Revision Sheet 2: Linear Systems and Finding Roots

*Hand in solutions to Problems only on Wednesday, **25 October**, in College House by **11am**, or as a pdf or Word document (single file) on Blackboard by **noon**.*

Theory

1. When does the system of linear equations $Ax = b$ have a unique solution? No solutions? Infinitely many solutions?
2. Let $M_k = I - m_k e_k^T$ be an elementary elimination matrix. Show that $M_k^{-1} = I + m_k e_k^T$. Show that $M_k^{-1} M_j^{-1} = I + m_k e_k^T + m_j e_j^T$ for any $k < j$.
3. Explain how Gaussian Elimination algorithm produces the LU factorisation of a matrix.
4. Show how LU factorisation of A can be used to solve the system $Ax = b$.
5. What is a *pivot*?
6. What is the difference between partial pivoting and complete pivoting? What is the difference between partial pivoting and scaled partial pivoting?
7. Derive an algorithm for computing the Cholesky factorisation LL^{T} of an $n \times n$ symmetric positive definite matrix A by equating the corresponding elements of A and LL^{T}.
8. Under what conditions an iterative method for solving a system of linear equations is faster than a direct method?
9. Let $g: \mathbb{R}^n \mapsto \mathbb{R}^n$ be a map. Define a *fixed point* of $g(x)$. What is a fixed point iteration?
10. State the theorem about the convergence of a fixed point iteration of a linear map on $\mathbb{R}^n$.
11. Write the general form of the iterative scheme for solving $Ax = b$. Describe what the splitting matrix Q is in relation to A for Richardson's iteration, Jacobi iteration, and Gauss-Seidel iteration.
12. State the definition of *convergence rate* of an iterative algorithm.
13. Describe the bisection algorithm. What is the convergence rate of the bisection algorithm?
14. Derive Newton's algorithm for finding a root of a function $f: \mathbb{R} \mapsto \mathbb{R}$.
15. Derive Newton's algorithm for finding a root of a function $f: \mathbb{R}^n \mapsto \mathbb{R}^n$.
16. Determine the convergence rate of Newton's algorithm to a simple root of a function $f: \mathbb{R} \mapsto \mathbb{R}$.
17. Derive the Secant Method for finding a root of the function $f: \mathbb{R} \mapsto \mathbb{R}$.

Problems

1. Compute the LU factorisation of the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 3 & 1 \\ -5 & -3 & 3/2 \end{bmatrix}$$

using naïve Gaussian elimination. Show your work (i.e. what are M_1 and M_2?). Your final answer should be two matrices L and U. Verify that $A = LU$.

2. Compute the LU factorisation of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \\ -5 & -3 & 3/2 \end{bmatrix}$$

using Gaussian elimination with scaled partial pivoting. Show your work. Your final answer should be matrices L, U, and P. Verify that $PA = LU$.

3. Compute Cholesky factorisation of a symmetric matrix

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

that is, find a lower triangular matrix L, such that $A = LL^{\mathrm{T}}$.

4. Consider the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

 a) Perform one step of Richardson iteration starting with $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

 b) Perform one step of Jacobi iteration starting from the same x_0.

5. For the linear system in Problem 4:
 a) Will the Richardson iteration converge to the solution? Justify your answer.
 b) Will the Jacobi iteration converge to the solution? Justify your answer.

6. Use Newton's method to design an iterative scheme for computing $\sqrt[p]{a}$, where $a > 0$ and $p \geq 2$.

7. Determine the convergence rate of the following iterative scheme for computing $\sqrt{a}$, where $a > 0$:

$$x_{k+1} = x_k \frac{x_k^2 + 3a}{3x_k^2 + a}.$$