ELEC 221 Lecture 09
DT Fourier series; filters

Thursday 03 October 2024

Announcements

- Assignment 2 due Saturday 23:59 (final question removed, deferred to A3)
- Midterm 1 on Tuesday (bring your student ID and writing implements)

Last time

We explored periodic DT complex exponential signals:

We found that these signals behave differently than CT signals...

Difference 1: we only need to consider ω in the range $[0, 2\pi)$.

Difference 2: there are additional criteria for periodicity.

Example: $x[n] = \sin(5\pi n/7)$ is periodic.

- In CT, period of $x(t) = \sin(5\pi t/7)$ is
- In DT, period of $x[n] = \sin(5\pi n/7)$ is

Example: $x[n] = \sin(5n/7)$ is NOT periodic in DT.

Last time

Difference 3: there are only finitely many harmonics.

Last time

We found DT complex exponential signals are also eigenfunctions of LTI systems.

We need a Fourier series representation of DT signals:

Today

Learning outcomes:

- Evaluate Fourier series coefficients of DT signals
- Define filter systems through the frequency response
- Distinguish between finite impulse response and infinite impulse response filters in DT

Leverage the following identity about complex numbers:

We will multiply on both sides, and sum.

DT Fourier coefficients

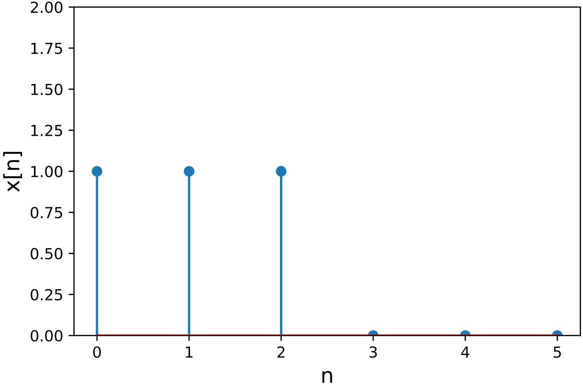
DT Fourier coefficients

DT Fourier synthesis equation

DT Fourier analysis equation

Exercise: the DT square wave

Compute the Fourier coefficients of this signal:



Exercise: the DT square wave

Properties of DT Fourier coefficients

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Exercise: the DT square wave

Let's try the same thing as we did in CT:

- shift the signal left by 1
- speed it up by 2

Where do we go from here?

We've showed a couple important things so far.

Signals can be expressed in terms of weighted, shifted impulses.

Where do we go from here?

If we know what an LTI system does to a unit impulse (the impulse response $h(t)$ or $h[n]$), we can learn what it does to any signal.

This was the convolution integral and sum:

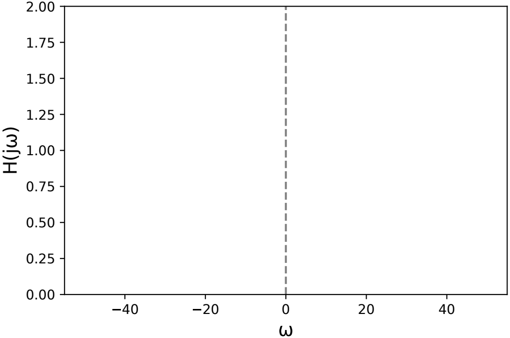
Where do we go from here?

Complex exponential signals are eigenfunctions of LTI systems:

$H(j\omega)$ in CT, and $H(e^{j\omega})$ in DT, are the **frequency response** of the system (more generally, system functions).

Through careful choice of $H(j\omega)$ or $H(e^{j\omega})$, we can change the behaviour of a system.

What does a system with the following frequency response do?



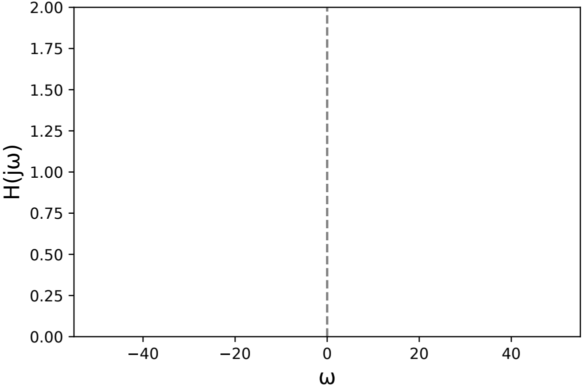
Filters are LTI systems that can be used to separate out, combine, or modify the components of a signal at specific frequencies.

Two key types:

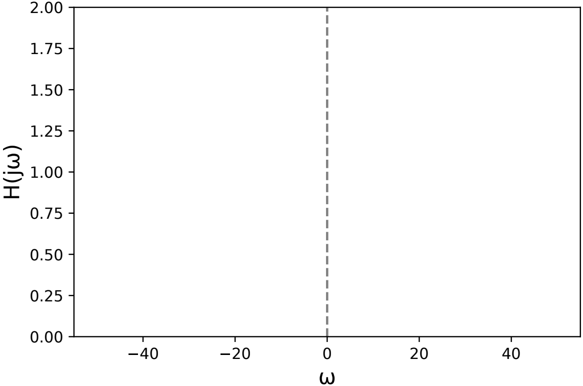
- **Frequency-shaping**: change the amplitudes of parts of a signal at specified frequencies
- **Frequency-selective**: eliminate or attentuate parts of a signal at specified frequencies

CT frequency-selective filters

We can also consider an ideal **highpass filter**:



Or an ideal **bandpass filter**:



Lowpass filters in practice

Some filters made of physical components are described by differential equations.

Example: a lowpass filter using a resistor and a capacitor.

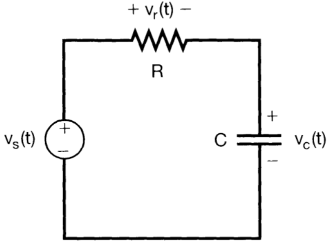
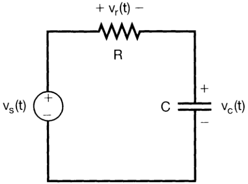


Image credit: Oppenheim chapter 3.10.

Exercise: lowpass filters in practice

What is the voltage across the capacitor if $v_s = e^{j\omega t}$?



Can derive two expressions for the current, using the resistor and capacitor respectively:

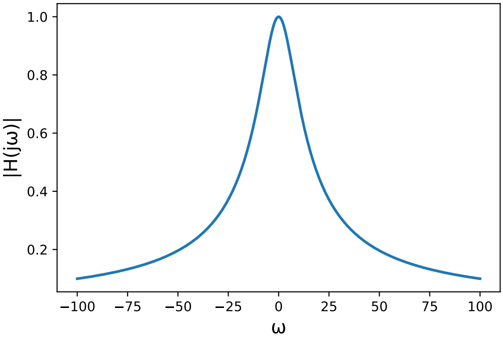
Image credit: Oppenheim chapter 3.10.

Exercise: lowpass filters in practice

Put these together to form a differential equation:

Exercise: lowpass filters in practice

Results in the following frequency response (setting $RC = 0.1$):



Adjusting the value of RC controls the frequency response.
Increasing RC cuts off more frequencies.

DT filters

Recall that in DT, the frequency increases up until $\omega = \pi$, then decreases as it approaches 2π.

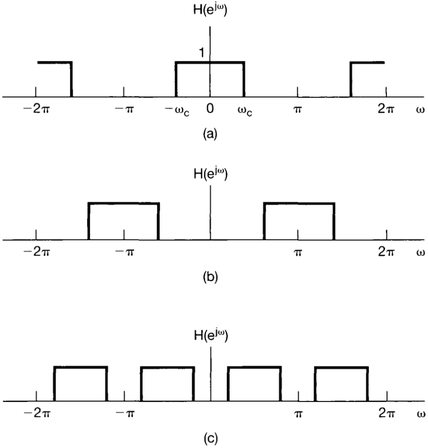
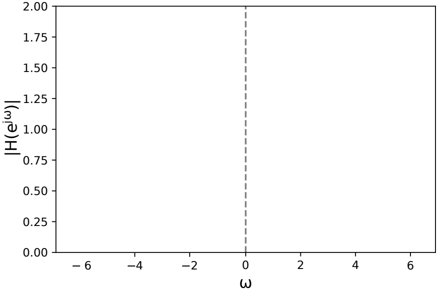


Image credit: Oppenheim chapter 3.9.

DT filters

Example: consider the DT filter with frequency response

Sketch below:



DT filters

There are two major categories of DT filters:

1. Infinite impulse response (IIR)
2. Finite impulse response (FIR)

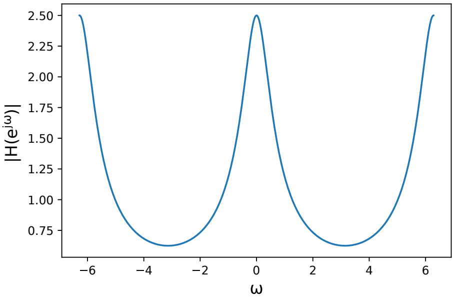
DT filters (IIR)

Example: we can generate a low- or high-pass filter using a system described by a first-order difference equation.

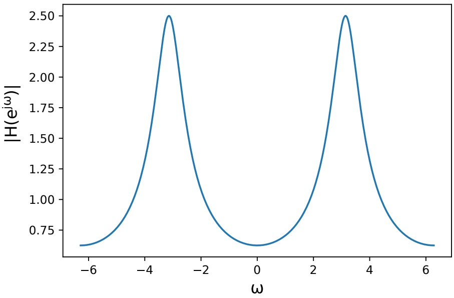
This system has infinite impulse repsonse:

DT filters (IIR)

Case: $a > 0$



Case: $a < 0$



DT filters (FIR)

An example of a DT filter with finite impulse response is the moving average with window of size M:

Impulse response: set $x[n] = \delta[n]$

This is clearly finite (as a result: these filters are stable systems).

Today's learning outcomes were:

- Evaluate Fourier series coefficients of DT signals
- Define filter systems through the frequency response
- Distinguish between finite impulse response and infinite impulse response filters in DT

For next time

Action items:

1. Assignment 2 due Saturday 23:59
2. Study for Midterm 1
3. Suggest tutorial topics on Piazza

Recommended reading:

- From today's class: Oppenheim 3.6-3.12
- Suggested problems: 3.2, 3.10-3.17, 3.27-3.31, 3.39