems described by difference equations~~
- ~~z-transforms and feedback system analysis~~

Action items:

1. Assignment 5 due Sunday 8 Dec at 23:59

Recommended reading:

- From this class: Oppenheim 9.7, 11.0-11.2, ~~10.1-10.3~~
- Suggested problems: 9.48, 11.1-11.4, ~~10.1-10.8, 10.21-10.23, 10.26~~
- For next class: ~~10.5-10.7~~, 11.2