ELEC 221 Lecture 24
The Laplace transform and feedback systems; introducing the z-transform

Thursday 1 December 2022

Announcements

- Midterms available for pickup
- Assignment 6 (computational) due tonight at 23:59 - submit via e-mail, but still fill out contributions on PL
- Assignment 7 available, due Tuesday at 23:59

We explored various properties of the Laplace transform.

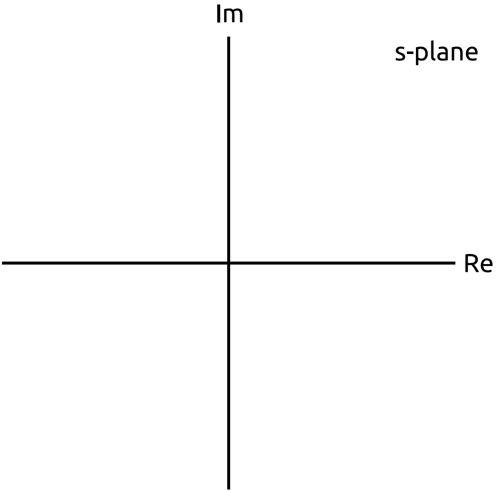
TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R

Image credit: Oppenheim 9.5

Last time

We used the ROC to reason about the stability and causality of systems with rational Laplace transforms.



Last time

We saw how to compute $H(s)$ for systems described by linear constant-coefficient ODEs.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

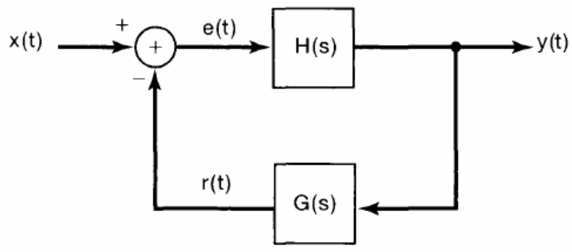
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Today

Learning outcomes:

- compute system functions for feedback systems
- use the Laplace transform to design an inverse system
- define the z-transform and compute it and its ROC for basic signals

An important application of Laplace transforms is the analysis of **feedback systems**.



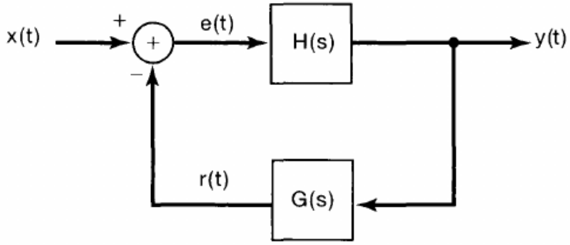
Some important applications of feedback include:

- reducing sensitivity to disturbance and errors
- stabilizing unstable systems
- designing tracking systems

Image credit: Oppenheim 11.1

Feedback systems

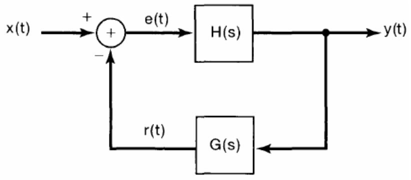
An important application of Laplace transforms is the analysis of **feedback systems**.



- $H(s)$ is the system function of the forward path
- $G(s)$ is the system function of the feedback path
- the combined function $Q(s)$ is the closed-loop system function

Try it: compute $Q(s)$ in terms of $H(s)$ and $G(s)$.

Image credit: Oppenheim 11.1



Solution: from the convolution property, know that

$$Q(s) = \frac{Y(s)}{X(s)}$$

From the diagram, find that

$$Y(s) = H(s)E(s)$$

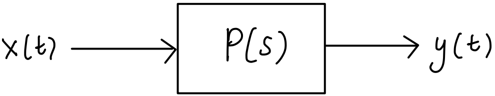
$$E(s) = X(s) - R(s) = X(s) - G(s)Y(s)$$

Thus:

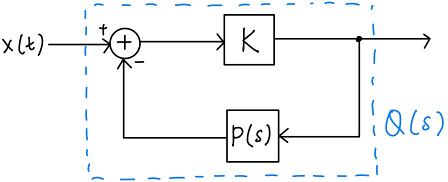
$$Q(s) = \frac{H(s)E(s)}{E(s) + R(s)} = \frac{H(s)E(s)}{E(s) + G(s)H(s)E(s)} = \frac{H(s)}{1 + G(s)H(s)}$$

Image credit: Oppenheim 11.1

Suppose we have some LTI system



Let's use it as part of a larger system:

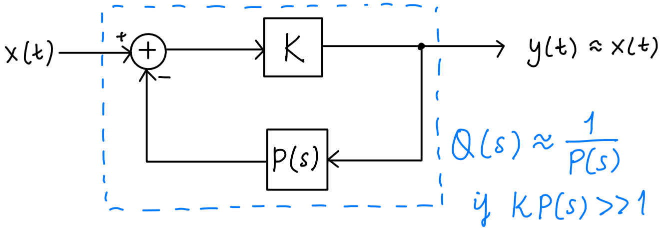


where the transfer function K is simply gain of strength K.

Exercise: What is $Q(s)$, and under what conditions can it act as the inverse of $P(s)$?

Solution: we can directly apply the expression for the closed-loop system function here

$$Q(s) = \frac{K}{1 + KP(s)}$$

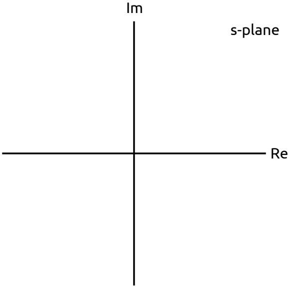


Application of feedback: stabilizing an unstable system

Consider a system described by the first order DE

$$\frac{dy(t)}{dt} - ay(t) = bx(t)$$

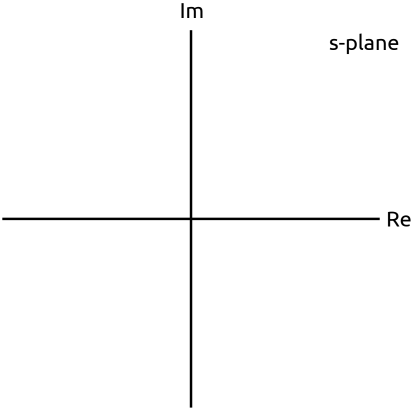
Exercise: compute the system function and draw the ROC. Under what conditions is it stable?



$$H(s) = \frac{b}{s-a}$$

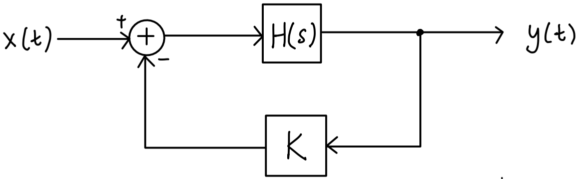
Application of feedback: stabilizing an unstable system

Suppose we have this setup ($a > 0$):

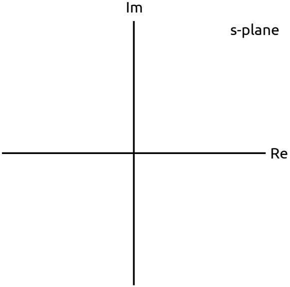


How can we make it stable?

Show that the following system will move the pole (under certain conditions on K):



$$
\begin{aligned}
Q(s) &= \frac{H(s)}{1 + KH(s)} \\
&= \frac{b}{(s-a)(1 + K\frac{b}{s-a})} \\
&= \frac{b}{s - a + Kb}
\end{aligned}
$$



Called a *proportional feedback system* since feeding back in a rescaled version of the output.

The z-transform

CT

Fourier series coefficients

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier transform (spectrum)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

DT

Fourier series coefficients

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi n}{N}}$$

Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Consider a DT complex exponential signal

$$x[n] = e^{j\omega n} = z^n$$

If we put this in a system with impulse response $h[n]$, obtain

$$y[n] = h[n] * x[n] = H(z)x[n]$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- $z = e^{j\omega}$: discrete-time Fourier transform
- $z = re^{j\omega}$: z-transform

The z-transform

For a general signal $x[n]$,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Just like in CT, this can be expressed with a DTFT involving $x[n]$:

$$\begin{aligned} X(re^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n} \\ &= \mathcal{F}(x[n]r^{-n}) \end{aligned}$$

The z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Exercise: compute the z-transform of

$$x[n] = a^n u[n]$$

For what values of z does it converge?

Exercise: compute the z-transform of

$$x[n] = a^n u[n]$$

For what values of z does it converge?

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \\ &= \frac{1}{1 - az^{-1}} \end{aligned}$$

Must be the case that $|az^{-1}| < 1$, or $|z| > |a|$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Exercise: compute the z-transform of

$$x[n] = -a^n u[-n-1]$$

For what values of z does it converge?

Exercise: compute the z-transform of

$$x[n] = -a^n u[-n-1]$$

For what values of z does it converge?

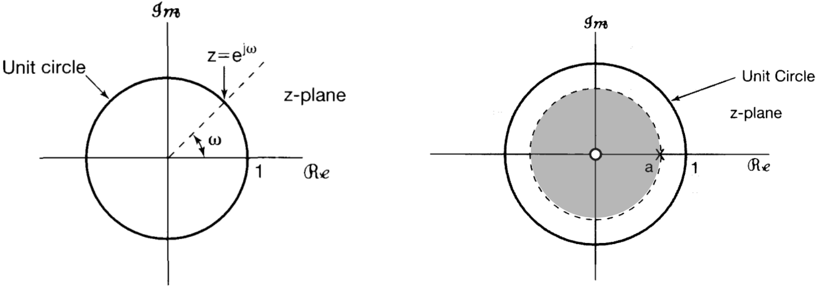
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \end{aligned}$$

Must have $|a^{-1}z| < 1$, or $|z| < a$. Then can write

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

Regions of convergence

To represent the ROC of the z-transform, we will use the z-plane and pole-zero plots:



Unit circle $z = e^{j\omega}$ ($|z| = 1$) corresponds to the DTFT case (like the vertical axis $s = j\omega$ for CT).

Image credit: Oppenheim 10.1

Exercise: compute the z-transform for

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

and sketch the pole-zero plot of its ROC.

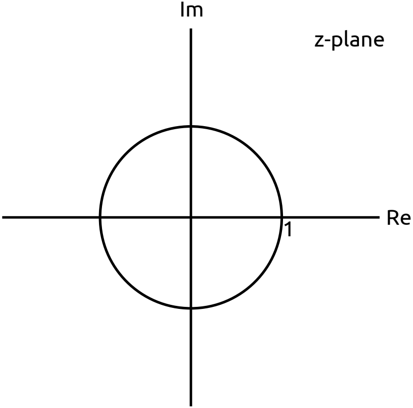
Regions of convergence

Exercise: compute the z-transform for

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

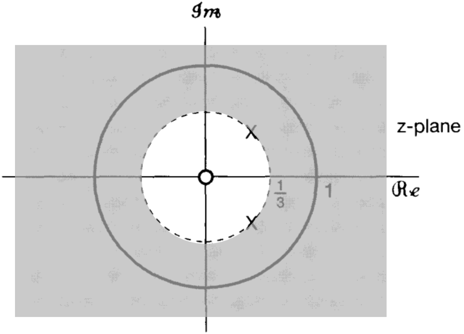
and sketch the pole-zero plot of its ROC.

$$\begin{aligned}
X(z) &= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} \\
&= \frac{7-\frac{7}{2}z^{-1}-6+\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}
\end{aligned}$$



ROC of the z-transform has many properties:

- if ROC doesn't contain unit circle, DTFT doesn't converge
- it is a ring in the z-plane centred around origin (for $z = re^{j\omega}$, does not depend on ω, only r)
- it does not contain any poles



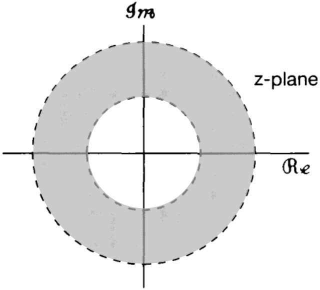


Image credit: Oppenheim 11.1, 11.2

If a signal $x[n]$ is of finite duration, its ROC is the entire z-plane *except possibly* $z = 0$ *and/or* $z = \infty$.

Exercise: compute the z-transform and ROC of

1. $z[n] = \delta[n]$
2. $z[n] = \delta[n-1]$
3. $z[n] = \delta[n+1]$

Regions of convergence

If a signal $x[n]$ is of finite duration, its ROC is the entire z-plane *except possibly* $z = 0$ *and/or* $z = \infty$.

Exercise: compute the z-transform and ROC of

1. $z[n] = \delta[n]$
2. $z[n] = \delta[n-1]$
3. $z[n] = \delta[n+1]$

Solution:

$$\begin{aligned}
\delta[n]: \quad X(z) &= \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1 \\
\delta[n-1]: \quad X(z) &= \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-n} \\
\delta[n+1]: \quad X(z) &= \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z
\end{aligned}$$

Regions of convergence

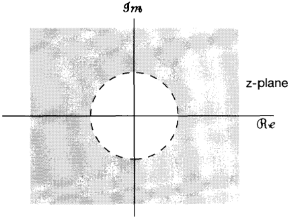
Right-sided signal: $X(z)$ has the form

$$X(z) = \sum_{n=N_1}^{\infty} x[n] z^{-n}$$

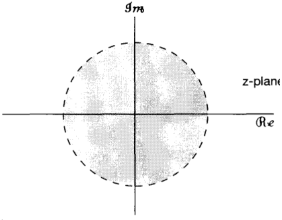
This may or may not include ∞ depending on the structure of the signal (in particular, if $N_1 < 0$, terms will become unbounded).

If $|z| = r_0$ is in the ROC for right-sided signal, then so are all *finite* z where $|z| > r_0$.

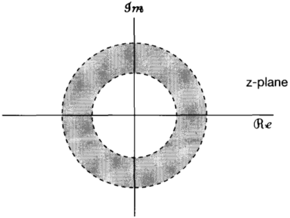
Similar argument for left-sided signals and the zero point.



Right-sided



Left-sided



Two-sided

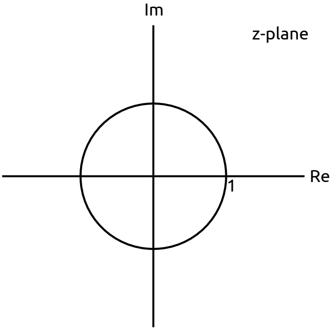
Image credit: Oppenheim 10.2

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

Draw the pole-zero plot and determine the possible ROCs.

Hint: this function has 2 zeros; express it in a different way to find them.

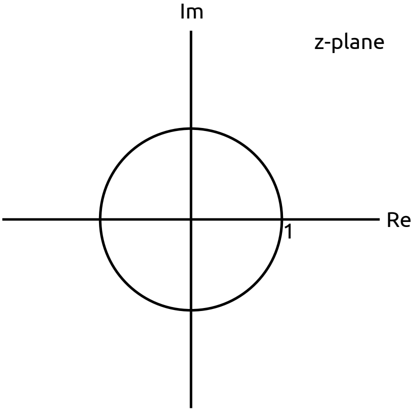


Regions of convergence

Exercise: how many signals could have produced the z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

Solution:



When the z-transform can be expressed as a rational function, we can compute the inverse using partial fractions. We still need the ROC to help us.

Exercise: compute the inverse z-transform of

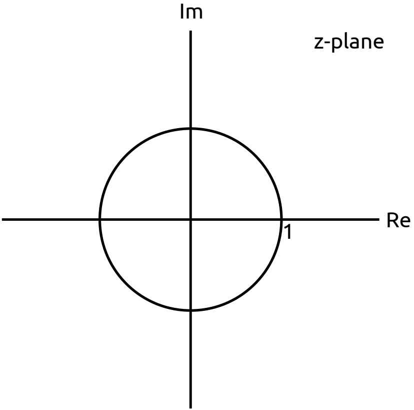
$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

if ROC is specified to be $|z| > 2$.

Exercise: compute the inverse z-transform of

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

if ROC is specified to be $|z| > 2$.



Use partial fractions:

$$\begin{aligned} X(z) &= \frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B}{1 - 2z^{-1}} \\ &= \frac{-1/5}{1 - \frac{1}{3}z^{-1}} + \frac{2/5}{1 - 2z^{-1}} \end{aligned}$$

From ROC, signal is right-sided:

$$x[n] = -\frac{1}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{2}{5}2^n u[n]$$

Take a closer look at the structure of $X(z)$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

This is a *power series in z*. If we can do the expansion, we can recover $x[n]$ from the coefficients.

Exercise 1: what is the inverse z-transform of

$$X(z) = 3z^2 - 1 + 2z^{-3}, \quad 0 < |z| < \infty$$

Solution:

$$x[n] = 3\delta[n+2] - \delta[n] + 2\delta[n-3]$$

Particularly helpful for non-linear cases.

Exercise 2 (Oppenheim 10.63a): what is the inverse z-transform of

$$X(z) = \log(1 - 2z), \quad |z| < \frac{1}{2}$$

Hint:

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \ |w| < 1$$

Solution:

$$\begin{aligned} X(z) &= \log(1 - 2z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i} = -\sum_{n=-\infty}^{-1} \frac{2^{-n}}{-n} z^{-n} \\ x[n] &= \begin{cases} \frac{2^{-n}}{n} & n \leq -1 \\ 0 & n > -1 \end{cases} = \frac{2^{-n}}{n} u[-n-1] \end{aligned}$$

Today

Learning outcomes:

- compute system functions for feedback systems
- use the Laplace transform to design an inverse system
- define the z-transform and compute it and its ROC for basic signals

Oppenheim practice problems: 9.48, 11.1-11.4, 10.1-10.8, 10.21-10.23, 10.26

For next time

Content:

- more properties of z-transforms
- systems described by difference equations
- z-transforms and feedback system analysis

Action items:

1. Assignment 6 due tonight at 23:59
2. Assignment 7 due Tuesday at 23:59

Recommended reading:

- From this class: Oppenheim 9.7, 11.0-11.2, 10.1-10.3
- For next class: 10.5-10.7, 11.2