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Demonstration of controller synthesis via SReachPoint

This example will demonstrate the use of SReachTools in controller synthesis for stochastic continuous-state discrete-time linear time-invariant (LTI) systems.

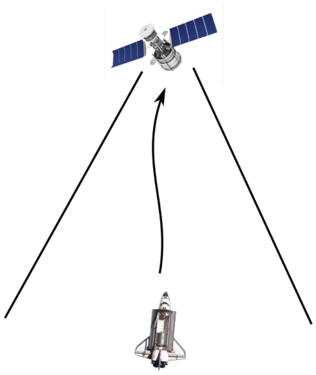
Specifically, we will discuss how SReachTools can use Fourier transforms (Genz's algorithm and MATLAB's patternsearch), particle filter, or convex chance constraints to synthesize open-loop controllers. We also synthesize an affine controller using difference-of-convex program.

Our approaches is grid-free and recursion-free resulting in highly scalable solutions, especially for Gaussian-perturbed LTI systems.


```
% Prescript running
close all;clc;clear;
srtinit
```

Problem formulation: Spacecraft motion via CWH dynamics

We consider both the spacecrafts, referred to as the deputy spacecraft and the chief spacecraft, to be in the same circular orbit. In this example, we will consider the problem of optimal controller synthesis for the deputy such that it can rendezvous with the chief while staying within the line-of-sight cone with maximium likelihood.



Dynamics model for the deputy relative to the chief spacecraft

The relative planar dynamics of the deputy with respect to the chief are described by the Clohessy-Wiltshire-Hill (CWH) equations,

$$\ddot{x} - 3\omega x - 2\omega\dot{y} = \frac{F_x}{m_d}$$

$$\ddot{y} + 2\omega\dot{x} = \frac{F_y}{m_d}$$

where the position of the deputy relative to the chief is $x, y \in \mathbf{R}$, $\omega = \sqrt{\frac{\mu}{R_0^3}}$ is the orbital frequency, μ is the gravitational constant, and R_0 is the orbital radius of the chief spacecraft. We define the state as $\overline{x} = [x\ y\ \dot{x}\ \dot{y}]^\top \in \mathbf{R}^4$ which is the position and velocity of the deputy relative to the chief along x- and y- axes, and the input as $\overline{u} = [F_x\ F_y]^\top \in \mathcal{U} \subset \mathbf{R}^2$.

We will discretize the CWH dynamics in time, via zero-order hold, to obtain the discrete-time linear time-invariant system and add a Gaussian disturbance to account for the modeling uncertainties and the disturbance forces,

$$\overline{x}_{k+1} = A\overline{x}_k + B\overline{u}_k + \overline{w}_k$$

with $\overline{w}_k \in \mathbf{R}^4$ as an IID Gaussian zero-mean random process with a known covariance matrix $\Sigma_{\overline{w}}$.

System definition

```
umax=0.1;
mean_disturbance = zeros(4,1);
covariance_disturbance = diag([1e-4, 1e-4, 5e-8, 5e-8]);
% Define the CWH (planar) dynamics of the deputy spacecraft relative
 to the
% chief spacecraft as a LtiSystem object
sys = getCwhLtiSystem(4, Polyhedron('lb', -umax*ones(2,1),...
                                    'ub',  umax*ones(2,1)),...
```

```
       RandomVector('Gaussian',
 mean_disturbance,covariance_disturbance));
```

Methods to run

```
ft_run = 1;
cc_open_run = 1;
cc_affine_run = 1;
pa_open_run = 1;
plot_traj_instead_of_ellipses = 0;
```

Target tube construction --- reach-avoid specification

```
time_horizon=5;          % Stay within a line of sight cone for 4 time
 steps and
                         % reach the target at t=5% Safe Set --- LoS
 cone
% Safe set definition --- LoS cone |x|<=y and y\in[0,ymax] and |vx|
<=vxmax and
% |vy|<=vymax
ymax=2;
vxmax=0.5;
vymax=0.5;
A_safe_set = [1, 1, 0, 0;
             -1, 1, 0, 0;
              0, -1, 0, 0;
              0, 0, 1,0;
              0, 0,-1,0;
              0, 0, 0,1;
              0, 0, 0,-1];
b_safe_set = [0;
              0;
              ymax;
              vxmax;
              vxmax;
              vymax;
              vymax];
safe_set = Polyhedron(A_safe_set, b_safe_set);
% Target set --- Box [-0.1,0.1]x[-0.1,0]x[-0.01,0.01]x[-0.01,0.01]
target_set = Polyhedron('lb', [-0.1; -0.1; -0.01; -0.01],...
                        'ub', [0.1; 0; 0.01; 0.01]);
target_tube = Tube('reach-avoid',safe_set, target_set, time_horizon);
```

Initial state definition

```
initial_state = [-1.15;         % Initial x relative position
                 -1.15;         % Initial y relative position
                 0;             % Initial x relative velocity
                 0];            % Initial y relative velocity
slice_at_vx_vy = initial_state(3:4);
```

Preparation for Monte-Carlo simulations of the optimal controllers

Monte-Carlo simulation parameters

```
n_mcarlo_sims = 1e5;
n_sims_to_plot = 5;      % Required only if
 plot_traj_instead_of_ellipses = 1
% Generate matrices for optimal mean trajectory generation
% Get H and mean_X_sans_input
[~, H, ~] = getConcatMats(sys, time_horizon);
sysnoi = LtvSystem('StateMatrix',sys.state_mat,'DisturbanceMatrix',...
    sys.dist_mat,'Disturbance',sys.dist);
[mean_X_sans_input, ~] = SReachFwd('concat-stoch', sysnoi,
 initial_state,...
    time_horizon);

if ft_run
    timer_ft = tic;
    [lb_stoch_reach_avoid_ft, optimal_input_vector_ft] =
 SReachPoint(...
        'term','genzps-open', sys, initial_state, target_tube);
    elapsed_time_ft = toc(timer_ft);
    if lb_stoch_reach_avoid_ft > 0
        % This function returns the concatenated state vector stacked
 columnwise
        concat_state_realization_ft =
 generateMonteCarloSims(n_mcarlo_sims,...
            sys, initial_state, time_horizon,
 optimal_input_vector_ft);
        % Check if the location is within the target_set or not
        mcarlo_result_ft = target_tube.contains(...
            [repmat(initial_state,1,n_mcarlo_sims);
             concat_state_realization_ft]);
        % Optimal mean trajectory generation
        optimal_mean_X_ft = mean_X_sans_input + H *
 optimal_input_vector_ft;

 optimal_mean_trajectory_ft=reshape(optimal_mean_X_ft,sys.state_dim,
[]);
    end
end
```

CC (Linear program approach)

We will use the default options

```
if cc_open_run
    timer_cc_pwl = tic;
    [lb_stoch_reach_avoid_cc_pwl, optimal_input_vector_cc_pwl] =
 SReachPoint(...
        'term','chance-open', sys, initial_state, target_tube);
```

```
    elapsed_time_cc_pwl = toc(timer_cc_pwl);
    if lb_stoch_reach_avoid_cc_pwl > 0
        % This function returns the concatenated state vector stacked
 columnwise
        concat_state_realization_cc_pwl = generateMonteCarloSims(...
            n_mcarlo_sims, sys, initial_state, time_horizon,...
            optimal_input_vector_cc_pwl);
        % Check if the location is within the target_set or not
        mcarlo_result_cc_pwl = target_tube.contains(...
            [repmat(initial_state,1,n_mcarlo_sims);
             concat_state_realization_cc_pwl]);
        % Optimal mean trajectory generation
        optimal_mean_X_cc_pwl = mean_X_sans_input +...
            H * optimal_input_vector_cc_pwl;

 optimal_mean_trajectory_cc_pwl=reshape(optimal_mean_X_cc_pwl,...
            sys.state_dim,[]);
    end
end
```

CC with affine controllers (Second order cone program approach)

We set $\Delta_U = 0.01$ and define verbosity level of 1.

```
max_input_viol_prob = 0.01;
if cc_affine_run
    timer_cc_affine = tic;
    options = SReachPointOptions('term','chance-affine',...
        'max_input_viol_prob', 1e-2, 'verbose', 1);
    [lb_stoch_reach_avoid_cc_affine,
 optimal_input_vector_cc_affine,...
        optimal_input_gain, risk_alloc_state, risk_alloc_input] =...
         SReachPoint('term','chance-affine', sys, initial_state,
 target_tube,...
            options);
    elapsed_time_cc_affine = toc(timer_cc_affine);
    if lb_stoch_reach_avoid_cc_affine > 0
        % This function returns the concatenated state vector stacked
 columnwise
        [concat_state_realization_cc_affine,...
            concat_disturb_realization_cc_affine] =...
                generateMonteCarloSims(n_mcarlo_sims, sys,
 initial_state,...
                    time_horizon,optimal_input_vector_cc_affine,...
                    optimal_input_gain);

        % Check if the location is within the target_set or not
        mcarlo_result_cc_affine = target_tube.contains(...
            [repmat(initial_state,1,n_mcarlo_sims);
             concat_state_realization_cc_affine]);
```

```
        % Check if the input is within the tolerance
        [concat_input_space_A, concat_input_space_b] =...
            sys.getConcatInputSpace(time_horizon);
        mcarlo_result_cc_affine_input = any(concat_input_space_A *...
            (optimal_input_gain * concat_disturb_realization_cc_affine
 +...

 optimal_input_vector_cc_affine)<=concat_input_space_b);

        % Optimal mean trajectory generation
        optimal_mean_X_cc_affine = mean_X_sans_input +...
            H * optimal_input_vector_cc_affine;
        optimal_mean_trajectory_cc_affine=reshape(...
            optimal_mean_X_cc_affine,sys.state_dim,[]);
    end
end
```

Particle filter approach

```
if pa_open_run
    timer_pa = tic;
    [lb_stoch_reach_avoid_pa, optimal_input_vector_pa] =
 SReachPoint(...
        'term','particle-open', sys, initial_state, target_tube);
    elapsed_time_pa = toc(timer_pa);
    if lb_stoch_reach_avoid_pa > 0
        % This function returns the concatenated state vector stacked
 columnwise
        concat_state_realization_pa = generateMonteCarloSims(...
            n_mcarlo_sims, sys, initial_state, time_horizon,...
            optimal_input_vector_pa);
        % Check if the location is within the target_set or not
        mcarlo_result_pa = target_tube.contains(...
            [repmat(initial_state,1,n_mcarlo_sims);
             concat_state_realization_pa]);
        % Optimal mean trajectory generation
        optimal_mean_X_pa = mean_X_sans_input +...
            H * optimal_input_vector_pa;
        optimal_mean_trajectory_pa=reshape(optimal_mean_X_pa,...
            sys.state_dim,[]);
    end
end
```

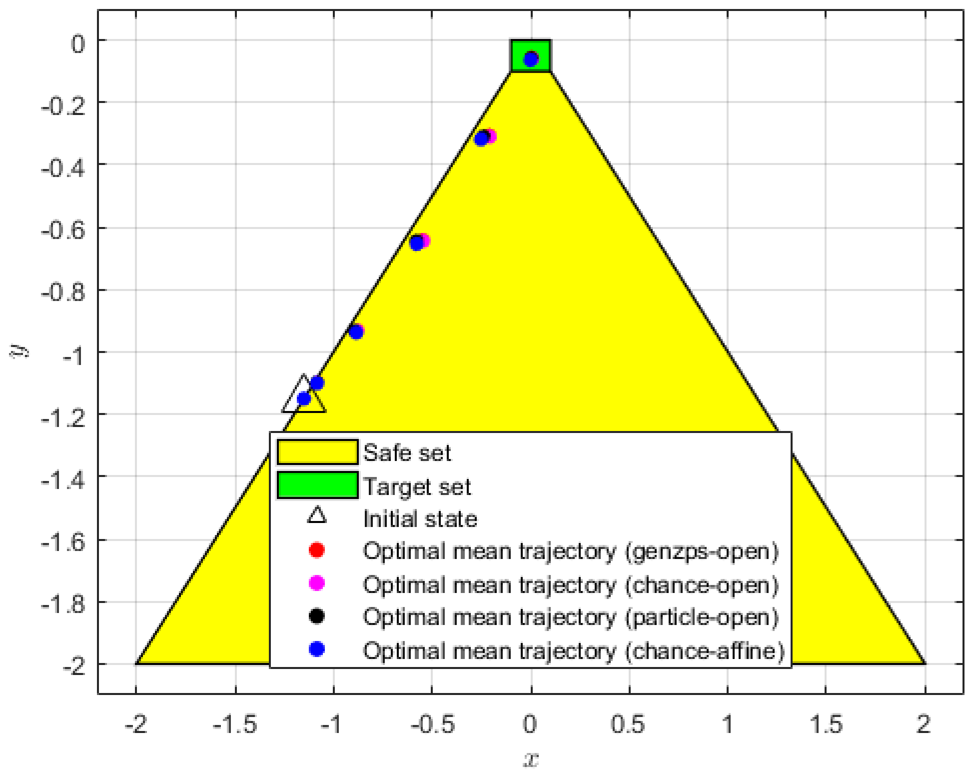
Plotting and Monte-Carlo simulation-based validation

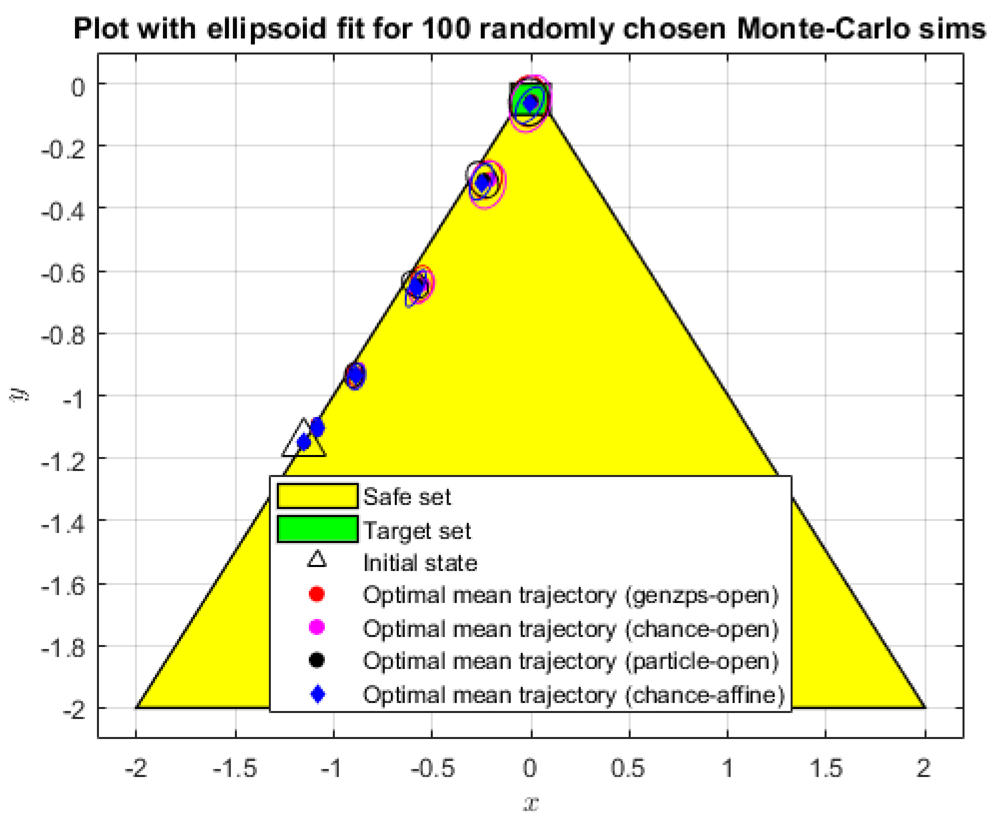
```
figure(1);
clf
box on;
hold on;
plot(safe_set.slice([3,4], slice_at_vx_vy), 'color', 'y');
```

```
plot(target_set.slice([3,4], slice_at_vx_vy), 'color', 'g');
scatter(initial_state(1),initial_state(2),200,'k^');
legend_cell = {'Safe set','Target set','Initial state'};
if exist('optimal_mean_trajectory_ft','var')
    scatter([initial_state(1), optimal_mean_trajectory_ft(1,:)],...
            [initial_state(2), optimal_mean_trajectory_ft(2,:)],...
            30, 'ro', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (genzps-open)';
end
if exist('optimal_mean_trajectory_cc_pwl','var')
    scatter([initial_state(1),
 optimal_mean_trajectory_cc_pwl(1,:)],...
        [initial_state(2), optimal_mean_trajectory_cc_pwl(2,:)],...
        30, 'mo', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (chance-open)';
end
if exist('optimal_mean_trajectory_pa','var')
    scatter([initial_state(1), optimal_mean_trajectory_pa(1,:)],...
        [initial_state(2), optimal_mean_trajectory_pa(2,:)],...
        30, 'ko', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (particle-open)';
end
if exist('optimal_mean_trajectory_cc_affine','var')
    scatter([initial_state(1),
 optimal_mean_trajectory_cc_affine(1,:)],...
        [initial_state(2), optimal_mean_trajectory_cc_affine(2,:)],...
        30, 'bo', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (chance-affine)';
end
legend(legend_cell, 'Location','South');
xlabel('$x$','interpreter','latex');
ylabel('$y$','interpreter','latex');

figure(2);
clf
box on;
hold on;
plot(safe_set.slice([3,4], slice_at_vx_vy), 'color', 'y');
plot(target_set.slice([3,4], slice_at_vx_vy), 'color', 'g');
scatter(initial_state(1),initial_state(2),200,'k^');
legend_cell = {'Safe set','Target set','Initial state'};
if exist('optimal_mean_trajectory_ft','var')
    scatter([initial_state(1), optimal_mean_trajectory_ft(1,:)],...
            [initial_state(2), optimal_mean_trajectory_ft(2,:)],...
            30, 'ro', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (genzps-open)';
    if ~isnan(concat_state_realization_ft)
        if plot_traj_instead_of_ellipses == 1
            [legend_cell] = plotMonteCarlo('(genzps-open)',
 mcarlo_result_ft,...
                concat_state_realization_ft, n_mcarlo_sims,
 n_sims_to_plot,...
                sys.state_dim, initial_state, legend_cell);
        else
```

```
 ellipsoidsFromMonteCarloSims(concat_state_realization_ft,...
                sys.state_dim, [1,2], {'r'});
        end
    end
else
    lb_stoch_reach_avoid_ft = NaN;
    mcarlo_result_ft = NaN;
    elapsed_time_ft = NaN;
end
if exist('optimal_mean_trajectory_cc_pwl','var')
    scatter([initial_state(1),
 optimal_mean_trajectory_cc_pwl(1,:)],...
        [initial_state(2), optimal_mean_trajectory_cc_pwl(2,:)],...
        30, 'mo', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (chance-open)';
    if ~isnan(concat_state_realization_cc_pwl)
        if plot_traj_instead_of_ellipses == 1
            [legend_cell] = plotMonteCarlo('(chance-open)', ...
                mcarlo_result_cc_pwl,
 concat_state_realization_cc_pwl,...
                n_mcarlo_sims, n_sims_to_plot, sys.state_dim,
 initial_state,...
                legend_cell);
        else

 ellipsoidsFromMonteCarloSims(concat_state_realization_cc_pwl,...
                sys.state_dim, [1,2], {'m'});
        end
    end
else
    lb_stoch_reach_avoid_cc_pwl = NaN;
    mcarlo_result_cc_pwl = NaN;
    elapsed_time_cc_pwl = NaN;
end
if exist('optimal_mean_trajectory_pa','var')
    scatter([initial_state(1), optimal_mean_trajectory_pa(1,:)],...
        [initial_state(2), optimal_mean_trajectory_pa(2,:)],...
        30, 'ko', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (particle-open)';
    if ~isnan(concat_state_realization_pa)
        if plot_traj_instead_of_ellipses == 1
            [legend_cell] = plotMonteCarlo('(particle-open)', ...
                mcarlo_result_pa, concat_state_realization_pa,...
                n_mcarlo_sims, n_sims_to_plot, sys.state_dim,
 initial_state,...
                legend_cell);
        else

 ellipsoidsFromMonteCarloSims(concat_state_realization_pa,...
                sys.state_dim, [1,2], {'k'});
        end
    end
else
```

```
    lb_stoch_reach_avoid_pa = NaN;
    mcarlo_result_pa = NaN;
    elapsed_time_pa = NaN;
end

if exist('optimal_mean_trajectory_cc_affine','var')
        scatter([initial_state(1),
 optimal_mean_trajectory_cc_affine(1,:)],...
        [initial_state(2), optimal_mean_trajectory_cc_affine(2,:)],...
        30, 'bd', 'filled');
    legend_cell{end+1} = 'Optimal mean trajectory (chance-affine)';
    if ~isnan(concat_state_realization_cc_affine)
        if plot_traj_instead_of_ellipses==1
            [legend_cell] = plotMonteCarlo('(chance-affine)',...
                mcarlo_result_cc_affine,...
                concat_state_realization_cc_affine, n_mcarlo_sims,...
                n_sims_to_plot, sys.state_dim, initial_state,
 legend_cell);
        else

 ellipsoidsFromMonteCarloSims(concat_state_realization_cc_affine,...
                sys.state_dim, [1,2], {'b'});
        end
    end
else
    lb_stoch_reach_avoid_cc_affine = NaN;
    mcarlo_result_cc_affine = NaN;
    elapsed_time_cc_affine = NaN;
end
legend(legend_cell, 'Location','South');
if plot_traj_instead_of_ellipses==1
    title(sprintf('Plot with %d Monte-Carlo sims', n_sims_to_plot));
else
    title('Plot with ellipsoid fit for 100 randomly chosen Monte-Carlo
 sims');
end
box on;
grid on;
xlabel('$x$','interpreter','latex');
ylabel('$y$','interpreter','latex');
```


0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
Safe set
Target set
Initial state
Optimal mean trajectory (genzps-open)
Optimal mean trajectory (chance-open)
Optimal mean trajectory (particle-open)
Optimal mean trajectory (chance-affine)


Plot with ellipsoid fit for 100 randomly chosen Monte-Carlo sims
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
Safe set
Target set
Initial state
Optimal mean trajectory (genzps-open)
Optimal mean trajectory (chance-open)
Optimal mean trajectory (particle-open)
Optimal mean trajectory (chance-affine)

Reporting the results

```
if any(isnan([lb_stoch_reach_avoid_cc_pwl, lb_stoch_reach_avoid_ft,...
        lb_stoch_reach_avoid_cc_affine, lb_stoch_reach_avoid_pa]))
    disp('Skipped items would show up as NaN');
end
fprintf(['FT: %1.3f | CC (Open): %1.3f | Scenario (Open): %1.3f |
 ',...
    'CC (Affine): %1.3f\n'],...
    lb_stoch_reach_avoid_ft,...
    lb_stoch_reach_avoid_cc_pwl,...
    lb_stoch_reach_avoid_pa,...
    lb_stoch_reach_avoid_cc_affine);
fprintf('MC (%1.0e particles): %1.3f, %1.3f, %1.3f, %1.3f\n',...
    n_mcarlo_sims,...
    sum(mcarlo_result_ft)/n_mcarlo_sims, ...
    sum(mcarlo_result_cc_pwl)/n_mcarlo_sims,...
    sum(mcarlo_result_pa)/n_mcarlo_sims,...
    sum(mcarlo_result_cc_affine)/n_mcarlo_sims);
fprintf('Elapsed time: %1.3f, %1.3f, %1.3f, %1.3f seconds\n',...
    elapsed_time_ft, elapsed_time_cc_pwl, elapsed_time_pa,...
    elapsed_time_cc_affine);
```

```
FT: 0.731 | CC (Open): 0.670 | Scenario (Open): 0.750 | CC (Affine):
 0.813
MC (1e+05 particles): 0.731, 0.710, 0.685, 0.831
Elapsed time: 54.734, 0.284, 6.230, 71.362 seconds
```

Helper functions

Plotting function

```
function [legend_cell] = plotMonteCarlo(method_str, mcarlo_result,...
    concat_state_realization, n_mcarlo_sims, n_sims_to_plot,
 state_dim,...
    initial_state, legend_cell)
% Plots a selection of Monte-Carlo simulations on top of the plot

    green_legend_updated = 0;
    red_legend_updated = 0;
    traj_indices = floor(n_mcarlo_sims*rand(1,n_sims_to_plot));
    for realization_index = traj_indices
        % Check if the trajectory satisfies the reach-avoid objective
        if mcarlo_result(realization_index)
            % Assign green triangle as the marker
            markerString = 'g^-';
        else
            % Assign red asterisk as the marker
            markerString = 'r*-';
        end
        % Create [x(t_1) x(t_2)... x(t_N)]
        reshaped_X_vector = reshape(...
```

```
            concat_state_realization(:,realization_index), state_dim,
[]);
        % This realization is to be plotted
        h = plot([initial_state(1), reshaped_X_vector(1,:)], ...
                 [initial_state(2), reshaped_X_vector(2,:)], ...
                 markerString, 'MarkerSize',10);
        % Update the legends if the first else, disable
        if strcmp(markerString,'g^-')
            if green_legend_updated
                h.Annotation.LegendInformation.IconDisplayStyle
 = 'off';
            else
                green_legend_updated = 1;
                legend_cell{end+1} = strcat('Good trajectory ',
 method_str);
            end
        elseif strcmp(markerString,'r*-')
            if red_legend_updated
                h.Annotation.LegendInformation.IconDisplayStyle
 = 'off';
            else
                red_legend_updated = 1;
                legend_cell{end+1} = strcat('Bad trajectory ',
 method_str);
            end
        end
    end
end
```

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