Revision Sheet 3

Hand in solutions to <u>*Problems only*</u> *on Wednesday,* ***8 November****, in College House by* ***11am****, or as a pdf or Word document (single file) on Blackboard by* ***noon****.*

Theory

1. State the interpolation problem for a set of data points (x_i, y_i), $i = 0,\dots, n$. Show that, when the interpolating function is chosen as a linear combination of basis functions, the interpolation problem reduces to the solution of a system of linear algebraic equations.
2. State and prove the polynomial interpolant Existence and Uniqueness Theorem (see Pav's notes p. 64).
3. Explain why monomial basis is not good for constructing interpolating polynomial of high degree.
4. List at least three reasons why the use of Newton's basis for polynomial interpolation is preferable to monomial or Lagrange basis.
5. State and prove the Interpolation Error Theorem.
6. State the property of Chebyshev nodes which makes them the optimal choice for approximating a smooth function on an interval $[-1, 1]$.
7. State the conditions that characterise $S(t)$ as a natural cubic spline on the interval $[a, b]$. Let the knots be $a = t_0 < t_1 < \dots < t_n = b$.
8. Derive the system of linear equations which needs to be solved in order to compute the natural cubic spline interpolating (t_i, y_i), $i = 0, \dots, n$.

Algorithms

1. The solution of upper triangular system $Ux = b$ is calculated by back-substitution as follows:

$$x_n = \frac{b_n}{u_{nn}}; \quad x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij}x_j\right)\Big/ u_{ii}, \quad i = n-1, \dots, 1.$$

   Write a Matlab programme which implements the above calculation. The programme should output an error message if $u_{ii} = 0$.

2. The naive Gaussian Elimination algorithm for the solution of the system of linear equations $\sum_{j=1}^{n} a_{ij}x_j = b_i,\ i = 1, \dots, n$ has the following basic structure:

   for k = ???
       for i = ???
           for j = ???
               $a_{???} = a_{???} - (a_{???}/a_{???})a_{???}$
           end
           $b_{???} = b_{???} - (a_{???}/a_{???})b_{???}$
       end
   end

   Replace "???" with appropriate statements in order to complete the algorithm.

3. Write Matlab function implementing Newton's algorithm for finding the root of $f(x)$ starting from x_0 and stopping when $|f(x_k)| <$ *tol*, or when the number of steps exceeds $k_{\max}$. The header line should be

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function x = newton(f, fp, x0, tol, kmax)
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Problems

1. Given the data points $(-2, 9)$, $(0, 1)$, and $(1, 6)$, find the interpolating polynomial of degree no greater than two:
   a) Using the monomial basis.
   b) Using the Lagrange basis. Show that you get the same polynomial as in a).

c) Using the Newton basis and solving the lower triangular system. Show that the result is the same as in a).

2. a) For the same data points as in Problem 1, determine the Newton interpolating polynomial using the divided differences table.
b) Augment the divided differences table in b) to find the Newton interpolating polynomial for points (–2, 9), (0, 1), (1, 6), and (–3, –26). Express the obtained polynomial in a nested form.

3. Let $p_n(x)$ interpolate the function $1/x$ at $n + 1$ Chebyshev nodes on the interval [1, 3]. Using the Interpolation Error Theorem, determine the error bound $\max_{x\in[1,3]} \left|\frac{1}{x} - p_n(x)\right|$ as a function of n. How large should n be in order for the bound to be smaller than 10^{-8}?

4. Derive (but do not attempt to solve) the system of linear equations for the parameters of the natural cubic spline interpolating the data points (0, 4), (2, 2), and (3, 7). Assume the spline has the form

$$S(x) = \begin{cases} a_1x^3 + b_1x^2 + c_1x + d_1, & 0 \le x \le 2, \\ a_2x^3 + b_2x^2 + c_2x + d_2, & 2 \le x \le 3. \end{cases}$$