Revision Sheet 1 : Solutions

Theory

1. See Heath, Ch. 1, slide 10.
2. Heath, Ch. 1, slide 9.
3. Heath, Ch. 1., slide 18.
4. Heath, Ch. 1, slides 10, 31, 32
5. If we sum this series directly, then at some point the partial sum, $S_n = \sum_{k=1}^{n} 1/k$, will become bigger than $1/(n+1)$ by more than ε_{mach}^{-1}, where ε_{mach} is machine precision.

   That is $\dfrac{1/(n+1)}{S_n} = \varepsilon < \varepsilon_{mach}$. So $fl(S_{n+1}) = fl\left(S_n + 1/(n+1)\right)$
   $= fl\left(S_n + \varepsilon S_n\right) = S_n\, fl\left(1+\varepsilon\right) = S_n$.
   Recall that by definition $fl(1+\varepsilon) = 1$ for $0<\varepsilon<\varepsilon_{mach}$.

6. Let $C = AB$, $C_{ij} = \sum_k A_{ik} B_{kj}$

   $$(C^T)_{ij} = C_{ji} = \sum_k A_{jk} B_{ki} = \sum_k A^T_{kj} B^T_{ik} = \sum_k B^T_{ik} A^T_{kj} = (B^T A^T)_{ij}$$

   So, $(AB)^T = B^T A^T$.

7. Heath, Ch. 2, slide 4.

8. Matrix C is symmetric if $C = C^T$.
   Obviously, $C^T = (A^T A)^T = A^T A^{TT} = A^T A = C$.

   Matrix is positive definite if for $\forall\, x \neq 0$, $x^T C x > 0$.

   $$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|_2^2 \geq 0$$

   Since A is nonsingular, $Ax \neq 0$ for any $x \neq 0$, so $\|Ax\|_2^2 > 0$.

9. Pav's lecture notes, p. 9.

10. p. 59 of Heath's book.

Problems

1. After each elementary operation, round the result to $p=3$ digits:

a) $b^2 = 3.88^2 = 15.0544 \approx 1.51 \times 10^1$

$4ac = 4 \cdot 1.22 \cdot 3.08 = 15.0304 \approx 1.50 \times 10^1$ $\rightarrow b^2 - 4ac = 0.1$

b) Exact: $b^2 - 4ac = 0.024$

c) Relative error: $\dfrac{|\hat{x} - x|}{|x|} = \dfrac{0.1 - 0.024}{0.024} \approx 3.17 = 317\%$

2. a) From Taylor's Thm:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(\xi_1), \quad \xi_1 \in [x, x+h]$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(\xi_2), \quad \xi_2 \in [x-h, x]$$

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x) + \frac{h^2}{24}\left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right]$$

$$\left| f''(x) - \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \right| \underset{\uparrow}{\leq} \frac{h^2}{24}\left[\left|f^{(4)}(\xi_1)\right| + \left|f^{(4)}(\xi_2)\right|\right] \leq \frac{h^2}{12} M,$$

triangle inequality $|a+b| \leq |a| + |b|$

where $|f^{(4)}(\xi)| \leq M$, for $\xi \in [x-h, x+h]$

If $|f(x)| < M$, $x \in [a,b]$, then $|f(y)| < M$, $y \in [c,d]$ if $[c,d] \subset [a,b]$

So, truncation error is bounded by $\frac{h^2}{12} M$.

b) $|\hat{f}(\hat{x}) - f(x)| < \varepsilon$ - rounding error

$$\left| \frac{\hat{f}(\hat{x}+h) + \hat{f}(\hat{x}-h) - 2\hat{f}(\hat{x})}{h^2} - \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \right|$$

$$= \frac{1}{h^2} \left| \hat{f}(\hat{x}+h) - f(x+h) + \hat{f}(\hat{x}-h) - f(x-h) - 2\left(\hat{f}(\hat{x}) - f(x)\right) \right|$$

$$\le \frac{1}{h^2}\left\{\left|\hat{f}(\hat{x}+h)-f(x+h)\right| + \left|\hat{f}(x-h)-f(x-h)\right| + 2\left|\hat{f}(\hat{x})-f(x)\right|\right\} \le \frac{4\varepsilon}{h^2}$$

c) Total error: $E(h) = \frac{h^2}{12}M + \frac{4\varepsilon}{h^2}$

Optimal $h = h^*$ when $E'(h^*) = 0$

$$E'(h^*) = \frac{2h^* M}{12} - \frac{8\varepsilon}{h^{*3}} = 0 \Rightarrow h^{*4} = \frac{48\varepsilon}{M} \Rightarrow h^* = 2\sqrt[4]{\frac{3\varepsilon}{M}}$$

$$E(h^*) = 4\sqrt{\frac{3\varepsilon}{M}}\cdot\frac{M}{12} + \frac{4\varepsilon}{4}\sqrt{\frac{M}{3\varepsilon}} = 2\sqrt{\frac{\varepsilon M}{3}}.$$

For $M = 1$, $\varepsilon = 10^{-16}$, $h^* = 2.63\cdot 10^{-4}$, $E(h^*) = 1.15\cdot 10^{-8}$.

3. $\operatorname{cond} A = \|A\|\,\|A^{-1}\| = \max\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|}\cdot\max\limits_{x\neq 0}\frac{\|A^{-1}x\|}{\|x\|}$.

Let $A^{-1}x = y$, so $x = Ay$

Let x^* be such that $\frac{\|A^{-1}x^*\|}{\|x^*\|} = \max\limits_{x\neq 0}\frac{\|A^{-1}x\|}{\|x\|} \ge \frac{\|A^{-1}x\|}{\|x\|}$, that is, the maximum value of $\frac{\|A^{-1}x\|}{\|x\|}$ is attained at $x = x^*$.

So $\|A^{-1}\| = \frac{\|A^{-1}x^*\|}{\|x^*\|}$. Let $A^{-1}x^* = y^*$.

Then we have $\|A^{-1}\| = \frac{\|y^*\|}{\|Ay^*\|} \ge \frac{\|y\|}{\|Ay\|}$, $\frac{1}{\|A^{-1}\|} = \frac{\|Ay^*\|}{\|y^*\|} \le \frac{\|Ay\|}{\|y\|}$

This is true for any $y \neq 0$. So, $\frac{1}{\|A^{-1}\|} = \frac{\|Ay^*\|}{\|y^*\|} = \min\limits_{y\neq 0}\frac{\|Ay\|}{\|y\|}$

or $\|A^{-1}\| = \left(\min\limits_{y\neq 0}\frac{\|Ay\|}{\|y\|}\right)^{-1}$. $\square$

4. $(A+E)\hat{x} = b \Rightarrow A\hat{x} - b = -E\hat{x}$

$\Delta x = \hat{x} - x = A^{-1}(A\hat{x} - b) = -A^{-1}E\hat{x}$.

$\|\Delta x\| \le \|A^{-1}\|\,\|E\|\,\|\hat{x}\| \Rightarrow \frac{\|\Delta x\|}{\|\hat{x}\|} \le \|A\|\,\|A^{-1}\|\,\frac{\|E\|}{\|A\|}$.

See also p. 60 of Heath's book.