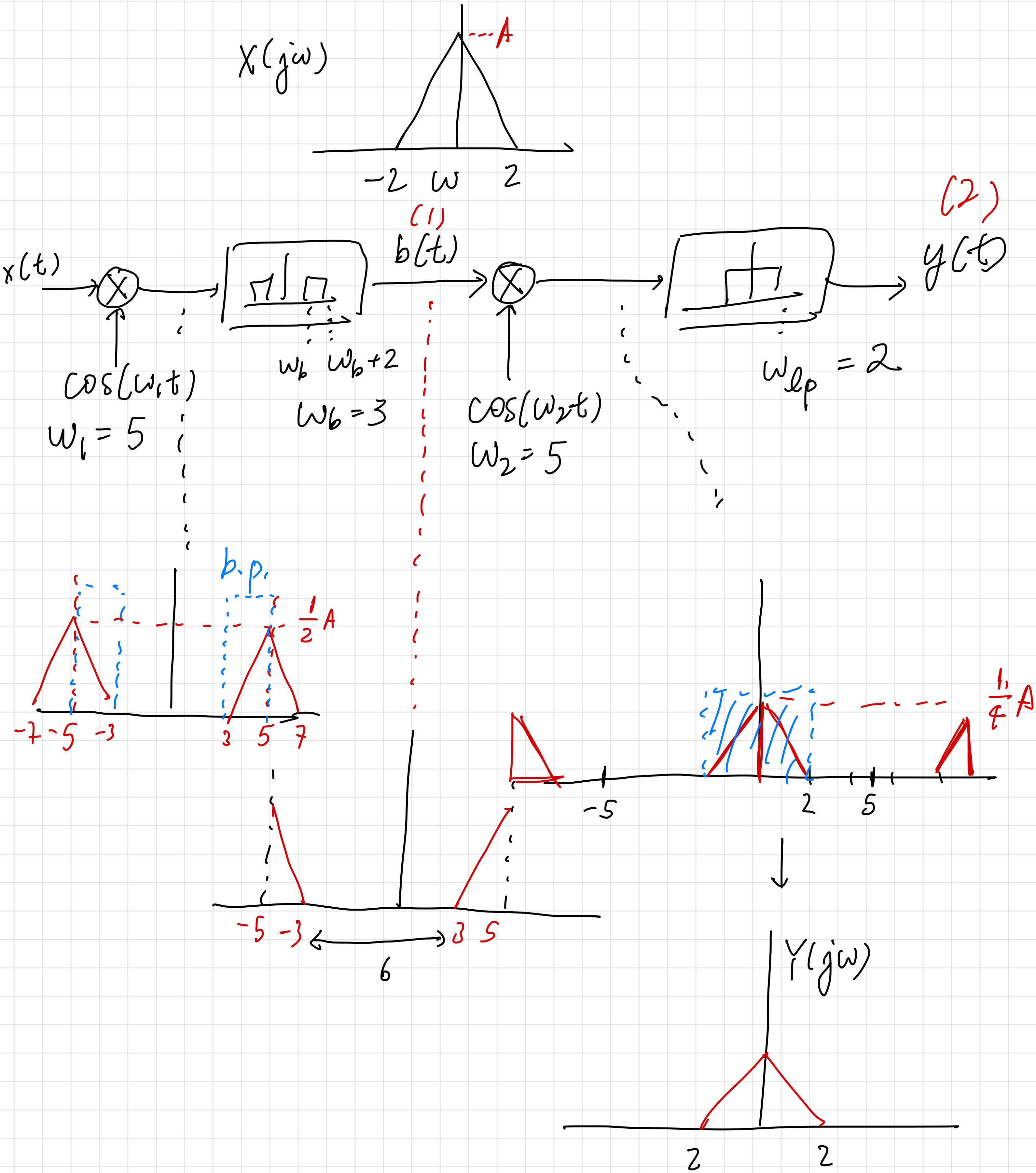
ELEC 221 Lecture 21
The Laplace transform

Tuesday 26 November 2024

Announcements

- Quiz 9 today
- Tutorial assignment 5 due Monday 23:59
- First part of A5 released; due 8 Dec 23:59

☆ please fill out Canvas student exp. survey


X(jω)
A
-2 ω 2
(1)
b(t)
(2)
x(t)
y(t)
cos(ω₁t)
ω₁ = 5
ω_b ω_b+2
ω_b = 3
cos(ω₂t)
ω₂ = 5
ω_lp = 2
b.p.
1/2 A
-7 -5 -3 3 5 7
-5 -3 3 5
6
1/4 A
-5 2 5
Y(jω)
2 2

Last time

We made frequency-division multiplexing more efficient with single-sideband modulation

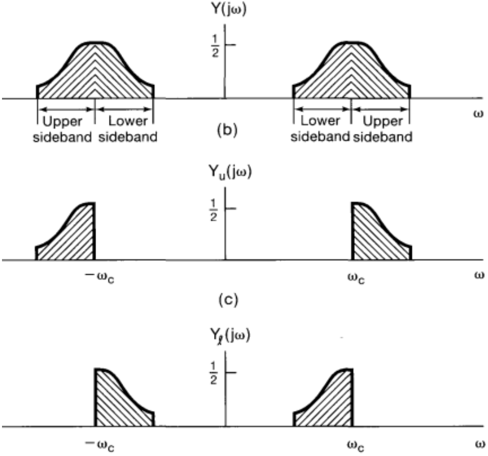


Image credit: Oppenheim 8.4

Last time

We saw how AM with a pulse-train carrier can be used for time-division multiplexing, and pulse-amplitude modulation.

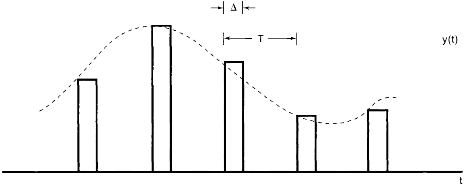
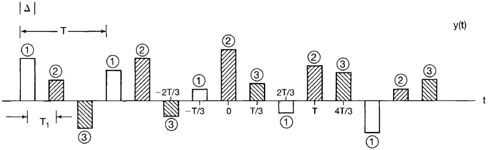


Figure 8.26 Transmitted waveform for a single PAM channel. The dotted curve represents the signal $x(t)$.

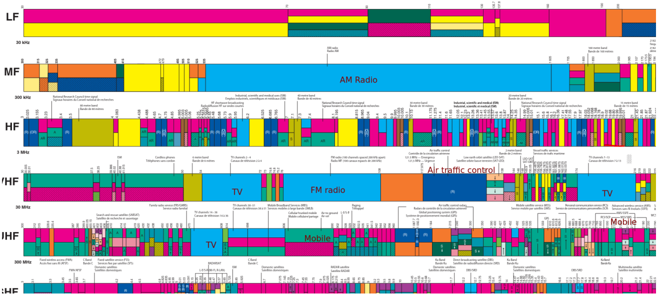


You will get to explore this more in A5.

Image credit: Oppenheim 8.6

Last time

We discussed how cell phones are radios and how the radio spectrum gets divided (and auctioned off).



https://ised-isde.canada.ca/site/spectrum-management-telecommunications/sites/default/files/attachments/2022/2018_Canadian_Radio_Spectrum_Chart.PDF

Wayyyyy back in lecture 5:

LTI systems and complex exponential functions

To summarize:

$$x(t) = e^{st} \quad \rightarrow \quad y(t) = H(s) \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

Wayyyyy back in lecture 5:

The Fourier series

Let's consider a special set of signals[1]:

$$x(t) = e^{st} = e^{j\omega t}$$

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

[1] We will see the general case at the end of the course.

Learning outcomes:

- distinguish between the Fourier transform and the Laplace transform
- compute the Laplace transform and its region of convergence (ROC) for some basic signals
- represent a ROC using a pole-zero plot
- compute the inverse Laplace transform of basic signals using the ROC

Input a signal into LTI system with impulse response $h(t)$:

$$x(t) = e^{st} \rightarrow y(t) = h(t) * x(t) = \underbrace{H(s)}_{\text{System function}} \cdot x(t)$$

If $s = j\omega$: **Fourier transform**

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} h(t)\, dt$$

If $s = \sigma + j\omega$: (bilateral) **Laplace transform**

$$H(s) = \int_{-\infty}^{\infty} e^{-st} h(t)\, dt$$

More generally,

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt \qquad x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

We can relate the Laplace and Fourier transforms.

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt \qquad s = \sigma + j\omega$$

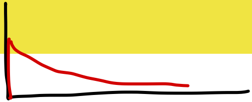
$$= \int_{-\infty}^{\infty} e^{-(\sigma + j\omega)t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\sigma t} \cdot e^{-j\omega t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \left[e^{-\sigma t} x(t) \right] dt$$

$$= \mathcal{F}\left[e^{-\sigma t} x(t) \right]$$

Fourier transform



Example: Let $x(t) = e^{-at}u(t)$. What is $X(j\omega)$?

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} e^{-at} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{-j\omega t} e^{-at}\, dt$$

$$= \frac{1}{j\omega + a}$$

Imagine $a<0$

$$\int_0^{\infty} e^{-j\omega t} e^{-at}\, dt$$

$$\downarrow$$

$$\int_0^{\infty} e^{+|a|t}$$ does not converge!

Recall: conditions on a? $a>0$ $(\mathrm{Re}(a)>0)$

Example: Let $x(t) = e^{-at}u(t)$. What is $X(s)$?

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-st} e^{-at} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{-st} e^{-at}\, dt \qquad s = \sigma + j\omega$$

$$= \int_{0}^{\infty} e^{-\sigma t} e^{-j\omega t} e^{-at}\, dt$$

$$= \int_{0}^{\infty} e^{-(\sigma + a)t} e^{-j\omega t}\, dt$$

Conditions on a?

$$\sigma + a > 0$$

$$\mathrm{Re}(s) + a > 0$$

$$\mathrm{Re}(s) > -a$$

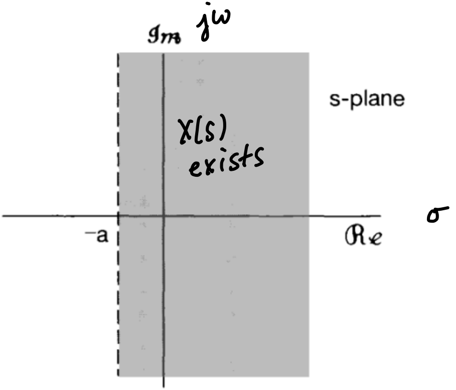
Laplace transform can exist where Fourier transform does not!

We must specify for which s the Laplace transform is valid.

$x(t) = e^{-at} u(t)$

$X(s) = \frac{1}{s+a}$

$Re(s) > -a$



This is called the **region of convergence** (ROC).

Image credit: Oppenheim 9.1

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\,dt$$

Exercise: what is the Laplace transform and ROC of

$$x(t) = -e^{-at}u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} (-e^{-at}u(-t))e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{\underbrace{-(a+s)t}} dt$$

$$= \frac{1}{a+s}$$

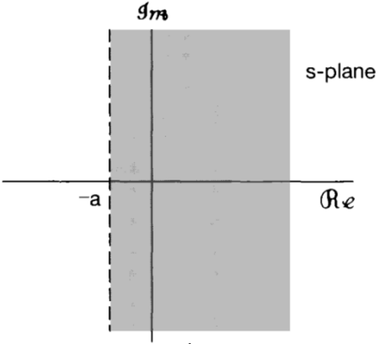
must be negative!

$$e^{-(\underline{a+\sigma})t}\ e^{-j\omega t}$$

$$a+\sigma = a+Re(s) < 0$$

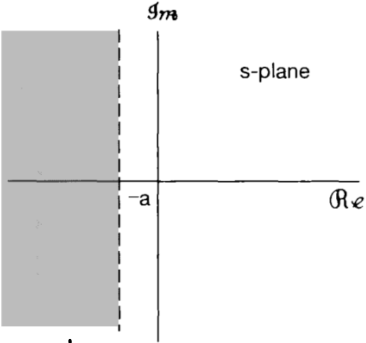
$$Re(s) < -a$$

Multiple signals can have the same algebraic Laplace transform, but different ROCs.



$e^{-at}u(t)$

$Re(s) > -a$



$-e^{-at}u(-t)$ $\quad Re(s) < -a$

Image credit: Oppenheim 9.1

Exercise: what is the Laplace transform and ROC of

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

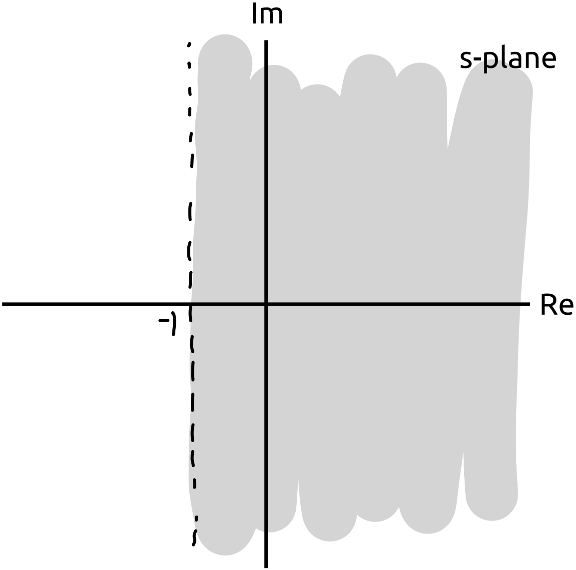
Hint: the Laplace transform is also linear!

$$X(s) = 3 \cdot \mathcal{L}\left(e^{-2t}u(t)\right) - 2 \cdot \mathcal{L}\left(e^{-t}u(t)\right)$$

$$= 3 \cdot \underbrace{\frac{1}{2+s}}_{\mathrm{Re}(s) > -2} - 2 \cdot \underbrace{\frac{1}{1+s}}_{\mathrm{Re}(s) > -1}$$

$\Rightarrow$ ROC: $\mathrm{Re}(s) > -1$

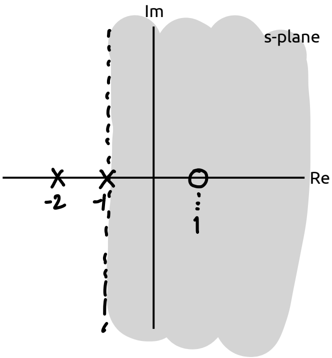
Let's draw the ROC:



Pole-zero plots

$X(s)$ are often rational polynomials of s. Indicate roots on the s-plane using $\times$ for denominator (poles), $\circ$ for numerator (zeros):

$$\frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{(s+2)(s+1)}$$



This is a **pole-zero plot**. (May also have poles/zeros at infinity if degree of polynomials is different)

Exercise: compute the Laplace transform of

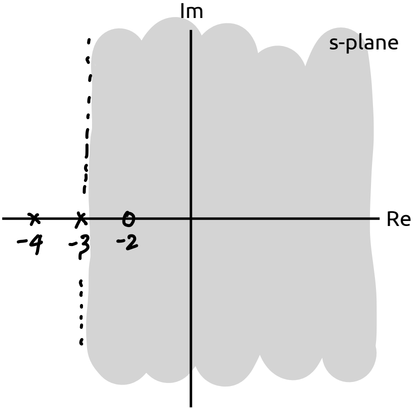
$$x(t) = -2e^{-3t}u(t) + 4e^{-4t}u(t)$$

and draw its pole-zero plot. $Re(s) > -3$ $Re(s) > -4$

$$X(s) = \frac{-2}{s+3} + \frac{4}{s+4}$$

$$= \frac{-2(s+4) + 4(s+3)}{(s+3)(s+4)}$$

$$= \frac{2(s+2)}{(s+3)(s+4)}$$



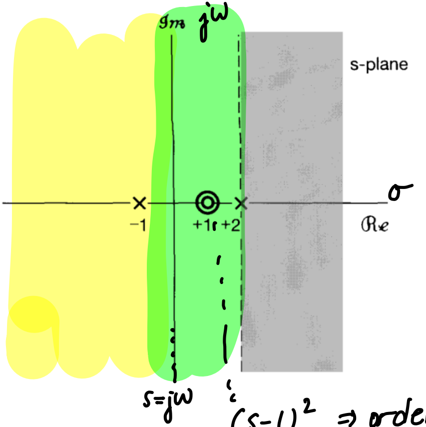
The ROC has many nice properties:

- if ROC doesn't contain $j\omega$ axis, FT does not converge
- ROC is strips parallel to $j\omega$ axis
- ROC of rational Laplace transform contains no poles

$s = \sigma + j\omega$

$\Downarrow$

$\sigma = 0 \quad s = j\omega$



$(s-1)^2 \Rightarrow$ order 2 zero

Image credit: Oppenheim 9.1

Regions of convergence

If $x(t)$ has finite duration and is absolutely integrable, the ROC is the entire s-plane.

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\,dt = \int_{T_1}^{T_2} e^{-st} x(t)\,dt$$

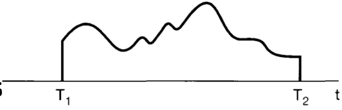


Figure 9.4 Finite-duration signal.

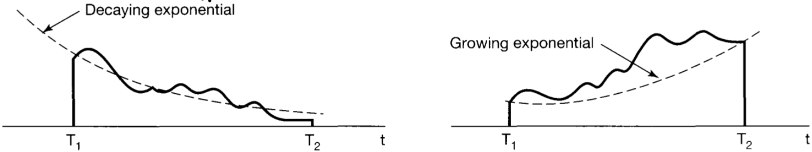
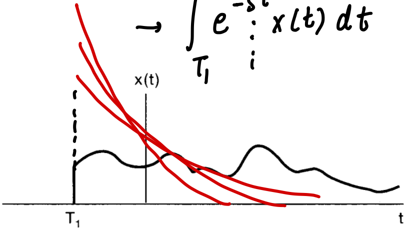


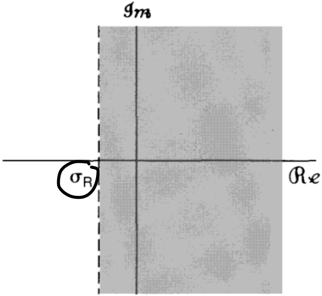
Image credit: Oppenheim 9.2

Right-sided signals

If $x(t)$ is right sided and $\mathrm{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\mathrm{Re}(s) > \sigma_0$ are also in the ROC.

$\rightarrow \int_{T_1}^{\infty} e^{-st} x(t)\, dt$





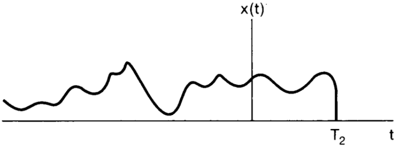
This ROC is called a **right-half plane**.

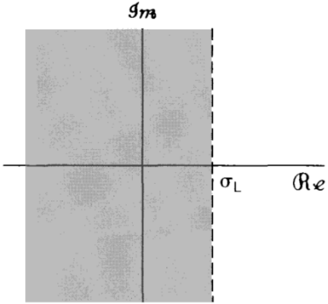
Intuition: if $\mathrm{Re}(s) = \sigma_1 > \sigma_0$ the exponential in $x(t)e^{-\sigma t}$ decays even faster and will still converge.

Image credit: Oppenheim 9.2

Left-sided signals

If $x(t)$ is left sided and $\mathrm{Re}(s) = \sigma_0$ is in the ROC, then all values s.t. $\mathrm{Re}(s) < \sigma_0$ are also in the ROC.



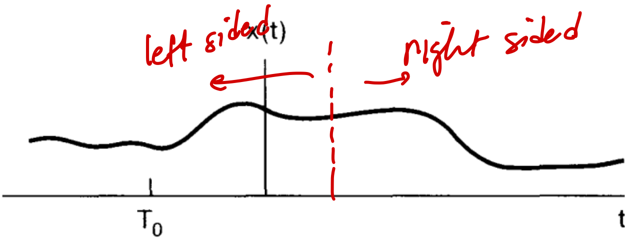


This ROC is called a **left-half plane**.

Image credit: Oppenheim 9.2

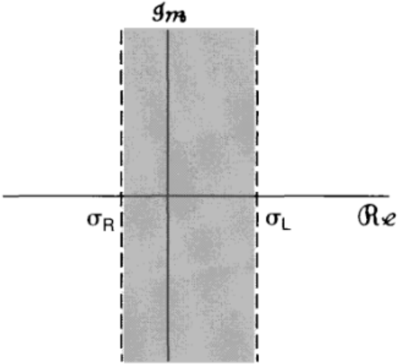
Two-sided signals

Any guesses?



Option 1: nowhere
Option 2: a band in middle
(intersection)

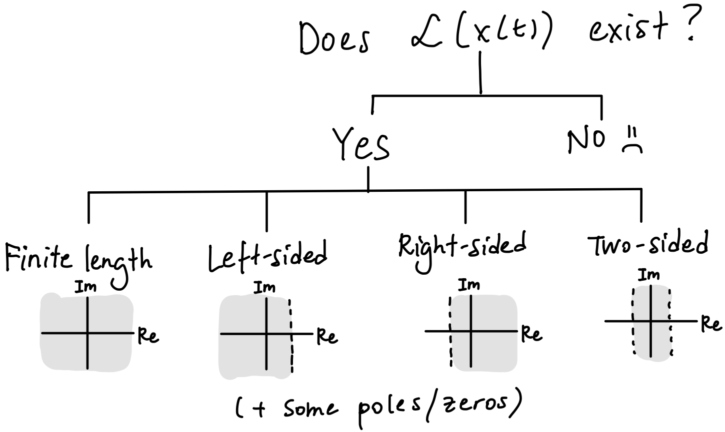
Image credit: Oppenheim 9.2



Only works if initial ROCs overlap - otherwise $X(s)$ doesn't exist!

Image credit: Oppenheim 9.2

For any signal $x(t)$...



For next time

Content:

- properties and system analysis with Laplace transform

Action items:

1. Thursday class back in person
2. Tutorial assignment 5 due Monday 23:59

Recommended reading:

- From this class: Oppenheim 9.0-9.3, 9.5 (skip 9.4)
- Suggested problems: Oppenheim 9.1-9.9, 9.21, 9.26
- For next class: 9.5-9.8 (skip 9.9)