Table of Contents

Forward stochastic reachability using Fourier transforms

This example will demonstrate the use of SReachTools in forward stochastic reachability analysis for stochastic continuous-state discrete-time linear time-invariant (LTI) systems.

Specifically, we will discuss how SReachTools uses Fourier transforms to efficiently compute

1. **Forward stochastic reach probability density**: The probability density function associated with the random vector describing the state at a future time of interest.
2. **Probability computations**: Probability that the state lies in a target set or the trajectory in a target tube at a future time of interest.

Our approach is grid-free and recursion-free resulting in highly scalable solutions, especially for Gaussian-perturbed LTI systems.

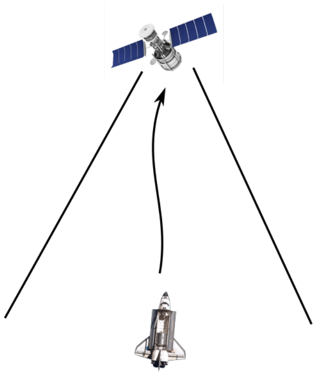
We will consider the case where the initial state is a known deterministic point in the state space, and the case where the initial state is a random vector.

This Live Script is part of the SReachTools toolbox. License for the use of this function is given in https://github.com/unm-hscl/SReachTools/blob/master/LICENSE.

```
% Prescript running
close all;clc;clear;
srtinit
```

Problem formulation: Spacecraft motion via CWH dynamics

We consider both the spacecrafts, referred to as the deputy spacecraft and the chief spacecraft, to be in the same circular orbit. In this example, we will consider the forward stochastic reachability analysis of the deputy.



Dynamics model for the deputy relative to the chief spacecraft

The relative planar dynamics of the deputy with respect to the chief are described by the [Clohessy-Wiltshire-Hill (CWH) equations,](#)

$$\ddot{x} - 3\omega x - 2\omega \dot{y} = \frac{F_x}{m_d}$$

$$\ddot{y} + 2\omega \dot{x} = \frac{F_y}{m_d}$$

where the position of the deputy relative to the chief is $x, y \in \mathbf{R}$, $\omega = \sqrt{\frac{\mu}{R_0^3}}$ is the orbital frequency, μ is the gravitational constant, and R_0 is the orbital radius of the chief spacecraft. We define the state as $\overline{x} = [x\ y\ \dot{x}\ \dot{y}]^\top \in \mathbf{R}^4$ which is the position and velocity of the deputy relative to the chief along x- and y- axes, and the input as $\overline{u} = [F_x\ F_y]^\top \in \mathcal{U} \subset \mathbf{R}^2$.

We will discretize the CWH dynamics in time, via zero-order hold, to obtain the discrete-time linear time-invariant system and add a Gaussian disturbance to account for the modeling uncertainties and the disturbance forces,

$$\overline{x}_{k+1} = A\overline{x}_k + B\overline{u}_k + \overline{w}_k$$

with $\overline{w}_k \in \mathbf{R}^4$ as an IID Gaussian zero-mean random process with a known covariance matrix $\Sigma_{\overline{w}}$.

SReachTools directly allows us to create a `LtiSystem` object with these dynamics. We will set the input space to be unbounded.

System definition

```
umax=Inf;
mean_disturbance = zeros(4,1);
covariance_disturbance = diag([1e-4, 1e-4, 5e-8, 5e-8]);
% Define the CWH (planar) dynamics of the deputy spacecraft relative
 to the
```

```
% chief spacecraft as a LtiSystem object
sys_CWH = getCwhLtiSystem(4, Polyhedron('lb', -umax*ones(2,1),...
                                         'ub',  umax*ones(2,1)),...
       RandomVector('Gaussian',
 mean_disturbance,covariance_disturbance));
```

Next, we close the control loop under the action of a linear feedback law (LQR) We will define a `LtiSystem` object to describe the dynamics when $\overline{u}_k = -K\overline{x}_k$ for some $K \in \mathbf{R}^{4\times 2}$. We will compute K using LQR theory with $Q = 0.01 I_4$ and $R = I_2$, i.e., $\overline{u}_k = -K\overline{x}_k$ will regulate the deputy spacecraft towards the origin.

```
% Create a discrete-time LQR controller that regulates the deputy to
 the origin
K = lqr(ss(sys_CWH.state_mat,sys_CWH.input_mat,[],
[],-1),0.01*eye(4),eye(2));
% Reuse the system definition in sys_CWH with appropriately defined
 state matrix
closed_loop_state_mat = sys_CWH.state_mat - sys_CWH.input_mat*K;
sys = LtiSystem('StateMatrix', closed_loop_state_mat,...
                'DisturbanceMatrix', sys_CWH.dist_mat,...
                'Disturbance', sys_CWH.dist);
disp(sys);
```

```
Linear time invariant system with 4 states, 0 inputs, and 4
 disturbances.
```

P1. Probability that the deputy rendezvous with the chief at some time k?

Since the chief is located at the origin in this coordinate frame (`sys` describes the relative dynamics of the deputy), we define the target set to be a small box centered at the origin (`target_set` is a box axis-aligned with side 0.2). We are interested in the probability that the deputy will meet the chief at `target_time` time steps in future.

```
% Time of interest
target_time = 23;
% Target set definition
target_set = Polyhedron('lb',-0.05 * ones(4,1),...
                        'ub', 0.05 * ones(4,1));
desired_accuracy = 1e-2;

% Problem 1a: Fixed initial state
% --------------------------------
% Initial state definition
initial_state = [-10;10;0;0];

% 1. Compute the mean and the covariance of the forward stochastic
 reach
% probability density of the state at time |target_time starting from
 this fixed
% initial state.|
```

```
[mean_x, cov_x] = SReachFwd('state-stoch', sys, initial_state,
 target_time);
disp(mean_x);
disp(cov_x);

   -0.0056
    0.0229
   -0.0001
    0.0003

   1.0e-03 *

    0.4935   -0.0000   -0.0026   -0.0003
   -0.0000    0.4937    0.0003   -0.0026
   -0.0026    0.0003    0.0001    0.0000
   -0.0003   -0.0026    0.0000    0.0001


% 2. Compute the probability of reaching a target set at a specified
 target
% time

% Integrate the FSRPD at time target_time over the target_set
prob = SReachFwd('state-prob', sys, initial_state, target_time,
 target_set,...
    desired_accuracy);
fprintf('Probability of x_{target_time} lying in target_set: %1.4f
\n',prob);

Probability of x_{target_time} lying in target_set: 0.8600

% 3. Validate this probability via Monte-Carlo simulations

n_mcarlo_sims = 1e5;
% This function returns the concatenated state vector stacked
 columnwise
concat_state_realization = generateMonteCarloSims(...
                                                n_mcarlo_sims,...
                                                sys,...
                                                initial_state,...
                                                target_time);
% Extract the location of the deputy at target_time
end_locations = concat_state_realization(end-sys.state_dim +1 :
 end,:);
% Check if the location is within the target_set or not
mcarlo_result = target_set.contains(end_locations);
fprintf('Monte-Carlo simulation using %1.0e particles: %1.3f\n',...
        n_mcarlo_sims,...
        sum(mcarlo_result)/n_mcarlo_sims);

Monte-Carlo simulation using 1e+05 particles: 0.861

% Problem 1b: Initial state is a Gaussian random vector
% -----------------------------------------------------
% Initial state definition
```

```
initial_state_rv = RandomVector('Gaussian', initial_state,
 0.1*eye(4));

% 1. Compute the mean and the covariance of the forward stochastic
 reach
% probability density of the state at time |target_time when the
 initial state
% is stochastic.|
[mean_x, cov_x] = SReachFwd('state-stoch', sys, initial_state_rv,
 target_time);
disp(mean_x);
disp(cov_x);
```

```
   -0.0056
    0.0229
   -0.0001
    0.0003

   1.0e-03 *

    0.9818   -0.0100   -0.0263   -0.0007
   -0.0100    0.9350    0.0010   -0.0254
   -0.0263    0.0010    0.0013    0.0000
   -0.0007   -0.0254    0.0000    0.0013
```

```
% 2. Compute the probability of reaching a target set at a specified
 target
% time
% Integrate the FSRPD at time target_time over the target_set
prob = SReachFwd('state-prob', sys, initial_state_rv, target_time,...
    target_set, desired_accuracy);
fprintf('Probability of x_{target_time} lying in target_set: %1.4f
\n',prob);
```

```
Probability of x_{target_time} lying in target_set: 0.7100
```

Notice how the probability of success is lower due to a random initial state.

```
% 3. Validate this reach probability via Monte-Carlo simulations

n_mcarlo_sims = 1e5;
% This function returns the concatenated state vector stacked
 columnwise
concat_state_realization = generateMonteCarloSims(...
                                                 n_mcarlo_sims,...
                                                 sys,...
                                                 initial_state_rv,...
                                                 target_time);
% Extract the location of the deputy at target_time
end_locations = concat_state_realization(end-sys.state_dim +1 :
 end,:);
% Check if the location is within the target_set or not
mcarlo_result = target_set.contains(end_locations);
fprintf('Monte-Carlo simulation using %1.0e particles: %1.3f\n',...
```

```
        n_mcarlo_sims,...
        sum(mcarlo_result)/n_mcarlo_sims);
```

Monte-Carlo simulation using 1e+05 particles: 0.708

P2. Probability that the deputy (safely) rendezvous with the chief?

Since the chief is located at the origin in this coordinate frame (`sys` describes the relative dynamics of the deputy), we define the target set to be a small box centered at the origin (`target_set` is a box axis-aligned with side 0.2). We are interested in the probability that the deputy will meet the chief at `target_time` time steps in future. **Additionally**, we desire that the deputy satellite stays within a line-of-sight cone for accurate sensing.

```
target_time = 10;                                       % Time of
 interest
target_set = Polyhedron('lb',-0.05 * ones(4,1),...
                        'ub', 0.05 * ones(4,1));        % Target set
 definition

% Create a target tube
% Safe set definition
% LoS cone |x|<=y and y\in[0,ymax] and |vx|<=vxmax and |vy|<=vymax
%
 ------------------------------------------------------------------------------
ymax=10;
vxmax=0.5;
vymax=0.5;
A_safe_set = [1, 1, 0, 0;
             -1, 1, 0, 0;
              0, -1, 0, 0;
              0, 0, 1,0;
              0, 0,-1,0;
              0, 0, 0,1;
              0, 0, 0,-1];
b_safe_set = [0;
              0;
              ymax;
              vxmax;
              vxmax;
              vymax;
              vymax];
safe_set = Polyhedron(A_safe_set, b_safe_set);
safety_tube = Tube('reach-avoid', safe_set, target_set, target_time);

% Problem 2a: Fixed initial state
% ------------------------------
initial_state = [0;
                 -1;
                 0;
                 0];                                   % Initial
 state definition
```

```
% 1. Compute the mean and the covariance of the forward stochastic
 reach
% probability density of the state at time |target_time starting from
 this fixed
% initial state.|

[mean_X, ~] = SReachFwd('concat-stoch', sys, initial_state,
 target_time);
mean_X_trajectory = reshape(mean_X,4,[]);
disp(mean_X_trajectory);
```

```
  Columns 1 through 7

   -0.0034   -0.0098   -0.0145   -0.0159   -0.0142   -0.0102   -0.0051
   -0.9486   -0.8201   -0.6587   -0.4943   -0.3451   -0.2204   -0.1232
   -0.0003   -0.0003   -0.0002    0.0000    0.0002    0.0002    0.0003
    0.0051    0.0077    0.0084    0.0080    0.0069    0.0056    0.0042

  Columns 8 through 10

    0.0001    0.0046    0.0079
   -0.0526   -0.0051    0.0237
    0.0002    0.0002    0.0001
    0.0029    0.0018    0.0010
```

```
% 2. Compute the probability of reaching a target set at a specified
 target
% time *while staying within a safe set*

% Integrate the FSRPD at time target_time over the target_set
prob = SReachFwd('concat-prob', sys, initial_state, target_time,
 safety_tube, desired_accuracy);
fprintf('Probability of x_{target_time} lying in target_set while
 staying inside line-of-sight cone: %1.4f\n',prob);
```

```
Probability of x_{target_time} lying in target_set while staying
 inside line-of-sight cone: 0.3100
```

```
% 3. Validate this reach probability via Monte-Carlo simulations
n_mcarlo_sims = 1e5;
% This function returns the concatenated state vector stacked
 columnwise
concat_state_realization = generateMonteCarloSims(...
                                                n_mcarlo_sims,...
                                                sys,...
                                                initial_state,...
                                                target_time);
% Check if the location is within the target_set or not
mcarlo_result =
 safety_tube.contains([repmat(initial_state,1,n_mcarlo_sims);
                                     concat_state_realization]);
prob_mc_estim = sum(mcarlo_result)/n_mcarlo_sims;
fprintf('Monte-Carlo simulation using %1.0e particles: %1.3f\n',...
        n_mcarlo_sims,...
```

```
        sum(mcarlo_result)/n_mcarlo_sims);
```

Monte-Carlo simulation using 1e+05 particles: 0.313

```
% Problem 2b: Initial state is a Gaussian random vector
% -----------------------------------------------------

initial_state_rv = RandomVector('Gaussian',...
                             initial_state,...
                             0.001*eye(4));        % Initial state
 definition

% 1. Compute the mean and the covariance of the forward stochastic
 reach
% probability density of the state at time |target_time starting from
 this fixed
% initial state.|

[mean_X, cov_X] = SReachFwd('concat-stoch', sys, initial_state_rv,
 target_time);
mean_X_trajectory = reshape(mean_X,4,[]);
disp(mean_X_trajectory);
```

```
  Columns 1 through 7

   -0.0034   -0.0098   -0.0145   -0.0159   -0.0142   -0.0102   -0.0051
   -0.9486   -0.8201   -0.6587   -0.4943   -0.3451   -0.2204   -0.1232
   -0.0003   -0.0003   -0.0002    0.0000    0.0002    0.0002    0.0003
    0.0051    0.0077    0.0084    0.0080    0.0069    0.0056    0.0042

  Columns 8 through 10

    0.0001    0.0046    0.0079
   -0.0526   -0.0051    0.0237
    0.0002    0.0002    0.0001
    0.0029    0.0018    0.0010
```

Note that the mean remains the same as Problem 2a.

```
% 2. Compute the probability of reaching a target set at a specified
 target
% time *while staying within a safe set*

prob = SReachFwd('concat-prob', sys, initial_state_rv, target_time,
 safety_tube, desired_accuracy);
fprintf('Probability of x_{target_time} lying in target_set while
 staying inside line-of-sight cone: %1.4f\n',prob);
```

Probability of x_{target_time} lying in target_set while staying inside line-of-sight cone: 0.0700

```
% However, the probability decreases drastically since the initial
 state
% is now random.
```

```
% 3. Validate this reach probability via Monte-Carlo simulations

n_mcarlo_sims = 1e5;
% This function returns the concatenated state vector stacked
 columnwise
concat_state_realization = generateMonteCarloSims(...
                                                n_mcarlo_sims,...
                                                sys,...
                                                initial_state_rv,...
                                                target_time);
% Check if the location is within the target_set or not
mcarlo_result =
 safety_tube.contains([repmat(initial_state,1,n_mcarlo_sims);
                                       concat_state_realization]);
fprintf('Monte-Carlo simulation using %1.0e particles: %1.3f\n',...
        n_mcarlo_sims,...
        sum(mcarlo_result)/n_mcarlo_sims);
```

```
Monte-Carlo simulation using 1e+05 particles: 0.067
```

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