Rev. Sheet 5.

a) 1) $\left(\sum_{i=0}^{n} [y_i - f(x_i)^2]\right)^2$; $\sum_{i=0}^{n} [c_0\sqrt{x_i} + c_1 \ell(x_i) + c_2 e^{x_i} - y_i]^2 = \phi$

$$0 = \frac{\partial \phi}{c_0} = \sum_{k=0}^{n} 2\sqrt{x_k}\left(c_0\sqrt{x_k} + c_1\ell(x_k) + c_2 e^{x_k} - y_k\right)$$

$$0 = \frac{\partial \phi}{c_1} = \sum_{k=0}^{n} 2\ell(x_k)\left(c_0\sqrt{x_k} + c_1\ell(x_k) + c_2 e^{x_k} - y_k\right)$$

$$0 = \frac{\partial \phi}{c_2} = \sum_{k=0}^{n} 2e^{x_k}\left(c_0\sqrt{x_k} + c_1\ell(x_k) + c_2 e^{x_k} - y_k\right)$$

1) $\sum c_0 x_k + \sum c_1 \ell(x_k)\sqrt{x_k} + \sum c_2 e^{x_k}\sqrt{x_k} = \sum \sqrt{x_k}\, y_k$

2) $\sum \ell(x_k) c_0\sqrt{x_k} + \sum c_1 (\ell(x_k))^2 + \sum c_2 e^{x_k}\ell(x_k) = \sum \ell(x_k) y_k$

3) $\sum e^{x_k} c_0\sqrt{x_k} + \sum e^{x_k} c_1 \ell(x_k) + \sum c_2 e^{2x_k} = \sum e^{x_k} y_k$

$$DC = E$$

$$D\begin{bmatrix} \sum x_k & \sum \ell(x_k)\sqrt{x_k} & \sum e^{x_k}\sqrt{x_k} \\ \sum \ell(x_k)\sqrt{x_k} & \sum \ell(x_k)^2 & \sum e^{x_k}\ell(x_k) \\ \sum e^{x_k}\sqrt{x_k} & \sum e^{x_k}\ell(x_k) & \sum e^{2x_k} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum \sqrt{x_k}\, y_k \\ \sum \ell(x_k) y_k \\ \sum e^{x_k} y_k \end{bmatrix}$$

b) $\ell(x_5) = 1 \quad \ell(x_j) = 0,\ j \neq 5$

$f^*(x_5) = y_5$

$$\begin{bmatrix} \sum x_k & \sqrt{x_5} & \sum e^{x_k}\sqrt{x_k} \\ \sqrt{x_5} & 1 & e^{x_5} \\ \sum e^{x_k}\sqrt{x_k} & e^{x_5} & \sum e^{2x_k} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum \sqrt{x_k}\, y_k \\ y_5 \\ \sum e^{x_k} y_k \end{bmatrix}$$

$$y_5 = c_0\sqrt{x_5} + c_1 + c_2 e^{x_5}$$

$$f^* = \{ f(x) = c_0\sqrt{x} + c_1\ell(x) + c_2 e^{x} : c_0, c_1, c_2 \in \mathbb{R} \}$$

$$f^*(x_5) = c_0\sqrt{x_5} + c_1\ell(x_5) + c_2 e^{x_5} = c_0\sqrt{x_5} + c_1 + c_2 e^{x_5} =$$

$$= y_5 .$$

2) $U_{n+1} = U_n + \frac{h}{2}[f(U_n) + f(U_{n+1})]$, $f(U) = f$, $\Psi_0' = f$

$\phi'(U) = \frac{dU}{dt} = f$. $\phi'' = \frac{df}{dU}\frac{dU}{dt} = f'f$

$\Psi_h = U_h + \frac{h}{2}[f(U) + f(U_{h+1})]$

$\Psi_h' = 0 + \frac{1}{2}[f + f(\Psi_h)] + \frac{h}{2}[0 + f'(\Psi_h)\cdot\Psi_h']$

$\Psi_0' = \frac{1}{2}[0 + f(\cancel{\Psi_h}\,U)] = \frac{1}{2}[f + f] = f = \phi_0'$

$\Psi_h'' = \frac{1}{2}[f'(\Psi_h)\cdot\Psi_h'] + \frac{1}{2}[f'(\Psi_h)\Psi_h'] + \frac{h}{2}[f''(\Psi_h)(\Psi_h')^2 +$

$+ f'(\Psi_h)\Psi_h'']$

$\Psi_0'' = \frac{1}{2}[f'(U)\cdot f] + \frac{1}{2}[f'(U)\cdot f] + 0 = \frac{1}{2}[f'\cdot f] + \frac{1}{2}[f'\cdot f] = f'f =$

$= \phi_0''$

$\Psi_h''' = \frac{1}{2}[f''(\Psi_h)\Psi_h'\cdot\Psi_h' + f'(\Psi_h)\Psi_h''] + \frac{1}{2}[f''(\Psi_h)\Psi_h'\cdot\Psi_h' + f(\Psi_h)\Psi_h''] +$

$+ \frac{1}{2}[f''(\Psi_h)\Psi_h'\cdot\Psi_h' + f(\Psi_h)\cdot\Psi_h''] + \frac{1}{2}[f''(\Psi_h)\Psi_h'\cdot\Psi_h'] +$

$+ \frac{h}{2}[f'''(\Psi_h)\Psi_h'\cdot\Psi_h' \ldots]$

$\Psi_0''' = \frac{1}{2}[f''(\Psi_0)(\Psi_0')^2 + f(\Psi_0)(\Psi_0'')] + \frac{1}{2}[f''(\Psi_0)(\Psi_0')^2 +$

$+ f(\Psi_0)\Psi_0''] + \frac{1}{2}[f''(\Psi_0)(\Psi_0')^2] + 0 = \frac{1}{2}[f''(U)f^2 + f(U)f''] +$

$+ \frac{1}{2}[f''(U)\cdot f^2 + f(U)f'] + \frac{1}{2}[f''(U)f^2] = \frac{1}{2}[f''f^2 + ff''] +$

$+ \frac{1}{2}[f''f^2 + ff'] + \frac{1}{2}[f''f^2] = [f''f^2 + f''f] + \frac{1}{2}[f''f^2] =$

$= \frac{3}{2}f''f^2 + f''f^2 \neq \phi_0'''$

$Le(h,U) = \phi_h''' - \Psi_h''' = -\frac{h^3}{6}[-\frac{1}{2}[f''(f)^2 + (f')^2 f]] =$

$= -\frac{h^3}{12}[f''(f')^2 + (f')^2 f] + O(h^4)$

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b) $\Psi_h = U_h + \frac{h}{2}[f + f(U + hf)]$

$\Psi_h' = \frac{1}{2}[f + f(v + hf)] + \frac{h}{2}[f' + f(v+hf) f]$

$\Psi_h' = \frac{1}{2}[f + f(v+hf)] + \frac{h}{2}[f'(v+hf)f]$

$\Psi_0' = \frac{1}{2}[f + f] = f = \phi_0'$

$\Psi_h'' = \frac{1}{2}[f(v+hf) \cdot f] + \frac{1}{2}[f'(v+hf)f] + \frac{h}{2}[f''(v+hf)f^2]$

$\Psi_0'' = \frac{1}{2}[f'(v) \cdot f] + \frac{1}{2}[f'(v) f] = f' \cdot f = \phi_0''$

$\Psi_h''' = \frac{1}{2}[f''(v)f^2] + \frac{1}{2}[f''(v) f^2] + \frac{1}{2}[f''(v) f^2] = \frac{3}{2}[f' f^2] \neq$
$\neq \phi_0'''$

$Le(h, v) = \phi_h''' - \Psi_h''' = \frac{h^3}{6}\left[f'' f^2 - (f')^2 f - \frac{3}{2} f'' f^2\right] = -\frac{1}{2}\cancel{f''f + (f')^2 f}$

$Le(h, v) = \frac{h^3}{6}\left[-\frac{1}{2} f'' f + (f')^2 f\right] = -\frac{h^3}{12} f'' f^2 + \frac{h^3}{6}(f')^2 f + O(h^4)$

3) $v(h) = \phi_h(v) = v + hf + \frac{h^2}{2} f' f + O(h^3)$

RK: $\tilde{v}_1 = v \quad ; \quad \tilde{v}_2 = v + h\alpha_{21} f(\tilde{v}_1)$

$v(h) = v + h[b_1 f(\tilde{v}_1) + b_2 f(\tilde{v}_2)]$

$\tilde{v}_2 = v + h \alpha_{21} f$

$v(h) = v + h[b_1 f + b_2 f(v + h\alpha_{21} f)] =$

$= v + hb_1 f + hb_2 [f + h\alpha_{21} f f' + \frac{1}{2}(h\alpha_{21} f)^2 f'' + O(h^3)] =$

$= v + h(b_1 + b_2) f + h^2 (b_2 \alpha_{21}) f' f$

Order cond:

1^{st} : $b_1 + b_2 = 1$

2^{nd} : $b_2 \alpha_{21} = \frac{1}{2}$

So we have 3 param. α_{21}, b_1, b_2 in two equations.