ELEC 221 Lecture 04
CT convolution integral; the impulse response and system properties

Tuesday 17 September 2024

Announcements

- Assignment 1 due Thursday 23:59 (final question moved to Assignment 2)
- Thursday class on Zoom (link in Canvas)
- Friday office hour cancelled this week

Start with Quiz 2.

Learning outcomes:

- Define the CT unit impulse and step functions
- Define the convolution integral and use it to compute the output of a system
- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems

The convolution sum

We expressed signals as weighted sums of impulses

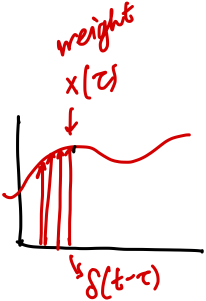
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

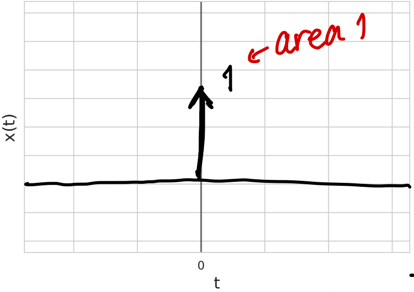
If we know what an LTI system does to a unit impulse (i.e., the impulse response $h[n]$), we know what it does to any other signal:

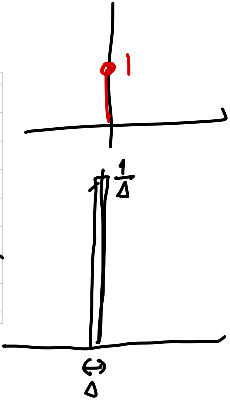
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

This is the **convolution sum**.
Today we will see the **convolution integral** in continuous time.

The CT unit impulse





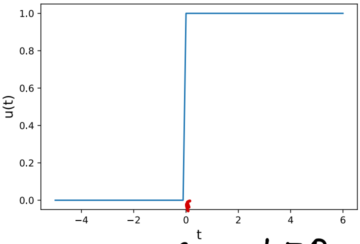


$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

The CT unit step



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Just like in DT, the unit impulse and step are related:

$$\delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\delta(t) \longrightarrow h(t)$$

The CT analogue of convolution sum is the **convolution integral**.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

where $h(t)$ is the **CT impulse response**.

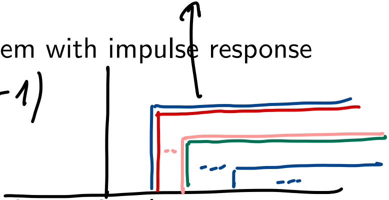
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

It has the same properties (commutative, associative, distributive).

See Piazza and demos folder for example

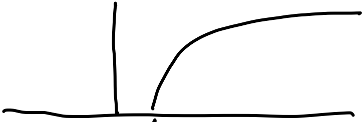
(Oppenheim Ex. 2.6 Var.) Consider system with impulse response

$$h(t) = u(t-1)$$

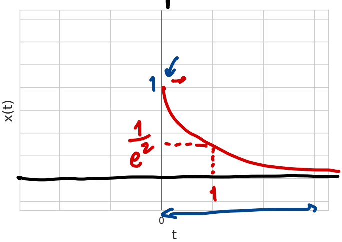


What is the output of the system for the input signal

$$x(t) = e^{-2t}u(t)$$



h(t)



$$x(t) = e^{-2t}\, u(t) \qquad h(t) = u(t-1)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \underline{\int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau}$$

$$= \int_{-\infty}^{\infty} e^{-2(t-\tau)}\, u(t-\tau)\, u(\tau-1)\, d\tau$$

$$= \begin{cases} 0 & \tau < 1 \\ \int_{1}^{\infty} e^{-2(t-\tau)}\, u(t-\tau)\, d\tau & \tau > 1 \end{cases}$$

$$v = t - \tau \quad \Rightarrow \quad d\tau = -dv$$

$$\int_{1}^{\infty} e^{-2(t-\tau)}\, u(t-\tau)\, d\tau = \int_{t-1}^{-\infty} e^{-2v}\, u(v)\, (-dv)$$

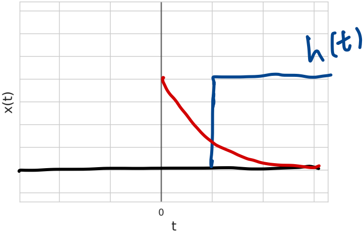
$$= \int_{-\infty}^{t-1} e^{-2v}\, u(v)\, dv$$

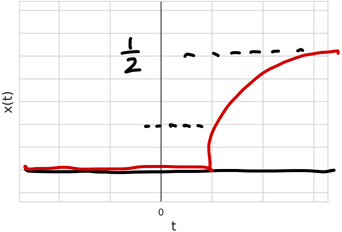
Example: convolution

$$\int_{-\infty}^{t-1} e^{-2\tau} \, u(\tau) \, d\tau = \int_{0}^{t-1} e^{-2\tau} \, d\tau = -\frac{1}{2} e^{-2\tau} \Big|_{0}^{t-1}$$

$$= \frac{1}{2}\left(1 - e^{-2(t-1)}\right)$$

$$y(t) = \frac{1}{2}\left(1 - \underbrace{e^{-2(t-1)}}_{\to 0,\ t \to \infty}\right) u(t-1)$$





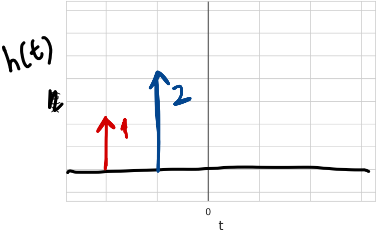
(Oppenheim 2.8) Consider system with impulse response

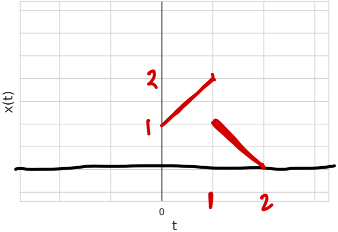
$$h(t) = \delta(t+2) + 2\delta(t+1)$$

What is the output of the system for the input signal

$$x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$x(t) * h(t)$





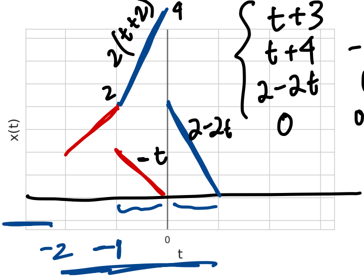
Direct integration:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

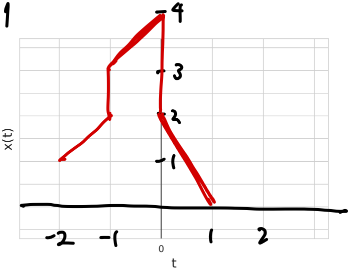
$$= \int_{-\infty}^{\infty} x(t-\tau) \left[\delta(\tau+2) + 2\,\delta(\tau+1) \right] d\tau$$

$$= \underline{x(t+2)} + \underline{2\,x(t+1)}$$

Visual intuition:



$$\begin{cases} t+3 & -2 \le t \le 1 \\ t+4 & -1 < t \le 0 \\ 2-2t & 0 < t \le 1 \\ 0 & \text{otw.} \end{cases}$$



To reiterate: the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

and convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

show that as long as we know how a system responds to a unit impulse, we can determine its response to any other signal.

The impulse response also allows us to reason about key system properties.

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

A system is memoryless if the output depends only on the input at the same time. This implies $h[n] = 0$ for $n \neq 0$, meaning

$$h[n] = K\,\delta[n]$$

(And analogous for CT case)

If a system is invertible, it has an inverse system.
Suppose impulse response of a system is $h(t)$. Then

$$x(t) \longrightarrow y(t) = h(t) * x(t)$$

$$h_1(t) * y(t) = x(t)$$

inv

$$\underbrace{[h_1(t) * h(t)]}_{= \delta(t)} * x(t) = x(t)$$

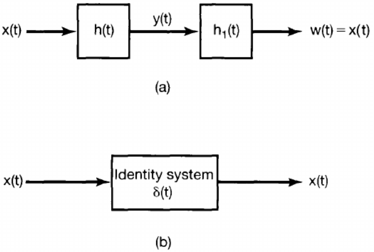


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

(And analagous for DT case. We will see this later in the course.)

Image: Oppenheim, Fig 2.26

Suppose $x(t)$ is boudned, $|x(t)| \leq B$. If the system is stable, the output should be bounded.

$$y(t) = \int_{-\infty}^{\infty} x(\overset{t-\tau}{\tau})\, h(\overset{\tau}{t-\tau})\, d\tau$$

$$\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right| \\
&= \left| \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau \right| \\
&\leq \int_{-\infty}^{\infty} |x(t-\tau)| \cdot |h(\tau)|\, d\tau \\
&\leq \int_{-\infty}^{\infty} B \cdot |h(\tau)|\, d\tau \\
&\leq \underbrace{B \int_{-\infty}^{\infty} |h(\tau)|\, d\tau}_{\text{if finite, then stable.}}
\end{aligned}$$

(And analogous for DT case)

We didn't get this far, but leaving notes for reference.

As long as

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

is bounded (i.e., $h(t)$ is absolutely integrable), the system is stable.

(And analogous for DT case)

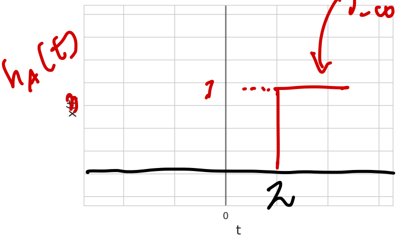
Consider systems A and B with impulse responses

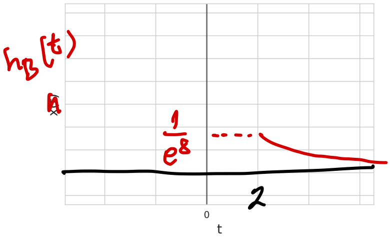
$$h_A(t) = u(t-2)$$
$$h_B(t) = e^{-4t}u(t-2)$$

Are they stable?

not stable ... $\int_{-\infty}^{\infty} 1 \, d\tau$

stable





Integrate $h_8(t)$ to confirm it is stable

$$\int_{-\infty}^{\infty} |e^{-4\tau} u(\tau-2)| \, d\tau$$

$$= \int_{2}^{\infty} |e^{-4\tau}| \, d\tau$$

$$= \int_{2}^{\infty} e^{-4\tau} \, d\tau$$

$$= -\frac{1}{4} e^{-4\tau} \Big|_{2}^{\infty}$$

$$= \frac{1}{4} e^{-8}$$

Recall definition of causal signal and consider the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

What properties does $h[n]$ need to have for system to be causal?

cannot pick up any $x[n-k]$ for negative k, otherwise $n-k$ would be greater than n

$\Downarrow$

need $h[k]=0$ for $k<0$.

(Analogous holds for CT systems)

Today's learning outcomes were:

- Define the CT unit impulse and step functions
- Define the convolution integral and use it to compute the output of a system
- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems

For next time

Content:

- CT Fourier series representation and properties
- Dirichlet conditions, and the Gibbs phenomenon
- Power and energy of signals and Parseval's relation

Action items:

1. Assignment 1 due Thursday 23:59

Recommended reading:

- From today's class: Oppenheim 1.4, 2.2-2.3
- practice problems: 2.8-2.12, 2.14-16, 2.22, 2.28, 2.29
- For next class: Oppenheim 1.3, 3.0-3.5