INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

Creating a HasCASL Library

G. M. Cabral *A. V. Moura*

Technical Report - IC-08-45 - Relatório Técnico

December - 2008 - Dezembro

Creating a HasCASL Library

Glauber Módolo Cabral*   Arnaldo Vieira Moura†

Abstract

1 Introduction

2 Languages

This section intended to introduce the languages involved in our work. We started with a presentation of the *CASL* specificatiton language, quickly describing its syntax and semantic. Next, we introduced the Haskell programming language with some interesting concepts that we had to deal latter in specifications. Following, described the *HasCASL* specification language, an *CASL* extension, which we used to write our specifications. We presented some main concepts of *HasCASL* and a small example. Latter, we introduced the *HasCASL* extention to *CASL* language and its related tool, namely *Hets*, which is responsible for parsing and translating our specifications to the theorem prover. Next, we introduced the *Isabelle* theorem prover with a quick presentation of its main features. Finally, we presented our proposal to this work.

2.1 CASL

The *Common Algebraic Specification Language* (*CASL*) emerged as the product of an international initiative to create an unified language for algebraic specifications containing the largest possible set of known language constructions. This section describes the *CASL* language [1].

With few exceptions, the characteristics of *CASL* are present in some form or another in other specifications languages. However, no previous single language had all the desired purposes: some sophisticated features require specific programming

*Instituto de Computação, Universidade Estadual de Campinas, 13081-970 Campinas, SP.
Pesquisa desenvolvida com suporte financeiro parcial do CNPq
†Instituto de Computação, Universidade Estadual de Campinas, 13081-970 Campinas, SP.

paradigms; on the other hand, methods for prototyping and generation of specifications work only in the absence of certain characteristics. For example, term rewriting requires specifications with equational or conditional equational axioms.

CASL was constructed to be the kernel of a family of languages: sub-languages are obtained through syntactic or semantic restrictions, while extensions are created to support the various programming paradigms. The language definition took into account previously planned extensions, such as the support to second order functions. *CASL* is divided into several parts that can be understood and used separately, namely:

- Basic Specifications: contain declarations (of types and operations), definitions (of operations) and axioms (related operations);
- Structured Specifications: allow Basic Specifications to be combined in larger specifications;
- Architectural Specifications: define how specifications should be separated in an implementation, allowing reuse of specifications with dependence relations;
- Specification libraries: similar specifications are joined together in these libraries; their syntax has constructions that allow version control and library distribution over the Internet.

Structured Specification language constructions are independent of the Basic Specifications, so, *CASL* sub-languages or extensions can be created by extending or restricting Basic Specification language constructions, without the need to change any of the other three language constructions. We now briefly describe the most important Basic Specification language constructions.

Basic Specification denotes a class of models which are many-sorted partial first order structures, i.e., many-sorted algebras with total and partial functions and predicates. These models are classified by signatures, which contain sort names, total and partial function names, predicate names and definitions (or profiles) for functions and predicates.

Specifications contain: declarations, which introduce components of the signature (operations or functions, and predicates), and axioms, which define properties of the structures that should be models of the specification. Operations may be declared total (by using '`->`') or partial (by using '`->?`'), and we can assign to this operations some common properties, such as associativity, avoiding the need for axiomatizing those properties for each different operation.

Partial operations are a simple way to treat errors (such as dividing by zero) and these errors are propagated to callers directly, as when an argument of an operation is not defined, the operation result is also not defined. The errors and exceptions can be treated by super-types and sub-types. The domain of a partial function can be defined as a sub-type of that function's argument type in order to make this partial function a total function over the sub-type. Functions can be declared total rather than making them total by axioms.

Predicates are similar to operations but have no return type; only parameter types are declared. Predicates may be declared and defined at the same time, instead of having their declaration and axiom in separate sections.

Axioms are written as atomic first-order formulas. Variables used in axioms may be declared in three different ways: globally, before axiom declarations; locally to a list of formulas; or individually for each formula, using explicit quantification.

Formula are interpreted in two-valued first order logic (with values true and false). Definedness assertions are used to indicate when a term is defined or not. Assertions may be declared explicitly by a keyword or implicitly by means of an existential equation. An existential equation, declared by using '`=e=`' between two terms of the same type, is valid when both terms are defined and are equal; in contrast, strong equations, declared by using '`=`' between terms, are also valid when both terms are undefined.

Sub-sort membership, indicated by '`in`', creates a predicate asserting the membership of an element to a sort. It's a good practice to use existential equations when defining properties and strong equations when defining partial functions inductively.

CASL uses loose semantic for Basic Specifications, i.e., all structures that meet the axioms are selected as models. This semantic is interesting during requirement analysis because it creates very restrictive specifications that may be refined later by other axioms.

A data type can be declared as free, changing its loose semantic into an initial semantic. Thus, values of the same type that differ only in the order of type constructor application are treated as different elements of that type.

The third semantic allowed in *CASL* forces data types to be generated only by type constructor applications. This eliminates the confusion between terms, i.e., unless axioms force a term equality, all the terms of that type all different from each other. When needed, axioms can be used to reintroduce term equality.

Linear visibility is used to control term declaration except for type declarations, i.e., except in type declarations a term must be declared before its use.

2.2 Haskell

This section presents some general elements of the Haskell programming language. Information provided here as well as further concepts can be found online [3] or in books [11].

Haskell is a pure, strong typed functional programming language with lazy evaluation that resulted from the need of standardization in the field of functional languages. The language is functional because it implements the concepts of $\lambda-Calculus$, so the programming is done through function and computation applications. The language is strongly typed, i.e., the types of functions and values must be explicitly defined on compile time; otherwise, the compiler will try to bind those types to the broadest possible ones in the current context.

Concepts of lazy evaluation and strict evaluation relates to interpretation of the parameters of a function. Languages with strict evaluation evaluates all parameters of a function call before running its body. In the case of languages with lazy evaluation, such as Haskell, parameters of a function are evaluated only when they become necessary inside the function body.

The language is called purely functional because it does not allow a function application to change the global state of the program, only changes to variables and values local to the function execution are allowed. Changing the global state of the program is a kind of side effect which is common in imperative languages. Functional programming languages that allow side effects are called non-pure.

To allow operations that may cause side effects to be executed without causing side effects to the hole program, Haskell performs side-effect actions through a mathematical entity called a monad. Monads can sequence side-effect computations passing a copy of the actual global state implicitly to those computations. It prevents the side effects to change the real global state of the program.

Haskell functions can be declared just as in $\lambda - Calculus$ by *Lambda Abstractions* or in the *Haskell* syntax; in both styles we can name function definitions to later reuse. If a function type is not defined, the compiler will compute the broader type in the context. The *Haskell* syntax is preferred because it's easier and more practical for writing larger programs. Here we show the function `add` for summing two numbers defined with *Lambda Abstractions* and in the *Haskell* syntax, respectively; the compiler will calculate type `Integer -> Integer` to the functions as we haven't declared the type of the functions:

```
add     = \x y -> x+y
add x y = x+y
```

It is necessary to differentiate functions, as the previously defined `add`, from operators, as the operator `+`. A function in Haskell is always defined in a prefix way and an operator has an infix definition. Besides these differences, it's possible to simulate a function with an operator and vice versa. Operators can be used as functions if enclosed by parenthesis; a function can be used as an operator if enclosed by back-quotes. We can use the operator `+` as a function like this: `(+) x y` and the function `add` as an operator like this: ``x `add` y``

Just as in other functional languages, the main data representation in Haskell are lists. There can be lists of primitive types, lists of tuples, lists of lists, lists of functions, etc.; the only requirement is that all elements of the list have the same type. The order and the quantity of elements within a list are taken into account when comparing them for equality.

Two basic operators to manipulate lists are the operator "`:`" (list construction) and the operator "`++`" (list concatenation). A list is always constructed from an empty list and some element by the list construction operator and two lists can be concatenated only if their elements have the same type.

Another feature largely explored in Haskell is pattern matching. Functions can be defined by pattern matching their parameters, as follows:

```
fat :: Int -> Int
fat 1 = 1
fat x = x * fat(x-1)
```

Each call to the function `fat` will pattern match against each line of its definition, from the first one to the last one, until the parameters of the function call match parameters from one of the definitions. Thus, the more specific definitions must come before the generic one. In Haskell Source Code 2.2.1 we can see pattern matching applied in case expressions, list constructors and let expressions.

A fundamental tool in Haskell is the data type construction. A data type must have at least one constructor that may be empty or may have type variables. Type variables are used to construct polymorphic data types; the constructor and its type variables may be enclosed by parenthesis in order to avoid ambiguity. In Haskell Source Code 2.2.1 we defined the polimorphic type `Split a b` with one constructor (`Split b [[a]]`).

We can collect functions and data types from similar contexts in libraries. Haskell libraries are called modules and can control which functions and data types from that module should be exposed to users. We've created a module in Haskell Source Code 2.2.1 and let all the functions to be exposed to the users. There is a standard library, called *Prelude*, which defines basic functions that operate on primitive types, such as: *Bool*, *Char*, *List*, *String*, numeric types and tuples involving those types. All Haskell compilers must implement the Prelude library as this implementation is part for the language definition.

Haskell Source Code 2.2.1 Haskell source code for GenSort sorting program.

```
module GenSort where
import Data.List
data Split a b = Split b [[a]]
genSort :: Ord a => ([a] -> Split a b) -> (Split a b -> [a]) -> [a] -> [a]
genSort split join l = case l of
  _ : _ : _ -> let Split c ls = split l in
               join $ Split c $ map (genSort split join) ls
  _ -> l
splitInsertionSort :: [a] -> Split a a
splitInsertionSort (a : l) = Split a [l]
joinInsertionSort :: Ord a => Split a a -> [a]
joinInsertionSort (Split a [l]) = insert a l
insertionSort :: Ord a => [a] -> [a]
insertionSort = genSort splitInsertionSort joinInsertionSort
splitQuickSort :: Ord a => [a] -> Split a a
splitQuickSort (a : l) =
  let (ls, gs) = partition (< a) l in Split a [ls, gs]
joinQuickSort :: Split a a -> [a]
joinQuickSort (Split a [ls, gs]) = ls ++ (a : gs)
quickSort :: Ord a => [a] -> [a]
quickSort = genSort splitQuickSort joinQuickSort
splitMergeSort :: [a] -> Split a ()
splitMergeSort l =
  let (l1, l2) = splitAt (div (length l) 2) l in Split () [l1, l2]
joinMergeSort :: Ord a => Split a () -> [a]
joinMergeSort (Split _ [l1, l2]) = merge l1 l2
merge :: Ord a => [a] -> [a] -> [a]
merge l1 l2 = case l1 of
  [] -> l2
  x1 : r1 -> case l2 of
     [] -> l1
     x2 : r2 -> if x1 < x2
       then x1 : merge r1 l2
       else x2 : merge l1 r2
mergeSort :: Ord a => [a] -> [a]
mergeSort = genSort splitMergeSort joinMergeSort
splitSelectionSort :: Ord a => [a] -> Split a a
splitSelectionSort l =
  let m = minimum l in Split m [delete m l]
joinSelectionSort :: Split a a -> [a]
joinSelectionSort (Split a [l]) = a : l
selectionSort :: Ord a => [a] -> [a]
selectionSort = genSort splitSelectionSort joinSelectionSort
```

2.3 HasCASL

This section presents the language *HasCASL*[10]. The formal language definition can be found in another document [8].

The language *HasCASL* is an extension of *CASL* with concepts of higher-order logic such as high order types and functions, polymorphism and type constructors. *HasCASL* was planned to have Haskell as its subset; this should make it possible to transform a *HasCASL* specification in a Haskell program in a simple way.

Standard higher-order logic does not allow recursive types and functions widely used in functional languages. *HasCASL* tries to solve this problem without using denotational semantic, by creating an internal logic to λ-abstractions which is not a primitive concept, but that emerges from the constructions. Thus, although higher-order properties can be obtained, *HasCASL* remains close to the *CASL* language.

The sentences in *HasCASL* differ from those in *CASL* in two respects:

- Quantifiers (universal, existential and unique existential) can be applied on type variables and have restrictions related to sub-types;
- *CASL* predicates are replaced by terms of the type `Unit`.

Unlike in functional programming languages, polymorphic operators must be explicitly instantiated, since it is not yet clear, theoretically, how they relate to resolution of sub-type overloads and implicit instantiation.

As *HasCASL* tries to keep as close as possible to *CASL*, its semantic is also based on set theory. Intentional Henkin models are chosen to model higher-order signatures in the *HasCASL* semantic. In this model, the types of functions are interpreted by arbitrary sets equipped with an application function of the appropriate type (opposed to a partial type `s ->? t` been interpreted by the complete set of all partial functions from `s` to `t`). The interpretation of the λ-terms is part of the model structure rather than just being an existential axiom.

The intensional Henkin model has some advantages, including: it eliminates the completeness problem; allows initial models of signatures containing partial functions; and allows the operational semantics of functional programming languages to be applied, instead of directly using higher-order logic operational semantic.

Unlike Haskell, in which function evaluation is lazy, the evaluation of functions in *HasCASL* is strict, i.e., undefined arguments always result in undefined values. One way to emulate the lazy evaluation is to move a parameter with type `a` to the unit type `Unit ->? a`.

To illustrate the language syntax, we'll take a look into the Specification 2.3.1. Types are defined by the reserved word `type`, which may be preceded by the qualifiers `free` and `generated`, as in *CASL*. Defining types which contain function types as constructor parameters and recursion only on the right side of the arrow should be done with the reserved word `cofree`; when recursion is present in both sides of the arrow, the types must be defined with the reserved word `free`. Type `Bool` was defined as `free` type with two constructors (`True` and `False`).

Functions may be defined by the word `fun`, which differs from the command `op` with relation to their behaviour over sub-typing [9]. A lazy type differs from a strict one by a question mark in front of the type, as in `?Bool`. Functions in mixfix notation have their parameters indicated by the placeholder `__` and the parameter types must be defined as tuples. Thus, the function `__&&__` expects two elements of type `Bool` (indicated by: `?Bool * ?Bool`) and returns one element of that type (indicated by: `-> ?Bool`). Curried functions are defined applying them to the parameters in opposite to using the placeholder and the types of the parameters should be separated by `->` in place of `*`.

Variables are introduced by the word `var` followed by a list of one or more variables and the type of these variables, separated from the list of variables by a colon.

Axioms and theorems are introduced by a final point. Annotations are included in front of axioms and theorems to make it easier to reference them and to allow their use by tools; the annotation should be a name between `%(` and `)%`.

Specification 2.3.1 Initial Bool Specification from scratch

```
spec Bool = %mono
     free type Bool ::= True | False
     fun Not__: ?Bool -> ?Bool
     fun __&&__: ?Bool * ?Bool -> ?Bool
     fun __||__: ?Bool * ?Bool -> ?Bool
     fun otherwise: ?Bool
     vars x,y: ?Bool
     . Not(False) = True                %(Not_False)%
     . Not(True) = False                %(Not_True)%
     . False && False = False           %(And_def1)%
     . False && True = False            %(And_def2)%
     . True  && False = False           %(And_def3)%
     . True  && True = True             %(And_def4)%
     . x || y = Not(Not(x) && Not(y))   %(Or_def)%
     . otherwise = True                 %(Otherwise_def)%
end
```

2.4 Heterogeneous Specifications: HetCASL and Hets

Nowadays, in the formal method area, different logics and methods are used to specify large systems because there isn't a single best solution to achieve all the desired functionalities. These heterogeneous specifications must have a formal interoperability between the languages involved in such a way that each language may have its own proof method and all formal proofs must be consistent when viewed in terms of the heterogeneous specification.

The various sub-languages and extensions of *CASL* maybe linked by the language *Heterogeneous CASL* (*HetCASL*) [12], which has the structural constructions of *CASL*. *HetCASL* extends the semantic properties of *CASL* by being institution independent with constructions that forcing the relationship between translations of specifications that occur in conjunction with translations of the logics. It is worth emphasizing that *HetCASL* preserves the fact that the logic used by individual specifications are orthogonal to *CASL*.

The *Heterogeneous Tool Set* (*Hets*) [12] is a syntactic analyzer and a proof manager for *HetCASL* specifications, implemented in Haskell, which combines the various proof tools for each individual logic used in various sub-languages and extensions of *CASL*. *Hets* is based on a graph of logics and languages, providing a clear semantic and a proof calculus for heterogeneous specifications.

Each logic of the graph is represented by a set of types and functions in Haskell. The syntax and semantics of the heterogeneous specifications in *HetCASL* and their implementations are parametrized by an arbitrary graph of logics inside *Hets*. This allows easily management of each *Hets* module implementation using software engineering techniques.

HasCASL specifications are translated to the Isar language, which is the language used by the *Isabelle* theorem prover [5], a semi-automatic theorem prover for higher-order logics. Hets supports other first-order theorem provers for proving *CASL* specifications. Other *CASL* sub-languages or extensions maybe proved by translating them to *CASL* or *HasCASL*.

The structure of proofs in *Hets* is based on the formalism of development graphs [4], widely used for specifications of industrial systems. The graph structure allows for a direct visualization of the specification structure and facilitates the management of specifications with many sub-specifications.

A development graph consists of a number of nodes (corresponding to complete specifications or parts+ of specifications) and a set of edges called definition links that indicate dependency between the various specifications and their sub-specifications. Each node is associated with a signature and a local set of axioms; these axioms are inherited by other nodes which depend on this node through definition links. Different types of edges are used to indicate when the logic is changed between two nodes.

A second type of edge, a theorem link, is used to indicate relations between different theories, serving to represent proof needs that arise during the specification development. Theorem links can be global or local (represented by edges with different shapes in the graph): global links indicate that all valid axioms in the source node are valid in the target node; local links indicate that only axioms defined in the source node are valid in target node.

Global theory links are broken down into simpler links (global or local) through proof calculus for development graphs. Local links may be proved by transforming them into local proof goals; this transformation will mark the node corresponding to that goal to be proved using the theorem prover for the logic represented on this node.

2.5 Isabelle

This section describes the theorem prover *Isabelle* [2]; a full description can be found on the manual [5].

Isabelle is a generic theorem prover that allows the use of several logics as formal calculus to assist in theorem proofs. *Hets* uses *Isabelle* to prove theorems in higher-order logic, but the prover allows, for example, the use of axiomatized set theory, among other logics. Support for multiple logic is one of the prominent features of the tool.

The prover has an excellent support for mathematical notation: new symbols may be included using common mathematical syntax and proofs can be described in a structured way or as a sequence of proof commands. Proofs may include TEX codes so that formatted documents can be generated directly from the proof source.

The major limitation of theorem provers is the usual need for an extensive previous experience from the users. In order to facilitate the process of proof construction, *Isabelle* has some tools that automate some proof contents, such as equations, basic arithmetic and mathematical formulas.

The higher-order language (HOL) is used to write theories. Its syntax is very similar to those of functional programming languages because it is based on the typed λ-calculus. This language allows construction of data types, types with functions as parameters and other constructions common in functional languages. Translation of *HasCASL* specifications to HOL theories are automatically made by the *Hets* tool.

Isabelle has an extension, called Isar, which allows one to describe proofs that can be read by humans and can be easily interpreted by computers. It has an extensive library of mathematical theories already proved (for example, in topics like algebra and set theory), and also many examples of proofs carried out in a formal verification context. In this work, proofs were written using proof commands, although they are less powerful than the notation used in Isar.

2.6 Proposal

A prerequisite for the practical use of a specification language is the availability of a set of standard specifications previously defined [7]. *CASL* language has such set o specifications defined in "CASL Basic Datatypes" [6]; instead of providing common blocks for reuse as programming languages usually do, this document provides complete specification examples that illustrate the use of *CASL* both in terms of Basic Specifications and Structured Specification. There are two groups of examples: one with basic data types and one with specifications that express properties of complex structures. In the first case, we can find simple data types, such as numbers and characters, as well as structured data types, such as lists, vectors and matrices; the second group contains algebraic structures such as rings and monoids and mathematical properties such as equivalence relation and partial order.

Currently, *HasCASL* language does not have a library along the lines of *CASL* library. According Scröder, data types described in "CASL Basic Datatypes" can serve as a basis for building a standard library to each *CASL* extensions. In case of *HasCASL*, it is suggested the inclusion of new specifications that involve higher order features such as completeness of partial orders as well as extending data types and changing parameterization for real type dependence. As example, higher order functions operating on lists, such as *map*, *filter* and *fold*, can be specified after importing already defined functions on List data type from *CASL* library to improve reuse.

Based on these suggestions, we propose to build a library for *HasCASL* based on *CASL* library and Haskell Prelude library. Creating such a library can be very useful to increase *HasCASL* usage in real projects by providing predefined specifications for reuse. As Prelude library must be implemented by all Haskell compilers, having its data types already specified in *HasCASL* can contribute to automatic code generation in the future as, once these data types are already specified, verified and refined to Haskell code, larger specifications using them can be created and translated to Haskell in an easier way.

Creation of such a library required studing how Haskell functions and types operate and finding solutions to include these elements on our library with maximum reuse of *CASL* library data types. Learning *CASL*, *HasCASL* and *Isabelle* and dealing with their peculiarities were the center of the project difficulties.

All generated specifications were verified by *Hets* tool and most of them were proved using *Isabelle* to ensure their correctness.

3 Specifying the library

In this section we intended to explain what we've been working on. We started by discussing the choices we'd to do when we started the project. Later, for each specification we'd done, we listed its source and explained some problems we've faced and some choices we've made when writing that specification.

3.1 Initial choices

To fully capture Haskell features, our library should use laziness, be refined to use continuous, this allowing infinite data types. Since starting with all these functionalities would require using the the most advanced constructions of the *HasCASL* language and would require deep know-how of *Isabelle* proof scripts, it would not be the best first approach to use as an algebraic specification methodology. Thus, we decided that the library should be specified using strict types and more advanced Haskell features should be left for a latter refinement.

Different from Haskell, *HasCASL* doesn't allow the same function to be used both in prefix and infix notation. Thus, all functions from the *CASL* library which were defined in a mixfix way (and thus expected tuples as parameters) wouldn't be compatible with Haskell curried functions. To solve this problem, we redefined functions from the *CASL* library in a mixfix way and, for each mixfix definition, we created a curried version whose name would be formed by enclosing the name of the mixfix function between brackets. This solution created a pattern for naming curried functions that was easy to remember and allowed all of our functions to be curried with other functions.

To write our library, we used names from Prelude functions and types. When importing, we changed the imported name to the one used by the Prelude version using *CASL* renaming syntax. When there was any function in Prelude that had no equivalent in an existent *CASL* specification, we included that function in our *HasCASL* type to match Prelude types and functions as much as possible.

3.2 Our first specification: Bool

We started our library by importing type Boolean from the *CASL* library, like show in Specification 3.2.1.

Specification 3.2.1 Initial Bool Specification importing *CASL* type

```
from Basic/SimpleDatatypes get Boolean
spec Bool = {Boolean with
         Boolean |-> Bool,
         Not__ |-> not__,
         __And__ |-> __&&__,
         __Or__ |-> __||__
        }
     then
     op otherwise: Bool
    . otherwise = True
```

As we were still pondering about using laziness, we decided that it should be better to specify Boolean from scratch, since the one imported from *CASL* had only total functions. This tentative is shown in Specification 2.3.1.

Specification 3.2.2 Initial Bool Specification from scratch

```
spec Bool = %mono
     free type Bool ::= True | False
     fun Not__: ?Bool ->? ?Bool
     fun __&&__: ?Bool * ?Bool ->? ?Bool
     fun __||__: ?Bool * ?Bool ->? ?Bool
     fun otherwise: ?Bool
     vars x,y: ?Bool
     . Not(False) = True                %(Not_False)%
     . Not(True) = False                %(Not_True)%
     . False && False = False           %(And_def1)%
     . False && True = False            %(And_def2)%
     . True  && False = False           %(And_def3)%
     . True  && True = True             %(And_def4)%
     . x || y = Not(Not(x) && Not(y))   %(Or_def)%
     . otherwise = True                 %(Otherwise_def)%
end
```

Next, we've decided to use only stric types, as we could, later, refine our specifications to use laziness. We have also included curried versions for both boolean operations that are mixfix in the *CASL* version, as well as some axioms that whoul be needed later in *Isabelle* proofs that couldn't be done automatically. As "*otherwise*" is an *Isabelle* reserved word, we appended an *H*, from *H*askell, to its name. We thus achieved Specification 3.2.3.

3.3 The Specification for Equality

After defining the `Bool` type, the next step was to specify equality functions. As we were working over `Bool`, we could not use *HasCASL* predicates and their related operations. We thus had to redefine all functions and operations related to element comparison to use our `Bool` type. As in the Haskell Prelude, equality functions were grouped in a class named `Eq`, giving us Specification 3.3.1.

Equality was defined including axioms for symmetry, reflexivity and transitivity. An axiom mapping *HasCASL* equality to our equality was created, namelly, `%(EqualTDef)%`, since the opposite map cannot be created because it would be too restrictive. Negation was defined by negating equality, as any equation involving

Specification 3.2.3 Boolean Specification

```
spec Bool = %mono
free type Bool ::= True | False
fun Not__ : Bool -> Bool
fun __&&__ : Bool * Bool -> Bool
fun <&&> : Bool -> Bool -> Bool
fun __||__ : Bool * Bool -> Bool
fun <||> : Bool -> Bool -> Bool
fun otherwiseH: Bool
vars x,y: Bool
. Not(False) = True                %(NotFalse)%
. Not(True) = False                %(NotTrue)%
. False && x = False               %(AndFalse)%
. True && x = x                    %(AndTrue)%
. x && y = y && x                  %(AndSym)%
. x || y = Not(Not(x) && Not(y))   %(OrDef)%
. otherwiseH = True                %(OtherwiseDef)%
. <&&> x y = x && y                %(AndPrefixDef)%
. <||> x y = x || y                %(OrPrefixDef)%
%%
. Not x = True <=> x = False       %(NotFalse1)% %implied
. Not x = False <=> x = True       %(NotTrue1)% %implied
. not (x = True) <=> Not x = True  %(notNot1)% %implied
. not (x = False) <=> Not x = False %(notNot2)% %implied
end
```

Specification 3.3.1 Equality specification

```
spec Eq = Bool then
class Eq {
var a: Eq
fun __==__ : a * a -> Bool
fun <==> : a -> a -> Bool
fun __/=__ : a * a -> Bool
fun </=> : a-> a-> Bool
vars x,y,z: a
. x = y => (x == y) = True                                %(EqualTDef)%
. x == y = y == x                                       %(EqualSymDef)%
. (x == x) = True                                       %(EqualReflex)%
. (x == y) = True /\ (y == z) = True => (x == z) = True  %(EqualTransT)%
. (x /= y) = Not (x == y)                                  %(DiffDef)%
. <==> x y = x == y                                  %(EqualPrefixDef)%
. </=> x y = x /= y                                   %(DiffPrefixDef)%
. (x /= y) = (y /= x)                            %(DiffSymDef)% %implied
. (x /= y) = True <=> Not (x == y) = True          %(DiffTDef)% %implied
. (x /= y) = False <=> (x == y) = True             %(DiffFDef)% %implied
. (x == y) = False => not (x = y)          %(TE1)% %implied
. Not (x == y) = True <=> (x == y) = False   %(TE2)% %implied
. Not (x == y) = False <=> (x == y) = True   %(TE3)% %implied
. not ((x == y) = True) <=> (x == y) = False %(TE4)% %implied
}
type instance Bool: Eq
. (True == True) = True              %(IBE1)% %implied
. (False == False) = True            %(IBE2)% %implied
. (False == True) = False            %(IBE3)%
. (True == False) = False            %(IBE4)% %implied
. (True /= False) = True             %(IBE5)% %implied
. (False /= True) = True             %(IBE6)% %implied
. Not (True == False) = True         %(IBE7)% %implied
. Not (Not (True == False)) = False  %(IBE8)% %implied
type instance Unit: Eq
. (() == ()) = True  %(IUE1)% %implied
. (() /= ()) = False %(IUE2)% %implied
end
```

negation could be translated to a negated equality and thus proved using the equality axioms. Curried versions for both functions were also defined. Seven auxiliary theorems were created and proved to be used by *Isabelle*, if needed.

Type instances were declared, as it's done in Prelude, for `Bool` and `Unit` data types. In the first case, although `Bool` is a free data type and, hence, `True` is different from `False`, this difference had to be axiomatized by the axiom `%(IBE3)%` because our equality is not mapped to the *HasCASL* equality; all the other theorems for `Bool` instance declarations should follow from `%(IBE3)%` and the other `Eq` axioms. In the second case, as `()` is the only element from type `Unit`, instance definitions should be theorems as they follow from the `Eq` axioms.

3.4 The Specification for Ordering

The next specification we defined was `Ord`, for Ordering relations. Our first approach was to import the partial order defined by the Ord specification inside the library *HasCASL/Metatheory/Ord*. As importing this library would cause problems to our strict library, because the imported one uses lazy types, we decided to specify our own version.

To create the Ord specification we defined the `Ordering` data type and declared this type as an instance of the Eq class: three axioms relate the three constructors and the other theorems follow from them. See Specification 3.4.1 for details. As in Haskell, we defined the Ord class to be a subclass of class Eq. We specified a total order function `__<__` and all the other ordering functions were defined using this function. Irreflexivity, asymmetry, transitivity and totality properties appear as theorems over the ordering functions plus `__<__`.

Next, four axioms defining equality in function of functions, four axioms to swap equal variables in the `__<__` function, and two axioms relating total and partial ordering involving equality were defined. Twenty one theorems relating ordering functions guarantee that these functions work as expected. Curried version for ordering functions were defined, followed by the definition of the `compare`, `min` and `max` functions. Next, two theorems relating `min` and `max` functions were specified and proved. Seven auxiliary theorems were included, as some of them were needed in *Isabelle* proofs later, specially `%(TO6)%`, which relates ordering functions and the funciton `Not__`.

The folloing types were declared instances of the `Ord` class: `Ordering`, `Bool`, `Nat` and `Unit`. For the first two data types we needed to axiomatically define how `__<__` works because they have more than one type constructor. For the type `Nat` we only declared the type to be an instance of Ord, but we didn't define the axioms. For the type `Unit` all functions can be proved because there is only one member of this type.

Specification 3.4.1 Ord Specification - Part 1

```
spec Ord = Eq and Bool then
free type Ordering ::= LT | EQ | GT
type instance Ordering: Eq
. (LT == LT) = True   %(IOE01)% %implied
. (EQ == EQ) = True   %(IOE02)% %implied
. (GT == GT) = True   %(IOE03)% %implied
. (LT == EQ) = False  %(IOE04)%
. (LT == GT) = False  %(IOE05)%
. (EQ == GT) = False  %(IOE06)%
. (LT /= EQ) = True   %(IOE07)% %implied
. (LT /= GT) = True   %(IOE08)% %implied
. (EQ /= GT) = True   %(IOE09)% %implied
class Ord < Eq {
 var a: Ord
 fun compare: a -> a -> Ordering
 fun __<__ : a * a -> Bool
 fun <<> : a -> a -> Bool
 fun __>__ : a * a -> Bool
 fun <>> : a -> a -> Bool
 fun __<=__ : a * a -> Bool
 fun <<=> : a -> a -> Bool
 fun __>=__ : a * a -> Bool
 fun <>=> : a -> a -> Bool
 fun min: a -> a -> a
 fun max: a -> a -> a
 var    x, y, z, w: a
 . (x == y) = True => (x < y) = False         %(LeIrreflexivity)%
 . (x < y) = True => y < x = False           %(LeTAsymmetry)% %implied
 . (x < y) = True /\ (y < z) = True => (x < z) = True  %(LeTTransitive)%
```

Specification 3.4.1 Ord Specification - Part 2

```
 . (x < y) = True \/ (y < x) = True
 \/ (x == y) = True                                  %(LeTTotal)%
 . (x > y) = (y < x)                                 %(GeDef)%
 . (x == y) = True => (x > y) = False         %(GeIrreflexivity)% %implied
 . (x > y) = True => (y > x) = False             %(GeTAsymmetry)% %implied
 . ((x > y)  && (y > z)) = True
 => (x > z) = True                                 %(GeTTransitive)% %implied
 . (((x > y) || (y > x)) || (x == y)) = True  %(GeTTotal)% %implied
 . (x <= y) = (x < y) || (x == y)                    %(LeqDef)%
 . (x <= x) = True                                %(LeqReflexivity)% %implied
 . ((x <= y) && (y <= z)) = True
 => (x <= z) = True                               %(LeqTTransitive)% %implied
 . (x <= y) && (y <= x) = (x == y)                   %(LeqTTotal)% %implied
 . (x >= y) = ((x > y) || (x == y))                  %(GeqDef)%
 . (x >= x) = True                                %(GeqReflexivity)% %implied
 . ((x >= y) && (y >= z)) = True
 => (x >= z) = True                               %(GeqTTransitive)% %implied
 . (x >= y) && (y >= x) = (x == y)                   %(GeqTTotal)% %implied
 . (x == y) = True <=> (x < y) = False /\ (x > y) = False  %(EqTSOrdRel)%
 . (x == y) = False <=> (x < y) = True \/ (x > y) = True   %(EqFSOrdRel)%
 . (x == y) = True <=> (x <= y) = True /\ (x >= y) = True  %(EqTOrdRel)%
 . (x == y) = False <=> (x <= y) = True \/ (x >= y) = True %(EqFOrdRel)%
 . (x == y) = True /\ (y < z) = True => (x < z) = True    %(EqTOrdTSubstE)%
 . (x == y) = True /\ (y < z) = False => (x < z) = False  %(EqTOrdFSubstE)%
 . (x == y) = True /\ (z < y) = True => (z < x) = True    %(EqTOrdTSubstD)%
 . (x == y) = True /\ (z < y) = False => (z < x) = False  %(EqTOrdFSubstD)%
 . (x < y) = True
 <=> (x > y) = False /\ (x == y) = False                  %(LeTGeFEqFRel)%
 . (x < y) = False
 <=> (x > y) = True \/ (x == y) = True                    %(LeFGeTEqTRel)%
 . (x < y) = True <=> (y > x) = True            %(LeTGeTRel)% %implied
 . (x < y) = False <=> (y > x) = False          %(LeFGeFRel)% %implied
 . (x <= y) = True <=> (y >= x) = True         %(LeqTGetTRel)% %implied
 . (x <= y) = False <=> (y >= x) = False       %(LeqFGetFRel)% %implied
 . (x > y) = True <=> (y < x) = True            %(GeTLeTRel)% %implied
 . (x > y) = False <=> (y < x) = False          %(GeFLeFRel)% %implied
 . (x >= y) = True <=> (y <= x) = True         %(GeqTLeqTRel)% %implied
 . (x >= y) = False <=> (y <= x) = False       %(GeqFLeqFRel)% %implied
 . (x <= y) = True <=> (x > y) = False          %(LeqTGeFRel)% %implied
 . (x <= y) = False <=> (x > y) = True          %(LeqFGeTRel)% %implied
```

Specification 3.4.1 Ord Specification - Part 3

```
 . (x > y) = True
 <=> (x < y) = False /\ (x == y) = False        %(GeTLeFEqFRel)% %implied
 . (x > y) = False
 <=> (x < y) = True \/ (x == y) = True          %(GeFLeTEqTRel)% %implied
 . (x >= y) = True <=> (x < y) = False            %(GeqTLeFRel)% %implied
 . (x >= y) = False <=> (x < y) = True            %(GeqFLeTRel)% %implied
 . (x <= y) = True
 <=> (x < y) = True \/ (x == y) = True         %(LeqTLeTEqTRel)% %implied
 . (x <= y) = False
 <=> (x < y) = False /\ (x == y) = False       %(LeqFLeFEqFRel)% %implied
 . (x >= y) = True
 <=> (x > y) = True \/ (x == y) = True         %(GeqTGeTEqTRel)% %implied
 . (x >= y) = False
 <=> (x > y) = False /\ (x == y) = False       %(GeqFGeFEqFRel)% %implied
 . (x < y) = True <=> (x >= y) = False            %(LeTGeqFRel)% %implied
 . (x > y) = True <=> (x <= y) = False            %(GeTLeqFRel)% %implied
 . (x < y) = (x <= y) && (x /= y)                  %(LeLeqDiff)% %implied
 . <<> x y = x < y                                  %(LePrefixDef)%
 . <<=> x y = x <= y                                %(LeqPrefixDef)%
 . <>> x y = x > y                                  %(GePrefixDef)%
 . <>=> x y = x >= y                                %(GeqPrefixDef)%
 . (compare x y == LT) = (x < y)                    %(CmpLTDef)%
 . (compare x y == EQ) = (x == y)                   %(CmpEQDef)%
 . (compare x y == GT) = (x > y)                    %(CmpGTDef)%
 . (max x y == y) = (x <= y)                        %(MaxYDef)%
 . (max x y == x) = (y <= x)                        %(MaxXDef)%
 . (min x y == x) = (x <= y)                        %(MinXDef)%
 . (min x y == y) = (y <= x)                        %(MinYDef)%
 . (max x y == y) = (max y x == y)                  %(MaxSym)% %implied
 . (min x y == y) = (min y x == y)                  %(MinSym)% %implied
}
. (x == y) = True \/ (x < y) = True <=> (x <= y) = True %(TO1)% %implied
. (x == y) = True  => (x < y) = False                  %(TO2)% %implied
. Not (Not (x < y)) = True \/ Not (x < y) = True       %(TO3)% %implied
. (x < y) = True => Not (x == y) = True                %(TO4)% %implied
. (x < y) = True /\ (y < z) = True /\ (z < w) = True
=> (x < w) = True                                      %(TO5)% %implied
. (z < x) = True => Not (x < z) = True                 %(TO6)% %implied
. (x < y) = True <=> (y > x) = True                    %(TO7)% %implied
```

Specification 3.4.1 Ord Specification - Part 4

```
type instance Ordering: Ord
. (LT < EQ) = True                          %(IO013)%
. (EQ < GT) = True                          %(IO014)%
. (LT < GT) = True                          %(IO015)%
. (LT <= EQ) = True                         %(IO016)% %implied
. (EQ <= GT) = True                         %(IO017)% %implied
. (LT <= GT) = True                         %(IO018)% %implied
. (EQ >= LT) = True                         %(IO019)% %implied
. (GT >= EQ) = True                         %(IO020)% %implied
. (GT >= LT) = True                         %(IO021)% %implied
. (EQ > LT) = True                          %(IO022)% %implied
. (GT > EQ) = True                          %(IO023)% %implied
. (GT > LT) = True                          %(IO024)% %implied
. (max LT EQ == EQ) = True                  %(IO025)% %implied
. (max EQ GT == GT) = True                  %(IO026)% %implied
. (max LT GT == GT) = True                  %(IO027)% %implied
. (min LT EQ == LT) = True                  %(IO028)% %implied
. (min EQ GT == EQ) = True                  %(IO029)% %implied
. (min LT GT == LT) = True                  %(IO030)% %implied
. (compare LT LT == EQ) = True              %(IO031)% %implied
. (compare EQ EQ == EQ) = True              %(IO032)% %implied
. (compare GT GT == EQ) = True              %(IO033)% %implied
type instance Bool: Ord
. (False < True) = True                     %(IBO5)%
. (False >= True) = False                   %(IBO6)% %implied
. (True >= False) = True                    %(IBO7)% %implied
. (True < False) = False                    %(IBO8)% %implied
. (max False True == True) = True           %(IBO9)% %implied
. (min False True == False) = True          %(IBO10)% %implied
. (compare True True == EQ) = True          %(IBO11)% %implied
. (compare False False == EQ) = True        %(IBO12)% %implied
type instance Nat: Ord
type instance Unit: Ord
. (() <= ()) = True                         %(IUO01)% %implied
. (() <  ()) = False                        %(IUO02)% %implied
. (() >= ()) = True                         %(IUO03)% %implied
. (() > ()) = False                         %(IUO04)% %implied
. (max () () == ()) = True                  %(IUO05)% %implied
. (min () () == ()) = True                  %(IUO06)% %implied
. (compare () () == EQ) = True              %(IUO07)% %implied
end
```

3.5 Maybe, Either, MaybeMonad and EitherFunctor Specifications

The data type `Maybe a`, where `a` is a type variable, has constructors: `Just a` and `Nothing`, as shown in Specification 3.5.1. It has an associated `maybe` function that applies a function to the value `x` of a constructor `Just x`, and returns this application's result or returns a default value, received as parameter.

We declare the type `Maybe` to be an instance of the class `Eq` by defining how equality works on two elements of the `Just` constructor. Next, we prove that it works as expected on two `Nothing` constructors and then define the result of comparing both `Just` and `Nothing` constructors.

The type instance declaration for class `Ord` defines how function `__<__` compares `Just` and `Nothing` constructors, and how it compares two different `Just` elements. Comparing two elements of the `Nothing` constructor doesn't need to be defined because they always compare two equal elements (two copies of the `Nothing` constructor). The theorems prove that the other comparing functions work as expected when comparing `Just` and `Nothing` constructors. More theorems involving two elements of the `Just` constructor could be proved just as we did for `Just` and `Nothing`. We decided not to write them because all of them should follow from the ordering theorems after applying some comparing axioms and the axioms `%(IMO12)%` and `%(IME03)%`. Unless *Isabelle* needs them later, writing these theorems would only take a lot of time and wouldn't change the way the specification is defined.

Data type `Either a b`, where `a` and `b` are types, has constructors `Left a` and `Right b`, as shown in Specification 3.5.2. The associated function `either` receives as parameters two functions and an `Either a b` element. Then function `either` applies the first function received to the element in case its constructor is the `Left a` constructor; the second functions is applied to the same element in case the constructor is `Right b`.

`Either` was declared an an instance of the class Eq by three equality comparisons: first, between two elements with the constructor `Left a`; next, between two elements with the constructor `Right b`; and last, between one element whith each of those constructors.

The type declaration for class Ord defines how the function `__<__` works with two different constructors and with two elements of each constructor. The theorems were, again, defined by relating two elements of distinct constructors with the ordering relations, as done in the `Maybe` data type specification.

We separated the functor and monadic functions for `Maybe` and `Either` data types in different specifications, as show in Specification 3.5.3 and in Specification 3.5.4, respectively. At this time, *Hets* cannot translate functions from constructor classes, as the Monad class. Thus, these specifications can only be syntactically checked by *Hets*, but not translated to and neither proved by *Isabelle*. Our approach was to

Specification 3.5.1 Maybe Specification

```
spec Maybe = Eq and Ord then
var a,b,c : Type;
    e : Eq;
    o : Ord;
free type Maybe a ::= Just a | Nothing
var x : a;
    y : b;
    ma : Maybe a;
    f : a -> b
fun maybe : b -> (a -> b) -> Maybe a -> b
. maybe y f (Just x: Maybe a) = f x                        %(MaybeJustDef)%
. maybe y f (Nothing: Maybe a) = y                      %(MaybeNothingDef)%
type instance Maybe e: Eq
var x,y : e;
. (Just x == Just y) = True <=> (x == y) = True        %(IME01)%
. ((Nothing : Maybe e) == (Nothing: Maybe e)) = True  %(IME02)% %implied
. Just x == Nothing = False                            %(IME03)%
type instance Maybe o: Ord
var x,y : o;
. (Nothing < Just x) = True                            %(IMO01)%
. (Just x < Just y) = (x < y)                          %(IMO02)%
. (Nothing >= Just x) = False                         %(IMO03)% %implied
. (Just x >= Nothing) = True                          %(IMO04)% %implied
. (Just x < Nothing) = False                          %(IMO05)% %implied
. (compare Nothing (Just x) == EQ)
     = (Nothing == (Just x))                          %(IMO06)% %implied
. (compare Nothing (Just x) == LT)
     = (Nothing < (Just x))                           %(IMO07)% %implied
. (compare Nothing (Just x) == GT)
     = (Nothing > (Just x))                           %(IMO08)% %implied
. (Nothing <= (Just x))
     = (max Nothing (Just x) == (Just x))             %(IMO09)% %implied
. ((Just x) <= Nothing)
     = (max Nothing (Just x) == Nothing)              %(IMO10)% %implied
. (Nothing <= (Just x))
     = (min Nothing (Just x) == Nothing)              %(IMO11)% %implied
. ((Just x) <= Nothing)
     = (min Nothing (Just x) == (Just x))             %(IMO12)% %implied
end
```

Specification 3.5.2 Either Specification

```
spec Either = Eq and Ord then
var a, b, c : Type; e, ee : Eq; o, oo : Ord;
free type Either a b ::= Left a | Right b
var x : a; y : b; z : c; eab : Either a b; f : a -> c; g : b -> c
fun either : (a -> c) -> (b -> c) -> Either a b -> c
. either f g (Left x: Either a b) = f x             %(EitherLeftDef)%
. either f g (Right y: Either a b) = g y           %(EitherRightDef)%
type instance Either e ee: Eq
var x,y : e; z,w : ee;
. ((Left x : Either e ee) ==
   (Left y : Either e ee)) = (x == y)                       %(IEE01)%
. ((Right z : Either e ee) ==
   (Right w : Either e ee)) = (z == w)                      %(IEE02)%
. ((Left x : Either e ee) ==
   (Right z : Either e ee)) = False                         %(IEE03)%
type instance Either o oo: Ord
var x,y : o; z,w : oo;
. ((Left x : Either o oo) < (Right z : Either o oo))  = True    %(IEO01)%
. ((Left x : Either o oo) < (Left y : Either o oo)) = (x < y)   %(IEO02)%
. ((Right z : Either o oo) < (Right w : Either o oo)) = (z < w)  %(IEO03)%
. ((Left x : Either o oo) >= (Right z : Either o oo))
    = False                                         %(IEO04)% %implied
. ((Right z : Either o oo) >= (Left x : Either o oo))
    = True                                          %(IEO05)% %implied
. ((Right z : Either o oo) < (Left x : Either o oo))
    = False                                         %(IEO06)% %implied
. (compare (Left x : Either o oo) (Right z : Either o oo) == EQ)
    = ((Left x) == (Right z))                       %(IEO07)% %implied
. (compare (Left x : Either o oo) (Right z : Either o oo) == LT)
    = ((Left x) < (Right z))                        %(IEO08)% %implied
. (compare (Left x : Either o oo) (Right z : Either o oo) == GT)
    = ((Left x) > (Right z))                        %(IEO09)% %implied
. ((Left x : Either o oo) <= (Right z : Either o oo))
    = (max (Left x) (Right z) == (Right z))         %(IEO10)% %implied
. ((Right z : Either o oo) <= (Left x : Either o oo))
    = (max (Left x) (Right z) == (Left x))          %(IEO11)% %implied
. ((Left x : Either o oo) <= (Right z : Either o oo))
    = (min (Left x) (Right z) == (Left x))          %(IEO12)% %implied
. ((Right z : Either o oo) <= (Left x : Either o oo))
    = (min (Left x) (Right z) == (Right z))         %(IEO13)% %implied
end
```

declare all functions from the Functor and Monad classes as theorems, so that later, if some of them must be redefined as axioms, we can remove the `%implied` directive and change the theorems into axioms.

Specification 3.5.3 MaybeMonad Specification

```
from HasCASL/Metatheory/Monad get Functor, Monad

spec MaybeMonad = Maybe and Monad then
var a,b,c : Type; e : Eq; o : Ord;
type instance Maybe: Functor
vars  x: Maybe a; f: a -> b; g: b -> c
. map (\ y: a .! y) x = x                              %(IMF01)% %implied
. map (\ y: a .! g (f y)) x = map g (map f x)          %(IMF02)% %implied
type instance Maybe: Monad
vars  x, y: a;
      p: Maybe a;
      q: a ->? Maybe b;
      r: b ->? Maybe c;
      f: a ->? b
. def q x => ret x >>= q = q x                         %(IMM01)% %implied
. p >>= (\ x: a . ret (f x) >>= r)
     = p >>= \ x: a . r (f x)                          %(IMM02)% %implied
. p >>= ret = p                                        %(IMM03)% %implied
. (p >>= q) >>= r = p >>= \ x: a . q x >>= r           %(IMM04)% %implied
. (ret x : Maybe a) = ret y => x = y                   %(IMM05)% %implied
var x : Maybe a; f : a -> b;
. map f x = x >>= (\ y:a . ret (f y))                   %(T01)% %implied
end
```

Specification 3.5.4 EitherFunctor Specification

```
from HasCASL/Metatheory/Monad get Functor, Monad

spec EitherFunctor = Either and Functor then
var a, b, c : Type; e, ee : Eq; o, oo : Ord;
type instance Either a: Functor
vars x: Either c a; f: a -> b; g: b -> c
. map (\ y: a .! y) x = x                         %(IEF01)% %implied
. map (\ y: a .! g (f y)) x = map g (map f x)   %(IEF02)% %implied
end
```

3.6 Composition and Function Specifications

To define Haskell functions, we had to define or import function composition. We preferred to define then, because the available definition used lambda expression. Later, we defined some auxiliary functions present in Prelude, as the identity function `id`, and functions to swap between curried and uncurried versions of other functions. These specifictions can be seen on Specification 3.6.1.

Specification 3.6.1 Composition and Function Specifications

```
spec Composition =
vars a,b,c : Type
fun __o__ : (b -> c) * (a -> b) -> (a -> c);
vars a,b,c : Type; y:a;
     f : b -> c;
     g : a -> b
. ((f o g) y) = f (g y)                 %(Comp1)%
end

spec Function = Composition then
var a,b,c: Type;
    x: a;
    y: b;
    f: a -> b -> c;
    g: (a * b) -> c
fun id: a -> a
fun flip: (a -> b -> c) -> b -> a -> c
fun fst: (a * b) -> a
fun snd: (a * b) -> b
fun curry: ((a * b) -> c) -> a -> b -> c
fun uncurry: (a -> b -> c) -> (a * b) -> c
. id x = x                    %(IdDef)%
. flip f y x = f x y          %(FlipDef)%
. fst (x, y) = x              %(FstDef)%
. snd (x, y) = y              %(SndDef)%
. curry g x y = g (x, y)      %(CurryDef)%
. uncurry f (x,y) = f x y     %(UncurryDef)%
end
```

3.7 List Specification

The list specification was the largest one and it still doesn't aggregate all the functions that the *Haskell Prelude* defines, specially those involving numeric types. Once again, we had to redefine our specification to remove laziness. We divided this specification in six parts in order to bring related functions together, in almost the same way as the *Haskell Prelude* does. See Specification 3.7.1.

The first step was to define the `free type List a`, depending on a type `a`, with constructors `Nil` and `Cons a (List a)`. The next step was to redefine basic functions to work without laziness. Two of these functions, `head` and `tail` must be partial, as they are not defined when applied on an empty list.

The second part of the specification contains the type instance declarations. To declare List as an instance of the class Eq we had to define how equality should work and to prove that comparing `Nil` lists worked as expected. To instanciate the declaration to class Ord, we proved that comparing `Nil` lists worked correctly. Next, we defined how function `__<__` compares two lists and, finally, we proved that all the other ordering functions respected their respective specifications.

The third part contains eight theorems involving some functions of the first part of the specification. These theorems are needed in order to specify how those functions interact. They should not be axioms because they must follow from the function definitions. As can be seen, we used the `%implies` directive after the `then` keyword in order to mark all the equations in this part as theorems.

The forth part contains five functions that are listed in the *Haskell Prelude* as List operations. They complete the function operations from the first part. Again, some of these functions had to be partial as they are not defined on empty lists. The fifth part aggregates some special kind of folding functions or functions that create sublists. The last part of this specification brings in functions related to Lists and that are not defined in the *Haskell Prelude*, but are implemented on every compiler and are necessary even to write basic programs.

Specification 3.7.1 List Specification - Part 1

```
spec List = Nat and Function and Ord then
var a : Type
free type List a ::= Nil | Cons a (List a)
var a,b : Type
fun length : List a -> Nat;
fun head : List a ->? a;
fun tail : List a ->? List a;
fun foldr : (a -> b -> b) -> b -> List a -> b;
fun foldl : (a -> b -> a) -> a -> List b -> a;
fun map : (a -> b) -> List a -> List b;
fun filter : (a -> Bool) -> List a -> List a;
fun __++__ : List a * List a -> List a;
fun <++> : List a -> List a -> List a;
fun zip : List a -> List b -> List (a * b);
fun unzip : List (a * b) -> (List a * List b)
vars a,b : Type;
     f : a -> b -> b;
     g : a -> b -> a;
     h : a -> b;
     p : a -> Bool;
     x,y,t : a;
     xs,ys,l : List a;
     z,s : b;
     zs : List b;
     ps : List (a * b)
. length (Nil : List a) = 0                             %(LengthNil)%
. length (Cons x xs) = suc(length xs)                   %(LengthCons)%
. not def head (Nil : List a)                           %(NotDefHead)%
. head (Cons x xs) = x                                  %(HeadDef)%
. not def tail (Nil : List a)                           %(NotDefTail)%
. tail (Cons x xs) = xs                                 %(TailDef)%
. foldr f s Nil = s                                     %(FoldrNil)%
. foldr f s (Cons x xs)
    = f x (foldr f s xs)                                %(FoldrCons)%
. foldl g t Nil = t                                     %(FoldlNil)%
. foldl g t (Cons z zs)
    = foldl g (g t z) zs                                %(FoldlCons)%
. map h Nil = Nil                                       %(MapNil)%
. map h (Cons x xs)
    = (Cons (h x) (map h xs))                           %(MapCons)%
. Nil ++ l = l                                          %(++Nil)%
. (Cons x xs) ++ l = Cons x (xs ++ l)                   %(++Cons)%
. <++> xs ys = xs ++ ys                                 %(++PrefixDef)%
```

Specification 3.7.1 List Specification - Part 2

```
. filter p Nil = Nil                                          %(FilterNil)%
. p x = True
    => filter p (Cons x xs) = Cons x (filter p xs)           %(FilterConsT)%
. p x = False
    => filter p (Cons x xs) = filter p xs                    %(FilterConsF)%
. zip (Nil : List a) l = Nil                                     %(ZipNil)%
. l = Nil
    => zip (Cons x xs) l = Nil                               %(ZipConsNil)%
. l = (Cons y ys)
    => zip (Cons x xs) l = Cons (x,y) (zip xs ys)           %(ZipConsCons)%
. unzip (Nil : List (a * b)) = (Nil, Nil)                      %(UnzipNil)%
. unzip (Cons (x,z) ps) = let (ys, zs) = unzip ps in
     (Cons x ys, Cons z zs)                                   %(UnzipCons)%
then
var a : Eq; x,y: a; xs, ys: List a
type instance List a: Eq
. ((Nil: List a) == (Nil: List a)) = True               %(ILE01)% %implied
. ((Cons x xs) == (Cons y ys)) = ((x == y) && (xs == ys))    %(ILE02)%
var b : Ord; z,w: b; zs, ws: List b
type instance List b: Ord
. ((Nil: List b) < (Nil: List b)) = False               %(ILO01)% %implied
. ((Nil: List b) <= (Nil: List b)) = True               %(ILO02)% %implied
. ((Nil: List b) > (Nil: List b)) = False               %(ILO03)% %implied
. ((Nil: List b) >= (Nil: List b)) = True               %(ILO04)% %implied
. (z < w) = True => ((Cons z zs) < (Cons w ws)) = True       %(ILO05)%
. (z == w) = True => ((Cons z zs) < (Cons w ws)) = (zs < ws) %(ILO06)%
. (z < w) = False /\ (z == w) = False
     => ((Cons z zs) < (Cons w ws)) = False                  %(ILO07)%
. ((Cons z zs) <= (Cons w ws)) = ((Cons z zs) < (Cons w ws))
    || ((Cons z zs) == (Cons w ws))                     %(ILO08)% %implied
. ((Cons z zs) > (Cons w ws))
    = ((Cons w ws) < (Cons z zs))                       %(ILO09)% %implied
. ((Cons z zs) >= (Cons w ws)) = ((Cons z zs) > (Cons w ws))
    || ((Cons z zs) == (Cons w ws))                     %(ILO10)% %implied
. (compare (Nil: List b) (Nil: List b) == EQ)
    = ((Nil: List b) == (Nil: List b))                  %(ILO11)% %implied
. (compare (Nil: List b) (Nil: List b) == LT)
    = ((Nil: List b) < (Nil: List b))                   %(ILO12)% %implied
. (compare (Nil: List b) (Nil: List b) == GT)
    = ((Nil: List b) > (Nil: List b))                   %(ILO13)% %implied
. (compare (Cons z zs) (Cons w ws) == EQ)
    = ((Cons z zs) == (Cons w ws))                      %(ILO14)% %implied
```

Specification 3.7.1 List Specification - Part 3

```
. (compare (Cons z zs) (Cons w ws) == LT)
    = ((Cons z zs) < (Cons w ws))                      %(ILO15)% %implied
. (compare (Cons z zs) (Cons w ws) == GT)
    = ((Cons z zs) > (Cons w ws))                      %(ILO16)% %implied
. (max (Nil: List b) (Nil: List b) == (Nil: List b))
    = ((Nil: List b) <= (Nil: List b))                 %(ILO17)% %implied
. (min (Nil: List b) (Nil: List b) == (Nil: List b))
    = ((Nil: List b) <= (Nil: List b))                 %(ILO18)% %implied
. ((Cons z zs) <= (Cons w ws))
    = (max (Cons z zs) (Cons w ws) == (Cons w ws))     %(ILO19)% %implied
. ((Cons w ws) <= (Cons z zs))
    = (max (Cons z zs) (Cons w ws) == (Cons z zs))     %(ILO20)% %implied
. ((Cons z zs) <= (Cons w ws))
    = (min (Cons z zs) (Cons w ws) == (Cons z zs))     %(ILO21)% %implied
. ((Cons w ws) <= (Cons z zs))
    = (min (Cons z zs) (Cons w ws) == (Cons w ws))     %(ILO22)% %implied
then %implies
vars a,b,c : Ord;
     f : a -> b;
     g : b -> c;
     h : a -> a -> a;
     i : a -> b -> a;
     p : b -> Bool;
     x:a;
     y:b;
     xs,zs : List a;
     ys,ts : List b;
     z,e : a;
     xxs : List (List a)
. foldl i e (ys ++ ts)
    = foldl i (foldl i e ys) ts                        %(FoldlDecomp)%
. map f (xs ++ zs)
    = (map f xs) ++ (map f zs)                         %(MapDecomp)%
. map (g o f) xs = map g (map f xs)                    %(MapFunctor)%
. filter p (map f xs)
    = map f (filter (p o f) xs)                        %(FilterProm)%
. length (xs) = 0 <=> xs = Nil                         %(LengthNil1)%
. length (Nil : List a) = length ys
    => ys = (Nil : List b)                             %(LengthEqualNil)%
. length (Cons x xs) = length (Cons y ys) =>
  length xs = length ys                                %(LengthEqualCons)%
. length xs = length ys
    => unzip (zip xs ys) = (xs, ys)                    %(ZipSpec)%
```

Specification 3.7.1 List Specification - Part 4

```
then
vars a,b : Type;
     x : a;
     xs : List a;
     f: a -> a -> a;
fun init: List a ->? List a;
fun last: List a ->? a;
fun null: List a -> Bool;
fun reverse: List a -> List a;
fun foldr1: (a -> a -> a) -> List a ->? a;
fun foldl1: (a -> a -> a) -> List a ->? a;
. not def init (Nil: List a)                                  %(InitNil)%
. init (Cons x (Nil: List a)) = (Nil:List a)              %(InitConsNil)%
. init (Cons x xs) = Cons x (init xs)                    %(InitConsCons)%
. not def last (Nil: List a)                                  %(LastNil)%
. last (Cons x (Nil: List a)) = x                         %(LastConsNil)%
. last (Cons x xs) = last xs                             %(LastConsCons)%
. null (Nil:List a) = True                                    %(NullNil)%
. null (Cons x xs) = False                                   %(NullCons)%
. reverse (Nil: List a) = (Nil: List a)                    %(ReverseNil)%
. reverse (Cons x xs)
   = (reverse xs) ++ (Cons x (Nil: List a))               %(ReverseCons)%
. not def foldr1 f (Nil: List a)                            %(Foldr1Nil)%
. foldr1 f (Cons x (Nil: List a)) = x                   %(Foldr1ConsNil)%
. foldr1 f (Cons x xs) = f x (foldr1 f xs)             %(Foldr1ConsCons)%
. not def foldl1 f (Nil: List a)                            %(Foldl1Nil)%
. foldl1 f (Cons x (Nil: List a)) = x                   %(Foldl1ConsNil)%
. foldl1 f (Cons x xs) = f x (foldr1 f xs)             %(Foldl1ConsCons)%
then
vars a,b,c : Type;
     d : Ord;
     x, y : a;
     xs, ys, zs : List a;
     xxs : List (List a);
     r, s : d;
     ds : List d;
     bs : List Bool;
     f : a -> a -> a;
     p, q : a -> Bool;
     g : a -> List b;
     n,nx: Nat;
```

Specification 3.7.1 List Specification - Part 5

```
fun andL : List Bool -> Bool;
fun orL : List Bool -> Bool;
fun any : (a -> Bool) -> List a -> Bool;
fun all : (a -> Bool) -> List a -> Bool;
fun concatMap : (a -> List b) -> List a -> List b;
fun concat : List (List a) -> List a;
fun maximum : List d -> d;
fun minimum : List d -> d;
fun takeWhile : (a -> Bool) -> List a -> List a
fun dropWhile  : (a -> Bool) -> List a -> List a
fun span : (a -> Bool) -> List a -> (List a * List a)
fun break : (a -> Bool) -> List a -> (List a * List a)
fun splitAt: Nat -> List a -> (List a * List a)
. andL bs = foldr <&&> True bs                              %(AndLDef)%
. orL bs = foldr <||> False bs                              %(OrLDef)%
. any p xs = orL (map p xs)                                 %(AnyDef)%
. all p xs = andL (map p xs)                                %(AllDef)%
. concat xxs = foldr <++> (Nil: List a) xxs               %(ConcatDef)%
. concatMap g xs = concat (map g xs)                   %(ConcatMapDef)%
. maximum ds = foldl1 max ds                             %(MaximumDef)%
. minimum ds = foldl1 min ds                             %(MinimumDef)%
. takeWhile p (Nil: List a) = Nil: List a               %(TakeWhileNil)%
. p x = True => takeWhile p (Cons x xs)
    = Cons x (takeWhile p xs)                         %(TakeWhileConsT)%
. p x = False => takeWhile p (Cons x xs) = Nil: List a   %(TakeWhileConsF)%
. dropWhile p (Nil: List a) = Nil: List a               %(DropWhileNil)%
. p x = True => dropWhile p (Cons x xs) = dropWhile p xs %(DropWhileConsT)%
. p x = False => dropWhile p (Cons x xs) = Cons x xs     %(DropWhileConsF)%
. span p (Nil: List a) = ((Nil: List a), (Nil: List a))    %(SpanNil)%
. p x = True => span p (Cons x xs)
     = let (ys, zs) = span p xs in
          ((Cons x ys), zs)                                 %(SpanConsT)%
. p x = False => span p (Cons x xs)
     = let (ys, zs) = span p xs in
          ((Nil: List a), (Cons x xs))                      %(SpanConsF)%
. span p xs = (takeWhile p xs, dropWhile p xs)        %(SpanThm)% %implied
. break p xs = let q = (Not__ o p) in span q xs           %(BreakDef)%
. break p xs = span (Not__ o p) xs                  %(BreakThm)% %implied
. splitAt 0 xs = ((Nil: List a), xs)                     %(SplitAtZero)%
. splitAt n (Nil: List a) = ((Nil: List a), Nil)         %(SplitAtNil)%
. def(pre(n)) /\ nx = pre(n) => splitAt n (Cons x xs)
     = let (ys,zs) = splitAt (nx) xs in (Cons x ys, zs)    %(SplitAt)%
```

Specification 3.7.1 List Specification - Part 6

```
then
vars a,b,c : Type;
     d : Ord;
     e: Eq;
     x, y : a;
     xs, ys : List a;
     q, r : d;
     qs, rs : List d;
     s,t: e;
     ss,ts: List e;
     p: a -> Bool
fun insert: d -> List d -> List d
fun delete: e -> List e -> List e
fun select: (a -> Bool) -> a -> (List a * List a) -> (List a * List a)
fun partition: (a -> Bool) -> List a -> (List a * List a)
. insert q (Nil: List d) = Cons q Nil                        %(InsertNil)%
. (q <= r) = True => insert q (Cons r rs)
    = (Cons q (Cons r rs))                                  %(InsertCons1)%
. (q > r) = True => insert q (Cons r rs)
    = (Cons r (insert q rs))                                %(InsertCons2)%
. delete s (Nil: List e) = Nil                               %(DeleteNil)%
. (s == t) = True => delete s (Cons t ts) = ts              %(DeleteConsT)%
. (s == t) = False => delete s (Cons t ts)
    = (Cons t (delete s ts))                                %(DeleteConsF)%
. (p x) = True => select p x (xs, ys) = ((Cons x xs), ys)    %(SelectT)%
. (p x) = False => select p x (xs, ys) = (xs, (Cons x ys))   %(SelectF)%
. partition p xs = foldr (select p) ((Nil: List a),(Nil)) xs %(Partition)%
. partition p xs
   = (filter p xs, filter (Not__ o p) xs)       %(PartitionProp)% %implied
end
```

3.8 Char and String Specifications

In order to create `Char` specification, we imported the *CASL* `Char` specification and then declared the `Char` type as an instance of the classes `Eq` and `Ord` See Specification 3.8.1. We defined, respectively for each of those type instances, the equality and the `__<__` function. Other theorems were proved just as in the previous specifications.

Specification 3.8.1 Char Specification

```
from Basic/CharactersAndStrings get Char |-> IChar

spec Char = IChar and Eq and Ord then
vars x, y: Char
type instance Char: Eq
. (ord(x) == ord(y)) = (x == y)                          %(ICE01)%
. Not(ord(x) == ord(y)) = (x /= y)                       %(ICE02)% %implied
type instance Char: Ord
. (ord(x) < ord(y)) = (x < y)                            %(ICO04)%
. (ord(x) <= ord(y)) = (x <= y)                          %(ICO05)% %implied
. (ord(x) > ord(y)) = (x > y)                            %(ICO06)% %implied
. (ord(x) >= ord(y)) = (x >= y)                          %(ICO07)% %implied
. (compare x y == EQ) = (ord(x) == ord(y))               %(ICO01)% %implied
. (compare x y == LT) = (ord(x) < ord(y))                %(ICO02)% %implied
. (compare x y == GT) = (ord(x) > ord(y))                %(ICO03)% %implied
. (ord(x) <= ord(y)) = (max x y == y)                    %(ICO08)% %implied
. (ord(y) <= ord(x)) = (max x y == x)                    %(ICO09)% %implied
. (ord(x) <= ord(y)) = (min x y == x)                    %(ICO10)% %implied
. (ord(y) <= ord(x)) = (min x y == y)                    %(ICO11)% %implied
end
```

The `String` specification was created importing our `Char` and `List` specifications. We defined `String` as a list of characters, just as the *Haskell Prelude* does. We declared `String` as an instance of the classes `Eq` and `Ord`. Because `Char` and `List` are also instances of those classes, we didn't need to define axioms to instanciate declarations. To prove this fact, we wrote five theorems involving the equality and ordering functions.

Specification 3.8.2 String Specification

```
spec String = %mono
     List and Char then
type String := List Char
type instance String: Eq
type instance String: Ord
vars a,b: String; x,y,z: Char; xs, ys: String
. x == y = True => ((Cons x xs) == (Cons y xs)) = True   %(StringT1)% %implied
. xs /= ys = True => ((Cons x ys) == (Cons y xs)) = False %(StringT2)% %implied
. (a /= b) = True =>  (a == b) = False                %(StringT3)% %implied
. (x < y) = True =>  ((Cons x xs) < (Cons y xs)) = True  %(StringT4)% %implied
. (x < y) = True /\ (y < z) = True => ((Cons x (Cons z Nil))
      < (Cons x (Cons y Nil))) = False            %(StringT5)% %implied
end
```

3.9 Example Specifications

To exemplify the use of our library, we created two example specifications involving ordering algorithms. In the first specification, seen at Specification 3.9.1, we used two sorting algorithms: *Quick Sort* and *Insertion Sort*. They were defined using functions from our library (`filter`, `__++__` and `insert`) and total lambda expressions as parameters for the `filter` functions; the lambda expressions are made total by using `!` just after the final point that separates variables from expressions. In order to prove the correctness of the specification, we created four theorems involving the sorting functions.

The second specification uses a new data type (`Split a b`), as an internal representation for the sorting functions. See Specification 3.9.2. We used the idea that we can split a list and then rejoin their elements, following each algorithm. We defined a general sorting function, `GenSort`, which is responsible for applying the splitting and the joining functions over a list.

The Insertion Sort algorithm in implemented by a joining function that uses the `insert` function to insert splited elements into the list. The Quick Sort algorithm uses a splitting function that separates the list in two new lists: the first containing elements smaller than the first element of the original list and the second with the other elements. The joining function inserts an element in the middle of two lists.

The Selection Sort algorithm uses a splitting function that relies on the `minimum` function to extract the smaller element from the rest of the list. The joining function just joins two lists. The Merge Sort algorithm is implemented by splitting the initial list in the middle, using the splitting function, and then merging the elements using joining function. The latter takes the smaller head of both lists and then merges the

Specification 3.9.1 ExamplePrograms Specification

```
spec ExamplePrograms = List then
var a: Ord;
    x,y: a;
    xs,ys: List a
fun quickSort: List a -> List a
fun insertionSort: List a -> List a
. quickSort (Nil: List a) = Nil                        %(QuickSortNil)%
. quickSort (Cons x xs)
     = ((quickSort (filter (\ y:a .! y < x) xs))
        ++ (Cons x Nil))
        ++ (quickSort (filter (\ y:a .! y >= x) xs))    %(QuickSortCons)%
. insertionSort (Nil: List a) = Nil                 %(InsertionSortNil)%
. insertionSort (Cons x Nil) = (Cons x Nil)          %(InsertionSortConsNil)%
. insertionSort (Cons x xs) = insert x (insertionSort xs)  %(InsertionSortConsCons)%
then %implies
var a: Ord;
    x,y: a;
    xs,ys: List a
. andL (Cons True (Cons True (Cons True Nil))) = True     %(Program01)%
. quickSort (Cons True (Cons False (Nil: List Bool)))
     = Cons False (Cons True Nil)                         %(Program02)%
. insertionSort (Cons True (Cons False (Nil: List Bool)))
     = Cons False (Cons True Nil)                         %(Program03)%
end
```

other list and the remaining elements of the list from which the head was taken.

We specified two predicates found in the *CASL* library repository (but not in the *CASL* Library itself). `isOrdered` guarantees that a list is correctly ordered; `permutation` guarantees that one list is a permutation of the other, i.e., both lists have the same elements. Finally, we created theorems to verify that the application of the algorithms, in pairs, resulted in the same list; to verify that applying each algorithm to a list results in an ordered list; and to verify that a list is a permutation of the list returned by the application of each algorithm.

Specification 3.9.2 SortingPrograms Specification - Part 1

```
spec SortingPrograms = List then
var a,b : Ord;
free type Split a b ::= Split b (List (List a))
var x,y,z,v,w: a;
    r,t: b;
    xs,ys,zs,vs,ws: List a;
    rs,ts: List b;
    xxs: List (List a);
    split: List a -> Split a b;
    join: Split a b -> List a;
    n: Nat
fun genSort: (List a -> Split a b) -> (Split a b -> List a) -> List a -> List a
fun splitInsertionSort: List b -> Split b b
fun joinInsertionSort: Split a a -> List a
fun insertionSort: List a -> List a
fun splitQuickSort: List a -> Split a a
fun joinQuickSort: Split b b -> List b
fun quickSort: List a -> List a
fun splitSelectionSort: List a -> Split a a
fun joinSelectionSort: Split b b -> List b
fun selectionSort: List a -> List a
fun splitMergeSort: List b -> Split b Unit
fun joinMergeSort: Split a Unit -> List a
fun merge: List a -> List a -> List a
fun mergeSort: List a -> List a
. xs = (Cons x (Cons y ys)) /\ split xs = Split r xxs
     => genSort split join xs
         = join (Split r (map (genSort split join) xxs))  %(GenSortT1)%
. xs = (Cons x (Cons y Nil)) /\ split xs = Split r xxs
     => genSort split join xs
         = join (Split r (map (genSort split join) xxs))  %(GenSortT2)%
. xs = (Cons x Nil) \/ xs = Nil
     => genSort split join xs = xs                          %(GenSortF)%
. splitInsertionSort (Cons x xs)
   = Split x (Cons xs (Nil: List (List a)))        %(SplitInsertionSort)%
. joinInsertionSort (Split x (Cons xs (Nil: List (List a))))
    = insert x xs                                  %(JoinInsertionSort)%
. insertionSort xs
   = genSort splitInsertionSort joinInsertionSort xs    %(InsertionSort)%
```

Specification 3.9.2 SortingPrograms Specification - Part 2

```
. splitQuickSort (Cons x xs)
     = let (ys, zs) = partition (<<> x) xs
      in Split x (Cons ys (Cons zs Nil))             %(SplitQuickSort)%
. joinQuickSort (Split x (Cons ys (Cons zs Nil)))
    = ys ++ (Cons x zs)                               %(JoinQuickSort)%
. quickSort xs = genSort splitQuickSort joinQuickSort xs   %(QuickSort)%
    => unzip (zip xs ys) = (xs, ys)                         %(ZipSpec)%
. splitSelectionSort xs = let x = minimum xs
 in Split x (Cons (delete x xs) (Nil: List(List a)))    %(SplitSelectionSort)%
. joinSelectionSort (Split x (Cons xs Nil)) = (Cons x xs)  %(JoinSelectionSort)%
. selectionSort xs
   = genSort splitSelectionSort joinSelectionSort xs    %(SelectionSort)%
. def((length xs) div 2) /\ n = ((length xs) div 2)
     => splitMergeSort xs = let (ys,zs) = splitAt n xs
      in Split () (Cons ys (Cons zs Nil))            %(SplitMergeSort)%
. xs = (Nil: List a) => merge xs ys = ys                   %(MergeNil)%
. xs = (Cons v vs) /\ ys = (Nil: List a)
    => merge xs ys = xs                               %(MergeConsNil)%
. xs = (Cons v vs) /\ ys = (Cons w ws) /\ (v < w) = True
    => merge xs ys = Cons v (merge vs ys)            %(MergeConsConsT)%
. xs = (Cons v vs) /\ ys = (Cons w ws) /\ (v < w) = False
    => merge xs ys = Cons w (merge xs ws)            %(MergeConsConsF)%
. joinMergeSort (Split () (Cons ys (Cons zs Nil)))
    = merge ys zs                                     %(JoinMergeSort)%
. mergeSort xs = genSort splitMergeSort joinMergeSort xs   %(MergeSort)%
```

Specification 3.9.2 SortingPrograms Specification - Part 3

```
then
vars a: Ord;
     x,y: a;
     xs,ys: List a
preds isOrdered: List a;
      permutation: List a * List a
. isOrdered (Nil: List a)                                 %(IsOrderedNil)%
. isOrdered (Cons x (Nil: List a))                       %(IsOrderedCons)%
. isOrdered (Cons x (Cons y ys))
   <=> (x <= y) = True /\ isOrdered(Cons y ys)        %(IsOrderedConsCons)%
. permutation ((Nil: List a), Nil)                      %(PermutationNil)%
. permutation (Cons x (Nil: List a), Cons y (Nil: List a))
    <=> (x==y) = True                                  %(PermutationCons)%
. permutation (Cons x xs, Cons y ys) <=>
     ((x==y) = True /\ permutation (xs, ys))
      \/ (permutation(xs, Cons y (delete x ys)))     %(PermutationConsCons)%
then %implies
var a,b : Ord;
    xs, ys : List a;
. insertionSort xs = quickSort xs                            %(Theorem01)%
. insertionSort xs = mergeSort xs                            %(Theorem02)%
. insertionSort xs = selectionSort xs                        %(Theorem03)%
. quickSort xs = mergeSort xs                                %(Theorem04)%
. quickSort xs = selectionSort xs                            %(Theorem05)%
. mergeSort xs = selectionSort xs                            %(Theorem06)%
. isOrdered(insertionSort xs)                                %(Theorem07)%
. isOrdered(quickSort xs)                                    %(Theorem08)%
. isOrdered(mergeSort xs)                                    %(Theorem09)%
. isOrdered(selectionSort xs)                                %(Theorem10)%
. permutation(xs, insertionSort xs)                          %(Theorem11)%
. permutation(xs, quickSort xs)                              %(Theorem12)%
. permutation(xs, mergeSort xs)                              %(Theorem13)%
. permutation(xs, selectionSort xs)                          %(Theorem14)%
end
```

4 Parsing and verifying the specifications

In this section we talked about *Hets* and *Isabelle* uses. We started by describing how the specifications were grouped to be parsed by *Hets*. Next, we describe how we did the parsing and showed the resulting graph or theories. Finally, we described how we made proofs with *Isabelle* and we related which proofs could not yet be finished.

4.1 Hets in action

All the specifications from the previous section were placed together in a single file. As the specification growed, we could separate the full specification in smaller sets of related specifications or even write one specification per file. *Hets* can deal with all these scenarios.

Although *Hets* is a command line program, it has also a mode integrated with the Emacs text editor, which can also be used to interact with *Isabelle* using the ProofGeneral interface. In that way, we could edit specifications in Emacs and parse them with *Hets* using the `CMD + r` keyboard shortcut. Another option is to parse the specifications with the `CMD + g` keyboard shortcut, which can generate the graph of theories based on the syntactic analysis. Parsing our specifications generated the graph shown in Figure 4.1.1.

As can be seen, all the red (dark gray) nodes indicate specifications that have one or more theorems. The green (light gray) ones don't have theorems or, either, their proofs are already done. The rectangular nodes indicate imported specifications and the elliptical ones indicate specifications taken from our file. Some nodes, as `ExamplePrograms` and `SortingPrograms`, do have theorems but are marked green because the theorems are inserted in sub-specifications.

We started our proofs by using the automatic proof mode of *Hets* (menu: Edit -> Proofs -> Automatic). This method analyzed the theories and directives (`%mono`, `%implies, etc`) and then revealed the nodes from sub-specifications that created theorems, for example, by the `%implied` directive.

The next step was to prove each red node. To do so, we did a right click on a node and chose the option *Prove* from the *node menu*. This opened the Emacs text editor. After *Isabelle* had parsed the full theory file (and proved it or not, according to *Isabelle* rules), we closed the Emacs window and thus the proof status for that theory was reported back to *Hets* by *Isabelle*. If the node was proved, its color was chanced to green; otherwise, it kept the red color. If sub-nodes were proved, they were omitted again by *Hets*. At this point, we could not yet prove all the theorems we had created. Most of the unproved nodes had yet one or two theorems to be proved. The actual status of our proofs can be seen in Figure 4.1.2.

Figure 4.1.1 Initial state of the proof graph.

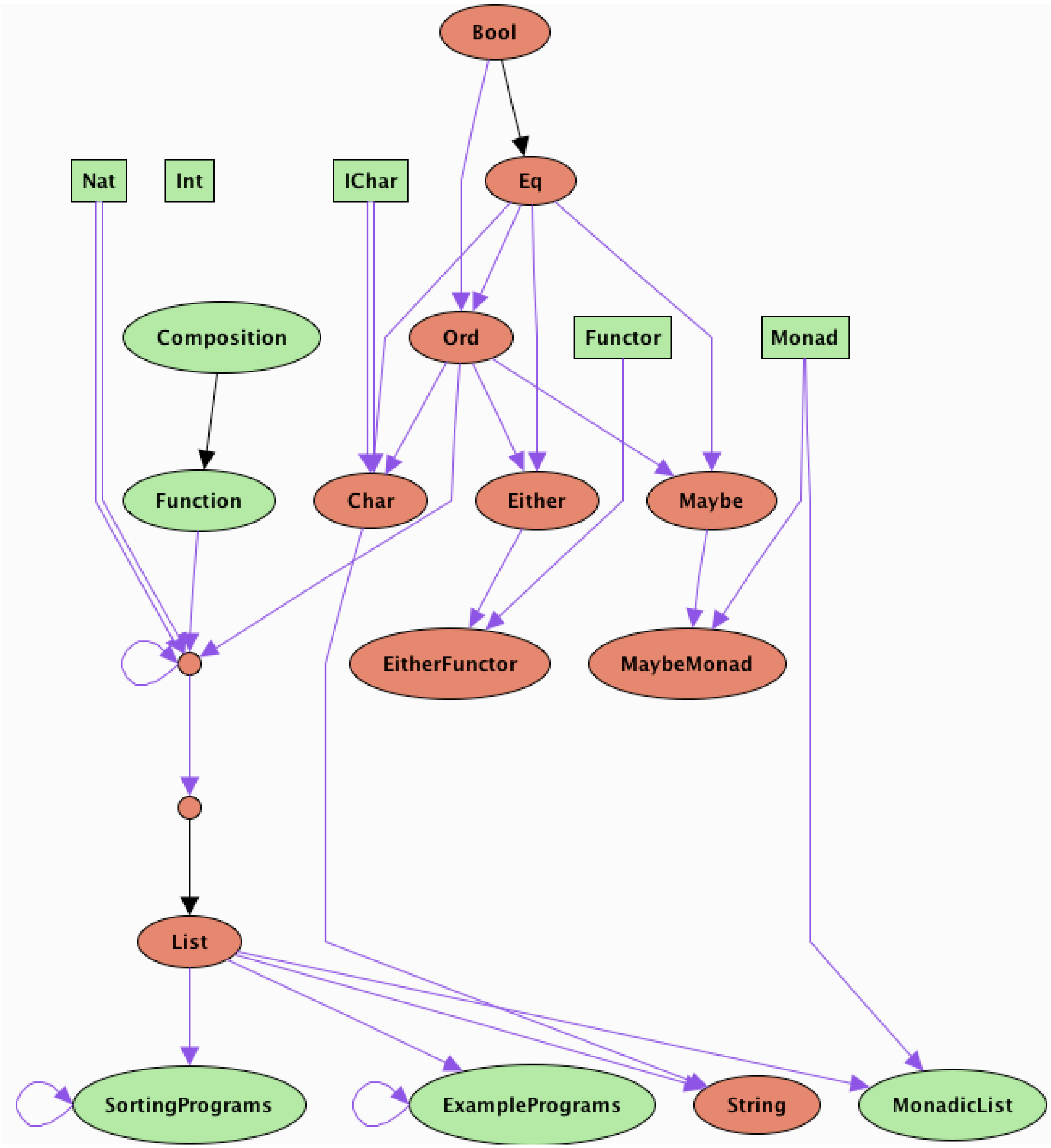
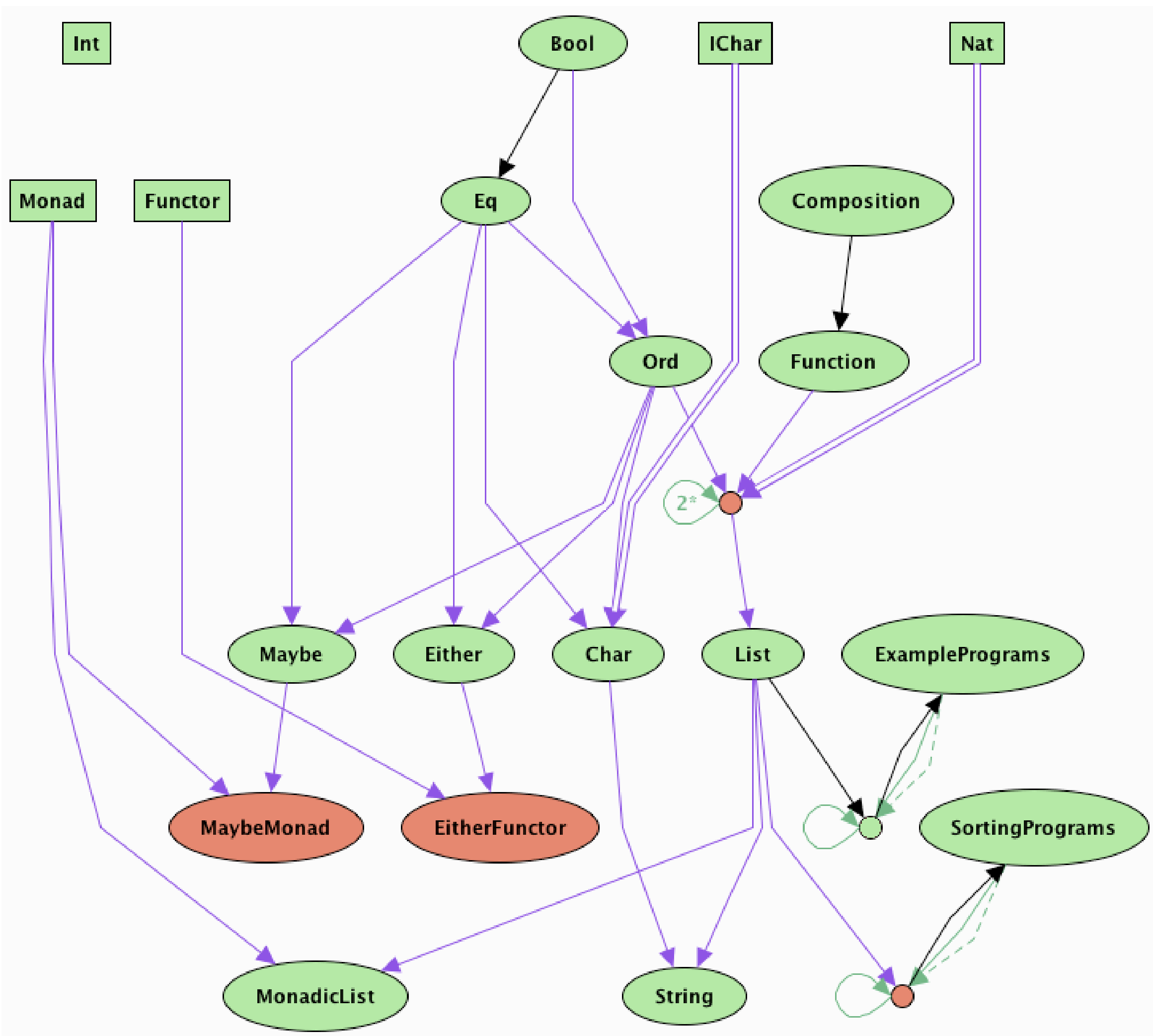


Figure 4.1.2 Actual state of the proof graph.



4.2 Doing proofs with Isabelle

As part of specifying our library, the task of proving its theorems were a major undertaking. Although some theorems remained unproved, we verified almost all of them. Next, we indicate how we constructed our proofs using excerpts from interesting proofs. Our full proof scripts can be found in Appendix A.

The four theorems form Specification 3.2.3 were translated by *Hets* to *Isabelle* theorems like the one shown at Isabelle Proof Script Excerpt 4.2.1.

Isabelle Proof Script Excerpt 4.2.1 Proof for theorem NotFalse1 from Bool specification

```
theorem NotFalse1 : "ALL x. Not' x = True' = (x = False')"
apply auto
apply (case_tac x)
apply auto
done
```

All the proofs for the theorems of `Bool` specification followed this pattern:

- *apply (auto)*:

  This command tries to simplify the actual goal automatically as deep as it can. In this case, the command could only eliminate the universal quantifier, getting the result:

  ```
  goal (1 subgoal):
   1. !!x. Not' x = True' ==> x = False'
  ```

- *apply (case_tac x)*:

  case_tac method executes a case distinction over all constructors of the data type of variable x. In this case, because the type of x is Bool, x was instantiated to True and False:

  ```
  goal (2 subgoals):
   1. !!x. [| Not' x = True'; x = False' |]
           ==> x = False'
   2. !!x. [| Not' x = True'; x = True' |]
           ==> x = False'
  ```

- *apply (auto)*:

  At this time, this command could finalize all the proof automatically.

```
goal:
No subgoals!
```

One example of a proof for an `Eq` theorem is shown in the Isabelle Proof Script Excerpt 4.2.2. In this proof, we used a new command: `simp add:`. This command expects a list of axioms and previously proved theorems as parameters to be used in an automatic tentative of proving the actual goal. This command uses other axioms from the theory, together with the theorems passed as parameters, when trying to simplify the goal. If the goal cannot be reduced, the command produces an error; otherwise, a new goal is received.

Isabelle Proof Script Excerpt 4.2.2 Equality proof

```
theorem DiffTDef :
"ALL x. ALL y. x /= y = True' = (Not' (x ==' y) = True')"
apply(auto)
apply(simp add: DiffDef)
apply(case_tac "x ==' y")
apply(auto)
apply(simp add: DiffDef)
done
```

Almost all Ord theorem proofs used the same commands and tactics from the previous proofs. One interesting proof was the the one for the axiom `%(LeTAsymmetry)%`, presented in the Isabelle Proof Script Excerpt 4.2.3. Sometimes, *Isabelle* expect us to rewrite axioms to match goals because it cannot change the axioms to all their equivalent forms. We applied the command `rule ccontr` to start a proof by contradiction. After some simplification, *Isabelle* was not able to use the axiom `%(LeIrreflexivity)%` to simplify the goal:

```
goal (1 subgoal):
 1. !!x y. [| x <' y = True'; y <' x = True' |] ==> False
```

We needed to define an auxiliary lemma, `LeIrreflContra`, which *Isabelle* automatically proved. This theorem is interpreted internally by *Isabelle* as:

```
?x <' ?x = True' ==> False
```

Hence, we could tell *Isabelle* to use this lemma, thus forcing it to attribute the variable `x` to each `?x` variable in the lemma using the command `rule_tac x="x" in LeIrreflContra`. The same tactic was used to force the use of the axiom `%(LeTTransitive)%`. The command `by auto` was used to finalize the proof.

Isabelle Proof Script Excerpt 4.2.3 Proof for the axiom LeTAsymmetry from specification Ord.

```
lemma LeIrreflContra : " x <' x = True' ==> False"
by auto

theorem LeTAsymmetry :
"ALL x. ALL y. x <' y = True' --> y <' x = False'"
apply(auto)
apply(rule ccontr)
apply(simp add: notNot2 NotTrue1)
apply(rule_tac x="x" in LeIrreflContra)
apply(rule_tac y="y" in LeTTransitive)
by auto
```

We started most of our proofs by applying the command `apply(auto)`, as we wanted *Isabelle* to act automatically as much as possible. Sometimes this command could do some reduction; sometimes it could only remove HOL universal quantifiers; sometimes it got into a loop.

An example of a loop occurred when proving theorems from the `Maybe` and `Either` specifications. To avoid the loop, we applied the universal quantifier rule directly, using the command `apply(rule allI)`. The command `rule` applies the specified theorem directly. When there were more then one quantified variable, we could use the `+` sign after the rule, in order to tell *Isabelle* to apply the command as many times as it could.

After we removed the quantifiers, we could use the command `simp only:` to do some simplification. Diferently from `simp add:`, the command `simp only:` tries to rewrite only the rules passed as parameters, when simplifying the actual goal. Most of the times they could be used interchangeably. Sometimes, however, `simp add:` got into a loop and `simp only:` had to be used with other proof commands. Two theorems from the`Maybe` specification exemplifie the use of the previous commands, as shown in the Isabelle Proof Script Excerpt 4.2.4.

The `List` specification still had two unproved theorems (`FoldlDecomp` and `ZipSpec`) inside one of its sub-nodes; the other nodes could have all its theorems proved. Almost all theorems in this specification needed induction to be proved. *Isabelle* executes induction over a specified variable using the command `induct_tac`. it expects as parameter an expression or a variable over which to execute the induction. In the Isabelle Proof Script Excerpt 4.2.5 we can see one example of proof by induction for a `List` theorem.

The specification `Char` was another case where we had to use the `rule` command to

Isabelle Proof Script Excerpt 4.2.4 Proof for theorems IMO05 and IMO08 from specification Maybe.

```
theorem IMO05 : "ALL x. Just(x) <' Nothing = False'"
apply(rule allI)
apply(case_tac "Just(x) <' Nothing")
apply(auto)
done

theorem IMO08 :
"ALL x. compare Nothing (Just(x)) ==' GT = Nothing >' Just(x)"
apply(rule allI)+
apply(simp add: GeDef)
done
```

Isabelle Proof Script Excerpt 4.2.5 Proof for theorem FilterProm from pecification List.

```
theorem FilterProm :
"ALL f.
 ALL p.
 ALL xs.
 X_filter p (X_map f xs) = X_map f (X_filter (X__o__X (p, f)) xs)"
apply(auto)
apply(induct_tac xs)
apply(auto)
apply(case_tac "p(f a)")
apply(auto)
apply(simp add: MapCons)
apply(simp add: FilterConsT)
apply(simp add: MapCons)
apply(simp add: FilterConsT)
done
```

remove universal quantification by hand in order to avoid loops. Besides this problem, all theorems needed only one or two applications of the command `simp add:` to be proved. An example can be seen in the Isabelle Proof Script Excerpt 4.2.6

Isabelle Proof Script Excerpt 4.2.6 Proof for theorem ICO07 from specification Char.

```
theorem ICO07 : "ALL x. ALL y. ord'(x) >='' ord'(y) = x >='' y"
apply(rule allI)+
apply(simp only: GeqDef)
apply(simp add: GeDef)
done
```

The specification for `String` also used few commands in order to have its theorems proved. Almost all proofs were done with combinations of the `auto` and the `simp add:` commands. the In Isabelle Proof Script Excerpt 4.2.7 we show the largest proof in the `String` theory.

Isabelle Proof Script Excerpt 4.2.7 Proof for theorem StringT2 from specification String.

```
theorem StringT2 :
"ALL x.
 ALL xs.
 ALL y.
 ALL ys. xs /= ys = True' --> X_Cons x ys ==' X_Cons y xs = False'"
apply(auto)
apply(simp add: ILE02)
apply(case_tac "x ==' y")
apply(auto)
apply(simp add: EqualSymDef)
apply(simp add: DiffDef)
apply(simp add: NotFalse1)
done
```

Proofs for the `ExamplePrograms` theorems were very long. They were done using basically three commands: `simp only:`, `case_tac` and `simp add:`. The latter was used as the last command to allow *Isabelle* finish the proofs with fewer commands. Before using `simp only:` applications, we tried the `simp add:` command without success. We then used the `simp only:` command directly when the theorem used as a parameter previously failed when using the `simp add:` command. In the Isabelle Proof Script Excerpt 4.2.8 we show the proof for an `insertionSort` function application.

Isabelle Proof Script Excerpt 4.2.8 Proof for theorem Program03, an example of insertionSort function application from specification ExamplePrograms.

```
theorem Program03 :
"insertionSort(X_Cons True' (X_Cons False' Nil')) =
 X_Cons False' (X_Cons True' Nil')"
apply(simp only: InsertionSortConsCons)
apply(simp only: InsertionSortNil)
apply(simp only: InsertNil)
apply(case_tac "True' >'' False'")
apply(simp only: GeFLeTEqTRel)
apply(simp add: LeqTLeTEqTRel)
apply(simp only: InsertCons2)
apply(simp only: InsertNil)
done
```

All the theorems from our last proof, `SortingPrograms`, still couldn't be proved. Although for all of them we could prove some goals, the last one, representing the general case, is yet unproved. To show our progress in the proofs, we show an example in the Isabelle Proof Script Excerpt 4.2.9, with some comments inserted. The command `prefer` is used to choose which goal to prove in *Isabelle* interactive mode and the command `oops` indicates that we could not prove the theorem, and that we gave up the proof.

Isabelle Proof Script Excerpt 4.2.9 Actual status of the proof for theorem Theorem07 of specification SortingPrograms.

```
theorem Theorem07 : "ALL xs. isOrdered(insertionSort(xs))"
apply(auto)
apply(case_tac xs)
(* Proof for xs=Nil *)
prefer 2
apply(simp only: InsertionSort)
apply(simp add: GenSortF)
(* Proof for general case *)
apply(simp only: InsertionSort)
apply(case_tac List)
apply(auto)
apply(case_tac "X_splitInsertionSort (X_Cons a (X_Cons aa Lista))")
(* Proof for xs= Cons a Nil *)
prefer 2
apply(simp add: GenSortF)
(* Proof for xs=Cons a as*)
apply(case_tac Lista)
apply(auto)
prefer 2
(* Proof for xs = Cons a (Cons b Nil)*)
oops
```

5 ~~Faced Problems~~ Discussion and difficulties

~~When starting this work~~ We faced some interesting difficulties that we will briefly discuss here. The first problem ~~we had~~ was dealing with the *HasCASL* and *CASL* languages. Although both languages can be used together, we intended to use the *HasCASL* features, but ~~and~~ separating both syntax were a little ~~difficult~~ troublesome. The *HasCASL* language doesn't yet have a manual as does the *CASL* language. So, we started ~~the work~~ reading the *CASL* manual and then the *HasCASL* definition. This created some difficulties ~~to use~~ when using the *HasCASL* syntax because some of the constructions may be used interchangeably between both languages.

definition and complete

difficulty we faced

Another ~~doubt we had~~ was to distinguish between ~~from~~ the logic relations of the *HasCASL* language and our functions. This is ~~These doubts were~~ related to the logical equivalence between some axioms. Although these axioms were ~~logic~~ equivalent, their uses as rewriting rules were different. Axioms could be defined by equality, as in

```
. (x > y) = (y < x)
```

or by *HasCASL* equivalence, as in

```
. (x > y) = True  <=> (y < x) = True
```

? .

The first case is a better choice when using ~~to~~ axioms and theorems To refine ~~that should be used as rewrite rules to refine rules~~ into basic axioms. The second case must be used when defining ~~to define~~ basic axioms. Otherwise, *Isabelle* will never be able to use these axioms. ~~to conclude that a goal is valid or not.~~ This relates to the fact that axioms defining relations should use the `Bool` type to allow *Isabelle* to conclude that rules are true or false and, then, prove goals.

proceed to

We had some problems dealing with *Isabelle* itself. ~~as this was the first time using a theorem prover.~~ We started using *HOL* in place of *Isar* and this seems to have complicated some proof scripts of ~~for~~ larger proofs. We also had ~~some problems~~ to get used to the way *Isabelle* uses axioms as rewriting rules. If a predicate P implies a ~~in~~ predicate Q (`P ==> Q`), *Isabelle* matches the predicate Q with the actual goal, constructing the proof in a bottom up manner that is ~~was~~ not ~~our~~ usual. ~~way of thinking.~~

As *HasCASL* is a work-in-progress project, the tool is not fully implemented and we got some errors because the tool could not translate some specifications to *HOL*. The solutions to those errors were ~~immediately~~ kindly explained by the *HasCASL* research team, thus ~~fully~~ minimizing our difficulties ~~problems or making them disappear.~~

6 Future Works

As presented before, our library still lacks some proofs as well as other *Haskell Prelude* functions. We are trying to finish open proofs in order to get a fully verified subset of the *Haskell Prelude* functions.

An open question is how to deal with numbers. The alternative of recreating all the lemmas needed by *Isabelle*, which are already written in HOL, definitely is not a good approach. One solution could be to create an isomorphism between the builtin *Isabelle* numeric types and the types specified in the *CASL* library. If we call this isomorphism `h`, we could prove a goal like `t1 = t2` by injecting the isomorphism using the rule `h x = h y ==> x = y`. This axiom would give us a new goal, `h t1 = h t2`, that would be written in terms of builtin *Isabelle* types and, thus, could be proved with the *Isabelle* axioms and builtin auxiliary lemmas. This isomorphism could be extended to the specification `List`, as most Haskell data types and functions rely on lists.

After solving the problem with numeric specifications, we could specify the *Haskell Prelude* functions that involve numbers. Many functions that should have been specified on the specification `List`, for example, are absent because importing the numeric specifications wouldn't allow their proofs to be constructed.

The next natural stage would be to use laziness in our library. This would require a rewrite of almost all the specifications. An alternative would be to study transformations that could help us to reuse the proofs we have already written.

Another point of interest would be to refine our library in order to use the *HasCASL* language subset. This subset contains structures like infinite data types and allows specifications to be converted to *Haskell* programs. This last step could also be used to verify existing *Haskell Prelude* implementations or to serve as a guide for new ones.

7 Conclusions

Appendices

A Isabelle Proofs

References

[1] Egidio Astesiano, Michel Bidoit, Hélène Kirchner, Bernd K. Brückner, Peter D. Mosses, Donald Sannella, and Andrzej Tarlecki. CASL: The Common Algebraic Specification Language. *Theoretical Computer Science*, 2002.

[2] Isabelle Comunity. Isabelle overview, 2007.

[3] The Haskell Team. Learning haskell, 2007.

[4] Till Mossakowski, Serge Autexier, and Dieter Hutter. Development graphs - proof management for structured specifications, 2006.

[5] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer, 2002.

[6] Markus Roggenbach, Till Mossakowski, and Lutz Schröder. CASL libraries. In CASL *Reference Manual*, LNCS Vol. 2960 (IFIP Series), part V. Springer, 2004.

[7] Lutz Schröder. Higher order and reactive algebraic specification and development. Feb 2006.

[8] Lutz Schröder, Till Mossakowski, and Christian Maeder. Hascasl - integrated functional specification and programming. language summary., 2003.

[9] Lutz Schröder and Till Mossakowski. HasCASL - integrated higher-order specification and program development.

[10] Lutz Schröder and Till Mossakowski. HasCASL: Towards Integrated Specification And Development Of Haskell Programs.

[11] Simon Thompson. *Haskell: The Craft Of Functional Programming*. Addison Wesley, Boston, USA, 2 edition, 1999.

[12] Klaus Lüttich Till Mossakowski, Christian Maeder and Stefan Wölfl. The Heterogeneous Tool Set.