ELEC 221 Lecture 05
CT Fourier series

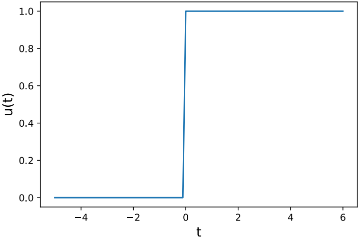
Thursday 19 September 2024

TA office hour 5pm

- Assignment 1 due tonight at 23:59
- Quiz 3 Tuesday
- Assignment 2 released next week
- Tutorial Assignment 2 on Monday

Last time

We defined the CT unit step, $u(t)$



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

And the CT unit impulse function, $\delta(t)$:

$$\delta(t) = \frac{du(t)}{dt}$$

Last time

We introduced the CT convolution integral,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

If we know the impulse response of an LTI system, we can determine its response to any other signal.

Last time

We saw how properties of the impulse response are related to the system's properties as a whole. $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

(**Memory**) For a system to be memoryless,

$$h(t) = K\,\delta(t) \qquad h[n] = K\,\delta[n]$$

(**Invertibility**) If a system with impulse response $h(t)$ is invertible, there exists an *inverse system* with impulse response $h_i(t)$ s.t.

$$x(t) \longrightarrow \underset{h_1(t)}{\boxed{S1}} \longrightarrow \underset{h_2(t)}{\boxed{S2}} \longrightarrow x(t) \qquad h(t) * h_i(t) = \delta(t)$$

$$\Rightarrow \quad x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow x(t) \qquad h[n] * h_i[n] = \delta[n]$$

"identity" = unit impulse

(Analogous for DT case)

Last time

(**Stability**) For a system to be stable, the impulse response $h(t)$ must be *absolutely integrable*, i.e.,

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \text{finite} \qquad \left(\sum_{k=-\infty}^{\infty} |h[k]| \text{ finite} \right)$$

is finite.

(**Causality**) The impulse response $h(t)$ of a causal system must have the property

$$h(t) = 0 \text{ for } t < 0$$

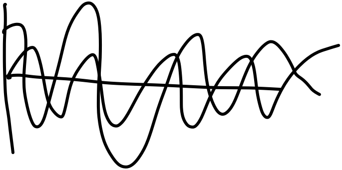
$$h[n] = 0 \text{ for } n < 0$$

(Analogous for DT case)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

$k < 0 \Rightarrow x[n-k] \Rightarrow$ future signal values

Today



Learning outcomes:

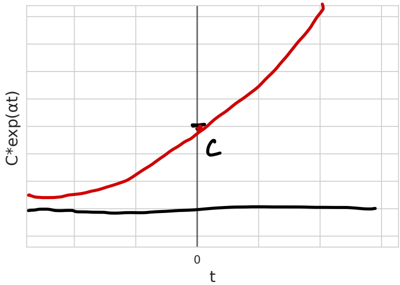
- Use the convolution integral to show that complex exponentials are eigenfunctions of LTI systems
- Define and compute the system function (frequency response) of an LTI system
- Express a periodic CT signal as a Fourier series, and compute its Fourier coefficients

Most general form: $x(t) = C e^{\alpha t}$

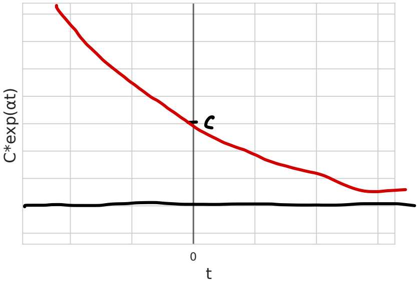
where α can be real or complex.

Case: both C and α are real-valued.

$\alpha > 0$



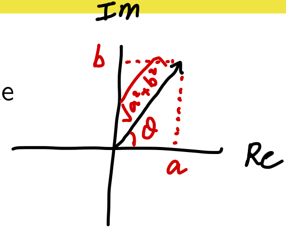
$\alpha < 0$



Case: α is complex. Most generally, we can write

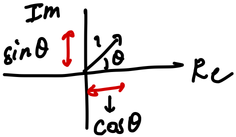
$$\alpha = a + bj \qquad \alpha = |\alpha| e^{j\cdot\theta}$$

$$|\alpha| = \sqrt{a^2+b^2}$$



Euler's relation allows us to write

$$e^{j\theta} = \cos\theta + j\sin\theta$$



As a result,

$$\begin{array}{rl} & e^{j\theta} = \cos\theta + j\sin\theta \\ + & e^{-j\theta} = \cos\theta - j\sin\theta \\ \hline & e^{j\theta} + e^{-j\theta} = 2\cos\theta \end{array}$$

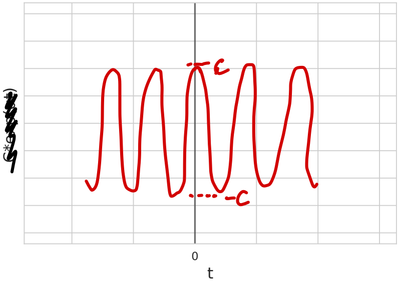
$$\rightarrow \quad \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

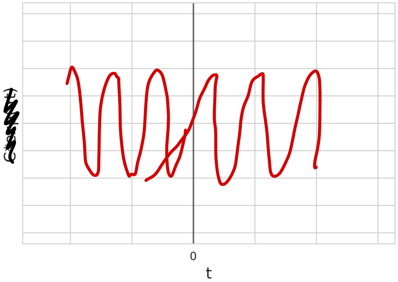
Case: $\alpha = j\omega$. Then,

$$x(t) = Ce^{j\omega t} = C\cos(\omega t) + jC\sin(\omega t)$$

$Re[x(t)] = C\cos(\omega t)$



$Im[x(t)] = C\sin(\omega t)$



ω : frequency

$x(t) = Ce^{j\omega t}$ is periodic:

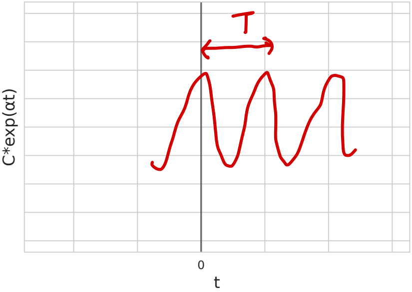
Suppose period = T

$$x(t+T) = Ce^{j\omega(t+T)} = Ce^{j\omega t} e^{j\omega T} = x(t)$$

$Ce^{j\omega t}$: $x(t)$

$$e^{j\omega T} = 1 \Rightarrow \omega T = 2\pi$$

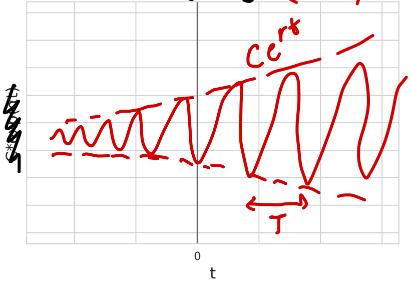
$$T = \frac{2\pi}{\omega}$$



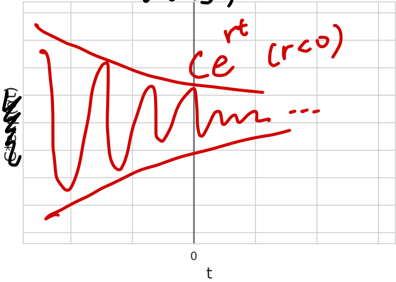
Case: $\alpha = r + j\omega$. Then,

$$x(t) = Ce^{(r+j\omega)t} = Ce^{rt}e^{j\omega t}$$
$$= Ce^{rt}\cos(\omega t) + jCe^{rt}\sin(\omega t)$$

$\mathrm{Re}[x(t)]$ $(r>0)$



$\mathrm{Re}[x(t)]$, $r<0$



Recall the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

What happens when $x(t)$ is a complex exponential signal?

Write $x(t) = e^{st}$. Then:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)}\, h(\tau)\, d\tau$$

$$= \underbrace{e^{st}}_{x(t)} \underbrace{\int_{-\infty}^{\infty} e^{-s\tau}\, h(\tau)\, d\tau}_{H(s)}$$

$$= x(t) \cdot H(s)$$

$\downarrow$

$x(t)$ is an eigenfunction of the system!

To summarize:

$$x(t) = e^{st} \rightarrow y(t) = H(s) \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is the **system function**.

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

Recall that for **LTI** systems,

$$x(t) = \sum_i a_i x_i(t) \longrightarrow y(t) = \sum_i a_i y_i(t)$$

If all $x_i(t)$ are complex exponential signals, and

$$x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$$

then

$$x(t) = \sum_k c_k e^{s_k t} \longrightarrow y(t) = \sum_k c_k \cdot H(s_k) e^{s_k t}$$

The response is a superposition **of those same signals**, scaled by the system function.

Consider the limited set of signals[1]: $s = j\omega$

$$x(t) = e^{st} = e^{j\omega t}$$

$x(t)$ has frequency ω and period $T = 2\pi/\omega$.

When such signals are input into an LTI system, the system function, $H(s) = H(j\omega)$, is called the *frequency response*.

[1]We will see the general case at the end of the course.

Suppose we have access to a system with impulse reponse

$$h(t) = \frac{1}{2}\delta(t-1)$$

1. What is the frequency response, $H(j\omega)$, of the system?

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)\, d\tau \quad = \frac{1}{2} e^{-j\omega}$$

2. What is the output of the system given input

$$x(t) = 2e^{j\omega t} - 3e^{5j\omega t}$$

3. What is the output of the system given input

$$x(t) = \frac{2}{3}\cos(3\omega t)$$

Exercise: frequency response

Suppose we have access to a system with impulse reponse

$$h(t) = \frac{1}{2}\,\delta(t-1)$$

1. What is the frequency response, $H(j\omega)$, of the system?

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} \cdot \frac{1}{2}\, \delta(\tau-1)\, d\tau$$

$$= \frac{1}{2} e^{-j\omega}$$

Suppose we have access to a system with impulse reponse

$$h(t) = \frac{1}{2}\delta(t-1) \qquad H(j\omega) = \underline{\frac{1}{2}}e^{-j\omega}$$

2. What is the output of the system given input

$$x(t) = 2e^{2j\omega t} - 3e^{5j\omega t}$$

$$y(t) = 2 \cdot H(2j\omega)e^{2j\omega t} - 3 \cdot H(5j\omega)e^{5j\omega t}$$

$$= 2 \cdot \frac{1}{2}e^{-2j\omega} \cdot e^{2j\omega t} - 3 \cdot \frac{1}{2}e^{-5j\omega}e^{5j\omega t}$$

$$= e^{2j\omega(t-1)} - \frac{3}{2}e^{5j\omega(t-1)}$$

Suppose we have access to a system with impulse reponse

3. What is the output of the system given input

$$x(t) = \frac{2}{3} \cos(3\omega t)$$

$\downarrow$

try it!

How can we express arbitrary signals using complex exponentials?

$$x(t) = \sum_k c_k e^{j\omega_k t} \longrightarrow y(t) = \sum_k c_k H(j\omega_k) e^{j\omega_k t}$$

Consider

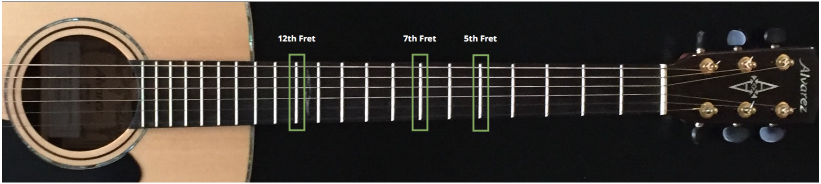
$$x(t) = e^{j\omega t}$$

It also has an infinite number of cousins called *harmonics*.

$$x_k(t) = e^{jk\omega t} = e^{jk\cdot\frac{2\pi}{T}t}, \quad k = 0, \pm 1, \pm 2 \ldots$$

ω is the **fundamental frequency**.

(Yes, these harmonics.)



The Fourier series

We can create a superposition of all harmonics,

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

This signal is also periodic with period $T = 2\pi/\omega$.

Given $x(t)$ with period T[2], we can write it as a **Fourier series**.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega k t}$$

The c_k (complex numbers) are its **Fourier coefficients**.

[2]There are some additional conditions; we will cover this next class

Usually dealing with $x(t)$ that is always *real*.

complex conjugate

$a+bj$

$\updownarrow$

$a-bj$

$$x(t) = x^*(t)$$

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega t}$$

This means $c_{-k}^* = c_k$

We can leverage this to express $x(t)$ in a different way.

Apply Euler's relation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega kt} = \sum_{k=-\infty}^{\infty} c_k \left(\cos(k\omega t) + j\sin(k\omega t)\right)$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(k\omega t) \cdot$$

$$+ \sum_{k=-\infty}^{\infty} j \cdot c_k \sin(k\omega t)$$

$$c_k = c_{-k}^{*}$$

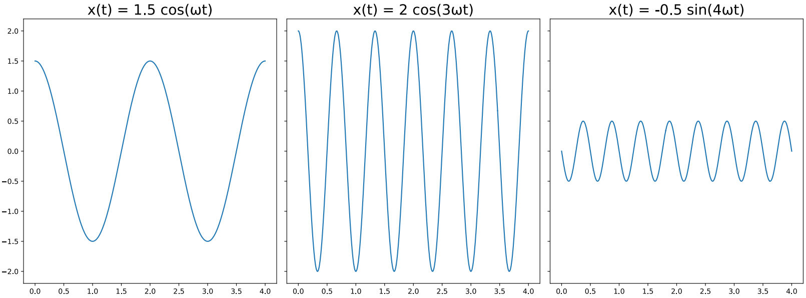
With a bit of work (i.e., in assignment 1!), you can find

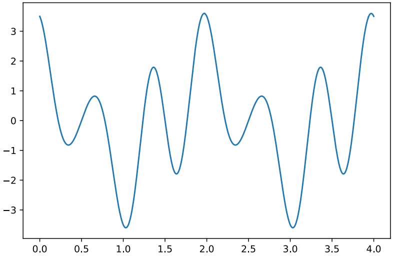
$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega t) + b_k \sin(k\omega t)\right)$$

where a_0, a_k, b_k are expressed in terms of the c_k.

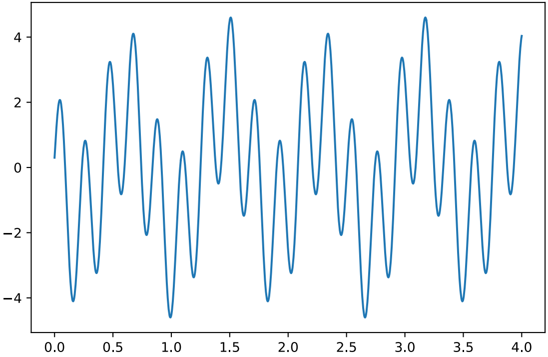
Note: sometimes a_0 is written $a_0/2$ for convenience

Example





What if we are given a signal like this:



we will start here on Tuesday!

Can we determine the c_k?

Recap

Today's learning outcomes were:

- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems
- Define and compute the system function (frequency response) of an LTI system
- Express a periodic CT signal as a Fourier series, and ~~compute its Fourier coefficients~~ next time!

For next time

Content:

- Dirichlet conditions and the Gibbs phenomenon
- Properties of Fourier series and Fourier coefficients

Action items:

1. Assignment 1 is due tonight
2. Tuesday quiz on this week's material

Recommended reading:

- From today's class: Oppenheim 1.3, 3.0-3.3
- Suggested problems: 1.6, 1.8, 1.10, 3.1, 3.3, 3.4, 3.13, 3.17, 3.22a,c
- For next class: Oppenheim 3.4-3.5