ELEC 221 Lecture 02
LTI systems, DT impulse response and the convolution sum

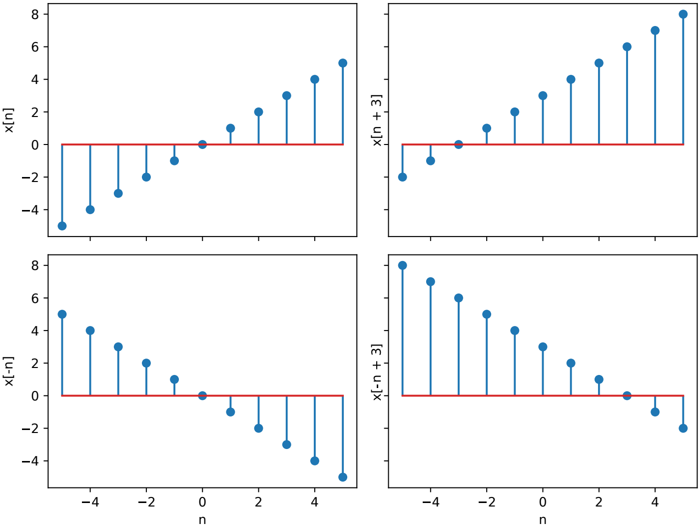
Tuesday 10 September 2024

Announcements

- Quiz 1 today
- Tutorial assignment 1 Monday 16 Sept 23:59
- Assignment 1 due Thursday 19 Sept 23:59

Last time

We saw continuous-time and discrete-time signals, and applied some simple transformations to them.



Last time

We introduced systems, which respond to signals, transform them, and output new signals.

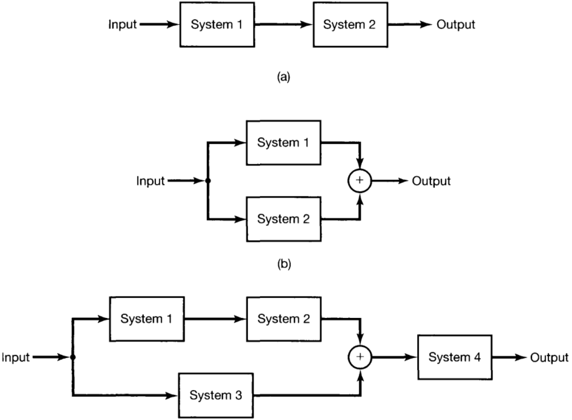


Image credits: Signals and Systems 2nd ed., Oppenheim

Last time

We explored some properties of systems:

1. Memory
2. Invertibility
3. Causality
4. Stability
5. Linearity
6. Time invariance

Quiz time

Available at `us.prairielearn.com`

- Work individually
- You may use lecture slides, notes, or the textbook.
- Please do not search online for answers

Once you start the quiz you will have 10 minutes to complete it.

You have one attempt per question.

Questions will re-enabled later for practice with random variants.

Today

Learning outcomes:

- Define what it means for a system to be LTI (linear, time-invariant)
- Define the DT unit impulse and unit step functions
- Define the convolution sum and use it to compute the output of a system

Consider a function f such that $y = f(x)$.

If f is linear, what key properties does it have?

$$f(ax) = af(x)$$ "homogeneity"

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$ "additivity"

Properties of systems: linearity

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t)$$

$(x[n])$

A **linear** system $x(t) \to y(t)$ sends

$$a\,x(t) \longrightarrow \boxed{S} \longrightarrow a\,y(t)$$

$$x_1(t) \longrightarrow \boxed{S} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{S} \longrightarrow y_2(t)$$

$$x_1(t) + x_2(t) \longrightarrow \boxed{S} \longrightarrow y_1(t) + y_2(t)$$

Thus, a linear system sends

$$a\,x_1(t) + b\,x_2(t) \xrightarrow{S} a\,y_1(t) + b\,y_2(t)$$

for arbitrary a, b (which may be complex).

Example: linearity

Is the following system linear? YES

$$x(t) \rightarrow y(t) = x(t+1) - x(t-1)$$

$x'(t) = a x(t) \Rightarrow y'(t) = x'(t+1) - x'(t-1)$

$= a x(t+1) - a x(t-1)$

$= a\left(x(t+1) - x(t-1)\right)$

$= a y(t)$

$x'(t) = x_1(t) + x_2(t)$

$y'(t) = x'(t+1) - x'(t-1)$

$= x_1(t+1) + x_2(t+1)$

$\quad - \left(x_1(t-1) + x_2(t-1)\right)$

$y_1(t)$ $y_2(t)$

$= y_1(t) + y_2(t)$

Exercise: linearity

not this kind of linear

$y = x + 1$

Is the following system linear? NO

$$x[n] \to y[n] = x[n] + 1$$

Homogeneity broke

$x'[n] = a x[n] \to y'[n] = x'[n] + 1$

$= a x[n] + 1$

$\neq a(x[n] + 1)$

ODM check "0 in 0 out"

$\hookrightarrow$ if system is linear, input $x[n] = 0$ (all 0s signal) leads to $y[n] = 0$ (all 0s output)

Properties of systems: time invariance

A system is **time invariant** if a time-shifted input leads to an output time-shifted by the same amount.

$$x(t) \longrightarrow y(t)$$
$$x(t-t_0) \longrightarrow y(t-t_0)$$

Intuition: behaviour of the system is fixed over time.

Example: time invariance

$x(t) \to y(t)$

$x(t-t_0) \to y(t-t_0)$

Is this system time-invariant? NO

$$y(t) = \cos(3t)x(t)$$

$x'(t) = x(t-t_0)$

$y'(t) = \cos(3t)x'(t) = \cos(3t)x(t-t_0)$

$\neq$

$y(t-t_0) = \cos[3(t-t_0)]x(t-t_0)$

Exercise: time invariance

$x(t) \to y(t)$

$x(t-t_0) \to y(t-t_0)$

Is this system time-invariant? YES

$$y(t) = x(t+1) - x(t-1)$$

expect if is TI

$y(t-t_0) = x(t-t_0+1) - x(t-t_0-1)$

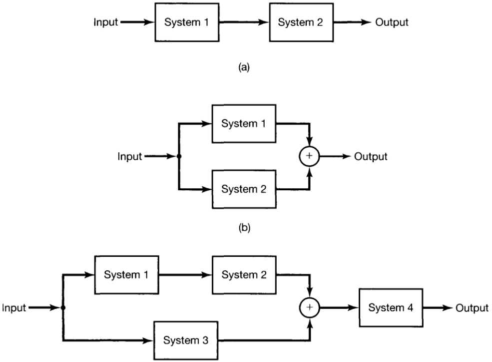
$x'(t) = x(t-t_0)$

$\Rightarrow y'(t) = x'(t+1) - x'(t-1)$

$= x(t-t_0+1) - x(t-t_0-1)$

LTI systems

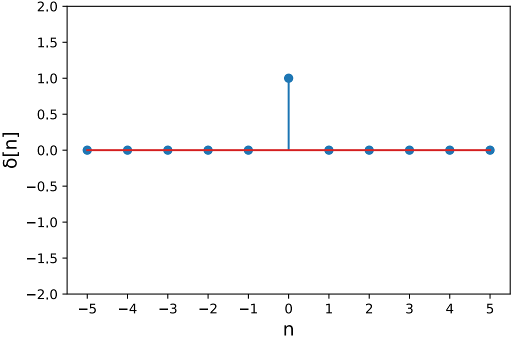
We are most interested in systems that are both **linear** and **time-invariant**, i.e., LTI systems.



How can we characterize the behaviour of LTI systems?

Image credits: Signals and Systems 2nd ed., Oppenheim

The DT unit impulse



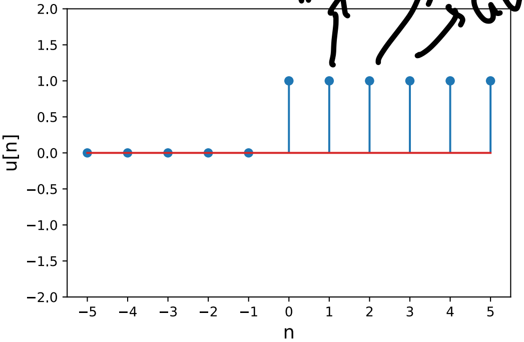
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

The DT unit step

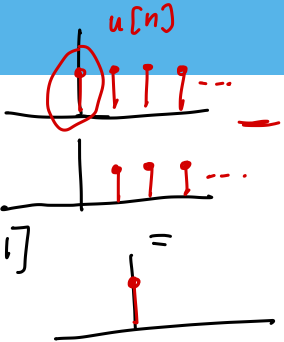
$\sum_{m=-\infty}^{1} \delta[m]$

$u[2] = 1$

$u[3] = 1$



$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

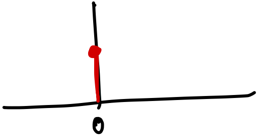


We can express these in terms of each other:

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

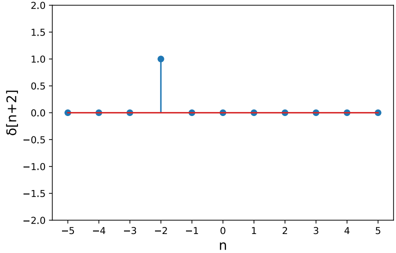
$$u[2] = \cancel{\delta[-\infty]} + \ldots + \cancel{\delta[-2]} + \cancel{\delta[-1]} + \delta[0] + \cancel{\delta[1]} + \cancel{\delta[2]}$$



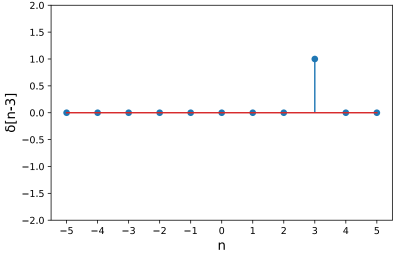
The unit impulse is an important tool for characterizing the behaviour of systems.

By considering unit impulses time-shifted as various points, we can pick out, or *sift* out specific parts of the signal.

$\delta[n+2]$



$\delta[n-3]$

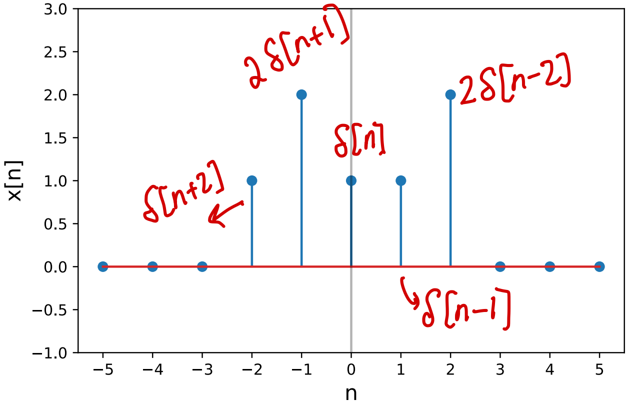


The sifting property

$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \cdots \delta[-2] + \delta[-1] + \delta[0] + \cdots + \delta[n]$

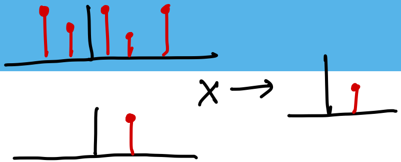
step

The value of a DT at every point is a *weighted, shifted impulse*.



$x[n] = \delta[n+2] + 2\delta[n+1] + \delta[n] + \delta[n-1] + 2\delta[n-2]$

The unit impulse as a sampler



Multiplying by a shifted impulse "samples" the signal at that point:

$$x[n]\,\delta[n-k] = x[k]$$

Any signal can be written as a **superposition of weighted impulses.**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Given a signal

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

how does an LTI system respond to it?

For a **linear** system $z[n] \to w[n]$,

$$a_1 z_1[n] + a_2 z_2[n] \to a_1 w_1[n] + a_2 w_2[n]$$

More generally,

$$\sum_k a_k z_k[n] \to \sum_k a_k w_k[n]$$

The impulse response

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{weight}} \underbrace{\delta[n-k]}_{\text{signal}}$$

Suppose the system sends $\delta[n-k] \to h_k[n]$. Then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h_k[n]$$

$h_k[n]$ is called the **impulse response**.

We will start from here on Thursday.

Real-world example: nerve conduction study

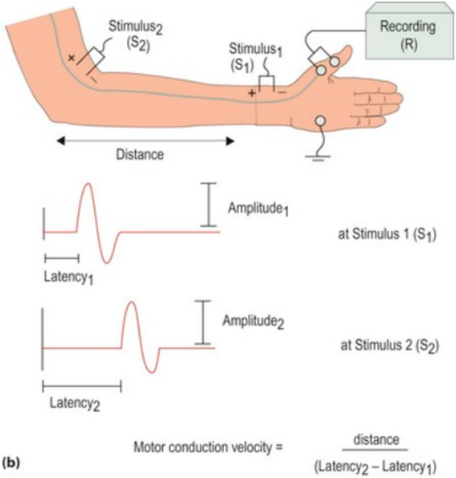


Image source: `https://neupsykey.com/nerve-conduction-studies-and-electromyography/`

The impulse response and time-invariance

What if the system is also time invariant?

Then

The convolution sum

If we know how a **linear** system responds to the unit impulse, we can learn how it responds to **any other signal**!

This is the **convolution sum**. We are "convolving" the sequences $x[n]$ and $h[n]$.

Exercise: impulse response

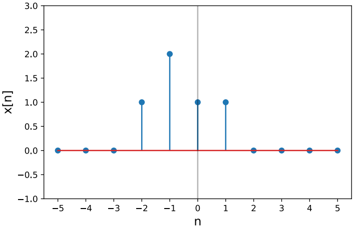
Consider an LTI system with input/output relationship

$$y[n] = 2x[n] + x[n-1]$$

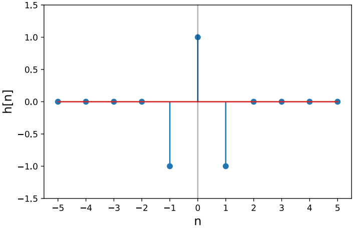
What is the impulse response of the system?

Example: convolution sum

Consider the signal

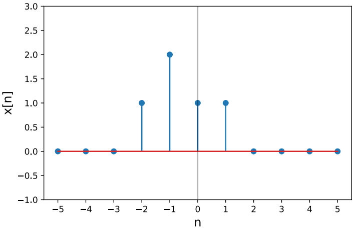


input to a system with impulse response



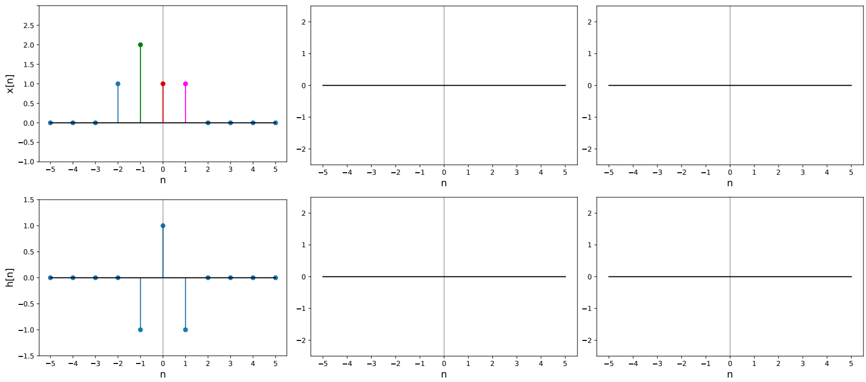
Example: convolution sum

To learn the system output, we must consider the contribution of each weighted impulse response:

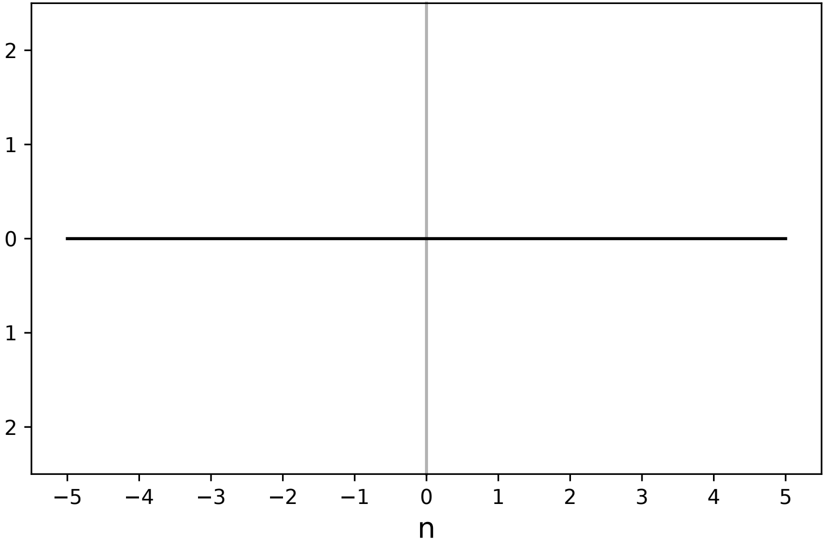


Only $x[k] \neq 0$ only for $k \in \{-2, -1, 0, 1\}$. So need to determine $x[k]h[n-k]$ for these cases, and sum them.

Example: convolution sum



Example: convolution sum



Properties of convolutions

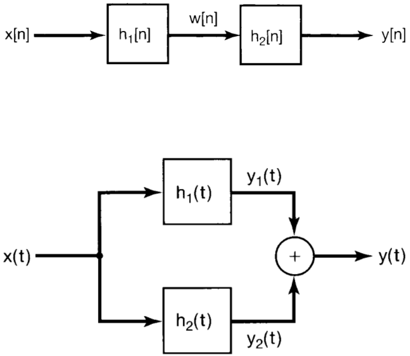


Image credits: Signals and Systems 2nd ed., Oppenheim

Convolution is:

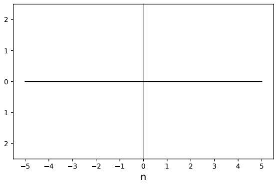
- Associative:
  $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Commutative:
  $x[n] * h[n] = h[n] * x[n]$
- Distributive:
  $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

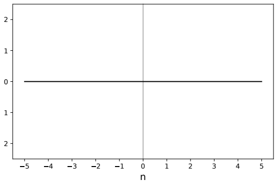
Consider an LTI system with impulse response

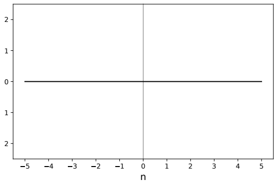
$$h[n] = 3\delta[n] + 2\delta[n+1]$$

What is output of the system if

$$x[n] = \left(\frac{2}{3}\right)^n u[n]$$







Example/exercise: convolution sum

What is output of the system

$$x[n] = \left(\frac{2}{3}\right)^n u[n], \quad h[n] = 3\delta[n] + 2\delta[n+1]$$

Example/exercise: convolution sum

What is output of the system

$$x[n] = \left(\frac{2}{3}\right)^n u[n], \quad h[n] = 3\delta[n] + 2\delta[n+1]$$

Recap

Today's learning outcomes were:

- Define what it means for a system to be LTI (linear, time-invariant)
- Define the DT unit impulse and unit step functions
- Define the convolution sum and use it to compute the output of a system

For next time

Content:

- Continuous-time unit impulse and step
- Convolution integral
- Characterizing systems with the impulse response

Action items:

1. Work on Tutorial Assignment 1
2. Work on Assignment 1

Recommended reading:

- from today's class: Oppenheim 1.6.5-6, 1.4, 2.1
- practice problems: 1.16-1.20, 2.1-2.7, 2.21
- for next class: Oppenheim 1.4, 2.2-2.3