BELL TREES

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ABSTRACT. This article presents a combinatorial structure that only formed by the Bell Numbers. Is also shown the definition and demonstration of this structure's properties.

INTRODUCTION

We can define a Bell Number as a number of partitions possibilities of a set with n elements. Such number is defined by:

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \tag{1}$$

Where $\left\{ {n \atop k} \right\}$ is the Stirling number of second kind, which is defined by:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n \tag{2}$$

With the definitions 1 and 2 above, this article proposes to show a combinatorial structure which is called here *Bell Tree* and the use of this structure to solve the *Partition of an n-Set* problem. It was used as basis the definition of a partition tree, shown in the algorithm *Next Partition of an n-Set*[1], associating the tree nodes with the equations which its terms are defined on the tree construction properties.

PROPERTIES OF BELL TREES

Let n be the number of elements of a set that will be partitioned and B the Bell Numbers that denotes a set of terms that can form a mathematical expression in a node of the tree, the structure here defined adopt the partition tree structure with n-Sets and added to it others properties. Follows below some properties of the structure:

- The Tree has n levels;
- The Tree nodes are non-negative integers numbers which is formed by expressions that evolves exclusively Bell Numbers and multiplicative integer constants;
- Let S be a subset contained in B which denotes the number of distinct terms of an expression with a node N, we determine the number of descendants produced by N from the number of elements of S increased one unit;

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- The number R of the nodes on a Bell Tree corresponds to:

$$R = \sum_{j=1}^{n} B_j \tag{3}$$

- Let k be the number of descendants of the actual node N, the value for all its $k-1$ are defined by:

  Let E_w be a mathematical expression with w terms from the ascendant node, the descendant node corresponds to E_{w-1}. To illustrate such fact, if we have a ascendant node represented by the expression:

$$E_w = (B_w - B_{w-1}) - 2(B_{w-1} - B_{w-2}) \tag{4}$$

  Then, the descendants nodes will correspond to the expression:

$$E_{w-1} = (B_{w-1} - B_{w-2}) - 2(B_{w-2} - B_{w-3}) \tag{5}$$

  For the k-tuple descendant of N, we have the expression:

$$E_k = E_w - (k-1)E_{w-1} \tag{6}$$

  That it is equivalent to say that the k-tuple descendant is equal to the ascendant node value minus $k-1$ times the value of the others descendant nodes.

We can now make a tree that follows this formation law. For $n = 5$, we have a tree illustrated on Figure 1. The three first levels expressions of this tree are in Table 1, which the number of nodes are entered from top to bottom and from left to right.

Table 1. Nodes values of the Tree illustrated on Figure 1

Nodes	Expressions
1	B_5
2	B_4
3	$B_5 - B_4$
4	B_3
5	$B_4 - B_3$
6	$B_4 - B_3$
7	$B_4 - B_3$
8	$(B_5 - B_4) - 2(B_4 - B_3)$

Search Analysis and Verification of Serials on Bell Tree

The Bell Tree stands the search of serial numbers placed between 1 and B_n. A typical search algorithm does n interactions on the number of levels that the tree can goes deep in maximum $k-1$ interactions inside of each tree level, where k represents the number of descendants of the visited node. This algorithm works on $O(nk)$ in the worst case. The use of search algorithms and serial numbers verification on this combinatorial structure is related to the production of the subset indexes

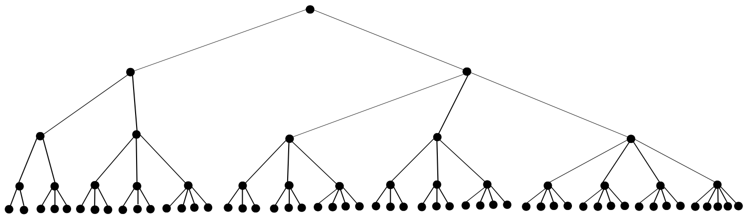

FIGURE 1. Bell Tree with level 5

partitioned from a set of n elements, giving as an input its position on the list of partitions[2]. Such set of indexes can be represented by vector p with size n. On each current level of the tree, the nodes are enumerated from zero and from left to right and each p component receives the value referring to the last index visited on each level of the Bell Tree.

DEMONSTRATION

(This section is not ready)

REFERENCES

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