Revision Sheet 2 : Solutions

Theory

3. GE algorithm converts a full matrix A into an upper-triangular matrix U through a series of multiplications by elementary transformation matrices M_j : $M_{n-1} M_{n-2} \cdots M_1 A = U$, $M_j = I - m_j e_j^T$

The lower triangular part L of LU factorisation is given by $L = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1} = I + \sum_{j=1}^{n-1} m_j e_j^T$.

4. See Par, top of p. 36.

8. The cost of direct (GE) method is proportional to n^3. The cost of one step of iterative method is prop. to n^2. If the number of required iterations k_{max} is smaller than n, then the iterative solver is faster (i.e., more efficient) than direct solver.

Problem 1

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 3 & 1 \\ -5 & -3 & \frac{3}{2} \end{pmatrix}; \quad m_1 = \begin{pmatrix} 0 \\ 2 \\ -\frac{5}{2} \end{pmatrix}, \quad M = I - m_1 e_1^T = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{5}{2} & 0 & 1 \end{pmatrix}$$

$$M_1 A = \begin{pmatrix} 2 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 2 & -1 \end{pmatrix}; \quad m_2 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \quad \Rightarrow \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$M_2 M_1 A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{pmatrix} = U, \quad L = M_1^{-1} M_2^{-1} = I + m_1 e_1^T + m_2 e_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -\frac{5}{2} & -2 & 1 \end{pmatrix}.$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{5}{2} & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 3 & 1 \\ -5 & -3 & \frac{3}{2} \end{pmatrix} = A.$$

Problem 2 GE with scaled partial pivoting

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \\ -5 & -3 & \frac{3}{2} \end{pmatrix} \quad s_1 = 3, \quad s_2 = 5, \quad s_3 = 5.$$

$$\frac{|a_{11}|}{s_1} = \frac{2}{3}, \quad \frac{|a_{21}|}{s_2} = \frac{4}{5} \quad \frac{|a_{31}|}{s_3} = 1 \quad \Rightarrow \quad P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_1 A = \begin{pmatrix} -5 & -3 & 3/2 \\ 4 & 5 & 0 \\ 2 & 3 & -1 \end{pmatrix}; \quad m_1 = \begin{pmatrix} 0 \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix} \quad \Rightarrow \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ \frac{2}{5} & 0 & 1 \end{pmatrix}$$

$$M_1 P_1 A = \begin{array}{cc} & \text{row} \\ \begin{pmatrix} -5 & -3 & 3/2 \\ 0 & \frac{13}{5} & \frac{6}{5} \\ 0 & \frac{9}{5} & -\frac{2}{5} \end{pmatrix} & \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \end{array} \quad \frac{|a_{12}|}{s_2} = \frac{13}{25}, \quad \frac{|a_{32}|}{s_1} = \frac{3}{5} \quad \Rightarrow \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

swap with rows!

$$P_2 M_1 P_1 A = \begin{pmatrix} -5 & -3 & 3/2 \\ 0 & \frac{9}{5} & -\frac{2}{5} \\ 0 & \frac{13}{5} & \frac{6}{5} \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{13}{9} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{13}{9} & 1 \end{pmatrix}$$

$$M_2 P_2 M_1 P_1 A = \begin{pmatrix} -5 & -3 & \frac{3}{2} \\ 0 & 9/5 & -\frac{2}{5} \\ 0 & 0 & 16/9 \end{pmatrix} = U, \quad \tilde{L} = P_1^T M_1^{-1} P_2^T M_2^{-1} = \begin{pmatrix} -\frac{2}{5} & 1 & 0 \\ -\frac{4}{5} & \frac{13}{9} & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P = P_2 P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad L = P\tilde{L} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ -\frac{4}{5} & \frac{13}{9} & 1 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ -\frac{4}{5} & \frac{13}{9} & 1 \end{pmatrix} \begin{pmatrix} -5 & -3 & \frac{3}{2} \\ 0 & \frac{9}{5} & -\frac{2}{5} \\ 0 & 0 & \frac{16}{9} \end{pmatrix} = \begin{pmatrix} -5 & -3 & \frac{3}{2} \\ 2 & 3 & -1 \\ 4 & 5 & 0 \end{pmatrix} = PA$$

Problem 3

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & 2 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} l_{11}^2 & & \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & \\ l_{31} l_{11} & l_{31} l_{21} + l_{32} l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

$l_{11}^2 = 4 \Rightarrow l_{11} = 2$ $\quad$ $l_{21}^2 + l_{22}^2 = 2 \Rightarrow l_{22} = 1$

$l_{21} l_{11} = -2 \Rightarrow l_{21} = -1$ $\quad$ $0 + l_{32} \cdot 1 = 2 \Rightarrow l_{32} = 2$

$l_{31} l_{11} = 0 \Rightarrow l_{31} = 0$ $\quad$ $0 + 2^2 + l_{33}^2 = 5 \Rightarrow l_{33} = 1$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Problem 4

$$A = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix} ; \quad D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} ; \quad L = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} ; \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

a) Richardson iteration : $x_{k+1} = (I - A) x_k + b$

$$x_1 = \begin{pmatrix} -3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} .$$

b) Jacobi iteration : $x_{k+1} = - D^{-1} (L + U) x_k + D^{-1} b$

$$x_1 = - \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} .$$

Problem 5

a) Need $\rho(I - A) < 1$ for convergence.

Calculate eigenvalues :

$$\det \begin{pmatrix} -3-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (-3-\lambda)(3-\lambda) + 1 = \lambda^2 - 8 = 0$$

$\lambda_{1,2} = \pm\sqrt{8}$. $\quad \rho(I - A) = \max |\lambda_{1,2}| = \sqrt{8} > 1$

Richardson iteration <u>does not converge</u>.

b) Need $\rho(D^{-1}(L+U)) < 1$.

$$\det \begin{pmatrix} -\lambda & \frac{1}{4} \\ \frac{1}{2} & -\lambda \end{pmatrix} = \lambda^2 - \frac{1}{8} = 0 , \quad \lambda_{1,2} = \pm \frac{1}{\sqrt{8}} .$$

$$\rho(D^{-1}(L+U)) = \frac{1}{\sqrt{8}} < 1$$

Jacobi iteration <u>converges</u>.

Problem 6.

$\sqrt[p]{a}$ is the root of the function $f(x) = x^p - a$.

Newton's method for finding this root is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^p - a}{p x_k^{p-1}} = \frac{(p-1)x_k^p + a}{p x_k^{p-1}}.$$

Problem 7

$$x_{k+1} = x_k \frac{x_k^2 + 3a}{3x_k^2 + a}, \quad \lim_{k\to\infty} x_k = \sqrt{a}.$$

Let $e_k = x_k - \sqrt{a}$. $\quad e_{k+1} + \sqrt{a} = (e_k + \sqrt{a}) \dfrac{(e_k+\sqrt{a})^2 + 3a}{3(e_k+\sqrt{a})^2 + a}$.

$$e_{k+1} = \frac{(e_k + \sqrt{a})^3 + 3a(e_k + \sqrt{a}) - 3(e_k+\sqrt{a})^2\sqrt{a} - a^{3/2}}{3(e_k+\sqrt{a})^2 + a}$$

$$= \frac{e_k^3 + \cancel{3e_k^2\sqrt{a}} + \cancel{3e_k a} + \cancel{a^{3/2}} + \cancel{3ae_k} + \cancel{3a^{3/2}} - \cancel{3e_k^2\sqrt{a}} - \cancel{6e_k a} - \cancel{3a^{3/2}} - \cancel{a^{3/2}}}{3e_k^2 + 6e_k\sqrt{a} + 4a}$$

$$= \frac{e_k^3}{4a + 6e_k\sqrt{a} + 3e_k^2} = \frac{1}{4a} e_k^3 + O(e_k^4)$$

So, $\displaystyle\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^3} = \frac{1}{4a}$. Convergence rate $r = 3$.