Table of Contents

Underapproximative verification of an automated anesthesia delivery system

This example will demonstrate the use of `SReachTools` for controller synthesis and verification of a stochastic continuous-state discrete-time linear time-invariant (LTI) systems. This example script is part of the `SReachTools` toolbox, which is licensed under GPL v3 or (at your option) any later version. A copy of this license is given in https://github.com/unm-hscl/SReachTools/blob/master/LICENSE.

In this example script, we discuss how to use `SReachSet` to synthesize open-loop controllers and verification for the problem of stochastic reachability of a target tube. Here, by verification, we wish to characterize a set of safe initial states with probabilistic safety above a threshold. We consider the verification of an automated anesthesia delivery model.

Automated anesthesia delivery systems have the potential to significantly reduce medical operation costs by allowing a single human-anestheologist to monitor multiple operations and delegate the low-level regulation of the patient's sedation level to the automation. Naturally, this system is safety critical, and we wish to ascertain the set of initial states (patient sedation levels) from which the automated anesthesia delivery system can continue to maintain within pre-specified safe bounds. If the patient sedation levels go outside these bounds, the patient may suffer from serious health consequences. This problem has been characterized as a benchmark problem in Abate et. al, ARCH 2018 paper (https://doi.org/10.29007/7ks7). To obtain a LTI system description, we consider Problem 2.1.1 with no anestheologist-in-the-loop, but an additive Gaussian disturbance to model the human patients. This script improves upon the Figures 6 and 7 of Abate et. al, ARCH 2018 paper (https://doi.org/10.29007/7ks7).

```
% Prescript running: Initializing srtinit, if it already hasn't been
 initialized
close all;clearvars;srtinit;srtinit --version;
```

```
SReachTools version 1.2.31
```

Problem Formulation

We first define a `LtiSystem` object corresponding to the discrete-time approximation of the three-compartment pharmacokinetic system model.

We bound the anesthesia the automation can deliver to $[0, 7]$ mg/dL and account for patient model mismatch via an additive Gaussian noise.

```
% System matrices: State matrix and input matrix
```

```
% ---------------------------------------------
systemMatrix = [0.8192, 0.03412, 0.01265;
                0.01646, 0.9822, 0.0001;
                0.0009, 0.00002, 0.9989];
inputMatrix = [0.01883;
               0.0002;
               0.00001];
% Input bounds
% ------------
auto_input_max = 7;

% Process disturbance with a specified mean and variance
% ------------------------------------------------------
dist_mean = 0;
dist_var = 5;
process_disturbance = RandomVector('Gaussian',dist_mean, dist_var);

% LtiSystem definition
sys = LtiSystem('StateMatrix', systemMatrix, ...
                'InputMatrix', inputMatrix, ...
                'DisturbanceMatrix', inputMatrix, ...
                'InputSpace', Polyhedron('lb', 0, 'ub',
 auto_input_max), ...
                'Disturbance', process_disturbance);
disp(sys)
```

Linear time invariant system with 3 states, 1 inputs, and 1 disturbances.

Safety specifications

We desire that the state remains inside a set $\mathcal{K} = \{x \in \mathbf{R}^3 : 0 \leq x_1 \leq 6, 0 \leq x_2 \leq 10, 0 \leq x_3 \leq 10\}$.

```
time_horizon = 5;
safe_set = Polyhedron('lb',[1, 0, 0], 'ub', [6, 10, 10]);
safety_tube = Tube('viability',safe_set, time_horizon);
```

Computation of the underapproximation of the stochastic viability set

We are interested in computing the stochastic viability set at probability 0.99.

For using `SReachSet` with `chance-open` option, we need a set of direction vectors and an affine hull (n-2 dimensional) intersecting the initial state. Since x_3 of the dynamics is slow, we fix it $x_3 = 5$ and analyze the rest of the system.

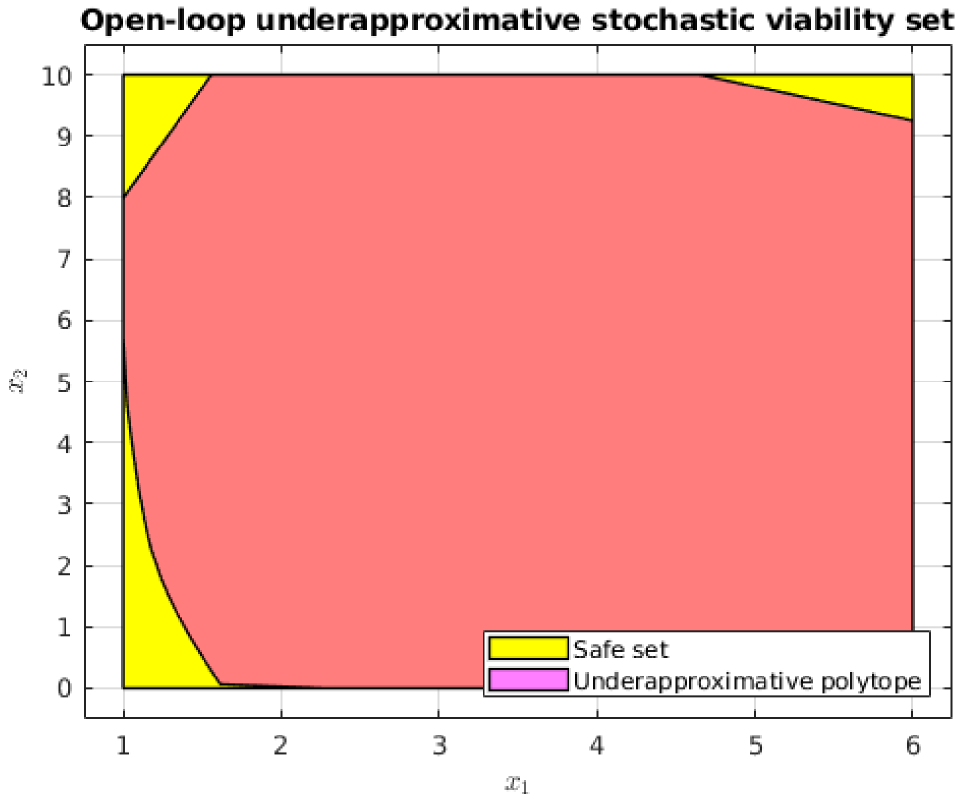
```
% Safety probability threshold of interest
% ----------------------------------------
prob_thresh = 0.99;      % Stochastic reach-avoid 'level' of interest
% Definition of the affine hull
```

```
% -----------------------------
x3_initial_state = 5;
init_safe_set_affine = Polyhedron('He',[0, 0, 1, x3_initial_state]);
% Definition of set of direction vectors
% -------------------------------------
no_of_dir_vecs = 32;
theta_vec = linspace(0,2*pi, no_of_dir_vecs);
set_of_dir_vecs =
 [cos(theta_vec);sin(theta_vec);zeros(1,no_of_dir_vecs)];
% Use SReachSet to compute the underapproximative set
% ---------------------------------------------------
% Use Ctrl + F1 to get the hints
options = SReachSetOptions('term','chance-open', ...
    'set_of_dir_vecs', set_of_dir_vecs, ...
    'init_safe_set_affine', init_safe_set_affine);
timer_val = tic;
[underapprox_stoch_viab_polytope, extra_info] = SReachSet('term', ...
    'chance-open', sys, prob_thresh, safety_tube, options);
elapsed_time = toc(timer_val);
disp(elapsed_time)
```

```
   19.7031
```

Plotting the stochastic viable set

```
figure(1);
hold on;
safe_set_2D = safe_set.intersect(init_safe_set_affine);
plot(safe_set_2D, 'color', 'y');
plot(underapprox_stoch_viab_polytope, 'color', 'm', 'alpha', 0.5);
leg=legend({'Safe set','Underapproximative polytope'});
set(leg,'Location','SouthEast');
xlabel('$x_1$','interpreter','latex')
ylabel('$x_2$','interpreter','latex')
box on;
grid on;
view([0,90]);
title('Open-loop underapproximative stochastic viability set');
```



Validate the underapproximative set and the controller using Monte-Carlo

We will now check how the optimal policy computed for one of the corners perform in Monte-Carlo simulations.

Note that we fail to obtain tight bounding ellipsoids in the initial time steps due to lack of spread in the trajectory.

```
n_mcarlo_sims = 1e5;              % How many Monte-Carlo simulations to
 use
vertex_indx = 8;                  % Index of the vertex corner of
 interest
if ~isEmptySet(underapprox_stoch_viab_polytope)
    % Obtain info about the vertices from extra_info struct given by
 SReachSet
    %
 -----------------------------------------------------------------------
    initial_state =
 extra_info(1).vertices_underapprox_polytope(:,vertex_indx);
    opt_input_vec =
 extra_info(1).opt_input_vec_at_vertices(:,vertex_indx);
    stoch_viab_prob_lb = extra_info(1).opt_reach_prob_i(vertex_indx);
    % Optimal mean trajectory generation
    % ----------------------------------
```

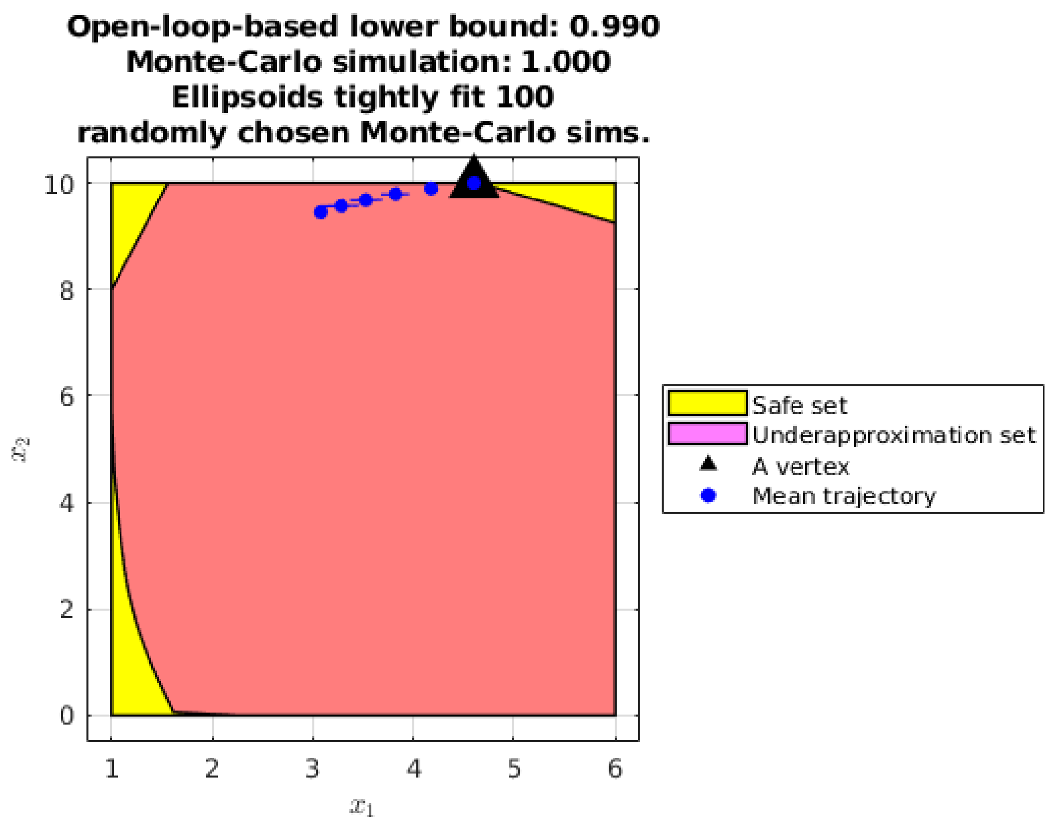
```
   [Z, H, G] = sys.getConcatMats(time_horizon);
   W = sys.dist.concat(time_horizon);
   optimal_X = (Z * initial_state + H * opt_input_vec) + G * W;
   optimal_mean_X = reshape(optimal_X.mean(), sys.state_dim,[]);
   % Monte-Carlo estimate of the safety probability
   % ---------------------------------------------
   concat_state_realization = generateMonteCarloSims(n_mcarlo_sims,
sys, ...
       initial_state, time_horizon, opt_input_vec);
   mcarlo_result = safety_tube.contains(concat_state_realization);
   stoch_viab_prob_mc_estim = sum(mcarlo_result)/n_mcarlo_sims;
   % Display the results
   % -------------------
   fprintf(['Open-loop-based lower bound and Monte-Carlo simulation
', ...
            '(%1.0e particles): %1.3f, %1.3f\n'], ...
           n_mcarlo_sims, ...
           stoch_viab_prob_lb, ...
           stoch_viab_prob_mc_estim);

   % Plotting
   % --------
   figure(2);
   hold on;
   % Plot the safe set at the fixed x_3
   plot(safe_set_2D.slice(3,x3_initial_state), 'color', 'y');
   % Plot the underapproximation of the stochastic reach set at the
fixed x_3

plot(underapprox_stoch_viab_polytope.slice(3,x3_initial_state), ...
       'color','m','alpha',0.5);
   % Plot the initial state (vertex) under test
   scatter(initial_state(1),initial_state(2), 300,'k^','filled');
   % Plot the optimal mean trajectory from the initial state (vertex)
under
   % test
   scatter([initial_state(1), optimal_mean_X(1,:)], ...
           [initial_state(2), optimal_mean_X(2,:)], ...
         30, 'bo', 'filled');
   legend_cell = {'Safe set', 'Underapproximation set', 'A
vertex', ...
       'Mean trajectory'};
   leg = legend(legend_cell,'Location','EastOutside');
   % Plot ellipsoids that tightly cover 100 randomly chosen
realizations
   ellipsoidsFromMonteCarloSims(concat_state_realization,
sys.state_dim, ...
       [1,2], {'b'});
   title(sprintf(['Open-loop-based lower bound: %1.3f\n Monte-Carlo
', ...
                  'simulation: %1.3f\nEllipsoids tightly fit
100\n',...
                  'randomly chosen Monte-Carlo sims.'],
stoch_viab_prob_lb, ...
```

```
                        stoch_viab_prob_mc_estim));
    box on;
    grid on;
    xlabel('$x_1$','interpreter','latex')
    ylabel('$x_2$','interpreter','latex');
end
```

```
Open-loop-based lower bound and Monte-Carlo simulation (1e+05
 particles): 0.990, 1.000
Warning: CVX failed to obtain the ellipsoid at 1, potentially due to
 numerical
issues.
Warning: CVX failed to obtain the ellipsoid at 2, potentially due to
 numerical
issues.
Warning: CVX failed to obtain the ellipsoid at 6, potentially due to
 numerical
issues.
```



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