

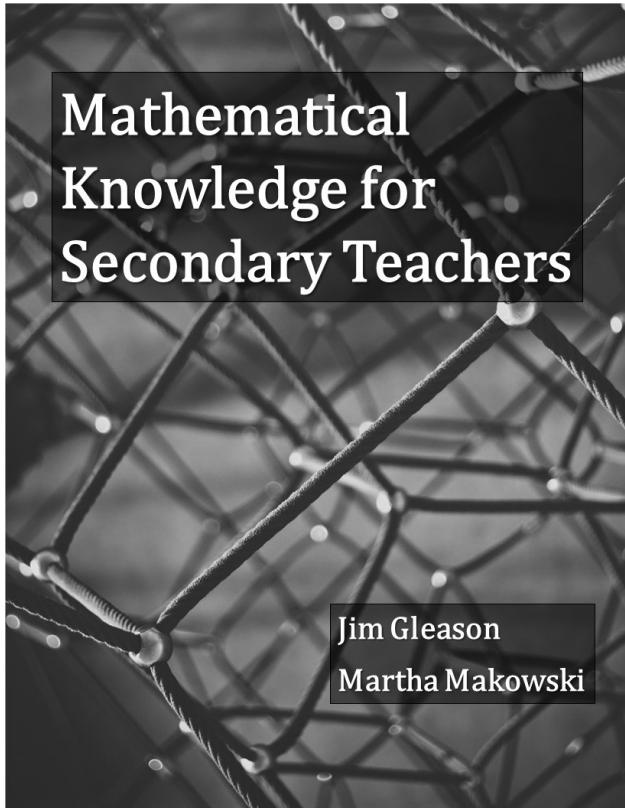
Mathematical Knowledge for Secondary Teachers

Jim Gleason and Martha Makowski

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Preface

The knowledge possessed by an excellent secondary mathematics teacher draws on a variety of skills, beliefs, and passions related to the practice of teaching. A large portion of this knowledge is acquired through experience working with students. However, excellent teachers develop certain aspects of knowledge through structured learning environments like a textbook or university course.

This textbook is inspired by Usiskin et al. [2003] and designed to help current and future teachers explore a unique blend of *mathematical content knowledge* used in the practice of teaching mathematics at the secondary and early post-secondary level. Content examining how K-12 students learn mathematics or on classroom management techniques related to mathematical teaching are left to other texts. There are many excellent resources for this pedagogical content knowledge, particularly from the National Council of Teachers of Mathematics (NCTM) and the Association of Mathematics Teacher Educators (AMTE).

For the Student

For the Instructor

One Semester Course

Two Semester Courses

Three Semester Courses

Focus on Middle School Content

Acknowledgements

Part I

FOUNDATIONS

Chapter 1

Mathematics Education Foundations

This first chapter focuses on developing the notion of Mathematical Knowledge for Teaching and examining how content knowledge supports the current standards for mathematical practice and the content standards developed by various educational researchers and mathematics education organizations. Understanding these foundations prior to jumping into the rest of the content provides both an organizational framework and a foundation for the topics covered. Some of these ideas may challenge your ways of thinking, while others reinforce what you already know. We encourage you to go beyond what is written here and to read some of the original research articles and organization standards cited in this chapter.



1.1 Mathematical Knowledge for Teaching

In his presidential address to the American Educational Research Association, Lee Shulman ([1986]) popularized the concept of knowledge for teaching that included a specialized content knowledge where “the teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification” (p. 9). Researchers in mathematics education further expanded on this idea in the development of domains of Mathematical Knowledge for Teaching. This textbook focuses on the Subject Matter Knowledge side of the Mathematical Knowledge for Teaching.

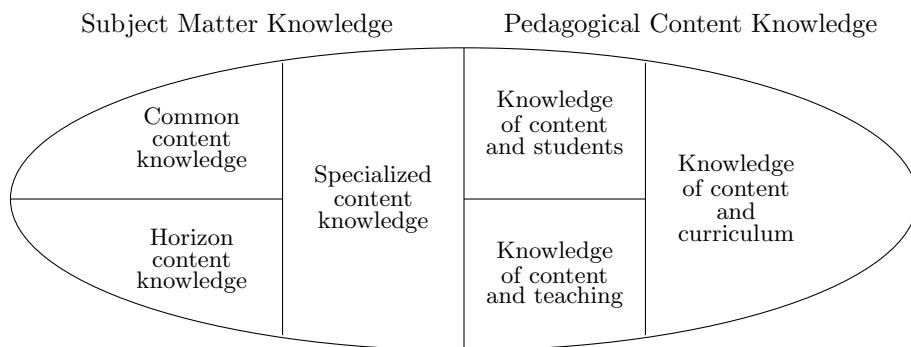


Figure 1.1: Domains of Mathematical Knowledge for Teaching [@Ball2008]

1.1.1 Common Content Knowledge

The foundation for much of the other aspects of mathematical knowledge for teaching is **common content knowledge**, defined as “the mathematical knowledge and skill used in settings other than teaching” [Ball et al., 2008, p. 399]. Common content knowledge provides the individual the ability to solve mathematical problems from the related curriculum and to apply the mathematical knowledge to other fields of knowledge outside of mathematics. Common content knowledge also includes understanding how some of the different mathematical subjects build upon one another, but does not require those studying the material to be able to explain broader patterns. Thus, common content knowledge represents the type of knowledge that we expect secondary and introductory post-secondary students to display.

For instance, common content knowledge related to rational algebraic expressions and functions includes understanding the relationships between rational expressions and rational numbers, how long-division of polynomials relates to long-division of integers, and the properties of the quotient and remainder of a rational expression and its effect on the graph of the related function. While this common content knowledge includes a deeper level of knowledge and a richer web of conceptual understanding by the teacher than is common for most K-12 students, the knowledge still falls within the expectations for the most advanced of these K-12 students. Thus, it is paramount that secondary mathematics teachers develop deep fluency in common content knowledge and the interconnectedness of the many different mathematical concepts in the curriculum. Such a deep and interconnected knowledge base of the teacher provides a requisite foundation in helping K-12 students learn mathematics. However, it is only a first step.

1.1.2 Specialized Content Knowledge

Specialized content knowledge incorporates mathematical knowledge that goes beyond knowledge expected of students, with a focus on the mathematical knowledge that improves the ability of the teacher to assist students learning mathematics [Ball et al., 2008, p. 400]. Thus, specialized content knowledge supplements the common content knowledge that all K-12 students of mathematics need. For example, while common content knowledge for rational algebraic expressions and rational functions include the relationships to the rational numbers, specialized content knowledge could include knowledge of rings and integral domains,

thereby improving the teacher's ability to help students to make the connections between various pieces of related content knowledge.

A teacher would also use this specialized content knowledge related to rational expressions when explaining the extent to which two rational expressions are equivalent when common factors of the numerator and denominator cancel. For instance the teacher could explain the ways in which the expressions

$$\frac{(x-1)(x+2)^2}{(x+1)(x+2)} \quad \text{and} \quad \frac{(x-1)(x+2)}{(x+1)}$$

are equivalent and distinct.

1.1.3 Horizon Content Knowledge

Excellent teachers use more than just a knowledge of the content in the current curriculum when teaching students. They also draw on knowledge of what their students have previously learned in mathematics, what mathematics content will be covered in the next few years, and how the current mathematical topic relates to applications outside of mathematics. That is, excellent mathematics teachers see their instruction as part of a continuum, of which their work is only a small part. We call this domain of knowledge **horizon content knowledge**.

For example, a teacher with horizon content knowledge might use students' familiarity with rational numbers to help high school students develop knowledge of rational algebraic expressions and functions. She could also use her knowledge from differential equations to know that the rational functions play a pivotal role in the Laplace transform, helping to determine the amount of time and detail appropriate for teaching about the quotient and remainder theorem for polynomials and how to rewrite rational expressions using partial fractions. The teacher could also use knowledge of the physics curriculum and Boyle's law to help students make connections between rational expressions and functions and other fields of study. While each of these pieces of knowledge are not essential for mathematics teachers, the more knowledge one has, the better that teacher can help students learn and apply the critical components of the content.

1.1.4 Exercises

1. Consider the general quadratic expression $ax^2 + bx + c$, where a , b , and c are real numbers such that $a \neq 0$.
 - a. Write a list of everything you **know** about the general quadratic expression.
 - b. Write a list of everything you can **do** to the general quadratic expression.
 - c. How are quadratic expressions different than quadratic equations?
 - d. How does the factorization of a quadratic expression relate to prime numbers?
 - e. How much do you know about the ways in which quadratic expressions are used outside of mathematics?
 - f. Review the mathematics standards for quadratic expressions for your state. How many of the things you have listed for parts (a) and (b) match up with those standards?
 - g. For each of your answers to a. through e., does the content you have listed or described align most with common content knowledge, specialized content knowledge, or horizon content knowledge?
2. Consider a triangle. A particular high school geometry textbook defines a triangle as “a polygon with three sides.” A second textbook defines a triangle as “the figure formed by connecting three non-co-linear points with straight segments.” A last textbook defines triangles as “A three-sided figure.”
 - a) Are all three definitions accurate, or do some allow for shapes that might not be triangles as you understand them to be included within the category of triangle?

Table 1.1: NCTM Process Standards and Common Core Standards for Mathematical Practice

Standards for Mathematical Practice	Process Standards
Make sense of problems and persevere in solving them.	Problem Solving
Reason abstractly and quantitatively.	Reasoning and Proof
Construct viable arguments and critique the reasoning of others.	Communication
Model with mathematics.	Connections
Use appropriate tools strategically.	Representations
Attend to precision	
Look for and make use of structure.	
Look for and express regularity in repeated reasoning.	

- b) In what ways might each definition be considered sloppy? That is, are there any parts of the definition that might not be well-defined? In what ways might each definition make using triangles in future lessons more difficult?
- c) What information do you think each textbook has presented prior to giving its definition for a triangle?
- d) How might horizon content knowledge help an instructor preparing a lesson on triangles decide whether a definition is appropriate or not for her students?

1.2 Mathematical Practice Standards

Just like almost all of the content taught in secondary schools, the development of procedural habits and pieces of information are important parts of a secondary education, the heart of learning lies in developing habits of thinking, perseverance techniques, and developing communication skills to improve the ability to interact with the world around them. For example, the United Nations Educational, Scientific, and Cultural Organization (UNESCO) and the United Nations Office on Drugs and Crime (UNODC) [2019] described cognitive learning outcomes for secondary education such that a student “Knows about local, national, and global governance and accountability systems and structures, understands issues affecting interaction and connectedness of communities at local, national and global levels, (and) develops skills for critical inquiry and analysis” (p. 16). Mathematics, despite it’s reputation, aligns well with these goals. Critical thinking, reasoning, and communicating are the most important components of the secondary mathematics curriculum, and are often the most ignored.

The National Council of Teachers of Mathematics [2000] describes these goals in the -Principles and Standards for School Mathematics-, giving them the name ‘Process Standards.’ The National Governors Association Center for Best Practices and the Council of Chief State School Officers [NGA-CCSSO, 2010] expanded upon these to create the ‘Common Core State Standards Standards for Mathematical Practice.’ These practice standards are listed in Table 1.1 and more details about these standards can be found in their corresponding publications.

It is worth noting that these practices are expected of students of mathematics at *all* grade levels. In order to help others develop these practices, we must first develop them in ourselves. Only then will we be in a position to create a learning environment that motivates and enables students to grow in these practices.

To support the users of this text in developing these practices, opportunities to practice them are woven into the text, exercises, and projects. However, as any good teacher knows, the learner needs to be actively engaged in order for an objective to be reached. As you work through the text, pay attention to how the arguments are presented and seek to understand the process behind the mathematical content. When completing the exercises and projects, do not just try to get an answer. Instead, take some time to grapple with the ideas, think of better ways to communicate what you do not understand, and seek to understand the deeper connections involved in the task.

For the purpose of this text we group these practice standards into four categories: mathematical problem solving, modeling with mathematics, communicating mathematically, and understanding mathematical structures. We briefly elaborate on each in the following sections.

1.2.1 Mathematical Problem Solving

Mathematical problem solving is perhaps the most widely cited application of mathematics “in the real world”. Although generally acknowledged as important, problem solving is a complex process that is difficult to teach. In his book *How to Solve It: A new aspect of mathematical method*, George Pólya described four phases of the problem solving process (See Figure 1.2 when approaching mathematical problems [Polya, 1957]. While many others have expanded on this problem solving process, much of the thinking around problem solving still traces back to the four phases that Pólya describes.

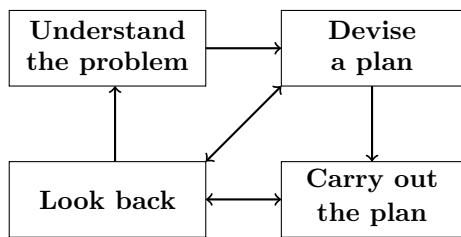


Figure 1.2: Pólya’s problem solving process

The first phase is the process of understanding the problem. What this phase looks like varies by problem, but often involves making sure you understand what information is given and relevant. It also includes knowledge of what an answer should look like for the question posed and, where relevant, being able to organize the provided information in a picture or diagram to better understand the situation. So, for example, if the problem is computational, understanding the problem may entail understanding whether positive or negative answers are valid solutions. If the problem is a theorem that needs proving, this phase may involve generating related examples or writing out relevant definitions to make sure you understand the given inputs and implications of the theorem.

The second phase of the problem solving process is to devise a plan. Many students struggle in the problem solving process because they only want to solve problems for which they have previously been given a template. The origin of this desire has roots in the fact that many teachers only assign students problems that are similar to those discussed in class. True mathematical problem solving involves confronting problems for which the solution is not immediately obvious to the problem solver. In problem solving, one may need to connect the current problem to previously solved problems or restate the problem in a different form with an easier solution process. Sometimes a full plan is not possible at the beginning and the solver needs to just plan initial steps and work through those to gather more information about the problem that will help them create a new plan later in the process.

Once an initial plan is devised, it needs to be carried out. This is often the simplest part of the problem solving process, and often the only component that students complete. With today’s technology, computer applications can often complete the details of this part of the problem solving process through statistical analyses, computer algebra systems, or graphical programs.

After the plan is carried out, it is important to look back, completing the final phase of the problem solving process. In this step the problem solver determines whether their solution makes sense in the context of the original problem. For example, in calculus, it is often possible to obtain negative solutions that are not appropriate to the problem posed. The final step of looking back also entails making sure that we have actually addressed the question asked or the problem posed, rather than a different, somewhat related, question or problem.

Although these four phases appear linear, the majority of mathematical problems require various iterations of these phases. In particular, it is often the case that the first plan created to solve a problem does not work,

so it has to be revised following an attempt to carry it out. Good problem solvers see these revisions as opportunities to learn more about the problem, rather than as failure at the problem solving process.

1.2.2 Modeling with Mathematics

Mathematical problem solving often relies on modeling with mathematics, particularly in the context of real-world phenomena. The Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) defines mathematical modeling as “a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena” [Garfunkel and Montgomery, 2019, p. 8]. This definition varies from other common connotations of “modeling” that are used in education, including in mathematics. For example in mathematics education, using manipulatives to model a mathematical idea; sketching graphs or pictures to communicate or understand concepts; and demonstrating how to solve certain types of problems are all referred to as modeling. However, as a process standard, modeling mathematics does not generally include these activities. Instead, it refers to activities of using mathematics to analyze, predict, and represent real-world data.

The GAIMME Report breaks the mathematical modeling process into the six components shown in Figure 1.3. Critically, these six components **do not** happen in a progression, but instead are iterative and sometimes run parallel to each other.

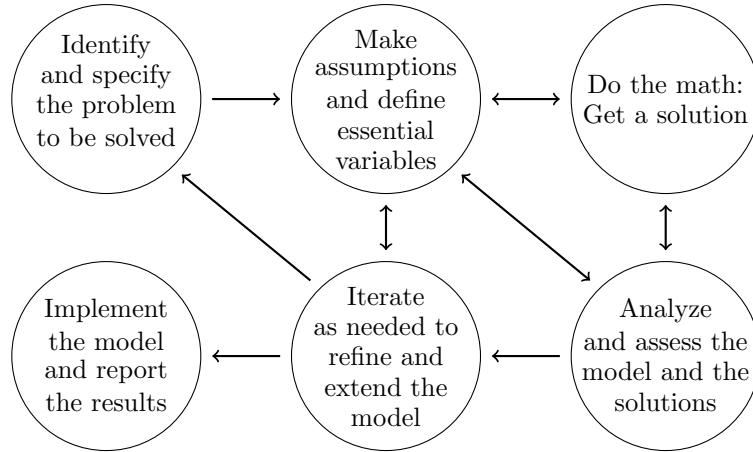


Figure 1.3: The Math Modeling Process [@GAIMME, p. 13]

The focus on trying to better understand real-world phenomena distinguishes mathematical modeling from application problems or word problems. Actual real-world phenomena are distinctly messy and arriving at a final solution requires interpretation and assumptions. Moreover, an individual engaged in modeling has to confront the challenge of determining whether the question, mathematical model, and data are well-aligned. Teachers face the additional challenge of ensuring that the situations they create for their students are appropriate for their students. For instance, an elementary school classroom might plant some seeds in the soil. The teacher would then assist the students in identifying specific questions that can be answered quantitatively about the seeds and plants, guiding them towards a set of questions and data that the children could collect, analyze, and interpret. A secondary teacher teaching about exponential functions may introduce the class to the concept of population growth and guide the students to questions that can be answered quantitatively with exponential models. In both these examples, the instructor introduces the students to a real-world situation and then guides them to questions that facilitate the students’ development of the mathematical modeling they want to discuss.

Another key aspect of the modeling process involves determining the relevance of various quantities in the situation and how to describe them with variables. This cyclical process involves identifying variables, assigning them labels and then assessing how they fit in with other variables. This process allows the modeler to create an idealized version of the original problem in order to create some type of solution. It

is important during the process to justify each assumption made and to clearly label each variable, along with its appropriate units. It is through this justification process that the modeler can communicate the applicability and interpretation of the results from the model to the consumer.

As the modeler defines the variables and the relationships between them, she can use the mathematical problem solving techniques to ‘do the math’ and come up with possible solutions to the problems posed. This section of ‘doing the math’ reflects the word problems of most math textbooks since the word problems rarely have extraneous information and have obvious variables defined with transparent relationships between them.

Similar to the ‘looking back’ phase of the mathematical problem solving process, the modeling process includes a stage during which the modeler steps back and assesses the model. During this stage he or she analyzes and tests the model and solution, examining the model created and determining the appropriateness and accuracy of the solution produced. This process often results in a revision to the original assumptions and adjustments to the model with improved variables and relationships between the variables.

Since the mathematical modeling process deals with messy real-world phenomena, an iterative process of reflection that results in the refinement and, where appropriate, extension, of the model is a key component that almost always appears in a modeling cycle. This process may run parallel to or between any of the other phases of the modeling process.

When the model gets to a state that satisfies the modeler in terms of the model justifications and appropriateness of the solutions, the implementation of the model and the reporting of the results occurs. The presentation of the results must include the justification for the model, a description of the various assumptions made to produce the model, and any limitations the model has in terms of the accuracy of the results. This presentation must be done in a way that is comprehensible to the desired audience. Because modeling deals with real-world situations, the solution is unlikely to be clear or definitive, but will instead be approximate and estimated.

1.2.3 Communicating Mathematically

Mathematical communication broadly relates to the ways and methods of representing, justifying, and interpreting mathematics. More specifically, the practice of mathematical communication “encompasses both listening and reading (comprehension) and both speaking and writing (expression)... and may also include representation of mathematical ideas in nonlinguistic ways” [Sammons, 2018, p. 7]. In order to better understand the practice of communicating mathematically, we examine six overlapping concepts that are at the core of this practice (see Figure 1.4).

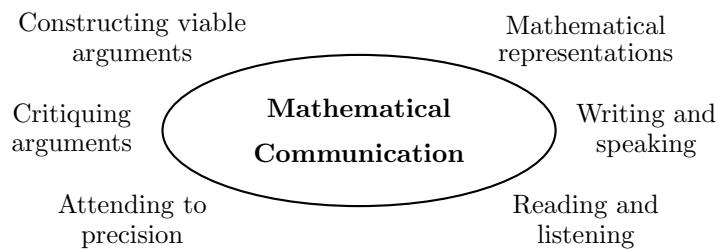


Figure 1.4: Components of Mathematical Communication

All types of communication require a language. Within a language there exists a shared meaning behind the words and symbols used when communicating with a particular language. Mathematics communication, in this sense, operates much as a language does. However, unlike a language such as English, mathematics is represented using a mixture of words that have very precise definitions and symbols that compress many different ideas and meanings into a very small amount of space (linguists call mathematics a “dense” language because of this property). Moreover, somewhat uniquely to mathematics, the meaning of symbols varies widely depending upon the context of the communication. Consider variables, which are usually represented using any number of symbols. While certain types of symbols will cue a well-informed mathematics student

that a variable has some implied meaning in terms of what type of number system to which it belongs, even in this case variables in different circumstances serve different roles. Sometimes the letter represents a fixed value (e.g., $x = 4$) and other times it represents a range of values that satisfy a particular condition (e.g., $x \leq 4$). Sometimes it indexes a set, while other times it represents a function. The differences between these uses are often subtle, making explicit training in these nuances mathematics critical.

Once the language of mathematics has been developed, the grammar is created. In mathematics, logic, argument, and proof form the core patterns of organization and presentation. Together, logic, argument, and proof boil down to the ability to construct viable arguments and critique the arguments of others using mathematical evidence.

Part of mathematical communication is learning how to consume and understand the ideas of others. In mathematics this usually occurs by reading or listening to the arguments of others. Both reading and listening to mathematics require a great deal of practice and are skills students must hone over many courses. In particular, reading mathematics requires a great deal of practice and discipline. While a novel can usually be consumed at a steady pace without much explicit effort by the reader, written mathematical ideas are often presented symbolically, with the reader left to fill in certain logical steps. Thus, the act of reading mathematics becomes one of translation of the symbolic work into a language he or she understands, followed by a effort to connect current parts of the argument to previously made points, and the absorption and evaluation of those ideas based on previous knowledge. As a result, reading mathematical texts and arguments requires a big investment in time. Sometimes it can take hours just to understand one line of a mathematical text! This slow, methodical way of reading is challenging to students of mathematics, but is a critical skill to develop as it is core to the practice of doing mathematics.

Previous paragraphs have described the difficulties and process of consuming mathematical texts. However, it is also important that students learn to produce their own mathematical communications in both oral and written forms. In communicating to others, it is essential that precise mathematical words are used, rather than slang, in order to reduce the confusion of potential readers. It is also useful to employ a variety of mathematical representations, including graphs, sketches, and diagrams to facilitate the communication of mathematical ideas.

As teachers of mathematics, we must continually emphasize the development of this practice of communicating mathematically by de-emphasizing simple answers to mathematical problems, teaching students how to read and write mathematics (not just about mathematics), and to promote classroom discourse to provide opportunities for students to experience different types of mathematical communication.

1.2.4 Understanding Mathematical Structures

While the definition of mathematics is not uniform or agreed on, most agree that mathematics is inherently about the study of underlying structures and logical reasoning. For example, the Encyclopædia Britannica [2019] defines mathematics as “the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects.” This means that an important practice in the doing of mathematics is to “look for and make use of structure” and to “look for and express regularity in repeated reasoning” [NGA-CCSSO, 2010].

The pursuit and use of structure and regularity appear throughout all of mathematics. Mathematical structure informs every part of mathematics teaching, from instruction on common mathematical procedures to searching for and making connections between various, apparently unrelated, mathematical content areas to provide new insight into a problem. As teachers, we need to point out these mathematical structures to the students and help them learn how to discover the structures and regularity on their own.

1.2.5 Exercises

1. Consider the following mathematical task:

An electricity company charges Kelly \$0.15 per kWh (kilowatt-hour) of electricity used, plus a basic connection charge of \$8.00 per month. Find a function that helps Kelly estimate her monthly electricity

bill for a given number of kWhs. Be explicit about the domain of the function you create.

- a) NCTM emphasizes multiple representation of functions using verbal, tabular, graphical, and symbolic representations. Using this word problem as the verbal representation, express the function in each of the remaining ways.
 - b) How does the ability to move between different representations of functions support students' development of the mathematical communication process standard?
 - c) The problem solving section discussed the fact that a mathematical task is only considered a problem if the person working on the task does not know in advance a solution method that will produce a correct answer. Would you say that for part (a) you were engaged in mathematical problem solving as described in the standards? Why or why not?
 - d) Under what conditions would the task be a problem for students you were teaching?
 - e) Based on the description of mathematical modeling, would you say that finding the function for the task in part (a) represents mathematical modeling? Why or why not?
 - f) A second part of the task asks students to estimate Kelly's electrical bill for the year, given that her monthly kWh usage ranges between 202 and 254 kWh.
 - g) The textbook you got the problem from lists the answer to the extension as \$506.40. What did the author do and what assumptions did she make in order to arrive at that answer? Do you agree with her process and assumptions? Why or why not?
 - h) In arriving at your answer for part (a), would you say that you were engaged in mathematical problem solving? Why or why not?
 - i) If you were using this task as a modeling activity for your students, what criteria would you use to evaluate whether their answer was reasonable?
2. There are five NCTM Process Standards and eight Common Core Standards for Mathematical Practice. While there is significant overlap between the two sets of standards, they are not the same. Read each set of standards. When there is overlap between the two sets of standards, create a map between them. Also identify the ways in which the two sets of standards differ. These regions are not at the heading level. You'll have to actually dig into the blurbs about each standard in their original documents.
 3. A basic theorem that students use from an early age is that the sum of two even integers is even. A typical proof of this theorem might look something like this:

Let a and b be even integers such that $a = 2m$ and $b = 2n$ where m and n are integers. Then $a + b = 2m + 2n = 2(m + n)$. Thus, the sum of two even integers is even.

 - a) The section on mathematical communication emphasizes the reliance of mathematical communication on precise definitions and symbols that compress complex ideas into short phrases. In examining this proof, identify the places where the problem statement or proof rely on a definition or compressed expression.
 - b) Identify the places where the proof writer has left the reader to fill in logical steps or rationales.
 - c) Rewrite the proof of the theorem without the use of any symbols, but with the same degree of precision and generalizability.
 - d) In comparing the symbolic proof to your verbal proof, do you think it is easier to understand (i.e., more like a novel?) or do you think it would still be difficult to read if all mathematics were presented without symbols? Why or why not?
 - e) If you reflect on the six parts of mathematical communication, which parts do you think you are best at? Which parts do you struggle the most with? What is one goal you have for yourself for mathematical communication that you want to develop during this course?

1.3 Mathematics Content Standards in the U.S.

The current standards for mathematics and assessment in the United States are derived from a variety of sources and past initiatives. David Klein [2003] gives a good summary of the major events and influential individuals during the 20th century. The current set of standards are strongly connected to a pair of reports from the 1980's: one by NCTM [1980], *An Agenda for Action*, and one by the U.S. National Commission on Excellence in Education [1983], *A Nation at Risk*. Each provided a different view about what should occur to improve mathematics education in the United States. While these two documents were used as the basis for the 'math wars' of the late 20th century, the biggest difference between them is that the NCTM document focused primarily on standards for teaching methodologies, while *A Nation at Risk* focused on standards of content knowledge.

The NCTM [1980] recommended "that

1. problem solving be the focus of school mathematics in the 1980s;
2. basic skills in mathematics be defined to encompass more than computational facility;
3. mathematics programs take full advantage of the power of calculators and computers at all grade levels;
4. stringent standards of both effectiveness and efficiency be applied to the teaching of mathematics;
5. the success of mathematics programs and student learning be evaluated by a wider range of measures than conventional testing;
6. more mathematics study be required for all students and a flexible curriculum with a greater range of options be designed to accommodate the diverse needs of the student population;
7. mathematics teachers demand of themselves and their colleagues a high level of professionalism;
8. public support for mathematics instruction be raised to a level commensurate with the importance of mathematical understanding to individuals and society" (p. 1).

The National Commission on Excellence in Education [1983] recommended that

- "The teaching of *mathematics* in high school should equip graduates to: (a) understand geometric and algebraic concepts; (b) understand elementary probability and statistics; (c) apply mathematics in everyday situations; and (d) estimate, approximate, measure, and test the accuracy of their calculations. In addition to the traditional sequence of studies available for college-bound students, new, equally demanding mathematics curricula need to be developed for those who do not plan to continue their formal education immediately" (p. 25).
- "Standardized tests of achievement (not to be confused with aptitude tests) should be administered at major transition points from one level of schooling to another and particularly from high school to college or work. The purposes of these tests would be to: (a) certify the student's credentials; (b) identify the need for remedial intervention; and (c) identify the opportunity for advanced or accelerated work. The tests should be administered as part of a nationwide (but not Federal) system of State and local standardized tests. This system should include other diagnostic procedures that assist teachers and students to evaluate student progress" (p. 28).
- "Persons preparing to teach should be required to meet high educational standards, to demonstrate an aptitude for teaching, and to demonstrate competence in an academic discipline. Colleges and universities offering teacher preparation programs should be judged by how well their graduates meet these criteria" (p. 30).
- "Substantial nonschool personnel resources should be employed to help solve the immediate problem of the shortage of mathematics and science teachers. Qualified individuals including recent graduates with mathematics and science degrees, graduate students, and industrial and retired scientists could, with appropriate preparation, immediately begin teaching in these fields. A number of our leading science centers have the capacity to begin educating and retraining teachers immediately. Other areas of critical teacher need, such as English, must also be addressed" (p. 31).

NCTM followed its 1980 report with the publication of the *Curriculum and Evaluation Standards for School Mathematics* [NCTM, 1989]. This document focused on the standards for teaching methodologies and mathematical practices. Content standards were sketched out, with overviews of the recommended content knowledge for students in 4-year grade bands. NCTM expanded this work with additional texts focusing on teaching standards and assessment: *Professional Teaching Standards* [1991] and *Assessment Standards* [1995]. In 2000, NCTM released an updated version of the 1989 standards with the *Principles and Standards for School Mathematics* [NCTM, 2000]. During this same time, many states developed more specific content standards reflecting the guidelines recommended by the 1989 document [Raimi and Braden, 1998]. Although the NCTM document did not make content recommendations by specific grade levels, many of the state standards still resided at the grade-band level [Reys and Lappan, 2007, p. 677].

1.3.1 Common Core State Standards-Math

In 2002, Congress passed the No Child Left Behind Legislation. This bill required that states determine measurable content standards for each grade level and develop assessments based on these standards to be given to all students at specific grade levels (generally fourth, eighth, and twelfth grade) in order to receive federal school funding. Since each state developed their own standards, the level and specificity of these standards varied greatly between States [Reys and Lappan, 2007].

It was in this environment that the National Governors Association Center for Best Practices and the Council of Chief State School Officers launched an effort in 2009 to develop “a common core of internationally benchmarked standards in math and language arts for grades K-12” [NGA-CCSSO, 2010]. These standards emphasize content knowledge, while also encouraging some of the teaching methodologies proposed by the NCTM documents.

In order to help teachers better understand how the topics in this book relate to the actual teaching and learning of their students, we include related content standards from the Common Core State Standards for Mathematics throughout the text. While not all states have adopted the Common Core State Standards for their school systems, these standards are a good representation of what is expected from students at different stages in their mathematical education.

Related Content Standards

- (8.F.1) Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.
- (HSF.IF.7) Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Standards that align with the content of each section are provided in blocks with the title “Related Content Standards.” One such box is given above as an example. The sequence of numbers and letters proceeding each standard is its standardized abbreviation. The first part of the abbreviation identifies the grade band. Since the Common Core State Standards for high school do not have a recommended grade level for each of the standards, they are denoted by “HS” and the general area of mathematics. The second part of the abbreviation refers to the general domain of mathematics, and the third part refers to the location of the standard in that grade band and domain.

1.3.2 NCTM-CAEP Standards

Another requirement of the No Child Left Behind legislation was that states had to create standards for teachers to receive certification. These standards for certification needed to be measured with a blend of standardized assessments, portfolios, and coursework. As a result, certification in secondary mathematics in most states requires a major in mathematics (or its equivalent), the teaching of and about standards during certification coursework, and achievement of certain scores on a content specific standardized test (such as the PRAXIS). In addition, the NCTM partnered with the Council for the Accreditation of Educator Preparation

(CAEP) to develop standards for what beginning secondary mathematics teachers should know and be able to do in the teaching of mathematics [NCTM, 2020].

The NCTM-CAEP content standards are built upon the standards developed by the Conference Board of Mathematical Sciences in [2001] and [2012]. The content of this book builds upon the contents of the first standard of “Knowing and Understanding Mathematics”. In particular:

Candidates demonstrate and apply, with the incorporation of mathematics technology, conceptual understanding, procedural fluency, and factual knowledge of major mathematical domains: Number; Algebra and Functions; Statistics and Probability; Geometry, Trigonometry, and Measurement; Calculus; and Discrete Mathematics.

We list the details of the NCTM-CAEP content standards below and discuss how this book connects to each of these standards.

Essential Concepts in Number. Candidates demonstrate and apply conceptual understanding, procedural fluency, and factual knowledge of number including flexibly applying procedures, using real and rational numbers in contexts, developing solution strategies, and evaluating the correctness of conclusions. Major mathematical concepts in Number include number theory; ratio, rate, and proportion; and structure, relationships, operations, and representations.

Number systems are primarily covered in Chapters 4, 6, and 7. We introduce initial concepts of the number systems in Chapter 4, focusing on operations and the basic properties of common number systems, including whole numbers, integers, rational, real, and complex numbers. Chapters 6 and 7 expand these ideas through the added structure of rings and fields.

Essential Concepts in Algebra and Functions. Candidates demonstrate and apply understandings of major mathematics concepts, procedures, knowledge, and applications of algebra and functions including how mathematics can be used systematically to represent patterns and relationships including proportional reasoning, to analyze change, and to model everyday events and problems of life and society. Essential Concepts in Algebra and Functions include algebra that connects mathematical structure to symbolic, graphical, and tabular descriptions; connecting algebra to functions; and developing families of functions as a fundamental concept of mathematics. Additional Concepts should include algebra from a more theoretical approach including relationship between structures (e.g., groups, rings, and fields) as well as formal structures for number systems and numerical and symbolic calculations.

Many of these concepts are spread throughout the text, as algebra and functions are fundamental to the secondary curriculum. That said, Chapters 5, 6, 7, and 8 specifically focus on topics of particular importance in understanding algebra and functions. In particular Chapter 8 combines the themes of the earlier chapters together using real-valued functions.

Essential Concepts in Calculus. Candidates demonstrate and apply understandings of major mathematics concepts, procedures, knowledge, and applications of calculus including the mathematical study of the calculation of instantaneous rates of change and the summation of infinitely many small factors to determine some whole. Essential Concepts in Calculus include limits; continuity; the Fundamental Theorem of Calculus; and the meaning and techniques of differentiation and integration.

The essential concepts in calculus are currently excluded from this text with the understanding that most pre-service teachers are already required to take at least two semesters worth of calculus courses that focus on this content standard.

Essential Concepts in Statistics and Probability. Candidates demonstrate and apply understandings of statistical thinking and the major concepts, procedures, knowledge, and applications of statistics and probability including how statistical problem solving and decision making depend on understanding, explaining, and quantifying the variability in a set of data to make decisions. They understand the role of randomization and chance in determining the

probability of events. Essential Concepts in Statistics and Probability include quantitative literacy; visualizing and summarizing data; statistical inference; probability; and applied problems.

The essential concepts of statistics and probability are the focus of Part IV of the text, Data Analysis. In that section we focus on statistics and a study of variability, looking at different ways to measure, communicate, and understand variability in the context of statistical problems.

Essential Concepts in Geometry, Trigonometry, and Measurement. Candidates demonstrate and apply understandings of major mathematics concepts, procedures, knowledge, and applications of geometry including using visual representations for numerical functions and relations, data and statistics, and networks, to provide a lens for solving problems in the physical world. Essential Concepts in Geometry, Trigonometry, and Measurement include transformations; geometric arguments; reasoning and proof; applied problems; and non-Euclidean geometries.

Part III on Geometry looks at the subject of Geometry from the perspectives of constructional, transformational, analytic, and algebraic. Each perspective helps us to better understand the essential concepts in geometry, trigonometry, and measurement, along with the interactions between geometry, algebra, functions, and number systems.

1.3.3 Exercises

1. Reflect on your K-12 mathematics education. Based on the brief history described, what documents were being used to guide your curriculum?
2. Those who study mathematics curriculum describe the changing focus of what is written in school standards for mathematics as a pendulum. Over time, the pendulum swings between a focus on procedural fluency such as *A Nation at Risk* [1983] describes and the more process oriented standards as outlined by NCTM (1989).
 - a) How does the Common Core State Standards attempt to unify the two extremes of the pendulum swing?
 - b) How does treating the process standards as separate from the content standards provide an opportunity to continue to let the pendulum swing?
 - c) Regardless of which documents were currently in vogue while you were in school, do you think your mathematics education was more procedurally focused or more process standard focused?
 - d) Is how you were taught what you hope your own teaching will be like? Why or why not?
 - e) K-12 mathematics students often think that mathematics is a set of unrelated procedures that they have to memorize how to do. In contrast, people who do mathematics for a living think mathematics is a conceptual and logical system where everything fits together. As you currently understand them, do you think the Common Core State Standards could support students in moving from thinking about mathematics as a set of unrelated procedures to a more conceptual system? Why or why not?

Chapter 2

Set Theory

In the mid-1800s the field of mathematics went through a major shift that ended up changing the very definition of mathematics. In 1847, George Boole wrote in the introduction to *The Mathematical Analysis of Logic* that up to that point, “the abstractions of the modern Analysis, not less than the ostensive diagrams of the ancient Geometry, have encouraged the notion, that Mathematics are essentially, as well as actually, the Science of Magnitude” [Boole, 1847]. Instead, Boole proposed a new definition, suggesting

We might justly assign it as the definitive character of a true Calculus, that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation... It is upon the foundation of this general principle, that I purpose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its object and in its instruments it must at present stand alone.



Figure 2.1: George Boole

At the time that Boole wrote the above passage, mathematics as a field shifted from the study of quantities to the study of abstract structures based on logic and set theory. This change in the definition of mathematics freed up those who studied it to move beyond structures tied to physical interpretations and applications and move to a more abstract field. The abstractification of mathematics was a powerful moment, resulting in the

development of fields such as quantum mechanics, relativity, cryptography, statistics.

In this chapter we will go through some of the basics of set theory needed to understand some of the later material and to develop a common vocabulary and set of notations. We do not offer a deep treatment of set theory as there are many textbooks, specifically in the area of Discrete Math, with a more detailed coverage of it.

2.1 Sets and Subsets

The foundation of modern mathematics is the theory of sets. Informally, sets can be thought of as collections of objects. While **set theory** is not specifically outlined in the content standards of most states, it is mentioned in the Common Core standards. The first reference to set theory occurs in the Introduction to Kindergarten, stating:

Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 - 2 = 5$. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away. [NGA-CCSSO, 2010]

Set theory also makes an appearance in the high school standards related to counting and probability, as knowledge of sets is critical to understanding basic formulas in probability.

Related Content Standards

- (HSS.CP.1) Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”).

We start our exploration of set theory by developing some basic notation and definitions. Cantor [1891] defined sets in the following way.

Unter einer ‘Menge’ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschabung oder unseres Denkens (welche die ‘Elemente’ von M genannt werden) zu einem Ganzen. (p. 481)

By a ‘Set’ we mean each collection M of certain well-differentiated objects m of our perception or our thinking (which are called the ‘elements’ of M) as a whole. (English translation)

This definition proved to not be precise enough to avoid certain paradoxes.

Example 2.1 (Russell’s Paradox). Let $R = \{x|x \notin x\}$, the set that contains all sets that are not members of themselves. Is R an element of R ? If it is, then it is a set that is an element of itself and so could not be an element of R , by its definition, thus leading to a contradiction. On the other hand, if one assumes that R is not an element of itself, then it satisfies the conditions of being an element of R , also leading to a contradiction. Thus defining the paradox.

Even though Cantor’s definition of a set leads to such paradoxes, we will use it as our working definition, with the detailed definition of set being a primitive notion in Zermelo-Fraenkel set theory. The Zermelo-Fraenkel axioms, combined with the axiom of choice, create the ZFC axioms upon which mathematics is built. Due to the level of abstraction involved in the axioms, we will not include a systematic coverage of the axioms in this text, but will reference them as needed.

Definition 2.1. • We will define a set by the collection of elements which belong to the set.

- If A is a set and a is an object that belongs to A , we say that a is an **element** of A and denote it as $a \in A$.
- If an object a is not an element of a set A , we denote that by $a \notin A$.
- Let A and B be sets. We say that $A = B$ if and only if every element of A is an element of B and every element of B is an element of A . (In the ZFC axioms, this is referred to as the axiom of extensionality.)

While the elements of a set are often written in a specific order, i.e. $\{1, 2, 3\}$, the members of a set have no particular order and the same set could be written as $\{3, 1, 2, 1\}$, with the repetition being irrelevant since the set is defined by its elements.

We also need to note that a and $\{a\}$ are two different mathematical objects (one is the element a and the other is the set that contains the element a). So $A = \{a, \{a\}\}$ defines a set with two distinct elements: a and $\{a\}$. Likewise, if $B = \{1, 2, 3, \{4\}, 5\}$, then $4 \notin B$ but $\{4\} \in B$. So the symbol 4 is not an element of B , but the symbol $\{4\}$ (the set that contains the element 4) does belong to B . These nuances mean that we have to be very careful with our notation and how we read the mathematical symbols.

Another way to describe sets is using set-builder notation. In this notation, we describe our new set using larger sets and a set of restrictions on (or description of) which objects in the larger set we are choosing to include. For instance, we can define C to be the set of all real numbers (denoted \mathbb{R}) greater than or equal to 3 . In this case, the larger set is the set of real numbers and the condition that the numbers are greater than or equal to three is the restriction or description. Of course we do not want to have to keep writing so much down, so we create a short-hand way of defining this set:

$$C = \{x \in \mathbb{R} | x \geq 3\}$$

where the $|$ separates the description of the larger set and the description of the restrictions. This notation is read “ C is the set of all of the elements x in the real numbers such that x is greater than or equal to 3 .” This set-builder notation is particularly useful when describing sets with more than just a few elements.

When sets are subsets of the real numbers, we can also describe them using interval notation. For the set C defined above, we can also write

$$C = [3, \infty)$$

where the closed bracket, $[$, denotes that the endpoint is contained in the set, while an open bracket, $($, denotes that the endpoint is not contained in the set. For real numbers a and b , with $a < b$, we have the following options as intervals from a to b :

$$(a, b) \quad (a, b] \quad [a, b) \quad [a, b].$$

Once we have created notation for sets with a few elements and a large number of elements, we can describe how to denote a set without any elements.

Definition 2.2. The set that does not contain any elements is called the **empty set** and is denoted by \emptyset . Any set that contains at least one element is then called **non-empty**.

In addition to determining how to describe sets, we must have a way to determine if two sets are the same or distinct.

A direct consequence of this definition is that the number of times an element is listed and the order of the elements is irrelevant for equality of sets. So $\{a, b, c, d\} = \{b, a, c, c, d, b\}$ and $\{1, 2\} \neq \{1, 2, 3\}$.

Definition 2.3. Let A and B be sets. We say that A is a subset of B , denoted $A \subseteq B$, if every element of A is also an element of B . If A is not a subset of B , we sometimes denote this by $A \not\subseteq B$. If there are elements of A that are not contained in B and there are elements of B that are not contained in A , then we say that A is a **proper subset** of B and denote it by $A \subset B$.

For example:

$$\{6, 7\} \subseteq \{5, 6, 7, 8\},$$

$$\{x \in \mathbb{R} | x > 5\} \subseteq \{x \in \mathbb{R} | x \geq 2\}, \text{ and}$$

$$\{a, b, c\} \subseteq \{a, b, c\}.$$

Notice that the first two examples are proper subsets.

It is important to distinguish between the phrases ‘element of’ and ‘contained in’ when discussing sets. If $A = \{a, b, c\}$, then we say that a is an element of A , while $\{a\}$ is contained in A .

Since the empty set has no elements, it is by default a subset of every set. Similarly, every set is a subset of itself.

Example 2.2. If $A = \{a, b, c\}$ we can describe all of the subsets of A ,

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

This set of subsets of A is called the **power set** of A and denoted by $\mathcal{P}(A)$.

We often need to prove that two sets are equal to one another, and we do not have the elements of the set listed out. In these situations the following theorem often proves useful.

Theorem 2.1. Let A and B be sets. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proof. Since the statement of the theorem is an ‘if and only if’ statement, there are actually two components that need to be proven. We need to first prove that if $A = B$ then $A \subseteq B$ and $B \subseteq A$. The second statement that needs proving is the converse: having $A \subseteq B$ and $B \subseteq A$ implies that $A = B$. We will complete both arguments using the corresponding definitions.

Assume that $A = B$. Then the definition of set equality states that every element of A is an element of B . This means that A meets the definition of a subset of B . That is: $A \subseteq B$. In addition, the assumption that $A = B$ gives us that every element of B is an element of A . As a result, $B \subseteq A$. Thus

$$A = B \Rightarrow A \subseteq B \text{ and } B \subseteq A.$$

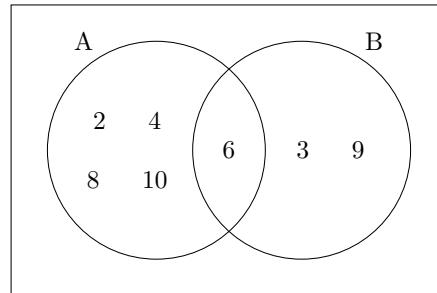
To prove the converse statement, we assume that $A \subseteq B$ and $B \subseteq A$. Thus, we have that every element of A is also an element of B and every element of B is also an element of A . This is the definition of set equality and so we have that $A = B$.

We have therefore proven both implications, and thus the theorem is proven. \square

2.1.1 Venn Diagrams

In order to better understand relationships between subsets of a larger set, it is sometimes helpful to represent the relationships with a **Venn diagram**. In these diagrams, we use circle-like figures to represent sets, with everything inside the circle being inside of the set and everything outside of the circle being outside of the set.

Example 2.3. Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9\}$. Then these sets could be represented by a Venn diagram such as the one below.



As previously discussed, sets do not have to just be composed of numbers. Moreover, sets can have many different relationships between them, as seen in Figure 2.2. If a set is a subset of a second set, then they are visualized as one region inside of the other region. If there are elements shared between two sets, but are not known to have a subset relationship, then they are visualized as overlapping regions. If no elements of the two sets are the same, then they are represented as two non-overlapping regions. In the next section we define vocabulary to describe these different types of relationships and how they are combined in different ways.

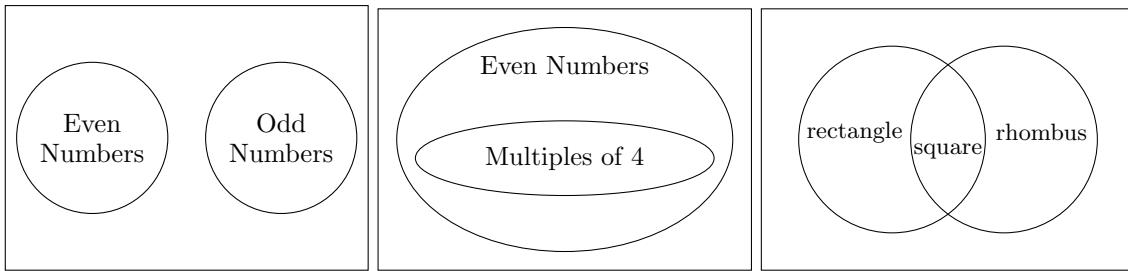


Figure 2.2: Sample Venn diagrams

2.1.2 List of Sets of Numbers

In order to help with terminology we will provide some notation for some of the basic sets used in the text.

We define the natural numbers as

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}.$$

This is distinct from some definitions of the natural numbers that do not include the number 0. When a text defines the natural numbers without the 0, it also defines the whole numbers to be our definition of the natural numbers. We are including 0 due to the method in which we define the natural numbers in Chapter 4.

We label the integers as

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\},$$

which we will define in detail in Section 4.1.

We label the set of rational numbers as \mathbb{Q} and the real numbers as \mathbb{R} , the detailed definitions of which are in Sections 4.3 and 4.5, respectively.

The complex numbers, \mathbb{C} , are defined and studied in Section 4.6.

For the integers, rationals and reals, we define \mathbb{Z}^+ , \mathbb{Q}^+ , and \mathbb{R}^+ to be the positive elements of the set, those that are greater than zero.

We visualize the nested nature of these number systems in the Venn diagram in Figure 2.3.

2.1.3 Exercises

1. Answer the following as true or false. If false, explain why the statement is not true.
 - a. $\emptyset \subseteq \{f, u, n, t, i, m, e, s\}$
 - b. $\{a, b\} \in \{a, b, c\}$
 - c. $\{0\} = \emptyset$
 - d. $\{f, u, n, f, u, n\} = \{f, u, n\}$
 - e. $\{0, 0\} \subseteq \{0, 0, 1, 1, 2, 2\}$
2. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{1, a, 2, b, 3, c\}$. Answer the following as true or false. If false, explain why the statement is not true.

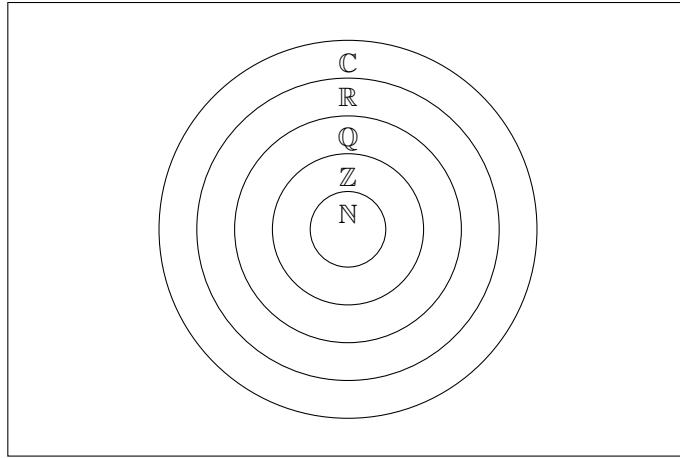


Figure 2.3: Sample Venn diagrams

- a) $A \subseteq A$
- b) $A \supseteq C$
- c) $A = B$
- d) $A \not\subseteq B$
3. Recall that for set A , $\mathcal{P}(A)$ is the power set of A .
- Let $A = \{a, b\}$. Write the set of $\mathcal{P}(A)$ by listing its elements.
 - Let A be a set with n elements in it.
 - How many elements are in $\mathcal{P}(A)$ if $n = 3$?
 - How many elements are in $\mathcal{P}(A)$ if $n = 4$?
 - How many elements are in $\mathcal{P}(A)$ if n is an unknown natural number?
4. Some textbooks describe a set as “a well-defined collection of objects” which means that the inclusion criteria that helps you decide what should be in the set is clearly specified. Classify each of the following sets as well-defined or not. If you identify a set as not well-defined, give two possible well-defined sets that would satisfy the original description.
- $\{x|x > 0\}$
 - The set of students at The University of Awesome who are currently enrolled in a class that has a 100-level designation.
 - $\{x|x \text{ is a letter in my first name}\}$
 - The set of my friends.
 - $A_n = \{x \in \mathbb{Z}|n \leq x \leq n + 3\}$
5. Let $A = \{1, 2, 3, 4\}$, $B = \{7, 8, 9\}$, and $C = \{3, 4, 5, 6, 7, 8, 9\}$. Using the most appropriate template of the Venn diagrams shown in Figure 2.2, fill in the regions with the elements from:
- Sets A and B .
 - Sets A and C .
 - Sets B and C .
 - Sets A , B , and C (you will have to make a new Venn diagram template).

2.2 Algebra of Sets

Now that we understand the basic definitions involving sets, we examine set operations. These will allow us to create new sets from given sets.

Definition 2.4. Let A and B be sets. We define the union of A and B , denoted $A \cup B$, to be the set of elements that are either in A or B , or possibly both. We define the intersection of A and B , $A \cap B$, to be the set of elements that are in both A and B . We say that A and B are **disjoint** if $A \cap B = \emptyset$.

The shaded regions in Figure 2.4 illustrate each set relationship in terms of general sets A and B . We can also write the union and intersection of sets in terms of set-builder notation:

$$A \cup B = \{x | x \in A \text{ or } x \in B\} \quad \text{and} \quad A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

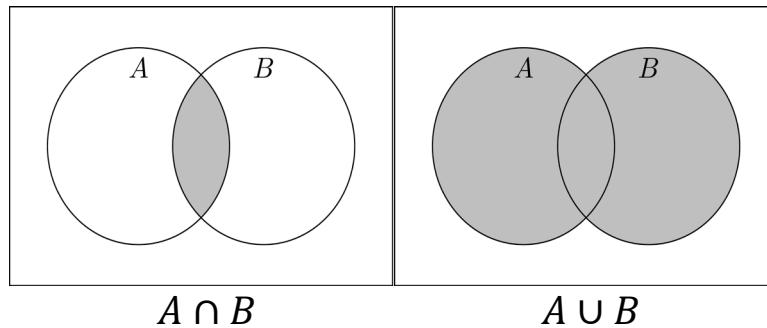


Figure 2.4: The union and intersection of two sets

Let's look at a couple of examples to better understand these unions and intersections.

Example 2.4. Let $A = \{a, b, c, d, e\}$ and $B = \{d, e, f, g\}$. Then

$$A \cup B = \{a, b, c, d, e, f, g\}$$

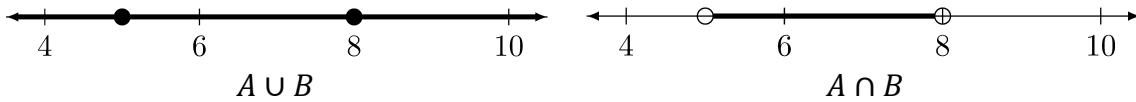
Example 2.5. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{2, 4, 6, 8\}$. Then

$$A \cup B = A \quad \text{and} \quad A \cap B = B.$$

Example 2.6. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Then

$$A \cup B = \{a, b, c, 1, 2, 3\} \quad \text{and} \quad A \cap B = \emptyset.$$

Example 2.7. Let $A = \{x \in \mathbb{R} | x > 5\}$ and $B = \{x \in \mathbb{R} | x < 8\}$. Then $A \cup B$ would be all real numbers, since any real number is either less than 8 or greater than 5. And $A \cap B$ would be the real numbers between 5 and 8. These can also be represented on number lines, where the shaded lines and filled dots are included in the set, with open dots and unshaded lines not being included in the set.



The following theorem identifies properties of the union as an operation on sets that follow from the definition.

Theorem 2.2. Let A , B , and C be sets. Then we have the following:

1. $A \cup \emptyset = A$
2. $A \cup A = A$
3. $A \cup B = B \cup A$
4. $A \cup (B \cup C) = (A \cup B) \cup C$
5. $A \subseteq A \cup B$
6. If $A \subseteq B$, then $A \cup B = B$

Proof. The first two statements follow directly from the definition of the union. The empty set has no elements, making its union with A equal to A . Because A and A have the same elements, the union must also be A .

The third statement suggests that the set operation “union” is commutative, in that the order of the operation does not matter. The proof follows from properties of symbolic logic in relation to the use of the “or” statement in the definition of the union.

The fourth statement in the theorem is used to expand the definition of union, which is only defined for a pair of sets, to more than two sets. A further consequence of statement four is that the method of pairing sets under the operation of union is irrelevant. We show this by proving the two statements $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ and $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ (see Theorem 2.1). In order to show the first containment, we let $x \in A \cup (B \cup C)$ be a generic element. Then by the definition of the union of sets, $x \in A$ or $x \in (B \cup C)$. We will then break this statement into two cases.

Case 1. If $x \in A$, then by the definition of unions $x \in (A \cup B)$. Then using the definitions of unions again, $x \in ((A \cup B) \cup C)$.

Case 2. If $x \in (B \cup C)$, then $x \in B$ or $x \in C$. If $x \in B$, then $x \in (A \cup B)$ and also $x \in ((A \cup B) \cup C)$. If $x \in C$, then $x \in ((A \cup B) \cup C)$.

So in either case, $x \in A \cup (B \cup C)$ implies that $x \in (A \cup B) \cup C$. So $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. The proof of the reverse containment is nearly identical.

We leave the last two statements as exercises. □

Similar to the union, the following theorem demonstrates how the intersection of sets works as an operation on sets. The proofs for each statement are similar to that of the unions so we will leave the proof of this theorem as an exercise.

Theorem 2.3. Let A , B , and C be sets. Then we have the following

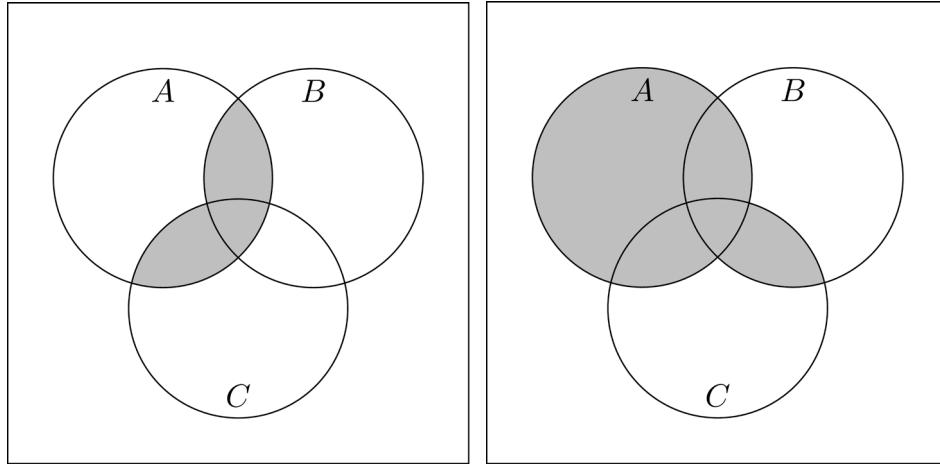
1. $A \cap \emptyset = \emptyset$
2. $A \cap A = A$
3. $A \cap B = B \cap A$
4. $A \cap (B \cap C) = (A \cap B) \cap C$
5. If $A \subseteq B$, then $A \cap B = A$

Now that we know how unions and intersections behave by themselves, we examine how they interact with each other.

Theorem 2.4. Let A , B , and C be sets. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{and} \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

These relationships are diagrammed in Figure 2.5.



$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Figure 2.5: Venn Diagrams for Set Distributions

Proof. As each of these are proofs of equality of sets, we will need to complete the proofs showing that each set is contained in the other (see Theorem 2.1). We prove Part 1 and leave Part 2 as an exercise.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$. This yields two cases: $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. This is equivalent to $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

If $x \in (A \cap B) \cup (A \cap C)$, then we have to examine two cases: $x \in (A \cap B)$ or $x \in (A \cap C)$.

Case 1. If $x \in (A \cap B)$, then $x \in A$ and $x \in B$. Since $x \in B$, we know that $x \in (B \cup C)$. Thus $x \in A \cap (B \cup C)$.

Case 2. If $x \in (A \cap C)$, then $x \in A$ and $x \in C$. Since $x \in C$, we know that $x \in (B \cup C)$. Thus $x \in A \cap (B \cup C)$.

These results imply that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Thus, the Part 1 of the theorem is proven. \square

2.2.1 Set Complements

When working on a problem, we usually describe several sets, with an underlying assumption that the sets referenced contain elements from some common, larger set. We call a set that contains all of the elements considered for a particular situation a **universal set**.

Definition 2.5. For every set A that is a subset of the universal set U , we define the **complement** of A to be the set of elements in the universal set that are not in A , denoted

$$A^c = \{x \in U | x \notin A\}.$$

For a given set A in universal set U , the complement of a set identifies everything that is in the universal set except for things in set A . This is often useful, however there are times when it is important to consider the elements that are in one set, but not in another, without reference to the universal set.

Definition 2.6. Let A and B be sets. The complement of A with respect to B , also called the **set difference** of B and A , is defined as

$$B \setminus A = B \cap A^c = \{x \in B | x \notin A\}.$$

Let's revisit our previous examples to understand this idea of a set difference.

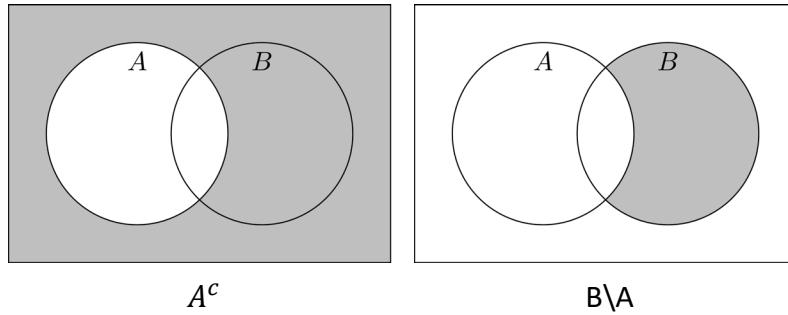


Figure 2.6: Set Complement and Set Difference

Example 2.8. Let $A = \{a, b, c, d, e\}$ and $B = \{d, e, f, g\}$. Then

$$A \setminus B = \{a, b, c\} \quad \text{and} \quad B \setminus A = \{f, g\}.$$

Example 2.9. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{2, 4, 6, 8\}$. Then

$$A \setminus B = \{1, 3, 5, 7, 9\} \quad \text{and} \quad B \setminus A = \emptyset.$$

Example 2.10. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Then

$$A \setminus B = A \quad \text{and} \quad B \setminus A = B.$$

2.2.2 De Morgan's Laws

In the same year as the seminal work of Boole [1847] that started mathematics as a theoretical discipline, Augustus De Morgan published a foundational work in logic [De Morgan, 1847]. In this book, De Morgan defines and describes symbolic mathematical logic. His work has become the foundation for our current mathematical system. One of the key components of this work examines the complements of intersections and unions [De Morgan, 1847, p. 69].

Theorem 2.5 (De Morgan's laws). *If A and B are subsets of the same universal set, then*

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c.$$

Proof. Because we are proving that two sets are equal we need to prove that the sets are subsets of each other. Here we will prove that $(A \cap B)^c = A^c \cup B^c$ and leave the other proof as an exercise.

(Proof that $(A \cap B)^c \subseteq A^c \cup B^c$):

Let $x \in (A \cap B)^c$. So x is not in $A \cap B$. x is not in both A and B .

Case 1. If $x \in A$, then $x \notin B$. So $x \in B^c$.

Case 2. If $x \notin A$, then $x \in A^c$.

So either way, x is in A^c or x is in B^c ($x \in A^c \cup B^c$). Therefore, $(A \cap B)^c \subseteq A^c \cup B^c$.

(Proof that $A^c \cup B^c \subseteq (A \cap B)^c$):

Let $x \in A^c \cup B^c$.

Case 1. $x \in A^c$. Then $x \notin A$. So $x \notin A \cap B$. So $x \in (A \cap B)^c$.

Case 2. $x \in B^c$. Then $x \notin B$. So $x \notin A \cap B$. So $x \in (A \cap B)^c$.

Therefore, $A^c \cup B^c \subseteq (A \cap B)^c$

Therefore, $(A \cap B)^c = A^c \cup B^c$. □

It is also helpful to understand De Morgan's laws by looking at the corresponding Venn diagrams, shown in Figure 2.7.

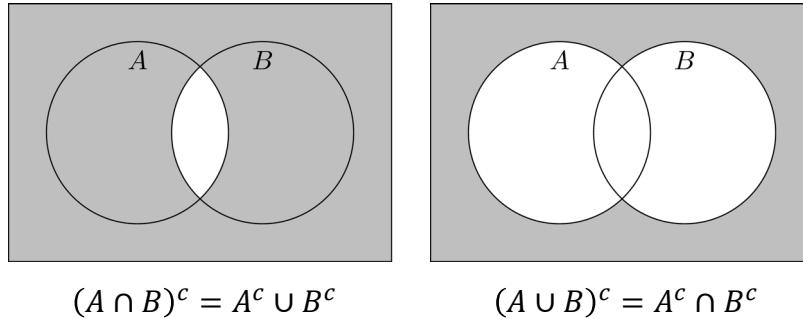


Figure 2.7: Venn diagrams for De Morgan's laws for pairs of sets

2.2.3 Cartesian Products

The previous sections have considered set operations between two sets that exist in the same universal set. Sets can also be combined to create new sets that exist in a universal set that differs from those of the sets used to create it. The collection of set operations that do this allows us to use sets to create multidimensional systems such as ordered pairs.

Definition 2.7. The **Cartesian Product** of two non-empty sets A and B is the set

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

of all ordered pairs of elements where the first coordinate in the pair comes from the set A and the second coordinate is an element of B .

The most common Cartesian product in the secondary mathematics curriculum is real plane, $\mathbb{R} \times \mathbb{R}$, which is often denoted by

$$\mathbb{R}^2 := \{(x, y) | x, y \in \mathbb{R}\}.$$

Related Content Standards

- (5.G.1) Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x -axis and x -coordinate, y -axis and y -coordinate).

The concept of the Cartesian product can be generalized to more than a pair of sets, for example

$$\mathbb{R}^3 := \{(x, y, z) | x, y, z \in \mathbb{R}\}$$

is the three dimensional Cartesian space where each coordinate is a real number.

Theorem 2.6. *It is sometimes helpful to summarize all of the properties of algebra on sets into a single location.*

Let all sets referred to below be subsets of a universal set U .

1. (*Commutative Laws*) For all sets A and B ,

$$(a) A \cup B = B \cup A \quad \text{and} \quad (b) A \cap B = B \cap A$$

2. (*Associative Laws*) For all sets A , B , and C ,

$$(a) (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (b) (A \cap B) \cap C = A \cap (B \cap C)$$

3. (*Distributive Laws*) For all sets A , B , and C ,

$$(a) A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and} \quad (b) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. (*Identity Laws*) For all sets A ,

$$(a) A \cup \emptyset = A \quad \text{and} \quad (b) A \cap U = A$$

5. (*Complement Laws*)

$$(a) A \cup A^c = U \quad \text{and} \quad (b) A \cap A^c = \emptyset$$

6. (*Double Complement Law*) For all sets A ,

$$(A^c)^c = A$$

7. (*Idempotent Laws*) For all sets A ,

$$(a) A \cup A = A \quad \text{and} \quad (b) A \cap A = A$$

8. (*Universal Bound Laws*) For all sets A ,

$$(a) A \cup U = U \quad \text{and} \quad (b) A \cap \emptyset = \emptyset$$

9. (*De Morgan's Laws*) For all sets A and B ,

$$(a) (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (b) (A \cap B)^c = A^c \cup B^c$$

10. (*Absorption Laws*) For all sets A and B ,

$$(a) A \cup (A \cap B) = A \quad \text{and} \quad (b) A \cap (A \cup B) = A$$

11. (*Complements of U and \emptyset*)

$$(a) U^c = \emptyset \quad \text{and} \quad (b) \emptyset^c = U$$

12. (*Set Difference Law*) For all sets A and B ,

$$A \setminus B = A \cap B^c$$

2.2.4 Exercises

1. Middle and high school students often struggle to remember the difference between union and intersection.
 - a) Describe a memory trick to help students remember which symbol goes with which of the two operations.
 - b) Review the definitions of the intersection and union of two sets. What key words separate the two definitions from each other?
 - c) Some students, when first learning the mathematical definition of union, think that the definition excludes objects that are in both sets. These students, when given two sets and asked to find $A \cup B$ will include the items that are in A only and B only. They exclude things in $A \cap B$. What might be the source of this misconception?
 - d) Define a non-numeric universe and two sets, A and B , in your universe such that $A \cap B \neq \emptyset$. Describe, using words, each of the following sets:
 - i. $A \cup B$
 - ii. $A \cap B$
 - iii. A^C
2. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4\}$, and $C = \{3, 6, 9\}$. List the elements of each of the specified sets.
 - a. $A \cap B$
 - b. $A \cup B$
 - c. $A \cup C$
 - d. $(A \cap B) \cup C$
 - e. $A \cap (B \cup C)$
 - f. $A \times B$
 - g. $B \times (A \cap C)$
3. For this exercise, assume that \mathbb{R} is the universal set. For any natural number, n , define $n\mathbb{Z} = \{nx | x \in \mathbb{Z}\}$. Answer the following as true or false. If false, explain why the statement is not true.
 - a. $(2\mathbb{Z})^C = \{2x + 1 | x \in \mathbb{Z}\}$
 - b. $\mathbb{R} \setminus \mathbb{Z} = \mathbb{Z}^C$
 - c. $5\mathbb{Z} \cap \{2x + 1 | x \in \mathbb{Z}\} = 5\mathbb{Z}$
 - d. $5\mathbb{Z} \cap 4\mathbb{Z} = 20\mathbb{Z}$
 - e. $2\mathbb{Z} \setminus (4\mathbb{Z} \cup 6\mathbb{Z}) = \emptyset$
 - f. $3\mathbb{Z} \setminus 2\mathbb{Z} = \{3(2x - 1) | x \in \mathbb{Z} \text{ and } x \geq 0\}$
4. Let A and B be sets. Prove that $A \subseteq A \cup B$.
5. Let A and B be sets. Prove that if $A \subseteq B$, then $A \cup B = B$.
6. Prove Theorem 2.3.
7. Prove Part 2 of Theorem 2.4.
8. Prove that for sets A and B , $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$
9. Write $(A \setminus B) \cup (B \setminus A)$ in terms of just unions, intersections, and complements, then simplify your expression.
10. Often when doing mathematics, the set you are working with or within is left unstated.
 - a. Under what conditions is it important to be explicit with students about the set you are working within?
 - b. What set do you assume you are working in when:
 1. You are figuring out how much something will cost?
 2. You are figuring out what proportion of a pizza to give everyone?
 3. You are determining what the temperature will be if it is predicted to drop 20 degrees overnight?

11. Construct an algebraic proof for the given statement. Cite a property from Theorem 2.6 for every step.

For all sets A and B ,

$$A \cup (B \setminus A) = A \cup B$$

2.3 Collections of Sets

Now that we know how to combine pairs of sets, we can inductively define unions and intersections for a finite or infinite number of sets.

When we are dealing with more than one or two related but distinct sets we often use another set as an **index set** in order to more easily describe and distinguish the sets in the collection.

Example 2.11. Let the natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$, be an indexing set. For each natural number, n , we define

$$A_n = [n, n+1),$$

which is an interval in the real numbers. In this situation, we have

$$A_1 = [1, 2), A_2 = [2, 3), A_3 = [3, 4), \dots$$

In this case, the use of \mathbb{N} tells us that we have an infinite set of sets. We use the indexing set to express the form of a general interval. When we assign an index value (such as $n = 3$), we are effectively identifying a particular interval in the sequence of intervals.

Using indexing sets, we can then define the union and intersection of a collection of sets.

Definition 2.8. Let S be an indexing set and let $\{A_i\}_{i \in S}$ be an indexed non-empty family of sets. Then we define the union and intersection of the family of sets as

$$\bigcup_{i \in S} A_i = \{x | x \in A_i \text{ for some } i \in S\} \quad \text{and} \quad \bigcap_{i \in S} A_i = \{x | x \in A_i \text{ for all } i \in S\}.$$

Example 2.12. Continuing with our previous example where $A_n = [n, n+1)$ for each $n \in \mathbb{N}$, we can write

$$\bigcup_{i \in \{1, 2, 3\}} A_i = [1, 4) \quad \text{and} \quad \bigcap_{i \in \{1, 2, 3\}} A_i = \emptyset$$

since the sets A_1 , A_2 , A_3 , and A_4 do not overlap. If we extend this to the entire indexing set, then

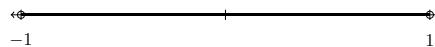
$$\bigcup_{i \in \mathbb{N}} A_i = [1, \infty) \quad \text{and} \quad \bigcap_{i \in \mathbb{N}} A_i = \emptyset.$$

An indexed collection of sets $\{A_i\}_{i \in S}$ is called **mutually disjoint** if, for any $i, j \in S$ with $i \neq j$, $A_i \cap A_j = \emptyset$. The sets in the previous example are mutually disjoint since $[n, n+1) \cap [m, m+1) = \emptyset$ if $m \neq n$.

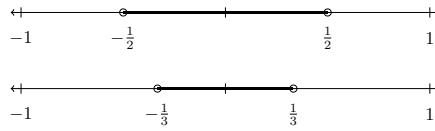
Example 2.13. For each positive integer n , ($n \in \mathbb{N}$), let

$$S_n = \left\{ x \in \mathbb{R} \mid \frac{-1}{n} < x < \frac{1}{n} \right\}.$$

$$S_1 = (-1, 1)$$



$$S_2 = \left(\frac{-1}{2}, \frac{1}{2} \right)$$



$$S_3 = \left(\frac{-1}{3}, \frac{1}{3} \right)$$

Then we can see that for any $i < j$, we have that $S_i \cap S_j = S_j$ and $S_i \cup S_j = S_i$.

We can also take the union and intersections over the entire collection of sets.

$$\bigcup_{n \in \mathbb{N}} S_n = (-1, 1) \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} S_n = \{0\}.$$

The above example is also an example of what is called a **nested** collection of sets.

Definition 2.9. An indexed collection of sets, $\{A_i\}_{i \in I}$, with an order on I is called **nested** if either

$$A_i \subseteq A_j \text{ whenever } i < j \quad \text{or} \quad A_i \supseteq A_j \text{ whenever } i < j.$$

When the indexing set is the natural numbers this is often denoted by

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \quad \text{or} \quad A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$

Using the ideas of indexing sets and families of sets, we can generalize De Morgan's Laws to a general collection of sets, the proofs of which are very similar to the proof in the case of two sets.

Theorem 2.7 (Generalized De Morgan's Laws). *Let I be an indexing set and let $\{A_i\}_{i \in I}$ be a collection of sets that are all subsets of the same universal set. Then*

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c$$

2.3.1 Exercises

1. For each of the following collections of sets:

$$\mathcal{A} = \left\{ \left[\frac{1}{n}, n \right) \right\}_{n=2,3,4,\dots}$$

$$\mathcal{B} = \{(n, \infty)\}_{n=0,1,2,3,4,5,\dots}$$

$$\mathcal{C} = \{[-n, n]\}_{n=0,1,2,3,4,5,\dots}$$

$$\mathcal{D} = \{[x, x+1)\}_{x \in \mathbb{R}}$$

$$\mathcal{E} = \{\{z \in \mathbb{C} \mid |z| = r\}\}_{r \in \mathbb{R}^+}$$

$$\mathcal{F} = \{\{n \in \mathbb{Z} \mid n = 3k + j \text{ for some } k \in \mathbb{Z}\}\}_{j=0,1,2}$$

- a. Determine if the sets are mutually disjoint
- b. Determine if the collection is nested
- c. Find the union of the collection

Chapter 3

Equality, Order, and Equivalence

The chapter title may look like just a list, but the notions of equality, order, and equivalence are a critically important trio of related but distinct ideas in mathematics. Together they help us define when two things are the “same.” Instruction in these ideas starts on almost the first day of school when students first learn how to count. Students continue study and expand these ideas in their study of mathematics: many of the theorems of mathematics labeled as “Fundamental” are statements of equivalence. These include the Fundamental Theorem of Arithmetic that every integer can be uniquely factored into primes, the Fundamental Theorem of Calculus describing the relationship between antiderivatives and infinite sums, and isomorphism theorems of abstract algebra and homeomorphisms of topology.

3.1 Partitions and Equivalence Relations



Figure 3.1: Sorting Candy

As a way to better understand the world, humans tend to sort information into categories or orderings. For instance, when some people see a pile of various types of M&M'sTM they have the urge to sort them in various ways. Some people sort the candies according to their color. Others may sort them according to their type (plain, peanut, etc.). This process of sorting is a type of “partitioning” the items into distinct categories. With this categorization we want to make sure that each candy is sorted into one, and only one, type. This type of sorting can be formalized with the following definition.

3.1.1 Partitions

Definition 3.1. A **partition** of a set A is a finite or infinite collection of non-empty, mutually-disjoint subsets whose union is A .

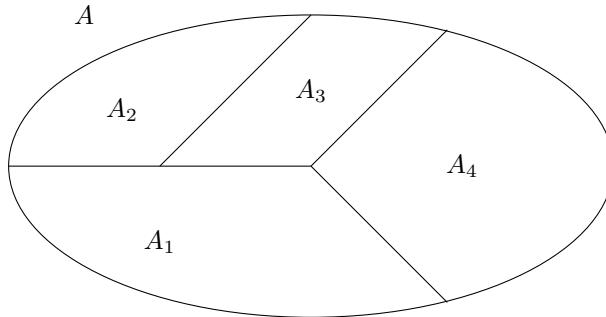


Figure 3.2: Partition of Sets

Thus, a partition of a set is a collection of subsets of the original set with several key properties. First, the subsets are non-empty, meaning that each subset in the partition contains at least one element. This allows us to restrict the partition to a collection of non-trivial subsets and means partitions have a type of uniqueness. The subsets in the partition are also mutually-disjoint which means that no element of the original set is in more than one of the subsets in the partition. Lastly, since the union of the subsets that form the partition is A , we have that every element in A is in one of these subsets. Thus, each component of the definition of a partition is essential to defining a way to split a set into a set of categories that we find useful.

Example 3.1. If we have a set

$$A = \{a, 2, \alpha, \text{apple}, \text{box}, \beta, 1, 3, \gamma, \text{cow}, c, b\}$$

there are some natural ways to partition the set. Here are three possible partitions.

$$A_1 = \{a, b, c\}, A_2 = \{1, 2, 3\}, A_3 = \{\alpha, \beta, \gamma\}, A_4 = \{\text{apple}, \text{box}, \text{cow}\}$$

$$A_1 = \{\text{apple}, \text{box}, \text{cow}\}, A_2 = \{a, b, c, 1, 2, 3, \alpha, \beta, \gamma\}$$

$$A_1 = \{a, 1, \alpha, \text{apple}\}, A_2 = \{b, 2, \beta, \text{box}\}, A_3 = \{c, 3, \gamma, \text{cow}\}$$

There are many other ways to partition the set, but we do not have the space to write out all of the possible partitions of this set.

Example 3.2. Early in the K-12 mathematics curriculum we ask students to determine which integers are even and which are odd. In this situation, the set A is the set of integers (\mathbb{Z}), while the subsets that form the partition are the subset of even numbers, ($2\mathbb{Z}$), and odd numbers, ($2\mathbb{Z} + 1$).

Example 3.3. If we let the set B be defined by the following list of elements,

$$B = \left\{ \begin{array}{lllll} \text{jar}, & \text{jellybean}, & \text{applesauce}, & \text{ape}, & \text{bear}, \\ \text{ball}, & \text{beanbag}, & \text{bag}, & \text{arm}, & \text{skate}, \\ \text{bike}, & \text{rock}, & \text{rowboat}, & \text{jigsaw}, & \text{saw} \end{array} \right\}$$

take some time to create various partitions of the set B , verifying the various properties in the definition of a partition.

In this text, we denote the partition \mathcal{P} of a set A as the union of subsets of A over some indexing set, I . For example, if the partition is formed using two subsets of A , then the indexing set could be $I = \{1, 2\}$. The indexing set can also be infinite. For instance,

$$\mathcal{P} = \{A_n\}_{n \in \mathbb{Z}}, \text{ where } A_n = \{x \in \mathbb{R} | n \leq x < n + 1\}$$

is a partition of the real numbers with an infinite number of sets.

It is important to note that there are usually many ways to partition a set, but the definition of partition does not impose any constraints on the uniformity or order of the subsets, only that the way we choose our subsets meets the three conditions described previously.

With our previous discussion and examples in mind, we can translate our verbal definition of a partition into a symbolic one. In particular: if $\mathcal{P} = \{U_i\}_{i \in I}$ is a partition of a set A , where I is some indexing set that is either finite or infinite, we have the following properties

$$U_i \bigcap U_j = \emptyset \quad \text{if } i \neq j, \quad \bigcup_{i \in I} U_i = A, \quad \text{and} \quad U_i \neq \emptyset \text{ for all } i \in I.$$

3.1.2 Relations

Up until this point we have focused on building a framework for understanding the notion of partition. In order to understand the main topics of this chapter, we also need to develop the idea of a relation. While we give the general definition of a relation and a few examples, we will focus on those relations that arise from partitions in this section. We will explore other types of relations in later sections.

Definition 3.2. Let A and B be sets. We define a relation R from A to B as a subset of $A \times B$. We say that for elements $a \in A$ and $b \in B$ that a is related to b , (aRb) , if and only if $(a, b) \in R$. If the two sets are the same set, A , then we say that R is a relation on A .

Here are some examples of relations in the K-12 curriculum.

- Let A and B be the set of integers. We could say that

$$R = \{(a, b) \in A \times B \mid a < b\},$$

which defines the ‘less than’ relation of $aRb \leftrightarrow a < b$.

- Let A be the real numbers and B be the non-negative real numbers. We can let

$$R = \{(x, y) \in A \times B \mid y = x^2\}.$$

This is the graph of the function $f(x) = x^2$.

- Let A and B be the set of integers. We can define

$$R = \{(m, n) \mid m - n = 2k, \text{ for some integer } k\}.$$

This relation is one where all the even numbers are related to each other and all of the odd numbers are related to each other.

3.1.3 Equivalence Relations Induced by Partitions

For the remainder of this section we will study a specific type of relation that is created by a partition, \mathcal{P} , of a set, A . If $\mathcal{P} = \{U_i\}_{i \in I}$ is a partition of A , we can write A as the union of a collection of non-empty, mutually disjoint subsets:

$$U_i \cap U_j = \emptyset \text{ if } i \neq j, \quad \text{for each } i \in I, \quad U_i \neq \emptyset, \quad \text{and} \quad A = \bigcup_{i \in I} U_i.$$

Consider one of the individual subsets in the partition, say U_i . Then the elements of U_i , by virtue of the fact they are in the same subset, have a relationship to each other. This idea extends to the whole partition. That is, the partition induces a relation on A in the following way. For each $a, b \in A$ we say that aRb if, and only if, there is a set U_i in the partition P such that both a and b are in U_i .

The relations we create through a partition behave in ways that are of general interest in the study of mathematics.

If we choose a generic element a in A , the property that $\bigcup_{i \in I} U_i = A$ tells us that there exists $i \in I$ such that $a \in U_i$. So for every element $a \in A$, we have that aRa . We will call this property of the relation the **reflexive property**.

For the second property, if we have $a, b \in A$ such that aRb , then there must be a $U_i \in P$ with $a, b \in U_i$. This also means that bRa . When you have the property that aRb implies that bRa , we say that the relation has the **symmetric property**.

The third property of relations that we define derives from the property of partitions that

$$U_i \cap U_j = \emptyset \Leftrightarrow i \neq j.$$

If $a, b, c \in A$ such that aRb and bRc , then we have that there are sets U_i and U_j in the partition such that a and b are both in U_i and b and c are both in U_j . Since b is in both U_i and U_j , $U_i \cap U_j \neq \emptyset$ and so $i = j$. This would then imply that aRc . This property that the combination of aRb and bRc implies that aRc is called the **transitive property** of relations.

Not all relations have these three properties of reflexive, symmetric, and transitive. Those relations that have all three properties are called **equivalence relations**.

We have therefore proven that a partition of a set induces an equivalence relation on that set.

3.1.4 Partitions Induced by Equivalence Relations

The concept of equivalence relation is one that transcends much of mathematics and we often consider things as being the “same” with respect to some property if there is an equivalence relation under which they are related. One location that this appears in the elementary school curriculum is in regards to rational numbers. We think of $\frac{1}{2}$ and $\frac{2}{4}$ as being the same rational number, but written in a different way. Along these lines we say that two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. This creates an equivalence relation between these rational expressions that are used to create the rational numbers.

Similarly, if a relation, R , on a set, A , is an equivalence relation, then one can generate an induced partition, \mathcal{P} , as follows. For each $a \in A$ we define the set

$$[a]_R := \{b \in A | aRb\}.$$

We call this set, $[a]_R$, the equivalency class of a under the relation R .

Since R is an equivalence relation, it is reflexive and so $a \in [a]_R$ causing $[a]_R \neq \emptyset$. Since $a \in [a]_R$, we have that

$$\bigcup_{a \in A} [a]_R = A.$$

Thus to show that $\mathcal{P} = \{[a]_R | a \in A\}$ is a partition of A , we need to show that for $a, b \in A$ that $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

If $[a]_R \cap [b]_R \neq \emptyset$, then we need to prove that two sets to be equal.

Let us assume that there exists a $c \in [a]_R \cap [b]_R$.

Let $d \in [a]_R$. So aRd . c is also in $[a]_R$ and so aRc . Since R is symmetric and aRd , dRa . We now have that dRc using transitive property. c is also in $[b]_R$ and so bRc . Using the symmetric property, we have cRb . Combining this with dRc and using the transitive property we have dRb . Using the symmetric property we have bRd . So $d \in [b]_R$. Therefore, $[a]_R \subseteq [b]_R$.

Using very similar arguments we can prove that $[b]_R \subseteq [a]_R$.

So if the intersection is nonempty, the two sets are the same set.

We have thus proven the following theorem showing the connection between equivalence classes and partitions.

Theorem 3.1. If R is an equivalence relation on a set A , then

$$P = \{[a]_R \mid a \in A\}$$

is a partition of A .

Similarly, if $P = \{U_i \mid i \in I\}$ is a partition on a set A , then the relation R defined by

$$aRb \Leftrightarrow \text{there exists } i \in I \text{ such that } a, b \in U_i$$

is an equivalence relation.

3.1.5 Exercises

1. Define a relation \sim on $\mathbb{R} \times (\mathbb{R} - \{0\})$ by

$$(x, y) \sim (z, w) \Leftrightarrow xw = yz.$$

- a. Show that this relation is an equivalence relation.
 - b. Give a description of the partition of $\mathbb{R} \times (\mathbb{R} - \{0\})$ induced by this equivalence relation.
 - c. Why do we not allow 0 in the second part of the ordered pairs?
2. Determine which of the following collections of sets form a partition of $\mathbb{R} \times \mathbb{R}$. For those that do form a partition, define the equivalence relation.
- a. For $c \in \mathbb{R}$, let $h_c = \{(x, y) \in \mathbb{R}^2 \mid y = c\}$, and let $\mathcal{C} = \{h_c\}_{c \in \mathbb{R}}$.
 - b. For $m \in \mathbb{R}$, let $s_m = \{(x, y) \in \mathbb{R}^2 \mid y = mx\}$, and let $\mathcal{D} = \{s_m\}_{m \in \mathbb{R}}$.
 - c. For $r \in [0, \infty)$, let $C_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$, and let $\mathcal{E} = \{C_r\}_{r \in [0, \infty)}$.
3. (For students who have knowledge of linear algebra) Let $M_{2,2}$ be the set of 2×2 matrices with real coefficients. For each of the following, prove that it is an equivalence relation, or show that one of the properties of equivalence relations does not hold.
- a. For matrices A and B in $M_{2,2}$, let $A \sim B$ if and only if $\det(A) = \det(B)$.
 - b. For matrices A and B in $M_{2,2}$, let $A \sim B$ if and only if $\text{tr}(A) = \text{tr}(B)$.
 - c. For matrices A and B in $M_{2,2}$, let $A \sim B$ if and only if the eigenvalues of A are the same as the eigenvalues of B .

3.2 Ordered Sets and Relations

In the previous section we defined a relation R from A to B as a subset of $A \times B$, and that a is related to b , (aRb), if and only if $(a, b) \in R$. If the two sets are the same set, A , then we say that R is a relation on A .

We also defined the following three properties of a relation, R , on a set A :

- **Reflexive.** The relation R is reflexive if aRa for all $a \in A$.
- **Symmetric.** The relation R is symmetric if aRb implies that bRa for all $a, b \in A$.
- **Transitive.** The relation R is transitive if aRb and bRc imply that aRc for $a, b, c \in A$.

A relation is an equivalence relation if it satisfies all three of these properties. Another type of relation is one that creates an order, or ranking, of the set. A common mathematical topic in this regards are the concepts of “less than” and “less than or equal to”.

We will first explore the relation of “less than”. Since a number cannot be less than itself we see that the relation is not reflexive. We also know that $a < b$ and $b < a$ cannot both be true and so the relation is not symmetric. However, if $a < b$ and $b < c$ we do have that $a < c$. So this relation of “less than” is transitive. Thus the familiar relation of “less than” is one which is transitive but not reflexive or symmetric.

When we expand the relation to that of “less than or equal to”, we add that $a \leq a$, resulting in the relation being reflexive. So this is a common relation that is reflexive and transitive, but not symmetric. For example, $1 \leq 2$, but 2 is not less than or equal to 1 .

Definition 3.3. We define an **order** on a set, A , to be a relation, denoted by $<$, with the following properties.

1. If x and y are two elements of A , then one, and only one, of the statements

$$x < y, \quad x = y, \quad , y < x$$

is true.

2. If $x, y, z \in A$, and if $x < y$ and $y < z$, then $x < z$.

If a set has an order defined on the set, then we say that it is an **ordered set**.

Such an order forms a relation that is transitive, but not symmetric or reflexive. As a convenient notation, we will write $x \leq y$ as the negation of $y < x$. We then see that it is equivalent to the statement that $(x < y \text{ or } x = y)$.

Definition 3.4 (Well-Ordering Property). A set, A , together with an order, $<$, satisfies the well-ordering property if every non-empty subset of A has a least element. Equivalently, if $S \subseteq A$ with $S \neq \emptyset$, then there exists an element $s_0 \in S$ such that $s_0 \leq s$ for all $s \in S$.

We will see in Chapter 4 that the natural numbers, $\{0, 1, 2, 3, \dots\}$, satisfy the well-ordering property, but the real numbers do not. For instance, $\{x \in \mathbb{R} | x > 1\}$ does not have a least element. This means that in some way, a set needs to be discrete in order to satisfy a well-ordering property.

For sets that are not discrete, we would like to have a similar property.

Definition 3.5. Let A be an ordered set and let $S \subseteq A$. If there exists an $\alpha \in A$ such that $x \leq \alpha$ for all $x \in S$, then we say that S is **bounded above** and that such an α is an **upper bound** of S .

We can similarly define **bounded below** and **lower bound** using opposite inequalities.

Definition 3.6 (Least-Upper-Bound Property). An ordered set A is said to have the **least-upper-bound property** if for every non-empty subset $S \subseteq A$ that is bounded above, there exists an element $s_0 \in S$ such that s_0 is an upper bound of S and if α is an upper bound of S , then $s_0 \leq \alpha$. Such an s_0 is called the **least-upper-bound** of S .

We will see in Chapter 4 that the real numbers satisfy the least-upper-bound property, while the rational numbers do not.

3.2.1 Exercises

1. For each of the following relations, determine if the relation is reflexive, symmetric, and/or transitive.
 - a. Let S be a relation defined on \mathbb{R} by

$$xSy \Leftrightarrow x^2 = y^2.$$

- b. Let T be a relation defined on \mathbb{R} by

$$xTy \Leftrightarrow x < y^2.$$

- c. Let U be a relation defined on \mathbb{Z} by

$$mTn \Leftrightarrow mn > 0.$$

- d. Let V be a relation defined on \mathbb{R}^2 by

$$(x, y)V(w, z) \Leftrightarrow x^2 + y^2 \leq w^2 + z^2.$$

2. Let $S = \{a, b\}$. List out all of the possible relations on S and determine which relations are reflexive, symmetric, and/or transitive.

3.3 Equality and Equivalence in the K-12 Curriculum

We can see from the discussion in the previous two sections that the concept of two things being equivalent mathematically is a very difficult and abstract concept. It is no surprise then that students of all grade levels have many different conceptions and misconceptions revolving around this concept of equality and equivalence.

Since all of the number systems that arise in the elementary and early secondary curriculum have an inherent order described by the \leq and $<$ symbols, the concept of equality is often described in terms of a balanced scale. Another conception around the equal sign is that two sets of objects have the same number of items, determined by counting the number of objects. This is likely since the concept of equality arises in the first grade where students are only working with whole numbers.

1. Solve the following:
 - a) $5 + 2 =$
 - b) $3 + 4 =$

Figure 3.3: Sample problems

One of the common misconceptions surrounding the equal sign is that the symbol implies an action of doing some type of computation. For instance, students usually see the equal sign in the context of the type of problems seen in Figure 3.3.

These type of problems promote an operational understanding of the equal sign, rather than a relational or substitutional understanding that is essential for later work [Gilmore et al., 2018, pp. 145-150]. Because secondary mathematics students maintain many misconceptions about the concept of equality and the meaning of the equal sign, we will take some time to reflect on some key content standards from first grade that have a very strong influence on student success of learning mathematics in the middle school.

Related Content Standards

- (1.OA.7) Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? $6 = 6$, $7 = 8 - 1$, $5 + 2 = 2 + 5$, $4 + 1 = 5 + 2$.
- (1.OA.8) Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations $8 + ? = 11$, $5 = ? - 3$, $6 + 6 = ?$.

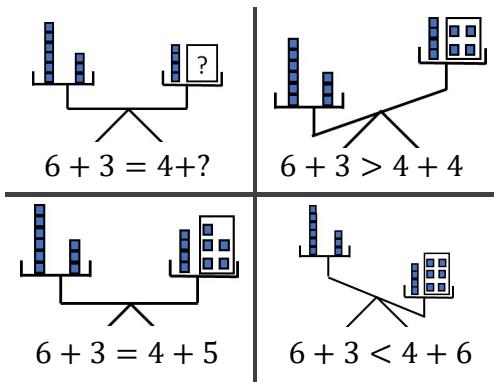


Figure 3.4: Pan balances

We should notice that in standard (1.OA.D.8) that the unknown should not always be at the end of a string of mathematical symbols and is based on educational research that compares students from different cultures

and their understanding of the equal sign [Gilmore et al., 2018, pp. 148-150]. Instead it should be at many different locations in order to make sure that students do not internalize the misconception that the equal sign means to compute. They should instead develop the concept of the equal sign representing two things that have the same value in some way. It is this understanding of two things looking different but representing the same underlying value that causes the equal sign to represent an equivalence.

Another method that may possibly contribute to a more relational understanding of the equal sign is the introduction of the equal sign alongside the greater than ($>$) and less than ($<$) symbols to emphasize the role of the equal sign in the comparison of the size of numbers. This comparison concept can be further reinforced with a pan balance (See Figure 3.4).

This concept of a pan-balance can give students a concrete idea to attach to the meaning of the equal and inequality signs. However, like all good analogies, the pan balance is challenging to use with negative numbers and breaks down if you are comparing numbers in an unordered number system such as the complex numbers.

3.3.1 Exercises

A student is given the following task.

- Sophia goes to the store with \$5.00 and buys two packs of gum for \$1.20. She then gives \$2.00 to her friend to buy a drink. How much money does she have left?
1. The student then writes the following on their paper.

$$5.00 - 1.20 = 3.80 - 2.00 = 1.80$$
, so Sophia has \$1.80 left.
 - a. What ideas from this section regarding the equal sign and equivalence would you use to provide feedback to this student?
 - b. What are other points of feedback that you would provide to this student?
 2. If a student asks you what the equal sign means in the context of $\frac{1}{2} = \frac{2}{4}$ since those symbols are not the same.
 3. What are the meanings of the equal signs in the task? >Evaluate the function $f(x) = 3x - 4$ for $x = 2$.

3.4 Expressions, Equations, and Inequalities

A **numerical expression** is a meaningful string of numbers, operation symbols, and/or grouping symbols. In early elementary grades, examples of numerical expressions include

$$2 + 6 + 4, \quad 13 - 3 + 2, \quad \text{or} \quad (4 + 6) + 3.$$

In upper elementary grades, students are expected to incorporate rational numbers and the operations of multiplication, division, and exponents in order to be comfortable working with such expressions as

$$3 \times (10 + 4), \quad \frac{2}{3} \times (12 + 8), \quad \text{or} \quad 2^3 + \frac{3+2}{7}.$$

In the transition from elementary school to secondary schools, students learn to add **variables**, symbols or letters that stand for any number within a specified range of numbers, to these expressions and thus work with **algebraic expressions** such as

$$100 - 10 \cdot (P - 15)^3, \quad 3xy + 5(x - 4)^2 - 7y^2, \quad 6 \cdot (4x + 3y), \quad \text{or} \quad \frac{3x + 2}{2x - 4}.$$

Related Content Standards

- (6.EE.2) Write, read, and evaluate expressions in which letters stand for numbers.
 - a. Write expressions that record operations with numbers and with letters standing for numbers.
 - b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity.
 - c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

In more advanced courses in secondary school mathematics, algebraic expressions can also include functions giving rise to expressions such as

$$f(x - h) + k, \quad \sin(3x) - 2, \quad \text{or} \quad \ln(x + 3).$$

The equivalence of numerical expressions is a key concept in the early elementary grades where students learn to compose and decompose numbers in different ways as they add and subtract numbers using different algorithms. In sixth grade, students are asked to determine the equivalence of two mathematical expressions. This definition that two algebraic expressions are equivalent if they generate the same number regardless of which number is substituted for the variable is one of the first places where students are pushed to move to the abstract realm of equivalence relations. We can verify that this definition of equivalent expressions is an equivalence relation using the fact that equality on the real numbers is an equivalence relation.

Related Content Standards

- (6.EE.3) Apply the properties of operations to generate equivalent expressions. %{For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.}
- (6.EE.4) Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).
- (6.EE.5) Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

While substituting in values to determine if two expressions are equivalent is the definition of the equivalence, it is almost always impossible to substitute in all possible values for a variable, as there are infinitely many possibilities in most situations. Instead, we determine the equivalency of two expressions by applying various properties of the operations. For example, the distributive property shows us that $3(x + y)$ is equivalent to $3x + 3y$. However, students have to be careful to consider the set of possible numbers for an expression when determining the equivalence of $\sqrt{x^2}$ and x , or $\frac{3x^2}{x}$ and $3x$.

An **equation** is a statement that a number or expression is equivalent to a different number or expression. An equation that is true for all values of the variable is called an **identity**. The associative, commutative, and distributive properties of the number systems

$$\begin{aligned} a + (b + c) &= (a + b) + c \\ a + b &= b + a \\ a \times (b + c) &= (a \times b) + (a \times c) \end{aligned}$$

are all common identities. Some other common identities in this regard are the trigonometric identity

$$\begin{aligned} \cos^2(\theta) + \sin^2(\theta) &= 1 \\ \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

that are used to simplify trigonometric expressions.

An equation that describes how multiple quantities vary together is usually called a **formula**. These can include the area formula for rectangles defined by $\text{Area} = \text{width} \times \text{height}$ or Boyle's law where pressure (P) is inversely proportional to volume (V), i.e. $PV = k$ for some constant k .

While the identity equations hold true irrespective of the values of the variables, it is often the case that equations hold for only some of the possible values for the variables, or possibly even none. We can then **solve** such equations by determining which values, called **solutions to the equation**, for the variables cause the equation to be a true statement.

Similar to equations, **inequalities** are statements that a number or expression is related to a different number or expression in a certain order relation. Some of these inequalities are identities or formulas, such as $x \leq |x|$, $x \in \mathbb{R}$, while some have a solution set, such as $x^2 + y^2 < 1$.

Definition 3.7. Two equations (inequalities) are equivalent if they are true statements for the same values of the variables.

In the process of finding solutions to equations or inequalities, it is almost always helpful to rewrite an equation or inequality as an equivalent equation or inequality. For instance in an effort to find the solutions to

$$\frac{x^2 - 10x + 21}{3x - 12} = \frac{x - 5}{x - 4}$$

we want to find an expression that is equivalent to the left side of the equation. Such an expression could be

$$\frac{(x - 3)(x - 7)}{3(x - 4)},$$

with the equivalency of the expressions verified using the distributive property. Since these two expressions are equivalent, they have the same values for every value of the variable. This means that the original equation is a true statement if, and only if,

$$\frac{(x - 3)(x - 7)}{3(x - 4)} = \frac{x - 5}{x - 4}$$

is a true statement. We also know that if two numbers are the same, then 3 times each of the numbers are also the same. So the original statement is true if, and only if, the equivalent equation

$$\frac{(x - 3)(x - 7)}{x - 4} = \frac{3(x - 5)}{x - 4}$$

is true. If $x = 4$, then the statement would have no meaning, as division by zero has no meaning. This means that we can eliminate 4 from the set of possible solutions. With $x \neq 4$ we can multiply both sides of the equation by $(x - 4)$ to generate the equivalent equation,

$$(x - 3)(x - 7) = 3(x - 5).$$

We can use the distributive property again to create an equivalent equation of

$$x^2 - 10x + 21 = 3x - 15.$$

Since we can also add the same expression, $-3x + 15$, to both sides of the equation and create an equivalent equation, we see that the original equation is equivalent to

$$x^2 - 13x + 36 = 0$$

as long as $x \neq 4$. Using the distributive property, we see that this equation is equivalent to

$$(x - 4)(x - 9) = 0.$$

So the original equation is a true statement if, and only if, $(x - 4)(x - 9) = 0$ is a true statement and $x \neq 4$. This only occurs when the variable, x , is equal to 9.

Through this process we can see how equivalence of equations is essential in the process of finding solutions to equations. This means that as students make the transition from elementary to secondary, it is important for teachers to be understanding of the challenge of thinking about equivalent expressions and to be explicit about this relationship.

3.4.1 Exercises

1. Most of this project will be in the context of algebraic reasoning and these questions should be answered with the expectations of your target student population.
 - a. What is the difference between an expression and equation?
 - b. Why can we not solve an expression?
 - c. What does it mean to “solve an equation”?
 - d. What makes two expressions equivalent to each other?
 - e. What is the difference between how the equality sign is used when solving an equation versus when working with expressions?
2. Prove that the definition of equivalent equations forms an equivalency relation.

Chapter 4

Number Systems

K-12 mathematics is rich in the study of number systems, with particular focus on the Natural Numbers, Integers, Rational Numbers, Real Numbers, and Complex Numbers. In this chapter we use the properties of set theory and equivalence classes discussed in Chapters 2 and 3 to construct these numbers systems. We follow this discussion with a study of some of the properties of number systems, connecting them to the K-12 curriculum. Lastly, we examine different representations of these number systems helpful for developing numerical fluency and conceptual understanding of the numerical operations.

This process of constructing the number systems is very abstract and challenging for those who have not previously studied these constructions. The process is included because it helps teachers to better understand the abstract nature of the number systems and some of the challenges of their students. While working with many of these number systems has become second nature for math teachers, the initial learning by students is much harder than expected for new teachers and so a deeper knowledge of the properties of the number system helps in this teaching situation.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

As we are working with the various number systems, we will use varying notation for our operations. With all of the constructions, subtraction and division are not defined. We instead define the additive inverse and multiplicative inverse and so we can think of subtraction $a - b$ as $a + (-b)$ and division $a \div b$ as $a \times \frac{1}{b}$. We will also use various symbols to represent multiplication so that $a \times b$, $a \cdot b$, and ab all represent the same thing.

4.1 Natural Numbers and Integers

As part of the effort to axiomatize mathematics in the late 19th and early 20th centuries, individuals such as Richard Dedekind (1831-1916), Georg Cantor (1845-1918), Giuseppe Peano (1858-1932), Alfred Whitehead (1861-1947), David Hilbert (1862-1943), Ernst Zermelo (1871-1953), Bertrand Russell (1872-1970), and Abraham Fraenkel (1891-1965) followed a program to start with the most basic of assumptions and build a solid foundation for the field of mathematics. One of the greatest works in this program is the *Principia Mathematica* by Whitehead and Russell [1910; 1912; 1913] that built such a foundation on logic and set theory that it took until page 86 of the second volume to prove that $1 + 1 = 2$. While we will not go to the full extent of abstraction, we will build up the natural numbers and the integers from the axioms of Zermelo-Fraenkel set theory, together with the axiom of choice (ZFC axioms).

4.1.1 Natural Numbers

We will study the concepts of counting and related ideas of the natural numbers in Chapter 5, but initially we will use John von Neumann's [1923] definition for the natural numbers which are defined inductively by

defining the symbols

$$0 := \emptyset, \quad 1 := \{\emptyset\}, \quad 2 := \{\emptyset, \{\emptyset\}\},$$

and so on. So for any natural number n , the next natural number is defined inductively as

$$n + 1 := S(n) := n \cup \{n\},$$

where you think of $S(n)$ as the successor of n .

This definition of the natural numbers is highly dependent upon the axiom of infinity from the ZFC axioms. In the formal language of the ZFC axioms, the axiom of infinity is stated as

$$\exists I(\emptyset \in I \wedge \forall x \in I((x \cup \{x\}) \in I)).$$

Or equivalently, there is a set I that contains the empty set as an element and that if $x \in I$, then $x \cup \{x\}$ is also in I . The set I is a larger set than \mathbb{N} , but allows for the existence of the natural numbers.

With our definition of the natural numbers, we also have that $S(n) \neq n$,

$$S(n) = S(m) \Leftrightarrow n = m,$$

and we can create an order on the set of symbols $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ by defining that

$$n \leq m \iff n \subseteq m.$$

One can prove that this set of symbols satisfies the well-ordering property (@ref(def:well_ordering)) using the ZFC axioms, which enables us to prove the principle of mathematical induction creating the mathematical machinery necessary for the study of the natural numbers.

Theorem 4.1 (Principle of Mathematical Induction). *Suppose that S is a subset of \mathbb{N} such that $0 \in S$, and for all $n \in \mathbb{N}$, $n \in S$ implies that $S(n) \in S$.*

Then S must be the set of natural numbers.

We will leave the detailed proof for a text on symbolic logic. However, the application of this theorem is that if we have a mathematical statement that varies along the natural numbers, $P(n)$. If the statement $P(0)$ is true, and if $P(k)$ being true implies that $P(k+1)$ is true, then $P(n)$ must be true for all natural numbers.

We inductively define the operation of addition on these symbols so that for any integer a ,

$$\begin{aligned} a + 0 &:= a, \\ a + 1 &:= a + S(0) = S(a + 0), \\ a + 2 &:= a + S(1) = S(a + 1), \\ &\vdots \\ a + S(b) &:= S(a + b) \end{aligned}$$

We similarly define multiplication inductively by

$$a \times 0 := 0, \quad a \times 1 := (a \times 0) + a = a, \quad \text{and} \quad a \times S(b) := (a \times b) + a.$$

With these operations we have the following properties of the natural numbers.

Theorem 4.2. *Let a , b , and c be natural numbers. Then*

- $a + b$ and $a \times b$ are natural numbers (closure)
- $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$ (associative property)
- $a + b = b + a$ and $a \times b = b \times a$ (commutative property)
- $a + 0 = a$ and $a \times 1 = a$ (additive and multiplicative identities)
- $a \times (b + c) = (a \times b) + (a \times c)$ (distributive property)
- *If $a \times b = 0$, then $a = 0$ or $b = 0$ (or both). (no non-zero divisors)*

In order to get a flavor for the techniques involved in proving these properties, we will prove the associative property of addition for the natural numbers. The proofs of the remainder of the statements will be left to the reader to work through.

Proof. Let $a, b \in \mathbb{N}$. Then, for each $c \in \mathbb{N}$, we let $P(c)$ be the statement that

$$a + (b + c) = (a + b) + c.$$

If $c = 0$, then $b + c = b$ and $(a + b) + c = a + b$, and so

$$a + (b + c) = a + b = (a + b) + c.$$

This implies that $P(0)$ is true.

Assume by that for some value of $c' \in \mathbb{N}$ that $P(c')$ is true, i.e. that $a + (b + c') = (a + b) + c'$ ($c' = 0$ is true and so such a c' exists). Then for $c = S(c')$,

$$\begin{aligned} a + (b + c) &= a + (b + S(c')) = a + S(b + c') \\ &= S(a + (b + c')) = S((a + b) + c') \quad (\text{by the induction hypothesis}) \\ &= (a + b) + S(c') = (a + b) + c. \end{aligned}$$

Therefore, $P(c)$ is true. and by the principle of mathematical induction we have that $a + (b + c) = (a + b) + c$ for all values of $a, b, c \in \mathbb{N}$. \square

We previously defined the order on the natural numbers using set inclusions. This order can be rewritten using operations by noting that $m < n$ if, and only if, $n = m + k$ for some natural number k . We will explore how this order interacts with the operations of addition and multiplication.

Lemma 4.1. *Let $m, n, k \in \mathbb{N}$. Then*

$$(1) (m = n) \Leftrightarrow (m + k = n + k) \quad \text{and} \quad (2) (m < n) \Leftrightarrow (m + k < n + k).$$

Proof. We will prove part (1) using an induction argument on k using the statement that given $m, n \in \mathbb{N}$, $P(k)$ is the statement that $(m = n) \Leftrightarrow (m + k = n + k)$. Since 0 is the additive identity, we know that $P(0)$ is true. We also have from the definition of $S(n)$ that $P(1)$ is true. If we assume for some $l \in \mathbb{N}$ that $P(l)$ is true, then $m + (l + 1) = n + (l + 1)$ is equivalent to $(m + l) + 1 = (n + l) + 1$, and since $P(1)$ is true this is equivalent to $(m + l) = (n + l)$. So $P(l)$ being true implies that $P(l + 1)$ is true. Therefore, $P(k)$ is true for all $k \in \mathbb{N}$.

In order to prove part (2), we note that

$$\begin{aligned} (m < n) &\Leftrightarrow (\text{There exists } j \in \mathbb{N} \text{ such that } (n = m + j)) \\ &\Leftrightarrow (\text{There exists } j \in \mathbb{N} \text{ such that } ((n + k) = (m + j) + k)) \\ &\Leftrightarrow (\text{There exists } j \in \mathbb{N} \text{ such that } ((n + k) = (m + k) + j)) \\ &\Leftrightarrow (m + k < n + k). \end{aligned}$$

\square

Lemma 4.2. *Let $m, n, k \in \mathbb{N}$. Then*

$$(1) (m = n) \Leftrightarrow (m \times k = n \times k) \quad \text{and} \quad (2) (m < n) \Leftrightarrow (m \times k < n \times k), \text{ for } k > 0.$$

The proof of this lemma is very similar to the previous lemma and is left as an exercise.

4.1.2 Integers

Now that we have defined the set of the natural numbers (\mathbb{N}) and the operations on \mathbb{N} of addition and multiplication, we would like to have additive inverses. However, we do not assume that the set of additive inverses exist, but instead construct the integers from the natural numbers.

Subsection 2.2.3 gives the existence of the set

$$\mathbb{N} \times \mathbb{N} = \{(a, b) | a, b \in \mathbb{N}\}.$$

On this set of ordered pairs, we will define an equivalence relation that

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c.$$

Since this is an equivalence relation we can define \mathbb{Z} to be the set of equivalence classes under this relation. We will denote by $[(a, b)]$ the equivalence class containing (a, b) .

We can now define the operations of addition and multiplication on this set of equivalence classes.

$$[(a, b)] + [(c, d)] := [(a + c, b + d)] \quad \text{and} \quad [(a, b)] \times [(c, d)] := [(ac + bd, ad + bc)]$$

We need to make sure that these definitions are well-defined in that the operations do not depend on which element of the equivalence class is chosen for the representation. Assume that (a, b) and (a', b') are both elements of $[(a, b)]$ and that (c, d) and (c', d') are both elements of $[(c, d)]$. This is equivalent to the statements that

$$a + b' = b + a' \quad \text{and} \quad c + d' = d + c'.$$

So, using properties of equivalent equations involving natural numbers we can add the two equations together and see that

$$(a + b') + (c + d') = (b + a') + (d + c').$$

Using the associative and commutative properties of addition for the natural numbers we see that this equation is equivalent to

$$(a + c) + (b' + d') = (b + d) + (a' + c')$$

and so $[(a + c, b + d)]$ is equivalent to $[(a' + c', b' + d')]$, proving that addition is well-defined. The proof that multiplication is well-defined is left as an exercise.

We can also put an order on this set by defining that

$$[(a, b)] < [(c, d)] \Leftrightarrow a + d < b + c.$$

We can think of the natural numbers, \mathbb{N} , as the subset of $\mathbb{N} \times \mathbb{N}$ of the form

$$\mathbb{N} := \mathbb{N} \times \{0\} = \{[(a, 0)] | a \in \mathbb{N}\},$$

since $[(a, 0)] + [(b, 0)] = [(a + b, 0)]$ and $[(a, 0)] \times [(b, 0)] = [(ab, 0)]$.

With this identification of the natural numbers with $\mathbb{N} \times \{0\}$, where $a = [(a, 0)]$, we are able to see that the negative integers can be associated with $\{0\} \times \mathbb{N}$, since $[(a, 0)] + [(0, a)] = [(a, a)] = [(0, 0)]$. We then label $[(0, a)]$ as $\$ - [(a, 0)] \$$, and usually write it as $-a$. We can then identify every element of $\mathbb{N} \times \mathbb{N}$ with an integer,

$$[(a, b)] \leftrightarrow a + (-b).$$

Thus we have constructed a new set \mathbb{Z} with the operations of addition and multiplication that satisfy the properties for $a, b, c \in \mathbb{Z}$ in Table 4.1.

Table 4.1: Properties of Integers

		Property
$a + b$ is an integer	$a \times b$ is an integer	Closure
$a + (b + c) = (a + b) + c$	$a \times (b \times c) = (a \times b) \times c$	Associative Property
$a + b = b + a$	$a \times b = b \times a$	Commutative Property
$a + 0 = a$	$a \times 1 = a$	Identities
$a \times (b + c) = (a \times b) + (a \times c)$	$(a + b) \times c = (a \times c) + (b \times c)$	Distributive Property
$a + (-a) = 0$		Additive Inverses
If $a, b > 0$, then $ab > 0$.	If $a, b < 0$, then $ab > 0$.	
$ab = 0$ if, and only if, $a = 0, b = 0$, or both		No zero divisors

While the integers do not satisfy the Well-Ordering Property (3.4), since the negative numbers are a non-empty set without a least element, it does satisfy the least-upper-bound property that any non-empty set that is bounded above has a least upper bound.

The integers do however, have generalizations of Lemmas 4.1 and 4.2, with the proofs following directly from various properties listed in Table 4.1.

Theorem 4.3. *Let $m, n, k \in \mathbb{Z}$. Then*

- $$(1) (m = n) \Leftrightarrow (m + k = n + k), \quad (2) (m < n) \Leftrightarrow (m + k < n + k),$$
- $$(3) (m = n) \Leftrightarrow (m \times k = n \times k), \text{ with } k \neq 0 \quad \text{and} \quad (4) (m < n) \Leftrightarrow (m \times k < n \times k), \text{ with } k > 0.$$

4.1.3 Properties of Exponents

Let a be an integer. We define a^n for each natural number n using the following recursive definition.

$$a^0 := 1, \quad a^1 := a, \quad \text{and} \quad a^{n+1} := a \cdot a^n \quad \forall n \in \mathbb{N}.$$

Notice that we have defined $0^0 = 1$. This is not a universally accepted definition, but is widely accepted in the context of algebraic and combinatoric contexts, while left as indeterminate in a limit context of analysis, since

$$\lim_{x \rightarrow 0^+} x^0 = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} 0^x = 0.$$

Related Content Standards

- (6.EE.1)] Write and evaluate numerical expressions involving whole-number exponents.

With this definition, we are able to prove many of the properties of exponents of natural numbers with a base of an integer. It is already given that $a^0 = 1$ and $a^1 = a$.

Lemma 4.3. *Let a be an integer, and let m and n be natural numbers. Then*

$$a^m \cdot a^n = a^{m+n}.$$

Proof. That we are trying to prove something to be true for all natural numbers points us to a proof by induction argument. In this case, we will fix $n \in \mathbb{N}$ and let $P(m)$ be the statement that $a^m \cdot a^n = a^{m+n}$.

Then $P(0)$ is the statement that $a^0 \cdot a^n = a^{0+n}$. Since $a^0 = 1$ and $0 + n = n$, we have that $1 \cdot a^n = a^n$, which is a true statement.

If we assume that $P(k)$ is true for some value of k (we already know that it is true for $k = 0$), then $a^k \cdot a^n = a^{k+n}$. This means that

$$\begin{aligned} a^{(k+1)} \cdot a^n &= (a \cdot a^k) \cdot a^n \quad (\text{by the definition of the exponent}) \\ &= a \cdot (a^k \cdot a^n) \quad (\text{since multiplication is associative for integers}) \\ &= a \cdot a^{k+n} \quad (\text{by the assumption that } P(k) \text{ is true}) \\ &= a^{(k+n)+1} \quad (\text{by the definition of the exponent}) \\ &= a^{(k+1)+n} \quad (\text{since addition is associative for the natural numbers}). \end{aligned}$$

So, $P(k+1)$ is true.

Therefore, by induction, $P(m)$ is true for all natural numbers m . \square

Lemma 4.4. *Let a and b be any integers, and let n be any natural number. Then*

$$(ab)^n = a^n \cdot b^n.$$

Proof. Since we are proving a statement to be true for all natural numbers, we will use a proof by induction with the statement $P(n)$ being that $(ab)^n = a^n \cdot b^n$.

Since $P(0)$ is the statement that $(ab)^0 = a^0 \cdot b^0$, we have that this is equivalent to the statement that $1 = 1$, and so is true.

If we assume that $P(k)$ is true for some value of k , then we have that $(ab)^k = a^k \cdot b^k$. We can then use properties of addition and multiplication of integers, and the definition of exponents, to see that

$$(ab)^{(k+1)} = (ab)^k \cdot (ab) = a \cdot (ab)^k \cdot b = (a \cdot a^k) \cdot (b^k \cdot b) = a^{(k+1)} \cdot b^{(k+1)}$$

and so $P(k+1)$ is true.

Therefore, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

Lemma 4.5. *Let a be an integer and n and m be any natural numbers. Then*

$$(a^m)^n = a^{mn}.$$

The proof of this lemma is similar to the proof of Lemma 4.3 and so will be left as an exercise.

Lemma 4.6. *Let a and b be integers with $0 < a < b$. Then for each natural number $n \geq 1$, $0 < a^n < b^n$.*

Proof. We will use an induction argument for fixed integers a and b , with $a < b$. We let $P(n)$ be the statement, $0 < a^n < b^n$. Since $a^1 = a$ and $b^1 = b$, we see that $P(1)$ is true. If for some $k \in \mathbb{N}$, with $k \geq 1$, $P(k)$ is true, then

$$a^{(k+1)} = a^k \cdot a^1 < b^k \cdot a^1 < b^k \cdot b^1 = b^{(k+1)}$$

using the induction hypothesis and Lemma 4.2. Since the natural numbers are closed under multiplication, and therefore also under exponents, we have that $P(k+1)$ is true. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$, with $n \geq 1$. \square

We will summarize these results in the following theorem and extend these properties to other number systems throughout the remainder of the chapter.

Theorem 4.4. *Let a and b be integers.*

$$a. \ a^0 = 1$$

$$b. \ a^1 = a$$

- c. $a^m \cdot a^n = a^{m+n}$ for each $m, n \in \mathbb{N}$
- d. $(ab)^n = a^n \cdot b^n$ for each $n \in \mathbb{N}$
- e. $(a^m)^n = a^{mn}$ for each $m, n \in \mathbb{N}$
- f. If $0 < a < b$, $0 < a^n < b^n$ for each $n \in \mathbb{N}$

4.1.4 Exercises

1. After introducing the associative property for addition, a student asks if there is something similar for subtraction so that

$$A - (B - C) = (A - B) - C.$$

How would you respond to the student?

2. Prove that the relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

is an equivalence relation.

3. Prove Lemma 4.2.
4. Prove Lemma 4.5.

4.2 Representations of Integers

Implicit to our discussion of number systems is the idea of representation. While the previous chapter focused on the concept of “3” in terms of equivalence and equality, representation is concerned with how we choose to communicate that we are currently using three in our mathematical process. As advanced students of mathematics, it is easy to take for granted the ways that we communicate fundamental notions such as number or operation using symbols like “3.” However, the numbers and symbols we take for granted are choices that have been made and agreed upon.

While the choices often allow for efficient communication, they are often not the most obvious choices for developing conceptual understanding of the abstract objects they represent. For example, when was the last time you saw negative 3 things in the natural world? While we understand the notion of “I owe 3 dollars,” the concept of owing money is an abstract concept, not a physical one. It takes a long time for students to develop the conceptual understanding of how to work with mathematical objects they cannot physically combine or otherwise manipulate. Fortunately, because the negative sign is a choice of one among many, we can choose different representations of negative numbers and other abstract mathematical objects to facilitate our teaching. This section examines some of the other choices we can make for representing our numeration system and the four common operations.

4.2.1 Numeration Systems

When we write the number 13,436, we are communicating how many of something we have. To understand just how many things 13,436 is, we need two fundamental ideas: digit and place value. For example, in 13,435 we used the *digit* 3 twice: once in the tens place value and once in the thousands place value. In these cases, the digit 3 communicates how many of each place value we have. Together, the set of digits and a shared understanding of place value gives us our current numeration system.

The Hindu-Arabic Numeration System is the numeration system we use in most of the world today. It is composed of three main ideas:

1. A set of 10 digits: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
2. A base 10 place value system
3. Reliance on a digit that means “none”

Table 4.2: Roman Numeration

Arabic Numeral	Roman Numeral
1	I
10	X
100	C
1,000	M

Note that these are choices: the base of a place value system determines how many digits you need. While our culture has settled on base 10 (based on the number of digits on our hands), the Mayan's developed a base-20 system (based on the number of digits on hands and feet) and computer machine language uses a base 2 system (based on a signal being on or off).

The place value base is used to build the place value by using exponential powers of the base value. For example:

$$1234.567 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 7 \cdot 10^{-3}.$$

Although this probably all sounds like review and very obvious, these are not obvious choices. It took a long time for human civilizations to develop the base number systems and children have to be indoctrinated into it. It is not an easy idea that “3” in one position does not have the same meaning as a “3” in another position. Although this section focuses mostly on representation of number, the use of symbols taking on multiple meanings in mathematics in different contexts is one with which mathematics teachers must repeatedly grapple.

To better understand the power of these systems, we will examine a couple numeration systems that have different combinations of the properties of the Hindu-Arabic system.

Example 4.1 (Egyptian Numeration). The Egyptian numeration system uses individual Hieroglyphs to hold value, with new symbols needed for each power of 10). To communicate multiples of a particular power of ten, the Egyptians used repeated Hieroglyphs. For example, the water lily or lotus Hieroglyphs means 1,000. A sequence of three water lily Hieroglyphs would mean 3,000. There are no requirements on how Hieroglyphs are presented, so the Hieroglyphs communicating “1” could appear at any place in the expression for the number, and did not even have to appear together. A clear con of this system is that new symbols are needed for each power of 10 would require a new Hieroglyph. In addition, the lack of an ordering or organizational system would make large numbers potentially cumbersome to read.

Value	1	10	100	1,000	10,000	100,000	many
Hieroglyph		o	ꝝ	ꝝꝝ	ꝝꝝꝝ	ꝝꝝꝝꝝ	ꝝꝝꝝꝝꝝ

Example 4.2 (Roman Numeration). The Roman numeration system is similar to the Egyptian system, in that the Roman numbers are each unique to a particular value (for example, X means 10). The Roman system improved on the Egyptian numeration system by imposing an order to how these numbers could be presented and partially solved the potential for a large number of repeated numbers by adding the subtraction element (so, IX would mean take 1 away from 10, while XI means add one to 10). However, as students of the Super Bowl know, reading these numbers still takes a fair amount of decoding and there is a still a problem with needing a new number for each new grouping.

Example 4.3 (Base 5). Base 5 is like our current standard numeration system, except it uses a set of five digits, $\{0, 1, 2, 3, 4\}$, and place values are based on powers of 5, not 10. In the United States we use base 5 in some of our coins, in that there are 5 pennies in a nickel and 5 nickels in a quarter. Then the number 14_5 (read as fourteen base five) is saying 1 nickel and 4 pennies, or 9 cents. Similarly,

$$231_5 = 2 \cdot 5^2 + 3 \cdot 5^1 + 1 \cdot 5^0 = 66_{10}.$$

While there is no obvious use for students to learn operations in a different base, working outside of our base 10 system can help teachers recognize and understand the inner workings of the base 10 system and its associated algorithms. As these algorithms are introduced throughout the section, you are encouraged to use the algorithms to perform operations on numbers in these additional base systems.

Example 4.4 (Base 12). The duodecimal (base 12) number system is a component of some languages in West Africa and Nepal and is used in our everyday measurements of dozen and gross. A duodecimal system requires twelve digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t , e , and place values are based on powers of 12. In this system $6 + 5 = e$, $7 + 8 = 13$, and

$$3t \times e4 = 2te4.$$

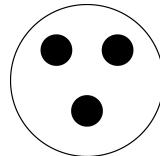
One of the implications of a base 12 system is additional tricks for multiplying numbers due to a larger number of factors of twelve than ten.

4.2.2 Representations of Number

In talking about numeration systems, we have exclusively been using one form of number representation. However, numeration systems are only one way that we communicate ideas about number. In this section, we explore different ways of representing numbers, each which helps develop different properties of numbers. For this introduction to representation, we focus first on the positive natural numbers, adding additional elements to these representations as we develop the various number systems.

Sets.

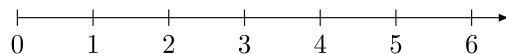
As discussed in Chapter 3, we can think of 3 as an equivalence class for all sets containing 3 objects. (We will go into this in more detail in Section 5.4.) As such, sets can be used to represent numbers, either pictorially as a circle containing generic objects or as a set with general elements. Using sets offers the most general representation of number we can develop, as it does not impose any structure on the representation beyond the property of “how many”. Relationships between numbers (such as inequality or equality) are fairly straightforward to model, but comparing a large number at one time would be difficult using this representation. As a result, most middle and high school textbooks do not represent numbers this way, although it is fairly common in the elementary grades.



or $\{a, b, c\}$.

Number Line.

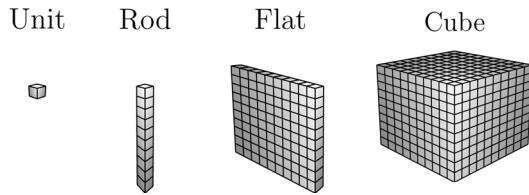
The number line imposes more structure on the number representation than the use of sets. In particular, it organizes numbers along a line, with integers spaced evenly along the line. This representation allows multiple numbers to be visualized at the same time while also showing relationships between the numbers represented. In this representation, any number that falls at the location of 3 is equivalent to three. Numbers of the left of 3 are less than it, while numbers above are greater.



Base-10 Blocks.

Base-10 blocks are usually a physical way (although virtual and pictorial versions exist) to represent numbers specifically designed to promote understanding of place value. The blocks come in four sizes: units (small,

approximately 1-centimeter cubes), rods (10, represented by a stack of 10 unit cubes), flats (100, represented by a 10 by 10 grid of unit cubes), and big cubes (1000, represented by a cube with side lengths equal to 10 unit cubes). Representing a number using base-10 blocks means drawing or collecting the correct number of blocks for each place value.



In order to write down the reasoning these blocks are sometimes transitioned into dots, lines, and squares. So that $13 + 24$ is represented by

$$\left| \begin{array}{c} \vdots \\ \vdots \end{array} \right| + \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right| = \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right|$$

Cuisenaire Rods.

Cuisenaire rods are another physical way that numbers are represented. They are a collection of long prisms of different colors and lengths. Rods of the same length are the same color, and each length is a natural number multiple of the smallest rod. These rods are used for a variety of reasons including factoring of natural numbers, but one of the most common is to help students understand concepts related to rational numbers and the operation of division.

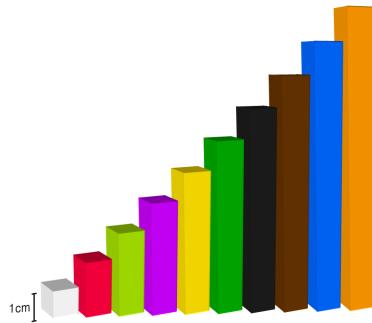


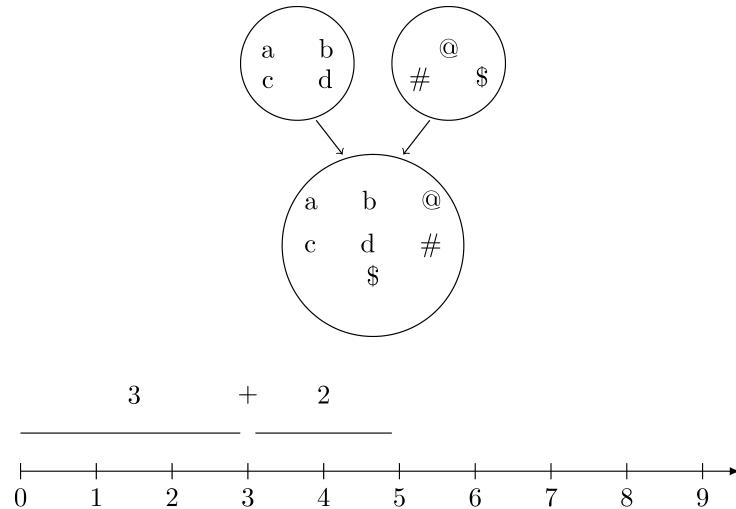
Figure 4.1: Cuisenaire Rods

4.2.3 Addition Models and Algorithms

Much like the concept of number, it is easy to overlook how quickly addition becomes a complex endeavor for young students of mathematics. Fundamentally, addition represents the notion of “combine” or finding the union of mutually disjoint sets.

Typically, we quickly move away from representing addition using sets and instead use the number line. For natural number addition, this process is fairly straight forward: Students start by identifying the first addend. Then, count up the number that is being added. The number you land on is the result.

Thinking of addition as the union of sets or on a number line of the natural numbers are good, conceptual starting places for understanding addition. However, doing so will lead some students to think that “when two numbers are added, the result will be the same size or bigger.” In the natural numbers this thinking will work, but in the integers the reasoning begins to fall apart. Note that idea of “combine” does not say anything



about the size of the result relative to the others, so the common error is a result of students focusing on the result rather than the concept of addition.

When asked to add something like $125 + 258$ without the use of a calculator, most adults are well-trained to rewrite the sum and apply the standard algorithm.

$$\begin{array}{r} & & 1 \\ & 1 & 2 & 5 \\ + & 2 & 5 & 8 \\ \hline & 3 & 8 & 3 \end{array}$$

While the work of teaching children these algorithms is usually done in elementary school, it is worth taking a little time to appreciate the ways these traditional methods take advantage of the properties of natural numbers to allow for quick and efficient calculation. To facilitate this discussion, we present several alternative algorithms. It is not our goal to instruct our readers in each of these algorithms: there is not a way to present all possible ways an addition problem can be solved. Rather, we choose algorithms that maximize the opportunity to explore the hidden work of the standard algorithm. A mathematics teacher who studies these examples in relation to the standard algorithm should be able to teach themselves unfamiliar algorithms that they encounter in the wild. For each of these, we use the example $125 + 258$.

Expanded form.

We can rewrite our numbers using place value as $(100 + 20 + 5)$ and $(200 + 50 + 8)$ and then use associative and commutative properties of addition for the natural numbers.

$$\begin{aligned} 125 + 258 &= (100 + 20 + 5) + (200 + 50 + 8) \\ &= (100 + 200) + (20 + 50) + (5 + 8) \\ &= (100 + 200) + (20 + 50) + (13) \\ &= (100 + 200) + (20 + 50) + (10 + 3) \\ &= (100 + 200) + (20 + 50 + 10) + (3) \\ &= 300 + 80 + 3 = 383 \end{aligned}$$

Lattice Algorithm.

The lattice algorithm described below provides another method of adding two numbers by keeping track of place value through some implicit methods.

- Stack the two numbers so that corresponding place values are in the same column. Draw the traditional equals bar.

$$\begin{array}{r} & 1 & 2 & 5 \\ + & 2 & 5 & 8 \\ \hline \end{array}$$

- Starting with the right-most column, add within each column. The result of the digit addition will be a number between 0 and 19. The resulting number is written into the square below the column, with the higher place value going into the higher triangle.

$$\begin{array}{r} & 1 & 2 & 5 \\ + & 2 & 5 & 8 \\ \hline \end{array}$$

- Add down each diagonal. You are in effect adding the total number of items in each place value.

$$\begin{array}{r} & 1 & 2 & 5 \\ + & 2 & 5 & 8 \\ \hline \end{array}$$

Expanded algorithm.

The expanded algorithm makes the implicit structure of the lattice algorithm more explicit.

You may have noticed that regardless of which method we used, there were several common elements that all of the algorithms created in one form or another. Perhaps most obviously, 13 occurred in all three calculations. The intermediate sums an algorithm produces are called **partial sums**. Note that although every algorithm used partial sums, they are not always the same between algorithms, even for the same addition problem. These differences in partial sums are a result of where the algorithm does the **regrouping** of the partial sums. Regrouping, generally, is the process of converting numbers expressed in terms of one place value into a different place value. For example, in the standard algorithm example at the beginning of this section, we got 13 when we added the ones digits in the right most column. Since 13 ones is not useful for continued adding, we regrouped 13 as 1 ten and 3 ones. We wrote the 3 in the ones place value and the 1 above the 10's place value column. You may have learned this process by the name of “carrying the one”. This language is no longer encouraged in teaching as it masks the fact that the value of the one changes as a person works through every place value.

4.2.4 Subtraction Models and Algorithms

Conceptually, we described addition as the process of combining. Subtraction has two different physical interpretations: the process of removal (often referred to as taking away) or the inverse operation of addition.

$$\begin{array}{r}
 125 \\
 +258 \\
 \hline
 13 \\
 70 \\
 + 300 \\
 \hline
 383
 \end{array}$$

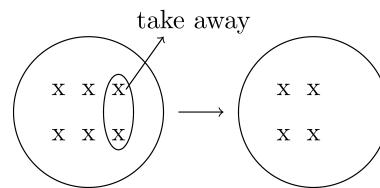
These are clearly related, but are developed separately, which we explore a bit more in the following paragraphs. Although both interpretations of subtraction seem straight forward when modeled in the physical world, students struggle with this operation a great deal more than addition. This may be at least partially because while the natural numbers have some great properties under addition (closed, commutative, and associative) that make addition easy to set up, but none of these properties under subtraction.

Using our different representations, we can model subtraction a couple different ways: the take away model or the missing addend model.

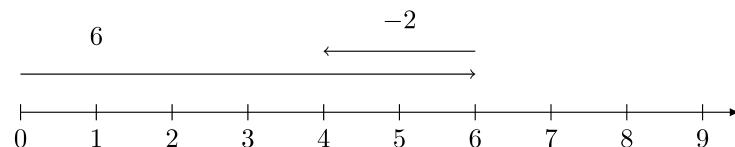
Take-Away Model.

The first interpretation of subtraction is as removal or the process of taking away. In the mathematics education literature, this is called the “take-away model”. This conceptualization of subtraction is a direct interpretation of the calculation $a - b$, where the translation of this expression is “ a take away b .”

To model $a - b$ using sets and the take-away model, we start with a set containing a elements and remove b elements from it. The composition of the set after the removal process is the resulting difference.



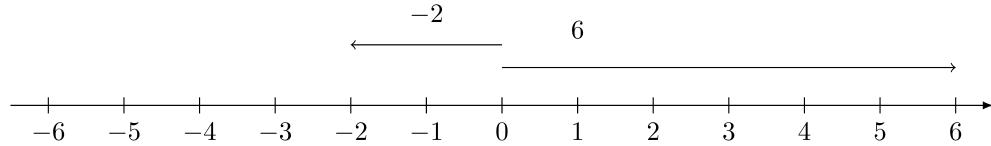
To model $a - b$ using the number line and the take-away model, we find a on the number line and then move back b units. The place where you land is the resulting difference.



Once we have introduced the integer number system, we extend the number line and think of $a - b$ as $a + (-b)$ where $-b$ is viewed as a vector of length $|b|$ pointed in the negative direction.

Related Content Standards

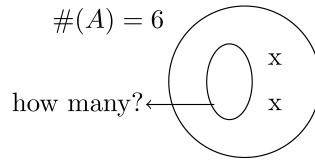
- (6.NS.6) Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.
 - a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.



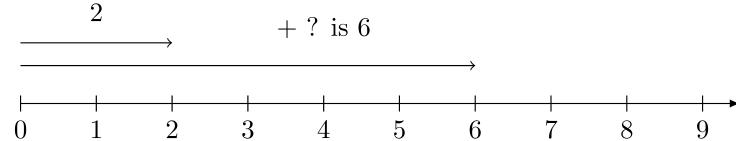
Missing Addend Approach.

The missing addend approach interpretation of subtraction is an explicit model of subtraction as the inverse operation of addition, while converting the computationally more nuanced subtraction problem into an addition problem where the number facts and such are easier. In particular, in the missing addend approach, the subtraction expression $a - b$ is assumed to equal some unknown (so $a - b = ?$), which we can rewrite as $b + ? = a$.

Using the set model and the missing addend approach, we take the statement $b + ? = a$ to mean that we know that we want a set to contain a elements and we have b . We must determine how many additional elements we need in order to bring the total number of things in our set up to a .



Using the number line, we reconceptualize $a - b = ?$ As $b + ? = a$ which means we start at b on the number line and then find a . We then ask, how much do I have to move (and in what direction) from b to get to a ? The answer to this question is the answer to the original subtraction problem. This process when formalized into an algorithm that counts up by place value is called **counting up**.



Note that in none of the conceptual work of this section have we used the language of larger number and smaller number, instead focusing on the expression of the difference, $a - b$. Early instruction in subtraction only presents cases of subtraction problems $a - b$ when $a \geq b$ and both are natural numbers. As a result of this presentation bias, students sometimes develop the impression that the results of subtraction will always be smaller than the original number. They may also believe that subtraction when $a < b$ cannot be done. The conceptual work we have developed in this section for the models of subtraction works for most sets of numbers students encounter in the K-12 mathematics curriculum.

As with addition, we have a standard subtraction algorithm. For example, consider the subtraction problem $423 - 368$. A well-trained student asked to do this problem without a calculator will produce work that looks something like the following:

As with addition, the traditional subtraction algorithm uses regrouping strategies and the properties of place value to allow for quick calculation in a small amount of space. (You may have learned the regrouping strategy under the name “borrowing” but, as with addition, this name is no longer used for similar reasons.)

We next provide several alternative algorithms for subtraction designed to help develop conceptual understanding of the subtraction algorithm.

$$\begin{array}{r}
 & 3 & 11 \\
 & 4 & 2 & 1 & 3 \\
 - & 3 & 6 & 8 \\
 \hline
 & 5 & 5
 \end{array}$$

Equal Addition.

The equal addition algorithm repeatedly adds zero to the problem in a way that makes the subtraction easier to compute. The goal is to write an equivalent problem that has at least one zero place value.

$$\begin{aligned}
 423 - 368 &= 423 - 368 + (2 - 2) = 425 - 370 \\
 &= (425 - 370) + (30 - 30) = 455 - 400 \\
 &= 55
 \end{aligned}$$

Partial Differences.

Another method of subtraction is to subtract from left to right with partial differences. In this case, we subtract the smaller digit from the larger digit, keeping track of positive or negative.

$$\begin{array}{cccc}
 \begin{array}{r}
 4 & 2 & 3 \\
 - & 3 & 6 & 8 \\
 \hline
 + & 1 & 0 & 0
 \end{array} &
 \begin{array}{r}
 4 & 2 & 3 \\
 - & 3 & 6 & 8 \\
 \hline
 + & 1 & 0 & 0 \\
 - & 4 & 0
 \end{array} &
 \begin{array}{r}
 4 & 2 & 3 \\
 - & 3 & 6 & 8 \\
 \hline
 + & 1 & 0 & 0 \\
 - & 4 & 0 \\
 - & 5
 \end{array} &
 \begin{array}{r}
 4 & 2 & 3 \\
 - & 3 & 6 & 8 \\
 \hline
 + & 1 & 0 & 0 \\
 - & 4 & 0 \\
 - & 5 \\
 \hline
 5 & 5
 \end{array}
 \end{array}$$

In the section on subtraction, we commented that when students first learn subtraction in the natural numbers, they are usually only given examples of the form $a - b$ where $b \leq a$. Part of the reason for only having students work on these types of problems is that children are not born knowing about negative numbers. The only place we “see” negative numbers in the real world is when humans have imposed a scale on something, such as temperature. However, this is not a true physical representation, since we have simply imposed a scale on a naturally occurring thing, where for one reason or another, we want to have a zero reference point.

To allow us to find answers to subtraction problems where $a < b$, we expand our number system to the integers. You may have noticed that we use “ $-$ ” in the above paragraph to mean additive inverse of a positive number. In the paragraph before that we used the same symbol to mean subtraction. In reality, these are fairly different concepts. Subtraction is an operation. Additive inverse is a relation. Long-time students of mathematics know that these interpretations are related and will move between them fairly seamlessly, but they are not the same thing.

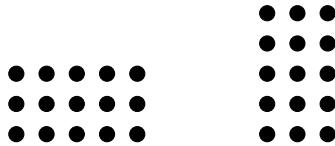
4.2.5 Multiplication Models and Algorithms**Repeated Addition.**

Most commonly, multiplication is described as **repeated addition**. For example, the product 2×4 can be understood as adding 2 four times ($2 + 2 + 2 + 2$), while the product 4×2 can be understood as adding 4 two times ($4 + 4$). While conceptually, this link between addition and multiplication is important, it quickly loses efficiency in practical applications (e.g., it would be cumbersome but possible to multiply 23×345 using repeated addition). That said, it is not uncommon to see secondary students apply this approach to multiplication problems.

Rectangular Array.

Another way to represent multiplication is through a **rectangular array**. One way to think about this is counting the number of chairs in a rows and b columns. Such arrays are discrete sets. We can see below how

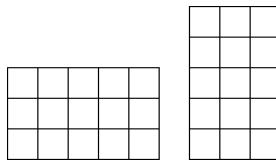
one can use two arrays to represent 3×5 and 5×3 .



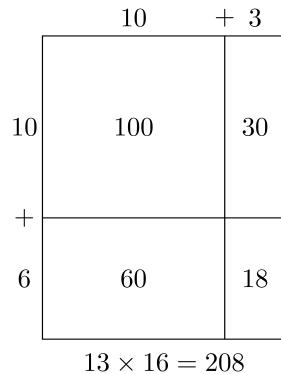
We can see that the two arrays are related to each other by a rotation. In other words, the order of multiplication is simply a different perspective on the same array.

Area Model.

While arrays work well for representing multiplication of natural numbers, they do not work well for integers, rational numbers, or real numbers. So we generalize the array model to an **area model**. If we change the dots in the array to blocks of 1 square unit measure, the above arrays become rectangles with square tiles helping students see the connection between the area of a rectangle and multiplication.



This area model can then be generalized to help students multiply multi-digit natural numbers and the relationship to the distribution property.



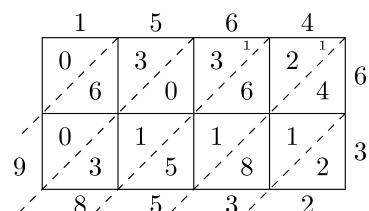
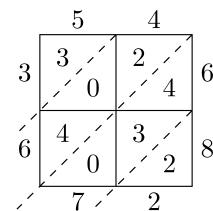
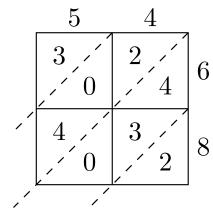
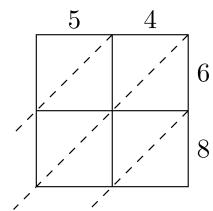
Lattice Multiplication.

The area model can become unwieldy with larger numbers. The lattice multiplication algorithm is a more abstract view of the area model by using a lattice to keep track of place values. Here we can see the progression of using a lattice model to multiply $54 \times 68 = 3672$.

After setting up the lattice with diagonals, we multiply all of the partial products and enter the product in the corresponding square with the one's digit in the bottom right and the ten's digit in the upper left. After all of the partial products are complete, we add along the diagonals in order to find the number of units of that place value. For some products, the sum along the diagonal is larger than 9 and so we have to regroup as is the case for 1564×63 .

Partial Products.

We continue the progression of creating more abstract algorithms for multiplication by removing some of the explicit structure using the **partial products algorithm**.



$$\begin{array}{r}
 & 1 & 5 & 6 & 4 \\
 \times & & & 6 & 3 \\
 \hline
 & 1 & 1 & 1 & 2 \\
 & 1 & 8 & 0 & \\
 & 1 & 5 & 0 & 0 \\
 & 3 & 0 & 0 & 0 \\
 & 2 & 4 & 0 & \\
 & 3 & 6 & 0 & 0 \\
 & 3 & 0 & 0 & 0 \\
 + & 6 & 0 & 0 & 0 \\
 \hline
 & 9 & 8 & 5 & 3 & 2
 \end{array}$$

Standard U.S. Multiplication Algorithm.

If we continue down the path of more abstract algorithms we come to what is considered the standard algorithm for multiplication. In this algorithm we compress the information in the partial products algorithm into many fewer lines.

$$\begin{array}{r}
 & 1 & 5 & 6 & 4 \\
 \times & & & 6 & 3 \\
 \hline
 & 4 & 6 & 9 & 2 \\
 & 9 & 3 & 8 & 4 \\
 \hline
 & 9 & 8 & 5 & 3 & 2
 \end{array}$$

However, with this reduction in the number of lines comes an increase in the number of things that one must keep track of implicitly.

4.2.6 Division Models and Algorithms

Division is the fourth major operation. Conceptually, the division problem $a \div b$ can be viewed several ways: as repeated subtraction, as the inverse of multiplication, and as a partition. Regardless of the model, all three of these representations draw on the notion that when we divide a by b ($a \div b$) the division operation produces two unique numbers: the **quotient**, q , and the **remainder**, r , where $0 \leq r < b$. (We will go into this division algorithm more in-depth in Section 7.1.) We call b the **divisor** and a the **dividend**. These four quantities are related by $a = bq + r$.

Repeated Subtraction Model.

In the repeated subtraction model of division, the division problem $a \div b$ conceptually is thought of as repeatedly subtracting b from a until you no longer have a number bigger than b . The quotient is the number of times you could subtract and the remainder is what is left over after the quotient has been determined.

Missing Factor Model.

In the missing factor model, the division problem defines division as the inverse operation of multiplication. The quotient and remainder are not separated. Instead, we reorganize the quantities in their reciprocal relationships in the equation $a = bq + r$. Generally, we use the missing factor approach when $r = 0$. In particular, if $a \div b = q$, where q is a unique value, then we rewrite the problem as the product $bq = a$.

Set (Partition) Model.

The set (partition) model frames the quotient as a partition of the whole, a , into b even parts. In particular, $a \div b$ is thought of as asking if I must evenly distribute a between b entities, how much will each entity get? While in practice, the finding of the quotient for both the partition and repeated subtraction models are similar, conceptually, the two methods handle the remainder differently. The repeated subtraction approach reports the remainder, while the partition model invites the person completing the problem to keep going until all parts of a , including r , have been evenly distributed.

4.2.6.1 Division Algorithms

The long division algorithm is the last major algorithm students learn. If we examine the work of the long division algorithm, presented below, a few things are apparent. First, the algorithm is the only one that does not begin work with the ones place value, making it an anomaly in students' learning. Secondly, it relies on students to have mastered the subtraction algorithm. And lastly, when we have multi-digit divisors, estimation is an important prerequisite skill. These factors all make the division algorithm particularly tricky to master.

With the widespread availability of cheap and accurate calculators, it is worth asking why we still teach the division algorithm (or really, most advanced arithmetic). And indeed, long division is not emphasized in the

$$\begin{array}{r}
 & \overset{36}{\overbrace{}} \text{ r } 34 \\
 43 \overline{)1582} \\
 129 \\
 \hline
 29 \\
 258 \\
 \hline
 34
 \end{array}$$

curriculum as much as it used to be for this very reason. We do provide a brief treatment here as the division algorithm, while not widely used in daily life, does lay a foundation for polynomial division, discussed in later chapters. In partial products algorithm presented below, pay particular attention to how the various division models are codified into the structure of the algorithm and how the partial quotients differ or are similar to those of the standard algorithm.

$$\begin{array}{r}
 43 \mid 1 & 5 & 8 & 2 \\
 - & 4 & 3 & 0 \\
 \hline
 1 & 1 & 5 & 2 \\
 - & 4 & 3 & 0 \\
 \hline
 7 & 2 & 2 \\
 - & 4 & 3 & 0 \\
 \hline
 2 & 9 & 2 \\
 - & 2 & 1 & 5 \\
 \hline
 7 & 7 \\
 - & 4 & 3 & 1 \\
 \hline
 3 & 4 & 36
 \end{array}$$

4.2.7 Exercises

1. Use the definition of base 5 in this section to answer the following questions.
 - a. What is the base 5 number following 4_5 ?
 - b. Express 234_5 in base 10.
 - c. Express 234_{10} in base 5.
2. Add $342_5 + 134_5$ using each of the four addition algorithms presented in this section.
3. Complete the product $234_5 \times 32_5$ using each of the three multiplication algorithms presented in this section (lattice, partial products, standard).
4. Complete the division problem $341_5 \div 41_5$ using any division algorithm discussed above.

5. Construct the addition and multiplication tables for the first twelve digits in a base 12 system. What patterns do you notice?
6. Read up on the Mayan numeration system, which is a base 20 system.
 - a. Describe the patterns used to develop the 20 digits.
 - b. How is this system comparable to the numeration systems discussed in this section.
7. For the two-number lattice addition, will adding along a diagonal ever require a student to regroup? Why or why not?
8. The standard addition algorithm requires that a person adds columns right to left. Many students just learning the algorithm will make the mistake of starting with the left most column and working towards right.
 - a. What conceptual understanding is a student missing when they make this type of error?
 - b. Which of the three non-standard algorithms will result in a correct sum regardless of whether a person adds right to left or left to right?
 - c. Which of the four algorithms presented in this section might help students develop the conceptual understanding you identified in part (a)?
9. Another common error students make with the standard addition algorithm is that students will line up the first digit of a number with the first digit of the second number, regardless of whether or not the numbers have the same number of digits.
 - a. What conceptual understanding is a student missing when they make this type of error?
 - b. Which of the four algorithms presented in this section might help students develop the conceptual understanding you identified in part (a)?
10. Using a set of base-ten blocks (or a virtual base-ten block simulation), model the addition problem $348 + 753$.
 - a. Write out the arithmetic steps you went through in completing the addition using the blocks.
 - b. Describe how base ten blocks might support conceptual understanding of the standard addition algorithm.
11. How does the standard multiplication algorithm utilize each of the four properties of the natural numbers (commutative, associative, distributive, identity)?
12. Why, in multiplication by multi-digit numbers in the standard multiplication algorithm, do you sometimes include zeros in the right most digit places?
13. For each model of division (e.g., repeated subtraction, partitions, missing factor), explain why the quotient $a \div 0$ does not make sense.

4.3 Rational Numbers

Since the only integers that have a multiplicative inverse are 1 and -1 , it would be beneficial to expand our number system to include symbols that can operate as multiplicative inverses. In this process we allow the ordered pairs in $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ to be written as a fraction,

$$\frac{a}{b} \cong (a, b).$$

But we also want the set to be closed under addition and multiplication. This results in a set that contains all possible rational expressions of integers whose denominator is non-zero. Therefore we create the set

$$\widehat{\mathbb{Q}} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\} \cong \mathbb{Z} \times (\mathbb{Z} - \{0\}).$$

Then, similar to our process of constructing the integers from the natural numbers, we define a relation of two rational expressions by

$$\frac{a}{b} \sim \frac{c}{d} \Leftrightarrow ad = bc.$$

We will now verify that this relation of rational expressions represents an equivalence relation.

Reflexive.

One can see that this relation is reflexive, since if $\frac{a}{b}$, with $b \neq 0$, is a rational expression of integers that $ab = ab$ and so $\frac{a}{b}$ is equivalent to itself.

Symmetric.

Since equality of integers is a symmetric relationship we have that for any integers a, b, c, d with $ad = bc$ that $bc = ad$ and so this equivalence of rational expressions is symmetric.

Transitive.

The more challenging property of equivalence relations to prove is the transitive property. To prove transitivity, we assume that we have three rational expressions,

$$\frac{a}{b}, \frac{c}{d}, \text{ and } \frac{m}{n},$$

with b, d , and n all non-zero, with

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{and} \quad \frac{c}{d} \sim \frac{m}{n}.$$

From the definition of equivalence of rational expressions we have that $ad = bc$ and $cn = md$. From the properties of multiplication of integers we have that

$$(ad) \cdot (cn) = (bc) \cdot (md),$$

or equivalently,

$$ancd = bmcd.$$

This can then be rewritten as

$$(an - bm) \cdot (cd) = 0.$$

Since the integers have the property that the product of two non-zero integers is also non-zero, we know that either $an - bm = 0$ or $cd = 0$. Since d is assumed to be non-zero, $cd = 0$ would imply that $c = 0$. This would then imply that both ad and md are zero, since $ad = bc$ and $cn = md$. Since $d \neq 0$ we would then have that $a = 0$ and $m = 0$, and consequently that $an = bm = 0$. Therefore, $an - bm = 0$ independently of $cd = 0$. Therefore, we have that $an = bm$ and that $\frac{a}{b} \sim \frac{m}{n}$, implying that equivalence of rational expressions is transitive.

Since this relation on rational expressions is an equivalence relation we should explore what the equivalence classes of these rational expressions look like.

We will first look at the rational expressions that are equivalent to $\frac{0}{1}$. We see that

$$\left[\begin{matrix} 0 \\ 1 \end{matrix} \right] = \left\{ \frac{a}{b} \mid b \neq 0, 0 \cdot b = 1 \cdot a \right\} = \left\{ \frac{0}{b} \mid b \neq 0 \right\}$$

Similarly,

$$\left[\begin{matrix} 1 \\ 1 \end{matrix} \right] = \left\{ \frac{a}{b} \mid b \neq 0, 1 \cdot b = 1 \cdot a \right\} = \left\{ \frac{a}{a} \mid a \neq 0 \right\}.$$

We can follow this process and see that we can embed the integers into this set of equivalence classes using the correspondence of

$$n \in \mathbb{Z} \leftrightarrow \left[\begin{matrix} n \\ 1 \end{matrix} \right].$$

4.3.1 Operations

Now that we understand the set of equivalence classes of rational expressions, we need to define our operations of addition and multiplication on this set and verify that these definitions are well-defined.

Definition 4.1. Let $[\frac{a}{b}]$ and $[\frac{c}{d}]$ be two equivalence classes of rational expressions with $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. We define addition and multiplication as

$$\left[\frac{a}{b} \right] + \left[\frac{c}{d} \right] := \left[\frac{ad + bc}{bd} \right] \quad \text{and} \quad \left[\frac{a}{b} \right] \times \left[\frac{c}{d} \right] := \left[\frac{ac}{bd} \right].$$

Since b and d are non-zero integers, the no non-zero zero divisors property tells us that $bd \neq 0$ and so the operations of addition and multiplication are closed. We can make sure that they operate appropriately on the equivalence classes by making sure that different representations for the original equivalence classes of rational expression generate a sum and product that are elements of the same equivalence class. The process is similar to the same process that was done with the integers. Since we did the proof for addition being well-defined on the integers, we will do the proof that multiplication is well-defined for the rational numbers.

Let $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$. Using the definition of the equivalence classes, we have that

$$a \cdot b' = b \cdot a' \quad \text{and} \quad c \cdot d' = d \cdot c'.$$

Therefore,

$$\begin{aligned} (bd) \cdot (a'c') &= (ba') \cdot (dc') && \text{by the commutative property of multiplication} \\ &= (ab') \cdot (dc') && \text{since } ab' = ba' \\ &= (ab') \cdot (cd') && \text{since } cd' = dc' \\ &= (ac) \cdot (b'd') && \text{by the commutative property of multiplication} \end{aligned}$$

and so we have that $\frac{ac}{bd} \sim \frac{a'c'}{b'd'}$.

So we have constructed a new number system that we call the rational numbers, \mathbb{Q} . In order to simplify our notation, we will no longer write each rational number as an equivalence class. We will instead just keep in mind the equivalence class structure and use any of the possible elements of the equivalence class to represent the rational number. We will also often write rational expressions of the form $\frac{n}{1}$ as the corresponding integer n . We will also write

$$\frac{-a}{b}, \quad \frac{a}{-b}, \quad \text{and} \quad -\frac{a}{b}$$

as equivalent expressions.

Related Content Standards

- (7.NS.2)]Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
 - Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.
 - Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real world contexts.
 - Apply properties of operations as strategies to multiply and divide rational numbers.

4.3.2 Order

We can define an order on \mathbb{Q} by first defining the relationship between a rational expression and 0, built on the order on the integers. We say that for a rational expression $\frac{a}{b}$, with $b \neq 0$ that

$$\left(\frac{a}{b} < 0 \Leftrightarrow ab < 0 \right), \quad \left(\frac{a}{b} = 0 \Leftrightarrow ab = 0 \right), \text{ and } \left(\frac{a}{b} > 0 \Leftrightarrow ab > 0 \right).$$

From the no non-zero divisors property of the integers, we know that one, and only one, of the three statements can be true for a single rational expression. If $\frac{c}{d} \sim \frac{a}{b}$, we use that $ad = bc$ and

$$(ad)^2 = (ad)(ad) = (ad)(bc) = (ab)(cd)$$

to see that ab and cd have the same relationship to 0. So this ordering with respect to 0 is well-defined.

We can then extend the ordering to the entire set of rational numbers by defining

$$\begin{aligned} \left(\frac{a}{b} < \frac{c}{d} \Leftrightarrow \left(\frac{a}{b} + \frac{-c}{d} \right) < 0 \right), \quad \left(\frac{a}{b} = \frac{c}{d} \Leftrightarrow \left(\frac{a}{b} + \frac{-c}{d} \right) = 0 \right), \\ \text{and } \left(\frac{a}{b} > \frac{c}{d} \Leftrightarrow \left(\frac{a}{b} + \frac{-c}{d} \right) > 0 \right). \end{aligned}$$

If we have rational numbers $\frac{a}{b}$, $\frac{m}{n}$, and $\frac{p}{q}$ with

$$\frac{a}{b} < \frac{m}{n}, \quad \text{and} \quad \frac{m}{n} < \frac{p}{q},$$

then

$$\frac{m}{n} + \frac{-a}{b} > 0, \quad \frac{p}{q} + \frac{-m}{n} > 0$$

and so

$$\frac{p}{q} + \frac{-a}{b} = \frac{p}{q} + \frac{-m}{n} + \frac{m}{n} + \frac{-a}{b} > 0$$

and this relation satisfies the properties of an order and we can extend the properties of inequalities from the integers.

Theorem 4.5. *Let $r, s, t \in \mathbb{Q}$. Then*

- (1) $(r = s) \Leftrightarrow (r + t = s + t)$,
- (2) $(r < s) \Leftrightarrow (r + t < s + t)$,
- (3) $(r = s) \Leftrightarrow (r \times t = s \times t)$, with $t \neq 0$
- (4) $(r < s) \Leftrightarrow (r \times t < s \times t)$, with $t > 0$.

Proof. In order to get the idea of the proof we will prove parts (1) and (4) and leave the proofs of the other two parts to the reader.

Let $r = \frac{a}{b}$, $s = \frac{c}{d}$, and $t = \frac{m}{n}$ be rational numbers with $a, b, c, d, m, n \in \mathbb{Z}$ and $b, d, n \neq 0$.

Part (1). Then

$$(r = s) \Leftrightarrow (ad = bc) \Leftrightarrow ((ad)n^2 = (bc)n^2) \Leftrightarrow ((an + bm)(dn) = (cn + dm)(bn))$$

since d and n are non-zero. We then see that

$$((an + bm)(dn) = (cn + dm)(bn)) \Leftrightarrow \left(\frac{an + bm}{bn} = \frac{cn + dm}{dn} \right) \Leftrightarrow (r + t = s + t)$$

by the definition of equality of rational numbers and properties of the operations on the integers.

Part (4). We can assume without any loss of generality that $b, d, n > 0$ by changing the signs in the numerators, if needed. Then we know that

$$(r < s) \Leftrightarrow ((s - r) > 0) \Leftrightarrow \left(\frac{cb - ad}{bd} > 0 \right) \Leftrightarrow ((cb - ad)(bd) > 0) \Leftrightarrow ((cb - ad) > 0)$$

since $bd > 0$.

Similarly,

$$(r \times t < s \times t) \Leftrightarrow \left(\frac{am}{bn} < \frac{cm}{dn} \right) \Leftrightarrow \left(\frac{bcmn - admn}{bdn^2} > 0 \right).$$

Since $b, d, n > 0$, we know that this is equivalent to

$$((bcmn - admn) > 0) \Leftrightarrow ((cb - ad)(mn) > 0) \Leftrightarrow ((cb - ad) > 0)$$

since $t > 0$ and $n > 0$ implies that $m > 0$ and $mn > 0$.

Therefore we have that

$$(r < s) \Leftrightarrow (r \times t < s \times t).$$

□

Related Content Standards

- (7.NS.1)] Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
 - a. Describe situations in which opposite quantities combine to make 0. For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.
 - b. Understand $p + q$ as the number located a distance $|q|$ from p , in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
 - c. Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.
 - d. Apply properties of operations as strategies to add and subtract rational numbers.

4.3.3 Algebraic Properties

Since the rational numbers were constructed from the integers, they maintain many of the corresponding properties of the integers. So if $\frac{a}{b}$, $\frac{m}{n}$, and $\frac{p}{q}$ are elements of \mathbb{Q} .

Table 4.3: Properties of Rational Numbers

Property
$\frac{a}{b} + \frac{m}{n} \in \mathbb{Q}$
$\frac{a}{b} + \left(\frac{m}{n} + \frac{p}{q} \right) = \left(\frac{a}{b} + \frac{m}{n} \right) + \frac{p}{q}$
$\frac{a}{b} + \frac{m}{n} = \frac{m}{n} + \frac{a}{b}$
$\frac{a}{b} + 0 = \frac{a}{b}$
$\frac{a}{b} \cdot \left(\frac{m}{n} + \frac{p}{q} \right) = \left(\frac{a}{b} \cdot \frac{m}{n} \right) + \left(\frac{a}{b} \cdot \frac{p}{q} \right)$
$\frac{a}{b} + \left(-\frac{a}{b} \right) = 0$
If $\frac{a}{b}, \frac{m}{n} > 0$, then $\frac{a}{b} \cdot \frac{m}{n} > 0$.
$\frac{a}{b} \cdot \frac{m}{n} = 0$ if, and only if, $\frac{a}{b} = 0$, $\frac{m}{n} = 0$, or both
$\frac{a}{b} \cdot \frac{m}{n} \in \mathbb{Q}$
$\frac{a}{b} \cdot \left(\frac{m}{n} \cdot \frac{p}{q} \right) = \left(\frac{a}{b} \cdot \frac{m}{n} \right) \cdot \frac{p}{q}$
$\frac{a}{b} \cdot \frac{m}{n} = \frac{m}{n} \cdot \frac{a}{b}$
$\frac{a}{b} \cdot 1 = \frac{a}{b}$
$\left(\frac{a}{b} + \frac{m}{n} \right) \cdot \frac{p}{q} = \left(\frac{a}{b} \cdot \frac{p}{q} \right) + \left(\frac{m}{n} \cdot \frac{p}{q} \right)$
If $a, b \neq 0$, $\frac{a}{b} \cdot \left(\frac{b}{a} \right) = 1$
If $\frac{a}{b}, \frac{m}{n} < 0$, then $\frac{a}{b} \cdot \frac{m}{n} > 0$.
No zero divisors

4.3.4 Properties of Exponents

Using our definition of exponents from Section 4.1, we will expand the number system for the base from the integers to the rational numbers.

If $\frac{p}{q}$ (with $q \neq 0$) is a rational number, for any natural number n we define

$$\left(\frac{p}{q}\right)^n := \frac{p^n}{q^n}.$$

Such a definition makes sense because we know from Section 4.1 that p^n and q^n are both well-defined integers, that $q^n \neq 0$, and the ratio of such integers in a rational number.

We can then extend the number system for the exponents from the natural numbers to the integers for $p, q \neq 0$ by defining

$$\left(\frac{p}{q}\right)^{-n} = \left(\frac{q}{p}\right)^n$$

for all $n \in \mathbb{N}$. In the case that $p = 0$, this definition would not be well-defined and so we define $0^n = 1$ for all $n \in \mathbb{Z}$.

Related Content Standards

- (8.EE.1) Know and apply the properties of integer exponents to generate equivalent numerical expressions.

With the extended definition of exponents we still have for all rational numbers, a , that $a^0 = 1$ and $a^1 = a$. We also have from our definition that for $a \neq 0$ and $n \in \mathbb{Z}$,

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n.$$

For $m, n \in \mathbb{N}$ and integers p and q , with $q \neq 0$,

$$\left(\frac{p}{q}\right)^m \cdot \left(\frac{p}{q}\right)^n = \left(\frac{p^m}{q^m}\right) \cdot \left(\frac{p^n}{q^n}\right) = \frac{p^m \cdot p^n}{q^m \cdot q^n} = \frac{p^{(m+n)}}{q^{(m+n)}} = \left(\frac{p}{q}\right)^{(m+n)}.$$

This can then be extended to $m, n \in \mathbb{Z}$ with

- [Case 1:] If $p = 0$, then $\frac{p}{q} = 0$ and

$$0^m \cdot 0^n = 0 \cdot 0 = 0 = 0^{(m+n)}.$$

- [Case 2:] If $p \neq 0$, we let $m, n \in \mathbb{Z}$. If m and n are both non-negative, then we have the property given above. If m and n are both negative, then

$$\left(\frac{p}{q}\right)^m \cdot \left(\frac{p}{q}\right)^n = \left(\frac{q}{p}\right)^{-m} \cdot \left(\frac{q}{p}\right)^{-n} = \left(\frac{q}{p}\right)^{(-m-n)} = \left(\frac{p}{q}\right)^{m+n}.$$

If one is negative, m , and the other is non-negative, n , then

$$\left(\frac{p}{q}\right)^m \cdot \left(\frac{p}{q}\right)^n = \frac{p^m}{q^m} \cdot \frac{q^{-n}}{p^{-n}} = \frac{p^{m+n}}{q^{m+n}} = \left(\frac{p}{q}\right)^{m+n}.$$

Therefore, we have that for all $a \in \mathbb{Q}$ and for all $m, n \in \mathbb{Z}$ that $a^m \cdot a^n = a^{m+n}$.

One can make similar arguments to extend the remainder of Theorem 4.4 so that $(ab)^n = a^n \cdot b^n$ for each $n \in \mathbb{Z}$ and $\$(a^m)^n = a^{\{mn\}}\$$ for each $m, n \in \mathbb{Z}$.

Since the rational numbers include values between integers, it makes sense to explore the relationships between exponents and the order of the rational numbers given by inequalities. If $a \in \mathbb{Q}$ with $a > 1$, then we can write a as $\frac{p}{q}$ for some integers p and q with $0 < q < p$, and from Theorem 4.4 we have $0 < q^n < p^n$ for any integer $n \geq 1$. So we can conclude that $a^n > 1$. If n is a negative integer, then $0 < q^{-n} < p^{-n}$ and so $a^{-n} > 1$, and hence $a^n < 1$.

If a and b are rational numbers with $0 < a < b$, then $\frac{a}{b} \in \mathbb{Q}$ with $0 < \frac{a}{b} < 1$ so $0 < \left(\frac{a}{b}\right)^n < 1$ and $0 < a^n < b^n$.

We combine all of the results into the following theorem.

Theorem 4.6. *Let a and b be rational numbers.*

1. $a^0 = 1$ and $a^1 = a$
2. If $a > 1$, then $a^n > 1$ for all integers $n > 0$, and if $a < 1$, then $0 < a^n < 1$ for all integers $n < 0$
3. $a^{-n} = \frac{1}{a^n}$, $a^m \cdot a^n = a^{m+n}$, $(ab)^n = a^n \cdot b^n$, and $(a^m)^n = a^{mn}$ for each $m, n \in \mathbb{Z}$
4. If $0 < a < b$ and $m \in \mathbb{N}$, then $0 < a^m < b^m$.
5. If $a > 1$ and $m < n$, then $a^m < a^n$.

4.3.5 Exercises

1. Prove that there exists a rational number between any two rational numbers. (If $a, b \in \mathbb{Q}$ with $a < b$, then there exists a $c \in \mathbb{Q}$ such that $a < c < b$.)
2. Let $x, y \in \mathbb{Q}$ such that $0 < x < y$. Prove that $0 < \frac{1}{y} < \frac{1}{x}$.
3. Look graphically at $\mathbb{Z} \times (\mathbb{Z} - \{0\})$, which is a different representation of $\widehat{\mathbb{Q}}$ defined at the beginning of this section.
 - a. How would you graphically describe the equivalence classes of the relation $(a, b) \sim (c, d) \Leftrightarrow ad = bc$?
 - b. How does this interact with the standards in Grades 7 and 8 regarding ratio and direct proportions?
4. Compare the definition of order of rational numbers in this section with other techniques of determining the order of rational numbers (i.e. cross-multiplying, finding lowest common denominator, ‘butterfly method’, etc.).
 - a. What types of (mis)conceptions arise with each of these methods?
 - b. Is there an order to presenting these various methods for determining the order of rational numbers that improves student understanding?

4.4 Representations of Rational Numbers

Each rational number can be represented in several different ways, with the most prominent being as a mixed number, a fraction, a decimal, or a percent. In this section we will discuss how to convert between these different representations and how to use other physical and graphical representations to understand and use rational numbers.

4.4.1 Decimal Representations

It is sometimes helpful to write rational numbers in a decimal representation, rather than fraction notation. This is particularly true in the comparison of two rational numbers. One process for finding the decimal representation from a fraction is long division, but it is often simpler to use technology to find the decimal representation. (It also greatly reduces the amount of error involved.) However, it is important to have a deeper understanding of the decimal representation of rational numbers to verify that the result obtained

from a calculator or computer is correct. For instance, if one uses a calculator to find $\frac{8}{17}$, a calculator with an 8 digit display will give the answer 0.4705882, and a calculator with a 16 digit display will give the answer 0.470588235294118. There is no apparent repetition in this decimal, leading to the question of if it repeats or not. It is the repeated use of the division algorithm that allows us to determine why a decimal representation of a rational number always repeats.

Related Content Standards

- (7.NS.2) Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
 - d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

We will state the division algorithm for integers here, but the proof is in Section 7.1.

Theorem 4.7 (The Division Algorithm for Integers). *If a and b are integers with $b > 0$, then there are unique integers q and r such that*

$$a = bq + r, \quad \text{with } 0 \leq r < b.$$

We will explain the process of long division and its relationship to the division algorithm to determine the decimal representation of rational numbers using an example.

In order to find the decimal expansion of $\frac{7}{12}$, we will first rewrite the rational expression as $\frac{1}{10} \cdot \frac{70}{12}$. We then note that $70 = 12 \cdot 5 + 10$ in the division algorithm. Thus,

$$\frac{7}{12} = \frac{1}{10} \cdot \frac{70}{12} = \frac{1}{10} \cdot \left(5 + \frac{10}{12}\right) = \frac{5}{10} + \frac{1}{10} \cdot \frac{10}{12}.$$

We continue this process with $\frac{10}{12}$ so that we have

$$\frac{7}{12} = \frac{5}{10} + \frac{1}{10^2} \left(\frac{100}{12}\right) = \frac{5}{10} + \frac{1}{10^2} \left(8 + \frac{4}{12}\right) = \frac{5}{10} + \frac{8}{10^2} + \frac{1}{10^2} \cdot \frac{4}{12}.$$

We continue the process, so that

$$\frac{7}{12} = \frac{5}{10} + \frac{8}{10^2} + \frac{1}{10^3} \cdot \frac{40}{12} = \frac{5}{10} + \frac{8}{10^2} + \frac{3}{10^3} + \frac{1}{10^3} \cdot \frac{4}{12}.$$

We notice that this remainder of 4 again tells us that we will get the same quotient again and again. Therefore, we see that $\frac{7}{12} = 0.58\bar{3}$.

As we can see with this process, a decimal will start repeating when a remainder in the division algorithm is repeated. Since the only possible remainders of a quotient by 12 are in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, we see that any decimal representation of a fraction whose denominator is 12 will repeat after at most 11 places.

Similarly, the decimal representation for any rational number will eventually repeat. This repetition may be of the digit 0, in which case we will call it a **finite decimal expansion**, or it may be a different number of digits, in which we will call it a **repeating decimal expansion**. In the case of $\frac{7}{12}$, we see that the repeating happens after a certain number of digits. These types of decimal representations will be called **delayed repeating decimal expansions**. One can determine the type of decimal expansion based on the prime factorization of the denominator, once the fraction is written in its simplified form.

Related Content Standards

- (8.NS.1) Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Now that we know that all rational numbers can be represented by decimal expansions that are finite, repeating, or delayed repeating, we will study the process of starting with a decimal representation and determining a fraction representation. We already know that if the decimal representation does not repeat that it does not correspond to a rational number.

If the number has a finite decimal expansion, then it can be converted to a fraction using the base ten representation. For instance the number 3627.854 can be rewritten as $\frac{3627854}{1000}$.

Let's look at a decimal expansion that is repeating like $0.\overline{35682}$. This could be written in the form

$$0.\overline{35682} = \frac{10^5}{10^5} 0.\overline{35682} = \frac{1}{10^5} \cdot 35682.\overline{35682} = \frac{1}{10^5} (35682 + 0.\overline{35682}).$$

We can rearrange this equation so that

$$10^5 \cdot 0.\overline{35682} - 0.\overline{35682} = 35682$$

or equivalently that

$$0.\overline{35682} = \frac{35682}{10^5 - 1}.$$

This same process can be used to find the fractional representation of any rational number with a repeating decimal expansion.

If a rational number has a delayed repeating decimal expansion, then one can rewrite the number in a way to generate a repeating decimal. For instance,

$$324.51\overline{89} = \frac{10^2}{10^2} \cdot 324.51\overline{89} = \frac{1}{10^2} (32451.\overline{89}) = \frac{1}{10^2} \left(32451 + \frac{89}{10^2 - 1} \right) = \frac{(32451 \cdot 99) + 89}{9900}.$$

4.4.2 Physical and Graphical Representations

Related Content Standards

- (6.NS.1) Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $\frac{2}{3} \div \frac{3}{4}$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$ because $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$. (In general, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.) How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{3}{4}$ -cup servings are in $\frac{2}{3}$ of a cup of yogurt? How wide is a rectangular strip of land with length $\frac{3}{4}$ mi and area $\frac{1}{2}$ square mi?

Consider the following problem from the Singapore Mathematics Curriculum:

Jenny, Bob, and Paul shared a sum of money. Jenny received $\frac{2}{5}$ of the money, Bob received $\frac{1}{4}$ of the money and Paul received the rest of the money. If Paul received \$1.50 more than Bob, how much more money did Jenny receive than Paul.

While this is not a modeling problem, in that this type of situation would never be described in this way, it does help generate fluency in working with different representations of rational numbers. We will walk through one possible solution that does not use variables, but does build the conceptual understanding necessary for successful work in algebra.

The first thing that we notice is that we need a way to represent $\frac{2}{5}$ and $\frac{1}{4}$ of the same whole. As such, we need our whole item to be broken into 20 equal sized pieces.

We then note that Jenny received $\frac{2}{5}$ of the money, which is equivalent to $\frac{8}{20}$ of the money. So we will mark 8 of the pieces with a J. Similarly, Bob received $\frac{1}{4}$ of the money and so we will mark 5 pieces with a B.

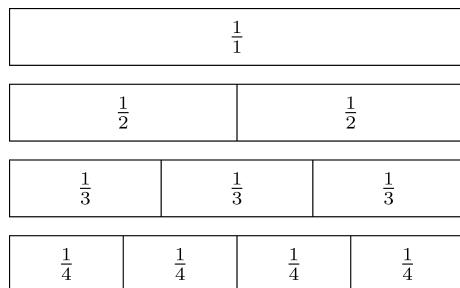
We then see that Paul received the equivalent of 7 blocks, while Bob received 5 blocks. Since Paul received \$1.50 more than Bob we know that 2 blocks is equivalent to \$1.50, making each block represent \$0.75. Since Jenny received the equivalent of 1 block more than Paul we see that she received \$0.75 more than Paul.

J	J	J	J	J
J	J	J		
B	B	B	B	B

There are many other ways to solve this problem that also develop numerical fluency with rational numbers, and the more ways that students can represent these numbers and move between the representations, the more fluent they will be with the rational number system.

4.4.2.1 Fraction Strips

One common tool used to help students determine equivalent fraction representations is through the use of fraction strips. These are pieces of paper (all the same size), but marked off in different numbers of parts. Then by setting the pieces of paper next to each other, students are able to notice equivalency of fractions. For instance, in the fraction strips below one can see that $\frac{1}{2}$ is equivalent to $\frac{2}{4}$.



One can also use fraction strips to represent the operation of division of fractions. For instance if we have the two fraction strips below we can use them to determine that $\frac{1}{2} \div \frac{1}{6} = 3$ because 3 of the $\frac{1}{6}$ regions can fit inside of the $\frac{1}{2}$ region.

4.4.2.2 Pattern Blocks

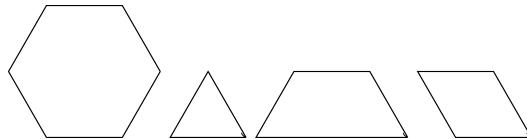
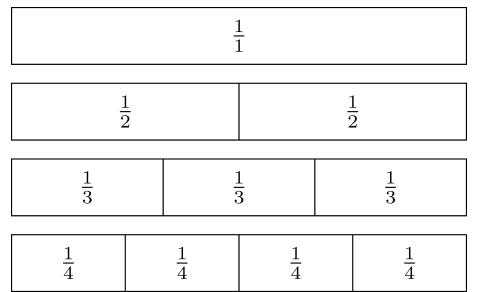
Another tool to help students work with fractions are pattern blocks. The pattern blocks most useful for many of the fractions that students use are the hexagon, triangle, trapezoid, and rhombus below.

Students can see that 6 triangles, 2 trapezoids, or 3 rhombi fit inside of the hexagon. So we can represent $\frac{2}{3}$ as two rhombi, with one hexagon representing the whole. One can then see that fractions whose denominator is 2, 3, or 6 are easily represented using one hexagon as the whole. However, if we allow two hexagons to represent the whole, we can represent fractions with denominators of 2, 3, 4, 6, or 12. One can also think about $\frac{1}{2} \div \frac{1}{3}$ as how many rhombi fit inside of one trapezoid.

4.4.2.3 Area Model

The area model described in Section 4.2 can help students to visualize multiplication of fractions. For instance, one can find the product of $\frac{2}{5}$ and $\frac{2}{3}$ using the following diagram. One can then see that $\frac{4}{15}$ of the squares are shaded. So $\frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$.

We are also able to build upon the area model to visualize the distribution property involved in the multiplication of mixed numbers. If we want to find $23\frac{2}{3} \times 32\frac{5}{7}$, then we have to remember the convention



that $23\frac{2}{3} = 23 + \frac{2}{3}$ and $32\frac{5}{7} = 32 + \frac{5}{7}$. (Note how confusing this convention can be for students, particularly since in algebra two numbers next to each other represent multiplication instead of addition.) We can then allow for an abstract area representation (the sizes are not proportional) to more easily use the distributive property.

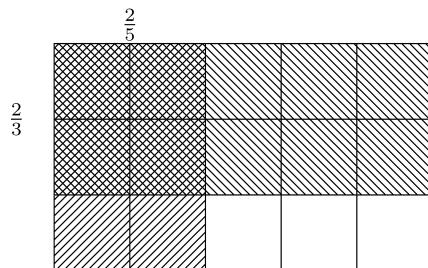
There are many other tools that can help students understand rational numbers and their operations, but we will leave those for more pedagogically oriented sources.

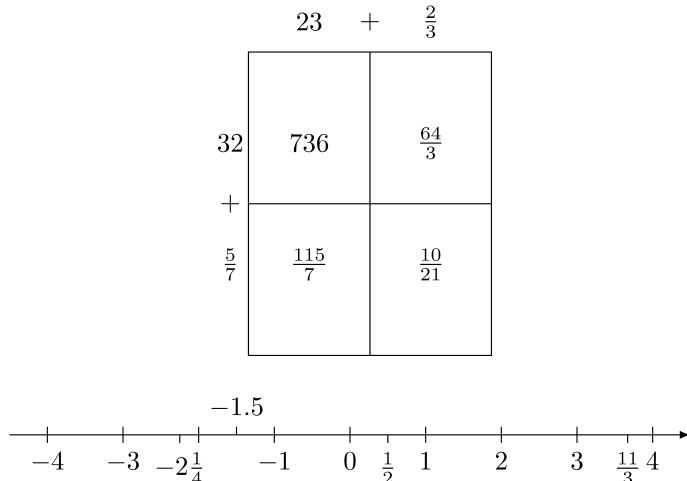
4.4.2.4 Number Line and Plane

It is also important for students to use rational numbers on a number line (and the associated plane). In particular, the number line can help students understand inequalities, relationships between different numbers, and the absolute value of a number. Some key factors to keep in mind are to not only have fractions between 0 and 1, or only mixed numbers outside of that interval. Students need to use a variety of different numbers and representations together to gain fluency.

Related Content Standards

- (6.NS.7) Understand ordering and absolute value of rational numbers.
 - a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram.
 - b. Write, interpret, and explain statements of order for rational numbers in real-world contexts.
 - c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation.
 - d. Distinguish comparisons of absolute value from statements about order.





4.4.3 Exercises

1. Explain, in a way that a middle school student can understand, how to determine the type of decimal expansion for any given fraction representation of a rational number and why it is that way. (i.e. when is the decimal expansion finite? when does it repeat? when is it delayed periodic? what do each of these mean?)
2. For each of the following, give an example of a rational number in the form of $\frac{p}{q}$ with the appropriate decimal representation and explain why it works.
 - a. has a terminating decimal with 4 decimal places
 - b. has a non-zero integer part and a delay of 3 decimal places before a periodic part with period of 3 digits.
 - c. has a periodic decimal representation with 5 digits.
3. For each of the following, give the representation of the rational number in the form of $\frac{p}{q}$, with p and q relatively prime (no common factors):
 - a. 0.156
 - b. 0.23̄
 - c. 23.1̄5
 - d. 3.14159
 - e. 2.5̄9
 - f. 2016
4. Write an algorithm in the programming language of your choice to input a decimal representation of a rational number and output a fraction representation.
5. Classify each of the following real numbers as
 - a terminating decimal,
 - simple periodic decimal (note its period),
 - delayed periodic decimal (note its delay and period), or
 - non-periodic decimal
 - a. $\frac{1}{6}$
 - b. $\frac{1}{2}$
 - c. $\frac{2}{14}$
 - d. $\frac{3}{20}$
 - e. $\frac{4}{63}$
 - f. π

- g. $\sqrt{2}$
- h. $\sqrt{5}$
- i. $\frac{18}{40}$
- j. 0.75
- k. $0.83\overline{45}$

4.5 Real Numbers

One of the properties of the integers that does not extend to the rational numbers is the least-upper-bound property (3.4). In order to prove this, we first need to prove a series of small lemmas (a smaller version of a theorem that is used to prove a larger theorem).

Lemma 4.7. *If n is an integer, then n can be written as $2k$ for some integer k or $2j+1$ for some integer j , but cannot be written both ways. (In other words, an integer is even or odd, but not both.)*

Proof. We will use the technique of a proof by contradiction and assume that there is an integer n such that $n = 2k$ and $n = 2j + 1$ for some integers j and k . Then $2j = 2k + 1$ and so the equivalent equation $2(j - k) = 1$ is true. However, since 2 does not have a multiplicative inverse in the integers, $j - k$ cannot be an integer. This means that our assumption is false and our lemma is proven. \square

Lemma 4.8. *If n is an integer such that n^2 is even, then n is even.*

Proof. We will prove the contrapositive of the statement, which is logically equivalent to the original statement. Let n be an odd integer, i.e. $n = 2j + 1$ for some integer j . Then $n^2 = (2j+1)^2 = 4j^2+4j+1 = 2(2j^2+2j)+1$ and so n^2 is odd. \square

Lemma 4.9. *There does not exist a rational number whose square is 2.*

Proof. We will again use the technique of a proof by contradiction. Assume that there is a rational number whose square is 2. We will denote the rational number by $\frac{a}{b}$ where a and b are not both even. (This can be accomplished by finding an equivalent representation without 2 as a common factor.) Since its square is 2, we have that $(\frac{a}{b})^2 = 2$ and equivalently that $a^2 = 2b^2$. Since a^2 is even, the previous lemma tells us that a is even. So there exists an integer k such that $a = 2k$ and so $a^2 = 4k^2$. We then have that $4k^2 = 2b^2$ and so $b^2 = 2k^2$. This means that b^2 is even and consequently that b is even. This contradicts our original assumption and so no such rational number exists. \square

We will now look at the set

$$S = \left\{ \frac{a}{b} \in \mathbb{Q} \mid \frac{a}{b} > 0 \text{ and } \left(\frac{a}{b}\right)^2 \geq 2 \right\}$$

which is bounded below in its definition and does not have a least element. Therefore, the rational numbers do not satisfy the least-upper-bound property.

Since the rational numbers do not satisfy the least-upper-bound property, the rational numbers have “holes” that need to be filled, and so are not complete. Building upon the work of Bertrand [1840] and in a parallel vein to Heine [1872], Dedekind [1872] developed a construction of the real numbers from the rational numbers using what are now referred to as Dedekind cuts using equivalence classes of such cuts. Another method of construction of the real numbers from the rational numbers follows the path of Cauchy [1821] and defines the real numbers as equivalence classes of Cauchy sequences of rational numbers. As each of these methods requires the development of a great deal of mathematical machinery, we will refer to other textbooks in analysis for the details. With Rudin [1976] providing an excellent treatment of Dedekind cuts in the Appendix of Chapter 1. Chapter I of Thurston [1956] provides an detailed treatment of the construction of the Cauchy numbers to generate the real numbers through equivalence classes.

Because the real numbers form a complete number system in that it satisfies the least-upper-bound property, for any bounded set of real numbers, S , we can define $\sup S$ to be the least element of the set $\{x \in \mathbb{R} \mid x \geq$

$a, \forall a \in S\}$ and $\inf S$ is the greatest element of the set $\{x \in \mathbb{R} | x \leq a, \forall a \in S\}$. These two attributes of a set play an important role in the development of calculus.

Since $\sqrt{2}$ is not rational, but we know it is a real number because of the completeness of the reals, and so there are real numbers that are not rational. We define these numbers to be **irrational**.

Definition 4.2. A real number, a , is called rational if there exist integers p and q such that $a = \frac{p}{q}$. A real number that is not rational is called irrational.

Related Content Standards

- (HSN.RN.3) Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a non-zero rational number and an irrational number is irrational.

We have already shown that the sum and product of two rational numbers are rational, but we would like to explore how rational and irrational numbers interact with each other.

Theorem 4.8. Let $u \in \mathbb{Q}$ and $v \in (\mathbb{R} - \mathbb{Q})$. Then $u + v \notin \mathbb{Q}$ and if $u \neq 0$, $uv \notin \mathbb{Q}$.

Proof. When trying to prove that something does not satisfy a certain property, then it is best to use a proof by contradiction or to prove the contrapositive.

Let $u \in \mathbb{Q}$ and so there exist integers p and q such that $u = \frac{p}{q}$. Let $v \in (\mathbb{R} - \mathbb{Q})$. We will now prove that $u + v$ is not rational using a proof by contradiction. If $u + v$ is rational, then there exist integers m and n such that $u + v = \frac{m}{n}$. This means that

$$v = \frac{m}{n} - u = \frac{m}{n} - \frac{p}{q} = \frac{mq - np}{nq}$$

making it a rational number, contradicting the assumption that it is irrational.

The proof that uv is irrational is very similar. □

One of the important properties to notice about the rational and irrational numbers is that they are dense in the real numbers in that between any two real numbers one can find both a rational and an irrational number.

Theorem 4.9 (Density of the Rational Numbers). *Let $x, y \in \mathbb{R}$ be any two real numbers where $x < y$. Then there exists a rational number $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. Since $x < y$, we know that $y - x > 0$. So there exists a positive natural number $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y - x$. Equivalently, $0 < 1 < ny - nx$ and $nx + 1 < ny$. From properties of the integers we know that there exists an integer $N \in \mathbb{Z}$ such that $N - 1 \leq nx < N$, or equivalently $N \leq nx + 1 < N + 1$. Therefore,

$$nx < N \leq nx + 1 < ny$$

and so $x < \frac{N}{n} < y$. So $q = \frac{N}{n}$ is a rational number such that $x < q < y$. □

Theorem 4.10 (Density of the Irrational Numbers). *Let $x, y \in \mathbb{R}$ be any two real numbers where $x < y$. Then there exists an irrational number $v \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < v < y$.*

Proof. Since x and y are real numbers, $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$ are also real numbers such that $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$. By the density of the rational number, there exists a $q \in \mathbb{Q}$ such that

$$\frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}},$$

or equivalently $x < q\sqrt{2} < y$. Since $q \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, $q\sqrt{2}$ is irrational and so if $v = q\sqrt{2}$, $v \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < v < y$. □

Since the operations of addition and multiplication on the real numbers are associative and commutative, for each natural number n we can define a^n recursively and we can prove the following properties using the same techniques as in Section 4.1, and Section 4.3 using limit ideas from the definition of the real numbers.

Theorem 4.11. *Let $a, b \in \mathbb{R}$ and let $m, n \in \mathbb{Z}$.*

- $a^0 = 1$ and $a^1 = a$
- *If $a > 1$ and $n > 0$, then $a^n > 1$. If $a < 1$ and $n < 0$, then $0 < a^n < 1$.*
- $a^{-n} = \frac{1}{a^n}$
- $a^m \cdot a^n = a^{m+n}$, $(ab)^n = a^n \cdot b^n$, and $\$(a^m)^n = a^{\wedge}\{mn\}\$$
- *If $0 < a < b$ and $m \in \mathbb{N}$, then $0 < a^m < b^m$.*
- *If $a > 1$ and $m < n$, then $a^m < a^n$.*

With these properties, we are able to define algebraic numbers, which is a number system between the rational numbers and the real numbers.

Definition 4.3. A real number, a , is called **algebraic** if

$$n_0 + n_1a + n_2a^2 + \cdots + n_ma^m = 0$$

for some integers $n_0, n_1, n_2, \dots, n_m$.

For all rational numbers in the form $\frac{p}{q}$, we know that

$$q \cdot \left(\frac{p}{q}\right)^1 + (-p) = 0$$

and so all rational numbers are algebraic numbers. We can also see that $\sqrt{2}$ is an algebraic number since $(\sqrt{2})^2 - 2 = 0$. We say that the algebraic numbers are a number system because they include the additive identity, 0; the multiplicative identity, 1; and are closed under addition and multiplication.

We will prove in Section 5.4 that there are an infinite number of real numbers that are not algebraic. We will call any real number that is not algebraic a **transcendental** number. Two such examples of transcendental numbers are e and π .

In the early 1930's, Gel'fond [1934] and Schneider [1935] independently solved Hilbert's seventh problem from his 1900 address to the International Congress of Mathematics showing that if a and b are algebraic numbers with $a \neq 0, 1$ and b irrational that a^b is a transcendental number. Since we have such a limited knowledge of these transcendental numbers, it is evident that the continuum of the real numbers is a very deep and complicated concept that we are only beginning to understand.

We include a graphical representation of how these different sets of numbers fit together. All of these sets of numbers are subsets of the complex numbers, and a set above another set represents an inclusion relationship.

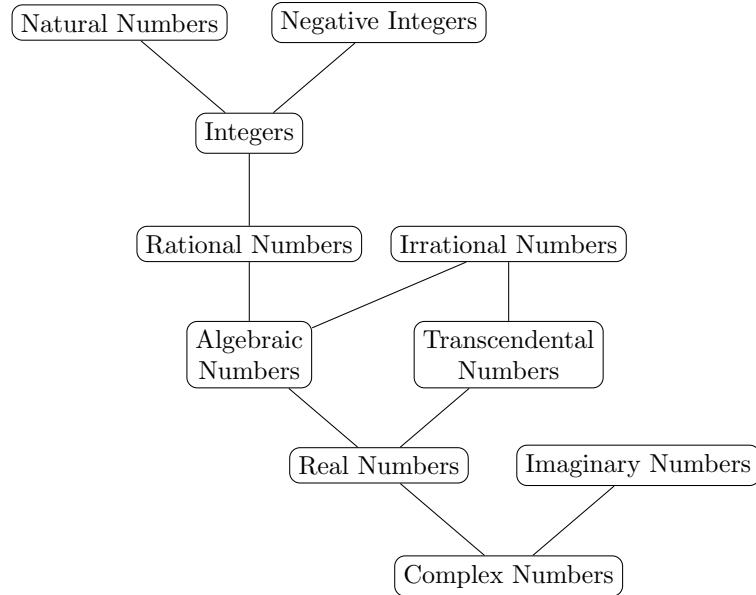
4.5.1 Properties of Exponents

Let $a > 0$ be a real number. For each integer $n > 0$, we define

$$S_n = \{x \in \mathbb{Q} | x > 0 \text{ and } x^n > a\}$$

and so we can define

$$\sqrt[n]{a} = a^{\frac{1}{n}} := \inf S_n.$$



Related Content Standards

- (8.EE.2) Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

If $a > 0$ is a real number, m, n, p, q are integers, $n > 0$, and $q > 0$ such that $\frac{m}{n} = \frac{p}{q} = r$, we have that $mq = pn$. So

$$\left(\left(a^{\frac{1}{n}}\right)^m\right)^{nq} = a^{mq} = a^{pn} = \left(\left(a^{\frac{1}{q}}\right)^p\right)^{nq}.$$

This means that we can define $a^r := (a^m)^{\frac{1}{n}}$.

Related Content Standards

- (HSN.RN.1) Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.
- (HSN.RN.2) Rewrite expressions involving radicals and rational exponents using the properties of exponents.

We can now prove various exponential properties of real numbers with rational exponents. In order to prove a generalization of Theorem 4.11 to rational exponents we will only need to prove the corresponding properties for rational numbers of the form $\frac{1}{n}$ and it directly follows from $(a^n = b^n) \Leftrightarrow a = b$ for real numbers a and b and positive integers n . As such, we will omit the majority of these proofs from this text. However, will include one lemma and proof in order to understand the flavor of the proofs.

Lemma 4.10. *Let $a \in \mathbb{R}^+$. For all $r, s \in \mathbb{Q}$, $a^{r+s} = a^r \cdot a^s$.*

Proof. Let $a \in \mathbb{R}^+$ and let $r, s \in \mathbb{Q}$ such that $r = \frac{m}{n}$ and $s = \frac{p}{q}$ for some integers m, n, p , and q .

Since $q \neq 0$ and $n \neq 0$,

$$(a^{r+s})^{qn} = \left(a^{\frac{mq+pn}{qn}}\right)^{qn} = a^{(mq+pn)}$$

and

$$(a^r \cdot a^s)^{qn} = \left(a^{\frac{mqn}{n}}\right) \cdot \left(a^{\frac{pqn}{q}}\right) = a^{(mq+pn)}.$$

Therefore, $a^{r+s} = a^r \cdot a^s$. □

For a real number y , define

$$S_y = \{a^r \mid r \leq y\}.$$

Furthermore, define

$$a^y = \sup S_y.$$

Using arguments from analysis and the properties of limits, one can extend the previous properties of exponents to real valued exponents as given in Theorem 4.12. Due to the mathematical machinery necessary for these proofs, we will refer the reader to an analysis textbook.

Theorem 4.12. *Let a and b be positive real numbers and let $x, y \in \mathbb{R}$.*

- $a^0 = 1$ and $a^1 = a$
- $a^x > 1$ for all $x > 0$, and $0 < a^x < 1$ for all $x < 0$
- $a^{-x} = \frac{1}{a^x}$
- $a^x \cdot a^y = a^{x+y}$, $(ab)^x = a^x \cdot b^x$, and $\$(a^x)^y = a^{\hat{x}\{xy\}}\$$
- If $0 < a < b$ and $x > 0$, then $a^x < b^x$.
- If $a > 1$ and $x < y$, then $a^x < a^y$.

4.5.2 Exercises

1. If $n = 2$, $a^{\frac{1}{2}}$ is often written as $\sqrt{2}$, but for $n = \{3, 4, 5, \dots\}$, $a^{\frac{1}{n}}$ is written as $\sqrt[n]{a}$. This difference in notation can cause many problems for students. What are some other notation differences in the secondary curriculum that have similar situations?
2. What can be said about the sum and product of two irrational numbers?
3. Suppose that $a \in \mathbb{R}$ with $a > 0$. If n is an even positive integer, show that there are exactly two real numbers that are solutions to the equation $x^n = a$.
4. Show that $\sqrt{2} + \sqrt{3}$ is an algebraic number by finding a polynomial for which it is a zero.
5. Write out a detailed proof that $\sqrt{3}$ is irrational using the techniques of the proof that $\sqrt{2}$ is irrational. For which numbers can this process be generalized?

4.6 Complex Numbers

Unlike all of the previous number system constructions, in order to construct the complex numbers, \mathbb{C} , we do not have to work through equivalence classes. Instead, we look at the set

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

and define the addition and multiplication operations as

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

We can see that elements of the form $(a, 0)$, $\mathbb{R} \times \{0\}$, act just like the real numbers in that

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0) \cdot (b, 0) = (ab, 0)$$

and so we can think of these as being the real numbers inside of the complex numbers.

Elements of the form $(0, a)$, $\{0\} \times \mathbb{R}$, operate very differently, particularly with regard to multiplication, since

$$(0, a) + (0, b) = (0, a + b) \quad \text{and} \quad (0, a) \cdot (0, b) = (-ab, 0).$$

4.6.1 Rectangular Representation

If we define scalar multiplication on this set by $k \cdot (a, b) = (ka, kb)$, we see that the elements of the set $\mathbb{R} \times \mathbb{R}$ operate as a vector space in that for all $(a, b) \in \mathbb{R} \times \mathbb{R}$, $(a, b) = a \cdot (1, 0) + b \cdot (0, 1)$. If we then replace $(1, 0)$ with its corresponding element of 1 in \mathbb{R} and label 0, 1 with a new symbol that we call i , we have the complex numbers,

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

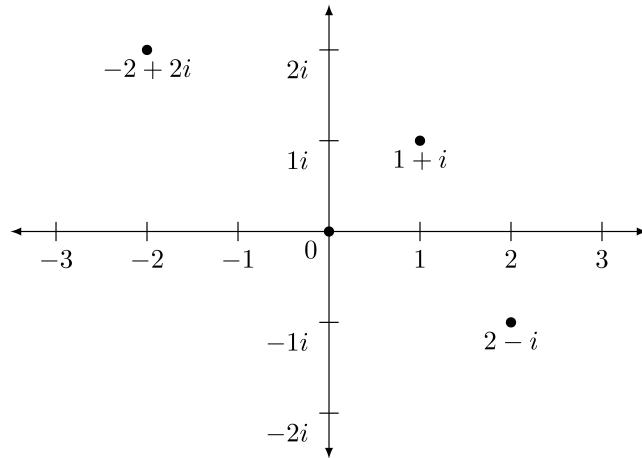
with addition and multiplication defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad \text{and} \quad (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

Related Content Standards

- (HSN.CN.1) Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
- (HSN.CN.2) Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

In the eighth chapter of the greatest work of mathematics in the 18th century, possibly even in the modern era, Leonard Euler [1748] described a geometric, algebraic, and analytic connection between ways of representing complex numbers that enabled many of the mathematical and scientific discoveries of the past 350 years. What Euler recognized is that in addition to representing complex numbers as $a + bi$, one can describe the complex numbers geometrically as points on the plane where the real part of the complex number provides the horizontal component and the imaginary part of the complex number provides the vertical component.



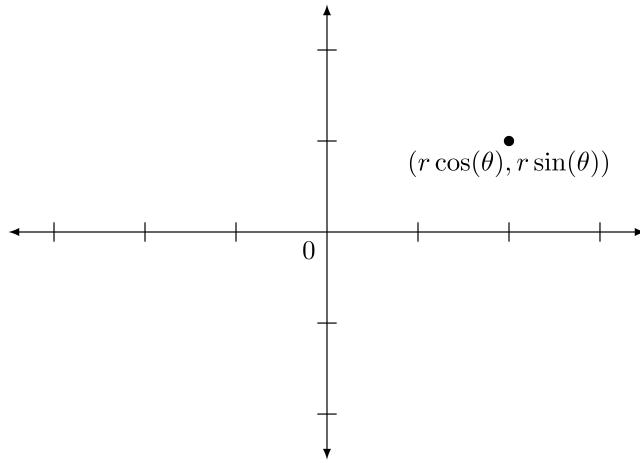
4.6.2 Complex Conjugation and Modulus

Related Content Standards

- (HSN.CN.4) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

Since points on the plane can also be described by the distance from the origin and the angle made with the positive horizontal axis, complex numbers can also be described by a magnitude and an angle. When we combine that with knowledge of trigonometric functions we have that every complex number can be written in the form

$$a + bi = r(\cos(\theta) + i \sin(\theta))$$



If we look at the correspondence of $a = r \cos(\theta)$ and $b = r \sin(\theta)$, we can derive formulas to solve for r and θ in terms of a and b ,

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{if } a \geq 0 \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{if } a < 0 \end{cases}.$$

In addition to this polar representation of a complex number, Euler defined the **complex conjugate** of a number $z = a + bi$ to be $\bar{z} = a - bi$. Geometrically, the complex conjugate of a number is its reflection across the horizontal axis. We also see that if we multiply a complex number $z = a + bi$ by its complex conjugate we have $(z\bar{z}) = a^2 + b^2 = r^2$, which is the square of the **modulus**, $|z|^2$, or distance from 0.

Notice that the modulus of a complex number is a real number. This makes it easier to divide by complex numbers. Using the complex conjugate we can write

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z\bar{w}}{|w|^2} \quad \text{or} \quad \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \left(\frac{ac+bd}{c^2+d^2} \right) + \left(\frac{bc-ad}{c^2+d^2} \right) i$$

for complex numbers $z = a + bi$ and $w = c + di$.

This modulus of a complex number generates a distance between two complex numbers, $|z_2 - z_1|$, in a similar manner to absolute value measuring the distance between two real numbers, $|x_2 - x_1|$.

Related Content Standards

- (HSN.CN.3) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.
- (HSN.CN.5) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.
- (HSN.CN.6) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

4.6.3 Euler's Equation

Euler's [1748] greatest discovery that one can use infinite series to find a strong connection between exponential functions and trigonometric functions and that

$$e^{x+yi} = e^x (\cos(y) + i \sin(y)) \quad \text{Euler's Equation,}$$

where e is the (Euler) constant

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

This polar representation of complex numbers simplifies multiplication and exponents in that

$$r_1 e^{i\alpha} \cdot r_2 e^{i\beta} = (r_1 \cdot r_2) e^{i(\alpha+\beta)} \quad \text{and} \quad (r e^{i\theta})^n = r^n e^{in\theta}.$$

Hence we can see that multiplication of complex numbers involves multiplying the moduli of the numbers and adding the angles, sometimes called an amplitwist.

We will explore these properties of the complex numbers in depth in Chapter 12.

4.6.4 Exercises

1. For each of the following complex numbers, plot the number on the complex plane and write the number in the form $a + bi$ and in the form $re^{i\theta}$. (You may leave θ as an arctangent of a number for angles that are not the standard angles.)
 - a. $-14 + 14i$
 - b. $\frac{1+i}{1-\sqrt{3}i}$
 - c. $3e^{\frac{-2\pi}{3}i}$
 - d. $1 + 3i$
 - e. $\overline{-2+5i}$
2. Use the Maclaurin series representations of e^x , $\cos(x)$, and $\sin(x)$ to prove that

$$e^{iy} = \cos(y) + i \sin(y).$$

3. Use Euler's equation to prove the angle addition identities for \sin and \cos .
4. Use Euler's equation to prove the double angle identities for \sin and \cos .
5. Use Euler's equation to show De Moivre's formula

$$(\cos(x) + i \sin(x))^n = \cos(nx) + i \sin(nx)$$

is true and use the formula to find identities for $\cos(3x)$ and $\sin(3x)$ in terms of $\cos(x)$ and $\sin(x)$.

Chapter 5

Functions

The next foundational concept of mathematics that we will study is the concept of function. Some of the early ideas of function involved describing how different quantities covaried, with the work of Oresme (1323-1382) being some of the earliest related work. Our current concept of function arose in the 17th century with the rise of analytic geometry and the foundation of calculus, with Leibniz first using the word the late 1600's [Ponte, 1992].

5.1 Definitions of Functions

Currently there are two dominant perspectives for defining a function. The first is the concept of function as a rule or procedure, as is denoted in the Common Core Standard (8.F.1).

Related Content Standards

- (8.F.1) Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

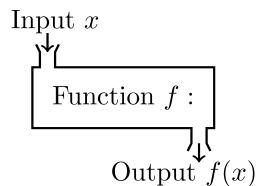


Figure 5.1: Machine analogy of a function

This definition is often represented as a box or machine with certain inputs and outputs as shown in Figure 5.1. This definition of the function, while generally very useful in understanding the use of functions, is not as mathematically sound as the generally accepted definition of function that arose in the early 1900's and appears in most theoretical mathematics textbooks. One of the main differences between the two definitions is the focus on the function being defined by three items: the domain, the codomain, and the relationship between elements of the two sets.

Definition 5.1. Let A and B be two sets, and let R be a relation from A to B . Then R is a function if for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in R$. The set A is called the **domain** of the function and B is called the **codomain** of the function.

One should note from this definition of a function that the function is the set of ordered pairs that is often referenced as the graph of the function, rather than the algebraic or tabular representation that the

input-output definition implies. Throughout the remainder of the text, we will bounce back and forth between the two views of a function, with the “input-output” or “rule” perspective often being reflected in an algebraic representation of the function and the “relation” or “ordered-pairs” perspective being reflected in a graphical representation of the function.

Related Content Standards

- (HSF.IF.1) Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x . The graph of f is the graph of the equation $y = f(x)$.

In both of these definitions we talk about the function as a relation between two sets, A and B , and we often denote the function as $f : A \rightarrow B$. The subset of the co-domain

$$\text{Ran}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$$

is called the **range** of the function. This is different than the usual definitions in the K-12 mathematics textbooks and in the Common Core State Standards where authors do not define the co-domain because it is almost always assumed to be the real numbers.

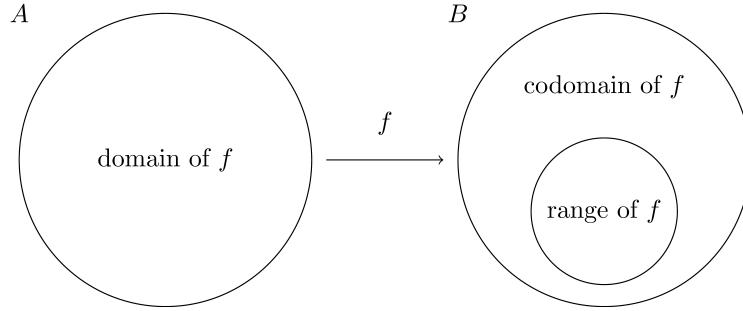


Figure 5.2: Concept sketch of $f : A \rightarrow B$

One of the key distinctions between relations that represent functions and those that do not has to do with the uniqueness of the element in the co-domain that is related to each element in the domain. For instance, with the relation from x to y given by $y = 5x$, each value of x has only one value of y related to it. However, with the relation on $[-1, 1] \times [-1, 1]$ given by $x^2 + y^2 = 1$, each value of x has two distinct values of y related to it and so y is not a function of x .

If one has the graphical representation of a relation available, one can determine if the relation represents a function by seeing if there is any value in the domain that has multiple values in the co-domain by looking for a vertical line that intersects the graph of the function at multiple points. Thus the relation on $\mathbb{R} \times \mathbb{R}$ given by $y^2 = x$ does not represent y as a function of x , but it does represent x as a function of y .

5.1.1 Function Representations

In order to understand the different representations for a function we will use the squaring function, $f : \mathbb{R} \rightarrow \mathbb{R}$, that is often represented by $f(x) = x^2$, or equivalently $x \mapsto x^2$.

If we use the sets and relation definition of the function, this function is defined to be

$$\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}.$$

This is often represented graphically as

If we restrict the domain and co-domain to be natural numbers, so that $f : \mathbb{N} \rightarrow \mathbb{N}$, we can represent the function in a tabular format

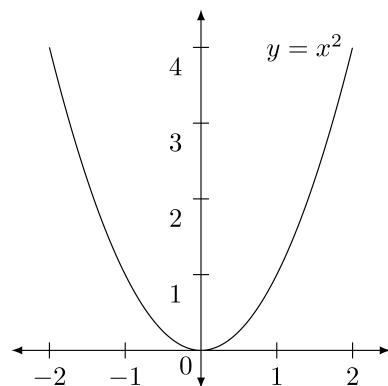
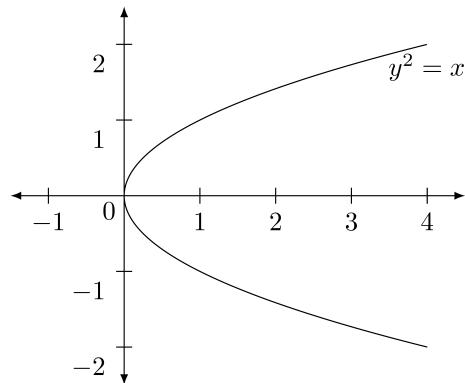
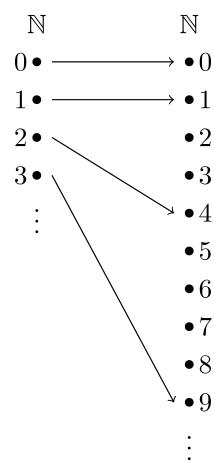


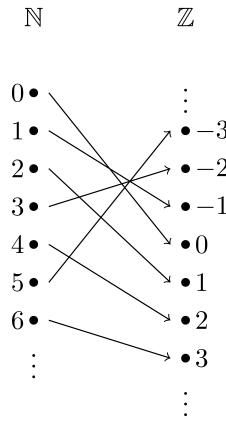
Table 5.1: The squaring function

$\$x\$$	$\$y\$$
0	0
1	1
2	4
3	9
$\$\\vdots\$$	$\$\\vdots\$$

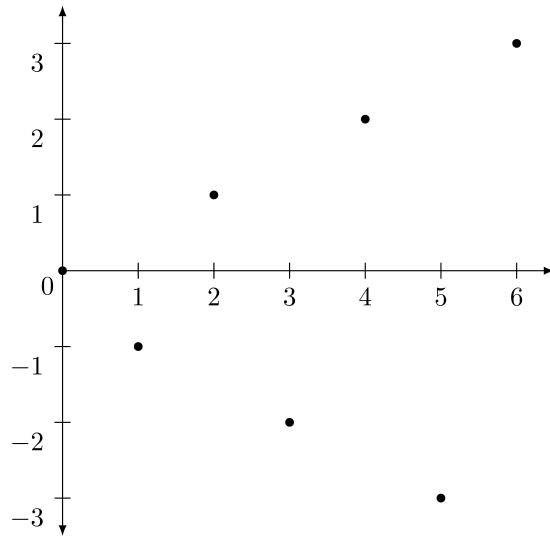


or by mapping elements from the domain to the co-domain

There are some times where it is easier to start describing the function by sketching how elements from the domain map to the co-domain. For example, if the function, $f : \mathbb{N} \rightarrow \mathbb{Z}$, is defined by the relationship



where each successive even natural number maps to the next larger positive integer and each successive odd natural number maps to the next larger negative integer. This function can also be represented by the graph



and it can be represented algebraically as

$$f(n) = \begin{cases} -\frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} \quad \text{or} \quad f(n) = (-1)^n \frac{2n+1 - (-1)^n}{4}.$$

5.1.2 Exercises

- Let $A = \{a, b, c\}$ and let $B = \{1, 2\}$. List all of the possible functions from A to B .
- Define $\text{sum} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by $(m, n) \mapsto m + n$. Show that sum is a function.
- Describe how the content of this section relates to the statement, “You can’t take the square root of a negative number.”
- Which of the following are functions? For those that are not functions, explain why not. For those that are functions, find the range of the function.

- a. $f : \mathbb{R} \rightarrow \mathbb{R}$, with $f(x) = \frac{1}{x}$.
 - b. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
 - c. $g : \mathbb{R}^+ \rightarrow \mathbb{R}$, with $g(x) = \sqrt{x}$.
 - d. $f : \mathbb{N} \rightarrow \mathbb{N}$, with $h(x) = x^2$.
5. Let $f : A \rightarrow B$ be a function. Define a relation on A by

$$a_1 \sim_f a_2 \Leftrightarrow f(a_1) = f(a_2)$$

for all $a_1, a_2 \in A$.

- a. Show that \sim_f is an equivalence relation.
- b. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \sin(2\pi x)$, find the equivalence class of 0 under this equivalence relation.

5.2 Injections, Surjections, and Bijections

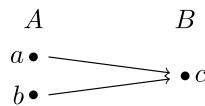
In this section we will define and explore several different properties that some functions have that provide useful information about the function.

Definition 5.2. A function $f : A \rightarrow B$ is called an **injection**, or **one-to-one**, if no two elements of the domain map to the same element of the co-domain. Or equivalently, for elements $a_1, a_2 \in A$,

$$(f(a_1) = f(a_2)) \Rightarrow (a_1 = a_2).$$

The primary benefit of a function being an injection is that for each element, b , in the range of the function there is a unique element, a , in the domain of the function such that $f(a) = b$. This provides the foundation for ‘undoing’ functions that we will explore further in the next section.

The simplest example of a function that is not one-to-one is given by a function from $A = \{a, b\}$ to $B = \{c\}$ represented by



since $f(a) = f(b)$, but $a \neq b$.

We can also see that the function $f(x) = |x|$ is not one-to-one because $f(-x) = f(x) = |x|$ for all $x \in \mathbb{R}$.

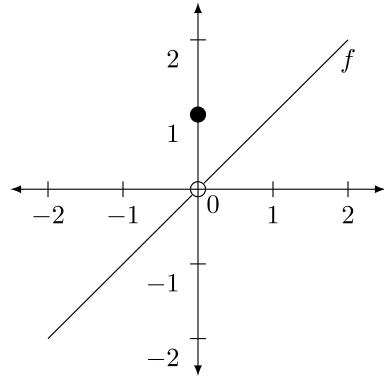
The function

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is not one-to-one since $f(1) = f(0)$. If we sketch the graph of this function, we can see that this

is true because if one draws a horizontal line at $y = 1$, it intersects the function at two places. This intersection at multiple places by a horizontal line is called the **horizontal line test**.

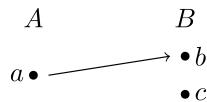
Sometimes a function cannot be graphed easily and we would like to prove that it is one-to-one. There is a fairly standard process for doing so using the one-to-one definition. In this process, we assume that a and b are in the domain of a function $f : A \rightarrow B$ such that $f(a) = f(b)$. Then using some logical arguments we prove that this implies that $a = b$. To prove that a function is not one-to-one we only need to find one case of two distinct elements in the domain mapping to the same element in the co-domain.



Definition 5.3. A function $f : A \rightarrow B$ is called a **surjection**, or **onto**, if every element of the co-domain is in the range of f . Or equivalently, if for all $b \in B$ there exists a $a \in A$ such that $f(a) = b$.

A primary consideration in a function being a surjection is to know that every element in the co-domain is ‘hit’ by an element in the domain. It is often more important to fully describe the range of the function, rather than if a function is a surjection, since any function is a surjection when the co-domain of the function is replaced by the range of the function in the definition of the function. For instance, $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is not a surjection, but $f : \mathbb{R} \rightarrow \mathbb{R}^+$ defined by $f(x) = x^2$ is a surjection.

The basic example of a function that is not a surjection is given by a function $f : A = \{a\} \rightarrow B = \{b, c\}$ represented by



Definition 5.4. A function $f : A \rightarrow B$ is called a **bijection**, or **one-to-one and onto**, if it is both an injection and a surjection.

One of the results of a function being a bijection is that there is a perfect correspondence between elements of the domain and elements of the co-domain that we will explore further in the next two sections.

5.2.1 Exercises

1. For each of the following functions, determine if it is an injection, a surjection, and/or a bijection.
 - a. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x$.
 - b. $f : \mathbb{R} \rightarrow \mathbb{R}^+$ defined by $f(x) = |x|$.
 - c. $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $f(x) = x^2$.
 - d. $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}$ defined by $f(x) = \frac{1}{x}$.
 - e. $f : \mathbb{N} \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{x}{1+x^2}.$$

5.3 Composition of Functions

We often want to look at how two functions work together.

Example 5.1. Suppose that oil is spilled from a ruptured tanker and spreads in a circular pattern. If the oil is leaving the tanker at a rate of 70 gallons an hour, then what is the radius of the 1 cm thick oil slick after t hours?

In order to find a solution to this word problem (note that this is not really a modelling problem) we will create some functions that we will compose together. We will start by letting g be the rate in gallons per hour that oil is leaking out of the boat.

Let $T(g) = \frac{1}{264.172}g$ be the cubic meters equivalent to g gallons.

Let $r(V) = \sqrt{\frac{100V}{\pi}}$ be the radius (in meters) of an oil slick with volume V (in cubic meters) that is 1 cm thick.

Then $r(V(T(g) \cdot t))$ gives us the radius of the oil slick after t hours.

Using this motivation, we will give a mathematical definition for the composition of functions.

Definition 5.5. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. Then we define the function $(g \circ f) : A \rightarrow C$ by

$$(g \circ f)(a) = g(f(a)), \quad \text{for all } a \in A.$$

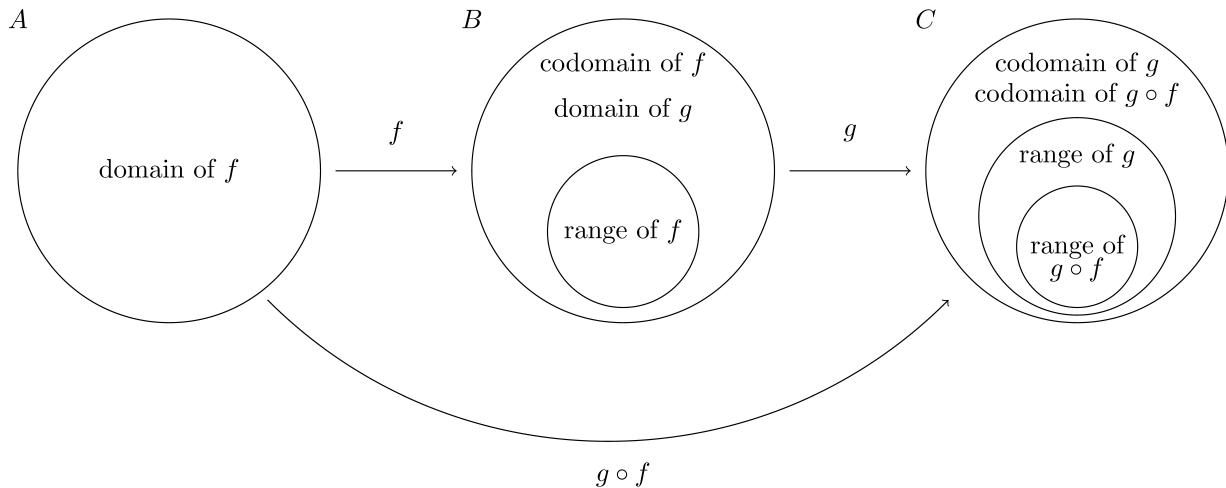


Figure 5.3: Sketch of $f : A \rightarrow B$, $g : B \rightarrow C$, and $g \circ f : A \rightarrow C$

The sketch in Figure 5.3 describes how the domains, ranges, and codomains of the three functions interact when describing function composition.

5.3.1 Inverse Functions

In addition to composing functions together, it is often useful to define a function that undoes another function. In the previous example, it is easiest to find the radius of the oil slick based on volume by first finding the volume as a function of the radius. In the case of a cylindrical cylinder, $V(r) = \pi r^2 h$ and so $r(V) = \sqrt{\frac{V}{\pi h}}$ is a function that undoes the volume. A more precise definition of these inverse functions is given below.

Definition 5.6. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions. If

$$(g \circ f)(a) = a \text{ for all } a \in A, \text{ and}$$

$$(f \circ g)(b) = b \text{ for all } b \in B,$$

then we say that f is **invertible**.

Note that in our example we define $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $f(r) = \pi r^2 h$ to be the volume function and $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $g(V) = \sqrt{\frac{V}{\pi h}}$ to be the radius function. Then for all $r \in \mathbb{R}^+$,

$$(g \circ f)(r) = g(f(r)) = g(\pi r^2 h) = \sqrt{\frac{(\pi r^2 h)}{\pi h}} = \sqrt{r^2} = r$$

and

$$(f \circ g)(V) = f(g(V)) = f\left(\sqrt{\frac{V}{\pi h}}\right) = \pi\left(\sqrt{\frac{V}{\pi h}}\right)^2 h = V.$$

If instead, we define $f : \mathbb{R} \rightarrow \mathbb{R}^+$ by $f(x) = x^2$ and $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $g(y) = \sqrt{y}$, then

$$(f \circ g)(y) = (\sqrt{y})^2 = y, \forall y \in \mathbb{R}^+$$

but

$$(g \circ f)(x) = \sqrt{x^2} = |x|$$

and so f and g are not invertible.

The next theorem is useful to verify that if a function is invertible, then the inverse function is unique.

Theorem 5.1. *If $f : A \rightarrow B$ is invertible, the function $g : B \rightarrow A$ satisfying the properties*

$$(g \circ f)(a) = a \text{ for all } a \in A, \text{ and}$$

$$(f \circ g)(b) = b \text{ for all } b \in B$$

is unique and so we say that that g is the inverse function of f . We then denote this inverse function as f^{-1} .

Furthermore, we have that f^{-1} is invertible.

Proof. In order to prove the uniqueness of g we will give a proof by contradiction. So we assume that $f : A \rightarrow B$ is an invertible function and that there are function $g : B \rightarrow A$ and $\tilde{g} : B \rightarrow A$ such that

$$(g \circ f)(a) = a \text{ and } (\tilde{g} \circ f)(a) = a \text{ for all } a \in A, \text{ and}$$

$$(f \circ g)(b) = b \text{ and } (f \circ \tilde{g})(b) = b \text{ for all } b \in B.$$

Then for each $b \in B$ we have that

$$\begin{aligned} \tilde{g}(b) &= \tilde{g}((f \circ g)(b)) \text{ (since } (f \circ g)(b) = b, \text{ for all } b \in B) \\ &= \tilde{g}(f(g(b))) = (\tilde{g} \circ f)(g(b)) \text{ (by the definition of function composition)} \\ &= g(b) \text{ (since } (\tilde{g} \circ f)(a) = a, \text{ for all } a \in A). \end{aligned}$$

Since g and \tilde{g} have the same output for each input, they are the same function. Therefore, the inverse function of f is unique. \square

Now that we have the uniqueness of inverse functions when a function is invertible, it would be helpful to further understand the properties of a function that make it invertible without having to find the inverse. This will save time looking for an inverse that doesn't exist and help us to adjust a function to make it invertible by restricting the domain and/or co-domain of the function.

Theorem 5.2. *The function $f : A \rightarrow B$ is invertible if and only if it is a bijection.*

Proof. Since this statement is an ‘if and only if’ statement we must prove both implications.

Invertible implies bijection. Assume that the function $f : A \rightarrow B$ is invertible. Then there exists a unique $f^{-1} : B \rightarrow A$ such that $(f \circ f^{-1})(b) = b$ for all $b \in B$ and $(f^{-1} \circ f)(a) = a$ for all $a \in A$.

To prove that f is an injection (one-to-one), we assume that there are elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ as an element of B . Since f^{-1} is a function from B to A , we know that $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$. From the definitions of function composition and inverse function, this implies that

$$a_1 = (f^{-1} \circ f)(a_1) = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = (f^{-1} \circ f)(a_2) = a_2$$

and so $a_1 = a_2$ proving that f is an injection.

To prove that f is a surjection (onto) we choose a generic element of the co-domain of f and prove that it is in the range of f . Let $b \in B$ be that generic element of the co-domain of f . Since

$$b = (f \circ f^{-1})(b) = f(f^{-1}(b))$$

we see that b is the image of f acting on $f^{-1}(b) \in A$ and so is in the range of f . Therefore, f is a bijection.

Bijection implies invertible. On the other hand, if we assume that $f : A \rightarrow B$ is a bijection, we can define a function $g : B \rightarrow A$ in the following way.

Since f is a surjection, for each $b \in B$ there is at least one element $a \in A$ such that $b = f(a)$. Since f is an injection, this element is unique and we will call the element a_b . Thus we can define $g : B \rightarrow A$ by $g(b) = a_b$ and this relation satisfies the definition of a function.

We can also see from this definition of g that $(f \circ g)(b) = f(g(b)) = f(a_b) = b$ for all $b \in B$.

Furthermore, for all $a \in A$, $(g \circ f)(a) = g(f(a)) = a$, thereby making f invertible. \square

This means that this concept of bijections is one that is very important in terms of the composition of functions. In particular it is important to understand what properties the composition of two bijections would have. This leads to the following theorem.

Theorem 5.3. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two bijections. Then the function $g \circ f : A \rightarrow C$ is also a bijection.*

Proof. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijections.

We will first prove that $(g \circ f) : A \rightarrow C$ is an injection.

Let a_1 and a_2 be elements of A such that

$$(g \circ f)(a_1) = (g \circ f)(a_2).$$

From the definition of the composition of functions this implies that

$$g(f(a_1)) = g(f(a_2)).$$

Since g is an injection on B , we know that $f(a_1) = f(a_2)$. Since f is an injection, this implies that $a_1 = a_2$ and thus proving that $g \circ f$ is an injection.

We will now prove that $(g \circ f) : A \rightarrow C$ is a surjection.

Let c be a generic element of C . Since $g : B \rightarrow C$ is onto, there exists an element $b_c \in B$ such that $g(b_c) = c$. Since $f : A \rightarrow B$ is onto, there exists an element $a_{b_c} \in A$ such that $f(a_{b_c}) = b_c$. We then have that

$$c = g(b_c) = g(f(a_{b_c})) = (g \circ f)(a_{b_c})$$

and so $g \circ f$ is a surjection.

Therefore, $(g \circ f) : A \rightarrow C$ is a bijection. \square

Note that in the proof that $(g \circ f) : A \rightarrow C$ is an injection that the only assumptions used are that f and g are both injections and the only sets referenced are the domains of the functions. This signifies that the property of a function being one-to-one is a property that focuses on the domain of the function and is independent of the co-domain defined for the function. Hence, one could enlarge the co-domain of an injective function to a new function that is also an injection.

Similarly, the surjective property of the composition of the functions depends only on the surjective properties of the individual functions. However, unlike the injective property, the surjective property involves a heavy reliance upon both the domain and co-domain of the function. While one could expand the domain to a larger set while maintaining the surjective nature of the function, one may not be able to reduce the size of the domain.

Since the composition of bijections is also a bijection, we know from Theorem 5.2 that the composition of a bijection is also invertible. Combining the results of these theorems we have the following theorem.

Theorem 5.4. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijections. Then we have the following:*

- f is invertible with inverse $f^{-1} : B \rightarrow A$
- g is invertible with inverse $g^{-1} : C \rightarrow B$
- $g \circ f$ is invertible with inverse $(g \circ f)^{-1} : C \rightarrow A$
- $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

We can illustrate this theorem with Figure 5.4.

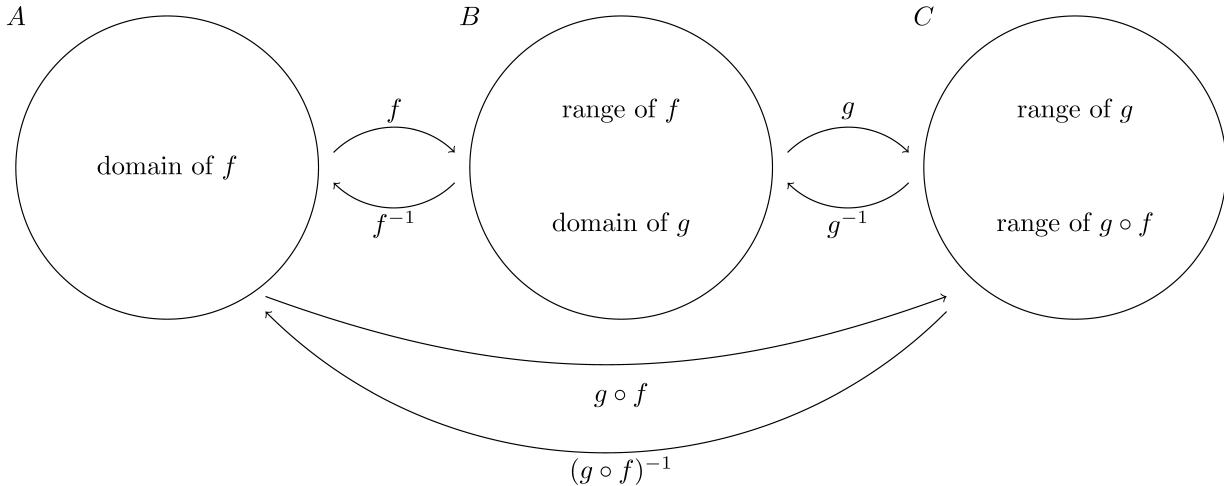


Figure 5.4: Composition of Bijections

5.3.2 Exercises

1. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. Prove that if $g \circ f : A \rightarrow C$ is a bijection that f must be an injection and g must be a surjection.
2. What adjustments can be made to the domain and/or the co-domain of $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ in order to make it an invertible function.
3. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions.
 - a. What properties hold for $g \circ f$ if both f and g are injections?
 - b. What properties hold for $g \circ f$ if both f and g are surjections?

- c. What properties hold for $g \circ f$ if f is an injection and g is a surjection?
- d. What properties hold for $g \circ f$ if f is a surjection and g is an injection?

5.4 Counting and Cardinality

The concept of cardinality is introduced in kindergarten where students are taught to have the abstract concept of a number associated to a certain number of objects such as in Figure 5.5 where we have five different representations of the same cardinality. We can give a more precise mathematical definition of cardinality using functions.

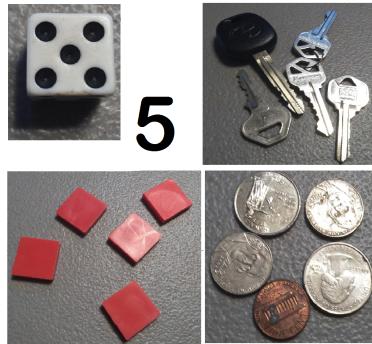


Figure 5.5: Multiple representations of the number five

Related Content Standards

- (K.CC.3) Write numbers from 0 to 20. Represent a number of objects with a written numeral 0 – 20 with (0 representing a count of no objects).
- (K.CC.4) Understand the relationship between numbers and quantities; connect counting to cardinality.
 - a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
 - b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
 - c. Understand that each successive number name refers to a quantity that is one larger.

Definition 5.7. A set A is said to have the same cardinality as the set B if there exists a bijection, $f : A \rightarrow B$.

This definition defines a relation on sets where A is related to B if A has the same cardinality as B .

Since the identity function, $\iota : A \rightarrow A$ defined by $\iota(a) = a$ for all $a \in A$, is a bijection we see that A has the same cardinality as itself and so the relation is reflexive.

In the previous section we proved that a bijection has a functional inverse, so if A has the same cardinality as B , there is a bijection $f : A \rightarrow B$ and a second bijection, $f^{-1} : B \rightarrow A$ showing that B has the same cardinality as A . So the cardinality relation is symmetric.

If we assume that A has the same cardinality as B and B has the same cardinality as C , there are bijections $f : A \rightarrow B$ and $g : B \rightarrow C$. In the previous section we proved that the compositions of bijections are bijections, so we know that $(g \circ f) : A \rightarrow C$ is a bijection and so A has the same cardinality as C , making the cardinality relation transitive.

These three properties of reflexive, symmetric, and transitive make the cardinality relation an equivalence relation and we define the abstract concept of a number with the equivalence class associated to that number,

as defined by our set theory definition of the natural numbers.

While this might seem like a very abstract way to define counting of finite sets, it is an explanation of what is happening in the mind of a child when he is learning to count. Teachers understanding this level of abstraction can feel more empathy for the struggles of students as they learn concepts as simple as counting.

Since cardinality is an equivalence relation, for sets with a finite number of elements we can define a function

$$\# : \{\text{sets with a finite number of elements}\} \rightarrow \mathbb{N}$$

by $\#(A)$ is the cardinality of A . For sets with an infinite number of elements, we will label $\#(A)$ as the equivalence class of A using the cardinality relation.

The pigeonhole principle states that if you are putting items into containers and there are more items than containers that at least one of the containers has more than one item. Some of the earliest references to this principle is in the 1620's where multiple editions of the same book from the Jesuit university at Pont-à-Mousson referenced that there must be two men on Earth with the same number of hairs on their heads [Leurechon, 1629]. The naming of the principle after pigeonholes was based on the reference to items being placed into boxes.



Figure 5.6: Illustration of pigeonholes

In the language of functions, if we have two sets, A and B , with $\#(A) > \#(B)$, then there cannot exist an injection $f : A \rightarrow B$. In the same vein, we can say that if $\#(A) < \#(B)$, then there cannot exist a surjection $f : A \rightarrow B$. With this principle we have the following theorem.

Theorem 5.5. Let $f : A \rightarrow B$ be a function.

- If f is an injection (one-to-one), then $\#(A) \leq \#(B)$.
 - If f is a surjection (onto), then $\#(A) \geq \#(B)$.
 - If f is a bijection, then $\#(A) = \#(B)$.

This means that if we want to prove that two sets, A and B , have the same cardinality, we can find a bijection between them. We could also find two different functions, $f : A \rightarrow B$ and $g : A \rightarrow B$, such that one is an injection and the other is a surjection. To prove that two sets do not have the same cardinality is usually done using a proof by contradiction to show that no bijection could exist.

We explore this further using subsets of the natural numbers and an idea introduced by David Hilbert in a series of lectures for the general public in 1924 [Ewald and Sieg, 2013]. In this example we imagine a hotel with an infinite number of rooms with numbers on the doors, $1, 2, 3, 4, \dots$, and that every room is full. When someone comes up to the registration desk, the person at the desk says that all the rooms are currently full, but we can make room for one more. The person checking in asks, “How can there be room for one more if all of the rooms are full?” The clerk says, “It’s easy. We will ask each person to just gather all of their belongings and they can move to the room with the next higher number.” (See Figure 5.7)

This idea is described with function notation by the function $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ such that $f(n) = n + 1$ as being an injective function that is not surjective. However, the function $g : \{1, 2, 3, 4, \dots\} \rightarrow \{2, 3, 4, 5, \dots\}$ defined by $g(n) = n + 1$ is a bijection and so the two sets have the same cardinality.

One can go a little farther in this direction by having everyone move to the room with a number that is twice their current room number. In function notation this describes a function $f : \{1, 2, 3, 4, 5, \dots\} \rightarrow$

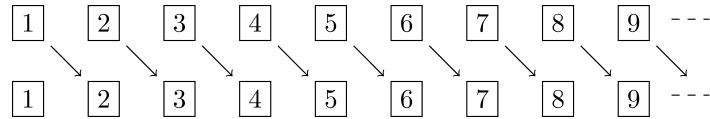


Figure 5.7: Hilbert's Hotel with one extra room.

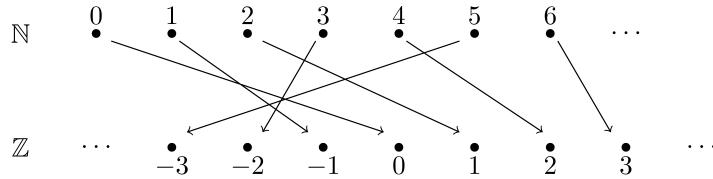
$\{2, 4, 6, 8, 10, 12, \dots\}$ such that $f(n) = 2n$, which is a bijection. So even though “half” of the numbers in the first set are removed, the two sets have the same cardinality.

This leads to the conjecture that all sets that are not of finite cardinality are the same cardinality. We will see that is not the case. We will start by dividing cardinality of infinite sets into two categories. If the set has the same cardinality as the natural numbers we will call the set **countable**, since the process of counting is a function that maps the natural numbers onto the set. If an infinite set is not countable, we will call it **uncountable**.

If a set A is countable, then we will say that $\#(A) = \aleph_0$ (called aleph-naught).

The function $f : \mathbb{N} \rightarrow 2\mathbb{N}$ such that $f(n) = 2n$ is a bijection and so both of these sets are countable.

Using the function described by



we can see that the integers are also countable.

We define \mathbb{Q}^+ to be the non-negative rational numbers and $f : \mathbb{N} \rightarrow \mathbb{Q}^+$ to be the function $f(n) = \frac{n}{1}$. We can see that f is an injection and so $\#(\mathbb{N}) \leq \#(\mathbb{Q}^+)$. However, if we write all of the non-negative rational expressions into a grid pattern as in Figure 5.8 we can see that we would write every possible non-negative rational expression of integers.

$\frac{0}{1}(0)$	$\frac{0}{2}(1)$	$\frac{0}{3}(3)$	$\frac{0}{4}(6)$	$\frac{0}{5}(10)$	$\frac{0}{6}(15)$	$\frac{0}{7}(21)$	$\frac{0}{8}(28)$	\dots
$\frac{1}{1}(2)$	$\frac{1}{2}(4)$	$\frac{1}{3}(7)$	$\frac{1}{4}(11)$	$\frac{1}{5}(16)$	$\frac{1}{6}(22)$	$\frac{1}{7}(29)$	$\frac{1}{8}(37)$	\dots
$\frac{2}{1}(5)$	$\frac{2}{2}(8)$	$\frac{2}{3}(12)$	$\frac{2}{4}(17)$	$\frac{2}{5}(23)$	$\frac{2}{6}(30)$	$\frac{2}{7}(38)$	$\frac{2}{8}(47)$	\dots
$\frac{3}{1}(9)$	$\frac{3}{2}(13)$	$\frac{3}{3}(18)$	$\frac{3}{4}(24)$	$\frac{3}{5}(31)$	$\frac{3}{6}(39)$	$\frac{3}{7}(48)$	$\frac{3}{8}(58)$	\dots
$\frac{4}{1}(14)$	$\frac{4}{2}(19)$	$\frac{4}{3}(25)$	$\frac{4}{4}(32)$	$\frac{4}{5}(40)$	$\frac{4}{6}(49)$	$\frac{4}{7}(59)$	$\frac{4}{8}(70)$	\dots
$\frac{5}{1}(20)$	$\frac{5}{2}(26)$	$\frac{5}{3}(33)$	$\frac{5}{4}(41)$	$\frac{5}{5}(50)$	$\frac{5}{6}(60)$	$\frac{5}{7}(71)$	$\frac{5}{8}(83)$	\dots
$\frac{6}{1}(27)$	$\frac{6}{2}(34)$	$\frac{6}{3}(42)$	$\frac{6}{4}(51)$	$\frac{6}{5}(61)$	$\frac{6}{6}(72)$	$\frac{6}{7}(84)$	$\frac{6}{8}(97)$	\dots
\vdots	\ddots							

Figure 5.8: Countability argument of non-negative rational numbers

We can now define a function g from the natural numbers onto this set of rational expressions by mapping 0 to the expression in the first row and first column, 1 to the expression in the first row and second column, 2 to the expression in the second row and first column, 3 to the expression in the first row and third column, 4 to the expression in the second row and second column, 5 to the expression in the third row and first column, and so on as written in the parentheses.

We can see then that g is a surjection and so $\#(\mathbb{N}) \geq \#(\mathbb{Q}^+)$. So \mathbb{Q}^+ must be countable.

We can then create a very similar map from the negative integers onto the negative rational expressions and combine these two functions to create a surjection from \mathbb{Z} onto \mathbb{Q} and so the entire set of rational numbers is countable.

One might think that since the rational numbers are countable that all infinite sets are the same size. Cantor [1891] prove this not to be true.

Theorem 5.6. *The open interval $(0, 1)$ in the real numbers is not countable.*

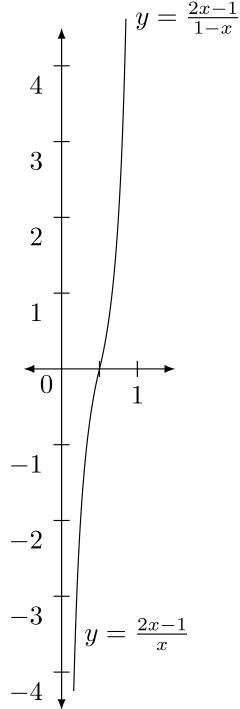
Proof. We will follow the general flow of a proof by Georg Cantor (1891). If we assume that the interval $(0, 1)$ is countable, then there is a bijection $f : \mathbb{N} \rightarrow (0, 1)$. For each natural number, n , we will write $f(n)$ in a decimal representation as

$$f(n) = 0.a_{1,n}a_{2,n}a_{3,n}\dots = \sum_{i=1}^{\infty} a_{i,n} \cdot 10^{-i},$$

where each of the $a_{i,n} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and in order to avoid two different decimal expansions representing the same real number we will assume that the $a_{i,n}$ are not eventually all 9.

For each $j \in \{1, 2, 3, 4, \dots\}$, let b_j be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $b_j \neq a_{j,j}$. Then $b = \sum_{i=1}^{\infty} a_{i,n} \cdot 10^{-i}$ is a real number in $(0, 1)$ such that $b \neq f(n)$ for any natural number n . This means that f was not a surjection and so no such bijection exists. \square

Now that we know that there are sets that are not countable, let's explore some other uncountable sets.



Let $f : (0, 1) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{2x-1}{1-x}, & \text{if } \frac{1}{2} \leq x < 1 \\ \frac{2x-1}{x}, & \text{if } 0 < x \leq \frac{1}{2} \end{cases}.$$

If we choose a generic real number and call it b , then if $b > 0$, we have that

$$f\left(\frac{1+b}{2+b}\right) = \frac{2\left(\frac{1+b}{2+b}\right) - 1}{1 - \left(\frac{1+b}{2+b}\right)} = \frac{(2+2b) - (2+b)}{(2+b) - (1+b)} = b,$$

and if $b < 0$,

$$f\left(\frac{1}{2-b}\right) = \frac{2\left(\frac{1}{2-b}\right) - 1}{\left(\frac{1}{2-b}\right)} = \frac{2 - (2-b)}{1} = b.$$

Since $f\left(\frac{1}{2}\right) = 0$, we have that f is onto and so $\#((0, 1)) \geq \#(\mathbb{R})$. Since $(0, 1) \subseteq \mathbb{R}$, we have the $\#((0, 1)) \leq \mathbb{R}$. With both inequalities being true, we have that the two sets have the same cardinality.

In fact, any subset of the real numbers is either finite, countable, or the same cardinality as \mathbb{R} . Cantor called the cardinality of \mathbb{R} the cardinality of the continuum and also proved that it is the same cardinality as the power set of the natural numbers $\#(\mathbb{R}) = \#(\mathcal{P}(\mathbb{N}))$.

5.4.1 Exercises

1. Prove that the following pairs of sets have the same cardinality.
 - a. $3\mathbb{Z} = \{n \in \mathbb{Z} | n = 3k \text{ for some } k \in \mathbb{Z}\}$ and $5\mathbb{Z} = \{n \in \mathbb{Z} | n = 5k \text{ for some } k \in \mathbb{Z}\}$
 - b. $[a, b]$ and $[c, d]$, where a, b, c , and d are real numbers such that $a < b$ and $c < d$.
 - c. A and $A \cup B$, where A and B are countable sets.
2. Prove that if A is a set of finite cardinality, the cardinality of the power set of A is $2^{\#(A)}$.
3. Look up a proof that the algebraic numbers are countable, and compare that to the proof that the rational numbers are countable.
4. Define two sets (not necessarily with finite cardinality) to be equivalent if they have the same cardinality. For a set A , let $\mathcal{P}(A)$ be the set of all subsets of A . Prove that A is not equivalent to $\mathcal{P}(A)$. [Hint: Suppose $f : A \rightarrow \mathcal{P}(A)$ and define

$$C = \{x | x \in A \text{ and } x \notin f(x)\}.$$

Show $C \notin \text{im}(f)$.]

Part II

ALGEBRA

Chapter 6

Groups, Rings, and Fields

Abstract (or modern) algebra studies basic mathematical structures that appear in many settings . In terms of groups, we study mathematical sets and a single operation (such as addition or multiplication). The study of how two operations interact is contained in studying rings and fields. In this chapter we intersperse the axioms and definitions of abstract algebra with the concrete examples that appear in the K-12 curriculum. The goal of the study of these basic groups, rings, and fields is to better understand the connections between different content in the K-12 curriculum with the goal of improving mathematics teaching by helping students learn new material by making connections with material they have previously learned.

6.1 Group Theory

Definition 6.1 (Group). A non-empty set G , together with a binary operation, $*$, is called a group if it satisfies the following conditions:

- $a * b \in G, \forall a, b \in G$ (Closure)
- $(a * b) * c = a * (b * c), \forall a, b, c \in G$ (Associative)
- There exists an element $e \in G$ such that for all $a \in G, e * a = a * e = a$ (Identity)
- For each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$ (Inverse)

This concept of a group is a generalization of most of the properties that we noticed in Chapter 4 as we constructed the various number systems.

Example 6.1. Some basic examples of groups that appear in the K-12 curriculum include:

1. $(\mathbb{Z}, +)$, the set of integers under addition
2. $(2\mathbb{Z}, +)$, the set of even integers under addition
3. $(\mathbb{Q}, +)$, the set of rational numbers under addition
4. $(\mathbb{R}, +)$, the set of real numbers under addition
5. $(\mathbb{Q} \setminus \{0\}, \cdot)$, the set of non-zero rational numbers under multiplication
6. (\mathbb{R}^+, \cdot) , the set of positive real numbers under multiplication
7. $(\mathbb{R}[x], +)$, the set of polynomials with real coefficients under addition
8. The set of one-to-one and onto functions, $f : A \rightarrow A$ for some set A , under function composition.
9. $(M_{(m,n)}(\mathbb{R}), +)$, the set of $m \times n$ matrices under addition.
10. $GL_n(\mathbb{R})$, the set of $n \times n$ invertible matrices under multiplication.

11. $SL(n, \mathbb{R})$, the set of $n \times n$ invertible matrices, with determinant of 1, under multiplication.

If we extend our examples to the calculus curriculum we can include the differentiable (or integrable) functions with the binary operation of addition. For the group of differentiable functions under addition, the identity is the function $f_0 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_0(x) = 0$ for all x . If $g : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, then its inverse in this group would be defined by $h(x) = -g(x)$.

The cardinality of each of the sets for the groups in Example 6.1 is at least countable. We can also have groups with a finite number of elements as shown in the following example.

Example 6.2. Let $\mathbb{Z}_2 = \{0, 1\}$ with addition defined by the following table (often referred to as a Cayley table)

+	0	1
0	0	1
1	1	0

Then the identity is 0 and each element is its own inverse.

Note that the complete definition of a group involves both the set G and the binary operation $*$ and so the group is officially labeled as $(G, *)$. However, the binary operation is often inferred from the set itself. So we often just refer to the group as G with the operation implied.

When showing that a set and operation form a group, the most challenging property to show directly is often the associative property. Since almost all of the groups that we will encounter are derived from the elements and operations in the number systems constructed in Chapter 4, we will gain the associative property from those number systems.

6.1.1 Abelian Groups

In most of the above mentioned groups, the order of the operation does not matter. However, for groups such as $GL_n(\mathbb{R})$, the order of the elements with the operation does matter.

Definition 6.2 (Abelian Groups). If a group $(G, *)$ also satisfies the property $a * b = b * a, \forall a, b \in G$ then $(G, *)$ is called **abelian**.

Example 6.3. The non-abelian group whose set has the smallest cardinality is the dihedral group of order 6 defined by the Cayley table:

*	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

We can verify that this group is not abelian since $a * b = d$ and $b * a = f$.

It will be important to notice whether or not a group is abelian, as abelian groups have a simpler structure than those for which the operation does not commute.

6.1.2 Uniqueness of Identities and Inverses

The existence of identity and inverse elements does not guarantee uniqueness, a priori. However, the following two theorems to verify the uniqueness of the identity and inverse elements within a group.

Theorem 6.1 (Uniqueness of the Identity). *For a group, $(G, *)$, the identity is unique.*

Proof. Assume that there are two distinct identity elements that we will call e and \tilde{e} . Then the following is true:

$$\tilde{e} = e * \tilde{e} = e$$

since both e and \tilde{e} are identities. Thus the identities are not distinct and so the identity is unique. \square

Theorem 6.2 (Uniqueness of the Inverse). *For a group, $(G, *)$, for each $a \in G$, the inverse of a is unique and so we call the inverse a^{-1} .*

Proof. Let $a \in G$ and assume that there are distinct elements $b, c \in G$ that are inverses of a . Then

$$b = b * (e) = b * (a * c) = (b * a) * c = (e) * c = c.$$

Therefore, the inverse of a is unique. \square

6.1.3 Homomorphisms

Now that we have a better understanding of the definitions and structures of groups we turn our attention to looking at when two groups are really the ‘same’ group but viewed from different perspectives. To do that we must create a way to determine equivalence classes of groups that are based on a relationship between the two sets and how they interact with their binary operations.

Definition 6.3. A function, ϕ , from a group $(G, *)$ to a group (G', \cdot) is called a **homomorphism** if

$$\phi(a * b) = \phi(a) \cdot \phi(b)$$

for every $a, b \in G$.

In other words, a homomorphism is a function between groups that maintains the structure of the binary operation. Notice that this restricts the types of functions quite a bit since a simple function like $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 1$ is not a homomorphism.

It will be helpful to understand this idea of maintaining the structure of the binary operation with a few examples of functions, some of which are homomorphisms and some are not.

Example 6.4. Let $\phi : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}, +)$ be defined by $\phi(n) = 2n$. In order to verify that ϕ is a homomorphism, we see that for any integers m, n that

$$\phi(m + n) = 2(m + n) = 2m + 2n = \phi(m) + \phi(n).$$

So ϕ is a homomorphism.

Example 6.5. Let $f : (\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$ be defined by $f(x) = 2^x$. Then for real numbers x and y we have that

$$f(x + y) = 2^{(x+y)} = 2^x \cdot 2^y = f(x) \cdot f(y)$$

making f a homomorphism.

Example 6.6. Let $g : (\mathbb{R}, +) \rightarrow (\mathbb{R}, +)$ be defined by $g(x) = 2x + 3$. Then for real numbers x and y ,

$$g(x + y) = 2 \cdot (x + y) + 3 = 2x + 2y + 3.$$

However,

$$g(x) + g(y) = (2x + 3) + (2y + 3) = 2x + 2y + 6$$

and so g is not a homomorphism.

For each of the examples above that generated a group homomorphism, notice that each of these homomorphisms mapped the identity element to the identity element and the mapped inverses to inverses. We see in the following two theorems that this is true for all homomorphisms.

Theorem 6.3. Let $f : (G, *) \rightarrow (G', \cdot)$ be a group homomorphism. If e and e' are the identity elements of $(G, *)$ and (G', \cdot) , respectively, then $f(e) = e'$.

Proof. Let a be any element of G and let e be the identity in G . Then

$$f(a) = f(a * e) = f(a) \cdot f(e)$$

and so

$$e' = f(a)^{-1} \cdot f(a) = f(a)^{-1} \cdot (f(a) \cdot f(e)) = (f(a)^{-1} \cdot f(a)) \cdot f(e) = e' \cdot f(e) = f(e).$$

□

Theorem 6.4. Let $f : (G, *) \rightarrow (G', \cdot)$ be a group homomorphism. Then for each $a \in G$, $f(a)^{-1} = f(a^{-1})$.

Proof. Since $f(a) \cdot f(a^{-1}) = f(a * a^{-1}) = f(e) = e'$ and since the inverse of an element is unique, we have that $f(a)^{-1} = f(a^{-1})$. □

Just like with the functions in Chapter 5, we often find it useful to compose two homomorphisms. We see in the following theorem that such a composition is also a homomorphism.

Theorem 6.5. Let $f : (A, *) \rightarrow (B, \times)$ and $g : (B, \times) \rightarrow (C, \cdot)$ be two group homomorphisms. Then

$$(g \circ f) : (A, *) \rightarrow (C, \cdot)$$

is a group homomorphism.

Proof. The proof of the theorem follows directly from the definition of f and g being group homomorphisms, with the difficulty lying in remembering the appropriate operation at each step in the process.

Let a_1 and a_2 be generic elements of A . Then

$$\begin{aligned} (g \circ f)(a_1 * a_2) &= g(f(a_1 * a_2)) = g(f(a_1) \times f(a_2)) \\ &= g(f(a_1)) \cdot g(f(a_2)) = (g \circ f)(a_1) \cdot (g \circ f)(a_2) \end{aligned}$$

and so $g \circ f$ is a homomorphism. □

6.1.4 Isomorphisms

When we combine the concept of a homomorphism with the idea of a bijection, we obtain a new type of function on groups.

Definition 6.4. A homomorphism between groups is called an **isomorphism** if it is one-to-one and onto.

Since isomorphisms are bijections, they satisfy all of the same properties of bijections, including having an inverse. We also can deduce that an isomorphism can only exist between groups whose sets have the same cardinality. So there cannot exist an isomorphism from $(\mathbb{Z}_2, +)$ in Example 6.2 to the dihedral group of order 6 in Example 6.3.

Just as a bijections create an equivalence relation in terms of cardinality, we will see that isomorphisms create an equivalence relation in terms of the group structure. In order to develop this equivalency it is helpful to understand the properties of the inverse function related to the group structures. As such, in the next theorem we prove that the inverse of an isomorphism is also an isomorphism.

Theorem 6.6. Let $f : (G, *) \rightarrow (G', \cdot)$ be an isomorphism. Then $f^{-1} : (G', \cdot) \rightarrow (G, *)$ is also an isomorphism.

Proof. Since f is a bijection, we know from Chapter 5 that f^{-1} is also a bijection. So in order to prove that f^{-1} is an isomorphism it suffices to prove that f^{-1} is a homomorphism.

If we let b_1 and b_2 be two generic elements of G' . Then since f is a bijection, there exist unique a_1 and a_2 in G such that $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Then

$$\begin{aligned} f^{-1}(b_1 \cdot b_2) &= f^{-1}(f(a_1) \cdot f(a_2)) \\ &= f^{-1}(f(a_1 * a_2)) \quad (\text{since } f \text{ is a homomorphism}) \\ &= (f^{-1} \circ f)(a_1 * a_2) = a_1 * a_2 \\ &= f^{-1}(b_1) * f^{-1}(b_2) \end{aligned}$$

and so f^{-1} is a homomorphism, and since it is a bijection it is an isomorphism. \square

Theorem 6.7. *We say that a group $(G, *)$ is isomorphic to a group (G', \cdot) if there is an isomorphism from $(G, *)$ to (G', \cdot) . This relation of $(G, *)$ isomorphic to (G', \cdot) is an equivalence relation.*

Proof. Recall that in order to prove that a relation is an equivalence relation we have to prove that the relation is reflexive, symmetric, and transitive.

Reflexive. Let $(A, *)$ be a group and define $\phi : (A, *) \rightarrow (A, *)$ by $\phi(a) = a$ for all $a \in A$. Since ϕ is the identity function on $(A, *)$ we see that ϕ is an isomorphism and so $(A, *)$ is isomorphic to itself.

Symmetric. Let $(A, *)$ and $(B, +)$ be groups such that $(A, *)$ is isomorphic to $(B, +)$. Let $f : (A, *) \rightarrow (B, +)$ be such an isomorphism. From the previous theorem we know that $f^{-1} : (B, +) \rightarrow (A, *)$ is also an isomorphism. Therefore, $(B, +)$ is isomorphic to $(A, *)$.

Transitive. The proof that isomorphisms are transitive follows from Theorem 5.3 that the composition of two bijections is a bijection and Theorem 6.5 that the composition of two homomorphisms is a homomorphism. Therefore, the composition of two isomorphisms is again an isomorphism. Therefore, group isomorphism creates a transitive relation. \square

The concept of isomorphism in algebra is similar to the concept of congruence in geometry where two triangles being congruent means that they share all of the related properties of triangles. Two groups are considered the “same” group if there is an isomorphism between them and are distinct only in the perspective in which one is viewing the groups. Such groups will be referred to as isomorphic groups. If G and G' are isomorphic groups, then we will denote this by $G \cong G'$.

6.1.5 Finite Groups

To better understand this relationship between isomorphic groups, let's look at groups with two elements. If $(G, *)$ is a group with two elements, the identity e , and an element $a \neq e$. We know that in order for $(G, *)$ to be a group, we know that each element must have a unique inverse. Since $a * e = a$, we know that e is not the inverse of a . So the inverse of a must be a and we have the Cayley table

*	e	a
e	e	a
a	a	e

We can then see that the function $\phi : (\mathbb{Z}_2, +) \rightarrow (G, *)$ defined by $\phi(0) = e$ and $\phi(1) = a$, that ϕ is an isomorphism and so every group with two elements is isomorphic to $(\mathbb{Z}_2, +)$.

In order to study groups with three elements, similarly to how we defined $(\mathbb{Z}_2, +)$ in Example 6.2, we can define $(\mathbb{Z}_3, +)$ as the set $\{0, 1, 2\}$ with the operation defined by the Cayley table,

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

If we let $(G, *)$ be another group with three distinct elements, $\{e, a, b\}$, with identity, e , we know that $e * a = a$, $a * e = a$, $e * b = b$, and $b * e = b$ giving the first row and first column of the Cayley table,

$*$	e	a	b
e	e	a	b
a	a		
b	b		

Since the group is closed under the operation, $a * b \in G$. If $a * b = a$, since a has an inverse, we have $b = a^{-1} * (a * b) = a^{-1} * a = e$. Similarly, if $a * b = b$, we would conclude that $a = e$. Since the identity is unique, neither of these could be true. So $a * b = e$ and $b * a = e$, and we have filled in another two positions in the Cayley table.

$*$	e	a	b
e	e	a	b
a	a		e
b	b	e	

We now need to determine $a * a$ and $b * b$. Since inverses are unique, we know that $a * a \neq e$ and $b * b \neq e$. If $a * a = a$, we have that $a = a * e = a * a * b = a * b = e$, which is not possible. So $a * a = b$, and similarly, $b * b = a$. So we see that $(G, *)$ has the Cayley table

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

We can now define a function $\phi : (\mathbb{Z}_3, +) \rightarrow (G, *)$ by the correspondence $0 \mapsto e$, $1 \mapsto a$, and $2 \mapsto b$, and one can verify that such a function is an isomorphism. So we can now conclude that all groups with three elements are isomorphic to $(\mathbb{Z}_3, +)$. One cannot prove the same type of result for groups with four elements, since $(\mathbb{Z}_4, +)$ is not isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

$(\mathbb{Z}_4, +)$				$(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$				
$+$	0	1	2	3	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
0	0	1	2	3	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
1	1	2	3	0	$(0, 1)$	$(0, 0)$	$(1, 1)$	$(1, 0)$
2	2	3	0	1	$(1, 0)$	$(1, 1)$	$(0, 0)$	$(0, 1)$
3	3	0	1	2	$(1, 1)$	$(1, 0)$	$(0, 1)$	$(0, 0)$

The classification of the groups with a finite number of elements was a significant endeavor in throughout the twentieth century, with a complete classification verified in the early twenty-first century using computers to check the validity of the proof.

6.1.6 Exercises

1. For each of the groups listed in Example 6.1:
 - a. What is the identity element?
 - b. For a generic element a , what would a^{-1} look like?
 - c. Is the group abelian?
 - d. In what ways are the groups related to one another? (Think about subsets or homomorphisms)
2. Find a homomorphism from $(\mathbb{Z}, +)$ to $(5\mathbb{Z}, +)$.
3. Are $(\mathbb{Z}, +)$ and $(\mathbb{R}, +)$ isomorphic groups? Why, or why not?
4. Let $(G, *)$ and $(H, *)$ be two groups such that $H \subseteq G$ and the operation is the same for both groups.
 - a. What are some possible homomorphisms $\phi : (H, *) \rightarrow (G, *)$?
 - b. What are some possible homomorphisms $\psi : (G, *) \rightarrow (H, *)$?

6.2 Relationship Between Linear and Exponential

In this section we will explore exponential and linear functions through the perspective of isomorphisms. Let a be a positive real number. Then we define the exponential function with base a as the function

$$\exp_a : (\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$$

with $\exp_a(x) = a^x$, where a^x is defined in Section 4.5.

Using the properties of exponents in Theorem 4.12 we have that \exp_a is an increasing function (i.e. $x < y \Rightarrow \exp_a(x) < \exp_a(y)$). This implies that \exp_a is an injection. Using properties of limits from analysis one can prove that \exp_a is a surjection. From Section 4.5 we have that

$$\exp_a(x+y) = a^{x+y} = a^x \cdot a^y = \exp_a(x) \cdot \exp_a(y)$$

and so \exp_a is an isomorphism.

We will now explore the isomorphism between $(\mathbb{R}, +)$ and (\mathbb{R}^+, \cdot) .

Property in $(\mathbb{R}, +)$	Property in (\mathbb{R}^+, \cdot)
$m + m = 2 \cdot m$	$a \cdot a = a^2$
$mk = m + m + m + \cdots + m, k \text{ times}$	$a^k = a \cdot a \cdot a \cdots a, k \text{ times}$
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$(b \cdot c)^a = (b^a) \cdot (c^a)$
$0 + m = m + 0 = m$ (additive identity)	$1 \cdot a = a \cdot 1 = a$ (multiplicative identity)
$m + (-m) = 0$ (additive inverse)	$a \cdot \frac{1}{a} = 1$ (multiplicative inverse)

One can see that repeated addition in $(\mathbb{R}, +)$ corresponds to repeated multiplication in (\mathbb{R}^+, \cdot) . One can also see that

$$\exp_a(m \cdot x + b) = \exp_a(b) \cdot (\exp_a(m))^x$$

and so we can see that linear expressions in $(\mathbb{R}, +)$ correspond to exponential expressions in (\mathbb{R}^+, \cdot) . An implication of this is that the teaching of linear and exponential functions should be intertwined. At a minimum, teachers should use their horizon content knowledge when initially teaching students about exponential functions to directly relate the properties of the functions to those of linear functions.

Related Content Standards

- (HSF.LE.1) Distinguish between situations that can be modeled with linear functions and with exponential functions.
 - a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.
 - b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
 - c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

In order to compare linear and exponential functions, we will study their properties side-by-side. Let

$$f(x) = mx + b \quad \text{and} \quad g(x) = b \cdot m^x$$

be linear and exponential functions, respectively. The graphs of the functions are then given by the set of points (x, y) such that

$$y = mx + b \quad \text{and} \quad y = b \cdot m^x.$$

Setting $x = 0$ in each equation implies that $(0, b)$ is the y -intercept for both functions. Exploring the relationship of $f(x)$ to $f(x + 1)$ and $g(x)$ to $g(x + 1)$ we see that

$$f(x + 1) - f(x) = m \quad \text{and} \quad \frac{g(x + 1)}{g(x)} = m$$

implying that in both cases the parameter m represents a constant rate of change. For the linear function m represents an additive rate of change, while in the exponential function m represents a multiplicative rate of change.

Related Content Standards

- (HSF.LE.5) Interpret the parameters in a linear or exponential function in terms of a context.

For the linear function, the function is increasing if, and only if, m is greater than the additive identity. With the exponential function, the values of m must be positive and the function is increasing if, and only if, m is greater than the multiplicative identity.

If we know that two points, (x_1, y_1) and (x_2, y_2) , are on the graph of the functions, then we know that

$$\begin{aligned} y_2 &= m \cdot x_2 + b & y_2 &= b \cdot m^{x_2} \\ &\text{or} \\ y_1 &= m \cdot x_1 + b & y_1 &= b \cdot m^{x_1}. \end{aligned}$$

Using subtraction for the linear function and division for the exponential function we find that

$$(y_2 - y_1) = m(x_2 - x_1) \quad \text{or} \quad \frac{y_2}{y_1} = m^{(x_2 - x_1)}$$

and so

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m = \left(\frac{y_2}{y_1} \right)^{\frac{1}{x_2 - x_1}}.$$

After determining the rate of change for the function one can now include that information in a point-'rate of change' form of the function as

$$y = m(x - x_1) + y_1 \quad \text{or} \quad y = y_1 \cdot m^{(x - x_1)}.$$

6.2.1 Exercises

1. Find the linear and exponential functions that pass through the points $(-3, -2)$ and $(2, 1)$.

6.3 Rings and Fields

In this section we are going to first explore several different examples of sets that have a standard pair of two different binary operations, addition and multiplication. We will create categorizations of these sets based upon their properties. Some properties to consider are associative, commutative, distributive, identities, and inverses.

Example 6.7. Consider the following sets and operations.

- \mathbb{N} with the usual addition and multiplication.
- \mathbb{Z} with the usual addition and multiplication.
- \mathbb{Q} with the usual addition and multiplication.
- \mathbb{R} with the usual addition and multiplication.
- \mathbb{C} with the usual addition and multiplication.
- $2\mathbb{Z}$ (the even integers) with the usual addition and multiplication.

For each of these sets with operations, determine which of the following properties hold.

1. Additive identity
2. Additive inverse
3. Additive commutativity
4. Multiplicative identity
5. Multiplicative inverse
6. Multiplicative commutativity
7. Distribution properties of multiplication over addition.

6.3.1 Finite Sets with Two Operations

While each of the sets listed previously have infinite cardinality, a large portion of abstract algebra studies finite sets, combined with appropriate operations. $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ with addition and multiplication defined by the remainder of the sum and product when divided by n . We provide some examples of these types of sets and operations below. For these finite sets, it is often easiest to just provide a table of the operations, rather than an algebraic definition. These tables are called Cayley tables, first presented in a work by the British mathematician, Arthur Cayley [1854].

Example 6.8. Let $\mathbb{Z}_2 = \{0, 1\}$ with the operations given in the following Cayley tables.

$+$	0	1	.	0	1
0	0	1	0	0	0
1	1	0	1	0	1

Example 6.9. Let $\mathbb{Z}_3 = \{0, 1, 2\}$ with the operations given in the following Cayley tables.

$+$	0	1	2	.	0	1	2
0	0	1	2	0	0	0	0
1	1	2	0	1	0	1	2
2	2	0	1	2	0	2	1

Example 6.10. Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ with the operations given in the following Cayley tables.

+	0	1	2	3		.	0	1	2	3
0	0	1	2	3		0	0	0	0	0
1	1	2	3	0		1	0	1	2	3
2	2	3	0	1		2	0	2	0	2
3	3	0	1	2		3	0	3	2	1

Example 6.11. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with the operations given in the following Cayley tables.

+	0	1	2	3	4	5		.	0	1	2	3	4	5
0	0	1	2	3	4	5		0	0	0	0	0	0	0
1	1	2	3	4	5	0		1	0	1	2	3	4	5
2	2	3	4	5	0	1		2	0	2	4	0	2	4
3	3	4	5	0	1	2		3	0	3	0	3	0	3
4	4	5	0	1	2	3		4	0	4	2	0	4	2
5	5	0	1	2	3	4		5	0	5	4	3	2	1

These different sets (and operations) satisfy various properties depending upon the number of elements in the set. It is a valuable exercise to explore how these are connected.

6.3.2 Matrices

We let $GL(n, \mathbb{R})$ be the set of $n \times n$ matrices with coefficients from the real numbers with the usual matrix addition and multiplication. We can see that this set has an additive identity, 0, and multiplicative identity, I .

$$0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

It also satisfies most of the properties of our number systems that we have explored, including the addition is commutative. However, we can discover that multiplication is not commutative, not every element has a multiplicative inverse, and we can have the product of two non-zero elements be zero. For example, if we look at $GL(2, \mathbb{R})$ and the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $A^2 = 0$, $B^2 = 0$,

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Sometimes we combine ideas of the finite sets with the additional structure of matrices. For example, we let $GL(2, \mathbb{Z}_2)$ be the two by two matrices with coefficients from \mathbb{Z}_2 , with the usual operations of matrix addition and multiplication.

Related Content Standards

- (HSN.VM.8) Add, subtract, and multiply matrices of appropriate dimensions.
- (HSN.VM.9) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

- (HSN.VM.10) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

6.3.3 Rings

Now that we have several examples of sets with two binary operations defined on them, we see that there are certain properties of these binary operations that are very useful for understanding the structure of the set.

Definition 6.5 (Rings). A **ring** $\langle R, +, \cdot \rangle$ is a set R together with two binary operations $+$ and \cdot , which we call addition and multiplication, defined on R such that the following are satisfied:

1. $(R, +)$ is an abelian group.
 - The binary operation $+$ is associative.
 - There is an element 0 in R such that $0 + a = a + 0 = a$ for all $a \in R$. (This element is called the additive identity of R .)
 - For each a in R there is an element $-a$ in R such that $(-a) + a = a + (-a) = 0$. (This element is called the additive inverse of a in R .)
 - For each a and b in R , $a + b = b + a$. (This means that the operation is commutative and so the group is abelian.)
2. The binary operation \cdot is associative.
3. For all $a, b, c \in R$, the left distribution law, $a(b + c) = (ab) + (ac)$, and the right distribution law, $(a + b)c = (ac) + (bc)$, hold.

We can see from this definition that the integers, $\langle \mathbb{Z}, +, \cdot \rangle$, form a ring but the natural numbers, $\langle \mathbb{N}, +, \cdot \rangle$, do not because they do not have additive inverses. Similarly, during the constructions of the rational numbers, $\langle \mathbb{Q}, +, \cdot \rangle$; the real numbers, $\langle \mathbb{R}, +, \cdot \rangle$; and the complex numbers, $\langle \mathbb{C}, +, \cdot \rangle$, we proved that these number systems satisfied all of the ring properties.

We can also see that the examples above of $\langle \mathbb{Z}_n, +, \cdot \rangle$ and $\langle GL(n, \mathbb{R}), +, \cdot \rangle$ are also rings. So in many ways, all of these sets and operations have some very similar properties. However, there are some properties that some of these rings have, that others do not. If a ring has certain additional properties, we will add some labels to these sets.

Definition 6.6 (Further definitions). We classify rings based on various properties of the ring.

1. A ring in which the multiplication is commutative is called a **commutative ring**.
2. A ring with a multiplicative identity is called a **ring with unity**.
3. If every non-zero element of a ring has a multiplicative inverse, then the ring is called a **division ring**.
4. A **field** is a commutative division ring.

Related Content Standards

- (7.NS2) Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
 - a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

6.3.4 Identities and Inverses

Now that we have created several categories and labels for the sets and operations, let's explore some of the consequences of the properties on the operations. As we do so, keep in mind which properties are used to prove different results. That way you can determine which properties are required of various sets and operations in order to get those results later on.

Theorem 6.8. *The additive identity for a ring is unique.*

This theorem is a direct consequence of the set with addition forming a group, whose identity is unique. Similarly, we have that the multiplicative identity is unique (if it exists). Since the proof is so similar to that of the uniqueness of the additive identity, the proof is left to the reader as an exercise.

Theorem 6.9. *If a ring has a multiplicative identity, then that multiplicative identity is unique.*

Now that we know that the additive and multiplicative identities of a ring are unique, we will usually label these identities as 0 and 1, respectively.

Our next theorem shows us that the property from the integers that multiplying any number by zero results in zero is not unique to our number systems, but is inherent in the properties of the additive identity in a ring.

Theorem 6.10. *If $\langle R, +, \cdot \rangle$ is a ring, and if 0 is the additive identity in R , $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$.*

Proof. Let $\langle R, +, \cdot \rangle$ be a ring with additive identity 0 and let $a \in R$. Then $a \cdot 0 = a \cdot (0 + 0)$, since 0 is the additive identity. From the distribution property we have that $a \cdot 0 = a \cdot 0 + a \cdot 0$ and if we add the additive inverse of $a \cdot 0$ to both sides of the equation, we have that $0 = a \cdot 0$. We can similarly show that $0 \cdot a = 0$. \square

Recall from Theorem 6.2 that we have uniqueness of the additive inverse since $\langle R, + \rangle$ is an abelian group. We can also prove the uniqueness of a multiplicative inverse, if such an inverse exists because in this situation, $\langle R, \cdot \rangle$ forms a group.

One question that students often ask when they are introduced to multiplication with negative numbers why a negative times a negative is a positive. We see from the theorem below that this property, and other related properties are inherited from the ring structure of the number system. However, when explaining this property to students, it is often better to focus on the symmetry about the additive identity. In particular, with integers and real numbers it is good to focus on the symmetry of the additive identities on the number line. For the complex numbers it is helpful to focus the students' attention on the symmetry about the additive identity when looking at the complex plane.

Theorem 6.11. *If $\langle R, +, \cdot \rangle$ is a ring and $a, b \in R$, then*

- $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$
- $(-a) \cdot (-b) = a \cdot b$

Proof. In order to prove the first part of the theorem we need to only show that

$$(a \cdot (-b)) + (a \cdot b) = 0 \quad \text{and} \quad ((-a) \cdot b) + (a \cdot b) = 0$$

because the additive inverse of $a \cdot b$ is unique.

We see that

$$\begin{aligned} (a \cdot (-b)) + (a \cdot b) &= a \cdot ((-b) + b) \quad (\text{from the distributive property}) \\ &= a \cdot 0 \quad (\text{since } -b \text{ is the additive inverse of } b) \\ &= 0 \end{aligned}$$

as a result of Theorem 6.10. Similarly,

$$\begin{aligned} ((-a) \cdot b) + (a \cdot b) &= ((-a) + a) \cdot b \quad (\text{from the distributive property}) \\ &= 0 \cdot b \quad (\text{since } -a \text{ is the additive inverse of } a) \\ &= 0 \end{aligned}$$

as a result of Theorem 6.10. For the second result we use the first result to see that

$$(-a) \cdot (-b) = - (a \cdot (-b)) = - (- (a \cdot b)) = a \cdot b$$

since the additive inverse of the additive inverse of an element is the original element. \square

6.3.5 Integral Domains

We often want to solve an equation such as $\$ax=b\$$ or $\$xa=b\$$ for x with $a \neq 0$. In order to do so we need to use the cancellation laws, $ab = ac$ with $a \neq 0$ implies $b = c$ and $ba = ca$ with $a \neq 0$ implies $b = c$.

We will see that the cancellation laws correspond with a property about zero divisors.

Definition 6.7. If a and b are two non-zero elements of a ring R such that $ab = 0$, then a and b are **divisors of 0** (or **zero divisors**). In particular, a is a **left divisor** of 0 and b is a **right divisor** of 0.

Theorem 6.12. *The cancellation laws hold in a ring R if and only if R has no left or right divisors of 0.*

Proof. Assume that for all elements a , b , and c of a ring R such that $a \neq 0$ we have that

$$(ab = ac) \Rightarrow (b = c).$$

We can then add the additive inverse of ac to both sides of the first equation and the additive inverse of c to both sides of the right equation to see that this implication is equivalent to

$$((ab - ac) = 0) \Rightarrow (b - c = 0).$$

Using the left distribution property of the ring, we have that the implication is equivalent to

$$(a \cdot (b - c) = 0) \Rightarrow ((b - c) = 0).$$

This implication is true if and only if a is not a left divisor of 0.

Similarly, one can show that the right cancellation law of

$$(ba = ca, \text{ with } a \neq 0) \Rightarrow (b = c)$$

is equivalent to a not being a right divisor of 0.

Therefore, we see that the cancellation laws holding for a ring is equivalent to the ring having no zero divisors. \square

Since this property of not having zero divisors is so important in the process of solving simple equations, we give rings with this property a label.

Definition 6.8. An **integral domain** is a commutative ring with unity containing no divisors of 0.

A very useful example of an integral domain is the set of integers with the usual addition and multiplication. We will see in the next chapter that the polynomials also form an integral domain, which allows us to factor polynomials in a similar way to factoring integers.

In order to better understand how all of these different types of rings fit together, Figure 6.1 is a graphical way to understand their relationships.

6.3.6 Ring Homomorphisms

Just like group homomorphisms in Section 6.1.3, we can study the functions between rings that maintain the ring structures.

Definition 6.9. Let $\langle R, +, \cdot \rangle$ and $\langle R', *, \times \rangle$ be rings. A map $\phi : R \rightarrow R'$ is a **ring homomorphism** if

- $\phi(a + b) = \phi(a) * \phi(b)$ for all a and b in R
- $\phi(a \cdot b) = \phi(a) \times \phi(b)$ for all a and b in R .

ϕ is called a **ring isomorphism** if it is also a bijection. If such an isomorphism exists from a ring $\langle R, +, \cdot \rangle$ to a ring $\langle R', *, \times \rangle$, then we say that $\langle R, +, \cdot \rangle$ is **isomorphic** to $\langle R', *, \times \rangle$.

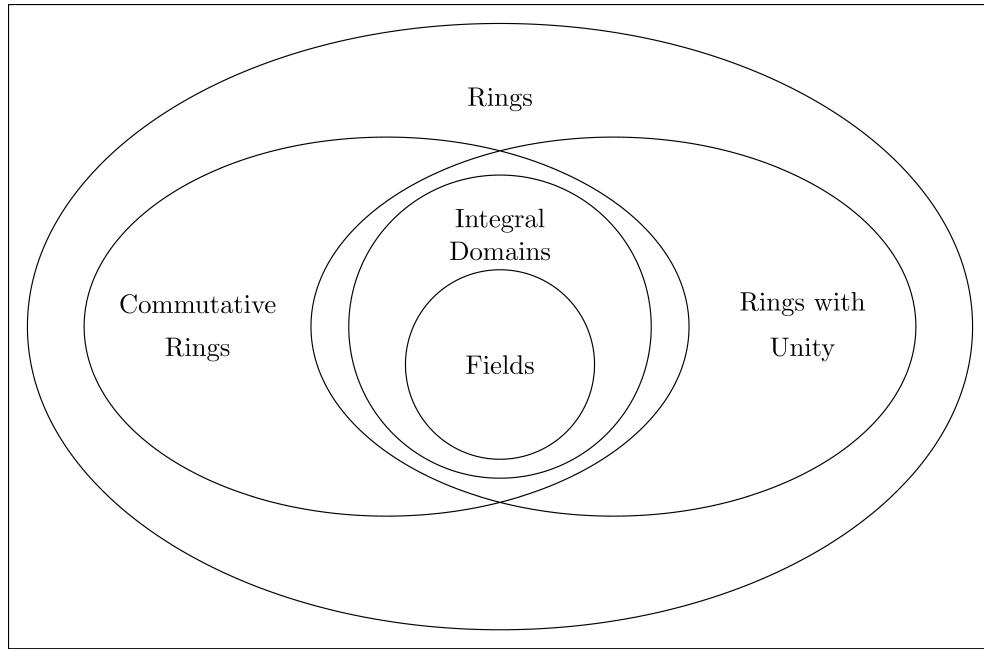


Figure 6.1: Venn diagram of ring relationships

When we talk about rings, we often refer to a ring $\langle R, +, \cdot \rangle$ as just R when the binary operations are easily inferred from the set. For instance, we often talk about \mathbb{R} as the ring of real numbers with the binary operations of addition and multiplication being inferred.

Similar to groups, it is useful to know when two rings are really the “same” ring with different labels. We see that just like with groups (Theorem 6.7) we see that ring isomorphisms create an equivalence relation and so two rings are the “same” if there is an isomorphism from one to the other. Therefore, in order to prove that two rings are isomorphic, it is usually done by finding the isomorphism. However, to prove that two rings are not isomorphic we usually use a proof by contradiction using certain properties of ring isomorphisms.

6.3.7 Exercises

1. Create Cayley tables for addition and multiplication for \mathbb{Z}_{12} . What patterns do you recognize inside of the table and how would you explain the reasons for those patterns?
2. What are the zero divisors of $GL(2, \mathbb{Z}_2)$?
3. For each of the following rings, determine the appropriate location for the ring in the Venn diagram of the ring relationships (Figure 6.1).
 - a. $\langle \mathbb{Z}, +, \cdot \rangle$, the integers with the usual addition and multiplication.
 - b. $\langle \mathbb{Q}, +, \cdot \rangle$, the rational numbers with the usual addition and multiplication.
 - c. $\langle \mathbb{R}, +, \cdot \rangle$, the real numbers with the usual addition and multiplication.
 - d. $\langle \mathbb{C}, +, \cdot \rangle$, the complex numbers with the usual addition and multiplication.
 - e. $\langle 2\mathbb{Z}, +, \cdot \rangle$, the even integers with the usual addition and multiplication.
 - f. $\langle \mathbb{Z}_2, +, \cdot \rangle$, as defined in Example 6.8.
 - g. $\langle \mathbb{Z}_4, +, \cdot \rangle$, as defined in Example 6.10.
 - h. $\langle GL(2, \mathbb{R}), +, \cdot \rangle$, the two by two matrices with real coefficients with the usual matrix addition and multiplication.
 - i. $\langle GL(2, 2\mathbb{Z}), +, \cdot \rangle$, the two by two matrices with even integer coefficients with the usual matrix addition and multiplication.
4. Prove Theorem 6.9.

5. Show that $a^2 - b^2 = (a + b)(a - b)$ for all a and b in R if and only if R is a commutative ring.
6. Prove that $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic as rings.
7. Let R be a ring and let R^R be the set of all functions mapping R into R . For $f, g \in R^R$, define the sum $f + g$ by

$$(f + g)(a) = f(a) + g(a)$$

and the product $f \cdot g$ by

$$(f \cdot g)(a) = f(a) \cdot g(a)$$

for each $a \in R$. Show that $\langle R^R, +, \cdot \rangle$ is a ring.

Chapter 7

Integral Domains and Polynomials

7.1 Properties of the Ring of the Integers

In this section we spend time better understanding the integral domain of the integers. We will look at some of the divisibility properties that are the underpinnings for our standard long division algorithms and build up to the prime factorization of the integers.

The first key component of the integers that we will look at is the Division Algorithm.

Theorem 7.1 (The Division Algorithm for Integers). *If a and b are integers with $b > 0$, then there are unique integers q and r such that*

$$a = bq + r, \quad \text{with } 0 \leq r < b.$$

We call the integer q in the division algorithm the quotient of a and b and we call r the remainder of the quotient of a and b .

Related Content Standards

- (4.NBT.6) Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
- (5.NBT.6) Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Proof. Let a and b be integers with $b > 0$. We will first prove the existence of integers q and r such that

$$a = bq + r, \quad \text{with } 0 \leq r < b$$

and then prove that these integers are unique.

Existence. If $a \geq 0$, then we can let

$$S = \{a - kb \mid a - kb \geq 0 \text{ and } k \text{ is a non-negative integer}\}.$$

Since $a \geq 0$, we see that $a = a - 0 \cdot b$ is an element of S and so S is non-empty. We also see that $S \subseteq [0, a]$.

So by the Well-Ordering Property of the Integers, there is a smallest element of S which we will call r . Since $r \in S$, there exists a non-negative integer q such that $r = a - qb$.

Since r is the smallest element of S and since $a - (q+1)b < a - qb = r$, we see that $a - (q+1)b < 0$, otherwise it would be a smaller element of S .

This implies that $r = a - qb < b$. Therefore we have the existence of the q and r .

If $a < 0$, using the above process we can find non-negative integers q and r , with $0 \leq r < b$, such that $-a = bq + r$. So $a = b(-q) + r$ satisfies the existence statement.

Uniqueness. In order to prove the uniqueness of the quotient and remainder, we assume that there are two pairs of integers (q_1, r_1) and (q_2, r_2) such that

$$a = bq_1 + r_1 \text{ and } a = bq_2 + r_2, \text{ with } 0 \leq r_1 < b \text{ and } 0 \leq r_2 < b.$$

This implies that $bq_1 + r_1 = bq_2 + r_2$ and by rearranging the terms we have that

$$b(q_1 - q_2) = r_2 - r_1.$$

If $0 \leq r_1 \leq r_2 < b$, then $0 \leq b(q_1 - q_2) = r_2 - r_1 < b$. So $q_1 - q_2$ is an integer in $[0, 1)$ and so must be 0. Therefore, $q_1 = q_2$ and so $r_1 = r_2$. Therefore, the quotient and remainder are unique. \square

It is through a repetition of a variation of this division algorithm that we are able to divide integers with the standard long division algorithm. Look at various ways that we can find the quotient of 284 by 13. In particular compare the division algorithm in Theorem 7.1, the standard long division algorithm, the area model, and other models with which you are familiar.

7.1.1 Composing and Decomposing Integers

Numerical fluency involves the ease at which one is able to break apart and recombined numbers under addition and/or multiplication. For addition, the fluency focuses on the tens using items like ten frames. For multiplication, it is a fluency with multiplication facts up to 12s, with flexibility and reversibility.

Related Content Standards

- (4.OA.4) Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.

Definition 7.1. Let a and b be integers with $b \neq 0$. We say that b divides a if there exists a $q \in \mathbb{Z}$ such that $a = bq$. In this situation we also say that

- a is a multiple of b
- b is a divisor (or factor) of a

An integer $n > 1$ is called prime if its only positive integer divisors are itself and 1.

Note that 1 is not a prime number, since prime numbers are defined to be integers greater than 1. The primary reason behind this is to make the prime factorization of the integers unique.

So the prime numbers less than 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

7.1.2 Greatest Common Divisor and Least Common Multiple

Definition 7.2. Let a and b be integers, not both zero. We say that the **greatest common divisor** (also called the **greatest common factor**) of a and b , denoted $\gcd(a, b)$, is the natural number d such that

- d divides both a and b , and
- if u divides both a and b for some integer u , then $u \leq d$.

Let a and b be integers, not both zero. We say that the **least common multiple** of a and b , denoted $\text{lcm}(a, b)$, is the natural number m such that

- m is a multiple of both a and b , and
- if u is a multiple of both a and b for some integer u , then $u \geq m$.

Related Content Standards

- (6.NS.4) Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

There are many different options for finding the greatest common divisor and least common multiple of two numbers. If the numbers are small, one can find the greatest common divisor by listing out all of the possible factors of each number and choosing the largest one in common. Another option is using the factor trees of the two numbers, based on the prime factorizations of the numbers. Similarly, for small numbers one can list out the multiples of each number until you find a common multiple, or you can use information from the numbers' prime factorizations. When the numbers are much larger, one can develop other techniques to create algorithms for finding the greatest common divisor and least common multiple. Many of these techniques are based on the following theorem.

Theorem 7.2. *If a, b, q, r are integers, with a and b not both zero, and if*

$$a = bq + r,$$

then $\text{gcd}(a, b) = \text{gcd}(b, r)$.

Proof. Since a and b are not both zero, we let $d = \text{gcd}(a, b)$. This means that there are integers m and n such that $a = md$ and $b = nd$. Therefore,

$$r = a - bq = md - (nd)q = d \cdot (m - nq)$$

and so d divides r and d divides b . Since d is a common divisor of b and r , we see that

$$\text{gcd}(a, b) \leq \text{gcd}(b, r).$$

Since $a = bq + r$, any common divisor of b and r must also be a divisor of a , we have that $\text{gcd}(b, r)$ is a common divisor of a and b and so

$$\text{gcd}(b, r) \leq \text{gcd}(a, b).$$

Therefore, $\text{gcd}(a, b) = \text{gcd}(b, r)$. □

Theorem 7.3 (Euclidean Algorithm). *Let a and b be two positive integers with $a > b$. Then the greatest common divisor of a and b can be found using the following algorithm.*

- [Step 1] Apply the division algorithm to find integers q and r such that $a = bq + r$ with $0 \leq r < b$.
- [Step 2] If $r = 0$, then b is $\text{gcd}(a, b)$.
- [Step 3] While $r > 0$, replace define $a := b$ and $b := r$ and apply the division algorithm to the new a and b to generate a new pair of integers q and r . Once $r = 0$, then the value of b is the greatest common divisor of the original integers a and b .

In order to better understand the algorithm, we will find the greatest common divisor of 924 and 260. We start by letting $a = 924$ and $b = 260$. From the division algorithm we find that

$$924 = 3 \cdot 260 + 144.$$

We now let $a = 260$ and $b = 144$ and use the division algorithm again to find that

$$260 = 1 \cdot 144 + 116.$$

Continuing this process we see that $\gcd(924, 260) = 4$.

$$\begin{aligned} 144 &= 1 \cdot 116 + 28 \\ 116 &= 4 \cdot 28 + 4 \\ 28 &= 7 \cdot 4 + 0 \end{aligned}$$

7.1.3 Fundamental Theorem of Arithmetic

The most common way to find the greatest common divisor and least common multiple of two numbers is dependent upon the numbers' prime factorizations. This ability to write a number uniquely as a product of primes is called the Fundamental Theorem of Arithmetic, which we will prove in this section.

We say that two numbers are **relatively prime** if their greatest common divisor is 1.

Theorem 7.4. *Natural numbers a and b are relatively prime if and only if there exist integers s and t such that $sa + tb = 1$.*

Proof. Suppose that a and b are relatively prime. From the proof of the Euclidean algorithm we can see that for integers a and b that there exists integers s and t such that $sa + tb = \gcd(a, b) = 1$.

Conversely, if there exist integers s and t such that $sa + tb = 1$, we know that $d = \gcd(a, b)$ divides a and b and so would also divide 1. Since the only integers that divide 1 are 1 and -1 and since $\gcd(a, b) \geq 1$, we have that $\gcd(a, b) = 1$. \square

Lemma 7.1. *If p is a prime number and a and b are integers such that p divides ab , then p divides a or p divides b .*

Proof. In order to prove the Lemma, we will prove that the negation is false. Suppose that a is relatively prime to p and b is relatively prime to p . Then from Theorem 7.4 we have integers s, t, u , and v such that $sa + tp = 1$ and $ub + vp = 1$. Then

$$1 = 1 \cdot 1 = (sa + tp) \cdot (ub + vp) = (su)(ab) + (sav + tub + tvp)p$$

and so ab and p are relatively prime, implying that p does not divide ab . \square

Theorem 7.5 (Fundamental Theorem of Arithmetic). *Every integer $n > 1$ can be factored as a product of prime numbers, and except for the order of the factors, this prime factorization is unique.*

Proof. Let $n > 1$ be an integer. If n is prime, we have a prime factorization of n . If n is not prime, then there are integers n_1 and n_2 such that $n = n_1n_2$ and we can assume without loss of generality that $1 < n_1 \leq n_2 < n$. If either of these integers are composite, then they also can be factored into factors strictly smaller than them. This process can be continued, but it cannot be continued indefinitely since n is a finite number. At the conclusion of the process, n will have a prime factorization. We now only need to show uniqueness of this factorization.

Assume that n has two distinct prime factorizations,

$$n = p_1p_2 \cdots p_j \quad \text{and} \quad n = q_1q_2 \cdots q_k$$

and we can assume without loss of generality that $j \leq k$.

Since p_1 divides n , a generalization of Lemma 7.1 gives us that p_1 must divide one of the q_i . By reordering the primes we can label this q_i as q_1 . Since both p_1 and q_1 are prime, they must be equal. We can continue this process with $p_2 \cdots p_j$ and $q_2 \cdots q_k$ to find that $p_2 = q_2$ and continuing the process to equate each of the p_i with a q_i . If some of the q_i remain after this process, these would be factors of 1 and therefore could not be prime. Therefore, $j = k$ and we see that the two prime factorizations are just a reordering of each other. \square

7.1.4 Exercises

1. Let $n \in \mathbb{Z}$. What are the possible values for the remainder of n^2 with a divisor of 3? Justify your result.
2. Find the q and r from the Division Algorithm for the given values of a and b .
 - a. $a = 0, b = 14$
 - b. $a = -42, b = 14$
 - c. $a = 9, b = 14$
3. Use the Division Algorithm to prove that for positive integers m and n , with $m > n$, then exactly one of the following statements must be true:
 - There is a positive integer k such that $m = kn$
 - There is a positive integer k such that $kn < m < (k + 1)n$
4. Use the Euclidean Algorithm to find $\gcd(4368, 31050)$.
5. Create a computer program or spreadsheet based on the Euclidean Algorithm that you can use to find the greatest common divisor of two integers.
6. Prove that for all integers that $\gcd(a, b) \cdot \text{lcm}(a, b) = |ab|$. (Hint: Use the prime factorizations of a and b .)

7.2 Polynomial Rings

Another set of examples of integral domains in the secondary curriculum are polynomial rings. We let

$$\mathbb{R}[x] = \left\{ a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{j=0}^n a_jx^j \mid a_j \in \mathbb{R} \right\}$$

The concept of a polynomial can be generalized to not just having real number coefficients. For instance, we will sometimes require that the polynomials have integer coefficients or possibly coefficients that are complex numbers. So we will generalize the definition of a polynomial to be able to be used in these cases.

Definition 7.3. Let R be a ring. We define a polynomial $f(x)$ with coefficients in R to be an infinite formal sum

$$\sum_{j=0}^{\infty} a_jx^j = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

where $a_j \in R$ for all j and $a_j = 0$ for all but finitely many values of j .

We denote the set of such polynomials as $R[x]$.

Since the $a_j = 0$ for all but finitely many values of j , by the well-ordering principle, there is a largest value of j for which $a_j \neq 0$. We call this largest value of j the **degree** of the polynomial. If all of the coefficients are 0, we say the polynomial is degree zero.

So we can define the function “degree” that maps the non-zero polynomials to the natural numbers by

degree $\left(\sum_{j=0}^{\infty} a_jx^j \right)$ is the largest value of j for which a_j is non-zero.

We call the a_j that corresponds to this value of j the **leading coefficient**.

Related Content Standards

- (HSA.APR.1) Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

If $a(x) = \sum_{j=0}^n a_j x^j$ and $b(x) = \sum_{j=0}^m b_j x^j$ are two elements of $R[x]$, with $n \leq m$, then we define

$$a(x) + b(x) := \sum_{j=0}^m (a_j + b_j) x^j$$

where $a_j := 0$ for $n < j \leq m$. From this definition, we see that addition is closed. We also see that addition is associative from the associativity of addition on R . Since R is a ring, addition is commutative on R and so addition is commutative on $R[x]$. We also see that the zero polynomial, $0(x) = 0$, is an additive identity for the set. So $(R[x], +)$ is an Abelian group.

Multiplication on $R[x]$ is more challenging to write in a compact form. We will assume that the indeterminate, x , commutes multiplicatively with every element in R , i.e. $ax = xa$, $\forall a \in R$ (the set of all elements of a ring that commute with every element of the ring is called the center of the ring). We can then define multiplication as

$$\begin{aligned} a(x) \cdot b(x) &:= \left(\sum_{j=0}^n a_j x^j \right) \cdot \left(\sum_{k=0}^m b_k x^k \right) = \sum_{j=0}^n \left(a_j x^j \left(\sum_{k=0}^m b_k x^k \right) \right) \\ &= \sum_{j=0}^n \sum_{k=0}^m (a_j x^j b_k x^k) = \sum_{j=0}^n \sum_{k=0}^m (a_j b_k) x^{(j+k)} \end{aligned}$$

and we see that $R[x]$ is closed under multiplication.

Unless R is a commutative ring, we must maintain the order of the coefficients. The associativity of multiplication and distributive properties are inherited from $R[x]$. If the ring R has a multiplicative identity, then $R[x]$ also has a multiplicative identity of the polynomial of the multiplicative identity. Finally, if R is a commutative ring, then multiplication on $R[x]$ is commutative.

Theorem 7.6. *The set $R[x]$ of polynomials with coefficients from a ring R with the binary operations of polynomial addition and multiplication is a ring.*

One of the main activities that we do with polynomials in the secondary curriculum is to treat the polynomials as functions and evaluate them for various values.

The following theorem gives us the mathematical justification for such evaluations.

In this theorem we use a field F and a subfield E , with E is a subset of F that also satisfies the field properties inherited from F . The reason that we use the field E and the subfield F is that we often have polynomials with whose coefficients are from one field and the point at which we are evaluating from a larger field. For instance, we often look at quadratic polynomials with real coefficients and determine that they have complex zeros. In this instance, the subfield would be \mathbb{R} and the larger field would be \mathbb{C} .

Theorem 7.7. *Let F be a subfield of a field E and let α be any element of E . The map $\phi_\alpha : F[x] \rightarrow E$ defined by*

$$\phi_\alpha(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_n \alpha^n$$

is a ring homomorphism from $F[x]$ into E .

We go ahead and include the proof of this theorem below to demonstrate how these evaluation homomorphisms work and to demonstrate certain properties of polynomial functions.

Proof. Let F be a subfield of a field E and let α be any element of E . We also let $a(x)$ and $b(x)$ be generic elements of $F[x]$ and label them as

$$a(x) = \sum_{j=0}^{\infty} a_j x^j \text{ and } b(x) = \sum_{j=0}^{\infty} b_j x^j$$

where the a_j and the b_j are zero for all but finitely many values of j .

Then $\phi_{\alpha}(a(x)) = \sum_{j=0}^{\infty} a_j \alpha^j$ and $\phi_{\alpha}((b(x))) = \sum_{j=0}^{\infty} b_j \alpha^j$, which are elements of E since addition and multiplication on E are closed and only finitely many elements of each of the sums are non-zero.

Using the definition of addition of two polynomials, we have that $a(x) + b(x)$ is a polynomial that is written as

$$a(x) + b(x) = \sum_{j=0}^{\infty} (a_j + b_j) x^j.$$

Therefore,

$$\phi_{\alpha}(a(x) + b(x)) = \sum_{j=0}^{\infty} (a_j + b_j) \alpha^j$$

and by rearranging the finite number of terms using the distributive property and that both addition and multiplication are commutative in E we have that

$$\phi_{\alpha}(a(x) + b(x)) = \sum_{j=0}^{\infty} a_j \alpha^j + \sum_{j=0}^{\infty} b_j \alpha^j = \phi_{\alpha}(a(x)) + \phi_{\alpha}(b(x)).$$

Similarly, one finds that

$$\phi_{\alpha}(a(x) \cdot b(x)) = \phi_{\alpha}(a(x)) \cdot \phi_{\alpha}(b(x)).$$

□

You should keep in mind that the operations are using the same symbol for operations in two different rings. For the case of $a(x) + b(x)$, the addition is in the polynomial ring $F[x]$, while $\phi_{\alpha}(a(x)) + \phi_{\alpha}(b(x))$ is addition in E .

We say that an element $a \in E$ is a **zero of a polynomial** $p(x) \in F[x]$ if

$$\phi_a(p(x)) = 0.$$

We would really like to have the property that for all $a, b \in R$, a and b are the only zeros of the polynomial $(x - a)(x - b)$. In order for that to be true, we need $R[x]$ to be an integral domain. However, we first need some properties related to the degree of polynomials.

Theorem 7.8. *Let R be an integral domain and let $a(x)$ and $b(x)$ be elements of $R[x]$. Then*

$$\deg(a(x) + b(x)) \leq \max(\deg(a(x)), \deg(b(x))), \text{ and}$$

$$\deg(a(x)b(x)) = \deg(a(x)) + \deg(b(x)).$$

Proof. Let $a(x) = \sum_{j=0}^n a_j x^j \in R[x]$ with degree n and $b(x) = \sum_{j=0}^m b_j x^j \in R[x]$ with degree m . We can assume without any loss of generality that $n \leq m$.

If $n < m$, $a(x) + b(x) = \sum_{j=0}^m (a_j + b_j) x^j$, where $a_j = 0$ for $n < j \leq m$, we see that $\deg(a(x) + b(x)) = m$. If $n = m$ the sum also has degree m , unless the two leading coefficients are additive inverses. Therefore,

$$\deg(a(x) + b(x)) \leq \max(\deg(a(x)), \deg(b(x))).$$

Since

$$a(x) \cdot b(x) = \sum_{j=0}^n \sum_{k=0}^m (a_j b_k) x^{(j+k)}$$

we see that the leading term of $a(x) \cdot b(x)$ is $a_n b_m$. This is non-zero because R is an integral domain. So the degree of $a(x) \cdot b(x)$ is $m + n$. And we have that

$$\text{degree}(a(x)b(x)) = \text{degree}(a(x)) + \text{degree}(b(x)).$$

□

We are now ready to prove that R being an integral domain implies the $R[x]$ is an integral domain.

Theorem 7.9. *If R is an integral domain, then $R[x]$ is an integral domain.*

Proof. Assume that there are two polynomials, $a(x) = \sum_{j=0}^n a_j x^j$ and $b(x) = \sum_{j=0}^m b_j x^j$ in $R[x]$, such that $a_n \neq 0$ and $b_m \neq 0$. Then the product of the polynomials has degree of $n + m$. If $n + m > 0$, then the product cannot be the zero polynomial. If $n + m = 0$, then n and m are both zero and $a(x) = a_0$, $b(x) = b_0$, and $a(x) \cdot b(x) = a_0 b_0$. Since $a_0 \neq 0$ and $b_0 \neq 0$, since R is an integral domain, the product is non-zero. □

7.2.1 Exercises

1. In \mathbb{Z}_{12} , what are the zeros of the polynomial $6 - 5x + x^2$?
2. Use the various models of multiplication presented in Section 4.2 to add and multiply the following pairs of polynomials:
 - a. $a(x) = -2x + 3$ and $b(x) = x - 1$
 - b. $a(x) = 2x^2 - 4x + 1$ and $b(x) = x^3 - 2$
 - c. Which models transition well to polynomials and which do not?

7.3 Properties of Polynomial Rings

The properties of irreducible (prime) elements and factorizations is very important in the study of the integers and is dependent upon the integral domain properties. We have also seen that certain polynomial rings are also integral domains. In this section we will see how the factorization of integers into prime factors can be extended to the ring of polynomials. Throughout this section we will let S be a general integral domain (you can think of it as \mathbb{Z} , \mathbb{Q} , \mathbb{R} , or \mathbb{C}) and let $S[x]$ be the ring of polynomials with coefficients from S .

7.3.1 Polynomial Division

We begin this generalization by finding a generalization of the division algorithm. We see from the Division Algorithm for Integers that we need to have some type of ordering for the elements of our integral domain. We will call such an ordering a Euclidean function.

Definition 7.4. Let R be an integral domain. A **Euclidean function** on R is a function f from $R \setminus \{0\}$ to the non-negative integers satisfying the fundamental division-with-remainder property.

- If a and b are in R and b is non-zero, then there exist q and r in R such that $a = bq + r$ and either $r = 0$ or $f(r) < f(b)$.

An integral domain for which a Euclidean function exists is called a Euclidean domain.

Since S is an integral domain, we see that $S[x]$ is also an integral domain (Theorem 7.9). We can now use the degree function of the polynomial as a Euclidean function and state a Division Algorithm for Polynomials that makes $S[x]$ a Euclidean domain, when S is a field.

Theorem 7.10 (Division Algorithm for Polynomials). *Let S be a field. Suppose that $a(x)$ and $b(x)$ are polynomials in $S[x]$ and that $b(x)$ is not the zero polynomial. Then there are unique $q(x)$ and $r(x)$ in $S[x]$ such that*

$$a(x) = b(x) \cdot q(x) + r(x)$$

and $r(x)$ is the zero polynomial or the degree of $r(x)$ is less than the degree of $b(x)$.

The reason that we need S to be a field is to allow for division of the coefficients in the division algorithm. For example in $\mathbb{Z}[x]$, if $a(x) = x^2$ and $b(x) = 3x$, the $r(x)$ in the division algorithm would need to be $3x$.

Proof. Let S be a field and let $a(x)$ and $b(x)$ be polynomials, with $b(x)$ not the zero polynomial.

Let

$$T = \{a(x) - b(x) \cdot s(x) | s(x) \in S[x]\}$$

and from the well-ordering property we can choose an $r(x)$ in T with minimal degree. So

$$a(x) = b(x)q(x) + r(x)$$

for some $q(x) \in S[x]$.

Let $m = \text{degree}(b(x))$ and $n = \text{degree}(r(x))$, and let b_m be the leading coefficient of $b(x)$ and r_n be the leading coefficient of $r(x)$. If $m \geq n$,

$$a(x) - b(x) \left(q(x) - \frac{r_n}{b_m} x^{m-n} \right) = r(x) - \frac{r_n}{b_m} x^{m-n} b(x)$$

which is an element of T with lower degree than $r(x)$, contradicting the choice of $r(x)$ to be an element of T with minimal degree. Therefore, $\text{degree}(r(x)) < \text{degree}(b(x))$.

For the uniqueness of $q(x)$ and $r(x)$, we assume that there are two pairs of such functions such that

$$a(x) = b(x)q_1(x) + r_1(x) \quad \text{and} \quad a(x) = b(x)q_2(x) + r_2(x)$$

with the degrees of $r_1(x)$ and $r_2(x)$ being less than the degree of $b(x)$. Subtracting the two equations and rearranging terms gives us

$$b(x)(q_1(x) - q_2(x)) = r_2(x) - r_1(x).$$

Since the degree of the polynomial on the right side of the equation is less than the degree of $b(x)$, $(q_1(x) - q_2(x))$ must be the zero polynomial. So $q_1(x) = q_2(x)$ and thus implying that $r_1(x) = r_2(x)$. And we have proven the uniqueness of the $q(x)$ and $r(x)$. \square

Related Content Standards

- (HSA.APR.6) Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Take some time and work through some examples of long division of polynomials and compare these with the procedures in the long division of integers. It is through these connections between the two processes that students are able to understand and retain the knowledge of the long division algorithm for polynomials.

We can now extend the properties of factors and prime (or irreducible) elements of the integers to the polynomial ring.

Definition 7.5. Let S be a ring and $S[x]$ be the corresponding ring of polynomials. Let $a(x)$ and $b(x)$ be polynomials in $S[x]$ with $b(x)$ not the zero polynomial. We say that $b(x)$ divides $a(x)$ if there exists a $q(x)$ in $S[x]$ such that $a(x) = b(x) \cdot q(x)$. In this situation we also say that

- $a(x)$ is a multiple of $b(x)$,

- $b(x)$ is a divisor of $a(x)$, and
- $b(x)$ is a factor of $a(x)$.

We can now state the prove the Remainder Theorem and Factor Theorem for polynomials.

Theorem 7.11 (Remainder Theorem). *Let S be a field ($\mathbb{Q}, \mathbb{R}, \mathbb{C}$) and suppose that $a(x)$ is a polynomial of degree greater than or equal to 1 in $S[x]$, and let $c \in S$. Then $a(c) := \phi_c(a(x))$ is the remainder obtained by dividing $a(x)$ by $x - c$, where ϕ_c is the evaluation function.*

Proof. Applying the Division Algorithm to $a(x)$ and $x - c$ we see that there is a unique $q(x)$ and $r(x)$ such that $a(x) = q(x) \cdot (x - c) + r(x)$ with the degree of $r(x)$ less than the degree of $x - c$. Therefore, $r(x)$ is a constant polynomial that we will call r . Since the evaluation function is a ring homomorphism we have that

$$a(c) := \phi_c(a(x)) = \phi_c(q(x)) \cdot \phi_c(x - c) + \phi_c(r).$$

Since $\phi_c(x - c) = c - c = 0$ and since $0 \cdot \phi_c(q(x)) = 0$, we have that $a(c)$ is the remainder. \square

This leads us directly to the Factor Theorem since $a(x) = q(x) \cdot (x - c) + a(c)$.

Theorem 7.12 (Factor Theorem). *Let S be a field and suppose that $a(x)$ is a polynomial of degree at least 1. Then $x - c$ is a factor of $a(x)$ if and only if $a(c) = 0$.*

Related Content Standards

- (HSA.APR.2) Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a , the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

7.3.2 GCF and LCM for Polynomials

Definition 7.6. Let $a(x)$ and $b(x)$ be in $S[x]$, not both the zero polynomial. We say that a **greatest common divisor** (also called the **greatest common factor**) of $a(x)$ and $b(x)$ is a polynomial $d(x)$ such that

- $d(x)$ divides both $a(x)$ and $b(x)$, and
- if $u(x)$ divides both $a(x)$ and $b(x)$ for some polynomial $u(x)$, then $u(x)$ divides $d(x)$.

Note that there is not a single greatest common divisor of two polynomials since constant multiples of a greatest common divisor may also be greatest common divisors. In the situation that S is a field, we can multiply a greatest common divisor by the multiplicative inverse of its leading coefficient and we now have a monic polynomial (a polynomial with leading coefficient of the multiplicative identity) that is a greatest common divisor. This monic polynomial is unique and so we define $\gcd(a(x), b(x))$ to be the **unique monic greatest common divisor of $a(x)$ and $b(x)$** .

Following the line of proofs of the integers, we can prove a theorem analogous to Theorem 7.2 for polynomials.

Theorem 7.13. *If $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials in $S[x]$ for a field S , with $a(x)$ and $b(x)$ not both the zero polynomial, and if*

$$a(x) = b(x)q(x) + r(x),$$

then $\gcd(a(x), b(x)) = \gcd(b(x), r(x))$.

This theorem is the basis for a generalization of the Euclidean Algorithm to the polynomials.

Theorem 7.14 (Euclidean Algorithm for Polynomials). *Let $a(x)$ and $b(x)$ be two polynomials in $S[x]$ for a field S , with $\deg(a(x)) > \deg(b(x))$. Then the greatest common divisor of $a(x)$ and $b(x)$ can be found using the following algorithm.*

- [Step 1] Apply the division algorithm to find polynomials $q(x)$ and $r(x)$ such that $a(x) = b(x)q(x) + r(x)$ with $0 \leq \deg(r(x)) < \deg(b(x))$.

- [Step 2] If $r(x)$ is the zero polynomial, then $\gcd(a, b)$ is $b(x)$ times the multiplicative inverse in S of its leading coefficient.
- [Step 3] While $r(x)$ is not the zero polynomial, replace $a(x) := b(x)$ and $b(x) := r(x)$ and apply the division algorithm to the new $a(x)$ and $b(x)$ to generate a new pair of polynomials $q(x)$ and $r(x)$. Once $r(x)$ is the zero polynomial, the value of $b(x)$ is a greatest common divisor of the original polynomials $a(x)$ and $b(x)$ and $\gcd(a, b)$ is this polynomial times the multiplicative inverse in S of its leading coefficient.

In order to better understand the Euclidean Algorithm for Polynomials we will use the algorithm to find the unique monic greatest common divisor of $a(x) = 2x^5 - x^4 - 41x^3 + 59x^2 - 43x + 60$ and $b(x) = x^3 + 2x^2 + x + 2$.

The first step is to use the division algorithm to find the quotient and remainder of $a(x)$ divided by $b(x)$. One of the most efficient methods of finding this quotient and remainder (other than using a computer algebra system) is through long division of polynomials.

$$\begin{array}{r} & \frac{2x^2 - 5x - 33}{x^3 + 2x^2 + x + 2) \overline{2x^5 - x^4 - 41x^3 + 59x^2 - 43x + 60}} \\ & \quad \frac{-2x^5 - 4x^4 - 2x^3 - 4x^2}{ - 5x^4 - 43x^3 + 55x^2 - 43x} \\ & \quad \frac{5x^4 + 10x^3 + 5x^2 + 10x}{ - 33x^3 + 60x^2 - 33x + 60} \\ & \quad \frac{33x^3 + 66x^2 + 33x + 66}{ 126x^2 + 126} \\ & \quad \end{array}$$

Therefore,

$$a(x) = b(x) \cdot (2x^2 - 5x - 33) + (126x^2 + 126)$$

and so we know that $\gcd(a(x), b(x)) = \gcd(x^3 + 2x^2 + x + 2, 126x^2 + 126)$. Using long division of polynomials we can find the new quotient and remainder for these two polynomials.

$$\begin{array}{r} & \frac{\frac{1}{126}x + \frac{1}{63}}{126x^2 + 126) \overline{x^3 + 2x^2 + x + 2}} \\ & \quad \frac{-x^3}{ - x} \\ & \quad \frac{2x^2}{ - 2x^2} \\ & \quad \frac{-2x^2}{ 0} \\ & \quad \end{array}$$

So,

$$x^3 + 2x^2 + x + 2 = (126x^2 + 126) \left(\frac{1}{126}x + \frac{1}{63} \right) + 0$$

and we see that $126x^2 + 126$ is a greatest common divisor of $a(x)$ and $b(x)$. Therefore, the unique monic greatest common divisor of $a(x)$ and $b(x)$ is $x^2 + 1$.

7.3.3 Irreducible Polynomials

We now turn our attention to studying the analogy to prime numbers in the polynomial ring.

Definition 7.7. An element in an integral domain, D , is called a **unit** of D if it has a multiplicative inverse. A non-zero element, p , of D is called an *irreducible* of D if any factorization $p = ab$ in D either a or b is a unit.

When S is a field, we see that the units of $S[x]$ are the constant polynomials since the degree of the product of two polynomials in $S[x]$ is equal to the sum of the degrees of the factors.

Notice that since $\sqrt{2}$ is irrational that $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ but has the factorization of $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ in $\mathbb{R}[x]$ and $\mathbb{C}[x]$. This means that the set of irreducible polynomials depends upon the field from which the coefficients reside.

The following is an essential theorem in understanding factorization of polynomials and is one of the main consequences of the concept of irreducible elements. However, the proof of the theorem requires building up significant machinery in abstract algebra and so we will not include it here.

Theorem 7.15. *Let $p(x)$ be an irreducible polynomial in $S[x]$. If $p(x)$ divides $a(x)b(x)$ for $a(x), b(x) \in S[x]$, then either $p(x)$ divides $a(x)$ or $p(x)$ divides $b(x)$.*

With this property involving the factorization of polynomials that we use on a regular basis we can also prove a property similar to that of the Fundamental Theorem of Arithmetic for integers.

Theorem 7.16. *Let S be a field. Then every polynomial of positive degree in $S[x]$ can be represented as a product*

$$cp_1(x)p_2(x) \cdots p_k(x)$$

where $c \in S$ and the $p_i(x)$ are monic irreducible polynomials in $S[x]$. This representation is unique up to reordering the irreducible polynomials.

The proof of this theorem is very similar to the method used to prove the Fundamental Theorem of Arithmetic, and so we will not include the details here. However, this theorem demonstrates the importance for understanding the irreducible polynomials in $\mathbb{R}[x]$ and $\mathbb{C}[x]$ in the process of understanding the properties of polynomials with real and complex coefficients.

The main result in this area is the Fundamental Theorem of Algebra.

Theorem 7.17 (The Fundamental Theorem of Algebra). *The irreducible monic polynomials in $\mathbb{C}[x]$ are of the form $x - c$.*

There are many different methods to prove this result, but the primary method uses the machinery of complex calculus.

Related Content Standards

- (HSN.CN.9) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

7.3.4 Exercises

1. Find a prime factorization of the following polynomials in $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, $\mathbb{R}[x]$, and $\mathbb{C}[x]$.

- $4x^3 - 2x^2 + 2x - 1$
- $x^3 - 1$
- $6x^2 - 13x - 5$
- $x^5 - 2x^4 - 2x^3 + 4x^2 - 3x + 6$

2. Explain how the domain of the variable x affects the number of solutions to the equation

$$(x^4 - 1)(4x^4 - 9x^2 + 2) = 0.$$

3. Find the unique **monic** prime factorization of the polynomials in $\mathbb{Q}[x]$, $\mathbb{R}[x]$, and $\mathbb{C}[x]$.

- $x^4 + 4x^2 + 4$
- $7x^4 - 28$
- $2x^4 - 5x^2 - 3$
- $9x^4 + 440x^2 - 49$
- $x^{106} - x^{100}$
- $4x^4 + 12x^3 - 86x^2 - 92x - 48$
- $x^4 - x^2 - 2$
- $x^2 - \pi$
- $2x^4 - x^3 - 19x^2 + 2x + 30$
- j. $\$a x^2 + bx + c \$$ based on the various possible values of a , b , and c with $a, b, c \in \mathbb{Z}$.

4. Prove Theorem 7.13 using the proof of Theorem 7.2 as an outline.
5. Prove that if f is a Euclidean function on an integral domain R , then $f(a) \leq f(b)$ for all non-zero a and b in R .
6. Find the value c so that $(x - 3)$ is a factor of the polynomial $p(x)$.

$$p(x) = cx^3 - 15x - 68$$

7. Find the monic greatest common divisor of the polynomials $a(x) = 4x^5 + 12x^4 + 20x^3 + 16x^2 + 16x + 4$ and $b(x) = 6x^5 + 12x^4 + 18x^3 + 12x^2 + 12x$.

7.4 Rational Expressions

In the same way that one can extend the integers to the field of rational numbers by defining addition and multiplication on pairs of integers, one can extend the polynomial ring by defining addition and multiplication of pairs of polynomials and then defining a set of equivalence classes on this set.

Related Content Standards

- (HSA.APR.7)] Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Let F be a field and let $F[x]$ be the integral domain of polynomials with coefficients from F . We define an equivalence relation on the set of rational expressions of polynomials as $\frac{p(x)}{q(x)}$ is equivalent to $\frac{a(x)}{b(x)}$ if and only if $p(x) \cdot b(x) = a(x) \cdot q(x)$.

Define $F(x)$ to be the set of equivalence classes of quotients of polynomials $\left[\frac{p(x)}{q(x)} \right]$ with $q(x)$ not being the zero polynomial under this equivalence relation. We then proceed with dropping the equivalence class notation to simplify the notation. We define the following operations on $F(x)$.

$$\frac{p(x)}{q(x)} + \frac{f(x)}{g(x)} = \frac{p(x) \cdot g(x) + q(x) \cdot f(x)}{q(x) \cdot g(x)} \quad \text{and} \quad \frac{p(x)}{q(x)} \cdot \frac{f(x)}{g(x)} = \frac{p(x) \cdot f(x)}{q(x) \cdot g(x)}$$

We call the set $F(x)$ the set of rational expressions with coefficients from F and we label the corresponding set with operations as $\langle F(x), +, \cdot \rangle$, but usually refer to it as just $F(x)$.

We see that the operations are well-defined because the polynomials are well-defined and integral domain and therefore closed under addition and multiplication and do not have any zero divisors. Thus if neither $q(x)$ nor $g(x)$ are the zero polynomial, then neither is their product. This leads us to our next result that is analogous to the rational numbers.

Theorem 7.18. *Let F be a field. Then $\langle F(x), +, \cdot \rangle$ is a field.*

Proof. We leave the proof as an exercise to help the reader become more familiar with this field. \square

This implies that the rational expressions with real valued coefficients, $\mathbb{R}(x)$ forms a field very similar to the rational numbers. We will explore this field in greater depth in Section 8.7.

7.4.1 Exercises

1. Prove that $\mathbb{R}(x)$ is a field.

Chapter 8

Real Valued Functions

This chapter builds upon the definitions of functions studied in Chapter 5, using the algebra material from Chapters 6 and 7, to study the interplay between the verbal, algebraic, tabular, and graphical representations of functions.

Related Content Standards

- (8.F.2) Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*
- (HSF.IF.9) Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

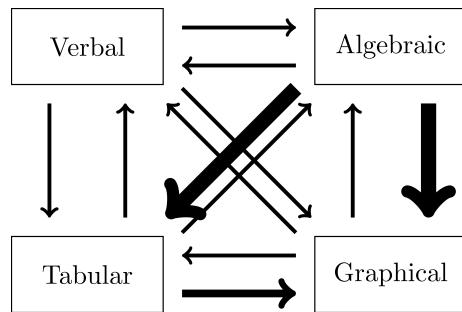


Figure 8.1: Emphases of translations between function representations

This process of translating between the different representations of a function is a key aspect of problem solving because different problems are more easily solved using different representations. For instance, if a set of data is given in a tabular format one would often use regression techniques to generate a graphical and algebraic representations of a function that models that data in order to interpolate or extrapolate information about the data within the context of the verbal situation given. Each of these different representations work together to help generate and interpret possible solutions to the problems determined. As such, it is important for students to be able to move from one representation to another.

In the secondary curriculum, students frequently move from algebraic expressions to graphical representations, either by hand or using technology, as stated in the Common Core Standard HSF.IF.7. However, students rarely translate from a graphical representation to an algebraic representation. We represent the different emphases between the translations by different sized arrows in Figure 8.1.

Related Content Standards

- (HSF.IF.7) Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
 - a. Graph linear and quadratic functions and show intercepts, maxima, and minima.
 - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
 - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
 - d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.
 - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

8.1 Function Properties

Throughout this chapter we will work to better understand the common functions in the secondary curriculum by studying certain properties of the functions.

Related Content Standards

- (8.F.5) Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.
- (HSF.IF.4) For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

8.1.1 Domain and Range

In this chapter we will deal with functions whose domain is a subset of the real numbers, \mathbb{R} . That domain may be inherited from the situation the function is modeling. For instance, a function that gives the cost of a purchase that is based on the number of items purchased would have a domain of the natural numbers. Where a function that models the number of feet above the ground that an object is located t seconds after being dropped would have a domain of the non-negative real numbers.

Frequently, a function is only defined by an algebraic representation. In this case, the implied domain for the function is the largest subset of \mathbb{R} for which the function is defined. For example, the function $f(x) = \sqrt{x}$ is only defined for non-negative real numbers and so that would be the implied domain for the function.

Related Content Standards

- (HSF.IF.5) Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.

Once the domain for a function is determined, it is important to understand all of the possible elements in the co-domain that correspond to elements in the domain with the function, the range. The process for determining the range often involves many different techniques. If the original function is an injection, then one may be able to use the algebraic representation of the inverse function to determine the domain of the inverse function, which is the range of the original function. One may also use other properties of the function, such as its extreme points and its intervals of the domain on which the function is increasing or decreasing.

When graphing a function, we usually have the horizontal axis represent the domain of the function and the vertical axis represent the co-domain.

8.1.2 Maxima (minima) or relative maxima (relative minima)

It is often valuable to find the largest or smallest values that can be obtained by the function. These appear most obviously in optimization problems in calculus, where one finds the greatest (or least) values of f , or finds the greatest (or least) values on a given interval. These extreme values often depend upon the subset of the domain considered. When looking for a maximum or minimum value we find that for differentiable functions that these extreme values occur when the derivative is zero or undefined or at the endpoints of intervals in the domain considered. Other methods for finding these extreme values include using graphing calculators to graph the functions and have the technology determine these extreme values.

8.1.3 Increasing or decreasing

It is also valuable to determine those intervals on which a function, f , is increasing and those intervals on which f is decreasing. A function f is **increasing** on an interval (a, b) if for every $a < x \leq y < b$, $f(x) \leq f(y)$. The function f is **decreasing** on (a, b) if $a < x \leq y < b$ implies that $f(x) \geq f(y)$. We call the function strictly increasing or decreasing if the \geq and \leq symbols are replaced by strict inequality. If a function is either increasing or decreasing on its entire domain, then we call the function **monotonic**. One can also see that a strictly monotonic function is an injection and so we can find an inverse function when restricting the co-domain to the range.

8.1.4 Intercepts

When a function is providing a relationship between two variables, it is often useful to find the points at which either of the variables is zero. This process helps with understanding the graph of a function and often provides insight into the interpretation of the situation that a function may be modeling. While we often say that we are looking for the x -intercept and y -intercept of a function, it is important to have the flexibility that the variables have different names, especially in modeling situations. So we will try to use the convention of horizontal and vertical intercepts.

8.1.5 End behavior, singularities, and asymptotes

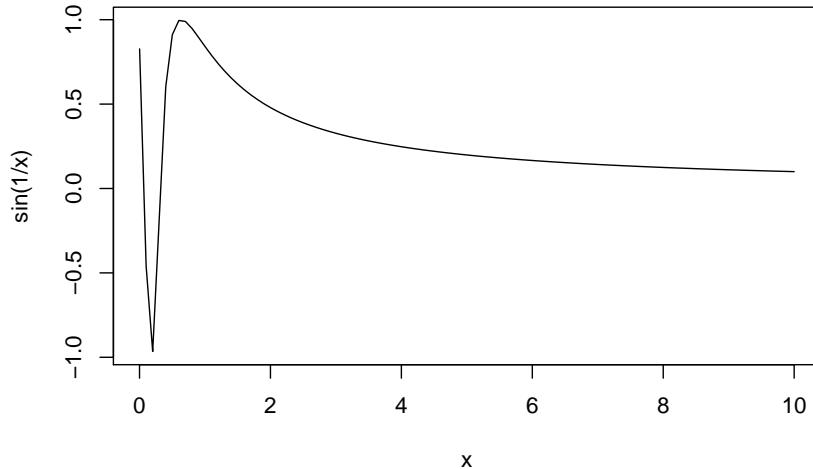
It is often helpful to understand what happens to the value of functions near the endpoints of a domain when the domain includes open intervals. For functions whose domain includes intervals of the form $(-\infty, c)$, (c, ∞) , and $(-\infty, \infty)$, this is understanding what happens to $f(x)$ as $x \rightarrow \infty$ and/or as $x \rightarrow -\infty$. The study of this often focuses on horizontal asymptotes, but we will expand that perspective to a comparison of the behavior of functions ‘near’ $\pm\infty$ to the behavior of known functions. For instance, does the function grow at a similar rate to a certain polynomial function, a certain exponential function, or possibly a certain logarithmic function?

If the domain of the function includes a bounded open interval, it is important that we understand what happens to the output of the function as the input approaches the endpoints of the open interval. For instance for $f(x) = \frac{1}{x}$, as $x \rightarrow 0^+$ (x approaches zero from the right) $f(x) \rightarrow +\infty$. While as $x \rightarrow 0^-$, $f(x) \rightarrow -\infty$.

The function $y = f(x) = \sin(\frac{1}{x})$ defined on the interval $(0, \infty)$ has a very unusual behavior near $x = 0$. The function oscillates between -1 and 1 an infinite number of times as x approaches 0 .

8.1.6 Special Properties

In addition to the properties listed above, it is important to look for other properties of the function.

Figure 8.2: Graph of $y = \sin(1/x)$ **Is f a composition of known functions?**

Consider $f(x) = \sqrt{-(x-1)(x+2)}$. This function is a composition of two functions that we already know and we can use knowledge about those functions to better understand this function. We will explore this process in Section 8.9.

Does the graph of f have a particular known shape?

When modeling a set of bivariate data, it is helpful to know if the graph of the data resembles the graph of a known function, i.e. linear, exponential, or quadratic. We can then use that information to determine the best options for choosing a function to model the situation.

Does the graph have symmetry?

If the graph is symmetric about a vertical line, then it is a transformation of an even function, and if it is symmetric about a point, it is a transformation of an odd function.

Definition 8.1. A function $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is **even** if $f(x) = f(-x)$ for all $x \in A$. The function is **odd** if $f(x) = -f(-x)$ for all $x \in A$.

Is f periodic?

It is also helpful to know if the function repeats itself over the domain. For instance, if someone wants to model the length of daylight for the day then it would be helpful to know that this is a periodic relationship that repeats itself every year.

Does f have an inverse function?

We often find it useful to ‘undo’ functions by using their inverse functions. When considering real-valued functions, this inverse function represents the same relationship between two variables, but viewed from a different perspective. Understanding this perspective also helps us to better understand the original function and to expand our ability to use such a function.

As we saw in Section 5.3, if a function is an injection it can be considered a bijection from its domain onto its range. For certain functions we may need to restrict the domain of the function in order to find an inverse.

Related Content Standards

- (HSF.BF.4) Find inverse functions.
 - a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.
 - b. Verify by composition that one function is the inverse of another.
 - c. Read values of an inverse function from a graph or a table, given that the function has an inverse.
 - d. Produce an invertible function from a non-invertible function by restricting the domain.

8.1.7 Exercises

1. For each of the following functions defined by their algebraic, verbal, tabular and/or graphical representation, describe the function according to the various properties described in this section. Furthermore, describe how these properties relate to the situation being modeled.
 - a. A factory is built to produce chairs and for each day of operation it has a fixed cost of \$2,000 per day for rent, salaries, and other expenses, and a cost of \$80 per chair built for materials. We let $C(n)$ be the function that represents the cost of building n chairs in a day.
 - b. A person is walking down a street at a constant rate. Let $D(t)$ be the distance from a certain lamppost t seconds from when the person starts walking down the street.
 - c. Let f be the days-in-the-month function in a non-leap year defined by the table

Month number, m	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D	31	28	31	30	31	30	31	31	30	31	30	31

- d. Let $G(L)$ be the grade points for a letter grade L given by

Letter grade, L	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Grade Points, G	4.33	4	3.67	3.33	3	2.67	2.33	2	1.67	1.33	1	.67	0

- e. The closing values of the S & P 500 Index each weekday for January 2000-December 2019.
 - a. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \sqrt[3]{x}$, whose graph is given by

8.2 Transformations of Functions

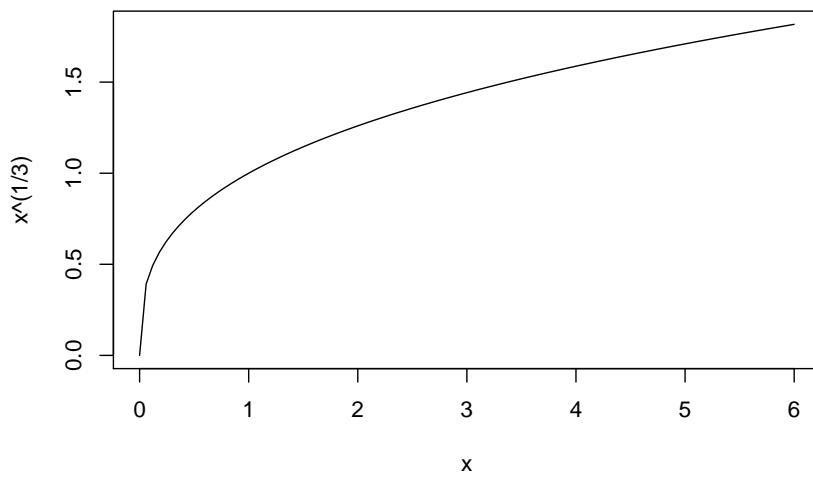
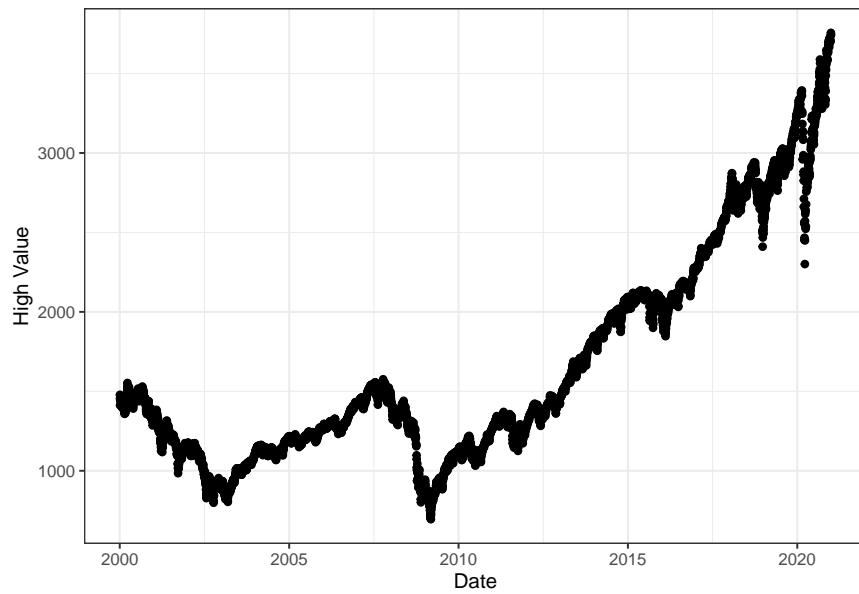
When modeling using a function it is usually helpful to transform the function by a combination translations, dilations, and reflections in order to have the function fit the situation being modeled.

As an example we will look at a model of the number of hours of daylight in London, England for each day during 2020. We notice that the hours of daylight are periodic and that they resemble a trigonometric function. This function has a period of 365.25 days and we calculate the average length of daylight to be 12 hours and 14 minutes, which first occurs on the 81st day of the year. So we can model the situation with the function

$$f(x) = \left(\frac{\max \text{ hours of daylight} - \min \text{ hours of daylight}}{2} \right) \cdot \sin \left(\frac{2\pi}{365.25} (x - 81) \right) + 12 : 14$$

As we can see in below, this function models the situation fairly well.

We will study the family of functions generated from a parent function using these transformations throughout the rest of the chapter, using the function f given by the graph in Figure 8.4 as an example.



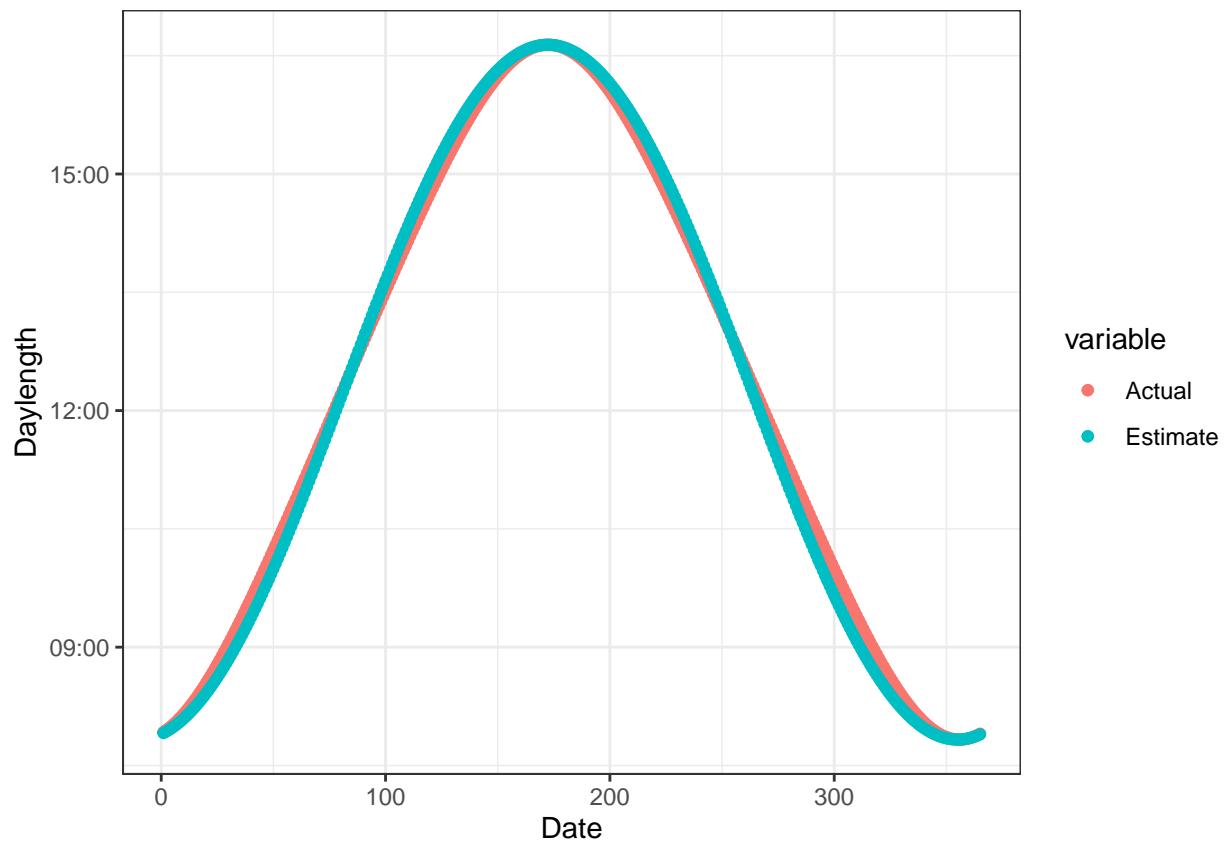


Figure 8.3: Hours of Daylight for London, 2020

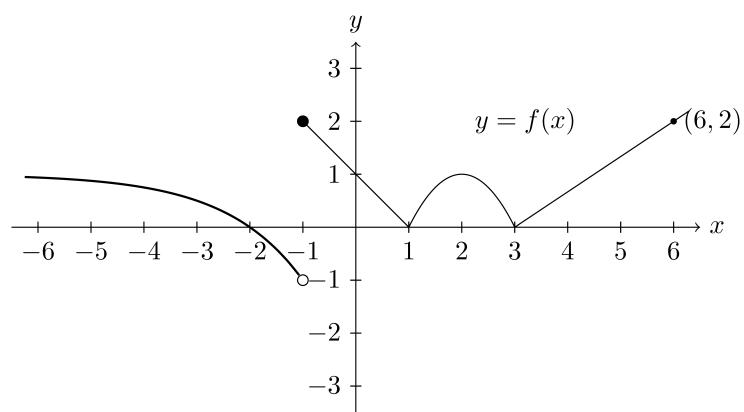


Figure 8.4: Transformations: Base Function

8.2.1 Horizontal Translations

Given a function f and a constant h , a new function $g(x) = f(x - h)$ is a **horizontal translation** of the function f . If h is positive, then $y = g(x)$ is a translation of $y = f(x)$ by h units to the right. If h is negative, it is a translation to the left.

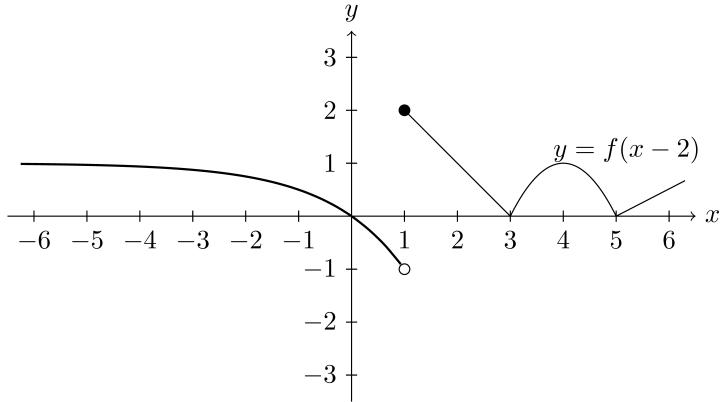


Figure 8.5: Transformations: Horizontal Translation

Related Content Standards

- (HSF.BF.3) Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

8.2.2 Vertical Translations

Given a function f and a constant k , a new function $g(x) = f(x) + k$ is a **vertical translation** of the function f . If k is positive, then $y = g(x)$ is a translation of $y = f(x)$ by k units up. If k is negative, it is a translation down.

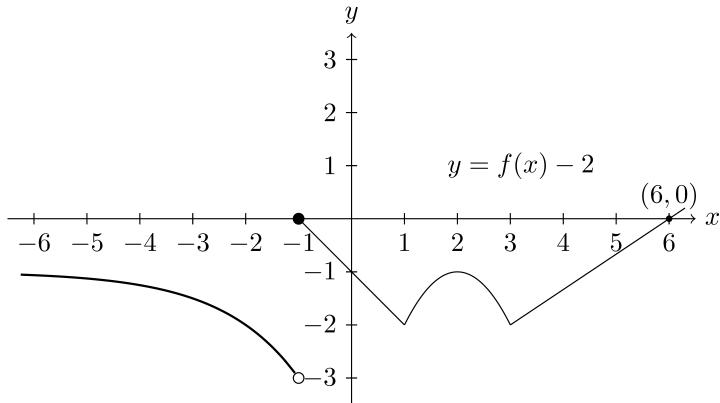


Figure 8.6: Transformations: Vertical Translation

8.2.3 Horizontal Dilations

In addition to the translations, we can dilate the function away from and closer to the vertical axis. Given a function f and a constant $b > 0$, a new function $g(x) = f(bx)$ is a **horizontal dilation** by a factor of b . If $b > 1$, the new function is a compression towards the vertical axis by a factor of b . If $0 < b < 1$, the new function is a dilation away from the vertical axis by a factor of $\frac{1}{b}$.

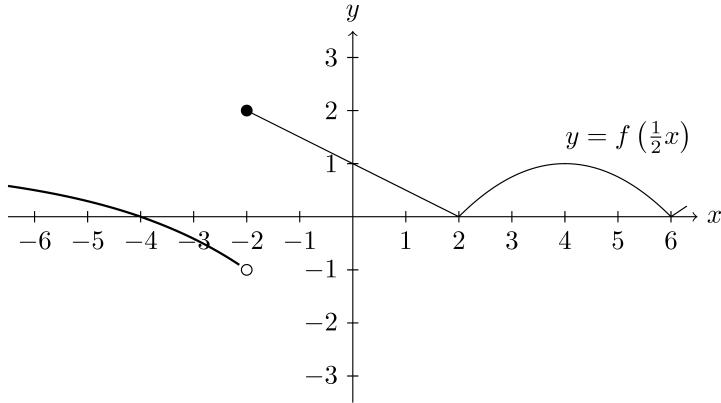


Figure 8.7: Tranformations: Horizontal Dilation

8.2.4 Vertical Dilations

Similarly to the horizontal dilations, multiplying the output of a function by a positive constant dilates the function vertically. Given a function f and a constant $a > 0$, the function $g(x) = af(x)$ is a dilation towards the horizontal axis if $0 < a < 1$ and away from the horizontal axis if $a > 1$.

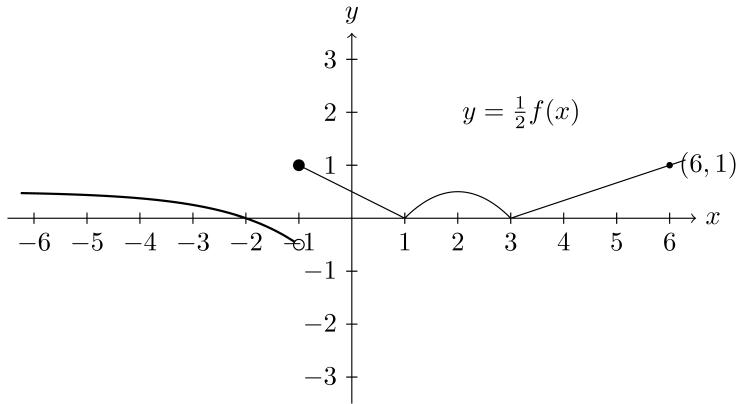


Figure 8.8: Tranformations: Vertical Dilation

8.2.5 Reflections

We are also able to reflect the graph of a function over the horizontal axis by changing the sign of the output of the function. So $g(x) = -f(x)$ is a **vertical reflection** of f about the horizontal axis.

By changing the sign of the variable in the domain, we can also reflect a function about the vertical axis so that the graph of $g(x) = f(-x)$ is a **horizontal reflection** of the graph of f about the vertical axis.

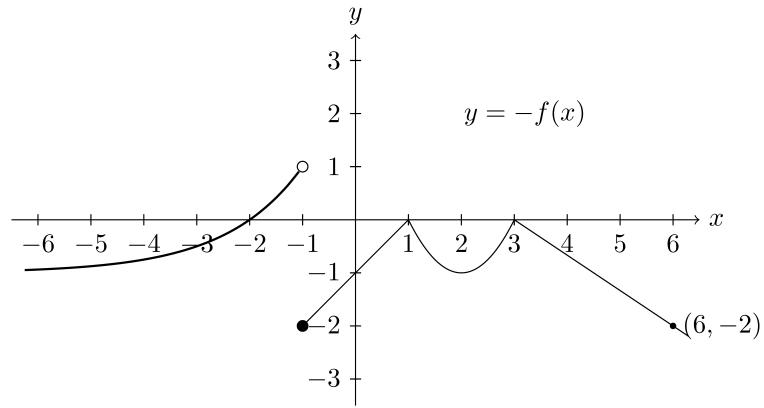


Figure 8.9: Transformations: Vertical Reflection

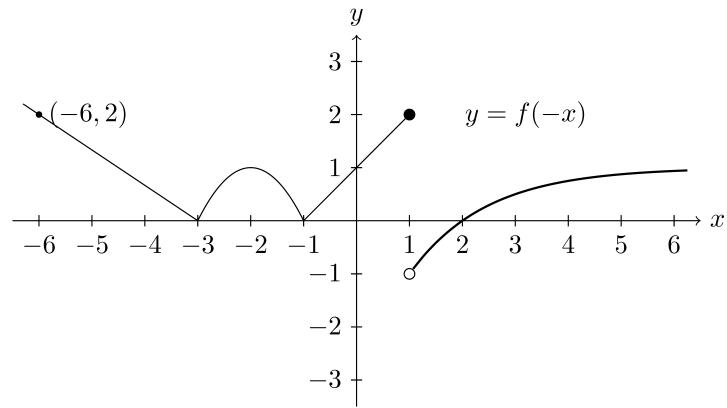


Figure 8.10: Transformations: Horizontal Reflection

8.2.6 Families of Functions

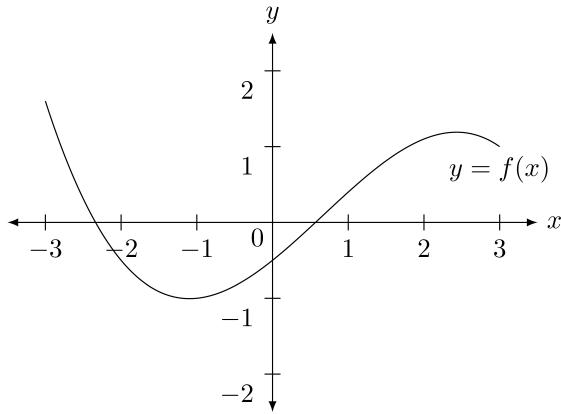
One can combine all of these transformations together in order to generate a family of functions related to the original function f ,

$$g(x) = a \cdot f(b(x - h)) + k.$$

In each of the next few sections we will study the family of functions related to the parent functions of $f(x) = x$, $f(x) = e^x$, $f(x) = \frac{1}{x}$, $f(x) = x^2$, and the basic trigonometric functions, along with each of their inverse functions when those functions exist.

8.2.7 Exercises

1. Let $f : [-3, 3] \rightarrow \mathbb{R}$ be given by the graph



- a. Sketch the graph of $y = f(2x)$.
- b. Sketch the graph of $y = 2f(x)$.
- c. Sketch the graph of $y = f(x + 2)$.
- d. Sketch the graph of $y = f(x) + 2$.
- e. Sketch the graph of $y = -f(-x)$.

8.3 Linear and Exponential Functions

8.3.1 Linear Functions

The first parent function that we will study is the identity function $f(x) = x$. With this function, we see that horizontal and vertical dilations are connected in that $f(ax) = af(x) = ax$, and that these functions describe proportional relationships between two variables. These proportional relationships are part of the core of the middle school mathematics curriculum, along with much of the physical science curriculum.

Related Content Standards

- (6.RP.1) Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.
- (6.RP.3) Use ratio and rate reasoning to solve real-world and mathematical problems.
- (7.RP.2) Recognize and represent proportional relationships between quantities.
- (8.EE.5) Graph proportional relationships, interpreting the unit rate as the slope of the graph.

While a direct proportional relationship fits some situations very well, such as Hooke's law that the force needed to compress a spring by a distance is directly proportional to that distance, many others can be modeled by a proportional relationship near certain values. In fact, one of the foundations of calculus is that most functions that we use in modeling can locally be approximated by a proportional relationship. However,

to model these proportional relationships we often need to translate the function to a point on the function. Such a translation gives us that for any ‘differentiable’ function, f , near a point $x = a$ we can approximate f by

$$f(x) \approx f'(a)(x - a) + f(a).$$

As such, we will study the family of linear functions of the form

$$f(x) = m(x - h) + k,$$

often described as the point-slope form of a line with slope m through the point (h, k) . These functions can also be written in a slope intercept form, $f(x) = mx + b$ where $b = k - mh$.

If $f(x) = mx + b$, then we can see that the domain and range of the function is all of the real numbers. If $m > 0$ the function is strictly increasing, while if $m < 0$ it is strictly decreasing. As such, the function has no extrema.

We also know that since linear functions are monotonic, they are invertible. If $y = f(x) = mx + b$, the inverse of f is $x = f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}$.

8.3.2 Exponential Functions

In Section 6.2 we found a strong relationship between linear functions and exponential functions with the additive relationship of linear functions corresponding with the multiplicative relationship of exponential functions. In the case of exponential functions we will have our parent function be the function $f(x) = e^x$, making the family of exponential functions to be written in the form

$$af(b(x - h)) + k = ae^{b(x-h)} + k = (ae^{-bh}) \cdot e^{bx} + k$$

which means that we can generate all possibilities with functions of the form $g(x) = ae^{bx} + k$

When introducing exponential functions in middle school, the base is usually either 2 or 10 since students have not yet been introduced to e , but we can see that each different base is actually a horizontal dilation of the function with base e .

We saw in Section 4.5 that if $b > 0$ and $x < y$, then $e^b > 1$ and $e^{bx} < e^{by}$, making e^{bx} a strictly increasing function. Similarly if $b < 0$, e^{bx} is a strictly decreasing function. So we have the following possible shapes for functions of the form $g(x) = ae^{bx}$.

We can see that with each of these cases that the function is monotonic and the range is either $(0, \infty)$ or $(-\infty, 0)$, depending on the sign of a . If we perform a vertical translation to achieve the entire family of exponential functions then we see that we have the following properties for $f(x) = ae^{bx} + k$:

- The domain is \mathbb{R} and the range is (k, ∞) if $a > 0$ and $(-\infty, k)$ if $a < 0$.
- The function is strictly monotonic.
- If $b > 0$, then $e^b > 1$, $f(x) \rightarrow k$ as $x \rightarrow -\infty$, and $f(x) \rightarrow \pm\infty$ as $x \rightarrow \infty$ (depending on the sign of a). These functions model exponential growth.
- If $b < 0$, then $0 < e^b < 1$, $f(x) \rightarrow k$ as $x \rightarrow \infty$, and $f(x) \rightarrow \pm\infty$ as $x \rightarrow -\infty$ (depending on the sign of a). These functions model exponential decay.

Related Content Standards

- (HSF.IF.8) Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
 - b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)12^t$, $y = \frac{(1.2)^t}{10}$, and classify them as representing exponential growth or decay.

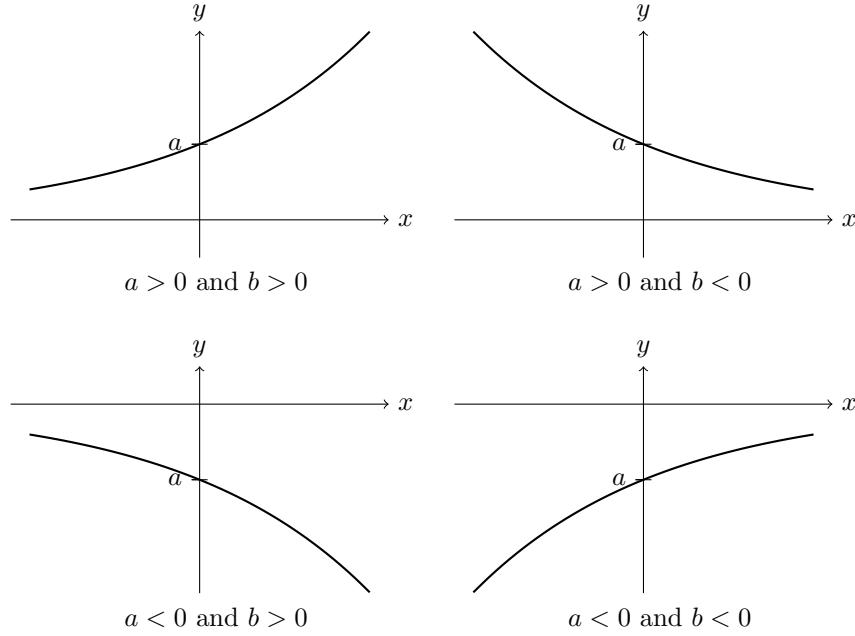


Figure 8.11: Graphs of Exponential Functions

8.3.3 Logarithmic Functions

Since the exponential functions are monotonic, they are bijections onto their range. If we let $f(x) = e^x$, then we know that $f : \mathbb{R} \rightarrow (0, \infty)$ is a bijection. From Chapter 5 we know that there is a unique function $g : (0, \infty) \rightarrow \mathbb{R}$ that is also a bijection with $g(f(x)) = x$ for all $x \in \mathbb{R}$ and $f(g(x)) = x$ for all $x \in (0, \infty)$. We will call this function the natural logarithm and denote it by \ln .

We can then use properties of the exponential function to determine properties of the natural logarithm.

If $\alpha, \beta > 0$, then there exist a and b so that $\alpha = e^a$ and $\beta = e^b$, and

$$\ln(\alpha\beta) = \ln(e^a \cdot e^b) = \ln(e^{a+b}) = a + b = \ln(\alpha) + \ln(\beta).$$

We also have that

$$\ln\left(\frac{\alpha}{\beta}\right) = \ln\left(\frac{e^a}{e^b}\right) = \ln(e^{a-b}) = a - b = \ln(\alpha) - \ln(\beta).$$

If $\alpha > 0$ and $x \in \mathbb{R}$, then there exists an $a \in \mathbb{R}$ such that $\alpha = e^a$. Then

$$\ln(\alpha^x) = \ln((e^a)^x) = \ln(e^{ax}) = ax = x \ln(\alpha).$$

Related Content Standards

- (HSF.BF.5) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Since a^x ($a > 0$) has the same properties as e^x it also has an inverse function that we will call logarithm base a and denote it by

$$\log_a : (0, \infty) \rightarrow \mathbb{R}.$$

We can find the relationships between the various bases. If $y = \log_a(x)$, then $x = a^y$. We can evaluate both sides of the equation in the natural log function so that

$$\ln(x) = \ln(a^y) = y \ln(a)$$

and so

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Using these properties together we are able to use logarithm functions to find equivalent equations to equations such as $ab^{ct} = d$. By inputting both sides of the equation into the \ln function, we have that

$$\ln(a) + (ct)\ln(b) = \ln(d).$$

We can then find the equivalent equation of

$$t = \frac{\ln(d) - \ln(a)}{c\ln(b)}.$$

Related Content Standards

- (HSF.LE.4) For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a , c , and d are numbers and the base b is 2, 10, or e ; evaluate the logarithm using technology.

8.3.4 Relationships between Linear and Exponential

Since linear functions have a constant additive rate of change and exponential functions have a constant multiplicative rate of change, it is important to recognize the types of situations that are more likely to be modeled with a linear function and which are more likely modeled with an exponential function.

Related Content Standards

- (HSF.LE.1) Distinguish between situations that can be modeled with linear functions and with exponential functions.
 - Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.
 - Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
 - Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

Some of the factors that point towards a linear model are adding a fixed amount over fixed intervals in the domain, such as adding \$200 to a savings account every month. Factors that direct the model to be exponential is that the amount of growth would depend upon the current value, such as interest rates or cell division in biology.

When given a set of data, the function used to model the data sometimes depend upon the question being asked about the data. Consider the following population data for the State of Alabama¹.

Year	1800	1850	1900	1950	2000
Population	1,250	771,623	1,828,697	3,061,743	4,447,100

This data can be modeled using either a linear regression or an exponential regression. However, if you include the population in 1800 in the exponential regression, it estimates the population in 2050 to be nearly 100 million people (a highly unlikely number). If one only uses the data points from 1900, 1950, and 2000, the exponential model predicts a population of around 7 million in 2050. A linear model with all of the data points estimates the population in 2050 to be around 5.3 million people (a much more likely scenario).

If, however, you are wanting to estimate the population in years prior to 1800, the linear model becomes useless as it predicts a negative population. So for this type of question, an exponential function would be more useful.

¹https://en.wikipedia.org/wiki/Demographics_of_Alabama, retrieved 6/25/2020

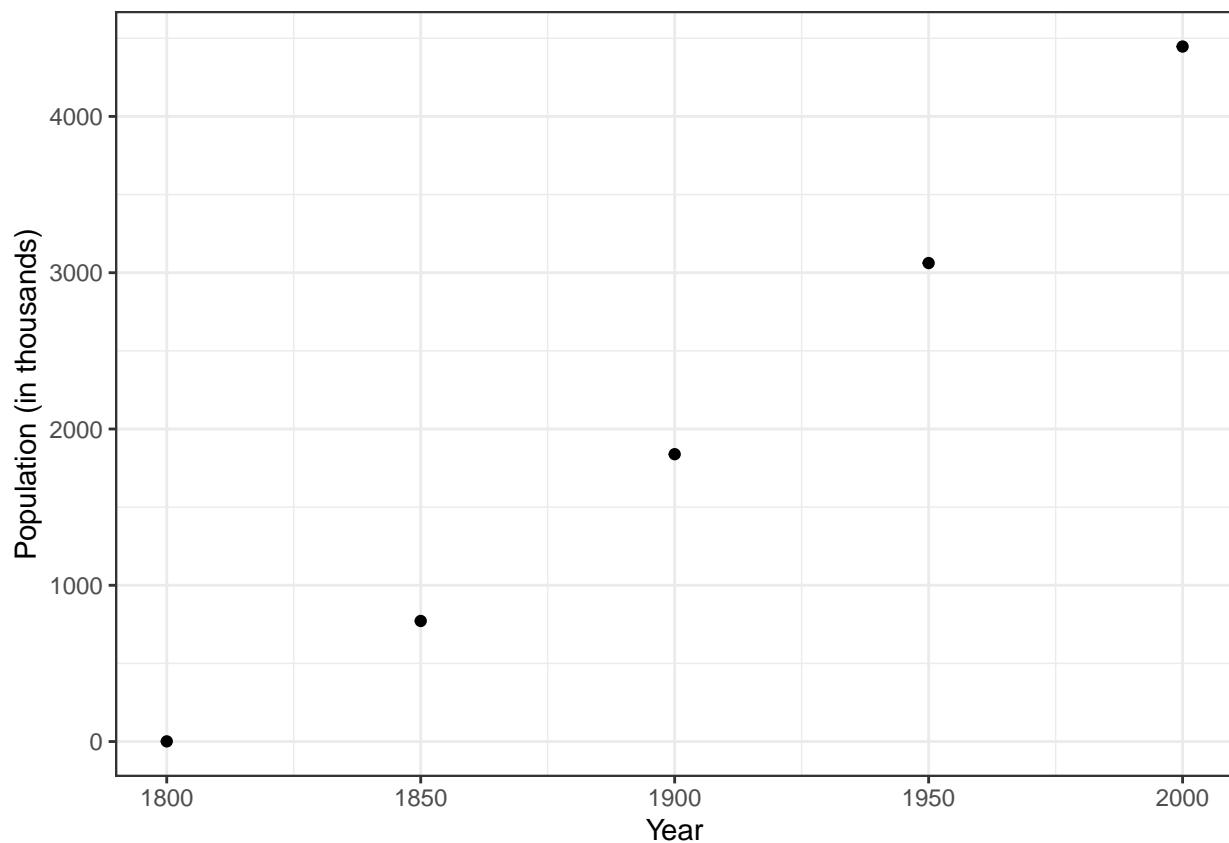


Figure 8.12: Population of Alabama

Table 8.1: Race Times for Olympic Men's 100m Dash

Year	Time (s)	Runner	Country
1948	10.30	Harrison Dillard	USA
1952	10.40	Lindy Remingino	USA
1956	10.50	Bobby Morrow	USA
1960	10.20	Armin Hary	Germany
1964	10.00	Bob Hayes	USA
1968	9.95	Jim Hines	USA
1972	10.14	Valery Borzov	USSR
1976	10.06	Hasely Crawford	Trinidad
1980	10.25	Allan Wells	Great Britain
1984	9.99	Carl Lewis	USA
1988	9.92	Carl Lewis	USA
1992	9.96	Linford Christie	Great Britain
1996	9.84	Donovan Bailey	Canada
2000	9.87	Maurice Greene	USA
2004	9.85	Justin Gatlin	USA
2008	9.69	Usain Bolt	Jamaica
2012	9.63	Usain Bolt	Jamaica
2016	9.81	Usain Bolt	Jamaica

8.3.5 Exercises

1. The salary scales in three school districts are as follows, for a teacher with a master's degree:
 - a. District P : \$30,000 plus \$1500 for each year of experience
 - b. District Q : \$30,000 plus \$1750 for each year of experience
 - c. District R : \$28,000 plus \$1750 for each year of experience
 - a. Give a formula for the salary in each district for a teacher with n years of experience at the start of the school year.
 - b. Use your formulas to indicate the number of years experience teachers in Districts P and R when they earn the same salary.
 - c. Use your formulas to indicate the number of years experience teachers in Districts P and Q when they earn the same salary.
 - d. Use your formulas to indicate the number of years experience teachers in Districts Q and R when they earn the same salary.
 - e. If in District T , teachers earn a salary S_T dollars plus E_T dollars for each year of experience, and in District U , teachers earn a salary of S_U dollars plus E_U dollars for each year of experience, $S_T > S_U$, and $E_U > E_T$, how many years will it take District U teachers to catch up to District T ?
 - f. For each of the previous scenarios, at what value of n will teachers in the two district have earned the same amount of total money during the previous n years?
2. The following are winning times for the Men's 100m dash at the Olympics.
 Find a mathematical function that model this behavior and use this function to predict the winning time in 2060. Give reasoning for your work, including the assumptions made for your models.
3. The number of bacteria in a certain population increases according to a continuous exponential growth model, with a growth rate parameter of 3.25% per hour. How many hours does it take for the size of the sample to double?
4. LaVonda is choosing between 2 jobs. For Job A, she would earn \$27,000 the first year and each year after that she would get a raise of \$3,000. For Job B, she would earn \$30,000 the first year and each

year after that she would get a raise of 4% of the previous year's salary.

- a. What is the first year during which LaVonda's salary for Job A would exceed that of Job B? Show your work, and explain the approach you used to find your answer.
- b. Which year would the total amount earned since starting Job A first exceed the total amount earned since starting Job B? Show your work, and explain the approach you used to find your answer.
- c. Using sequence and series formulas, what would LaVonda's yearly salary and total amount of money earned be after 20 years at Job A? What would her yearly salary and total amount of money earned be after 20 years at Job B?
5. A business owner spent \$500 on start-up fees to produce and sell candles. Each candle costs an additional \$3.00 to produce. What is the minimum number of candles that the owner must produce for the average cost per candle to be less than \$3.75?

8.4 Linear Fractional Transformations

The function $f(x) = \frac{1}{x}$ is another function that appears in physical applications such as Boyle's law that the pressure of a given mass of an ideal gas is inversely proportional to its volume at a fixed temperature. In this section we will explore the family of functions that derive from this function. If $f(x) = \frac{1}{x}$, then we see that for $\alpha \neq 0$ and $\beta \neq 0$

$$g(x) = \alpha \cdot f(\beta(x - h)) + k = \frac{\alpha}{\beta} \cdot \frac{1}{x - h} + k$$

which can be rewritten as

$$g(x) = \frac{\alpha + k(\beta x - \beta h)}{\beta x - \beta h} = \frac{(k\beta)x + (\alpha - \beta hk)}{\beta x - (\beta h)} = \frac{kx + \left(\frac{\alpha}{\beta} - hk\right)}{x - h}.$$

On the other hand, any function of the form

$$h(x) = \frac{ax + b}{cx + d}$$

can be rewritten in the form

$$h(x) = \frac{a}{c} - \left(\frac{ad - bc}{c^2}\right) \frac{1}{x + \frac{d}{c}}$$

and so we see that it is a transformation of the inverse proportional function if, and only if, $ad - bc \neq 0$ and $c \neq 0$.

If we allow $c = 0$, then the functions are transformations of both the linear and inversely proportional parent functions.

Therefore, the family of functions whose parent function is either $f(x) = \frac{1}{x}$ or $f(x) = x$ we call linear fractional transformations (also called a Möbius transformations) as a real-valued function of the form

$$f(x) = \frac{ax + b}{cx + d}$$

where a, b, c , and d are real-valued constants with $(ad - bc) \neq 0$.

8.4.1 Domain

If $c \neq 0$, then these functions are not defined at the point $\frac{-d}{c}$ and so the natural domain for the function would be $\mathbb{R} \setminus \{\frac{-d}{c}\}$.

If $c = 0$, then $d \neq 0$ and so we see that the function becomes a linear function of the form

$$f(x) = \frac{a}{d}x + \frac{b}{d},$$

and so the domain of the function in \mathbb{R} .

8.4.2 Range

If $c \neq 0$, the parent function for these functions is inversely proportional function so that

$$f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} - \left(\frac{ad-bc}{c^2} \right) \frac{1}{x+\frac{d}{c}}$$

So for this case that $c \neq 0$, we see that this function is a transformation of the parent function that takes transforms the plane so that the horizontal axis maps to the horizontal line of $y = \frac{a}{c}$, and so the range of the function is $\mathbb{R} \setminus \{\frac{a}{c}\}$.

If $c = 0$ the linear fractional transformation reduces to a linear function whose range is \mathbb{R} .

8.4.3 Increasing or decreasing

Using the graph of $y = \frac{1}{x}$ we can sketch a graph of the transformation of this function.

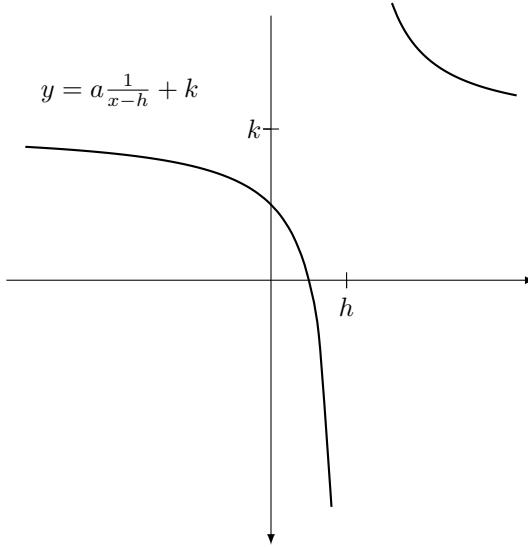


Figure 8.13: Transformations of $f(x) = 1/x$

By using the expression of the linear fractional transformation of

$$f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} - \left(\frac{ad-bc}{c^2} \right) \frac{1}{x+\frac{d}{c}}$$

we see that if $ad - bc > 0$ that f is increasing on $(-\infty, \frac{-d}{c})$ and on $(\frac{-d}{c}, \infty)$. We also see that if $ad - bc < 0$ that the function is decreasing on these intervals.

8.4.4 Intercepts

From the primary algebraic expression of the function one can see that if $a \neq 0$ the horizontal intercept of the function is at $\frac{-b}{a}$, and if $a = 0$ that there is no horizontal intercept.

One can also see that if $d \neq 0$ that the vertical intercept is at $\frac{b}{d}$ with no vertical intercept if $d = 0$.

8.4.5 End behavior and asymptotes

One can see from the graph of $f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} - \left(\frac{ad-bc}{c^2} \right) \frac{1}{x+\frac{d}{c}}$ that

$$\lim_{x \rightarrow \pm\infty} f(x) = \frac{a}{c}$$

and that the behavior near the vertical asymptote of $x = -\frac{d}{c}$ depends upon the sign of $ad - bc$.

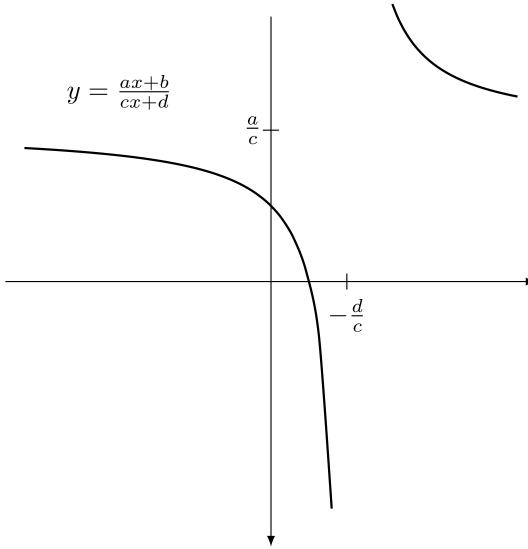


Figure 8.14: General Linear Fractional Transformation

8.4.6 Extensions

By adding a single element to the set of real numbers that we will call ∞ , we can redefine these functions to have both a domain and range of $\mathbb{R} \cup \{\infty\}$. If $c \neq 0$ we define f on this extended domain to be

$$f(x) = \begin{cases} \frac{ax+b}{cx+d}, & x \in \mathbb{R} \setminus \{-\frac{d}{c}\} \\ \frac{a}{c}, & x = \infty \\ \infty, & x = -\frac{d}{c} \end{cases}$$

and if $c = 0$,

$$f(x) = \begin{cases} \frac{a}{d}x + \frac{b}{d}, & x \in \mathbb{R} \\ \infty, & x = \infty \end{cases}.$$

For the ease of notation, we will denote these extended functions by the algebraic representation of the original function and imply the extension to the extended real numbers.

From the results above about range and monotonicity of the function we can see that the linear fractional transformations on these extended domains and co-domains are bijections. That leads us to want to understand the forms of the inverse function and the composition of two of these linear fractional transformations.

In order to understand more about the composition of two linear fractional transformations we will define

$$f(x) = \frac{ax+b}{cx+d} \text{ and } g(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Then we have that

$$\begin{aligned} (g \circ f)(x) &= \frac{\alpha \left(\frac{ax+b}{cx+d} \right) + \beta}{\gamma \left(\frac{ax+b}{cx+d} \right) + \delta} \\ &= \frac{\alpha ax + \alpha b + \beta cx + \beta d}{\gamma ax + \gamma b + \delta cx + \delta d} \\ &= \frac{(\alpha a + \beta c)x + (\alpha b + \beta d)}{(\gamma a + \delta c)x + (\gamma b + \delta d)} \end{aligned}$$

Since the composition of two bijections is again a bijection, this composition of two linear fractional transformations is again a linear fractional transformation.

In order to determine the form of the inverse function of a linear fractional transformation, $f(x) = \frac{ax+b}{cx+d}$ we will use the form of the composition of two linear fractional transformations given above and let

$$(\alpha b + \beta d) = 0, (\gamma a + \delta c) = 0, \text{ and}$$

$$(\alpha a + \beta c) = (\gamma b + \delta d)$$

so that the composition becomes the identity function.

If $c = 0$, then

$$f(x) = \frac{a}{d}x + \frac{b}{d}$$

and the second equation gives us that $\gamma = 0$, and then using the third equation we have that $\alpha a = \delta d$. This then implies that

$$f^{-1}(x) = \frac{d}{a}x + \frac{\beta}{\delta}$$

and since $\beta = \frac{-\alpha b}{d}$ we see that $\frac{\beta}{\delta} = -\frac{b}{a}$ and so

$$f^{-1}(x) = \frac{d}{a}x - \frac{b}{a}.$$

If $c \neq 0$, then $\delta = -\frac{\gamma a}{c}$ and $\beta = \frac{-\alpha ac + \gamma bc + \gamma ad}{c^2}$.

Another method to determine the form of the inverse function is based on the graph of the original function. Since, if $c \neq 0$,

$$f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} - \left(\frac{ad-bc}{c^2} \right) \frac{1}{x+\frac{d}{c}}$$

we see that f is undefined for $x = -\frac{d}{c}$ giving that f^{-1} will not have $-\frac{d}{c}$ in its range. Similarly, since $\frac{a}{c}$ is not in the range of f , it is not in the domain of f^{-1} . So if f^{-1} is a linear fractional transformation it is of the form

$$f^{-1}(x) = \frac{-d}{c} + \omega \frac{1}{x - \frac{a}{c}} = \frac{-dx + \frac{ad}{c} + c\omega}{cx - a} = \frac{-dx + \beta}{cx - a}$$

for some $\omega \in \mathbb{R}$ and where $\beta \in \mathbb{R}$ is dependent upon this ω . We can then use the algebraic representation of the composition of functions to see that $-db + \beta d = 0$ showing that $\beta = b$ and so

$$f^{-1}(x) = \frac{-dx + b}{cx - a}.$$

A third method of determining the inverse function of

$$f(x) = \frac{ax+b}{cx+d}$$

when $c \neq 0$ is to rewrite the equation

$$y = \frac{ax+b}{cx+d}$$

so that we have

$$x = \frac{dy - b}{-cy + a}$$

giving

$$f^{-1}(x) = \frac{dx - b}{-cx + a}.$$

With each of these methods, we see that the inverse function of a linear fractional transformation is another linear fractional transformation.

8.4.7 Exercises

1. Prove that the set of linear fractional transformations, together with the operation of function composition forms a group.
2. Is the group of linear fractional transformations with function composition abelian?
3. Compare and contrast the group of linear functional transformations under function composition and the group of invertible 2×2 matrices under matrix multiplication.

8.5 Quadratic Polynomials

Related Content Standards

- (HSF.IF.8) Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
 - a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Quadratic functions are polynomials whose highest non-zero coefficient is of degree 2 and are often written in the form

$$f(x) = a_0 + a_1x + a_2x^2.$$

In order to avoid the use of indices for students when they are working with these quadratic functions, we write them in the form

$$f(x) = ax^2 + bx + c.$$

8.5.1 Completing the Square

To better understand the graph of these quadratics we will modify this form of the quadratic function using a process called completing the square. In this process we rewrite the polynomial in order to find a monic polynomial without a constant coefficient,

$$f(x) = a \left(x^2 + \frac{b}{a}x \right) + c.$$

We now look at the area model for the polynomial $x^2 + \frac{b}{a}x$ by splitting the $\frac{b}{a}x$ into two pieces.

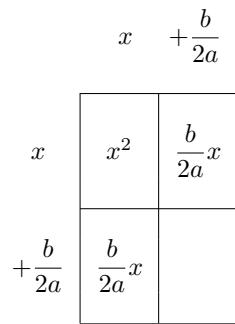


Figure 8.15: Completing the Square: Area Model

Then we see that the area needed to complete the square is $\frac{b^2}{4a^2}$ and so we adjust the polynomial accordingly to

$$f(x) = a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} \right) + c.$$

We are now able to distribute the a through the parenthesis to get

$$f(x) = a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) - \frac{b^2}{4a} + c$$

and we see that the expression inside of the parenthesis is the area of the square. Hence,

$$f(x) = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c.$$

Therefore, we can write all quadratic polynomials in the form

$$f(x) = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}.$$

We can also look at quadratic polynomials from the perspective of transformations of the graph. Beginning with $f(x) = x^2$, we can see that the family of functions generated from this function are of the form

$$h(x) = a(x - h)^2 + k$$

which we will call the vertex form of the quadratic function. Equating this with the prior expression we see that all quadratic polynomials can be written in this way and that the vertex of a polynomial of the form

$$f(x) = ax^2 + bx + c$$

is at the point

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a} \right).$$

We also see from properties of the transformations of the graph that if $a > 0$ the parabola opens up and if $a < 0$ the parabola opens down. Thus the range of the function $f(x) = ax^2 + bx + c$ is

$$\left[-\frac{b^2 - 4ac}{4a}, \infty \right) \text{ if } a > 0 \quad \text{and} \quad \left(-\infty, -\frac{b^2 - 4ac}{4a} \right] \text{ if } a < 0.$$

Related Content Standards

- (HSA.SSE.3) Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
 - b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

8.5.2 Factoring

In addition to the standard form of $f(x) = ax^2 + bx + c$ and the vertex form of $f(x) = a(x - h)^2 + k$, it is useful to write quadratic functions in their factored form of $f(x) = (x - d_1)(x - d_2)$ as this form helps to determine the horizontal intercepts of the graph of the function.

To move from the standard form to the factored form, one uses the vertex form of

$$f(x) = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}$$

and then find the values of x for which the function is equal to zero. These points are the values d such that

$$a \left(d + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a} = 0.$$

Using some algebraic properties we can rewrite this equation as

$$\left(d + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

We then see that there are two values for d that make this equation true.

$$d_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad d_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Therefore, the factored form of the quadratic can be written as

$$f(x) = \left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

Related Content Standards

- (HSA.SSE.3) Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
 - a. Factor a quadratic expression to reveal the zeros of the function it defines.
- (HSN.CN.7) Solve quadratic equations with real coefficients that have complex solutions.

8.5.2.1 Polynomial Identities

While the quadratic formula works every time for factoring a quadratic polynomial, there are a couple polynomial identities that are very useful when factoring specific types of quadratics. The first is when we see a difference of squares.

$$x^2 - b^2 = (x - b)(x + b)$$

This difference of squares can also be used to factor $x^4 - 16$ as $(x^2 - 4)(x^2 + 4)$. We can then rewrite this to find more differences of squares so that

$$\begin{aligned} x^4 - 16 &= (x^2 - 4)(x^2 + 4) = (x^2 - 4)(x^2 - (-4)) \\ &= (x - 2)(x + 2)(x - 2i)(x + 2i). \end{aligned}$$

Another quadratic form that creates a simpler factorization is a perfect square.

$$x^2 + 2ax + a^2 = (x + a)^2$$

This can be recognized when the absolute value of the middle term of a quadratic is twice the product of the square roots of the other two terms. Some examples of uses include $4x^2 + 12x + 9$ or $12y^2 - 8\sqrt{3}y + 4$.

Related Content Standards

- (HSN.CN.8) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*

8.5.3 Inverse Function

We notice that a quadratic polynomial does not have an inverse. However, if we restrict the domain of the quadratic polynomial to an interval on which it is monotonic, we can find an inverse. So if we look at the function

$$f : [0, \infty) \rightarrow [0, \infty) \text{ with } f(x) = x^2$$

we see that f is a bijection. We define the inverse function as

$$g : [0, \infty) \rightarrow [0, \infty) \text{ with } g(x) = \sqrt{x}.$$

We will look at this process of restricting the domain in order to find the inverse function in more detail in the section on trigonometric functions.

Table 8.2:

\$n\$	\$d(n)\$
3	0
4	2
5	5
6	9
7	14
8	20
9	27
10	35
11	44

8.5.4 Exercises

1. Let m and n be the zeros of the function $f(x) = ax^2 + bx + c$.
 - a. Find a formula for $m + n$ in terms of a , b , and c .
 - b. Find a formula for $m^2 + n^2$ in terms of a , b , and c .
 - c. Find a formula for $m^3 + n^3$ in terms of a , b , and c .
2. Given $f(x) = -ax^2 + 2ahx - ah^2 + k$, find the domain and range of $f(x)$ where $a > 0$, $h > 0$, and $k > 0$.
3. Given $(k-1)x^2 + kx + 1 = 0$, where -1 is one solution, what is the other solution?
4. Let $y = f(x)$ be a parabola with vertex $(4, 1)$ that passes through the point $(3, -7)$.
 - a. What are the domain and range of $f(x)$? Explain how you found your answers based on just the vertex and the point given.
 - b. What is the equation of this parabola in vertex form? Show your work algebraically. Explain how you used the points indicated on the graph to determine the equation.
5. Let $d(n)$ stand for the number of diagonals of a polygon of n sides. Here is a table of values of $d(n)$.
 - a. Use the table of values to find a polynomial formula for $d(n)$ in terms of n .
 - b. Give a geometric argument to show that your formula is true for all n .
6. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of the printed material on the poster is fixed at 384 cm^2 , find the dimensions of the poster with the smallest area.

8.6 General Polynomial Functions

Now that we understand the graphs of quadratic polynomials we turn our attention to polynomials of higher degree. For all polynomials, the domain is the set of all real numbers. The range, however, are much harder to define and are much easier to determine using numerical methods in computer software.

8.6.1 Short-Term Behavior

Related Content Standards

- (HSA.APR.3) Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

We know that the graph of a polynomial p intercepts the horizontal axis at every point $x = c$ such that $p(c) = 0$. We also know from the factor theorem that this corresponds to $x - c$ being a factor of $p(x)$. We say

that p has a zero of order m at a point $x = c$ if the prime factorization of $p(x)$ includes $(x - c)^m$ as a factor, or equivalently

$$p(x) = (x - c)^m \cdot q(x)$$

where $q(x)$ is a polynomial for which $x - c$ is not a factor.

Theorem 8.1. *Let p be a polynomial with a zero of order m at $x = c$. Then*

$$p(x) = (x - c)^m \cdot q(x)$$

where $q(x)$ is a polynomial with $q(c) \neq 0$. Furthermore, the **Taylor polynomial** of degree m for p centered at $x = c$ is

$$T_m(x) = q(c) \cdot (x - c)^m.$$

Before we prove this theorem, we will prove the following lemma.

Lemma 8.1 (General Leibniz Rule). *Let f and g be n -times differentiable functions. Then fg is also n -times differentiable and*

$$(fg)^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. Using a proof by induction argument we have the base case that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

which is the standard product rule for derivatives.

Now assume that the statement is true for some natural number j . Then

$$\begin{aligned} (fg)^{(j+1)}(x) &= \frac{d}{dx} \left((fg)^{(j)}(x) \right) \\ &= \frac{d}{dx} \left(\sum_{k=0}^j \binom{j}{k} f^{(j-k)}(x)g^{(k)}(x) \right) \\ &= \left(\sum_{k=0}^j \binom{j}{k} \left(\frac{d}{dx} f^{(j-k)}(x)g^{(k)}(x) \right) \right) \\ &= \left(\sum_{k=0}^j \binom{j}{k} \left(f^{(j-k+1)}(x)g^{(k)}(x) + f^{(j-k)}(x)g^{(k+1)}(x) \right) \right) \\ &= \left(\sum_{k=0}^j \binom{j}{k} f^{(j-k+1)}(x)g^{(k)}(x) \right) + \left(\sum_{k=0}^j \binom{j}{k} f^{(j-k)}(x)g^{(k+1)}(x) \right). \end{aligned}$$

We can then re-index the second summation in order to have the order of the derivatives of f and g to align*

between the two summations and we have that

$$\begin{aligned}(fg)^{(j+1)}(x) &= \left(\sum_{k=0}^j \binom{j}{k} f^{(j-k+1)}(x)g^{(k)}(x) \right) + \left(\sum_{k=1}^{j+1} \binom{j}{k-1} f^{(j+1-k)}(x)g^{(k)}(x) \right) \\ &= \binom{j}{0} f^{(j+1)}(x)g^{(0)}(x) + \left(\sum_{k=0}^j \left(\binom{j}{k} + \binom{j}{k-1} \right) f^{(j+1-k)}(x)g^{(k)}(x) \right) \\ &\quad + \binom{j}{j} f^{(0)}(x)g^{(j+1)}(x)\end{aligned}$$

Since

$$\binom{j}{0} = \binom{j+1}{0} = 1 \quad \text{and} \quad \binom{j}{j} = \binom{j+1}{j+1} = 1$$

and since

$$\begin{aligned}\binom{j}{k} + \binom{j}{k-1} &= \frac{j!}{k!(j-k)!} + \frac{j!}{(k-1)!(j-(k-1))!} \\ &= \frac{j! \cdot (j-k+1)}{k!(j-k+1)!} + \frac{j!(k)}{k!(j-k+1)!} \\ &= \frac{j!(j+1)}{k!((j+1)-k)!} = \frac{(j+1)!}{k!((j+1)-k)!} \\ &= \binom{j+1}{k}\end{aligned}$$

we have that

$$(fg)^{(j+1)}(x) = \sum_{k=0}^{j+1} \binom{j+1}{k} f^{((j+1)-k)}(x)g^{(k)}(x)$$

and so the statement is true for $j+1$.

Therefore, by induction we have that

$$(fg)^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)$$

is true for all n . □

Proof. Proof of Theorem 8.1

The factorization of p into $p(x) = (x - c)^m \cdot q(x)$ is a direct consequence of Theorem 7.16.

Let $f(x) = (x - c)^m$. Then multiple applications of the chain rule gives us that

$$f^{(j)}(x) = \begin{cases} \frac{m!}{(m-j)!} (x - c)^{m-j} & \text{if } 0 \leq j \leq m \\ 0 & \text{if } j > m \end{cases}$$

Therefore, using the General Leibniz Rule we have that for $0 \leq n \leq m$,

$$p^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} \frac{m!}{(m-n+k)!} (x - c)^{m-n+k} q^{(k)}(x)$$

and so

$$p^{(n)}(c) = \begin{cases} 0 & \text{if } 0 \leq n < m \\ q(c) & \text{if } n = m \end{cases}$$

giving us the result. \square

The key component of this theorem is that we can further understand how the graphs of polynomials behave near their zeros. If we let

$$p(x) = x^6 + x^5 - 11x^4 - 13x^3 + 26x^2 + 20x - 24,$$

we can write this in factored form and see that

$$p(x) = (x - 1)^2(x + 2)^3(x - 3)$$

has a zero of order 1 at $x = 3$, a zero of order 2 at $x = 1$, and a zero of order 3 at $x = -2$. From the theorem we have that near $x = 3$,

$$p(x) \approx (3 - 1)^2(3 + 2)^3(x - 3) = 500(x - 3).$$

Similarly, near $x = 1$,

$$p(x) \approx -54(x - 1)^2$$

and near $x = -2$,

$$p(x) \approx -45(x + 2)^3.$$

By graphing these polynomials using computer software we can see that these approximations for the polynomial behavior near the zeros is much more accurate than the standard linear, quadratic, or cubic descriptions.

8.6.1.1 Complex Zeros

If $p(x)$ is a polynomial in $\mathbb{R}[x]$, the factors of this polynomial that correspond to complex zeros come in conjugate pairs. This means that if $(x - (a + bi))^j$ is a factor, then so is $(x - (a + bi))^j = (x - (a - bi))^j$. While there is no direct correspondence between complex zeros and properties of the graph of a polynomial, If there are local minimum and maximum values for a polynomial for which the graph of the polynomial does not cross the horizontal axis, it implies that there are complex zeros of the polynomial.

8.6.2 Long-Term Behavior

When we study the long-term behavior of polynomials we focus on the standard form of the polynomial

$$p(x) = a_0 + a_1x + \cdots + a_nx^n \quad \text{with } a_n \neq 0.$$

When we look at the ratio of $p(x)$ and its leading term a_nx^n we have

$$\frac{p(x)}{a_nx^n} = \frac{a_0}{a_n}\frac{1}{x^n} + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{a_n}\frac{1}{x} + 1.$$

So we see that

$$\lim_{x \rightarrow \pm\infty} \frac{p(x)}{a_nx^n} = 1$$

meaning that these two functions act very similarly as $|x|$ gets large.

This means that the graph of our polynomial

$$p(x) = x^6 + x^5 - 11x^4 - 13x^3 + 26x^2 + 20x - 24$$

will look very similar to the graph of $y = x^6$ when zoomed out to study its long-term behavior.

8.6.3 Polynomial Identities

Just like the difference of squares and perfect square for quadratic polynomials, there are some identities of polynomials that are very useful.

If $p(x) = 1 + x + x^2 + \cdots + x^{n-1}$ is multiplied by $(x - 1)$ then one sees that

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

This is the basis for the series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

along with understanding the roots of unity for complex numbers.

The identity

$$(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2 - y^2)^2 + (2xy)^2$$

can be used to generate Pythagorean triples. For instance, if $x = 2$ and $y = 1$ the identity becomes $5^2 = 3^2 + 4^2$. If $x = 5$ and $y = 4$ we have the triple $41^2 = 9^2 + 40^2$.

Finally, we have the identity

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}, \quad \text{with } \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

which can be found using Pascal's triangle.

Related Content Standards

- (HSA.APR.4) Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*
- (HSA.APR.5) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

8.6.4 Exercises

1. For each of the following polynomials,

- What are the zeros of f ?
 - For what values of x is $f(x) > 0$?
 - For what values of x is $f(x) < 0$?
 - What is $\lim_{x \rightarrow \infty} f(x)$?
 - What is $\lim_{x \rightarrow -\infty} f(x)$?
- $f(x) = (3x + 5)(x - 1)(7x + 3)(x - 5)$
 - $f(x) = (2x - 1)(3x + 5)(x + 1)$
 - $f(x) = (x - 1)(4x + 3)(3x - 8)$
 - $f(x) = (2x + 1)(3x - 6)(x - 6)$

2. For which values of k does the equation

$$x^4 - 4x^2 + x + k = 0$$

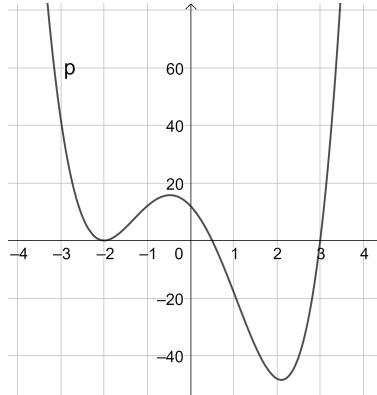
have four distinct real solutions?

3. If $f(x) = 3x^2$, what are all real values of a and b for which the graph of $g(x) = ax^2 + b$ is below the graph of $f(x)$ for all values of x ?
4. Consider the function $f(x) = x^4 - 25x^2 - 36x$, which has one x -intercept at $(-4, 0)$. Find all the other zeros of the function algebraically.
5. Determine the long-term behavior of the following polynomials.

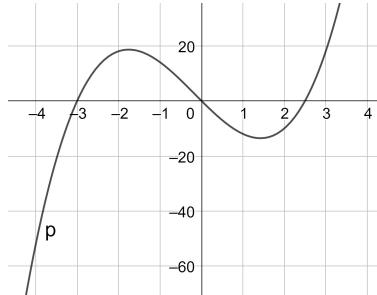
(Find $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.)

- a. $f(x) = 2x^3 - 3x^2 + 4x - 1$
- b. $f(x) = -3x^4 + 2x^2 - 3x + 2$
- c. $f(x) = -5x^5 + 3x^4 + 2x^3 - 10x + 15$
- d. $f(x) = 10x^6 + 7x^3 - 1$
6. For each of the zeros of the following polynomials, find the polynomial approximation of the same degree as the zero near the zero. (If $f(x)$ has a zero of order m at $x = c$, find the Taylor polynomial of degree m centered at $x = c$, $a_c(x - c)^m$.)
- a. $f(x) = (x - 1)(x + 2)^2(2x - 3)$
- b. $f(x) = (2x + 1)(x + 3)^3(x - 2)^2$
7. For each of the following graphs, give a possible algebraic representation for $p(x)$.

a. Graph 1

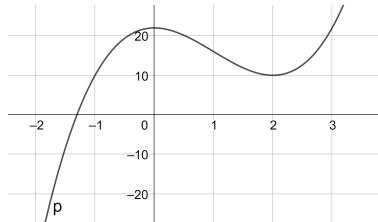


a. Graph 2



a. Graph 3

1. Assume the polynomial f has exactly two local maxima and one local minimum, and that these are the only critical points of f .
 - a. Sketch a possible graph of f .



- b. What is the largest number of zeros f could have? (Explain your answer.)
c. What is the least number of zeros f could have? (Explain your answer.)
d. What is the least number of inflection points f could have?
e. What is the smallest degree f could have?
f. Find two possible formulas for $f(x)$ with two different degrees.
2. Given the function $f(x) = -x^4 - 3x^3 + 101x^2 + 543x + 940$, use a graphing calculator to do the following:
- Find the zeros of the function. Round any zeros you find to the nearest tenth. Explain how you found your answer step-by-step, as if you were explaining to a student who does not know how to use a graphing calculator.
 - Identify any local minima or maxima of the graph of the function as ordered pairs. Indicate which are local minima and which are local maxima. Round any minima or maxima to the nearest tenth. Explain how you found your answer step-by-step.
 - Find the range of the function. Round the numbers in your answer to the nearest tenth. Explain how you found your answer step-by-step.
 - Sketch the graph of the function. List a viewing window that shows all the information you found in the other parts of the problem. Explain why you chose your viewing window.

8.7 Rational Functions

A rational function is of the form

$$f(x) = \frac{p(x)}{q(x)}$$

for some polynomials p and q with real-valued coefficients. If the degree of q is zero, then the rational function is just a polynomial. So we will assume throughout this section that the degree of q is at least one. In the same way that the different representations of a polynomial help us to understand different aspects of the polynomial, different representations of the rational function will help us to better understand properties of the rational function.

8.7.1 Short-Term Behavior

In order to understand the short-term behavior of a rational function, it is helpful to write the function in its factored form

$$f(x) = \frac{p(x)}{q(x)} = A \frac{(x - a_1)^{j_1}(x - a_2)^{j_2} \cdots (x - a_n)^{j_n}}{(x - b_1)^{k_1}(x - b_2)^{k_2} \cdots (x - b_m)^{k_m}}$$

where the $a_i \in \mathbb{C}$ are the zeros of degree j_i of the numerator (and thus the zeros of the function), the $b_i \in \mathbb{C}$ are zeros of the denominator, and A is a real number that allows the numerator and denominator to both be monic polynomials. Since the polynomials have real-valued coefficients, any factors corresponding to non-real zeros come in conjugate pairs. The b_i that are real-valued correspond with values that have been removed from the domain of the function. Similarly, the a_i that are real-valued correspond to the horizontal intercepts for the rational function.

8.7.1.1 Common Factors

When the numerator and denominator share a factor, $(x - a_0)$, we need to understand how this affects the rational function.

If the degree of the shared factor is larger in the numerator than in the denominator, the rational function is undefined at the zero of the factor, but has a limit of zero near this value. We can also see that the rational function is equivalent to the rational function where $(x - a_0)$ is only a factor in the numerator whose degree is the difference between the degrees of the factor in the numerator and denominator of the original rational function, with the zero of the factor removed from the domain of the new function. For example,

$$f(x) = \frac{(x-1)^3}{(x-1)^1} \text{ is equivalent to } g(x) = (x-1)^2, \text{ for } x \in \mathbb{R} \setminus \{1\}.$$

If the degree of the shared factor is larger in the denominator than in the numerator, the rational function is equivalent to the rational function where $(x - a_0)$ is only a factor in the denominator whose degree is the difference between the degrees of the factor in the denominator and numerator of the original rational function. For example,

$$f(x) = \frac{(x-1)^1}{(x-1)^3} \text{ is equivalent to } g(x) = \frac{1}{(x-1)^2}.$$

If the shared factor has the same degree in the numerator and denominator, then this is a removable singularity and the factor can be removed from both the numerator and denominator with the removal of the zero of the factor from the new rational function.

For the remainder of the section we will assume that the numerator and denominator do not have a common factor.

8.7.1.2 Behavior near Zeros

The factors of the numerator of the rational function determine the zeros of the polynomial. If $(x - a_0)$ is a factor of the numerator of degree j_0 , and if $a_0 \in \mathbb{R}$, then a_0 is a zero of the rational function (thus a_0 is a horizontal intercept of the graph of the rational function) and we can rewrite the rational function as

$$f(x) = (x - a_0)^{j_0} g(x)$$

where g is a rational function without $(x - a_0)$ as a factor of either the numerator or denominator (See Theorem 8.1). Using properties from analysis (similarly to Theorem 8.1) we can determine that near $x = a_0$ that

$$f(x) \approx g(a_0)(x - a_0)^{j_0}.$$

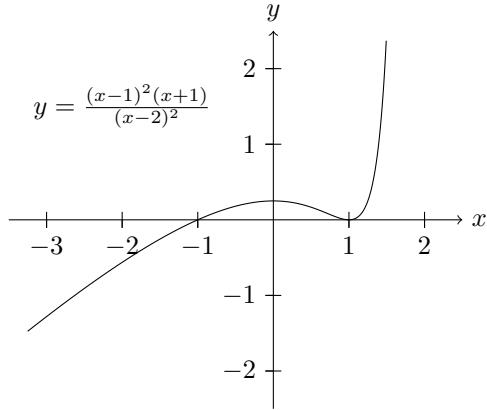
Example 8.1. To better understand this concept let us consider

$$f(x) = \frac{(x-1)^2(x+1)}{(x-2)^2}.$$

We see that the function has horizontal intercepts at -1 and 1 . We can also see that near $x = -1$, the function is very similar to

$$g(x) = (x+1) \cdot \frac{((-1)-1)^2}{((-1)-2)^2} = \frac{4}{9}(x+1).$$

Similarly, near $x = 1$ the function behaves very similarly to $h(x) = 2(x-1)^2$.



8.7.1.3 Behavior near Singularities

In order to determine the domain of the rational functions we consider the factors of the denominator of the function. If $(x - b_0)$ is a factor in the denominator of the rational function, and $b_0 \in \mathbb{R}$, then b_0 is not in the domain of the function. If the degree of the factor is k_0 , then we see that the function can be rewritten as

$$f(x) = \frac{1}{(x - b_0)^{k_0}} g(x)$$

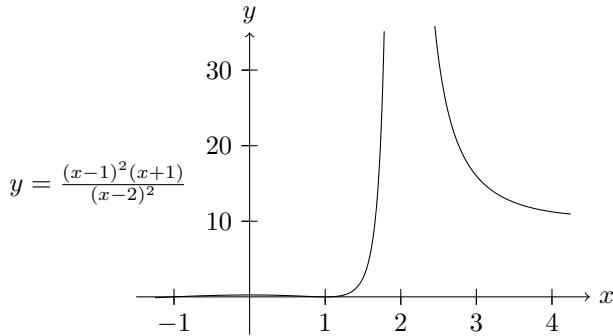
for a rational function g with b_0 in the domain of g and $g(b_0) \neq 0$. So near $x = b_0$,

$$f(x) \approx g(b_0) \frac{1}{(x - b_0)^{k_0}}.$$

Example 8.2. Returning to our example of $f(x) = \frac{(x-1)^2(x+1)}{(x-2)^2}$, we can see that near $x = 2$ that

$$f(x) \approx \frac{3}{(x - 2)^2},$$

as evidenced in the following graph.



This understanding of short-term behavior can help us to simplify certain models near given points, choose appropriate window sizes to have technology graph a function, and be able to more easily determine if the graph of a function generated by a program is accurate.

8.7.2 Long-Term Behavior

In order to understand the long-term behavior of rational functions we start by looking at rational functions where the degree of the numerator is less than the degree of the denominator. Such functions can be

generalized as

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_nx^n + \cdots + b_{n+r}x^{n+r}}.$$

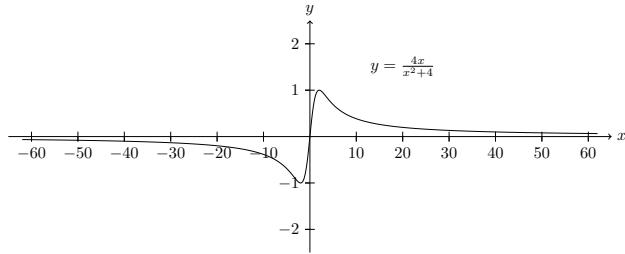
Then by multiplying the numerator and denominator by $x^{-(n+r)}$ we have

$$f(x) = \frac{a_0 \frac{1}{x^{n+r}} + a_1 \frac{1}{x^{n+r-1}} + \cdots + a_n \frac{1}{x^r}}{b_0 \frac{1}{x^{n+r}} + b_1 \frac{1}{x^{n+r-1}} + \cdots + b_n \frac{1}{x} + \cdots + b_{n+r}}$$

and we can see that as $|x|$ gets large that most of the terms converge to 0. Thus,

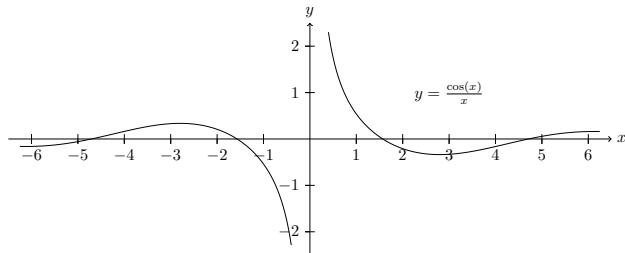
$$\lim_{x \rightarrow \pm\infty} f(x) = \frac{0}{b_{n+r}} = 0.$$

So we see that if the degree of the numerator is less than the degree of the denominator that the graph of the function approaches the horizontal axis as $|x|$ gets large. This is often called a horizontal asymptote of the function since the function asymptotically approaches the horizontal axis. A challenge that many students have with this terminology is that they have been told that a function cannot cross an asymptote. While that is true for vertical asymptotes, it is not true for horizontal asymptotes as seen in the graph of the function below.



In fact, a function can cross its “horizontal asymptote” an infinite number of times. For example,

$$f(x) = \frac{\cos(x)}{x}.$$



We can then build on this understanding to study the long-term behavior of rational functions with the degree of the numerator greater than or equal to the degree of the denominator. So if $f(x)$ is a rational function with numerator $a(x)$ and denominator $b(x)$ we have from the Division Algorithm for Polynomials 7.10 that there exist unique polynomials $q(x)$ and $r(x)$ such that

$$a(x) = b(x)q(x) + r(x)$$

with the degree of $r(x)$ less than the degree of $b(x)$. Dividing both sides of this equation by $b(x)$ we have

$$f(x) = \frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}.$$

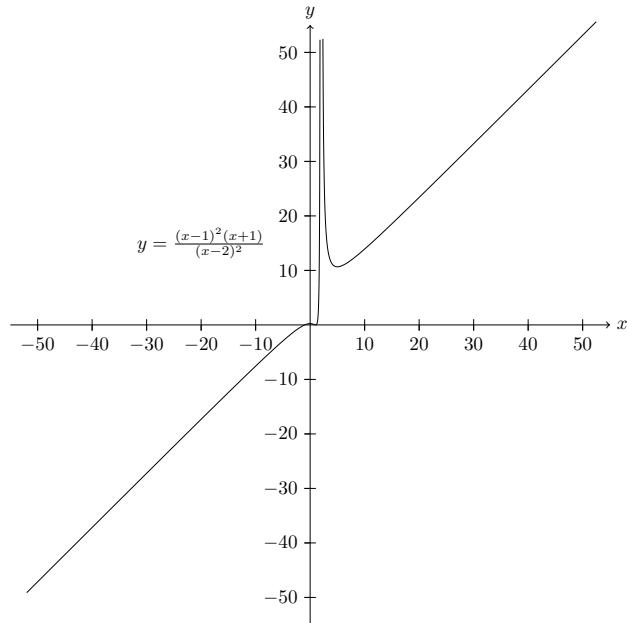
From the conversation above we can see that

$$\lim_{x \rightarrow \pm\infty} \frac{r(x)}{b(x)} = 0$$

and that $f(x)$ asymptotically approaches $q(x)$ as $|x| \rightarrow \infty$. So if the degree of the numerator is one plus the degree of the denominator then the quotient would be a linear function. Such asymptotes are often called “skew asymptotes.”

Example 8.3. Returning to our example of $f(x) = \frac{(x-1)^2(x+1)}{(x-2)^2}$, we can rewrite the function using the division algorithm as

$$f(x) = x + 3 + \frac{7x - 11}{(x-2)^2}.$$



8.7.3 Partial Fraction Decomposition

Another way to represent rational functions whose numerator has a lower degree than the denominator is a **partial fraction decomposition**. This decomposition allows the rational function to be written as the sum of rational functions whose denominators are irreducible and whose numerators are of lower degree than the denominator. For instance,

$$\frac{3x-1}{x^2-2x-3} = \frac{1}{x+1} + \frac{2}{x-3} \quad \text{or} \quad \frac{3x^2-x+1}{x^3+x} = \frac{2x-1}{x^2+1} + \frac{1}{x}.$$

There are limited uses of the partial fraction decomposition in the K-12 curriculum. Hence, it was not included in the Common Core standards. However, this representation of a rational function is essential in the area of integration of rational functions. It is also useful to find Taylor (and Laurent) series expansions, and to solve problems in differential equations involving the Laplace transform. Because of this, we recommend this content first being taught during the section of integration of rational functions in Calculus. In the high school curriculum, this is during the BC portion of an Advanced Placement Calculus AB/BC course.

8.7.4 Exercises

1. Consider the polynomials $a(x) = 2x^3 + 3x^2 - 5x - 7$ and $b(x) = x^2 + 1$.

- a. Perform polynomial division of $a(x)$ by $b(x)$. Call the quotient polynomial $q(x)$ and the remainder polynomial $r(x)$.
- b. Use the quotient and remainder to describe the behavior of $f(x)$ when $|x|$ is large.
2. Let $f(x) = \frac{x^3 - 1}{x^3 + 1}$.
- What is the domain of f ?
 - Determine the singular points of f (that is, the points at which f is undefined) and whether f has any vertical asymptotes at these points.
 - What is the range of f ?
 - At what points does the graph of f intersect the horizontal-axis?
 - Find any global or local maxima or minima.
 - Determine the intervals on which f is increasing and those intervals on which f is decreasing.
 - Describe what happens to $f(x)$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.
 - What is a power series of f ?
3. List and describe the behavior near each of the zeros and singularities for the following rational functions.
- $f(x) = \frac{x + 3}{x^2 + 2x - 3}$
 - $f(x) = \frac{x^2 - 2x - 24}{x^2 + 10x + 24}$
 - $f(x) = \frac{25x^2 - 80x + 64}{192 - 75x^2}$
 - $f(x) = \frac{12x^7 - 3x^3}{4x^4 - 16x}$
 - $f(x) = \frac{6x - 12x^9}{3x^3 + 7}$
 - $f(x) = \frac{x^2 + 15x + 56}{x + 5}$
 - $f(x) = \frac{x^2 + 2x - 15}{x^2 + 10x + 25}$
 - $f(x) = \frac{x^2 + 8x + 16}{x^2 + 6x + 8}$
 - $f(x) = \frac{x^5 + 3x^3 + 2x^2 + 4x + 1}{x^4 + 5}$
4. Describe the long-term behavior for the following rational functions.
- $f(x) = \frac{x + 3}{x^2 + 2x - 3}$
 - $f(x) = \frac{x^2 - 2x - 24}{x^2 + 10x + 24}$
 - $f(x) = \frac{25x^2 - 80x + 64}{192 - 75x^2}$
 - $f(x) = \frac{12x^7 - 3x^3}{4x^4 - 16x}$
 - $f(x) = \frac{6x - 12x^9}{3x^3 + 7}$
 - $f(x) = \frac{x^2 + 15x + 56}{x + 5}$
 - $f(x) = \frac{x^2 + 2x - 15}{x^2 + 10x + 25}$
 - $f(x) = \frac{x^2 + 8x + 16}{x^2 + 6x + 8}$
 - $f(x) = \frac{x^5 + 3x^3 + 2x^2 + 4x + 1}{x^4 + 5}$

5. Find an algebraic representation for a function, f , whose graph has all of the following attributes.

- $f(2) = 0$
- $f(\frac{1}{2}) = 0$
- $f(-3) = 0$
- $\lim_{x \rightarrow \frac{2}{3}^+} f(x) = \infty$
- $\lim_{x \rightarrow 1^-} f(x) = \infty$
- As $x \rightarrow \pm\infty$, $f(x) \approx \frac{10}{3}x^2 + \frac{77}{9}x - \frac{242}{27}$

6. Write the rational functions below in their partial fraction decomposition over the real numbers.

- a. $f(x) = \frac{x^3 + x^2 + 1}{x^2 - x}$
- b. $f(x) = \frac{x^4}{x^4 - 1}$
- c. $f(x) = \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}$
- d. $f(x) = \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}$
- e. $f(x) = \frac{2x^2 - x + 4}{x^3 + 4x}$

8.8 Trigonometric Functions

The area of trigonometry comes from the combination of τριγωνός (triangle) and μετρού (measure) and is the study of the measurements of triangles. The study of the measurements of sides of triangles was known to the ancient Egyptians and Babylonians for the purpose of constructions and land surveys [Boyer and Merzbach, 1991, p. 158]. The Greeks expanded on this work with many of our standard trigonometric properties included in Euclid's Elements. With the growth of astronomy and navigation in the 1500's trigonometry was expanded to include more identities. The current uses of trigonometric functions include optics, acoustics, electrical engineering, image compression, and computer-generated imagery (CGI).

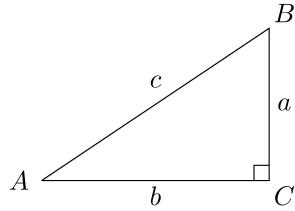
8.8.1 Definitions

Beginning in fourth grade, students are introduced to right triangles, with a cyclical study of these triangles continuing through middle school, and into high school.

Related Content Standards

- (4.G.2) Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.
- (6.G.1) Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
- (8.G.5) Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.
- (HSG.SRT.6) Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

Our first definitions for the trigonometric functions are based on the properties of right triangles. As such, the initial domain of the trigonometric functions are defined in degrees on $(0^\circ, 90^\circ)$. For an angle A , we define



the sine of angle A to be the ratio of the measure of the opposite side, a , and the hypotenuse, c ,

$$\sin(A) = \frac{a}{c}.$$

The cosine of angle A is the ratio of the measures of the adjacent side, b , and the hypotenuse, c , and the tangent of the angle A is the ratio of the measures of the opposite side, a , and the adjacent side, b ,

$$\cos(A) = \frac{b}{c} \quad \text{and} \quad \tan(A) = \frac{a}{b}.$$

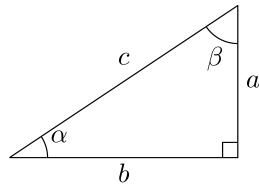
Similarly for angle B ,

$$\sin(B) = \frac{b}{c}, \quad \cos(B) = \frac{a}{c}, \quad \text{and} \quad \tan(B) = \frac{b}{a}.$$

We also define the additional trigonometric functions as follows:

$$\begin{aligned} \text{secant of } A &= \sec(A) = \frac{1}{\cos(A)} = \frac{c}{b} \\ \text{cosecant of } A &= \csc(A) = \frac{1}{\sin(A)} = \frac{c}{a} \\ \text{cotangent of } A &= \cot(A) = \frac{1}{\tan(A)} = \frac{b}{a} \end{aligned}$$

While these definitions are sufficient, it is less ambiguous to define the angle, not by the vertex, but by the amount between two line segments.



$$\alpha + \beta = 90^\circ$$

$$\alpha = 90^\circ - \beta$$

$$\beta = 90^\circ - \alpha$$

$$\cos(\alpha) = \sin(\beta) = \frac{b}{c}$$

$$\tan(\alpha) = \cot(\beta) = \frac{a}{b}$$

$$\sec(\alpha) = \csc(\beta) = \frac{c}{b}$$

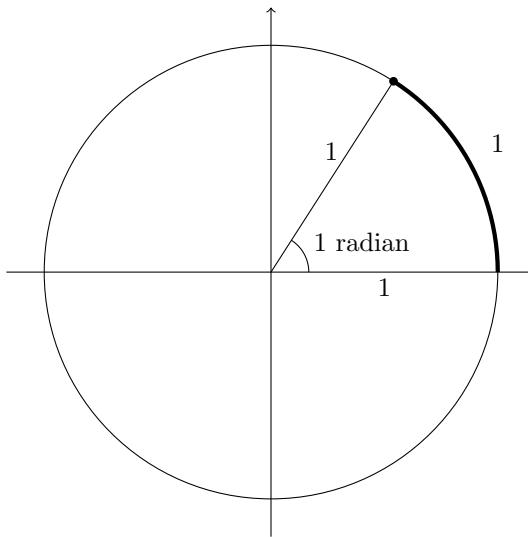
$$\sin(\alpha) = \cos(\beta) = \frac{a}{c}$$

$$\cot(\alpha) = \tan(\beta) = \frac{b}{a}$$

$$\csc(\alpha) = \sec(\beta) = \frac{c}{a}$$

8.8.1.1 Radians and Degrees

The use of 360° to measure angles is likely related to the number of days (365) it takes for the Earth to rotate around the sun. With the advent of calculus it became apparent that a different technique of measuring an



angle based on the arc length subtended by the angle was needed. Shortly after his death in 1716 at the age of 33, Roger Cotes published the first definition of the radian angle measurement [1722]. (It was in this same work that he wrote $ix = \ln(\cos(x) + i\sin(x))$, the precursor to Euler's formula.)

A radian is defined to be the angle subtended from by a circular arc whose length is the same as the radius. This means that there are 2π radians in a circle, a right angle is $\frac{\pi}{2}$ radians, and a straight line is π radians.

In order to convert an angle from degrees to radians, one would multiply by $\frac{\pi}{180^\circ}$. And to convert from radians to degrees would be necessitate multiplying by the reciprocal.

One of the primary benefits of using radians instead of degrees is that for angles near 0, $\sin(\alpha) \approx \alpha$. This simplifies the derivatives of the trigonometric functions and makes for a cleaner series expansion for the trigonometric functions.

In addition, to find the arclength of an angle in radians, one only needs to multiply by the radius, ($l = \alpha r$). To find the arclength in for an angle given in degrees we have $l = \alpha \frac{\pi r}{180^\circ}$.

Related Content Standards

- (HSF.TF.1) Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

8.8.1.2 Unit Circle

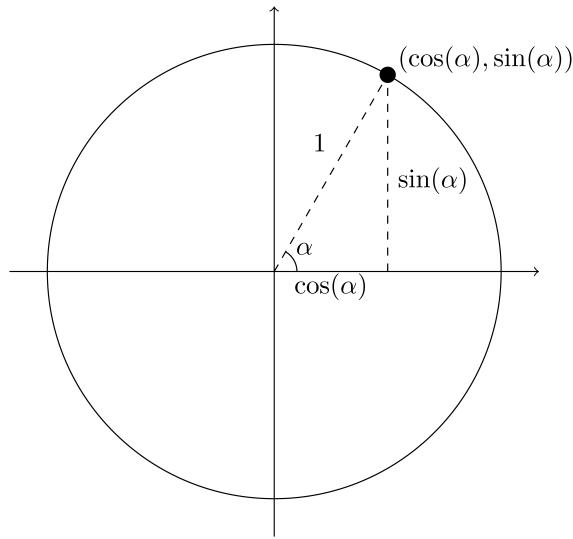
Combining the definitions of the trigonometric functions for right triangles with the angles defined by radians we can extend the domain of trigonometric functions to all possible angles using the unit circle. For angles $\alpha \in (0, \frac{\pi}{2})$, we plot a point on the unit circle that is swept out by that angle from the positive horizontal axis. We then form a right triangle from that point by drawing a line segment from the point that is perpendicular to the horizontal axis and draw a line segment from the point to the origin. We can then see that the coordinates of the point are $(\cos(\alpha), \sin(\alpha))$. This also means that the distance along the unit circle from $(1, 0)$ to this point is α .

Then, for any angle α we define $\sin(\alpha)$ and $\cos(\alpha)$ to be given by the coordinates on the unit circle whose distance from the point $(1, 0)$ along the circle in a counter-clockwise direction is α .

Related Content Standards

- (HSF.TF.2) Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed

counterclockwise around the unit circle.



With this extended definition of sine and cosine, we can define all of our trigonometric functions on a much larger domain. We can see from these definitions that

$$\sin : \mathbb{R} \rightarrow [-1, 1], \cos : \mathbb{R} \rightarrow [-1, 1], \text{ and}$$

$$\tan : \left(\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \right) \rightarrow \mathbb{R}.$$

Since $(\cos(\alpha), \sin(\alpha))$ is defined as a point on the unit circle, we know from the Pythagorean theorem that $(\cos(\alpha))^2 + (\sin(\alpha))^2 = 1$. In order to reduce the notation we often write $\cos^2(\alpha)$ and $\sin^2(\alpha)$ to represent the square of the function. With this notation we have

$$\cos^2(\alpha) + \sin^2(\alpha) = 1.$$

This equation can then be rearranged to create the equivalent equations of

$$\sec^2(\alpha) - \tan^2(\alpha) = 1 \quad \text{and} \quad \csc^2(\alpha) - \cot^2(\alpha) = 1.$$

Related Content Standards

- (HSF.TF.8) Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Using the properties of the unit circle we find that the values of the cosine and sine functions repeat with every cycle around the unit circle and so we find that these are periodic functions with a period of 2π ,

$$\cos(\alpha) = \cos(\alpha + 2\pi) \quad \text{and} \quad \sin(\alpha) = \sin(\alpha + 2\pi).$$

Since the negative angles are the rotation in the clockwise direction instead of the counter-clockwise direction we have some additional symmetry,

$$\sin(-\alpha) = -\sin(\alpha) \quad \text{and} \quad \cos(-\alpha) = \cos(\alpha),$$

and so we see that the sine function is an odd function and the cosine function is an even function.

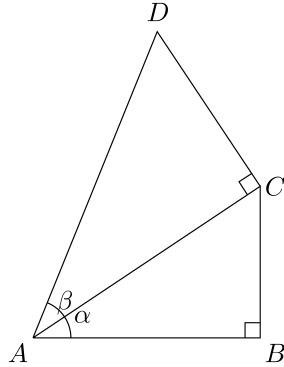
Related Content Standards

- (HSF.TF.4) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

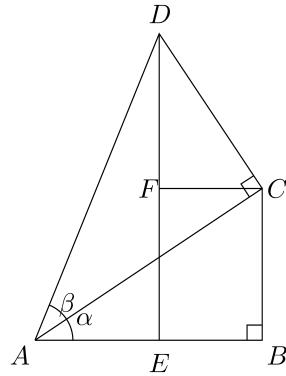
8.8.2 Additional Properties

8.8.2.1 Angle Addition Formulas

In order to prove the angle addition formula for the sine function we begin by choosing two acute (less than right) angles, α and β , and draw a pair of right triangles so that the hypotenuse of the triangle ABC is a leg of the triangle ACD , as in the figure below.



We can then draw a line segment through D perpendicular to AB , and another line segment perpendicular to this new line and through the point C .



We now have that

$$\sin(\alpha + \beta) = \frac{DE}{AD}$$

and that $DE = DF + EF$, and since $EF = BC$ we have that

$$\sin(\alpha + \beta) = \frac{DF + BC}{AD} = \frac{DF}{AD} + \frac{BC}{AD}$$

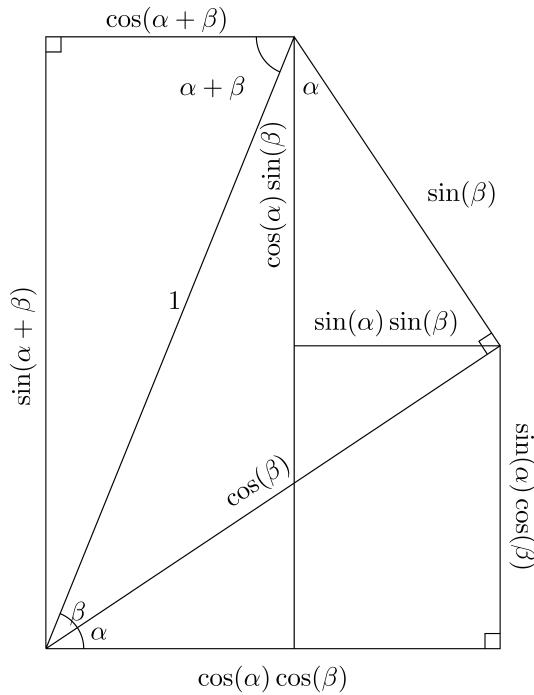
We can also view AC as a transversal to the parallel lines AB and CF so that the angle $\angle ACF = \alpha$ and also $\angle FDC = \alpha$. This means that $\cos(\alpha) = \frac{DF}{CD}$. We can then rewrite the angle addition equation as

$$\sin(\alpha + \beta) = \frac{DF}{CD} \frac{CD}{AD} + \frac{BC}{AC} \frac{AC}{AD} = \cos(\alpha) \sin(\beta) + \sin(\alpha) \cos(\beta).$$

In order to prove the angle addition formula for the cosine function we can see that

$$\begin{aligned}
\cos(\alpha + \beta) &= \sin\left(\frac{\pi}{2} - \alpha - \beta\right) \\
&= \sin\left(\left(\frac{\pi}{2} - \alpha\right) + (-\beta)\right) \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\sin(-\beta) + \sin\left(\frac{\pi}{2} - \alpha\right)\cos(-\beta) \\
&= \sin(\alpha)(-\sin(\beta)) + \cos(\alpha)\cos(\beta) \\
&= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)
\end{aligned}$$

Another proof of both of these identities involves can be constructed from the following diagram.



If the angles are obtuse, then one can use the symmetry of the sine and cosine functions to change the angle addition into a sum of angles that are acute.

Related Content Standards

- (HSF.TF.9) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

If we use Euler's equation, $e^{iy} = \cos(y) + i\sin(y)$, we can prove the angle addition formulas using properties of exponents and the addition and multiplication operations. We can write $e^{i(\alpha+\beta)}$ in two ways:

$$\begin{aligned}
e^{i(\alpha+\beta)} &= e^{i\alpha} \cdot e^{i\beta} = (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\
&= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \\
e^{i(\alpha+\beta)} &= \cos(\alpha + \beta) + i\sin(\alpha + \beta)
\end{aligned}$$

Then by equating the real part and imaginary parts of these two representations we have the angle addition formulas.

Related Content Standards

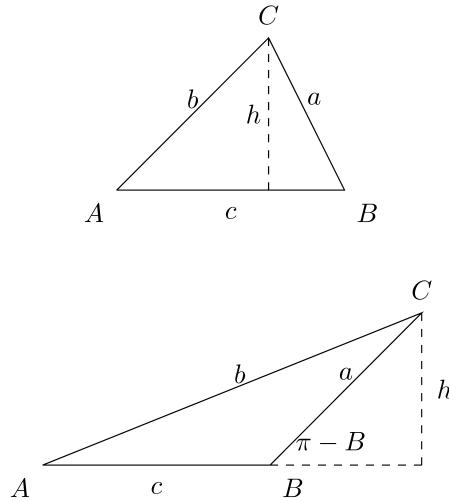
- (HSG.SRT.10) Prove the Laws of Sines and Cosines and use them to solve problems.
- (HSG.SRT.11) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Theorem 8.2 (Law of Sines). *If a triangle has sides of lengths a , b , and c opposite angles A , B , and C , respectively, then*

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Proof. If the triangle has a right angle, then the law of sines follows directly from the definition of the sine function. So we may assume that we have a triangle, $\triangle ABC$, without any right angles.

Then $\angle ABC$ can be acute or obtuse.



In each case, draw the altitude from the vertex at C to the side AB . In the acute triangle the altitude lies inside the triangle, while in the obtuse triangle the altitude lies outside the triangle.

Let h be the height of the altitude. For each triangle we see that

$$\frac{h}{b} = \sin(A) \quad \text{and} \quad \frac{h}{a} = \sin(B),$$

since $\frac{h}{a} = \sin(\pi - B) = \sin(B)$ because of the symmetry of the sine function. Thus, solving for h and substituting that into the previous equation gives

$$a \sin(B) = b \sin(A) \quad \text{or equivalently} \quad \frac{\sin(A)}{a} = \frac{\sin(B)}{b}.$$

By a similar argument, drawing the altitude from A to BC gives

$$\frac{\sin(C)}{c} = \frac{\sin(B)}{b}.$$

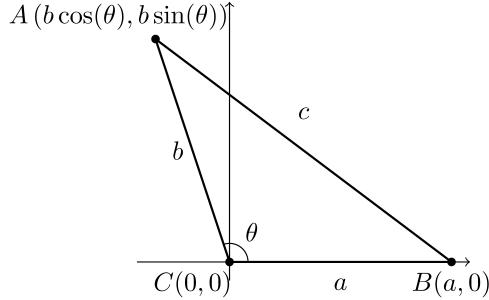
We then combine these to complete our proof. □

Theorem 8.3 (Law of Cosines). *If a triangle has sides of lengths a , b , and c , opposite angles A , B , and C , respectively, then*

$$c^2 = a^2 + b^2 - 2ab \cos(C)$$

There are many different proofs of the Law of Cosines. We include one from the perspective of analytic geometry.

Proof. We are able to create a system of coordinates on the plane in order to align the point C with the origin and point B on the positive real axis at the point $(a, 0)$.



We can then use the distance formula to find the distance from A to C as

$$c^2 = (b \cos(\theta) - a)^2 + (b \sin(\theta))^2 = b^2 \cos^2(\theta) - 2ab \cos(\theta) + a^2 + b^2 \sin^2(\theta)$$

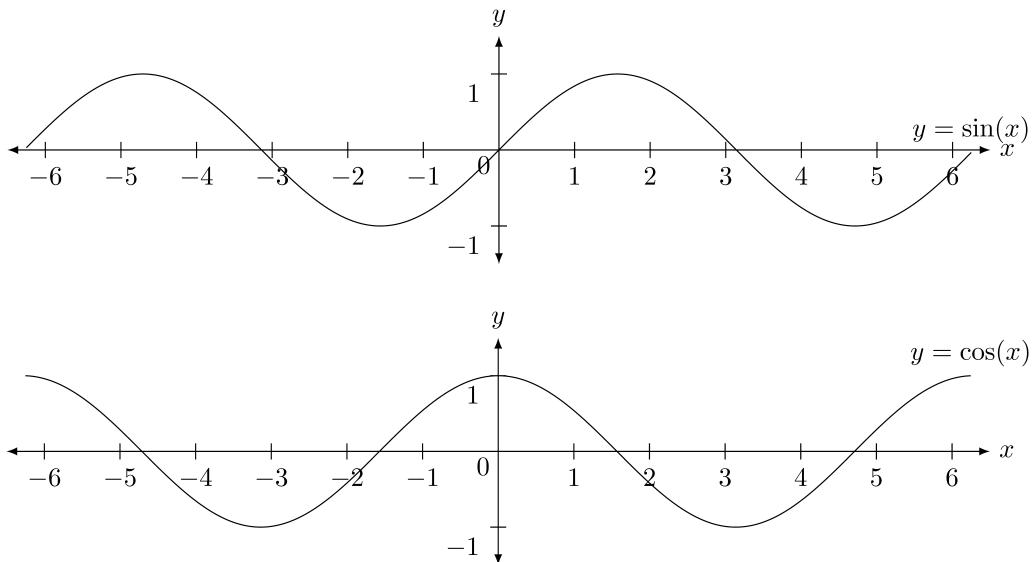
We can then rearrange the terms and use the property that $b^2 \cos^2(\theta) + b^2 \sin^2(\theta) = b^2$ to see that

$$c^2 = a^2 + b^2 - 2ab \cos(C)$$

□

8.8.3 Graphs of Trigonometric Functions

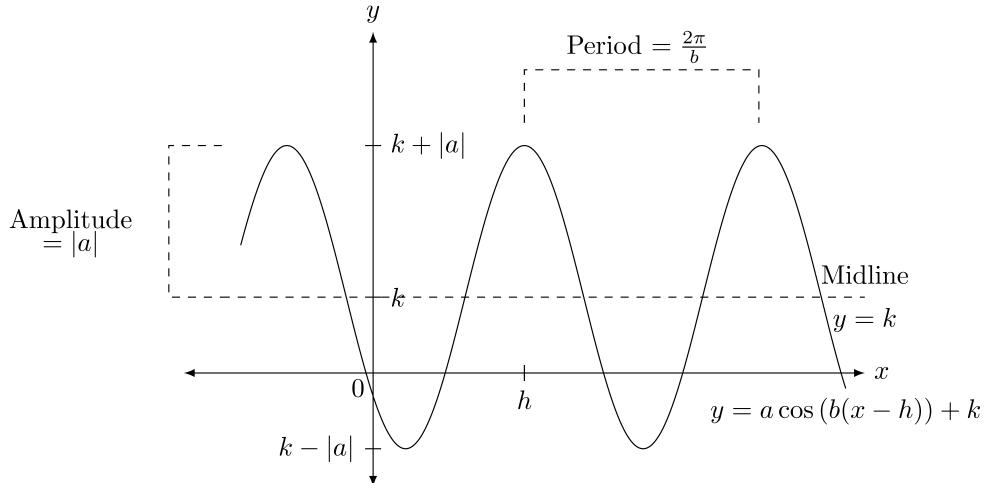
Using the definitions of the sine and cosine functions from the unit circle we can sketch their graphs as a function of the angle in radians.



We can then look at the family of functions generated from these two functions. One thing to notice is that the sine function is a horizontal shift of the cosine function, $\sin(x) = \cos(x - \frac{\pi}{2})$. Because of this, we will only look at the family of functions generated from the cosine function. This family of functions is all of the functions of the form

$$f(x) = a \cos(b(x - h)) + k$$

for real numbers a , b , h , and k .



Related Content Standards

- (HSF.TF.5) Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

We can also sketch the base graphs of the other trigonometric functions noticing that the domain of these functions will not include certain values as we cannot divide by zero and have a real number. We will give the graphs of the tangent and secant functions, with the graphs of the cotangent and cosecant functions being similar.

8.8.4 Inverse Trigonometric Functions

From Section 5.3, we see that a function is invertible if, and only if, it is a bijection. We can look at the graphs of the sine and cosine functions and see that they are definitely not bijections when thought of as functions from \mathbb{R} to \mathbb{R} .

However, if we restrict the domain and co-domain of the functions, these new functions are invertible.

For the cosine function, we restrict the domain to $[0, \pi]$ in order to find the maximum region on which the function is an injection and is onto the range of the original function, $[-1, 1]$. While there are many options for this choice, we choose the option that includes the first quadrant to correspond with the definitions of the functions for right triangles.

This function is now invertible and so we call its inverse function arccosine and denote it

$$\arccos : [-1, 1] \rightarrow [0, \pi] \quad \text{or} \quad \cos^{-1} : [-1, 1] \rightarrow [0, \pi].$$

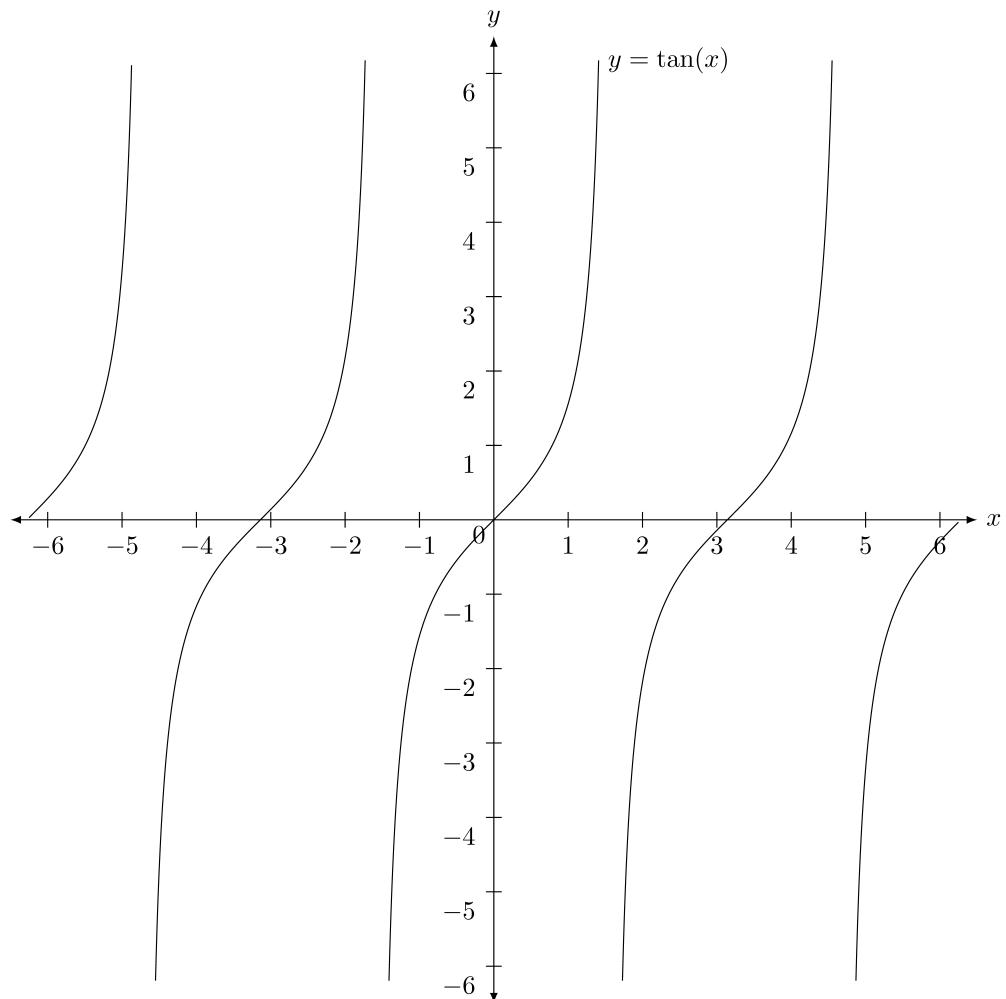
Similarly we will restrict the sine function to the largest domain that includes the first quadrant for which the function is an injection.

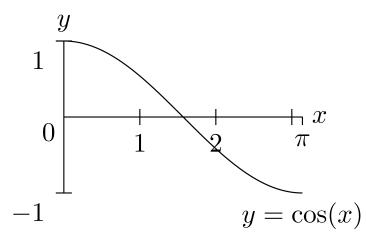
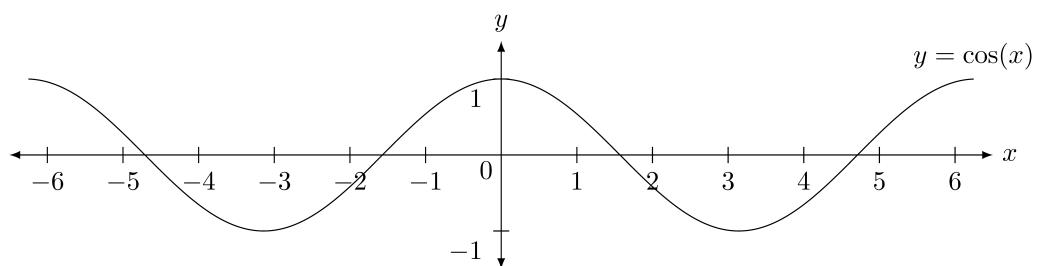
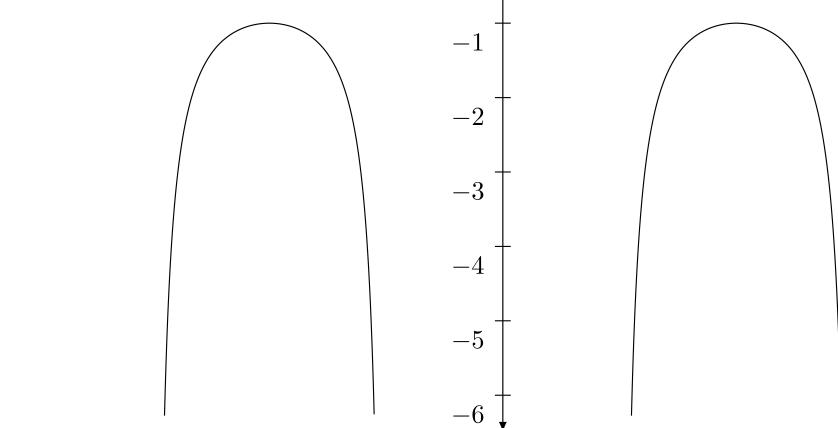
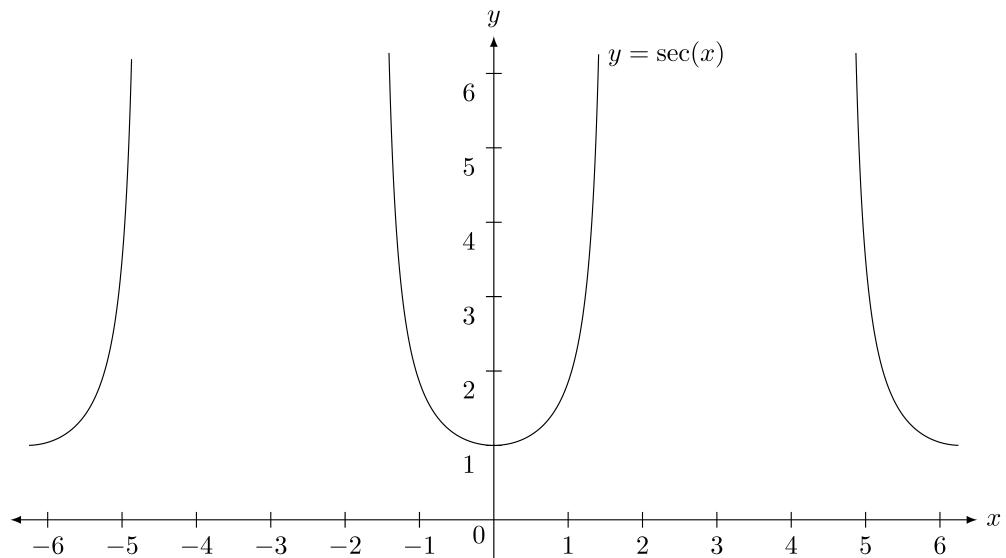
We then call the inverse function of this restricted domain function the arcsine and denote it

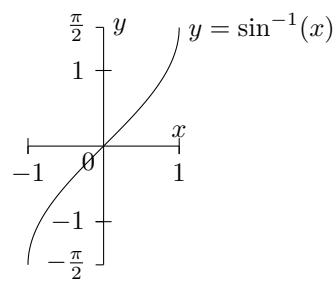
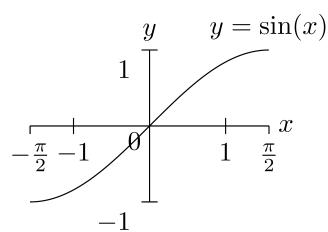
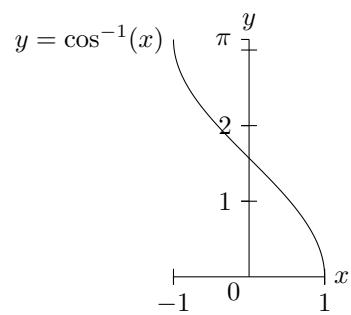
$$\arcsin : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \quad \text{or} \quad \sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

We can then say that the arcsine of a number, x , between -1 and 1 is the angle, θ , between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ so that $\sin(\theta) = x$. With a similar interpretation for the arccosine.

We can similarly restrict the domain and co-domain of the tangent, cotangent, secant, and cosecant functions to be invertible. Our choice of domains for these functions depend upon including the origin, if possible, and to have the function be increasing on the intervals of its domain.







$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow (-\infty, \infty)$$

$$\cot : (0, \pi) \rightarrow (-\infty, \infty)$$

$$\sec : \left[0, \frac{pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \rightarrow (-\infty, 1] \cup [1, \infty)$$

$$\csc : \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \rightarrow (-\infty, 1] \cup [1, \infty)$$

$$\tan^{-1} : (-\infty, \infty) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cot^{-1} : (-\infty, \infty) \rightarrow (0, \pi)$$

$$\sec^{-1} : (-\infty, 1] \cup [1, \infty) \rightarrow \left[0, \frac{pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

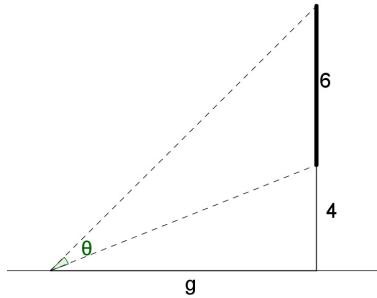
$$\csc^{-1} : (-\infty, 1] \cup [1, \infty) \rightarrow \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

Related Content Standards

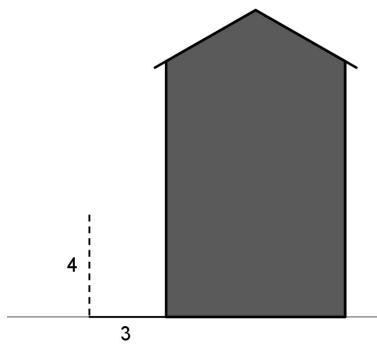
- (HSF.TF.6) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
- (HSF.TF.7) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

8.8.5 Exercises

1. A billboard with a height of 6 feet is mounted on the side of a building with its bottom edge at a 4 feet above the street. At what distance g along the ground should an observer stand from the building to get the best view?



2. Find the shortest length of a ladder that can be placed against the wall of the house above the 4 foot fence that is placed 3 feet from the house.



8.9 Combinations of Functions

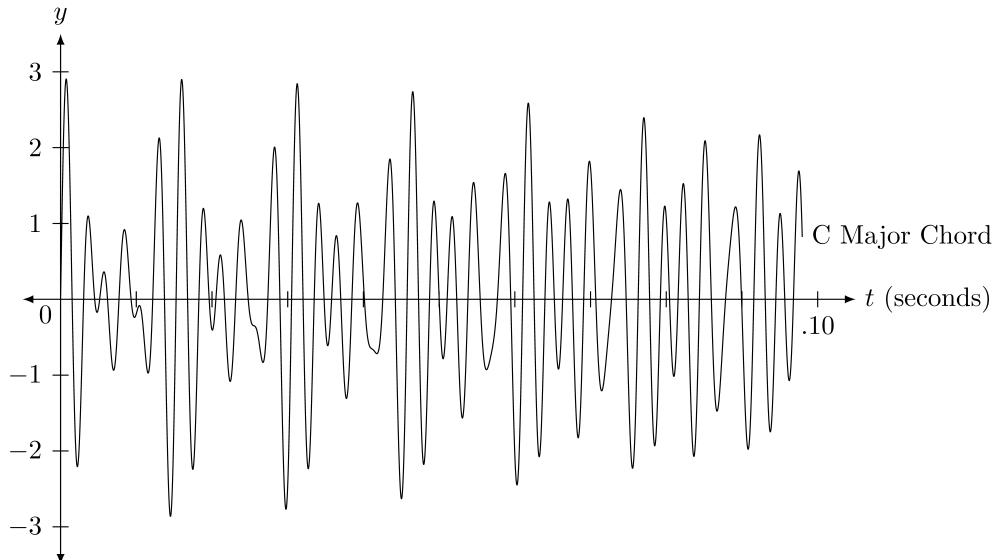
Now that we have an understanding of the main functions in the K-12 curriculum we will look at some of the ways that these functions are combined to create new functions or to model various phenomena.

8.9.1 Addition, Subtraction, Multiplication, and Division of Functions

We will explore how functions can be combined through addition and subtraction by looking at an example from music. The note of middle C has a frequency of 261.63 Hz (cycles per second) and can be modeled by the function $C(t) = \sin(261.63 \cdot (2\pi t))$. If one doubles the number of cycles per second we obtain the next octave higher of the note of C, $\tilde{C}(t) = \sin(523.25 \cdot (2\pi t))$. If we play these notes together, we have

$$C(t) + \tilde{C}(t) = \sin(261.63 \cdot (2\pi t)) + \sin(523.25 \cdot (2\pi t))$$

When the octave is divided into 8 parts, one finds that playing the 1st, 3rd, and 5th notes of the scale creates a very pleasing sound. We can see from the graph of the sum of the three sine functions corresponding to these three notes that the chord has a very complex structure.



The use of noise canceling headphones is another example of the adding and subtracting of functions. In this application, there is a microphone obtaining the noise that is in the surrounding environment and then subtracting that signal from the sound the person is hearing.

$$\begin{aligned} \text{Sound produced by the headphones} &= \text{Sound of the music or speaking desired} \\ &\quad - \text{Sound of surrounding environment} \end{aligned}$$

There are many additional applications that involve the addition and subtraction of functions including

$$\text{Net Profit} = \text{Revenue} - \text{Expenses}.$$

Some applications that involve the multiplication and/or division of functions include

$$\text{Work}(\text{time}) = \text{Force}(\text{time}) \cdot \text{Distance}(\text{time})$$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

$$\text{Surface Area of a Cylinder} = 2\pi(\text{radius}) \cdot (\text{height}) + 2\pi(\text{radius})^2$$

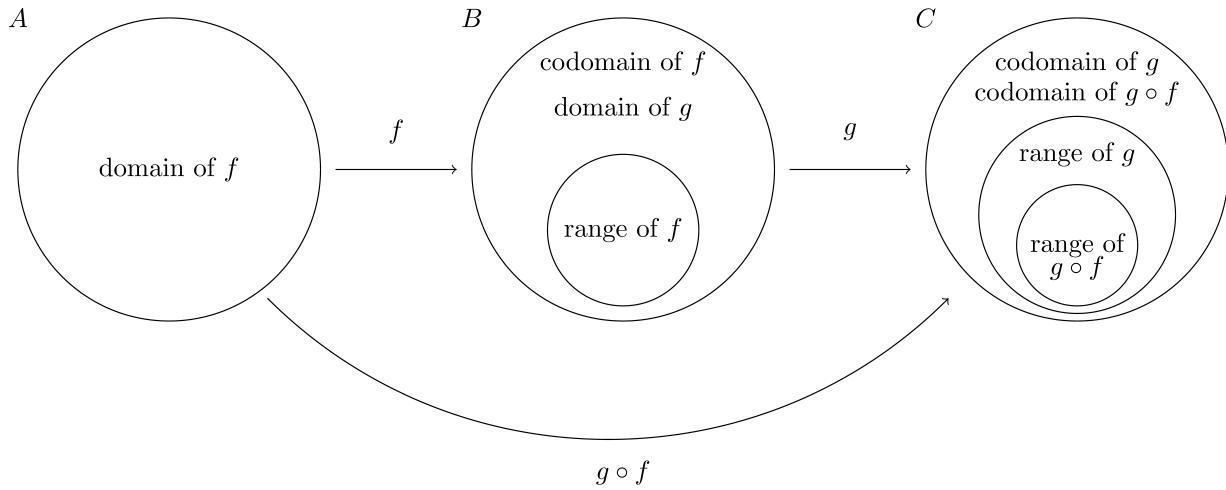
When functions are combined using the basic operations of addition, subtraction, multiplication, and division, the individual functions must have the same type of domain and the domain of the combined function would be the intersection of the domains of the individual functions, excluding any values for which a denominator may be undefined.

8.9.2 Composition of Functions

Another common way to combine functions is through function composition. An example of such a model is finding the area covered by a circular oil slick when the radius of the oil slick is changing with time.

$$\text{Area}(\text{time}) = \pi \cdot (\text{radius}(\text{time}))^2$$

Recall from Chapter 5 that with the composition of functions, for most cases, the domain of the second function corresponds with the domain of the first function.



However, we sometimes have compositions of functions for which the domain of the second function is a subset of the co-domain of the first function. For example,

$$g(x) = \sqrt{x}, \quad f(x) = x - 1, \quad \text{and} \quad (g \circ f)(x) = \sqrt{x - 1}.$$

In this case, the domain of g is $[0, \infty) \subseteq \mathbb{R}$ and so the domain of the composition of the two functions, $g \circ f$, is $[1, \infty)$.

In order to determine the domain and range of a composition of functions where the domain of the second function does not correspond to the co-domain of the first function, we need to define the pre-image of a function.

If $\phi : U \rightarrow V$ and $W \subseteq V$, we define the pre-image of W under the function ϕ as,

$$\phi^{-1}(W) = \{a \in U | \phi(a) \in W\}.$$

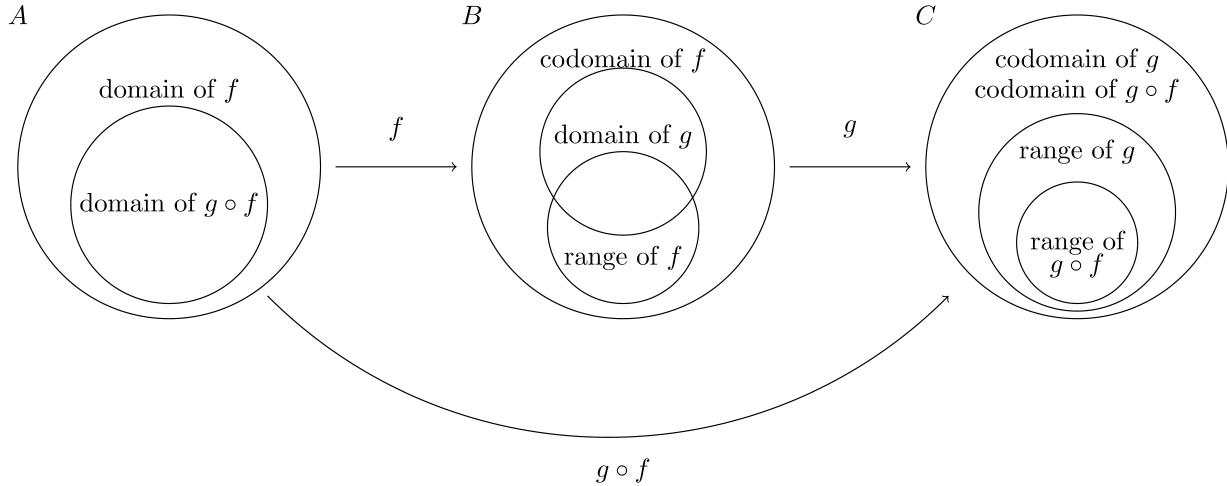
Then if

$$f : A \rightarrow B \quad \text{and} \quad g : D \subseteq B \rightarrow C$$

are functions, the domain of $g \circ f$ is

$$f^{-1}((\text{Domain of } g) \cap (\text{Range of } f)).$$

Let's look at how this applies to an example of compositions of linear fractional transformations.



Example 8.4. Let $f : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{2x - 1}{x - 3} \quad \text{and} \quad g(x) = \frac{5x + 2}{x + 1}.$$

Then the range of f is $\mathbb{R} \setminus \{2\}$ and the domain of g is $\mathbb{R} \setminus \{-1\}$. So the intersection is $\mathbb{R} \setminus \{-1, 2\}$ and

$$f^{-1}(\mathbb{R} \setminus \{-1, 2\}) = \mathbb{R} \setminus \left\{ \frac{4}{3}, 3 \right\},$$

since $f\left(\frac{4}{3}\right) = -1$.

So the domain of $g \circ f$ is $\mathbb{R} \setminus \left\{ \frac{4}{3}, 3 \right\}$ and we can determine the range of $g \circ f$ by removing $g(2)$ from the range of g . So

$$g \circ f : \mathbb{R} \setminus \left\{ \frac{4}{3}, 3 \right\} \rightarrow \mathbb{R} \setminus \{4, 5\}, \quad (g \circ f)(x) = \frac{12x - 11}{3x - 4}$$

is a surjection.

Once we have determined the domain and range of the composition of functions, we can use properties of each of the individual functions to determine properties of the composition of the functions.

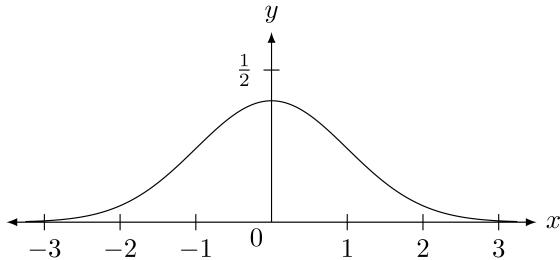
Example 8.5. The function

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

gives the standard normal distribution. If we let $h(x) = -\frac{x^2}{2}$ and $g(x) = \frac{e^x}{\sqrt{2\pi}}$, we see that $f = g \circ h$. Since h is an even function, f must be an even function and so we will have symmetry about the vertical axis. We also know that h is decreasing on $[0, \infty)$ and g is always increasing. So $f = g \circ h$ must be decreasing on $[0, \infty)$, and since $\lim_{x \rightarrow \infty} h(x) = -\infty$ and $\lim_{x \rightarrow -\infty} g(x) = 0$ we know that $\lim_{x \rightarrow \infty} f(x) = 0$. Knowing that $f(0) = \frac{1}{\sqrt{2\pi}}$ we have enough knowledge about the function to sketch a graph of the function.

8.9.3 Systems of Equations

The final method of combining multiple functions that we will study are determining the values for which a set of functions have the same input and output. The most common type of problems for which this occurs are systems of linear equations.



Sometimes it is valuable to know the values of x and y for which two equations,

$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta, \end{aligned}$$

are true. In terms of functions, this can be viewed in multiple ways. One option is to think of the two equations as two different linear functions, $f(x) = -\frac{a}{b}x + \frac{\alpha}{b}$ and $g(x) = -\frac{c}{d}x + \frac{\beta}{d}$. We are then determining the values of x for which $f(x) = g(x)$. With this perspective, one can think of this graphically as the points (x, y) for which the two graphs, $y = f(x)$ and $y = g(x)$, intersect. Another method is to view the situation algebraically and determine the values of x for which $-\frac{a}{b}x + \frac{\alpha}{b} = -\frac{c}{d}x + \frac{\beta}{d}$.

Another way to view this situation is through a linear algebra lens where we can write the system of equations in terms of a single function,

$$L \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}.$$

We are then trying to find the values for which $L(x, y) = (0, 0)$. These are the values

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Each of these perspectives has strengths and weaknesses when generalized to different situations. If we generalize to a larger system of equations for which the variables are all linear, $a_1x_1 + a_2x_2 + \dots + a_nx_n = b_n$, then the methods of linear algebra are often most efficient. If we generalize the system of two linear equations to different types of equations with two variables,

$$\begin{aligned} 3x + 2y &= 3 & \text{or} & & x + y &= 8 \\ x^2 + y^2 &= 4 & & & 3 \cdot 2^x &= 0 \end{aligned}$$

then graphical methods are often the more efficient. If we combine these ideas to systems of non-linear equations with more than two variables, numerical analysis techniques are often employed to find numerical approximations to the system of equations.

Related Content Standards

- (8.EE.8) Analyze and solve pairs of simultaneous linear equations.
 - a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
 - b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection.
 - a. Solve real-world and mathematical problems leading to two linear equations in two variables.
- (HSA.REI.11) Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

8.9.4 Exercises

1. Domain and Range.²
 - a. Find the domain and range of the following functions.
 - b. $f(x) = \frac{x^2}{(x^2 + 1)(x + 3)}$
 - c. $f(x) = \sqrt{x^2 - 16}$
 - d. $f(x) = \frac{1}{\sqrt{2x - 5}}$
 - e. $f(x) = \frac{x^2 - 9}{x\sqrt{x^2 - 3x + 2}}$
 - f. Find functions with the following domains.
 1. The set of all real numbers less than or equal to 6
 2. The set of all real numbers between -2 and 2 (including both -2 and 2)
 3. The set of all real numbers between 1 and 4 (including both 1 and 4).
2. Give an algebraic representation for a function with the indicated domain and range.
 - a. domain of \mathbb{R} and range of $\{y \in \mathbb{R} | a \leq y \leq b\}$.
 - b. domain of $\{x \in \mathbb{R} | x > 2\}$ and range of $\{y \in \mathbb{R} | y > 1\}$.
3. Find all solutions of the equation

$$x^2 - 3 - \frac{1}{x^2 - 3} = 0.$$
4. Determine the zeros, vertical asymptotes, and end behavior of the function defined by the given algebraic expression.
 - a. $f(x) = 2 \cdot 3^{x+2}$
 - b. $g(y) = \frac{y^2+1}{y^2-1}$
5. For each of the following pairs of functions, f and g , give the domain, range, and a simplified formula for $f \circ g$ and for $g \circ f$.
 - a. $f(x) = \cos(x)$ and $g(x) = \sin^{-1}(x)$
 - b. $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x^2 - 3x + 2}$
 - c. $f(x) = \ln(x - 3)$ and $g(x) = x^2 + 4$
 - d. $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{1}{x-3}$
6. Find an algebraic representation of a single function, $f(x)$ that has all of the following characteristics.
 - The domain of f is $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$
 - The range of f is $(-\infty, \infty)$
 - $\lim_{x \rightarrow -1^-} f(x) = \infty$
 - $\lim_{x \rightarrow -1^+} f(x) = -\infty$
 - $\lim_{x \rightarrow 4^-} f(x) = -\infty$
 - $\lim_{x \rightarrow 4^+} f(x) = \infty$
 - $f(1) = 0$ and near $x = 1$, $f(x) \approx \frac{3}{10}(x - 1)$
 - $f(-2) = 0$ and near $x = -2$, $f(x) \approx \frac{1}{4}(x + 2)^2$
 - $f(3) = 0$ and near $x = 3$, $f(x) \approx \frac{-5}{4}(x - 3)$
 - As $x \rightarrow \pm\infty$, $f(x) \approx \frac{x^2 + 3x + 4}{10}$

²From *The World of Functions*, part of the Interactive Mathematics Program

7. Consider the functions

$$f(x) = 1 + \sqrt{7 - 2x} \quad \text{and} \quad g(x) = \frac{1}{3 - \sqrt{x+5}}.$$

- Let $h(x) = f(x) \cdot g(x)$. What is the domain and range of h ? (Round any answers to the nearest tenth.)
8. If $f(x) = |x - 1| - 2$, what is the vertex of $y = f(x + 2) - 1$?
9. Let $f(x) = 3\sqrt[4]{x} - \sqrt{x} - 2$. Describe all of the features of the graph of $y = f(x)$ without using technology and then verify with graphing technology.

Part III

GEOMETRY

Chapter 9

Axiomatic Geometry

Chapter 10

Measurement

Chapter 11

Groups and Geometry

Chapter 12

Euclidean Transformational Geometry

Part IV

DATA ANALYSIS

Chapter 13

Data Analysis Foundations

Chapter 14

Exploring Data

Chapter 15

Samples, Simulations, and Probability

Chapter 16

Estimating Parameters and Testing Hypotheses

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