Table of Contents

Double Integrator Reach-Avoid Via Dynamic Programming

This example demonstrates how to use the SReachTools toolbox to solve a terminal-hitting time reach-avoid problem using dynamic programming.

In this example, we analyze the following problems via dynamic programming for a stochastic system with known dynamics:

1. **stochastic viability problem**: Compute a controller to stay within a safe set with maximum likelihood
2. **the terminal-hitting time stochastic reach-avoid problem**: Compute a controller that maximizes the probability of reaching a target set at a time horizon, `N`, while maintaining the system in a set of safe states
3. **stochastic reachability of a moving target tube**: Compute a controller that maximizes the probability of staying within a target tube (a collection of time-varying safe sets)

SReachTools has a dynamic programing implementation that can analyze systems upto three dimensions. For efficient implementation, we require the input set to be an axis-aligned hypercuboid, and define the grid the smallest hypercuboid containing all the target sets.

This Live Script is part of the SReachTools toolbox. License for the use of this function is given in https://github.com/unm-hscl/SReachTools/blob/master/LICENSE.

```
% Prescript running
close all;
% clc;
clearvars;
srtinit
```

Problem setup

In this example we use a discretized double integrator dynamics given by:

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k + w_k$$

with state x_k, input u_k, disturbance w_k, and sampling time T.

The double integrator system can be used as a simple model of a vehicle on a lane.

1. The state is position and velocity of the vehicle.
2. The vehicle is acceleration-controlled with restrictions on the maximum and minimum acceleration provided
3. The stochastic disturbance affects its position and velocity accounting for unmodelled phenomena like slipping or engine model mismatch.

```
% System definition
% -----------------
% discretization parameter
T = 0.1;

% define the system
sys = LtiSystem('StateMatrix', [1, T; 0, 1], ...
    'InputMatrix', [T^2/2; T], ...
    'InputSpace', Polyhedron('lb', -0.1, 'ub', 0.1), ...
    'DisturbanceMatrix', eye(2), ...
    'Disturbance', RandomVector('Gaussian', zeros(2,1), 0.01*eye(2)));

% Parameters for dynamic programming and visualization
% ----------------------------------------------------
% Step sizes for gridding
dyn_prog_xinc = 0.05;
dyn_prog_uinc = 0.1;

% Additionally, we need to specify thresholds of interest to compute
 the
% stochastic viability set
reach_set_thresholds = [0.2 0.5 0.9];

% For plotting purposes, we specify legend and axis limits
axis_vec = [-1 1 -1 1];
legend_str={'Safety tube at k=0', 'Safety Probability $\geq 0.2$', ...
    'Safety Probability $\geq 0.5$', 'Safety Probability $\geq 0.9$'};
```

Case 1: Stochastic viability problem

We are interested in assessing the safety of this double integrator system to stay within the safe set of $[-1,1]^2$.

For the vehicle in a lane example, we require controllers that maximize the probability of ensuring that the position stay within $[-1,1]$ and the velocity with $[-1,1]$ when the car at $k=0$ has some initial position and velocity in $[-1,1]^2$.

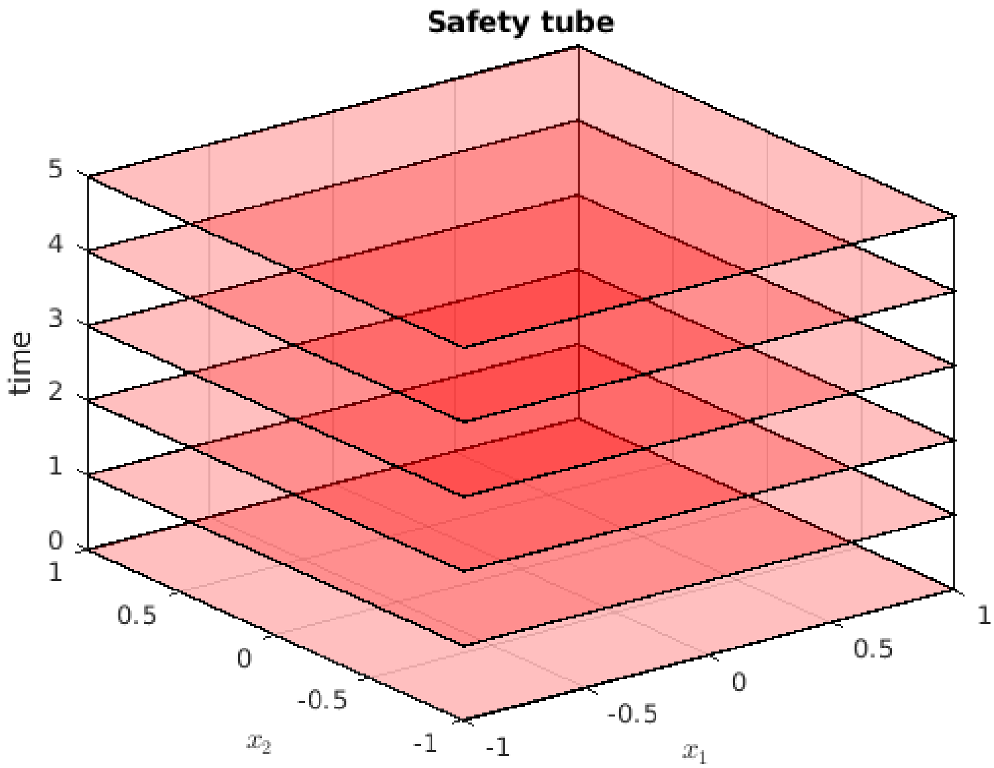
To perform this safety analysis, we need to do the following steps

1. Provide parameters for dynamic programming like grid size and thresholds (if stochastic viability sets are of interest) [*Reused from above*]
2. Construct the safety tube using 'viability' option (and plot it as well)

3. Obtain the dynamic programming solution via SReachDynProg

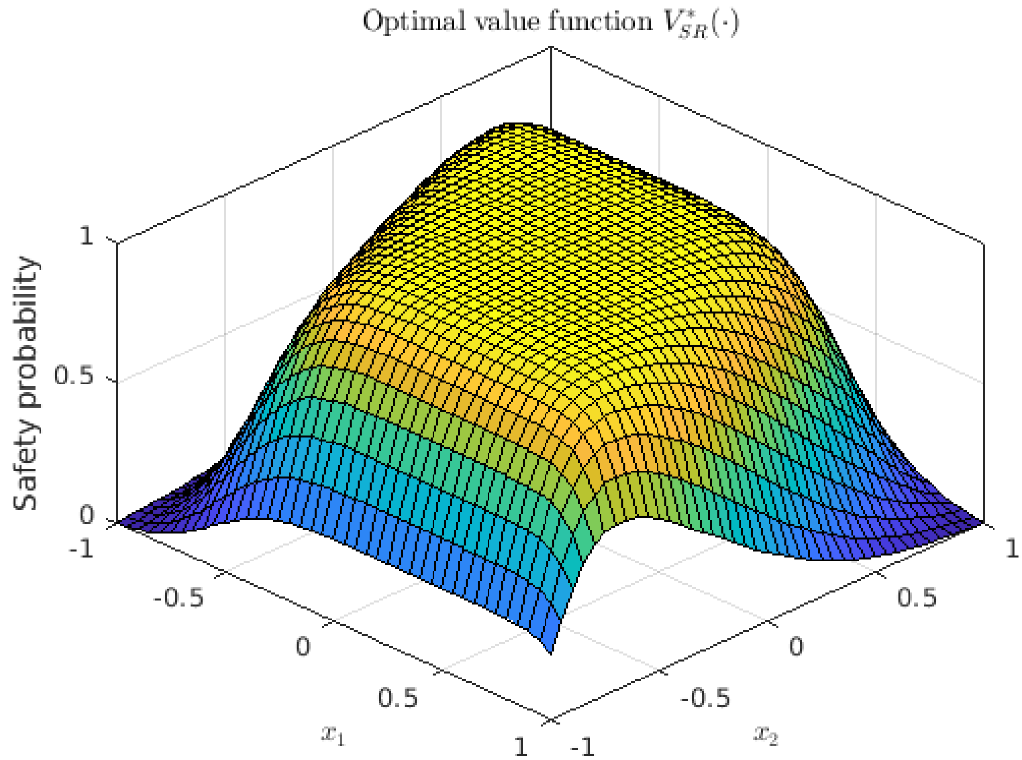
4. Obtain the stochastic viability sets at desired thresholds using getDynProgLevelSets2D

```
% Safety tube definition
% ----------------------
% time horizon
N = 5;
% The viability problem is equivalent to a stochastic reachability of
 a target
% tube of repeating safe sets
safe_set = Polyhedron('lb', [-1, -1], 'ub', [1, 1]);
safety_tube1 = Tube('viability', safe_set, N);

% Plotting of safety tube
% -----------------------
figure()
hold on
for time_indx = 0:N
    % Embed the 2-D safety sets in a 3-D space which is state-space x
 [0, N]
    safety_tube_at_time_indx = Polyhedron('H', ...
        [safety_tube1(time_indx+1).A, ...
            zeros(size(safety_tube1(time_indx+1).A,1),1), ...
            safety_tube1(time_indx+1).b], ...
        'He',[0 0 1 time_indx]);
    plot(safety_tube_at_time_indx, 'alpha',0.25);
end
axis([axis_vec 0 N])
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube');
```

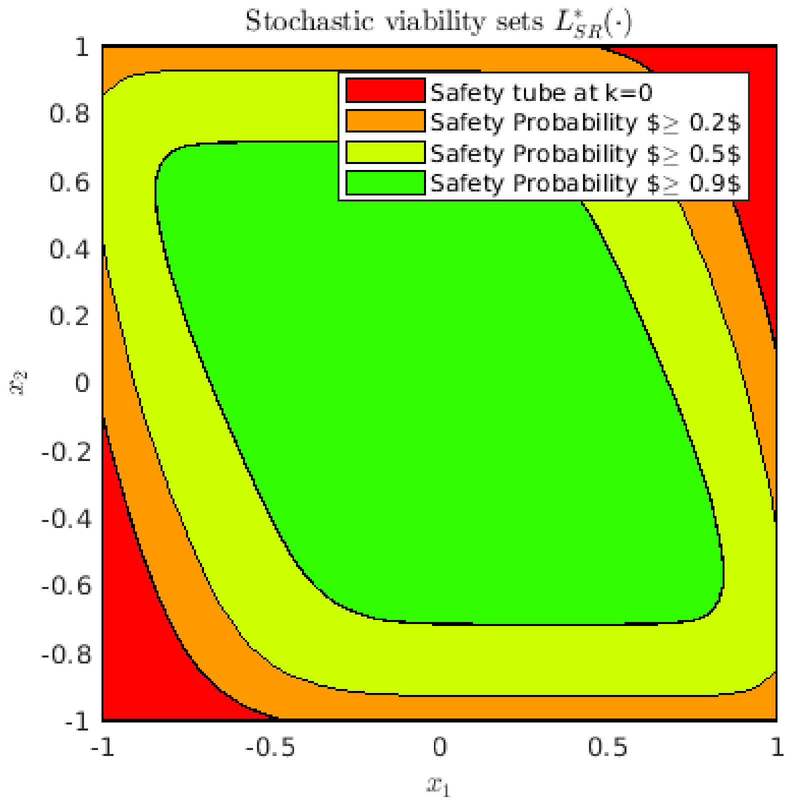



```
% Dynamic programming recursion via gridding
% ------------------------------------------
tic;
[prob_x1, cell_of_xvec_x1] =  SReachDynProg('term', sys,
 dyn_prog_xinc, ...
    dyn_prog_uinc, safety_tube1);
toc

% Plotting of the optimal value function
% ---------------------------------------
% Visualization of the value function at k=0 (safety probability)
figure();
x1vec = cell_of_xvec_x1{1};
x2vec = cell_of_xvec_x1{2};
surf(x1vec,x2vec,reshape(prob_x1,length(x2vec),length(x1vec)));
axis([axis_vec 0 1])
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
title('Optimal value function $V_{SR}^
\ast(\cdot)$','interpreter','latex');
box on
view(45, 45)
```

Elapsed time is 4.466950 seconds.




```
% Obtain the stochastic viability sets
% ------------------------------------
poly_array1 = getDynProgLevelSets2D(cell_of_xvec_x1, prob_x1, ...
    reach_set_thresholds, safety_tube1);

% Plotting of the stochastic viability sets
% -----------------------------------------
% Visualization of the safe initial states --- Superlevel sets of
 safety
% probability
figure();
hold on;
plot([safety_tube1(1), poly_array1])
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
box on
axis(axis_vec)
axis equal
legend(legend_str)
title('Stochastic viability sets $L_{SR}^
\ast(\cdot)$','interpreter','latex');
```



Case 2: Terminal hitting-time stochastic reach-avoid

Stochastic reach-avoid problems have a time-invariant safety set uptil time horizon (uptill $k = N-1$), and a different target set at time horizon $k = N$.

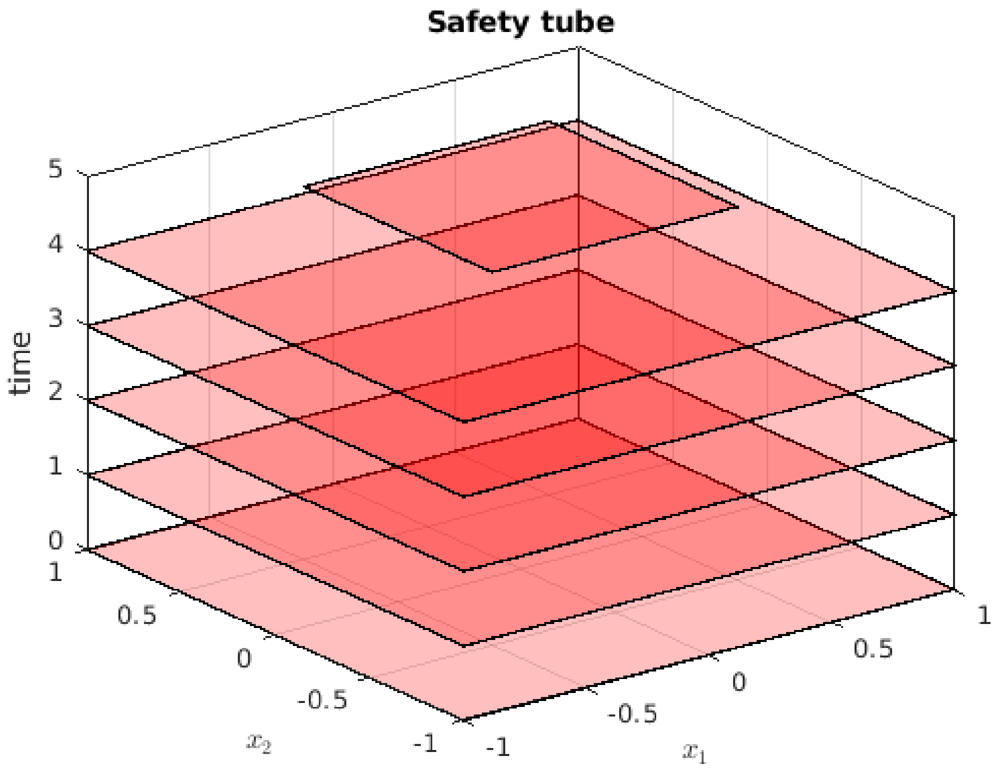
Specifically, we are interested in assessing the safety of this double integrator system to stay within the safe set of $[-1,1]^2$ for 4 time steps and then reach $[-0.5,0.5]^2$. Here, the car at $k=0$ has some initial position and velocity in $[-1,1]^2$ as before.

To perform this safety analysis, we need to do the following steps

1. Provide parameters for dynamic programming like grid size and thresholds (if stochastic reach-avoid sets are of interest) [*Reused from above*]
2. Construct the safety tube using 'reach-avoid' option (and plot it as well)
3. Obtain the dynamic programming solution via SReachDynProg
4. Obtain the stochastic reach-avoid sets at desired thresholds using getDynProgLevelSets2D

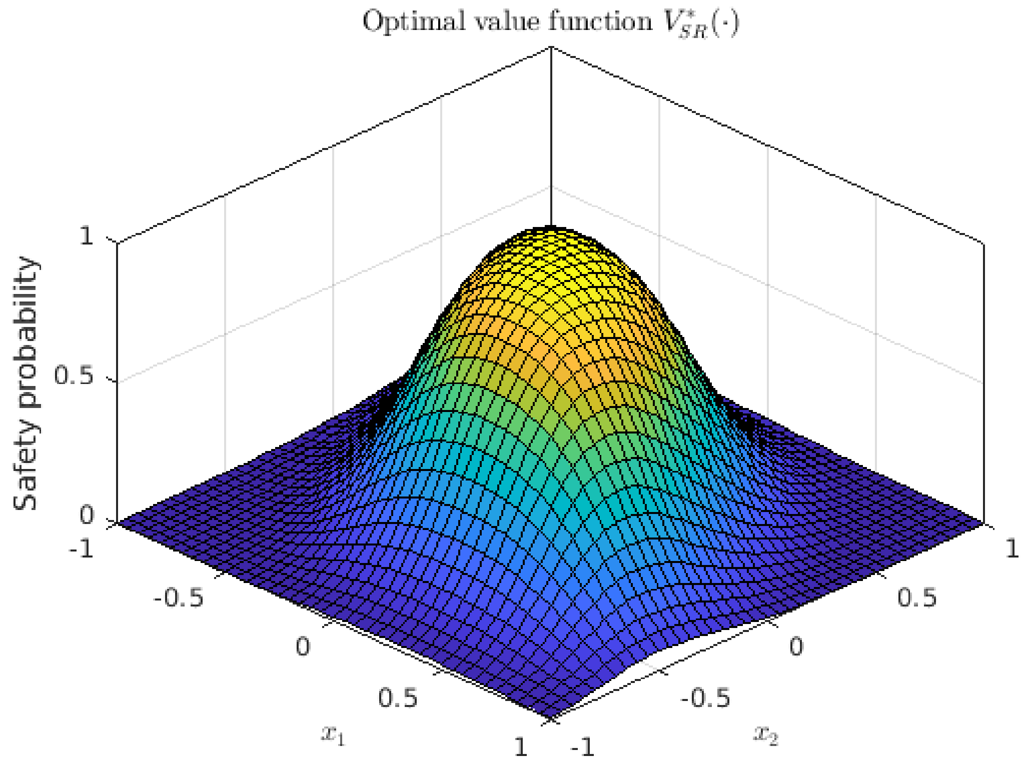
```
% Safety tube definition
% ----------------------
% Safety tube is a generalization of the reach problem. The reach
 avoid
```

```
% safety-tube is created by setting the first $N$ sets in the tube as
 the
% |safe_set| and the final set as the |target_set|.
%
% time horizon
N = 5;
% The viability problem is equivalent to a stochastic reachability of
 a target
% tube of repeating safe sets
safe_set = Polyhedron('lb', [-1, -1], 'ub', [1, 1]);
target_set = Polyhedron('lb', [-0.5, -0.5], 'ub', [0.5, 0.5]);
safety_tube2 = Tube('reach-avoid', safe_set, target_set, N);

% Plotting of safety tube
% -----------------------
figure()
hold on
for time_indx = 0:N
    % Embed the 2-D safety sets in a 3-D space which is state-space x
 [0, N]
    safety_tube_at_time_indx = Polyhedron('H', ...
        [safety_tube2(time_indx+1).A, ...
            zeros(size(safety_tube2(time_indx+1).A,1),1), ...
            safety_tube2(time_indx+1).b], ...
        'He',[0 0 1 time_indx]);
    plot(safety_tube_at_time_indx, 'alpha',0.25);
end
axis([axis_vec 0 N])
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube');
```

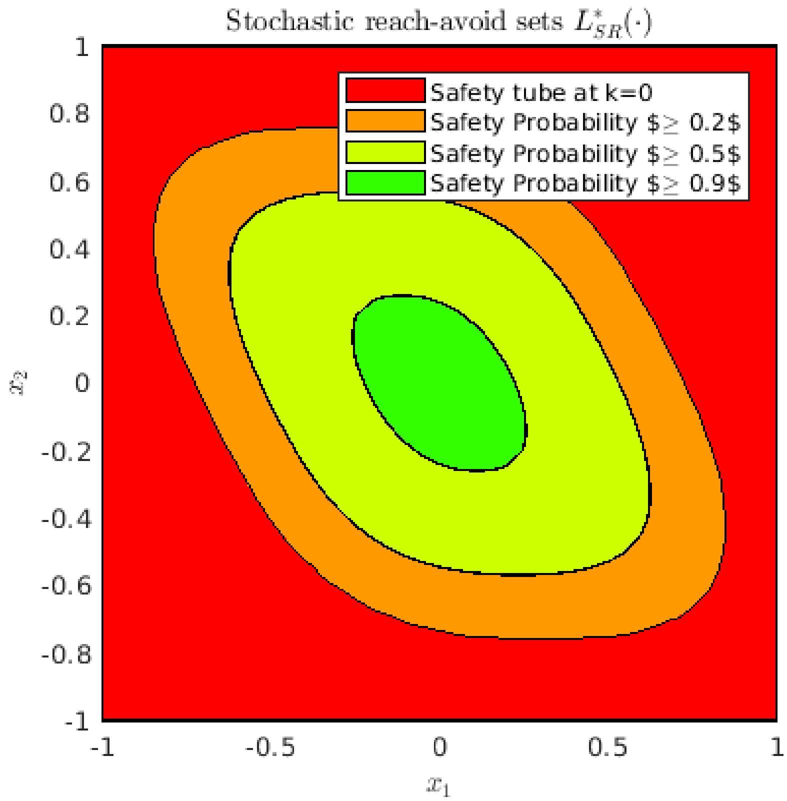



```
% Dynamic programming recursion via gridding
% -------------------------------------------
tic;
[prob_x2, cell_of_xvec_x2] =  SReachDynProg('term', sys,
 dyn_prog_xinc, ...
    dyn_prog_uinc, safety_tube2);
toc
% Plotting of the optimal value function
% --------------------------------------
% Visualization of the value function at k=0 (safety probability)
figure();
x1vec = cell_of_xvec_x2{1};
x2vec = cell_of_xvec_x2{2};
axis([axis_vec 0 N]);
surf(x1vec,x2vec,reshape(prob_x2,length(x2vec),length(x1vec)));
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
box on
view(45, 45)
title('Optimal value function $V_{SR}^
\ast(\cdot)$','interpreter','latex');
```

Elapsed time is 3.558113 seconds.




```
% Obtain the stochastic reach-avoid sets
% ---------------------------------------
poly_array2 = getDynProgLevelSets2D(cell_of_xvec_x2, prob_x2, ...
    reach_set_thresholds, safety_tube2);

% Plotting of the stochastic reach-avoid sets
% --------------------------------------------
% Visualization of the safe initial states --- Superlevel sets of
 safety
% probability
figure();
hold on;
plot([safety_tube2(1) poly_array2])
box on
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
axis(axis_vec);
box on
axis equal
legend(legend_str)
title('Stochastic reach-avoid sets $L_{SR}^
\ast(\cdot)$','interpreter','latex');
```



Case 3: Time-varying safety sets (Stochastic reachability of a target tube)

Stochastic reachability of a target tube has time-varying safety sets.

To perform this safety analysis, we need to do the following steps

1. Provide parameters for dynamic programming like grid size and thresholds (if stochastic reach sets are of interest) [*Reused from above*]

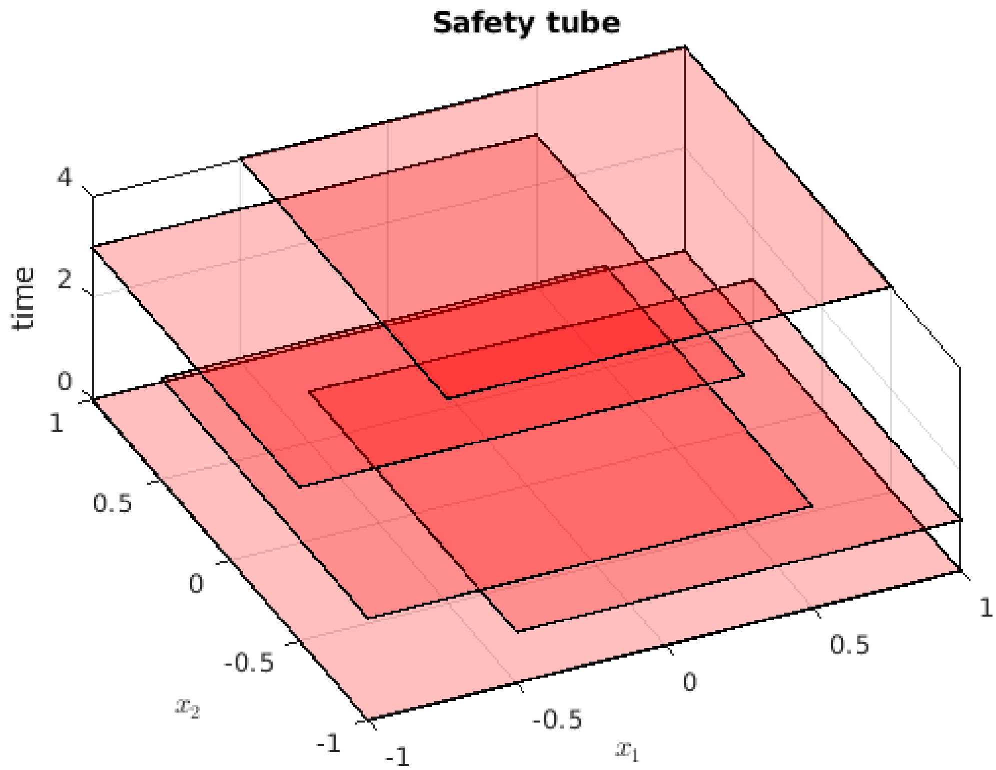
2. Construct the safety tube by constructing at piecemeal (and plot it as well)

3. Obtain the dynamic programming solution via SReachDynProg

4. Obtain the stochastic reach sets at desired thresholds using getDynProgLevelSets2D

```
% Safety tube definition
% ----------------------
safety_tube3 = Tube(Polyhedron('lb', [-1, -1], 'ub', [1, 1]), ...
    Polyhedron('lb', [-0.5, -1], 'ub', [1, 0.5]), ...
    Polyhedron('lb', [-1, -1], 'ub', [0.5, 0.5]), ...
    Polyhedron('lb', [-1, -0.5], 'ub', [0.5, 1]), ...
    Polyhedron('lb', [-0.5, -0.5], 'ub', [1, 1]));
N = length(safety_tube3)-1;

% Plotting of safety tube
```

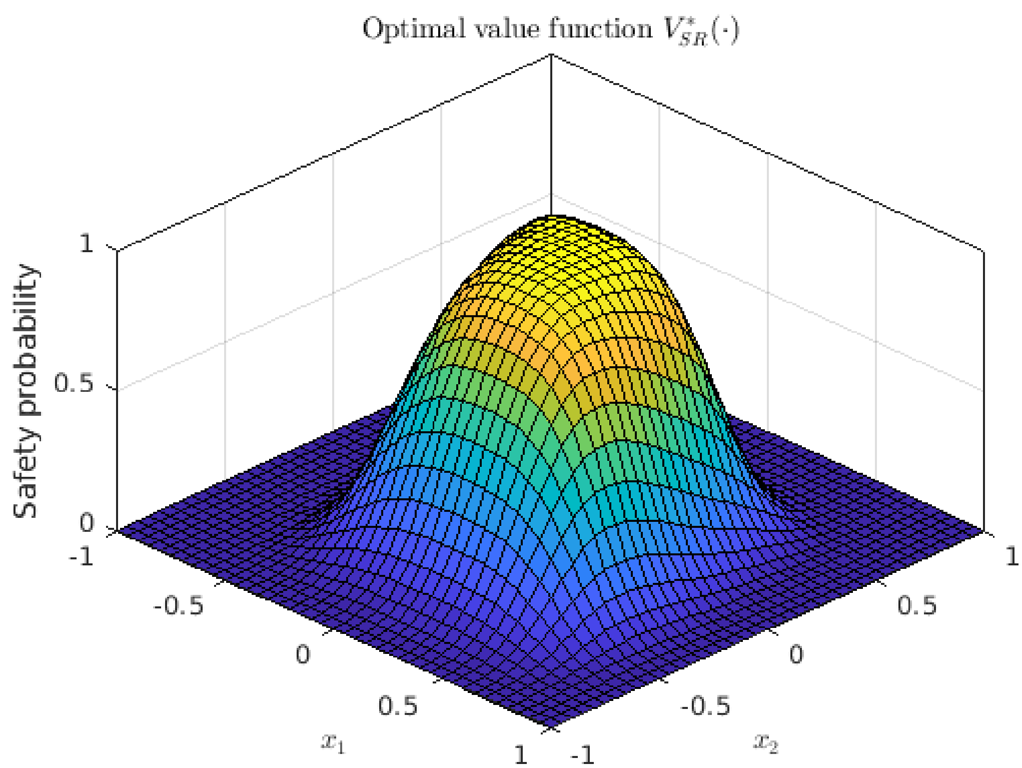
```
% -----------------------
figure()
hold on
for time_indx = 0:N
    % Embed the 2-D safety sets in a 3-D space which is state-space x
 [0, N]
    safety_tube_at_time_indx = Polyhedron('H', ...
        [safety_tube3(time_indx+1).A, ...
            zeros(size(safety_tube3(time_indx+1).A,1),1), ...
            safety_tube3(time_indx+1).b], ...
        'He',[0 0 1 time_indx]);
    plot(safety_tube_at_time_indx, 'alpha',0.25);
end
axis([axis_vec 0 N])
box on;
grid on;
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('time');
title('Safety tube');
view([-25 60]);
```




```
% Dynamic programming recursion via gridding
% ----------------------------------------
tic;
[prob_x3, cell_of_xvec_x3] =  SReachDynProg('term', sys,
 dyn_prog_xinc, ...
```

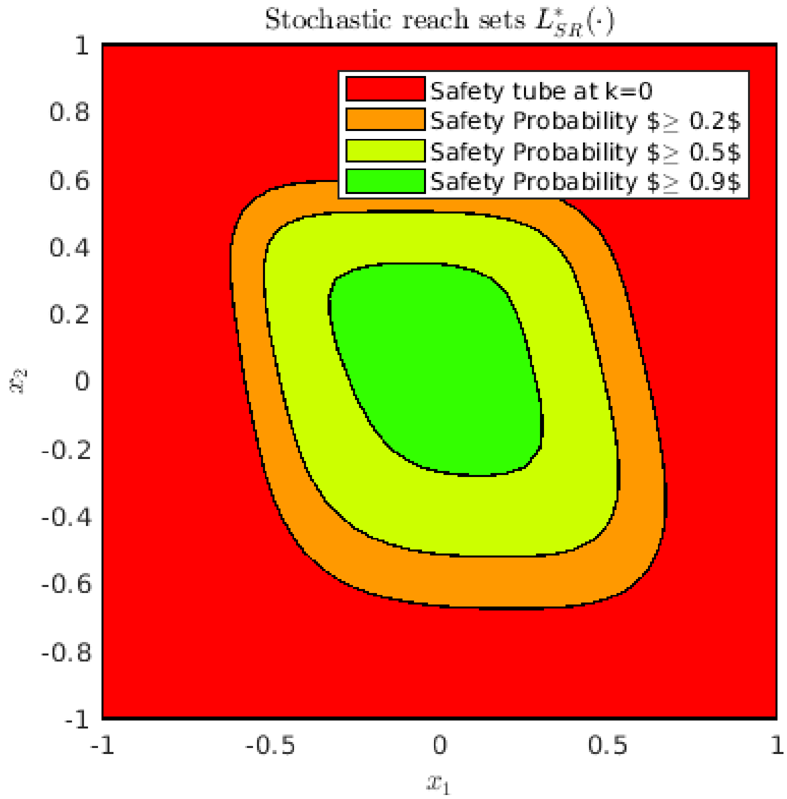
```
    dyn_prog_uinc, safety_tube3);
toc
% Plotting of the optimal value function
% ---------------------------------------
% Visualization of the value function at k=0 (safety probability)
figure();
x1vec = cell_of_xvec_x3{1};
x2vec = cell_of_xvec_x3{2};
axis([axis_vec 0 N]);
surf(x1vec,x2vec,reshape(prob_x3,length(x2vec),length(x1vec)));
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
zlabel('Safety probability')
box on
view(45, 45)
title('Optimal value function $V_{SR}^
\ast(\cdot)$','interpreter','latex');
```

Elapsed time is 3.536022 seconds.




```
% Obtain the stochastic reach sets
% --------------------------------
poly_array3 = getDynProgLevelSets2D(cell_of_xvec_x3, prob_x3, ...
    reach_set_thresholds, safety_tube3);

% Plotting of the stochastic reach sets
% -------------------------------------
```

```
% Visualization of the safe initial states --- Superlevel sets of
 safety
% probability
figure();
hold on;
plot([safety_tube3(1) poly_array3])
box on
xlabel('$x_1$','interpreter','latex');
ylabel('$x_2$','interpreter','latex');
axis(axis_vec);
box on
axis equal
legend(legend_str)
title('Stochastic reach sets $L_{SR}^
\ast(\cdot)$','interpreter','latex');
```



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