Revision Sheet 4 : Solutions

Algorithms

1.
```
function p = eval_newton(xi, ck, x)
n = length(ck);
p = ck(n);
for i = n-1:-1:1
    p = p*(x - xi(i)) + ck(i);
end
```

2. See solution to 1(a) in CA3.

3.
```
function y = dft(x)
n = length(x);
y = zeros(n, 1);
w = exp(-2i*pi/n);
for k = 0:n-1
    wk = w.^k;
    for l = 0:n-1
        y(k+1) = y(k+1) + x(l+1).*wk.^l;
    end
end
```

or, instead of the l-loop:

```
y(k+1) = y(k+1) + sum(x.*wk.^(0:n-1)');
```

Problems

1. Need to show that $F_n F_n^{-1} = I$. So,

$$(F_n F_n^{-1})_{jk} = \sum_{\ell=0}^{n-1} (F_n)_{j\ell} (F_n^{-1})_{\ell k} = \sum_{\ell=0}^{n-1} \omega_n^{j\ell} \frac{1}{n} \omega_n^{-\ell k} = \frac{1}{n} \sum_{\ell=0}^{n-1} \omega_n^{(j-k)\ell}$$

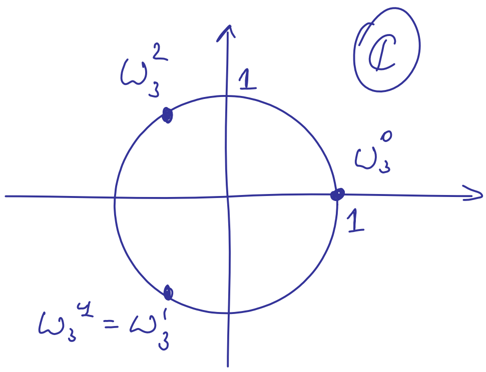
When $j=k$, $\sum_{\ell=0}^{n-1} \omega_n^0 = n$.

When $j \neq k$, let $j-k=m$. $\sum_{\ell=0}^{n-1} \omega_n^{m\ell} \overset{\substack{\text{geometric} \\ \text{series}}}{=} \frac{\omega_n^{mn} - 1}{\omega_n^m - 1} = 0$,

because $\omega_n^{mn} = e^{-i\frac{2\pi}{n}mn} = e^{-i2\pi m} = 1$, $\forall m \in \mathbb{Z}$.

2. a) $x = (1,2,0)^T$, $n=3$. $y = F_3 x$, where

$$F_3 = \begin{pmatrix} \omega_3^0 & \omega_3^0 & \omega_3^0 \\ \omega_3^0 & \omega_3^1 & \omega_3^2 \\ \omega_3^0 & \omega_3^2 & \omega_3^4 \end{pmatrix}, \text{ where } \omega_3 = e^{-i\frac{2\pi}{3}}$$



$\omega_3^0 = 1$, $\omega_3^1 = e^{-i\frac{2\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\omega_3^2 = e^{-i\frac{4\pi}{3}} = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\omega_3^4 = \omega_3^1$

$$y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -i\sqrt{3} \\ i\sqrt{3} \end{pmatrix}.$$

b) $x = (1\ 2\ 0\ 1)^T$, $n=4$, $\omega_4 = e^{-i\frac{2\pi}{4}} = -i$

$$y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1-i \\ -2 \\ 1+i \end{pmatrix}.$$

3. $$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) \quad \frac{h^3}{6} f'''(x) + O(h^4)$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(x) + O(h^4)$$

Eliminate f'' term:

$$4f(x-h) - f(x+2h) = 3f(x) - 6hf'(x) - 2h^3 f'''(x) + O(h^4)$$

$$f'(x) = \frac{f(x+2h) + 3f(x) - 4f(x-h)}{6h} - \underbrace{\frac{1}{3}h^2 f'''(x) + O(h^3)}_{O(h^2)}$$

Or writing $g(x,h) = \sum_{i=1}^{n} c_i f(x+a_i h) \approx f'(x)$,

where $a_i = \{0, -1, 2\}$.

$$\begin{aligned} \sum c_i &= 0 \\ \sum a_i c_i &= \frac{1}{h} \\ \sum a_i^2 c_i &= 0 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & \frac{1}{h} \\ 0 & 1 & 4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & \frac{1}{h} \\ 0 & 0 & 6 & \frac{1}{h} \end{array}\right)$$

$$\Rightarrow c_3 = \frac{1}{6h}, \quad -c_2 + 2c_3 = \frac{1}{h} \Rightarrow c_2 = -\frac{2}{3h}, \quad c_1 + c_2 + c_3 = 0 \Rightarrow c_1 = \frac{1}{2h}$$

$$f'(x) = \frac{1}{6h} f(x) - \frac{2}{3h} f(x-h) + \frac{1}{2h} f(x+2h) + O(h^2)$$

$(k = 1, \; n = 3)$

4. $$\phi(h) = L + a_2 h^2 + a_4 h^4 + \cdots$$

$$\phi\left(\frac{h}{3}\right) = L + a_2 \frac{h^2}{9} + a_4 \frac{h^4}{81} + \cdots$$

Eliminate h^2 term: $\phi(h) - 9\phi\left(\frac{h}{3}\right) = -8L + \frac{8}{9} a_4 h^4 + \cdots$

$$L = \frac{9}{8}\phi\left(\frac{h}{3}\right) - \frac{1}{8}\phi(h) + \frac{1}{9} a_4 h^4 + O(h^6).$$

5.

$$\int_0^1 f(x)\,dx \approx w_1 f\left(\tfrac{1}{4}\right) + w_2 f\left(\tfrac{1}{2}\right) + w_3 f\left(\tfrac{3}{4}\right)$$

$$\int_0^1 1\,dx = 1 = w_1 + w_2 + w_3$$

$$\int_0^1 x\,dx = \frac{1}{2} = \frac{1}{4} w_1 + \frac{1}{2} w_2 + \frac{1}{4} w_3$$

$$\int_0^1 x^2\,dx = \frac{1}{3} = \frac{1}{16} w_1 + \frac{1}{4} w_2 + \frac{9}{16} w_3$$

Linear system:

$$\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{16} & \frac{1}{4} & \frac{9}{16} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \Rightarrow \begin{aligned} w_1 &= \frac{2}{3} \\ w_2 &= -\frac{1}{3} \\ w_3 &= \frac{2}{3} \end{aligned}$$

$$\int_0^1 f(x)\,dx \approx \frac{2}{3} f\left(\tfrac{1}{4}\right) - \frac{1}{3} f\left(\tfrac{1}{2}\right) + \frac{2}{3} f\left(\tfrac{3}{4}\right)$$

Check degree:

$$\int_0^1 x^3\,dx = \frac{1}{4}; \quad \frac{2}{3}\left(\frac{1}{4}\right)^3 - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{2}{3}\left(\frac{3}{4}\right)^3 = \frac{1}{4} \quad \checkmark$$

$$\int_0^1 x^4\,dx = \frac{1}{5}; \quad \frac{2}{3}\left(\frac{1}{4}\right)^4 - \frac{1}{3}\left(\frac{1}{2}\right)^4 + \frac{2}{3}\left(\frac{3}{4}\right)^4 = \frac{37}{192} \quad \times$$

Exact up to degree 3.

6. $$\int_0^1 \frac{1}{(1+x)^2}\,dx = \frac{1}{2} \quad (\text{exact})$$

a) Simpson's rule $\displaystyle\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right]$

$$\int_0^1 \frac{1}{(1+x)^2}\,dx \approx \frac{1}{6}\left[\frac{1}{1^2} + 4\frac{1}{(1+\frac{1}{2})^2} + \frac{1}{2^2}\right] = \frac{109}{216} \approx 0.50463$$

Relative error $\displaystyle\frac{0.50463 - 0.5}{0.5} = 0.00926$

b) 3-node Gaussian quadrature

$$\int_{-1}^{1} f(x) \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

Let $y(x) = 2x - 1$, so that $y(0) = -1$, $y(1) = 1$

$$x = \frac{y+1}{2}, \quad dx = \frac{1}{2} dy$$

$$\int_0^1 \frac{1}{(1+x)^2} dx = \int_{-1}^{1} \frac{1}{\left(1 + \frac{y+1}{2}\right)^2} \frac{1}{2} dy = \int_{-1}^{1} \frac{2}{(3+y)^2} dy$$

$$\approx \frac{5}{9} \frac{2}{\left(3 - \sqrt{3/5}\right)^2} + \frac{8}{9} \cdot \frac{2}{3^2} + \frac{5}{9} \frac{2}{\left(3 + \sqrt{3/5}\right)^2} \approx 0.499874$$

Relative error: $\dfrac{0.499874 - 0.5}{0.5} = -0,000252.$