ELEC 221 Lecture 20
Amplitude modulation

Thursday 17 November 2022

- Quiz 9 on Tuesday
- Karplus-Strong bonus assignment due Sunday at 23:59

We have seen convolution and multiplication properties in a few different contexts now:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \leftrightarrow \quad Y(j\omega) = X(j\omega)H(j\omega)$$

$$y(t) = x(t)p(t) \quad \leftrightarrow \quad Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

The latter is sometimes known as the *modulation property*.

Recap

We have used modulation in a few different contexts already.

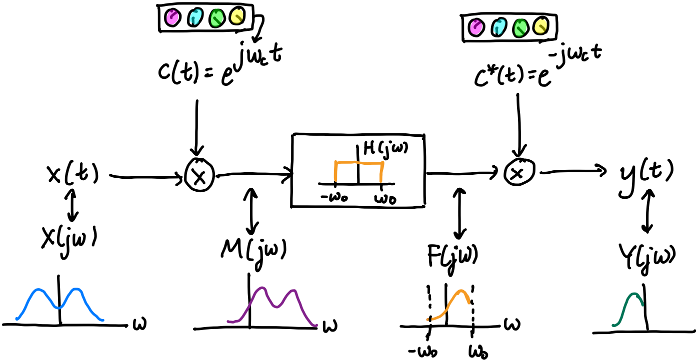
On Tuesday we did some simple amplitude modulation:

```python
def amplitude_modulation(signal, time_range, carrier_frequency):
    """Apply sinusoidal amplitude modulation.

    Args:
        signal (array[float]): The modulating signal.
        time_range (array[float]): The explicit times over which the signal has
            been sampled (in seconds).
        carrier_frequency (int): The frequency (in Hz) of the carrier signal.

    Returns:
        array[float]: The modulated signal.
    """
    return signal * cos_wave(time_range, carrier_frequency)
```

In lecture 10 (and homework 4) we explored how modulation could be leveraged for frequency-selective filtering.



We modeled sampling as multiplication by a periodic impulse train.

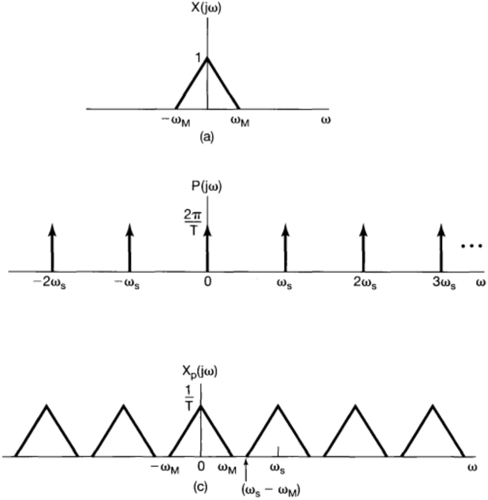
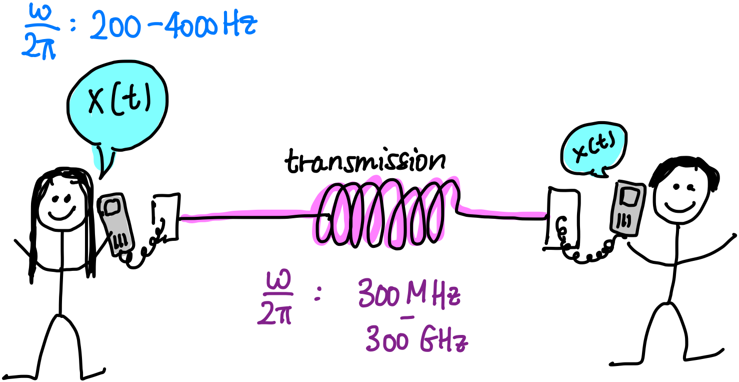


Image credit: Oppenheim 7.1

We briefly discussed the application of phone signal transmission.



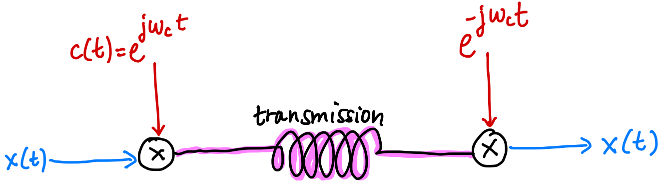
Today

Learning outcomes:

- perform sinusoidal amplitude modulation (AM) and demodulation
- describe the process of frequency-domain multiplexing
- differentiate between double- and single-sideband modulation

Modulation

The process of embedding an information-bearing signal into a second signal. (Extracting the signal: demodulation)



We will discuss two types:

- amplitude modulation (AM) (today)
- frequency modulation (FM) (next time)

Visualization: https://global.oup.com/us/companion.websites/fdscontent/uscompanion/us/static/companion.websites/9780199922963/images/AM_FM.gif

Sinusoidal amplitude modulation

We focus on two types of carrier signals:

- complex exponential signal

$$c(t) = e^{j(\omega_c t + \theta_c)}$$

- sinusoidal signal

$$c(t) = \cos(\omega_c t + \theta_c)$$

$$x(t)\, c(t)$$

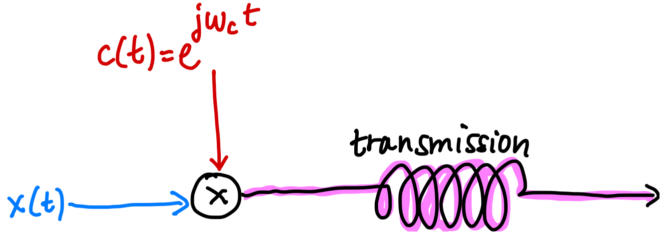
Complex exponential amplitude modulation

We've already seen what happens with the first one.

Suppose

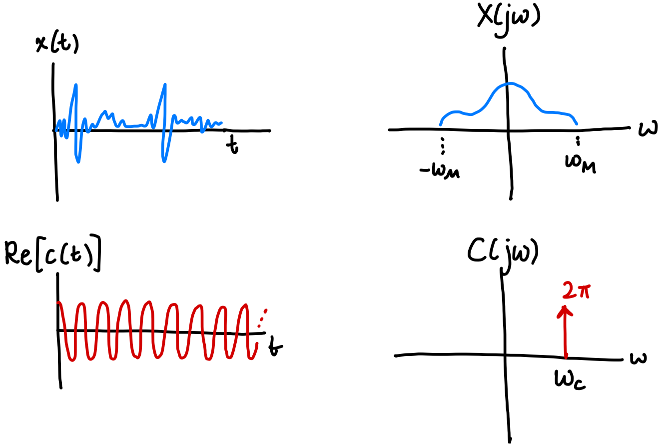
$$c(t) = e^{j\omega_c t}$$

(set $\theta_c = 0$, we will deal with it later).

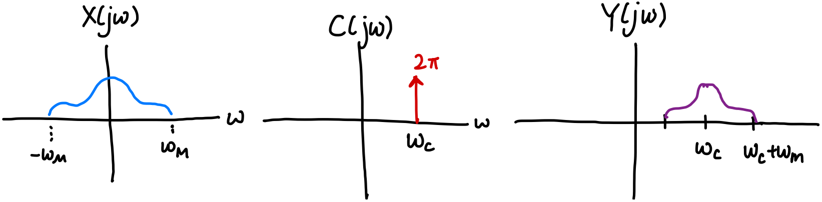


Complex exponential amplitude modulation

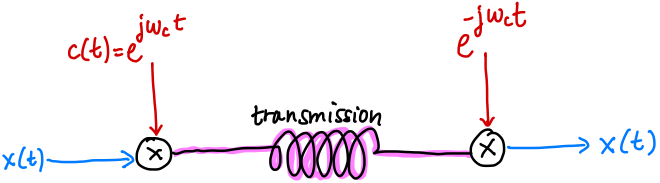
Consider the Fourier spectrum of both signals:



Convolution of the spectra leads to the original spectrum being moved into a different frequency regime.

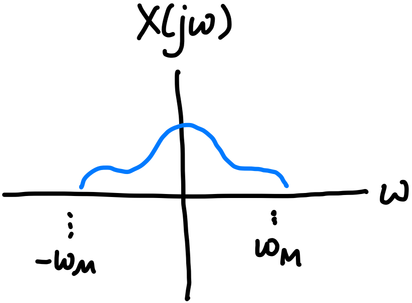


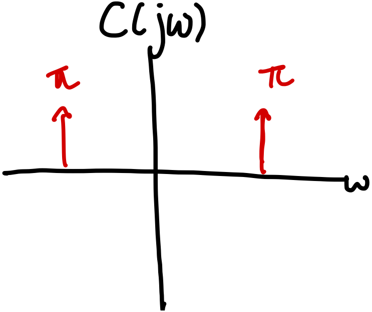
Demodulation is straightforward.



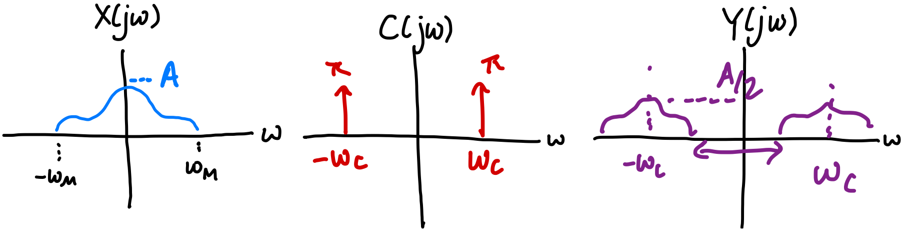
What if we modulate instead by a sinusoidal signal?

$$c(t) = \cos(\omega_c t)$$



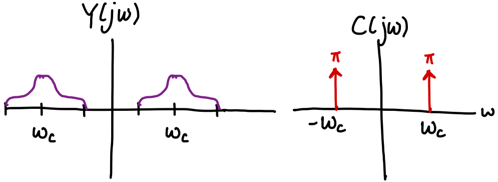


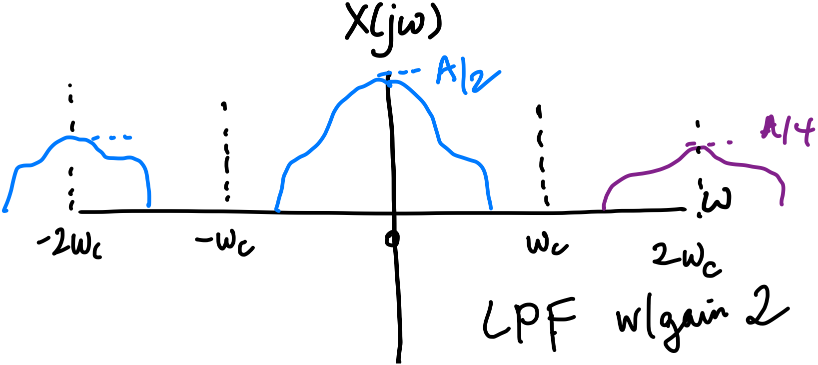
Sinusoidal amplitude modulation

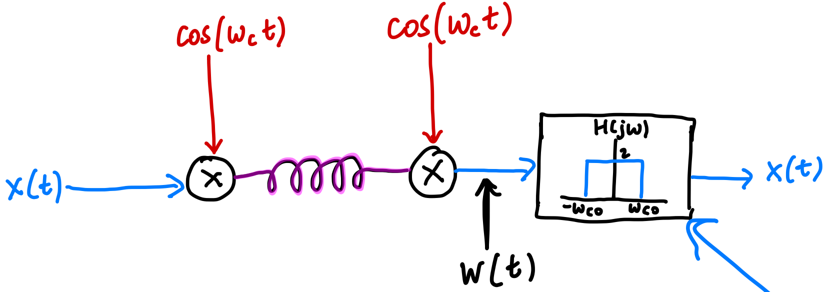


- Any foreseeable problems with this?
- How do we recover the signal?

Sinusoidal amplitude modulation



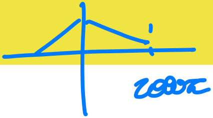




Let's check the math:

$$w(t) = x(t)\cos(w_c t)\cos(w_c t)$$
$$= x(t)\left[\frac{1}{2} + \frac{1}{2}\cos(2w_c t)\right]$$
$$= \frac{1}{2}x(t) + \frac{1}{2}\cos(2w_c t)\,x(t)$$

Exercise: sinusoidal amplitude modulation



8.3. Let $x(t)$ be a real-valued signal for which $X(j\omega) = 0$ when $|\omega| > 2{,}000\pi$. Amplitude modulation is performed to produce the signal

$$g(t) = x(t)\sin(2{,}000\pi t).$$

A proposed demodulation technique is illustrated in Figure P8.3 where $g(t)$ is the input, $y(t)$ is the output, and the ideal lowpass filter has cutoff frequency $2{,}000\pi$ and passband gain of 2. Determine $y(t)$.

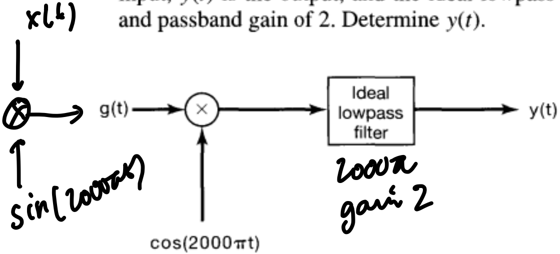


Figure P8.3

$\sin 2x = 2\sin x \cos x$

$g(t) = x(t)\sin(2000\pi t)$

Solution option 1: mathematical

Evaluate $w(t) = g(t)\cos(2000\pi t)$ (input to filter):

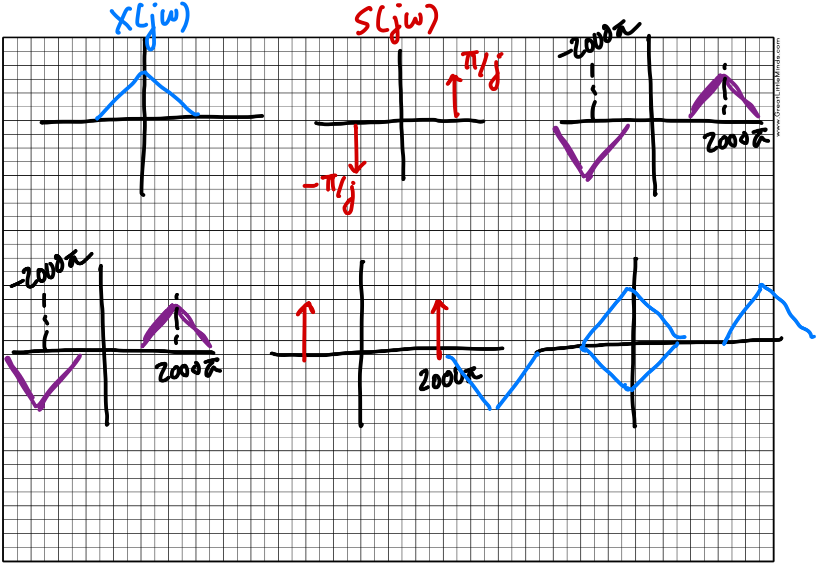
$$w(t) = x(t)\sin(2000\pi t)\cos(2000\pi t)$$
$$= x(t)\,\frac{1}{2}\sin(4000\pi t)$$

$$\Rightarrow y(t) = 0$$

$y(t) = 0$

Exercise: sinusoidal amplitude modulation

Solution option 2: analyze the spectra



 $c(t) = \cos(\omega_c t + \theta_c)$

More generally, we need to consider the phases in both the modulating and demodulating signals:

$$w(t) = \cos(\omega_c t + \theta_c)\cos(\omega_c t + \phi_c)\, x(t)$$

$$= \left[\frac{1}{2}\cos(\theta_c - \phi_c) + \frac{1}{2}\cos(2\omega_c t + \theta_c + \phi_c)\right] x(t)$$

Output after the lowpass filter is

$$x_r(t) = \frac{1}{2}\cos(\theta_c - \phi_c)\, x(t)$$

If $\phi_c = \theta_c$ we call this synchronous demodulation. What could go wrong?

Asynchronous demodulation

Suppose the following is true:

- $x(t)$ is positive
- ω_c is much larger than ω_m

The transmitted signal will look something like this:

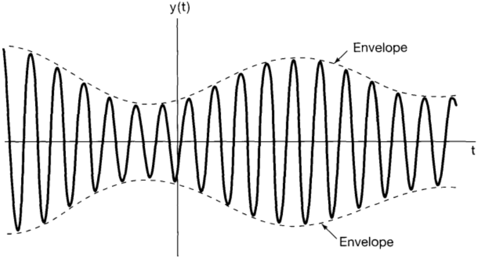


Image credit: Oppenheim 8.2

Design a system to track the envelope (we won't go into details).

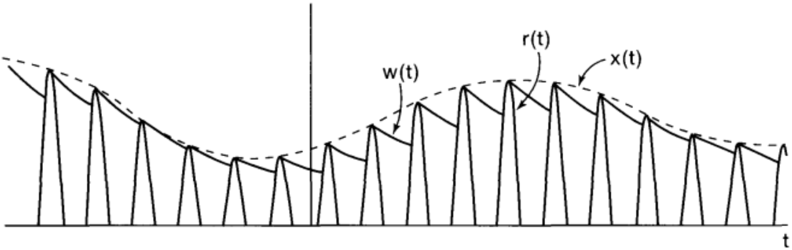
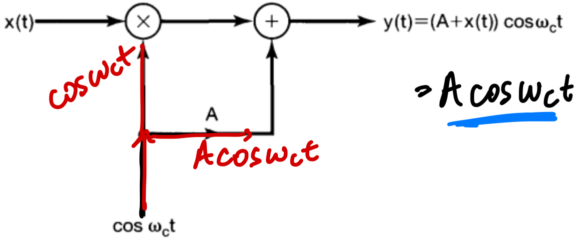


Image credit: Oppenheim 8.2

Asynchronous demodulation

If $x(t)$ is not positive, choose A sufficiently large and add to signal:



$= \underline{A\cos\omega_c t} + x(t)\cos\omega_c t$

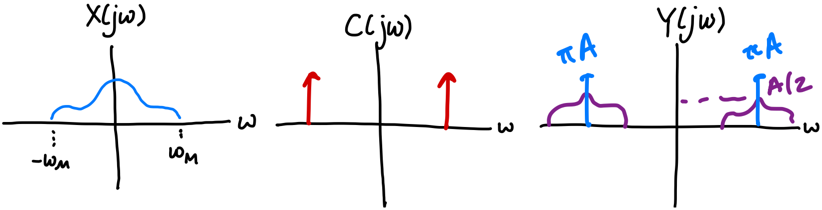
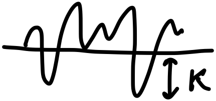
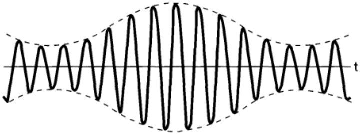


Image credit: Oppenheim 8.2

Asynchronous demodulation

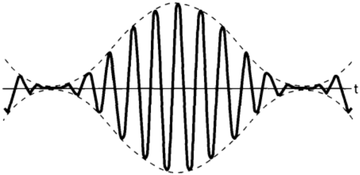
Suppose $|x(t)| \leq K$. Must have $A > K$. Then $m = K/A$ is known as the *modulation index*.





(a)

$m = 0.5$



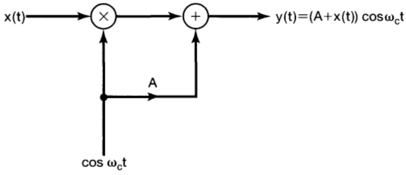
(b)

$m = 1.0$

Image credit: Oppenheim 8.2

Exercise: modulation index

(Oppenheim 8.27)



There is inefficiency because we are sending the carrier signal too. Suppose $x(t) = \cos(\omega_m t)$, $\omega_m < \omega_c$ and $A + x(t) > 0$.

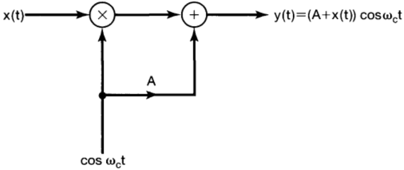
1. What is the modulation index m?
2. What is the average power as a function of m?
3. What is the *efficiency* (ratio of power in sidebands to total power)?

Exercise: modulation index

$m = \frac{K}{A}$

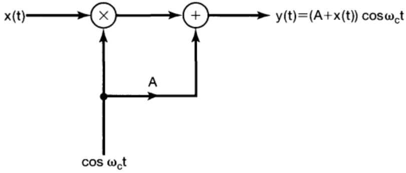
$$x(t) = \cos(\omega_m t)$$

$|x(t)| \leq K$



What is the modulation index m?

$$x(t) = \cos(\omega_m t)$$



What is the modulation index m?

$|x(t)| \leq 1$ $\quad m = \frac{1}{A}$

$|x(t)| \leq 1 \quad \Rightarrow \quad m = \frac{1}{A}$

 $m = \frac{1}{A}$

$$y(t) = (A + \cos(\omega_m t))\cos(\omega_c t)$$

What is the average power as a function of m? For periodic signal,

$$P = \frac{1}{T}\int_T |y(t)|^2 dt$$

Hint: use Parseval's theorem.

(aperiodic) $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

(periodic) $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$

Exercise: modulation index

$$m = \frac{1}{A}$$

$$y(t) = (A + \cos(\omega_m t))\cos(\omega_c t)$$

First let's expand the signal:

$$y(t) = A\cos(\omega_c t) + \cos(\omega_m t)\cos(\omega_c t)$$

$$= A\cos(\omega_c t) + \frac{1}{2}\left[\cos((\omega_m - \omega_c)t) + \cos((\omega_m + \omega_c)t)\right]$$

For a single sinusoid w/frequency ω, $\cos(\omega t)$

$$P = \frac{1}{T}\int_T |y(t)|^2 dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \cos^2(\omega t)\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2}\left[1 + \cos(2\omega t)\right] dt = \frac{\omega}{2\pi}\cdot\frac{1}{2}\cdot\frac{2\pi}{\omega} = \frac{1}{2}$$

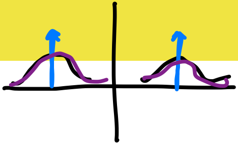
Then,

$$P = \frac{A^2}{2} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{A^2}{2} + \frac{1}{4} = \boxed{\frac{1}{2m^2} + \frac{1}{4}}$$

Exercise: modulation index

$P = \frac{1}{2m^2} + \frac{1}{4}$

$m = \frac{1}{A}$



$$y(t) = (A + \cos(\omega_m t))\cos(\omega_c t)$$

What is the *efficiency* (ratio of power in sidebands to total power)?

$$y(t) = \underline{A\cos(\omega_c t)} + \frac{1}{2}\underline{\cos((\omega_m - \omega_c)t)} + \frac{1}{2}\underline{\cos((\omega_m + \omega_c)t)}$$

Total power: $\frac{1}{2m^2} + \frac{1}{4}$

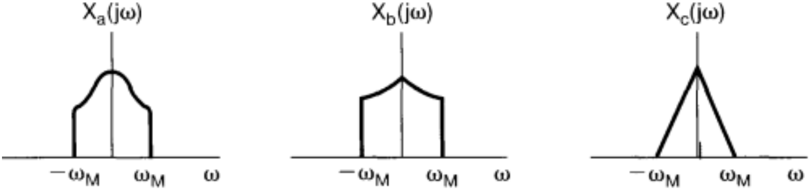
Power in sidebands: $1/4$

$$\text{Eff} = \frac{1/4}{1/2m^2 + 1/4}$$

$$= \frac{1}{2/m^2 + 1}$$

$$= \frac{m^2}{2 + m^2}$$

Suppose we have more than one signal we wish to transmit:



Modulation can help us send them at the same time!

Image credit: Oppenheim 8.3

Frequency-division multiplexing (FDM)

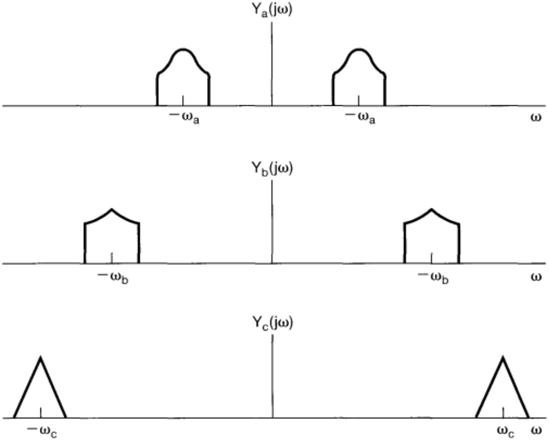


Image credit: Oppenheim 8.3

Frequency-division multiplexing (FDM)

This is called *frequency-division multiplexing*.

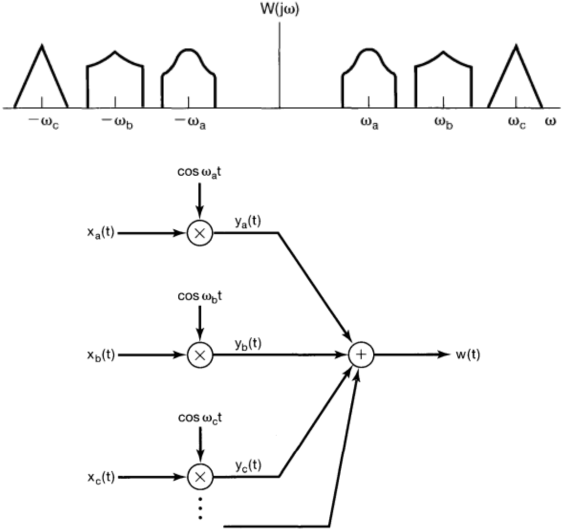


Image credit: Oppenheim 8.3

How to separate out the channels?

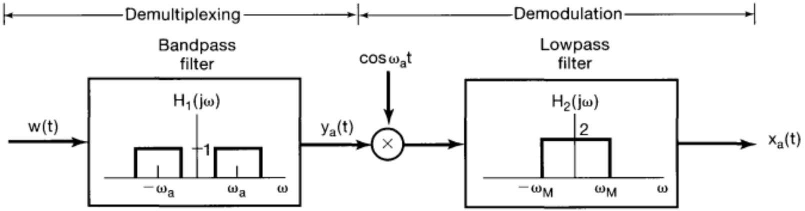


Image credit: Oppenheim 8.3

See Oppenheim problem 8.36.

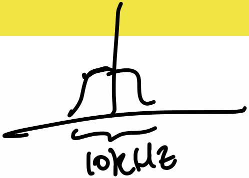
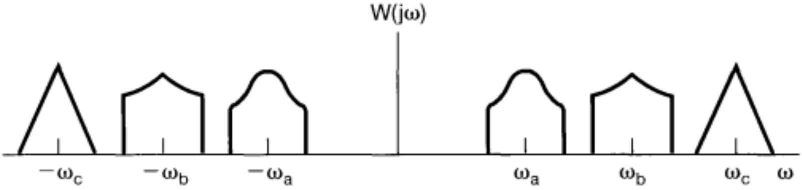


Image credit: https://www.comforthouse.com/analog-tabletop-radio.html

Single-sideband modulation (SSB)

This is inefficient...



We are using twice as much bandwidth as we need to!

Image credit: Oppenheim 8.3

Single-sideband modulation (SSB)

A modulated signal's spectrum can be divided into upper/lower *sidebands*:

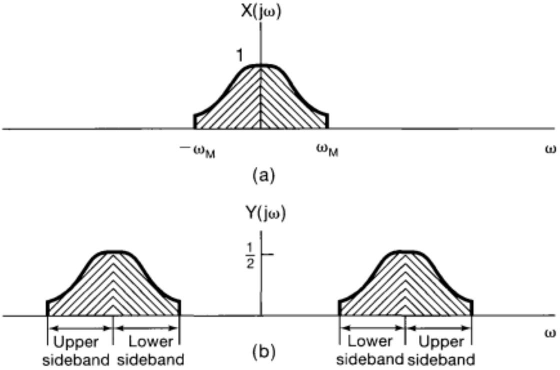
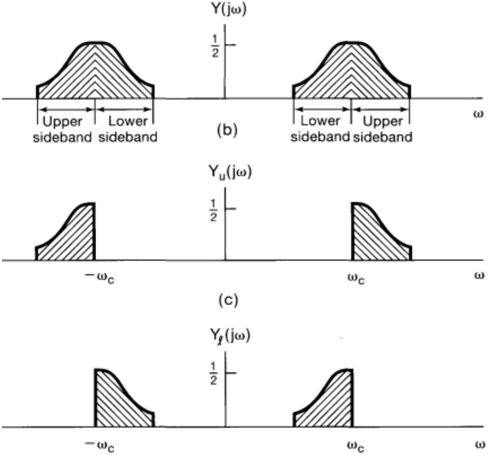


Image credit: Oppenheim 8.4

Single-sideband modulation (SSB)

In single-sideband modulation, keep and transmit only one band:

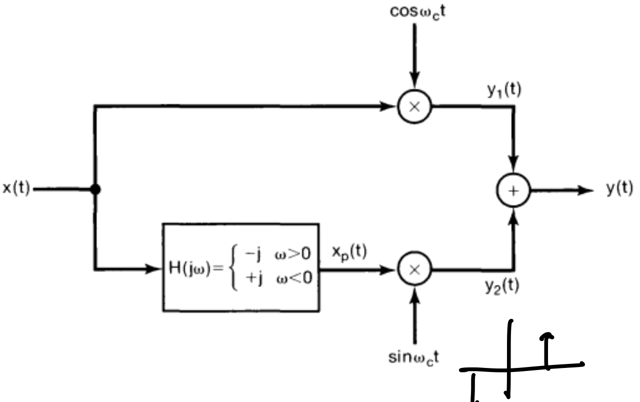


How do you think this is done?

Image credit: Oppenheim 8.4

Single-sideband modulation (SSB)

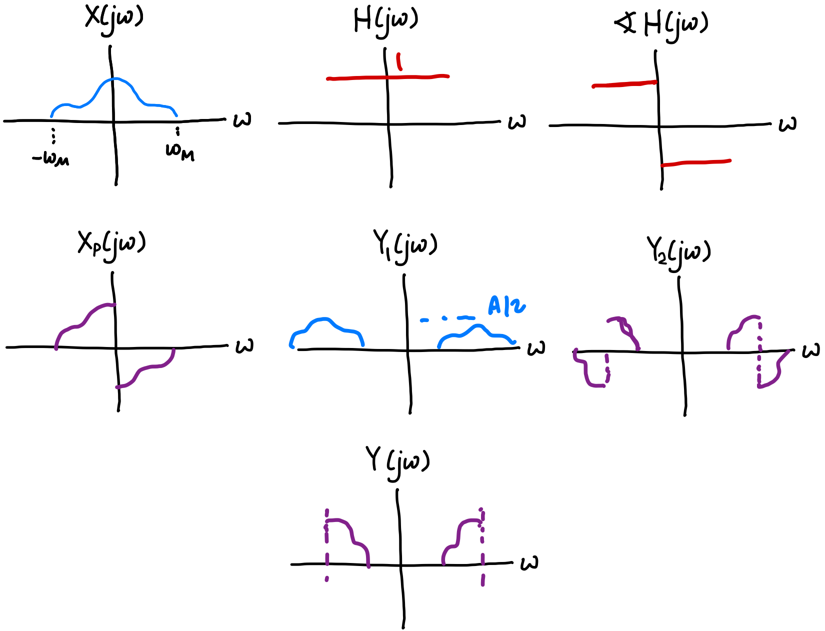
Alternative: "90* phase-shift network"



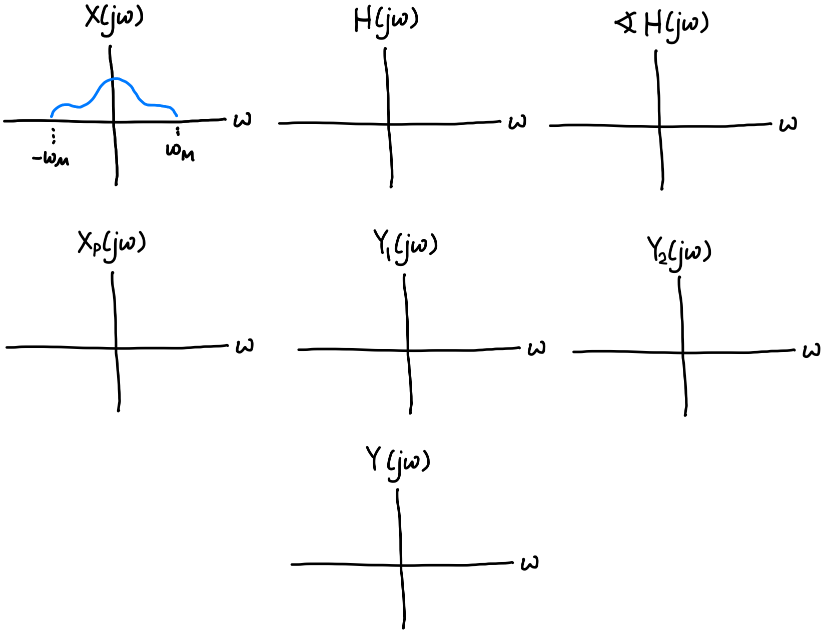
Exercise: see how this works by analyzing what happens to the spectra of $x_p(t)$, $y_1(t)$, $y_2(t)$, and $y(t)$ in the frequency domain.

Image credit: Oppenheim 8.4

Single-sideband modulation (SSB)



Single-sideband modulation (SSB)



Learning outcomes:

- perform sinusoidal amplitude modulation (AM) and demodulation
- describe the process of frequency-domain multiplexing
- differentiate between double- and single-sideband modulation

Oppenheim practice problems: 8.1, 8.2, 8.4, 8.7-8.9, 8.21, 8.22, 8.26, 8.28

For next time

Content:

- pulse-amplitude modulation, frequency modulation
- Quiz 9 on Tuesday

Action items:

1. Assignment 6 (computational) available soon

Recommended reading:

- From this class: Oppenheim 8.0-8.4
- For next class: Oppenheim 8.5-8.9