Revision Sheet 4

*Hand in solutions to Problems only on Wednesday, **22 November**, in College House by **11am**, or as a pdf or Word document (single file) on Blackboard by **noon**.*

Theory

1. What does it mean for a function $f(t)$ to be *square-integrable* on $[a, b]$? Define the L^2 norm of a function on $[a, b]$.
2. State the theorem about the convergence of the partial Fourier sum $S_n(t)$ in the L^2 norm.
3. State Dirichlet's theorem about pointwise convergence of a Fourier series.
4. Let $S_n(t) = \sum_{k=-n}^{n} c_k e^{iq_k t}$ be a trigonometric polynomial, where $q_k = \frac{\pi k}{T}$ and $t \in [-T, T]$. Derive expression for the coefficients c_k such that $S_n(t)$ interpolates a function $f(t)$ at points $t_l = \frac{2Tl}{2n+1}$, $l = -n, \dots, n$.
5. Show that the Discrete Fourier Transform (DFT) of length n can be written as a sum of two DFTs, each of length $n/2$, provided n is even.
6. Derive the system of equations for the coefficients c_i in the k-th derivative approximation formula
$$f^{(k)}(x) \approx \sum_{i=1}^{n} c_i f(x + a_i h).$$
7. Let $\phi(h) = L + a_p h^p + O(h^r)$, $r > p$. Derive Richardson extrapolation formula for $O(h^r)$ approximation of L which uses $\phi(h)$ and $\phi(h/q)$, where $q > 1$. *Hint:* See Heath's Ch. 8 slides 54, 55.
8. What is the difference between the Newton-Cotes and Gaussian quadrature rules?
9. State and prove the Gaussian Quadrature theorem.

Algorithms

1. Write a Matlab function which evaluates the Newton interpolating polynomial $p_n(x)$ in a nested form at a given point x. Assume that nodes x_i are given in a vector `xi` and coefficients c_k are given in a vector `ck`.
2. Write a Matlab function which computes, by divided differences, coefficients of Newton's interpolating polynomial for the points (x_i, y_i), $i = 0, \dots, n$. Your function should output a vector $c = (c_0, \dots, c_n)^T$ of Newton coefficients.
3. Given a vector of n data points $x = (x_0, x_1, \dots, x_{n-1})^T$, the discrete Fourier transform (DFT) is defined as
$$y_k = \sum_{l=0}^{n-1} x_l \omega_n^{kl}, \quad k = 0, \dots, n-1.$$
where $\omega_n = e^{-2\pi i/n}$ is the primitive n^{th} root of unity. Write a Matlab function which calculates y_k directly according to the above formula.

Problems

1. Given complex-valued vectors $x = (x_0, x_1, \dots, x_{n-1})^T$ and $y = (y_0, y_1, \dots, y_{n-1})^T$, $x_k, y_k \in \mathbb{C}$, the discrete Fourier transform (DFT) can be written as a matrix-vector product in the form $y = F_n x$, where $F_n \in \mathbb{C}^{n\times n}$ is an n-by-n matrix with elements $(F_n)_{jk} = \omega_n^{jk}$, $j, k = 0, \dots, n-1$, with $\omega_n = e^{-2\pi i/n}$ being the primitive n^{th} root of 1. Show that the matrix elements of the inverse DFT, $x = F_n^{-1} y$, are given by $(F_n^{-1})_{jk} = \frac{1}{n}\omega_n^{-jk}$.
2. Using the definition of the DFT from Problem 1
   a) Calculate the DFT of $x = (1, 2, 0)^T$. Express your answer in radicals. Show your work.
   b) Calculate the DFT of $x = (1, 2, 0, 1)^T$. Show your work.
3. Given a sufficiently smooth function $f: \mathbb{R} \mapsto \mathbb{R}$, derive the best possible approximation to $f'(x)$ in terms of $f(x)$, $f(x-h)$, and $f(x+2h)$.

4. Let $\phi(h)$ be some computable approximation to the quantity L such that $\phi(h) = L + a_2 h^2 + a_4 h^4 + \cdots$. Combine evaluations of $\phi(h)$ and $\phi(h/3)$ to devise a $O(h^4)$ approximation to L.

5. Use the method of undetermined coefficients to derive the quadrature rule

$$\int_0^1 f(x)\,dx \approx w_1 f(1/4) + w_2 f(1/2) + w_3 f(3/4)$$

which is exact for polynomials of the highest possible degree. What is the degree of this quadrature rule?

6. a) Find the approximate value of the integral

$$\int_0^1 \frac{dx}{(1+x)^2}.$$

using Simpson's rule. What is the relative error of this approximation?

b) Same as a) but using the three-node Gaussian Quadrature (see the table on p. 112 of Pav's lecture notes). You will need to change the integration variable as discussed on the same page (Example Problem 8.18). What is the relative error of this approximation?