Lab 2: Mine Crafting
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Introduction

In this study, we test the vertical depth of one of our deepest mines, with a depth approaching 4 km. Our method of computing this depth to the highest precision involves dropping a 1-kg test mass and measuring the time to hit the bottom.

The crux of this method lies in the force of gravity. After we drop our test mass down the shaft, gravity from the Earth's mass causes the mass to accelerate downward, acquiring speed over time until it hits the bottom of the well. If we can accurately model the acceleration for a test mass, we can calculate its velocity and displacement over a given amount of time. Inversely, we can also use the time elapsed to find the length traveled if we also know and account for all the forces that accelerate the test mass. To ensure the greatest precision, we take into account several key force that affect total fall time: gravity as a function of depth into the well and the density of the Earth, drag force, the Coriolis effect from the rotation of the planet.

In general, to calculate forces, we use the following relation to model differential equations:

$$F = ma = m\frac{dv}{dt} = m\frac{d^2y}{dt^2}$$

We use the force of gravity as:

$$F_{\text{gravity}} = \frac{GMm}{R^2}$$

Note that the force of gravity between two objects is inversely proportional to the square of their relative distance R, i.e. $F_g \propto 1/R^2$. Simply put, the force of gravity gets stronger as objects get closer and weaker when they move farther apart.

Another major factor considered is drag, which is the force acting opposite an object moving through a fluid (in this case, air). Drag force depends on the velocity of the object, which is variable over time, making it necessary to use numerical solutions from Python to calculate drag.

Additionally, we take into account the Coriolis force. Since the Earth is rotating as the object falls, the walls of the shaft are moving with respect to the object. If the Earth is taken as the reference frame, the object appears to move to the side as it falls, resulting in the Coriolis “force” which seems to "push" an object sideways.

Lastly, as an additional complication to gravity, we must consider that the Earth's mass is not uniformly distributed throughout its volume, which affects the force of gravity felt at different depths. In this paper, we consider different density configurations of the Earth and how they affect the fall.

To mathematically calculate the effects of these forces, we enlist the help of Python numeric differential equation methods to get quantitative results, as well as plotting programs to visualize them.

Calculation of Fall Time

To calculate the fall time, we start with an idealized model and gradually add in the different factors that impact real-world measurements. First, we assume no drag and a constant (i.e., unchanging) force of gravity. Then, we consider how the force of gravity varies depending on depth into the shaft (i.e., distance from the Earth's center). Finally, we add in the drag force, which is a function of velocity.

Constant Gravity: We start with our most idealized model of constant gravity, where the only force, and thus the only source of acceleration, is due to a constant $g = 9.8$ m/s². This model is simple enough to solve theoretically using kinematics equations:

$$y = \frac{1}{2}at^2, \quad t = \sqrt{\frac{2y}{a}}$$

With $y = 4000$ m and $a = g = 9.8$ m/s², we calculate that a 1 kg mass would fall to the bottom in **28.6 seconds**. We confirm our results numerically using the differential equation:

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -g$$

We reduce this into two coupled first-order differential equations:

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = -g$$

Using Python's solve_ivp function from the scipy.integrate library to numerically solve these equations, we again find that the time to fall is **28.6 seconds**. From this point on, all calculations of fall time utilize this numeric method, as further calculations become too complex to solve theoretically.

Position-Dependent Gravity: We next account for how gravity changes as a function of distance from the Earth's center. We let gravity vary with radius as, leading to the differential equation:

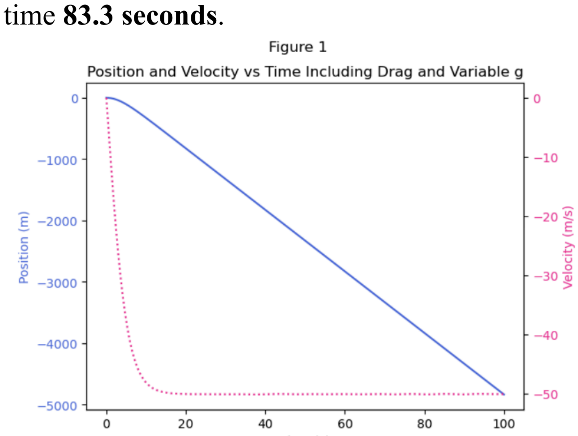
$$\frac{d^2y}{dt^2} = -g_0(\frac{r}{R_e}), \quad y = R_e - r$$

We again reduce this to two first-order equations and solve numerically. The resulting fall time is still **28.6 seconds** (rounded), but it is slightly less than with constant gravity. Theoretically, this makes sense — gravity becomes stronger as the object approaches the center, making it fall faster overall.

Adding Drag: Finally, we include drag force in our model. The updated equation becomes:

$$\frac{d^2y}{dt^2} = -g_0(\frac{r}{R_e}) + \alpha\,|\frac{dy}{dt}|^\gamma$$

Here, $\gamma = 2$, which is typical for free-fall in air. The coefficient α depends on air properties and the object's shape, and is difficult to determine theoretically. So, we use the fact that the terminal velocity of most objects on Earth is about 50 m/s, and calibrate $\alpha \approx 0.039$ (1/m) based on this. Solving this numerically using the same Python framework gives us the new fall time **83.3 seconds**.



Plot relating position (with the top of the shaft at 0) and velocity to time

Feasibility of Depth Measurement Approach

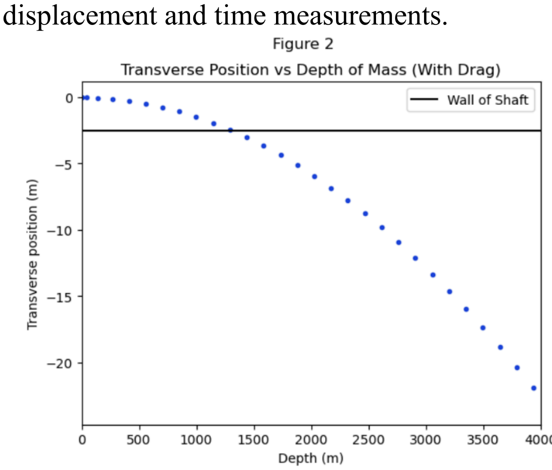
Accounting for the Coriolis effect, the test-mass drop approach is not feasible, as the mass will hit the side wall of the shaft before it hits the bottom. As detailed in the introduction, the Coriolis effect essentially moves the mass sideways. We incorporate the Coriolis force mathematically:

$$\frac{dy^2}{dt^2} = -g(y) + \alpha|\frac{dy}{dt}|^\gamma + \frac{F_{cy}}{m}$$

$$F_{c_y} = -2m\,\Omega v_x$$

And use our usual framework of solving differential equations to model the motion of the test mass with the Coriolis force included. However, in this case we also want to calculate when or if the mass hits the wall; using the solve ivp function's event-detecting abilities, we numerically calculate the time before the mass hits the wall if dropped from the center, which turns out to be **29.6 seconds**. Compared to the fall time of **83.3 seconds**, the object hits the wall before hitting the ground, thus disrupting our fall time measurement, and making our depth measurement approach unfeasible.

Drag makes a difference in these calculations, as it slows down the object's sideways motion; however, even accounting for drag, the object hits the wall before it hit the bottom of the well, which disrupts our displacement and time measurements.



Plot relating transverse position to depth; visibly, the object hits the wall of the shaft at y=-2.5 m before hitting the bottom of the shaft

Crossing Times for Homogeneous and

Non-Homogeneous Earth

In reality, the density of the Earth is not uniform throughout its volume, which has profound impacts on the force of gravity at different depths. We can relate the mass of the Earth at a particular radius given the equation:

$$M_n(R) = 4\pi \int_0^R \rho_n (1 - \frac{r^2}{R_E^2})^n \, r^2 \, dr$$

Which we can plug into the equation for gravity to find an acceleration:

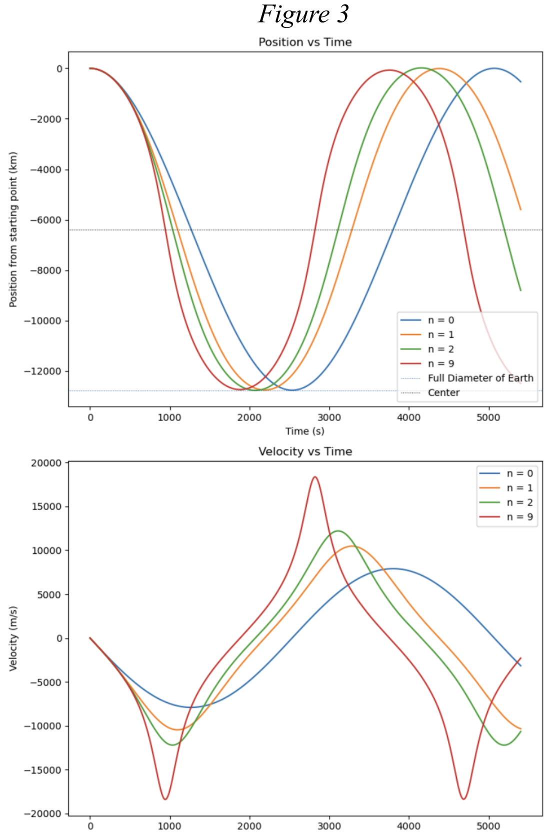
$$\frac{F}{m} = \frac{GM(R)}{R^2} = |\vec{a}|$$

Which we can then sub into to create a new differential equation:

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = \frac{GM(R)}{R^2}$$

Subbing in different values for n, (n = 0, 1, 2, 9), we see how different density configurations impact velocity and position, and how they differ from the uniform density Earth assumption made in previous calculations:

Figure 3



We can calculate the *crossing time*, which is the hypothetical time to cross a shaft going all the way across the diameter of a body due to gravity. For n=0,1,2,9, this crossing time was 2534.4, 2534.4, 2070.3, and 1887.6 seconds respectively. The crossing time relates the overall "gravitational strength" of a density configuration, even if the total mass is the same.

In particular, if we compare uniform density (n=0) to highly variable density (n=9), we see how density profile can impact the gravity of an object.

We can also compare the crossing time for the Earth and the Moon (assuming constant density for both) with the Earth's at 5069.4 seconds, and the Moon's at 6500.5 seconds. The ratio between crossing times is around 0.8. Theoretically, we know that crossing time half the orbital period, which can relate to density by:

$$T_{\text{orb}} \propto \frac{1}{\sqrt{\rho_n}}$$

Overall, although we calculated all previous sections using constant density for the Earth, it is very apparent that density has non-negligible effects.

Discussion and Future Work

In conclusion, our most accurate model gives us a predicted fall time of **83.3 seconds**, if the shaft is exactly 4000 m deep. However, we also point that this method of measurement is **not feasible** due to the Coriolis force causing the test mass to hit the wall. We also explore the limitations of assuming constant density, and find density to higher powers of "n" result in faster accelerations and crossing times.

Our model includes several approximations of physical values. Notably, we approximate the mass and radius of the Earth and Moon to 3-4 significant figures each; but since they are such big numbers (in SI units), these approximations can actually be quite large.

Future work may include refinements on these measurements, as well as considering a non-spherical Earth, Earth's shape distribution as a function of time (effect of tidal forces), modeling a shaft that is not perfectly aligned with the Earth's radius, allowing for variable drag due to changes in air inside the shaft, and more accurate treatment of the Earth's rotation.