Note Title

Revision Sheet 5 : Solutions

Problems

1. a) The normal equations have the form $A^TAc = A^Ty$, where

$$A = \begin{pmatrix} \sqrt{x_0} & \ell(x_0) & e^{x_0} \\ \sqrt{x_1} & \ell(x_1) & e^{x_1} \\ \vdots & \vdots & \vdots \\ \sqrt{x_n} & \ell(x_n) & e^{x_n} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$D = A^TA = \begin{pmatrix} \sum_{i=0}^n x_i & \sum_{i=0}^n \sqrt{x_i}\,\ell(x_i) & \sum_{i=0}^n \sqrt{x_i}\,e^{x_i} \\ \sum_{i=0}^n \sqrt{x_i}\,\ell(x_i) & \sum_{i=0}^n \ell^2(x_i) & \sum_{i=0}^n \ell(x_i)e^{x_i} \\ \sum_{i=0}^n \sqrt{x_i}\,e^{x_i} & \sum_{i=0}^n \ell(x_i)e^{x_i} & \sum_{i=0}^n e^{2x_i} \end{pmatrix}, \quad E = A^Ty = \begin{pmatrix} \sum_{i=0}^n y_i\sqrt{x_i} \\ \sum_{i=0}^n y_i\,\ell(x_i) \\ \sum_{i=0}^n y_i e^{x_i} \end{pmatrix}$$

b) Let $\ell(x_5) = 1$ and $\ell(x_i) = 0$, $i \neq 5$. Then

$$D = \begin{pmatrix} \sum_{i=0}^n x_i & \sqrt{x_5} & \sum_{i=0}^n \sqrt{x_i}\,e^{x_i} \\ \sqrt{x_5} & 1 & e^{x_5} \\ \sum_{i=0}^n \sqrt{x_i}\,e^{x_i} & e^{x_5} & \sum_{i=0}^n e^{x_i} \end{pmatrix}, \quad E = \begin{pmatrix} \sum_{i=0}^n y_i\sqrt{x_i} \\ y_5 \\ \sum_{i=0}^n y_i e^{x_i} \end{pmatrix}$$

The normal equations are $Dc = E$, where $c = (c_0, c_1, c_2)^T$, so the 2^{nd} equation reads

$$c_0\sqrt{x_5} + c_1 + c_2 e^{x_5} = y_5$$

On the other hand, $f^*(x) = c_0\sqrt{x} + c_1\ell(x) + c_2e^x$, where (c_0, c_1, c_2) solve the normal equations. Therefore

$$f^*(x_5) = c_0\sqrt{x_5} + c_1\underbrace{\ell(x_5)}_{=1} + c_2e^{x_5} = y_5 \qquad \square$$

2\. a) In order to derive the expression for the local error, we need to expand the numerical method map $\Psi_h(u)$ in the Taylor series around $h=0$ and compare term by term to the same expansion for the exact solution $\Phi_h(u)$.

$$\Psi_h(u) = \Psi_0(u) + h\,\Psi_0'(u) + \frac{h^2}{2}\Psi_0''(u) + \frac{h^3}{6}\Psi_0'''(u) + O(h^4)$$

The Implicit Trapezoidal Rule method can be written as

$\Psi_h = u + \frac{h}{2}\left[f + f(\Psi_h)\right]$. So $\Psi_0 = u$.

$\Psi_h' = \frac{1}{2}\left[f + f(\Psi_h)\right] + \frac{h}{2} f'(\Psi_h)\Psi_h'$; $\underline{\Psi_0' = f}$.

$$\Psi_h'' = \frac{1}{2} f'(\Psi_h)\Psi_h' + \frac{1}{2} f'(\Psi_h)\Psi_h' + \frac{h}{2} f''(\Psi_h)\Psi_h'^2 + \frac{h}{2} f(\Psi_h)\Psi_h''$$

$$= f'(\Psi_h)\Psi_h' + \frac{h}{2}\left[f''(\Psi_h)\Psi_h'^2 + f'(\Psi_h)\Psi_h''\right]; \quad \underline{\Psi_0'' = f'f}.$$

$$\Psi_h''' = \frac{3}{2}\left[f''(\Psi_h)\Psi_h'^2 + f'(\Psi_h)\Psi_h''\right] + \frac{h}{2}\left[f''' \dots\right]; \quad \underline{\Psi_0''' = \frac{3}{2}\left[f''f^2 + f'^2 f\right]}.$$

$$le(h;u) = \frac{h^3}{6}\left(f''f^2 + f'^2 f\right) - \frac{h^3}{6}\left(\frac{3}{2} f''f^2 + \frac{3}{2} f'^2 f\right) + O(h^4)$$

$$= \underline{-\frac{h^3}{12}\left(f''f^2 + f'^2 f\right)} + O(h^4).$$

b) For Heun's method, we have

$$\Psi_h = u + \frac{h}{2}\left[f + f(\underbrace{u + hf}_{\tilde{u}})\right]; \qquad \Psi_0 = u.$$

$$\Psi_h' = \frac{1}{2}\left[f + f(u+hf)\right] + \frac{h}{2} f'(u+hf)f; \quad \Psi_0' = f.$$

$$\Psi_h'' = f'(u+hf)f + \frac{h}{2} f''(u+hf)f^2; \qquad \Psi_0'' = f'f.$$

$$\Psi_h''' = \frac{3}{2} f''(u+hf)f^2 + \frac{h}{2} f'''(u+hf)f^3; \quad \Psi_0''' = \frac{3}{2} f''f^2.$$

$$le(h;u) = \frac{h^3}{6}\left[f''f^2 + f'^2 f - \frac{3}{2} f''f^2\right] + O(h^4)$$

$$= \underline{\frac{h^3}{6}\left[f'^2 f - \frac{1}{2} f''f^2\right]} + O(h^4).$$

3. The two-stage explicit RK method is given by

$$\tilde{u}_1 = u; \qquad \tilde{u}_2 = u + h a_{21} f(\tilde{u}_1);$$

$$\begin{aligned}\Psi_h(u) &= u + h[b_1 f(\tilde{u}_1) + b_2 f(\tilde{u}_2)] \\ &= u + h[b_1 f + b_2 f(u + h a_{21} f)]\end{aligned}$$

$$\Psi_h' = b_1 f + b_2 f(u + h a_{21} f) + h b_2 f'(u + h a_{21} f) a_{21} f;$$

$$\Psi_0' = (b_1 + b_2) f = \Phi_0' = f \Rightarrow \underline{b_1 + b_2 = 1}.$$

$$\Psi_h'' = 2 b_2 f'(u + h a_{21} f) a_{21} f + h b_2 f''(u + h a_{21} f)(a_{21} f)^2;$$

$$\Psi_0'' = 2 b_2 a_{21} f' f = \Phi_0'' = f' f \Rightarrow \underline{b_2 a_{21} = \tfrac{1}{2}}.$$

<u>One</u> free parameter in the 2^{nd} order 2-stage RK.

Alternatively, one can expand Ψ_h in the Taylor series:

$$\begin{aligned}\Psi_h(u) &= u + h\left[b_1 f + b_2 \left\{ f + h a_{21} f f' + \frac{(h a_{21})^2}{2} f^2 f'' + O(h^3) \right\}\right] \\ &= u + h(b_1 + b_2) f + h^2 b_2 a_{21} f f' + h^3 \frac{1}{2} b_2 a_{21}^2 f^2 f'' + O(h^4)\end{aligned}$$

Comparing with the Taylor expansion of $\Phi_h(u)$:

$$b_1 + b_2 = 1, \qquad b_2 a_{21} = \frac{1}{2}.$$