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Controller synthesis using `SReachPoint` for a spacecraft rendezvous problem

This example will demonstrate the use of `SReachTools` for controller synthesis in a stochastic continuous-state discrete-time linear time-invariant (LTI) systems. This example script is part of the `SReachTools` toolbox, which is licensed under GPL v3 or (at your option) any later version. A copy of this license is given in https://github.com/unm-hscl/SReachTools/blob/master/LICENSE.

In this example script, we discuss how to use `SReachPoint` to synthesize open-loop controllers and affine-disturbance feedback controllers for the problem of stochastic reachability of a target tube. We demonstrate the following solution techniques:

- `chance-open`: Chance-constrained approach that uses risk allocation and piecewise-affine approximations to formulate a linear program to synthesize an open-loop controller (See Vinod and Oishi, Hybrid Systems: Computation and Control, 2019 (submitted), Lesser et. al., Conference on Decision and Control, 2013)
- `genzps-open`: Fourier transforms that uses Genz's algorithm to formulate a nonlinear log-concave optimization problem to be solved using MATLAB's patternsearch to synthesize an open-loop controller (See Vinod and Oishi, Control System Society- Letters, 2017)
- `particle-open`: Particle control filter approach that formulates a mixed-integer linear program to synthesize an open-loop controller (See Lesser et. al., Conference on Decision and Control, 2013)
- `voronoi-open`: Particle control filter approach that formulates a mixed-integer linear program to synthesize an open-loop controller. In contrast to `particle-open`, `voronoi-open` permits a user-specified upper bound on the overapproximation error in the maximal reach probability and has significant computational advantages due to its undersampling approach. (See Sartipizadeh et. al., American Control Conference, 2019 (submitted))
- `chance-affine`: Chance-constrained approach that uses risk allocation and piecewise-affine approximations to formulate a difference-of-convex program to synthesize a closed-loop (affine disturbance feedback) controller. The controller synthesis is done by solving a series of second-order cone programs. (See Vinod and Oishi, Hybrid Systems: Computation and Control, 2019 (submitted))

All computations were performed using MATLAB on an Intel Xeon CPU with 3.4GHz clock rate and 32 GB RAM. The simulation times for individual methods are reported in each section along with a Monte-

Carlo simulation validation. The overall simulation time was 16 minutes. For sake of clarity, all commands were asked to be verbose (via SReachPointOptions). In practice, this can be turned off.

```
% Commands to ensure clean setup
close all;clc;clearvars;
```

Problem formulation: Stochastic reachability of a target tube

Given an initial state x_0, a time horizon N, a linear system dynamics $x_{k+1} = A_k x_k + B_k u_k + F w_k$ for $k \in \{0, 1, ..., N-1\}$, and a target tube $\{\mathcal{T}_k\}_{k=0}^N$, we wish to design an admissible controller that maximizes the probability of the state staying with the target tube. This maximal reach probability, denoted by $V^*(x_0)$, is obtained by solving the following optimization problem

$$V^*(x_0) = \max_{\overline{U} \in \mathcal{U}^N} P_X^{x_0,\overline{U}} \{\forall k, x_k \in \mathcal{T}_k\}.$$

Here, $\overline{U}$ refers to the control policy which satisfies the control bounds specified by the input space $\mathcal{U}$ over the entire time horizon N, $X = [x_1\ x_2\ \ldots\ x_N]^\top$ is the concatenated state vector, and the target tube is a sequence of sets $\{\mathcal{T}_k\}_{k=0}^N$. Here, X is a random vector with probability measure $P_X^{x_0,\overline{U}}$ which is a parameterized by the initial state x_0 and policy $\overline{U}$.

In the general formulation requires $\overline{U}$ is given by a sequence of (potentially time-varying and nonlinear) state-feedback controllers. To compute such a policy, we have to resort to dynamic programming which suffers from the curse of dimensionality. See these papers for details Abate et. al, Automatica, 2008, Summers and Lygeros, Automatica, 2010, and Vinod and Oishi, IEEE Trans. Automatic Control, 2018 (submitted).

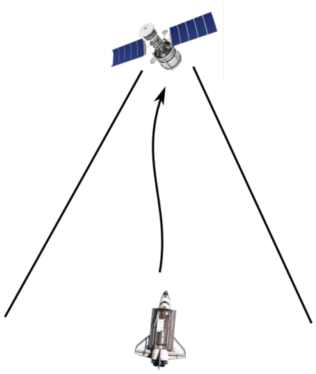
`SReachPoint` provides multiple ways to compute an **underapproximation** of $V^*(x_0)$ by restricting the search to the following controllers:

- open-loop controller: The controller provides a sequence of control actions $\overline{U} = [u_0\ u_1\ \ldots\ u_{N-1}]^\top \in \mathcal{U}^N$ parameterized only by the initial state. This controller does not account for the actual state realization and therefore can be conservative. However, computing this control sequence is easy due to known convexity properties of the problem. See Vinod and Oishi, IEEE Trans. Automatic Control, 2018 (submitted) for more details. Apart from `particle-open`, all approaches provide guaranteed underapproximations or underapproximations to a user-specifed error.
- affine-disturbance feedback controller: The controller is a characterized by an affine transformation of the concatenated disturbance vector. The gain matrix is forced to be lower-triangular for the causality, resulting in the control action at k be dependent only the past disturbance values. Here, the control action at time $k \in \{0, 1, \ldots, N-1\}$ is given by $u_k = \sum_{i=0}^{k-1} M_{ki} w_i + d_k$. We optimize for M_{ki} and d_k for every k, i, and the controller is given by $\overline{U} = MW + d \in \mathcal{U}^N$, with $W = [w_0\ w_1\ \ldots\ w_{N-1}]$ denoting the concatenated disturbance random vector. By construction, $\overline{U}$ is now random, and it can not satisfy hard control bounds with non-zero M_{ki} and unbounded W. Therefore, we relax the control bound constraints $\overline{U} \in \mathcal{U}^N$ to a chance constraint, $P_W\{MW + d \in \mathcal{U}^N\} \geq 1 - \Delta_U$ permitting the user to specify the probabilistic violation $\Delta_U \in [0, 1)$ of the control bounds. We then construct a lower

bound for the maximal reach probability when the affine disturbance feedback controller is used under saturation to meet the hard control bounds. In contrast to the open-loop controller synthesis, affine disturbance feedback controller synthesis is a non-convex problem, and we obtain a locally optimal solution using difference-of-convex programming. See Vinod and Oishi, Hybrid Systems: Computation and Control, 2019 (submitted) for more details.

All of our approaches are grid-free resulting in highly scalable solutions, especially for Gaussian-perturbed linear systems.

In this example, we perform controller synthesis that maximizes the probability of a deputy spacecraft to rendezvous with a chief spacecraft while staying within a line-of-sight cone.



Dynamics model for the deputy relative to the chief spacecraft

We consider both the spacecrafts to be in the same circular orbit. The relative planar dynamics of the deputy with respect to the chief are described by the Clohessy-Wiltshire-Hill (CWH) equations,

$$\ddot{x} - 3\omega x - 2\omega\dot{y} = \frac{F_x}{m_d}$$

$$\ddot{y} + 2\omega\dot{x} = \frac{F_y}{m_d}$$

where the position of the deputy relative to the chief is $x, y \in \mathbf{R}$, $\omega = \sqrt{\frac{\mu}{R_0^3}}$ is the orbital frequency, μ is the gravitational constant, and R_0 is the orbital radius of the chief spacecraft. We define the state as $\overline{x} = [x\ y\ \dot{x}\ \dot{y}]^\top \in \mathbf{R}^4$ which is the position and velocity of the deputy relative to the chief along x- and y- axes, and the input as $\overline{u} = [F_x\ F_y]^\top \in \mathcal{U} \subset \mathbf{R}^2$.

We will discretize the CWH dynamics in time, via zero-order hold, to obtain the discrete-time linear time-invariant system and add a Gaussian disturbance to account for the modeling uncertainties and the disturbance forces,

$$\overline{x}_{k+1} = A\overline{x}_k + B\overline{u}_k + \overline{w}_k$$

with $\overline{w}_k \in \mathbf{R}^4$ as an IID Gaussian zero-mean random process with a known covariance matrix $\Sigma_{\overline{w}}$.

```
umax = 0.1;
```

```
mean_disturbance = zeros(4,1);
covariance_disturbance = diag([1e-4, 1e-4, 5e-8, 5e-8]);
% Define the CWH (planar) dynamics of the deputy spacecraft relative
 to the
% chief spacecraft as a LtiSystem object
sys = getCwhLtiSystem(4, Polyhedron('lb', -umax*ones(2,1), ...
                                        'ub',  umax*ones(2,1)), ...
       RandomVector('Gaussian',
 mean_disturbance,covariance_disturbance));
```

Target tube definition

We define the target tube to be a collection of time-varying boxes $\{\mathcal{T}_k\}_{k=0}^N$ where N is the time horizon.

In this problem, we define $\mathcal{T}_k$ to be line-of-sight cone originating from origin (location of the chief spacecraft) for $k \in \{0, 1, \ldots, N-1\}$ and the terminal target set $\mathcal{T}_N$ as a box around the origin. This special sequence of target sets allows us to impose a reach-avoid specification of safety.

```
time_horizon = 5;   % Stay within a line of sight cone for 4 time
 steps and
                    % reach the target at t=5% Safe Set --- LoS cone
% Safe set definition --- LoS cone |x|<=y and y\in[0,ymax] and |vx|
<=vxmax and
% |vy|<=vymax
ymax = 2;
vxmax = 0.5;
vymax = 0.5;
A_safe_set = [1, 1, 0, 0;
             -1, 1, 0, 0;
              0, -1, 0, 0;
              0, 0, 1,0;
              0, 0,-1,0;
              0, 0, 0,1;
              0, 0, 0,-1];
b_safe_set = [0;
              0;
              ymax;
              vxmax;
              vxmax;
              vymax;
              vymax];
safe_set = Polyhedron(A_safe_set, b_safe_set);
% Target set --- Box [-0.1,0.1]x[-0.1,0]x[-0.01,0.01]x[-0.01,0.01]
target_set = Polyhedron('lb', [-0.1; -0.1; -0.01; -0.01], ...
                        'ub', [0.1; 0; 0.01; 0.01]);
target_tube = Tube('reach-avoid',safe_set, target_set, time_horizon);
```

Specifying initial states and which options to run

```
chance_open_run = 1;
```

```
genzps_open_run = 1;
particle_open_run = 1;
voronoi_open_run = 1;
chance_affine_run = 1;
% Initial state definition
initial_state = [-0.75;         % Initial x relative position
                 -0.75;         % Initial y relative position
                 0;             % Initial x relative velocity
                 0];            % Initial y relative velocity
slice_at_vx_vy = initial_state(3:4);
% Initial states for each of the method
init_state_chance_open = initial_state;
init_state_genzps_open = initial_state;
init_state_particle_open = initial_state;
init_state_voronoi_open = initial_state;
init_state_chance_affine = initial_state;
```

Quantities needed to compute the optimal mean trajectory and Monte-Carlo sims

We first compute the dynamics of the concatenated state vector $X = Zx_0 + HU + GW$, and compute the concatentated random vector W and its mean.

```
[Z,H,G] = sys.getConcatMats(time_horizon);
% Compute the mean trajectory of the concatenated disturbance vector
muW = sys.dist.concat(time_horizon).parameters.mean;
n_mcarlo_sims = 1e5;
```

`SReachPoint`: `chance-open`

This method is discussed in Vinod and Oishi, Hybrid Systems: Computation and Control, 2019 (submitted). It was introduced for stochastic reachability in Lesser et. al., Conference on Decision and Control, 2013.

This approach implements the chance-constrained approach to compute an optimal open-loop controller. It uses risk allocation and piecewise-affine overapproximation of the inverse normal cumulative density function to formulate a linear program for this purpose. Naturally, this is one of the fastest ways to compute an open-loop controller and an underapproximative probabilistic guarantee of safety. However, due to the use of Boole's inequality for risk allocation, it provides a conservative estimate of safety using the open-loop controller.

```
if chance_open_run
    fprintf('\n\nSReachPoint with chance-open\n');
    % Set the maximum piecewise-affine overapproximation error to 1e-3
    opts = SReachPointOptions('term', 'chance-
open','pwa_accuracy',1e-3);
    tic;
    [prob_chance_open, opt_input_vec_chance_open] =
 SReachPoint('term', ...
        'chance-open', sys, init_state_chance_open, target_tube,
 opts);
```

```
    elapsed_time_chance_open = toc;
    if prob_chance_open
        % Optimal mean trajectory construction
        % mean_X = Z * x_0 + H * U + G * \mu_W
        opt_mean_X_chance_open = Z * init_state_chance_open + ...
            H * opt_input_vec_chance_open + G * muW;
        opt_mean_traj_chance_open =
 reshape(opt_mean_X_chance_open, ...
            sys.state_dim,[]);
        % Check via Monte-Carlo simulation
        concat_state_realization_ccc =
 generateMonteCarloSims(n_mcarlo_sims, ...
            sys, init_state_chance_open, time_horizon,...
            opt_input_vec_chance_open);
        mcarlo_result =
 target_tube.contains(concat_state_realization_ccc);
        simulated_prob_chance_open = sum(mcarlo_result)/n_mcarlo_sims;
    else
        simulated_prob_chance_open = NaN;
    end
    fprintf('SReachPoint underapprox. prob: %1.2f | Simulated prob:
 %1.2f\n',...
        prob_chance_open, simulated_prob_chance_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_chance_open);
end
```

```
SReachPoint with chance-open
SReachPoint underapprox. prob: 0.87 | Simulated prob: 0.87
Computation time: 0.391
```

SReachPoint: genzps-open

This method is discussed in Vinod and Oishi, Control System Society- Letters, 2017.

This approach implements the Fourier transform-based approach to compute an optimal open-loop controller. It uses Genz's algorithm to compute the probability of safety and optimizes the joint chance constraint involved in maximizing this probability. To handle the noisy behaviour of the Genz's algorithm, we rely on MATLAB's `patternsearch` for the nonlinear optimization. Internally, we use the `chance-open` to initialize the nonlinear solver. Hence, this approach will return an open-loop controller with safety at least as good as `chance-open`.

```
if genzps_open_run
    fprintf('\n\nSReachPoint with genzps-open\n');
    opts = SReachPointOptions('term', 'genzps-open', ...
        'PSoptions',psoptimset('display','iter'));
    tic
    [prob_genzps_open, opt_input_vec_genzps_open] =
 SReachPoint('term', ...
        'genzps-open', sys, init_state_genzps_open, target_tube,
 opts);
    elapsed_time_genzps = toc;
```

```
    if prob_genzps_open > 0
        % Optimal mean trajectory construction
        % mean_X = Z * x_0 + H * U + G * \mu_W
        opt_mean_X_genzps_open =  Z * init_state_genzps_open + ...
            H * opt_input_vec_genzps_open + G * muW;
        opt_mean_traj_genzps_open= reshape(opt_mean_X_genzps_open, ...
            sys.state_dim,[]);
        % Check via Monte-Carlo simulation
        concat_state_realization_genz =
 generateMonteCarloSims(n_mcarlo_sims,...
            sys, init_state_genzps_open, time_horizon,...
            opt_input_vec_genzps_open);
        mcarlo_result =
 target_tube.contains(concat_state_realization_genz);
        simulated_prob_genzps_open = sum(mcarlo_result)/n_mcarlo_sims;
    else
        simulated_prob_genzps_open = NaN;
    end
    fprintf('SReachPoint underapprox. prob: %1.2f | Simulated prob:
 %1.2f\n',...
        prob_genzps_open, simulated_prob_genzps_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_genzps);
end
```

```
SReachPoint with genzps-open


Iter     Func-count       f(x)      MeshSize     Method
    0           1       0.14387             1
    1          15       0.14387           0.5     Refine Mesh
    2          29       0.14387          0.25     Refine Mesh
    3          43       0.14387         0.125     Refine Mesh
    4          65       0.14387        0.0625     Refine Mesh
    5          92       0.14387       0.03125     Refine Mesh
    6         119       0.14387       0.01563     Refine Mesh
    7         146       0.14387      0.007813     Refine Mesh
    8         173       0.14387      0.003906     Refine Mesh
    9         200       0.14387      0.001953     Refine Mesh
   10         227       0.14387     0.0009766     Refine Mesh
   11         261       0.14387     0.0004883     Refine Mesh
   12         295       0.14387     0.0002441     Refine Mesh
   13         329       0.14387     0.0001221     Refine Mesh
   14         363       0.14387     6.104e-05     Refine Mesh
   15         397       0.14387     3.052e-05     Refine Mesh
   16         431       0.14387     1.526e-05     Refine Mesh
   17         465       0.14387     7.629e-06     Refine Mesh
   18         499       0.14387     3.815e-06     Refine Mesh
   19         533       0.14387     1.907e-06     Refine Mesh
   20         567       0.14387     9.537e-07     Refine Mesh
Optimization terminated: mesh size less than options.MeshTolerance.
SReachPoint underapprox. prob: 0.87 | Simulated prob: 0.87
Computation time: 5.133
```

SReachPoint: particle-open

This method is discussed in Lesser et. al., Conference on Decision and Control, 2013.

This approach implements the particle control approach to compute an open-loop controller. It is a sampling-based technique and hence the resulting probability estimate is random with its variance going to zero as the number of samples considered goes to infinity. Note that since a mixed-integer linear program is solved underneath with the number of binary variables corresponding to the number of particles, using too many particles can cause an exponential increase in computational time.

```
if particle_open_run
    fprintf('\n\nSReachPoint with particle-open\n');
    opts = SReachPointOptions('term','particle-open','verbose',1,...
        'num_particles',50);
    tic
    [prob_particle_open, opt_input_vec_particle_open] =
 SReachPoint('term', ...
        'particle-open', sys, init_state_particle_open, target_tube,
 opts);
    elapsed_time_particle = toc;
    if prob_particle_open > 0
        % Optimal mean trajectory construction
        % mean_X = Z * x_0 + H * U + G * \mu_W
        opt_mean_X_particle_open =  Z * init_state_particle_open + ...
            H * opt_input_vec_particle_open + G * muW;
        opt_mean_traj_particle_open =...
            reshape(opt_mean_X_particle_open, sys.state_dim,[]);
        % Check via Monte-Carlo simulation
        concat_state_realization_pa =
 generateMonteCarloSims(n_mcarlo_sims, ...
            sys, init_state_particle_open,time_horizon,...
            opt_input_vec_particle_open);
        mcarlo_result =
 target_tube.contains(concat_state_realization_pa);
        simulated_prob_particle_open = sum(mcarlo_result)/
n_mcarlo_sims;
    else
        simulated_prob_particle_open = NaN;
    end
    fprintf('SReachPoint approx. prob: %1.2f | Simulated prob: %1.2f
\n',...
        prob_particle_open, simulated_prob_particle_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_particle);
end
```

```
SReachPoint with particle-open
Creating Gaussian random variable realizations....Done
Setting up CVX problem....Done
Parsing and solving the MILP....Done
SReachPoint approx. prob: 0.92 | Simulated prob: 0.83
Computation time: 0.924
```

SReachPoint: voronoi-open

This method is discussed in [Sartipizadeh et. al., American Control Conference, 2019 (submitted)]

This approach implements the undersampled particle control approach to compute an open-loop controller. It computes, using k-means, a representative sample realization of the disturbance which is significantly smaller. This drastically improves the computational efficiency of the particle control approach. Further, because it uses Hoeffding's inequality, the user can specify an upper-bound on the overapproximation error. The undersampled probability estimate is used to create a lower bound of the solution corresponding to the original particle control problem with appropriate (typically large) number of particles. Thus, this has all the benefits of the `particle-open` option, with additional benefits of being able to specify a maximum overapproximation error as well being computationally tractable.

```
if voronoi_open_run
    fprintf('\n\nSReachPoint with voronoi-open\n');
    opts = SReachPointOptions('term','voronoi-open','verbose',1,...
        'max_overapprox_err', 1e-3, 'undersampling_fraction', 0.001);
    tic
    [prob_voronoi_open, opt_input_vec_voronoi_open] =
 SReachPoint('term', ...
        'voronoi-open', sys, init_state_voronoi_open, target_tube,
 opts);
    elapsed_time_voronoi = toc;
    if prob_voronoi_open > 0
        % Optimal mean trajectory construction
        % mean_X = Z * x_0 + H * U + G * \mu_W
        opt_mean_X_voronoi_open =  Z * init_state_voronoi_open + ...
            H * opt_input_vec_voronoi_open + G * muW;
        opt_mean_traj_voronoi_open =...
            reshape(opt_mean_X_voronoi_open, sys.state_dim,[]);
        % Check via Monte-Carlo simulation
        concat_state_realization_vo =
 generateMonteCarloSims(n_mcarlo_sims, ...
            sys, init_state_voronoi_open,time_horizon,...
            opt_input_vec_voronoi_open);
        mcarlo_result =
 target_tube.contains(concat_state_realization_vo);
        simulated_prob_voronoi_open = sum(mcarlo_result)/
n_mcarlo_sims;
    else
        simulated_prob_voronoi_open = NaN;
    end
    fprintf('SReachPoint approx. prob: %1.2f | Simulated prob: %1.2f
\n',...
        prob_voronoi_open, simulated_prob_voronoi_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_voronoi);
end
```

```
SReachPoint with voronoi-open

Required number of particles: 9.2110e+03 | Samples used:   30
```

```
Creating Gaussian random variable realizations....Done
Using k-means for undersampling....Done
Setting up CVX problem....Done
Parsing and solving the MILP....Done
Undersampled probability (with 30 particles): 0.464
Underapproximation to the original MILP (with 9211 particles): 0.851
SReachPoint approx. prob: 0.85 | Simulated prob: 0.85
Computation time: 0.745
```

SReachPoint: chance-affine

This method is discussed in Vinod and Oishi, Hybrid Systems: Computation and Control, 2019 (submitted).

This approach implements the chance-constrained approach to compute a locally optimal affine disturbance feedback controller. In contrast to `chance-open`, this approach optimizes for an affine feedback gain for the concatenated disturbance vector as well as a bias. The resulting optimization problem is non-convex, and `SReachTools` formulates a difference-of-convex program to solve this optimization problem to a local optimum. Since affine disturbance feedback controllers can not satisfy hard control bounds, we relax the control bounds to be probabilistically violated with at most a probability of 0.01. After obtaining the affine feedback controller, we compute a lower bound to the maximal reach probability in the event saturation is applied to satisfy the hard control bounds. Due to its incorporation of state-feedback, this approach typically permits the construction of the highest underapproximative probability guarantee.

```
if chance_affine_run
    fprintf('\n\nSReachPoint with chance-affine\n');
    opts = SReachPointOptions('term', 'chance-affine',...
        'max_input_viol_prob', 1e-2, 'verbose',2);
    tic
    [prob_chance_affine, opt_input_vec_chance_affine,...
        opt_input_gain_chance_affine] = SReachPoint('term', 'chance-
affine',...
            sys, init_state_chance_affine, target_tube, opts);
    elapsed_time_chance_affine = toc;
    if prob_chance_affine > 0
        % mean_X = Z * x_0 + H * (M \mu_W + d) + G * \mu_W
        opt_mean_X_chance_affine = Z * init_state_chance_affine +...
            H * opt_input_vec_chance_affine + ...
            (H * opt_input_gain_chance_affine + G) * muW;
        % Optimal mean trajectory construction
        opt_mean_traj_chance_affine =
 reshape(opt_mean_X_chance_affine, ...
            sys.state_dim,[]);
        % Check via Monte-Carlo simulation
        concat_state_realization_cca =
 generateMonteCarloSims(n_mcarlo_sims, ...
            sys, init_state_chance_affine, time_horizon,...
            opt_input_vec_chance_affine,
 opt_input_gain_chance_affine);
        mcarlo_result =
 target_tube.contains(concat_state_realization_cca);
        simulated_prob_chance_affine = sum(mcarlo_result)/
n_mcarlo_sims;
    else
```

```
        simulated_prob_chance_affine = NaN;
    end
    fprintf('SReachPoint underapprox. prob: %1.2f | Simulated prob:
 %1.2f\n',...
        prob_chance_affine, simulated_prob_chance_affine);
    fprintf('Computation time: %1.3f\n', elapsed_time_chance_affine);
end
```

```
SReachPoint with chance-affine
Setting up the CVX problem
 0. CVX status: Solved | Max iterations : <200
Current probabilty: 0.010 | tau_iter: 1
DC slack-total sum --- state: 5.54e+02 | input: 5.35e+02

Setting up the CVX problem
 1. CVX status: Solved | Max iterations : <200
Current probabilty: 0.792 | tau_iter: 2
DC slack-total sum --- state: 8.33e-04 | input: 2.07e-11 | Acceptable:
 <1.000e-08
DC convergence error: 1.09e+03 | Acceptable: <1.000e-04

Setting up the CVX problem
 2. CVX status: Solved | Max iterations : <200
Current probabilty: 0.893 | tau_iter: 4
DC slack-total sum --- state: 7.96e-10 | input: 4.08e-10 | Acceptable:
 <1.000e-08
DC convergence error: 1.03e-01 | Acceptable: <1.000e-04

Setting up the CVX problem
 3. CVX status: Solved | Max iterations : <200
Current probabilty: 0.930 | tau_iter: 8
DC slack-total sum --- state: 5.97e-12 | input: 3.41e-12 | Acceptable:
 <1.000e-08
DC convergence error: 3.73e-02 | Acceptable: <1.000e-04

Setting up the CVX problem
 4. CVX status: Solved | Max iterations : <200
Current probabilty: 0.953 | tau_iter: 16
DC slack-total sum --- state: 7.55e-12 | input: 3.76e-12 | Acceptable:
 <1.000e-08
DC convergence error: 2.22e-02 | Acceptable: <1.000e-04

Setting up the CVX problem
 5. CVX status: Solved | Max iterations : <200
Current probabilty: 0.967 | tau_iter: 32
DC slack-total sum --- state: 8.55e-12 | input: 3.82e-12 | Acceptable:
 <1.000e-08
DC convergence error: 1.46e-02 | Acceptable: <1.000e-04

Setting up the CVX problem
 6. CVX status: Solved | Max iterations : <200
Current probabilty: 0.977 | tau_iter: 64
```

```
DC slack-total sum --- state: 1.20e-13 | input: 6.66e-14 | Acceptable:
 <1.000e-08
DC convergence error: 9.56e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
 7. CVX status: Solved | Max iterations : <200
Current probabilty: 0.984 | tau_iter: 128
DC slack-total sum --- state: 1.08e-12 | input: 6.00e-13 | Acceptable:
 <1.000e-08
DC convergence error: 6.77e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
 8. CVX status: Solved | Max iterations : <200
Current probabilty: 0.988 | tau_iter: 256
DC slack-total sum --- state: 4.11e-14 | input: 2.19e-14 | Acceptable:
 <1.000e-08
DC convergence error: 4.70e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
 9. CVX status: Solved | Max iterations : <200
Current probabilty: 0.991 | tau_iter: 512
DC slack-total sum --- state: 3.65e-13 | input: 2.03e-13 | Acceptable:
 <1.000e-08
DC convergence error: 3.18e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
10. CVX status: Solved | Max iterations : <200
Current probabilty: 0.994 | tau_iter: 1024
DC slack-total sum --- state: 7.78e-15 | input: 4.31e-15 | Acceptable:
 <1.000e-08
DC convergence error: 2.28e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
11. CVX status: Solved | Max iterations : <200
Current probabilty: 0.995 | tau_iter: 2048
DC slack-total sum --- state: 6.96e-14 | input: 3.77e-14 | Acceptable:
 <1.000e-08
DC convergence error: 1.62e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
12. CVX status: Solved | Max iterations : <200
Current probabilty: 0.996 | tau_iter: 4096
DC slack-total sum --- state: 3.91e-15 | input: 2.16e-15 | Acceptable:
 <1.000e-08
DC convergence error: 1.08e-03 | Acceptable: <1.000e-04

Setting up the CVX problem
13. CVX status: Solved | Max iterations : <200
Current probabilty: 0.997 | tau_iter: 8192
DC slack-total sum --- state: 4.43e-15 | input: 2.42e-15 | Acceptable:
 <1.000e-08
DC convergence error: 7.85e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
```

```
14. CVX status: Solved | Max iterations : <200
Current probabilty: 0.998 | tau_iter: 16384
DC slack-total sum --- state: 9.36e-15 | input: 5.17e-15 | Acceptable:
 <1.000e-08
DC convergence error: 5.56e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
15. CVX status: Solved | Max iterations : <200
Current probabilty: 0.998 | tau_iter: 32768
DC slack-total sum --- state: 9.94e-16 | input: 5.52e-16 | Acceptable:
 <1.000e-08
DC convergence error: 3.52e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
16. CVX status: Solved | Max iterations : <200
Current probabilty: 0.998 | tau_iter: 65536
DC slack-total sum --- state: 1.07e-15 | input: 5.92e-16 | Acceptable:
 <1.000e-08
DC convergence error: 2.70e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
17. CVX status: Solved | Max iterations : <200
Current probabilty: 0.999 | tau_iter: 100000
DC slack-total sum --- state: 4.48e-16 | input: 2.49e-16 | Acceptable:
 <1.000e-08
DC convergence error: 2.01e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
18. CVX status: Solved | Max iterations : <200
Current probabilty: 0.999 | tau_iter: 100000
DC slack-total sum --- state: 1.31e-16 | input: 7.28e-17 | Acceptable:
 <1.000e-08
DC convergence error: 1.23e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
19. CVX status: Solved | Max iterations : <200
Current probabilty: 0.999 | tau_iter: 100000
DC slack-total sum --- state: 8.65e-17 | input: 4.81e-17 | Acceptable:
 <1.000e-08
DC convergence error: 1.00e-04 | Acceptable: <1.000e-04

Setting up the CVX problem
20. CVX status: Solved | Max iterations : <200
Current probabilty: 0.999 | tau_iter: 100000
DC slack-total sum --- state: 3.14e-16 | input: 1.74e-16 | Acceptable:
 <1.000e-08
DC convergence error: 7.50e-05 | Acceptable: <1.000e-04

SReachPoint underapprox. prob: 1.00 | Simulated prob: 1.00
Computation time: 10.688
```

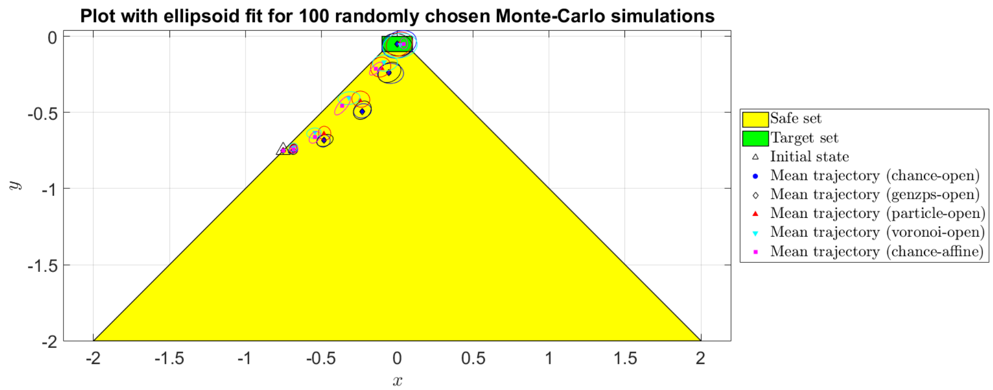
Plot of the optimal mean trajectories

```
dims_to_consider = [1,2];
figure(101);
clf;
hold on;
h_safe_set = plot(safe_set.slice([3,4],
 slice_at_vx_vy), 'color', 'y');
h_target_set = plot(target_set.slice([3,4],
 slice_at_vx_vy), 'color', 'g');
h_init_state = scatter(initial_state(1),initial_state(2),200,'k^');
legend_cell = {'Safe set','Target set','Initial state'};
axis equal
h_vec = [h_safe_set, h_target_set, h_init_state];
% Plot the optimal mean trajectory from the vertex under study
if chance_open_run
    h_opt_mean_ccc = scatter(...
          [init_state_chance_open(1),
 opt_mean_traj_chance_open(1,:)], ...
          [init_state_chance_open(2),
 opt_mean_traj_chance_open(2,:)], ...
          30, 'bo', 'filled','DisplayName', 'Mean trajectory (chance-
open)');
    legend_cell{end+1} = 'Mean trajectory (chance-open)';
    h_vec(end+1) = h_opt_mean_ccc;
    ellipsoidsFromMonteCarloSims(...
        concat_state_realization_ccc(sys.state_dim+1:end,:),
 sys.state_dim,...
        dims_to_consider, {'b'});
    disp('>>> SReachPoint with chance-open')
    fprintf('SReachPoint underapprox. prob: %1.2f | Simulated prob:
 %1.2f\n',...
        prob_chance_open, simulated_prob_chance_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_chance_open);
end
if genzps_open_run
    h_opt_mean_genzps = scatter(...
          [init_state_genzps_open(1),
 opt_mean_traj_genzps_open(1,:)], ...
          [init_state_genzps_open(2),
 opt_mean_traj_genzps_open(2,:)], ...
          30, 'kd','DisplayName', 'Mean trajectory (genzps-open)');
    legend_cell{end+1} = 'Mean trajectory (genzps-open)';
    h_vec(end+1) = h_opt_mean_genzps;
    ellipsoidsFromMonteCarloSims(...
        concat_state_realization_genz(sys.state_dim+1:end,:),
 sys.state_dim,...
        dims_to_consider, {'k'});
    disp('>>> SReachPoint with genzps-open')
    fprintf('SReachPoint underapprox. prob: %1.2f | Simulated prob:
 %1.2f\n',...
        prob_genzps_open, simulated_prob_genzps_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_genzps);
```

```
end
if particle_open_run
    h_opt_mean_particle = scatter(...
          [init_state_particle_open(1),
 opt_mean_traj_particle_open(1,:)], ...
          [init_state_particle_open(2),
 opt_mean_traj_particle_open(2,:)], ...
          30, 'r^', 'filled','DisplayName', 'Mean trajectory
 (particle-open)');
    legend_cell{end+1} = 'Mean trajectory (particle-open)';
    h_vec(end+1) = h_opt_mean_particle;
    ellipsoidsFromMonteCarloSims(...
        concat_state_realization_pa(sys.state_dim+1:end,:),
 sys.state_dim,...
        dims_to_consider, {'r'});
    disp('>>> SReachPoint with particle-open')
    fprintf('SReachPoint approx. prob: %1.2f | Simulated prob: %1.2f
\n',...
        prob_particle_open, simulated_prob_particle_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_particle);
end
if voronoi_open_run
    h_opt_mean_voronoi = scatter(...
          [init_state_voronoi_open(1),
 opt_mean_traj_voronoi_open(1,:)], ...
          [init_state_voronoi_open(2),
 opt_mean_traj_voronoi_open(2,:)], ...
          30, 'cv', 'filled','DisplayName', 'Mean trajectory (voronoi-
open)');
    legend_cell{end+1} = 'Mean trajectory (voronoi-open)';
    h_vec(end+1) = h_opt_mean_voronoi;
    ellipsoidsFromMonteCarloSims(...
        concat_state_realization_vo(sys.state_dim+1:end,:),
 sys.state_dim,...
        dims_to_consider, {'c'});
    disp('>>> SReachPoint with voronoi-open')
    fprintf('SReachPoint approx. prob: %1.2f | Simulated prob: %1.2f
\n',...
        prob_voronoi_open, simulated_prob_voronoi_open);
    fprintf('Computation time: %1.3f\n', elapsed_time_voronoi);
end
if chance_affine_run
    h_opt_mean_chance_affine = scatter(...
          [init_state_chance_affine(1),
 opt_mean_traj_chance_affine(1,:)], ...
          [init_state_chance_affine(2),
 opt_mean_traj_chance_affine(2,:)], ...
          30, 'ms', 'filled','DisplayName', 'Mean trajectory (chance-
affine)');
    legend_cell{end+1} = 'Mean trajectory (chance-affine)';
    h_vec(end+1) = h_opt_mean_chance_affine;
    ellipsoidsFromMonteCarloSims(...
        concat_state_realization_cca(sys.state_dim+1:end,:),
 sys.state_dim,...
```

```
        dims_to_consider, {'m'});
    disp('>>> SReachPoint with chance-affine')
    fprintf('SReachPoint underapprox. prob: %1.2f | Simulated prob:
 %1.2f\n',...
        prob_chance_affine, simulated_prob_chance_affine);
    fprintf('Computation time: %1.3f\n', elapsed_time_chance_affine);
end
legend(h_vec,
 legend_cell, 'Location','EastOutside', 'interpreter','latex');
title(['Plot with ellipsoid fit for 100 randomly chosen Monte-Carlo
 ',...
    'simulations']);
axis equal
box on;
grid on;
xlabel('$x$','interpreter','latex');
ylabel('$y$','interpreter','latex');
hf = gcf;
hf.Units = 'inches';
hf.Position = [0    0.4167   18.0000   10.0313];
set(gca,'FontSize',20);
```

```
>>> SReachPoint with chance-open
SReachPoint underapprox. prob: 0.87 | Simulated prob: 0.87
Computation time: 0.391
>>> SReachPoint with genzps-open
SReachPoint underapprox. prob: 0.87 | Simulated prob: 0.87
Computation time: 5.133
>>> SReachPoint with particle-open
SReachPoint approx. prob: 0.92 | Simulated prob: 0.83
Computation time: 0.924
>>> SReachPoint with voronoi-open
SReachPoint approx. prob: 0.85 | Simulated prob: 0.85
Computation time: 0.745
>>> SReachPoint with chance-affine
SReachPoint underapprox. prob: 1.00 | Simulated prob: 1.00
Computation time: 10.688
```



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