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Lemma 0.1. *Let $S \subseteq \mathbb{Z}^d$ be a finite set. A elements $s \in S$ is bad if there exists $1 \le i \le d$ such that $s + e_i \notin S$. Let b be the number of bad elements. We have*

$$b^d \ge |S|^{d-1}.$$

Proof. Follows from Loomis-Whitney. □

Lemma 0.2. *Let $X \subseteq \mathbb{N}$ be a finite set. For any $k \ge 2$, we have*

$$|X \cap 2X \cap 3X \cap \cdots \cap kX| \le |X|\Big(1 - |X|^{-\frac{1}{\pi(k)}}\Big),$$

where $\pi(k)$ denotes the number of primes smaller or equal than k.

Proof. Follows from the previous lemma when we transform the problem from $\mathbb{N}$ to $\mathbb{Z}^d$ by looking at the factorization of the elements of S. □

Theorem 0.3. *We say that a subset of $\mathbb{N}$ is nice if it consists of $\{a, 2a, 3a, \dots, ab\}$ for some $a \ge 1$ and $b \ge 1$.*

Let $X \subseteq \mathbb{N}$ be a finite set of size $n \coloneqq |X|$. The number of nice subsets of X is bounded by $(1 + o(1))n \log^3 n$.

Proof. Follows from the prime number theorem together with the previous lemma. □

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