Double Integrator Reach-Avoid Via Dynamic Programming

This example demonstrates how to use the SReachTools toolbox to solve a terminal-hitting time reach-avoid problem using dynamic programming.

In this example, we analyze the following problems via dynamic programming for a stochastic system with known dynamics:

1. **stochastic viability problem:** Compute a controller to stay within a safe set with maximum likelihood
2. **the terminal-hitting time stochastic reach-avoid problem:** Compute a controller that maximizes the probability of reaching a target set at a time horizon, `N`, while maintaining the system in a set of safe states
3. **stochastic reachability of a moving target tube:** Compute a controller that maximizes the probability of staying within a target tube
4. **the first-hitting time stochastic reach-avoid problem:** Compute a controller that maximizes the probability of reaching a target set within the time horizon, `N`, while maintaining the system in a set of safe states

SReachTools has a dynamic programing implementation that can analyze systems upto three dimensions. For efficient implementation, we require the input set to be an axis-aligned hypercuboid, and define the grid the smallest hypercuboid containing all the target sets.

Notes about this Live Script:

1. **MATLAB dependencies**: This Live Script uses MATLAB's Statistics and Machine Learning Toolbox.
2. **External dependencies**: This Live Script uses Multi-Parameteric Toolbox (MPT).
3. Make sure that `srtinit` is run before running this script.

Problem setup

Double Integrator

In this example we use a discretized double integrator dynamics given by:

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k + w_k$$

where T is the discretization time-step, and w_k is the stochastic disturbance.

Setup the system

```
% discretization parameter
T = 0.1;

% define the system
sys = LtiSystem('StateMatrix', [1, T; 0, 1], ...
    'InputMatrix', [T^2/2; T], ...
```

```
    'InputSpace', Polyhedron('lb', -0.1, 'ub', 0.1), ...
    'DisturbanceMatrix', eye(2), ...
    'Disturbance', StochasticDisturbance('Gaussian', zeros(2,1), 0.01*eye(2)));
```

Setup the dynamic programming and visualization parameters

```
dyn_prog_xinc = 0.05;
dyn_prog_uinc = 0.1;
reach_set_thresholds = [0.2 0.5 0.9];
legend_str={'Safety tube at t=0', 'Safety Probability $\geq 0.2$', 'Safety Probability $\geq 0
```

Case 1: Stochastic viability problem

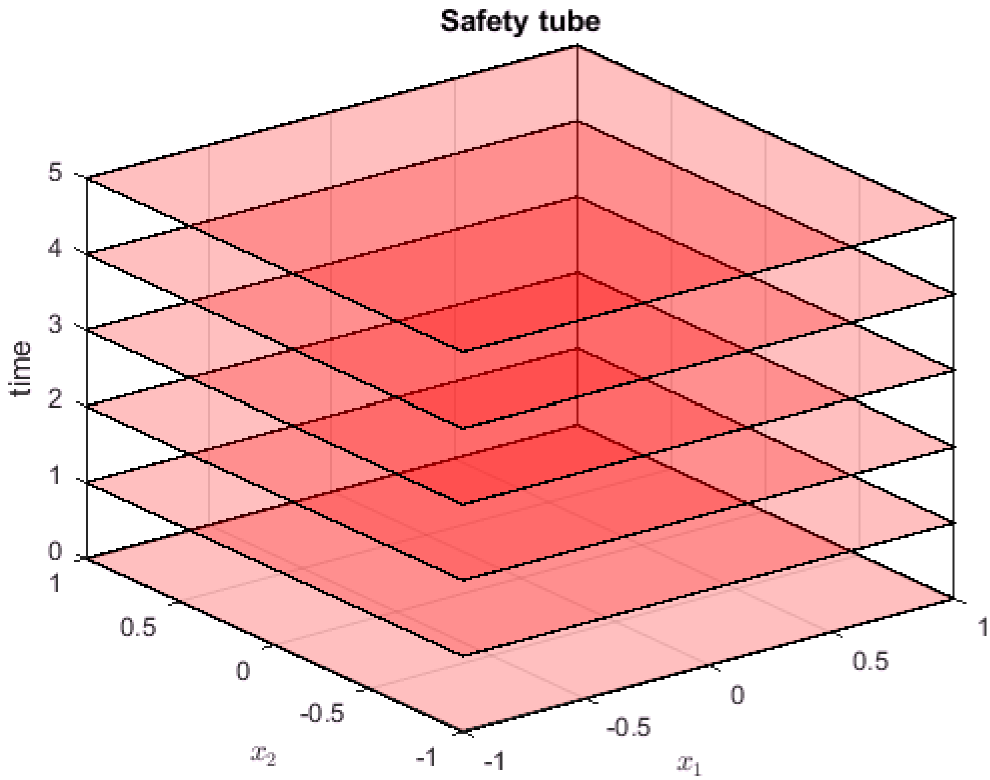
Setup the target and safe sets

```
safe_set = Polyhedron('lb', [-1, -1], 'ub', [1, 1]);
axis_vec1 = [-1 1 -1 1];
```

Setup the target tube

Safety tube is a generalization of the reach problem. The reach avoid target-tube is created by setting the first $N-1$ sets in the tube as the `safe_set` and the final set as the `target_set`.

```
% time horizon
N = 5;
% in target tube for the viability problem is equivalent to a tube of repeating
% safe sets
safety_tube1 = TargetTube('viability', safe_set, N);

% Plotting of safety tube
figure()
hold on
for time_indx=0:N
    safety_tube_at_time_indx = Polyhedron('H',[safety_tube1(time_indx+1).A,zeros(size(safety_t
    plot(safety_tube_at_time_indx, 'alpha',0.25);
end
axis([axis_vec1 0 N])
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube');
```



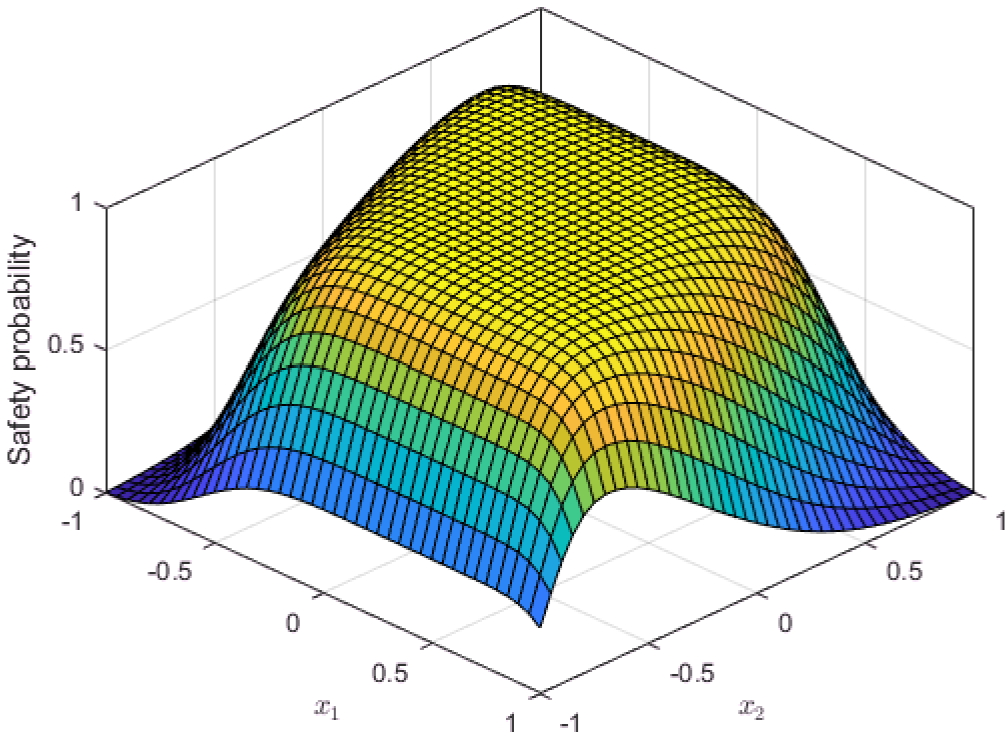
Dynamic programming recursion via gridding

```
tic;
[prob_x1, cell_of_xvec_x1] =  SReachDynProg('term', sys, dyn_prog_xinc, dyn_prog_uinc, safety_
toc
```

```
Elapsed time is 3.117079 seconds.
```

Visualization of the value function at t=0 (safety probability)

```
figure();
x1vec = cell_of_xvec_x1{1};
x2vec = cell_of_xvec_x1{2};
surf(x1vec,x2vec,reshape(prob_x1,length(x2vec),length(x1vec)));
axis([axis_vec1 0 1])
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
box on
view(45, 45)
```

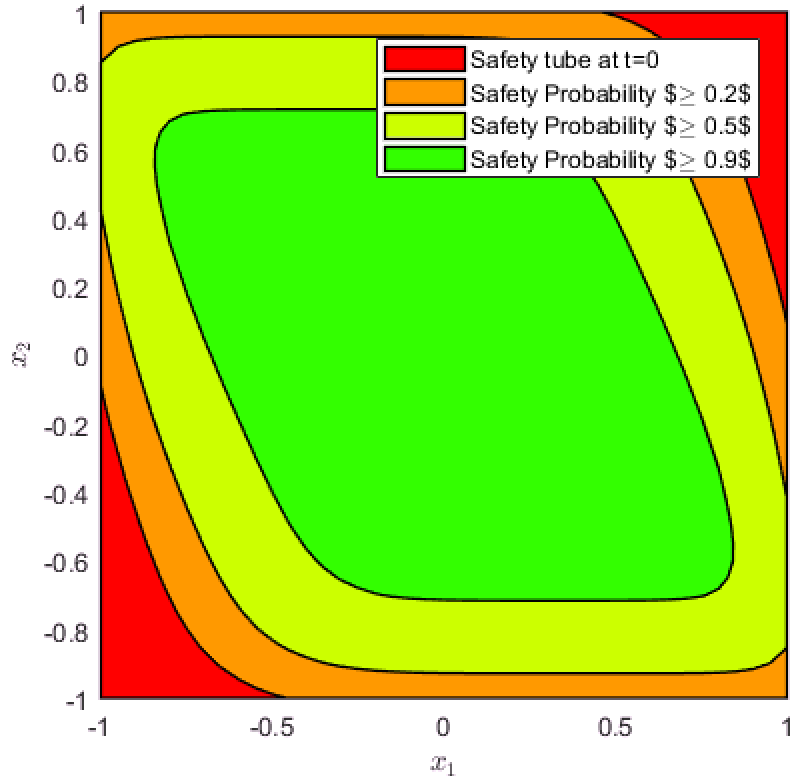


Visualization of the safe initial sets --- Superlevel sets of safety probability

```
figure();
poly_array1 = getDynProgLevelSets2D(cell_of_xvec_x1, prob_x1, reach_set_thresholds, safety_tub
```

```
Warning: MATLAB's contour matrix missed a corner!
Adding (1, -1) to polytope vertex list for level=0.20.
Warning: MATLAB's contour matrix missed a corner!
Adding (-1, 1) to polytope vertex list for level=0.20.
```

```
hold on;
plot([safety_tube1(1), poly_array1])
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
box on
axis(axis_vec1)
axis equal
legend(legend_str)
```



Case 2: Terminal hitting-time stochastic reach-avoid (Constant safety sets uptil time horizon - 1 and a different target set at time horizon)

Setup the time-varying safety tube

The advantage of safety tube is that it allows for problem formulations in which we would like to reach a moving target.

Setup the target and safe sets

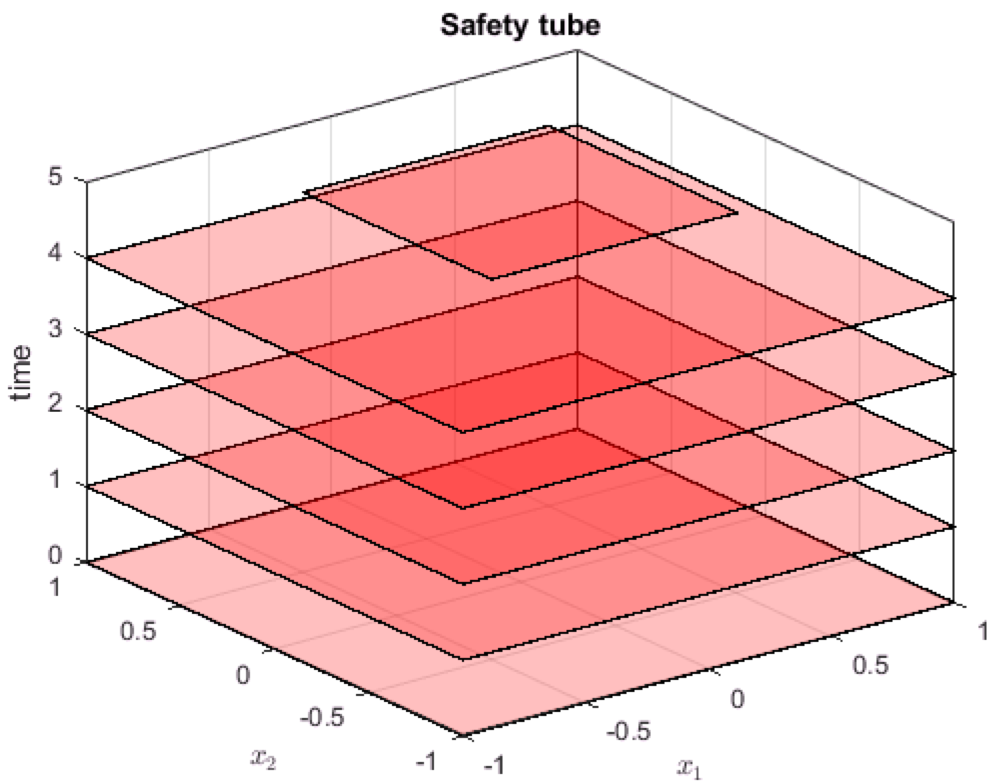
```
safe_set = Polyhedron('lb', [-1, -1], 'ub', [1, 1]);
target_set = Polyhedron('lb', [-0.5, -0.5], 'ub', [0.5, 0.5]);
axis_vec2 = [-1 1 -1 1];
```

Setup the safety tube

Safety tube is a generalization of the reach problem. The reach avoid safety-tube is created by setting the first $N - 1$ sets in the tube as the `safe_set` and the final set as the `target_set`.

```
% time horizon
N = 5;
% in target tube for the viability problem is equivalent to a tube of repeating
% safe sets
safety_tube2 = TargetTube('reach-avoid', safe_set, target_set, N);
```

```
% Plotting of safety tube
figure()
hold on
for time_indx=0:N
    safety_tube_at_time_indx = Polyhedron('H',[safety_tube2(time_indx+1).A,zeros(size(safety_t
    plot(safety_tube_at_time_indx, 'alpha',0.25);
end
axis([axis_vec2 0 N])
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube');
```



Dynamic programming solution on safety tube

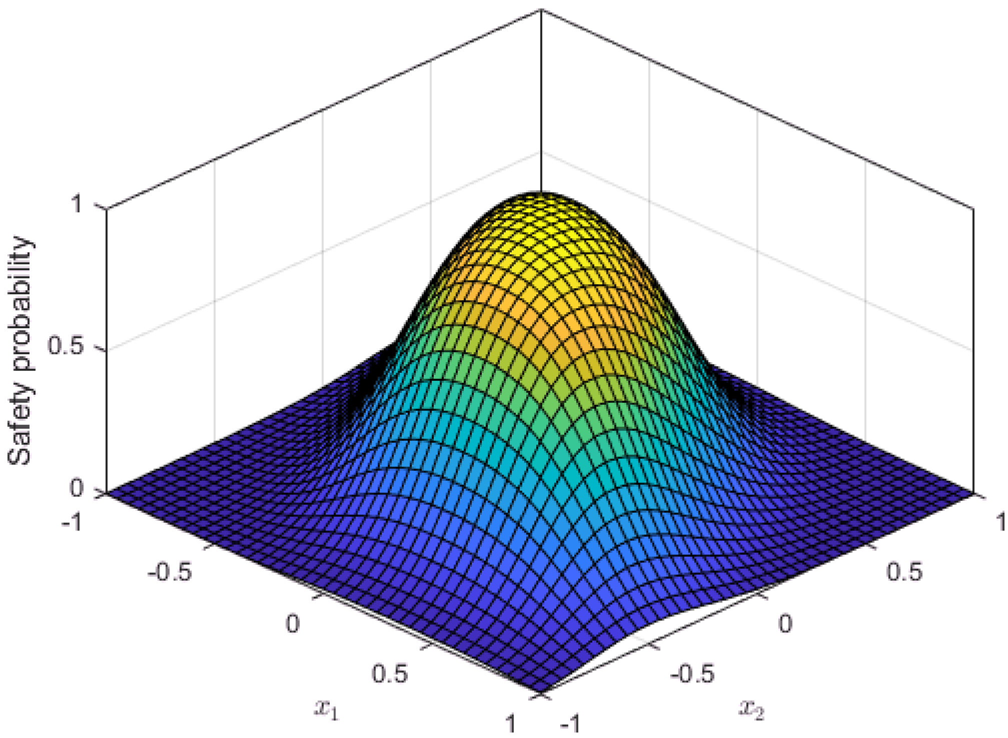
```
tic;
[prob_x2, cell_of_xvec_x2] = SReachDynProg('term', sys, dyn_prog_xinc, dyn_prog_uinc, safety_t
toc
```

```
Elapsed time is 3.028386 seconds.
```

Visualization of the value function at t=0 (safety probability)

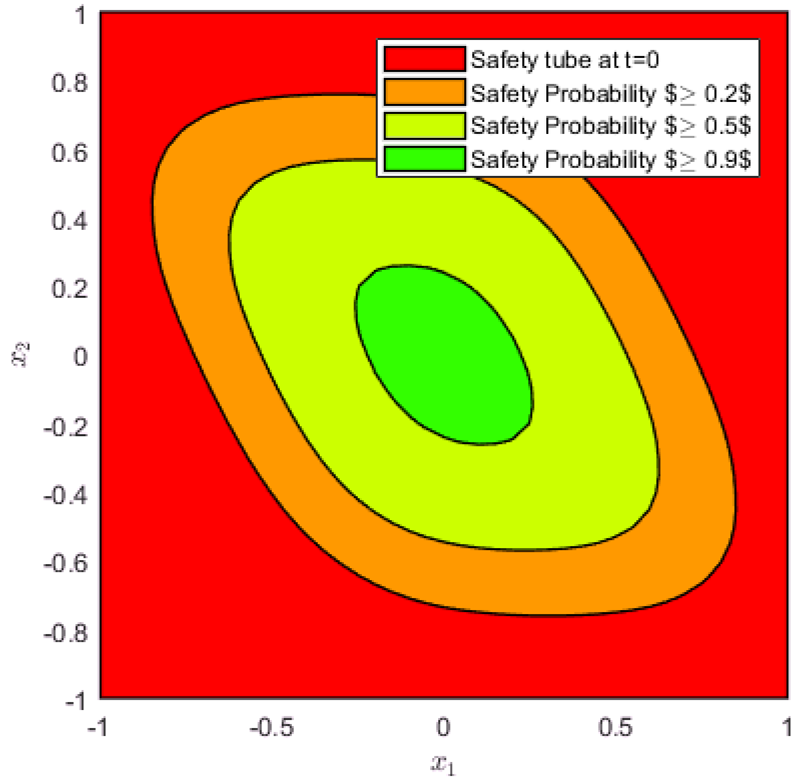
```
figure();
x1vec = cell_of_xvec_x2{1};
x2vec = cell_of_xvec_x2{2};
axis([axis_vec2 0 N]);
```

```
surf(x1vec,x2vec,reshape(prob_x2,length(x2vec),length(x1vec)));
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
box on
view(45, 45)
```



Visualization of the safe initial sets --- Superlevel sets of safety probability

```
figure();
poly_array2 = getDynProgLevelSets2D(cell_of_xvec_x2, prob_x2, reach_set_thresholds, safety_tub
hold on;
plot([safety_tube2(1) poly_array2])
box on
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
axis(axis_vec2);
box on
axis equal
legend(legend_str)
```

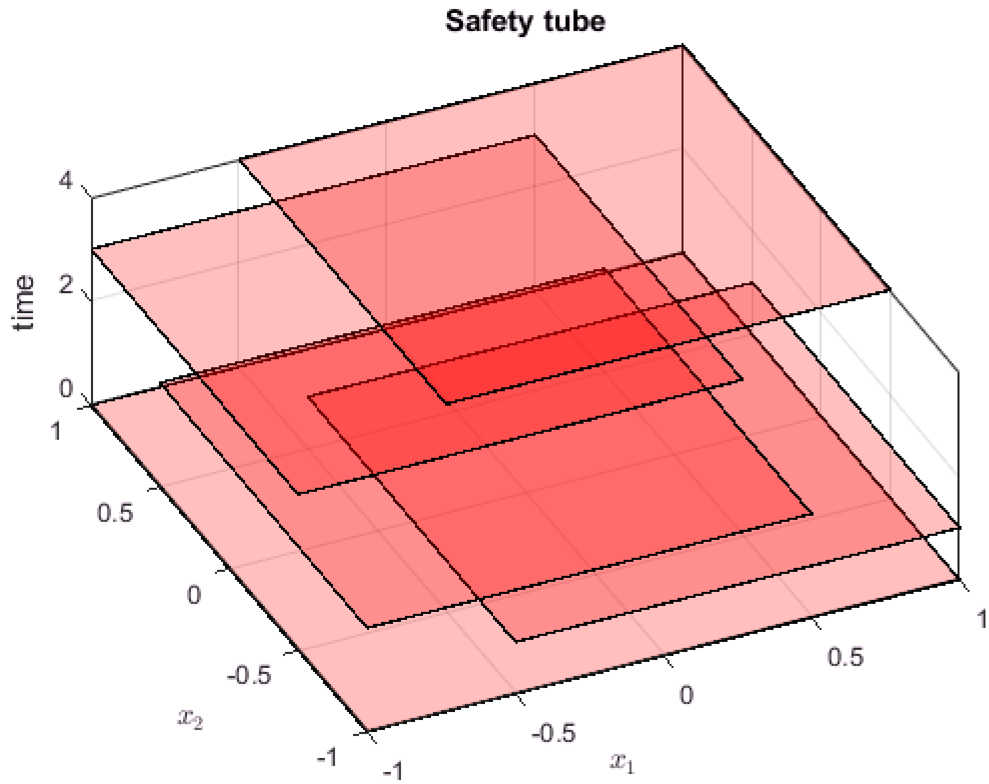


Case 3: Time-varying safety sets

Setup the time-varying safety tube

The advantage of safety tube is that it allows for problem formulations in which we would like to reach a moving target.

```
safety_tube3 = TargetTube(Polyhedron('lb', [-1, -1], 'ub', [1, 1]), ...
    Polyhedron('lb', [-0.5, -1], 'ub', [1, 0.5]),...
    Polyhedron('lb', [-1, -1], 'ub', [0.5, 0.5]), ...
    Polyhedron('lb', [-1, -0.5], 'ub', [0.5, 1]), ...
    Polyhedron('lb', [-0.5, -0.5], 'ub', [1, 1]));
axis_vec3 = [-1 1 -1 1];
N=length(safety_tube3)-1;
% Plotting of safety tube
figure()
hold on
for time_indx=0:N
    safety_tube_at_time_indx = Polyhedron('H',[safety_tube3(time_indx+1).A,zeros(size(safety_t
    plot(safety_tube_at_time_indx, 'alpha',0.25);
end
axis([axis_vec3 0 N]);
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube');
view([-25 60]);
```



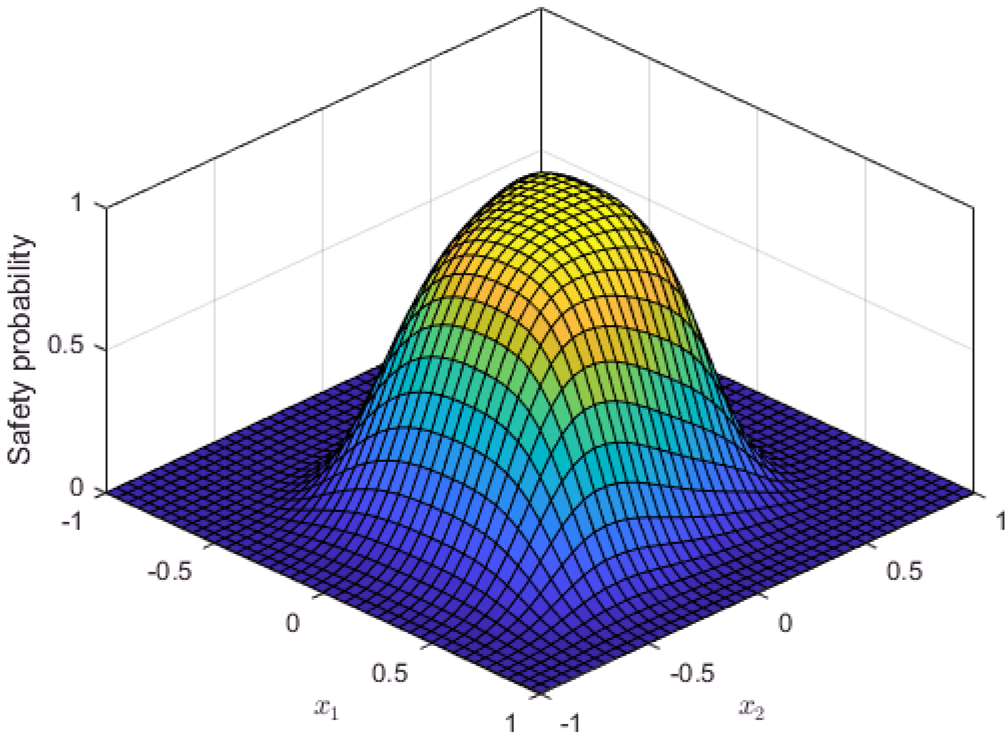
Dynamic programming solution on safety tube

```
tic;
[prob_x3, cell_of_xvec_x3] = SReachDynProg('term', sys, dyn_prog_xinc, dyn_prog_uinc, safety_t
toc
```

```
Elapsed time is 2.943270 seconds.
```

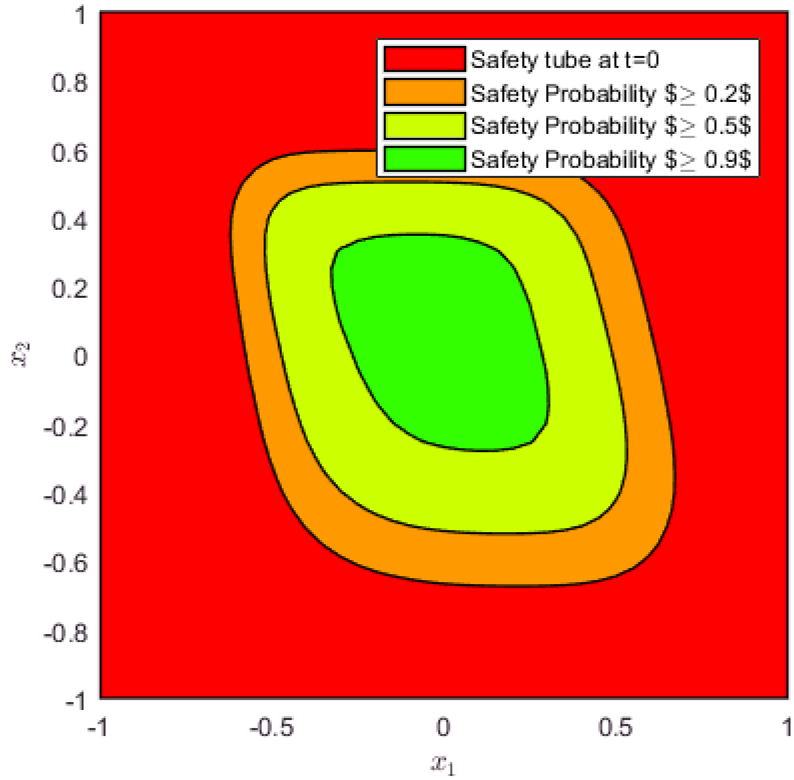
Visualization of the value function at t=0 (safety probability)

```
figure();
x1vec = cell_of_xvec_x3{1};
x2vec = cell_of_xvec_x3{2};
axis([axis_vec3 0 N]);
surf(x1vec,x2vec,reshape(prob_x3,length(x2vec),length(x1vec)));
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
box on
view(45, 45)
```



Visualization of the safe initial sets --- Superlevel sets of safety probability

```
figure();
poly_array3 = getDynProgLevelSets2D(cell_of_xvec_x3, prob_x3, reach_set_thresholds, safety_tub
hold on;
plot([safety_tube3(1) poly_array3])
box on
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
axis(axis_vec3);
box on
axis equal
legend(legend_str)
```



Case 4: First-hitting time stochastic reach-avoid problem

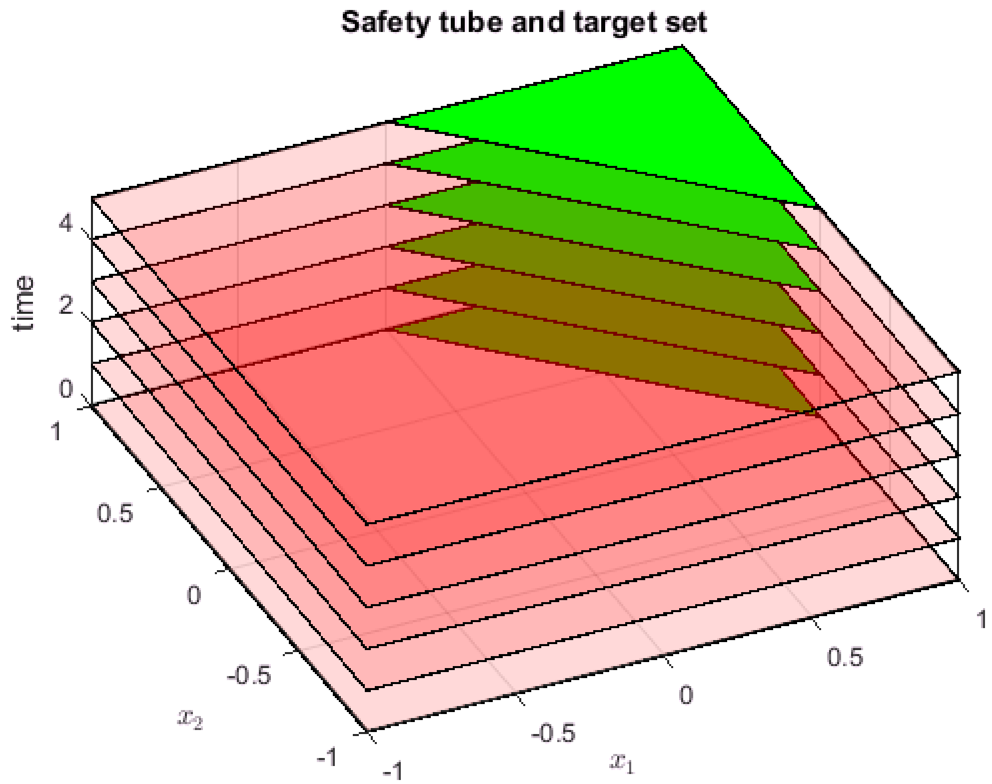
Define the double integrator with larger sampling time.

```
T = 0.25;

% define the system
sys = LtiSystem('StateMatrix', [1, T; 0, 1], ...
    'InputMatrix', [T^2/2; T], ...
    'InputSpace', Polyhedron('lb', -0.1, 'ub', 0.1), ...
    'DisturbanceMatrix', eye(2), ...
    'Disturbance', StochasticDisturbance('Gaussian', zeros(2,1), 0.001*eye(2)));
%% Setup the dynamic programming and visualization parameters

dyn_prog_xinc = 0.025;
dyn_prog_uinc = 0.1;
reach_set_thresholds = [0.2 0.5 0.9];
safe_set = Polyhedron('lb', [-1, -1], 'ub', [1, 1]);
% time horizon
N = 5;
% in target tube for the viability problem is equivalent to a tube of repeating
% safe sets
safety_tube4 = TargetTube('viability', safe_set, N);
```

Setup the time-varying safety tube

The advantage of safety tube is that it allows for problem formulations in which we would like to reach a moving target.

```
target_set4 = Polyhedron('H',[-1 -1 -1]).intersect(safety_tube1(1));

N=length(safety_tube4)-1;
% Plotting of safety tube and target sets
figure()
hold on
for time_indx=0:N
    target_set_at_time_indx = Polyhedron('H',[target_set4.A,zeros(size(target_set4.A,1),1), ta
    plot(target_set_at_time_indx, 'color','g');
    safety_tube_at_time_indx = Polyhedron('H',[safety_tube4(time_indx+1).A,zeros(size(safety_t
    plot(safety_tube_at_time_indx, 'alpha',0.15,'color','r');
end
axis([axis_vec1 0 N]);
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube and target set');
view([-25 60]);
```



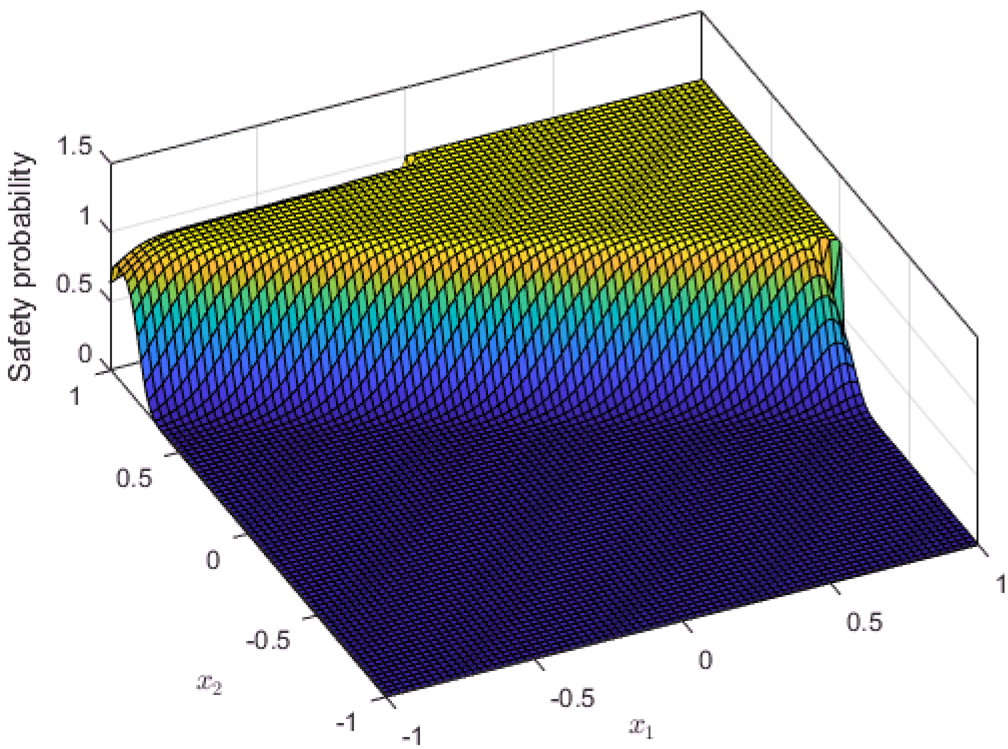
Dynamic programming solution on safety tube

```
tic;
[prob_x4, cell_of_xvec_x4] = SReachDynProg('first', sys, dyn_prog_xinc, dyn_prog_uinc, safety_
toc
```

```
Elapsed time is 17.175079 seconds.
```

Visualization of the value function at t=0 (safety probability)

```
figure();
x1vec = cell_of_xvec_x4{1};
x2vec = cell_of_xvec_x4{2};
axis([axis_vec1 0 N]);
prob_x4_mat = reshape(prob_x4,length(x2vec),length(x1vec));
surf(x1vec,x2vec,prob_x4_mat);
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
box on
view([-25 60]);
```



Visualization of the safe initial sets --- Superlevel sets of safety probability

We can't use getDynProgLevelSets2D here since the level sets are no longer guaranteed to be convex.

```
figure();
plot(safety_tube4(1));
hold on
contourf(x1vec,x2vec, prob_x4_mat, reach_set_thresholds);
colorbar;
hold on;
box on
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
axis(axis_vec1);
box on
axis equal
```

```
title('Contour plots of the different level sets and the original safe set');
```