ELEC 221 Lecture 22
The Laplace transform: properties and system analysis

Thursday 28 November 2024

Announcements

- Midterm 2 available for pickup at my office (some remaining MT1 as well)
- Quiz 10 Tuesday (last quiz)
- Final tutorial on Monday (problem solving - post suggestions on Piazza @226)
- Tutorial Assignment 5 due Monday 23:59
- Assignment 5 due Sunday 8 December 23:59
- Final exam details available late next week

Last time

We introduced the Laplace transform,

where $s = \sigma + j\omega$.

If $s = j\omega$, reduces to **Fourier transform**

$X(s)$ can exist in regions that $X(j\omega)$ does not (allows us to analyze more kinds of systems), but still doesn't exist everywhere.

We introduced the s-plane and pole-zero plots. We used them to plot the region of convergence (ROC) of $X(s)$.

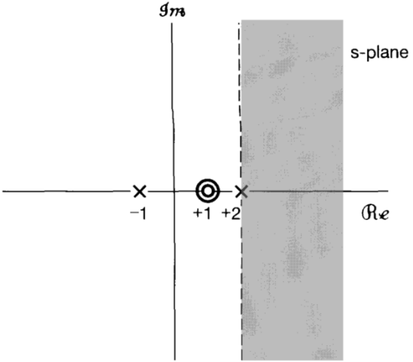
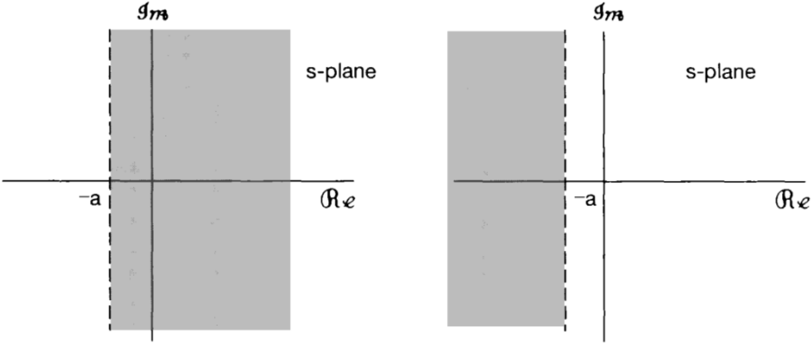


Image credit: Oppenheim 9.1

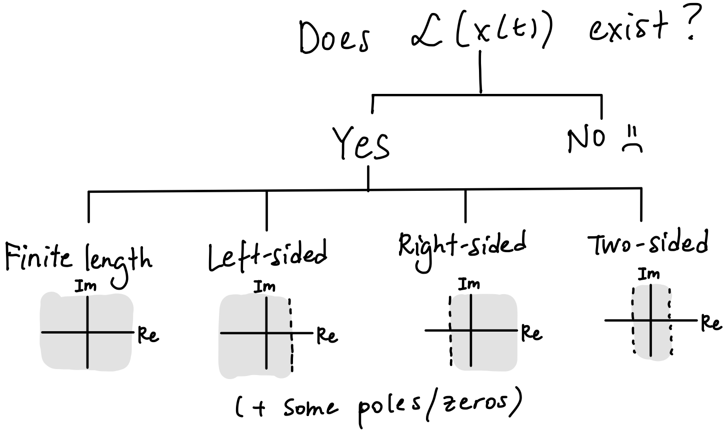
The ROC is essential for computing inverse Laplace transforms.



Both ROC associated to algebraic expression $X(s) = \frac{1}{s+a}$, but came from different signals.

Image credit: Oppenheim 9.1

We distinguished between types of signals and their ROCs.



Today

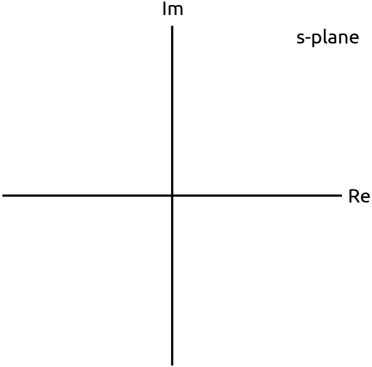
Learning outcomes:

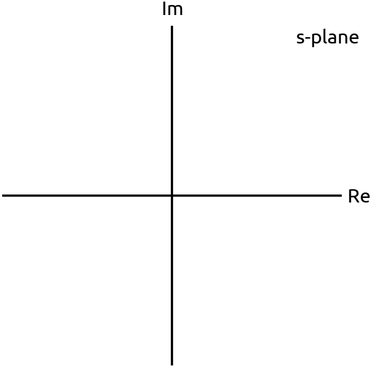
- apply key properties of the Laplace transform to its computation
- use the Laplace transform to determine whether a system is causal or stable
- compute the Laplace transform of systems described by constant-coefficient DEs

(Oppenheim 9.7) How many signals have a Laplace transform that may be expressed as

$$\frac{s-1}{(s+2)(s+3)(s^2+s+1)}$$

Hint: draw pole-zero plot and identify possible ROCs.





From this, we can invert:

Make a change of variables $ds = jd\omega$:

... we are not going to integrate this.

Suppose

where degree of denominator is higher than numerator.

To invert, we can use our handy identities, BUT the ROC matters.

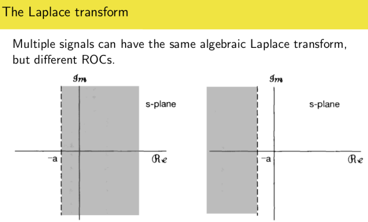


Image credit: Oppenheim 9.1

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s+2}{s^2+7s+12}, \quad -4 < \mathrm{Re}(s) < -3$$

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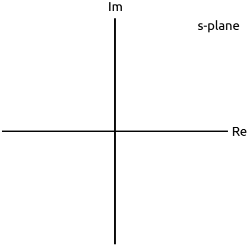
Draw the s-plane:



We've made use of many nice properties of the Fourier transform:

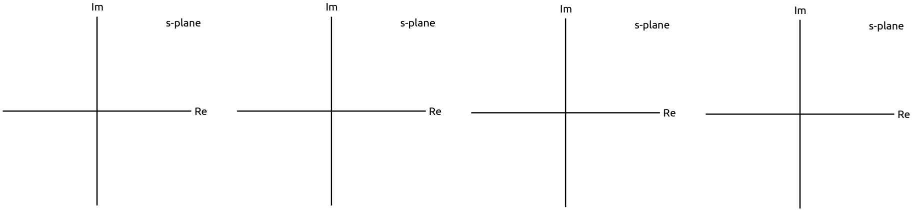
- linearity
- time shift/scale
- differentiation
- conjugation
- convolution

All have analogs with Laplace transform, but factor in the ROC.

Linearity

Example: $x(t) = e^{-b|t|}$.

Properties of the Laplace transform

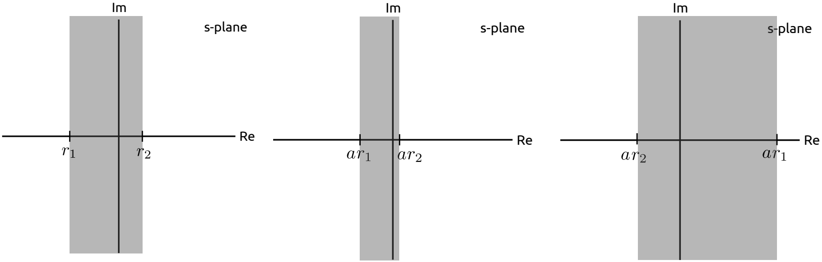


Properties of the Laplace transform

Time shifting.

Time scaling.

Time reversal.



$0 < a < 1$

$a < -1$

Properties of the Laplace transform

Exercise: what is the inverse Laplace transform of

$$X(s) = \frac{s}{s^2 + 9}, \quad \mathrm{Re}(s) < 0$$

Hint:

$$\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + \omega_0^2}, \quad \mathrm{Re}(s) > 0$$

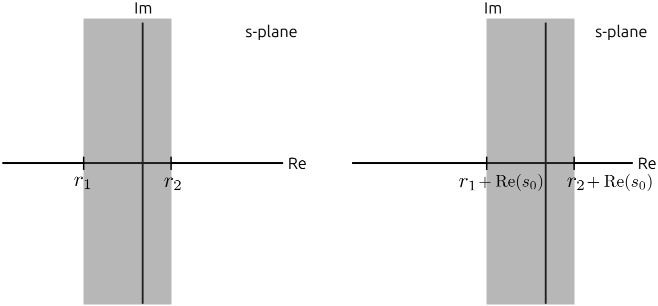
Properties of the Laplace transform

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Image credit: Oppenheim 9.6

s shifting



Differentiation in time.

Differentiation in s.

Exercise: what is the Laplace transform and ROC of

$$x(t) = te^{-2|t|}$$

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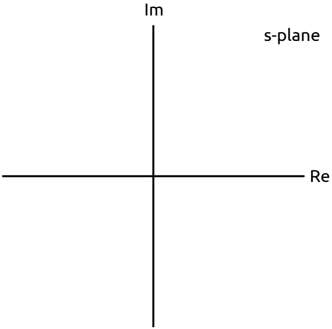


Conjugation.

Initial/final-value theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or singularities at the origin,

Furthermore if $x(t)$ has finite limit as $t \to \infty$,

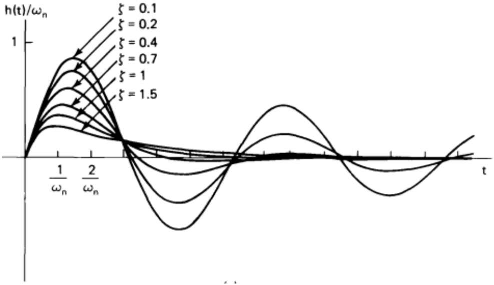
Convolution.

Recall the convolution property:

The ROC of the system (transfer) function can tell us a lot about a system, including systems whose Fourier transforms don't exist.

$H(s)$ and causality

Recall that a system is causal if $h(t) = 0$ for $t < 0$.

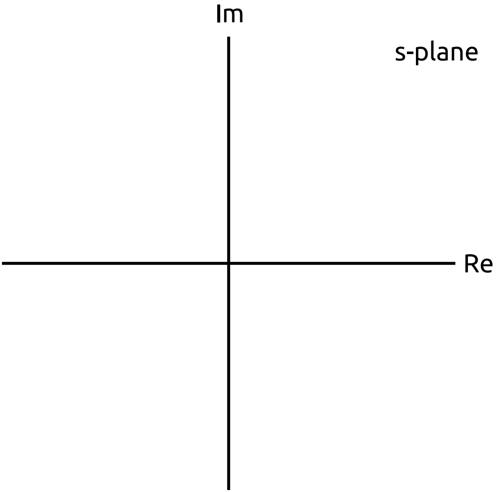


Means $h(t)$ is right-sided, so its ROC is a right-half plane.

Image credit: Oppenheim 6.5

$H(s)$ and causality

Note that the converse is not necessarily true! But if $H(s)$ is rational, the ROC is the right-half plane to right of right-most pole.



$H(s)$ and stability

Our original criteria for stability in terms of impulse response was if

Related to Dirichlet conditions: if a signal is absolutely integrable, its **Fourier transform** converges.

An LTI system with **rational** $H(s)$ is stable iff its ROC includes the entire $j\omega$ axis ($\mathrm{Re}(s) = 0$), and there aren't more zeros than poles.

9.28. Consider an LTI system for which the system function $H(s)$ has the pole-zero pattern shown in Figure P9.28.

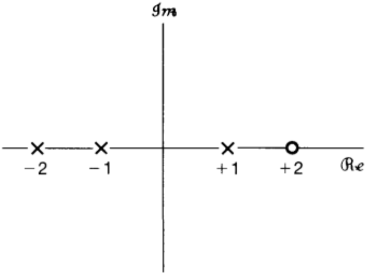


Figure P9.28

(a) Indicate all possible ROCs that can be associated with this pole-zero pattern.
(b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

Image credit: Oppenheim 9.28

Systems described by constant-coefficient differential equations

Recall the situation with the Fourier transform:

Fourier transforms and systems described by differential equations

The representation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

allows us to write down frequency response of systems described by ODEs **by inspection**! (and vice versa)

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Systems described by constant-coefficient differential equations

Same deal here. If system is described by the DE

then its system function is

Placement of zeros and poles is dictated by solutions of $x(t)$ and $y(t)$ stuff respectively.

(Oppenheim 9.31) Consider a CT LTI system described by the DE

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

- Determine $H(s)$ as a ratio of polynomials in s and sketch the pole-zero plot.
- Determine $h(t)$ for each of the following cases:
  1. The system is stable
  2. The system is causal
  3. The system is neither causal nor stable

Systems described by constant-coefficient differential equations

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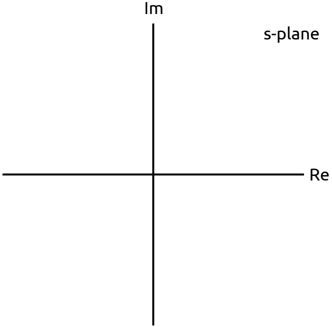

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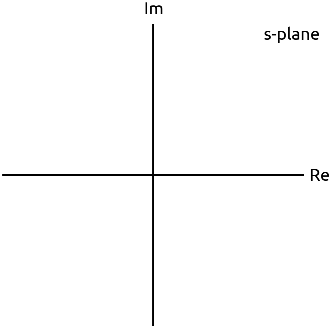

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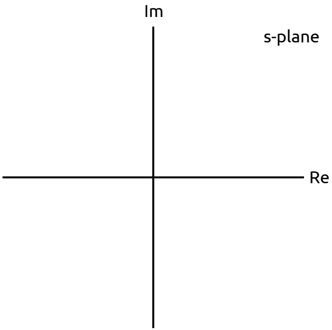

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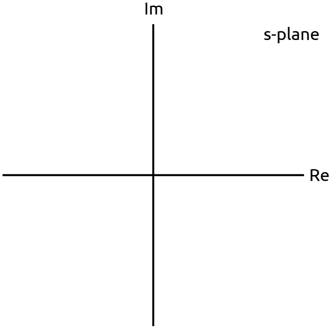

For next time

Content:

- the Laplace transform and feedback systems
- introducing the z-transform

Action items:

1. Suggest problems for tutorial
2. Tutorial assignment 5 due Monday 23:59
3. Assignment 5 due 8 Dec 23:59

Recommended reading:

- From this class: Oppenheim 9.5-9.7
- Suggested problems: 9.13-9.16, 9.21, 9.22, 9.26, 9.29, 9.32, 9.33
- For next class: 9.7, 11.0-11.2, 10.1-10.3