Revision Sheet 1: Introduction and Preliminaries

*Hand in solutions to Problems only on Wednesday, **11 October**, in College House by **11am**, or as a pdf or Word document (single file) on Blackboard by **noon**.*

Theory

1. What is a truncation error? Give an example. What is a round-off error? Give an example.
2. Explain the distinction between computational error and propagated data error.
3. What is a condition number of a computational problem? If the *condition number* of a problem is 10^6, is this good or bad? Why?
4. Explain the concept of a rounding error. What is *machine precision*?
5. Explain why a divergent infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ can have a finite sum in floating-point arithmetic. A what point will the partial sum cease to change when computed in floating-point arithmetic with machine precision ϵ_{mach}?
6. Define a transpose matrix. Let A^T denote the transpose of A. Show that $(AB)^T = B^T A^T$.
7. When is a matrix non-singular?
8. Let A be a non-singular matrix. Show that the matrix $C = A^T A$ is symmetric and positive definite.
9. Prove that $\|Ax\| \leq \|A\|\|x\|$. Prove that $\|AB\| \leq \|A\|\|B\|$.
10. Let x be the solution to the non-singular linear system $Ax = b$, and let $\hat{x}$ be the solution to the system $A\hat{x} = b + \Delta b$ with a perturbed right-hand side. Show that $\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\Delta b\|}{\|b\|}$, where $\Delta x = \hat{x} - x$.

Problems

1. Assume that you are solving the quadratic equation $ax^2 + bx + c = 0$ with $a = 1.22$, $b = 3.88$, and $c = 3.08$, using a normalised floating point system with $\beta = 10$ and $p = 3$ and *rounding to nearest*.
   a. What is the computed value of the discriminant $b^2 - 4ac$?
   b. What is the correct value of the discriminant in real (exact) arithmetic?
   c. What is the relative error in the computed value of the discriminant?

2. For a smooth function, $f: \mathbb{R} \mapsto \mathbb{R}$, consider the finite difference approximation to the second derivative,

$$f''(x) \approx \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

   a. Using Taylor's Theorem, determine the bound on the truncation error of this approximation in terms of h and a bound on the function derivative M of appropriate order (to be determined).
   b. Assuming the error in function evaluation is bounded by ε, determine the rounding error in evaluating the finite difference approximation formula.
   c. Determine the optimum choice of h for which the total error is minimized. What is the value of the minimum error? First, express your answers in terms of M and ε, and then obtain numerical values given that M = 1 and $\varepsilon = 10^{-16}$.

3. Show that $\text{cond}(A) = \left(\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \right) \left(\min_{x \neq 0} \frac{\|Ax\|}{\|x\|} \right)^{-1}$. Explain the meaning of this expression.

4. Let $\hat{x}$ be the solution to the system $(A + E)\hat{x} = b$ with a perturbed matrix. Show that $\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \text{cond}(A) \frac{\|E\|}{\|A\|}$.