Revision Sheet 3 : Solutions

Theory

3. In monomial basis, the basis functions $\varphi_j(t) = t^j$, $j = 0, 1, \ldots$ are very similar for large j.
this means that the Vandermonde matrix $a_{ij} = \varphi_j(t_i)$ is nearly singular, and cond (A) is very large.
So, the obtained solution will have low accuracy.

4. 1) Better than monomial because cond(A) is smaller.
   2) Better than monomial because A is triangular, so faster to solve.
   3) Better than Lagrange because it is faster to evaluate (especially in a nested form).
   4) Better than both because additional data points can be easily added.

5. See Sec. 5.2.1 in Pav's lecture notes

6. See middle of p. 72 in Pav's notes.

Algorithms

1.
```
function x = backsubs(U, b)
  n = length(b);
  if U(n,n)==0,
    error('Matrix U is singular'); end
  x(n) = b(n)/U(n,n);
  for i = n-1:-1:1,
    for j = i+1:n,
      b(i) = b(i) - U(i,j)*x(j);
    end
    if U(i,i)==0,
      error('Matrix U is singular'); end
    x(i) = b(i)/U(i,i);
  end
```

2.
for k = 1 to n-1 (loop over pivot row)
  for i = k+1 to n (loop over rows below k)
    for j = k+1 to n (loop over columns in row i)
      $a_{ij} = a_{ij} - (a_{ik}/a_{kk})\, a_{kj}$ (should get 0 when j=k)
    end
    $b_i = b_i - (a_{ik}/a_{kk})\, b_k$
  end
end

3.
```
function x = newton(f, fp, x0, tol, kmax)
  x = x0;
  for n = 1:kmax
    fx = f(x);
    if abs(fx) < tol
      return;
    end
```

```
    x = x - fx/fp(x);
end
warning('Maximum number of iterations exceeded')
```

Problems

1. a) Monomial Basis. $p(x) = \sum_{j=0}^{2} a_j x^j$

$(x_0, y_0) = (-2, 9)$, $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (1, 6)$,

$p(x_i) = y_i \Rightarrow \sum_{j=0}^{2} a_j x_i^j = y_i, \quad i = 0, 1, 2, 3$

$$\begin{pmatrix} x_0^0 & x_0^1 & x_0^2 \\ x_1^0 & x_1^1 & x_1^2 \\ x_2^0 & x_2^1 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 6 \end{pmatrix}$$

Solve in Matlab: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$

$$p(x) = 1 + 2x + 3x^2$$

b) Lagrange basis: $p(x) = \sum_{j=0}^{2} \ell_j(x) \cdot y_j$

$$\ell_0(x) = \frac{(x-0)(x-1)}{(-2-0)(-2-1)} = \frac{1}{6} x(x-1)$$

$$\ell_1(x) = \frac{(x+2)(x-1)}{(0+2)(0-1)} = -\frac{1}{2}(x+2)(x-1)$$

$$\ell_2(x) = \frac{(x+2)(x-0)}{(1+2)(1-0)} = \frac{1}{3} x(x+2)$$

$$p(x) = 9 \cdot \frac{1}{6} x(x-1) - 1 \cdot \frac{1}{2}(x+2)(x-1) + 6 \cdot \frac{1}{3} x(x+2)$$

$$= \frac{3}{2}x^2 - \frac{3}{2}x - \frac{1}{2}x^2 - \frac{1}{2}x + 1 + 2x^2 + 4x$$

$$= 1 + 2x + 3x^2$$ - same as in a).

c) Newton basis $\quad P(x) = \sum_{k=0}^{3} c_k \prod_{i=0}^{k-1} (x - x_i)$

$$\begin{pmatrix} 1 & x_0 - x_0 & (x_0 - x_1)(x_0 - x_0) \\ 1 & x_1 - x_0 & (x_1 - x_1)(x_1 - x_0) \\ 1 & x_2 - x_0 & (x_2 - x_1)(x_2 - x_0) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 6 \end{pmatrix} \Rightarrow \begin{matrix} c_0 = 9 \\ c_1 = -4 \\ c_2 = 3 \end{matrix}$$

$$P(x) = 9 - 4(x+2) + 3(x+2)x = 9 - 4x - 8 + 3x^2 + 6x$$
$$= 1 + 2x + 3x^2 \text{ - same as in a).}$$

2. a)

x	y	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
-2	9			
		$\frac{1-9}{0+2} = -4$		
0	1		$\frac{5+4}{1+2} = 3$	
		$\frac{6-1}{1-0} = 5$		$\frac{-1-3}{-3+2} = 4$
1	6		$\frac{8-5}{-3-0} = -1$	
		$\frac{-26-6}{-3-1} = 8$		
b) -3	-26			

$$p(x) = 9 - 4(x+2) + 3(x+2)x + 4(x+2)x(x-1)$$
$$= 9 + (x+2)\left[-4 + x\left[3 + 4(x-1)\right]\right] \text{ - nested form.}$$

3. $$|f(x) - p_n(x)| \le \frac{(\beta-\alpha)^{n+1}}{2^{2n+1}(n+1)!} \max_{\xi \in [\alpha,\beta]} |f^{(n+1)}(\xi)|$$

$$f(x) = \frac{1}{x}, \quad \alpha = 1, \quad \beta = 3.$$

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = 2x^{-3}, \quad f'''(x) = -2 \cdot 3\, x^{-4}, \ \ldots,$$

$$f^{(n+1)}(x) = (-1)^{n+1}(n+1)!\, x^{-(n+2)}.$$

$$\max_{\xi \in [1,3]} |f^{(n+1)}(\xi)| = (n+1)! \quad (\text{at } \xi = 1).$$

$$\left|\frac{1}{x}-p_n(x)\right| \leq \frac{(3-1)^{n+1}}{2^{2n+1}(n+1)!}\cdot(n+1)! = \frac{2^{n+1}}{2^{2n+1}} = 2^{-n}$$

$$2^{-n} \leq 10^{-8} \Rightarrow n \geq 27.$$

4. $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, \quad i = 1, 2.$

$S_i'(x) = 3a_i x^2 + 2b_i x + c_i$

$S_i''(x) = 6a_i x + 2b_i$

$S_1(0) = 4 \Rightarrow d_1 = 4$

$S_1(2) = 2 \Rightarrow 8a_1 + 4b_1 + 2c_1 + d_1 = 2$

$S_2(2) = 2 \Rightarrow 8a_2 + 4b_2 + 2c_2 + d_2 = 2$

$S_2(3) = 7 \Rightarrow 27a_2 + 9b_2 + 3c_2 + d_2 = 7$

$S_1'(2) = S_2'(2) \Rightarrow 12a_1 + 4b_1 + c_1 = 12a_2 + 4b_2 + c_2$

$S_1''(2) = S_2''(2) \Rightarrow 12a_1 + 2b_1 = 12a_2 + 2b_2$

$$\left.\begin{array}{l} S_1''(0) = 0 \Rightarrow 2b_1 = 0 \\ S_2''(3) = 0 \Rightarrow 18a_2 + 2b_2 = 0 \end{array}\right\} \text{conditions for natural cubic spline.}$$