ELEC 221 Lecture 08
DT Fourier series

Tuesday 01 October 2024

Announcements

- Quiz 4 today
- Assignment 2 due Saturday 23:59 (solutions posted immediately after deadline)
- Midterm details (+past midterm) available on PrairieLearn

Midterm 1 details

See details in "Practice" section on PrairieLearn:

- List of learning outcomes available; covers up to end of L9 (less emphasis on L8-L9);
- Formula sheet provided; no calculators (they aren't needed)
- Should understand *how* and *why* things are done in A1/A2 questions (midterm questions are less involved)
- Practice w/textbook questions and 2022 midterm (ignore content about Fourier transform)

Office hours:

- Tuesday 12:30-1:30pm KAIS 3047 (TA)
- Wednesday 3:30-4:30pm KAIS 3065 (TA)
- Thursday 5:00-6:00pm KAIS 3047 (TA)
- Friday 2:30-3:30pm KAIS 3043 (prof; also by appointment)

Monday 7 Oct tutorial:

- problem solving with TAs
- can request focus on specific topics in advance

- You need to show your scratch work for full marks. Just the final graph/expression is not enough
- "Evaluating the convolution" means finding an expression, not just computing the value of the convolution for a few points.
- Explain why you are doing something/what you are doing. This way an arithmetic mistakes can be awarded partial marks
- Read the entire question. Often they ask for multiple things or reflection/commentary on your work.

Fourier synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Fourier analysis equation:

$$c_k = \frac{1}{T} \int_T e^{-jk\omega t} x(t)\, dt$$

Last time

Dirichlet conditions: given a periodic function, if over one period it

1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

then the Fourier series converges to

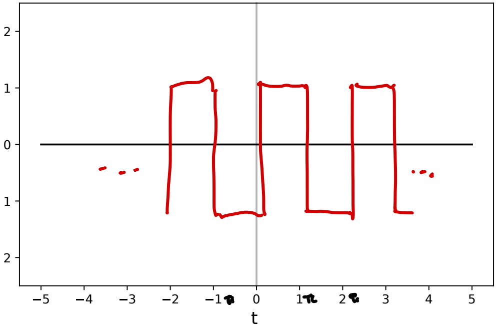
- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Last time

We evaluated the Fourier series coefficients of a square wave:

$$x(t) = \begin{cases} 1, & 0 \le t < \pi, \\ -1, & \pi \le t < 2\pi \end{cases}$$

$$C_0 = \frac{1}{T}\int_T x(t)\,dt$$



$$x(t) = \sum_{k=1}^{\infty} b_k \sin(kt), \quad b_k = \begin{cases} 0, & k \text{ is even} \\ 4/k\pi, & k \text{ is odd} \end{cases}$$

We determined how Fourier series coefficients transform.

Superposition of two signals with same ω:

$$x(t) \overset{F}{\longleftrightarrow} a_k \quad y(t) \overset{F}{\longleftrightarrow} b_k$$

$$z(t) = Ax(t) + By(t) \quad \overset{F}{\longleftrightarrow} \quad c_k = Aa_k + Bb_k$$

Time shift

$$x(t) \leftrightarrow c_k \qquad x(t - t_0) \leftrightarrow e^{-jk\omega t_0} c_k$$

Time scale

$$x(t) \leftrightarrow c_k \quad x(\alpha t) \leftrightarrow c_k \quad x(\alpha t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\alpha\omega t}$$

Multiplication leads to convolution:

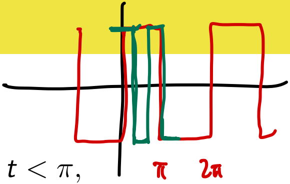
$$z(t) = x(t)\,y(t) \quad \leftrightarrow \quad c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} \rightarrow z = \sum c_k e^{\cdots}$$

Learning outcomes:

- Determine Fourier coefficients of a signal after transformation
- Compute the fundamental period and frequency of DT signals
- Evaluate Fourier series coefficients of DT signals

Exercise

Go back to the square wave



$$x(t) = \begin{cases} 1, & 0 \le t < \pi, \\ -1, & \pi \le t < 2\pi \end{cases}$$

We obtained

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(kt), \quad b_k = \begin{cases} 0, & k \text{ is even} \\ 4/k\pi, & k \text{ is odd} \end{cases}$$

What are the Fourier coefficients of the square wave

$$x'(t) = \begin{cases} 1, & -\frac{\pi}{4} \le t < \frac{\pi}{4}, \\ -1, & \frac{\pi}{4} \le t < \frac{3\pi}{4} \end{cases}$$

$x'(t) = ???(x(t))$

$= x\left(2t + \frac{\pi}{2}\right)$

Exercise

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(kt) \quad , k \text{ odd} \quad b_k = \frac{4}{\pi k}$$

Step 1: express the b_k as the "original" coefficients c_k

$$= \sum_{\substack{k=1 \\ (k \text{ odd})}}^{\infty} \frac{\cancel{4}\,2}{\pi k} \left[\frac{e^{jkt} - e^{-jkt}}{\cancel{2}j} \right]$$

$$= \frac{2}{\pi j} \left[\sum_{k=1}^{\infty} \frac{1}{k} e^{jkt} + \sum_{k=1}^{\infty} \left(-\frac{1}{k}\right) e^{-jkt} \right] \quad k \text{ odd}$$

k	$-k = k'$
1	-1
3	-3
5	-5
$\vdots$	$\vdots$
∞	$-\infty$

$$= \frac{2}{\pi j} \left[\sum_{k=1}^{\infty} \frac{1}{k} e^{jkt} + \sum_{k'=-1}^{-\infty} \frac{1}{k'} e^{jk't} \right] \quad k' = -k$$

$$= \frac{2}{\pi j} \left[\sum_{k=1}^{\infty} \frac{1}{k} e^{jkt} + \sum_{k'=-\infty}^{-1} \frac{1}{k'} e^{jk't} \right]$$

$$= \frac{2}{\pi j} \sum_{k=-\infty}^{\infty} \frac{1}{k} e^{jkt} \longrightarrow c_k = \frac{2}{\pi j k} \, , \, k \text{ odd}$$

Exercise

$$C_k = \frac{2}{j\pi k} \quad k \text{ odd}$$

shift left $\frac{\pi}{2}$

scale $t \to 2t$

Step 2: apply the transformations

shift left: $C_k \to e^{-jk\omega t_0} C_k$ $\qquad C_k \to e^{\frac{jk\pi}{2}} C_k$

$x(t) \to x(t-t_0)$ $\qquad x(t) \to x\left(t+\frac{\pi}{2}\right)$

(k odd!!) $\qquad C_k' = e^{\frac{jk\pi}{2}} \cdot C_k = \left[\cancel{\cos\left(\frac{k\pi}{2}\right)} + j\sin\left(\frac{k\pi}{2}\right)\right] \cdot \frac{2}{j\pi k}$

$$= \cancel{j}\sin\left(\frac{k\pi}{2}\right) \cdot \frac{2}{\cancel{j}\pi k}$$

$(-1)^{k+1}$

1	+1
3	-1
5	+1

$$= \frac{2}{\pi k}\sin\left(\frac{k\pi}{2}\right)$$

Scale: $k \to 2k$

$$x'(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left[\frac{2}{\pi k}\sin\left(\frac{k\pi}{2}\right)\right]}_{C_k} e^{jk2t}$$

(k odd)

DT complex exponential signals

Recall our CT representation of complex exponential signals:

$$x(t) = C e^{\alpha t}$$

where α could be real or complex.

In DT, we write $$x[n] = C \beta^n = C(e^{\alpha})^n$$

where β can be real or complex.

Case: $x[n] = C\beta^n$, where β is real.

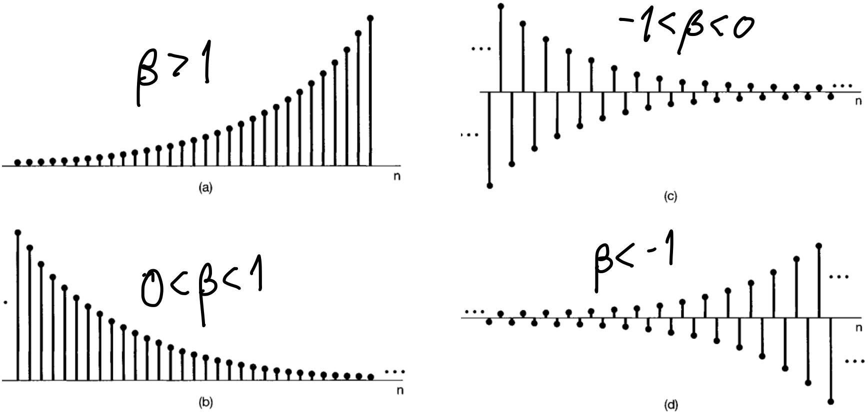


Image credit: Oppenheim chapter 1.3.

Case: $x[n] = C\beta^n$, where β is purely complex:

$$x[n] = Ce^{j\omega n} = C\cos(\omega n) + Cj\sin(\omega n)$$

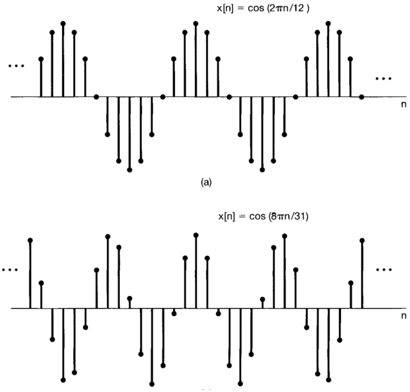


Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

While these might look similar to their CT counterpoints, there is a **very important difference** relating to frequency.

In CT, $x(t) = A\cos(\omega t + \phi) \quad x(t) = Ce^{j\omega t}$

This is periodic with period $T = \frac{2\pi}{\omega}$

The bigger the frequency (ω) gets, the faster it oscillates!

Exercise: consider the DT signal

$$x[n] = e^{j\omega n}$$

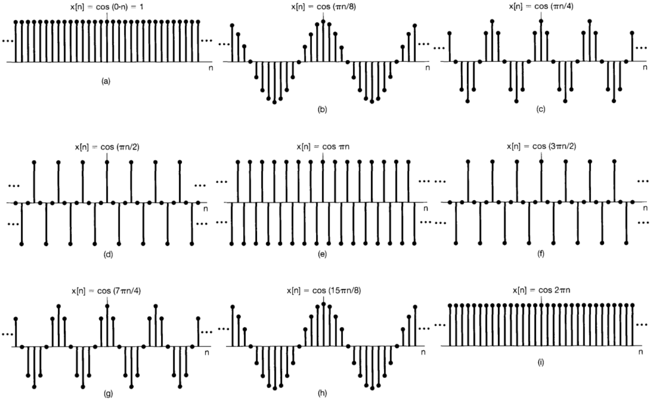
Does bigger ω always mean faster oscillation? If yes, why? If no, when does it stop getting faster?

$$e^{j(\omega+2\pi)n} = e^{j\omega n} e^{2\pi j n}$$

$$= e^{j\omega n}$$

Frequency and period of DT complex exponential signals

For a DT signal with frequency ω, the signals with frequencies $\omega \pm 2n\pi$ are the same



only need to consider ω in 0 to 2π (or $-\pi$ to π)

Image credit: Oppenheim chapter 1.3.

Exercise: What are the fundamental periods of

$$x(t) = \cos(3t), \quad \text{and} \quad x[n] = \cos(3n)$$

$T = \frac{2\pi}{3}$

$N = ???$

Not periodic!

Suppose the period is N:

$$
\begin{aligned}
x[n] &= e^{j\omega(n+N)} \\
&= e^{j\omega n} \cdot \underbrace{e^{j\omega N}}_{=1} \\
&= e^{j\omega n}
\end{aligned}
$$

This implies

$$e^{j\omega N} = 1 \implies \omega N = 2\pi m \text{, } m \text{ integer}$$

$\frac{\omega}{2\pi} = \frac{m}{N}$ must be rational for the signal to be periodic.

Frequency and period of DT complex exponential signals

Exercise: what is the fundamental period of

$$x[n] = \cos(5\pi n/6) + \sin(2\pi n/3)$$

$\cos\left(\frac{5\pi}{6} n\right)$

$N = 3$

$N = 12$

$n=6$ $\cos(5\pi)$

$n=12$ $\cos(10\pi)$

total period: 12

Harmonics of DT complex exponential signals

$N \leftrightarrow \omega = \frac{2\pi}{N}$

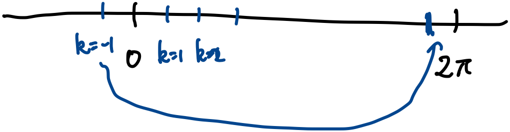
$$x_k(t) = e^{jk\omega t}$$

What about harmonics?

$$x_k[n] = e^{jk\omega n} = e^{\frac{jk2\pi n}{N}}$$

In CT we had an infinite number of these. What about DT?

$$x_k[n] = e^{\frac{jk2\pi n}{N}}, \quad k = 0, 1, \ldots, N-1$$



Consider a system with impulse response $h[n]$ and DT signal $x_m[n] = e^{jm\omega n}$. Use the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

$$= \sum_{k=-\infty}^{\infty} e^{jm\omega(n-k)}\, h[k]$$

$$= e^{jm\omega n} \sum_{k=-\infty}^{\infty} e^{-jm\omega k}\, h[k]$$

$$= x[n] \underbrace{\sum_{k=-\infty}^{\infty} e^{-jm\omega k}\, h[k]}_{H(e^{jm\omega})}$$

If we know how a system responds to complex exponential signals, we can learn its response to signals expressed in terms of them.

We need a Fourier series representation of DT signals:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{k 2\pi n}{N}}$$

How do we find the c_k?

Recap

Today's learning outcomes were:

- Determine Fourier coefficients of a signal after transformation
- Compute the fundamental period and frequency of DT signals
- Evaluate Fourier series coefficients of DT signals

next time!

For next time

Content:

- Using the frequency response to design filter systems

Action items:

1. Assignment 2 due Saturday 23:59

Recommended reading:

- From today's class: Oppenheim 3.5-3.7
- Suggested problems: 3.2, 3.10-3.12, 3.14, 3.17, 3.23-3.26, 3.28, 3.30, 3.31
- From today's class: Oppenheim 3.8-3.12