Revision Sheet 5

*Hand in solutions to Problems only on Wednesday, **6 December**, in College House by **11am**, or as a pdf or Word document (single file) on Blackboard by **noon**.*

Theory

1. Define what it means for the function $f(x) \in \mathcal{F}$ to be the linear least-squares best approximant to the data (x_0, y_0), (x_1, y_1), …, (x_n, y_n). *Hint:* See bottom of p. 117 in Pav's notes.
2. Let $f(x) = \sum_{j=0}^{m} c_j g_j(x)$ be a linear combination of linearly independent functions. Derive *normal equations* which coefficients c_j need to satisfy in order for $f(x)$ to be the least-squares best approximant to the data: (x_0, y_0), (x_1, y_1), …, (x_n, y_n).
3. Show that a third order non-autonomous ODE $y''' = g(t, y, y', y'')$ can be written as a system of first order autonomous ODEs.
4. State the definitions of the local and global errors of the numerical method $\Psi_h(u)$.
5. State the theorem about the global error of Euler's method.
6. How do the local and global errors depend on the step size h for a numerical method of order p? *Hint*: Recall the relevant definition and theorem from the lecture notes.
7. Derive order conditions for a three-stage explicit Runge-Kutta method. What is the highest possible order of such a method? How many free parameters does the highest order method have?

Algorithms

1. The composite trapezoidal quadrature rule is given by the formula
$$\int_a^b f(x)dx \approx h\left[\frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1} f(x_k)\right],$$
where $h = (b - a)/n$ and $x_k = a + kh$. Write a Matlab function which calculates the approximate value of the integral according to this formula.
2. Simpson's composite quadrature rule is given by the formula
$$\int_a^b f(x)dx \approx \frac{h}{6}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})],$$
where $h = (b - a)/n$, and $x_k = a + kh/2$, $k = 0, \ldots, 2n$. Write a Matlab function which calculates the approximate value of the integral according to this formula.
3. Write a Matlab function which calculates the approximate solution of the IVP $\frac{du}{dt} = f(u)$, $u(0) = u_0$, using the explicit midpoint method:
$$\tilde{u} = u_n + \frac{h}{2} f(u_n),$$
$$u_{n+1} = u_n + hf(\tilde{u}).$$
The function header should be `function u = midpoint(f, u0, h, N)`, where `u` is the vector of approximate solutions at times $t = 0, h, 2h, \ldots, Nh$.

Problems

1. Let $\mathcal{F} = \{f(x) = c_0\sqrt{x} + c_1\ell(x) + c_2 e^x \colon c_0, c_1, c_2 \in \mathbb{R}\}$, where $\ell(x)$ is some function. We need to find the least-squares best $f^* \in \mathcal{F}$ to approximate the data (x_0, y_0), (x_1, y_1), …, (x_n, y_n).
   a) Set up (but do not attempt to solve) the *normal equations* to find the vector $c = (c_0, c_1, c_2)^T$. In other words, determine the matrix D and vector E such that $Dc = E$ are the normal equations.

b) Suppose that the function $\ell(x)$ is the Lagrange Polynomial associated with x_5 among the values $x_0, x_1, \ldots, x_n$. That is, $\ell(x_5) = 1$ and $\ell(x_j) = 0$ for $j \neq 5$. Prove that $f^*(x_5) = y_5$, where f^* is the least squares best approximant to the data.

2. By using a Taylor series expansion of the numerical method $\Psi_h(u)$ and comparing it to the Taylor series for the exact flow map $\Phi_h(u)$, determine the leading term in the local error for the following numerical methods:
   a) Implicit trapezoidal method: $u_{n+1} = u_n + \frac{h}{2}[f(u_n) + f(u_{n+1})]$.
   b) Heun's "Improved Euler" method: $u_{n+1} = u_n + \frac{h}{2}[f(u_n) + f(u_n + hf(u_n))]$.

3. Derive the order conditions for the 2nd order two-stage explicit Runge-Kutta (RK) method. How many free parameters does the method have?