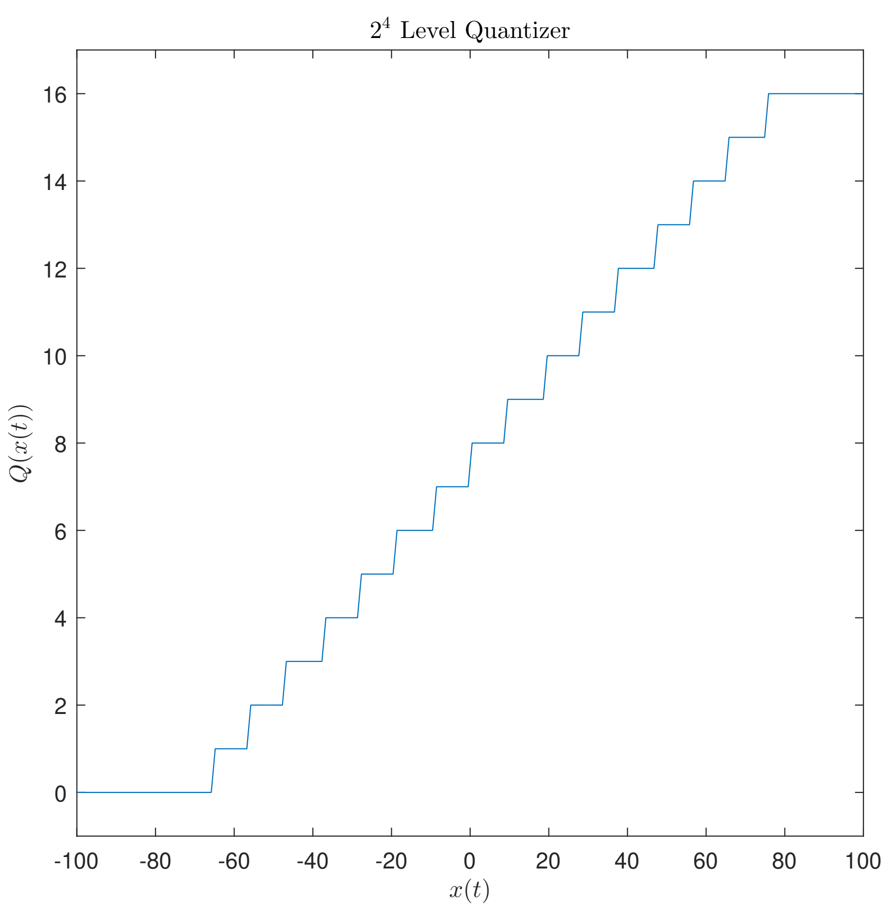
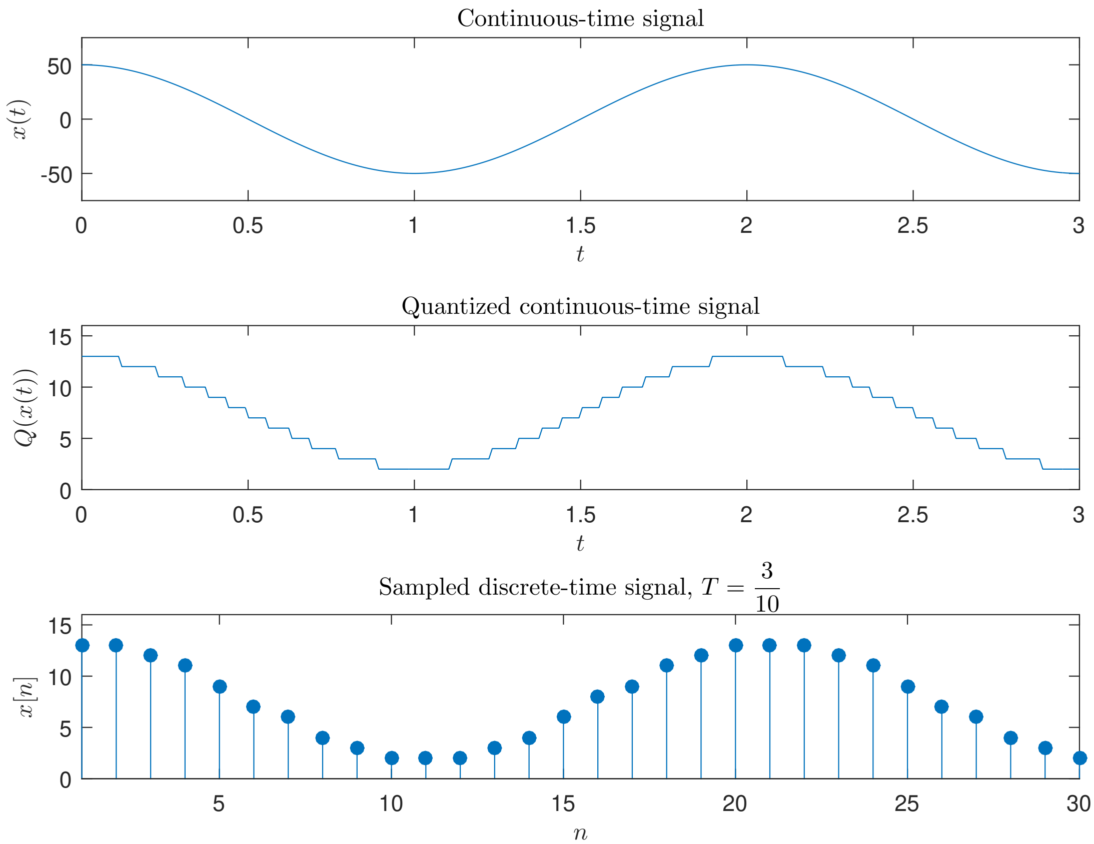
1.5 Chapter 1 Lecture Examples

Example 1.1: 2^4-level Quanitzer

Let's plot $Q(x(t))$ for a 4-bit quantizer. Let's then pass a simple continuous-time signal through the quantizer and sampling process to observe the results.





Example 1.2: Is $x(n) = e^{j\frac{2\pi n}{10}}$ periodic?

$\alpha = \frac{2\pi}{10} = \frac{2\pi\ell}{N}$. Thus, $\frac{\ell}{N} = \frac{1}{10}$. Hence, $\ell = 1$ and $N = 10$ and $x(n)$ is periodic with periodicity 10.

Example 1.3: Is $x(n) = e^{j3n}$ periodic?

$\alpha = 3 = \frac{2\pi\ell}{N}$. Thus, $\frac{\ell}{N} = \frac{3}{2\pi}$. Take $\ell = 3$ and $N = 2\pi$, which is not a rational number and thus $x(n)$ is **not** periodic.

Example 1.4: Is $\cos(n)$ periodic?

To be periodic, $n + 2\pi$ must be an integer value. Because π is irrational, $n + 2\pi \notin \mathbf{Z}$, and $x[n] = cos(n)$ is aperiodic.

Example 1.5: Is $x(n) = e^{j\frac{10\pi n}{3}}$ periodic?

$$\alpha = \frac{10\pi}{3} = \frac{2\pi\ell}{N}$$

Thus, $\ell = 5$ and $N = 3$. Thus, $e^{j\frac{10\pi n}{3}}$ is the 5th harmonic and its periodicity is 3.

Example 1.6:

Let's look at some examples of simple digital systems:

$$\begin{aligned}
y[n] &= \frac{1}{2}(x[n] + x[n-1]) \quad \text{(simple digital low-pass filter)} \\
y[n] &= x[n] - x[n-1] \quad \text{(simple digital high-pass filter)} \\
y[n] &= x^2[n] \quad \text{(nonlinear system)} \\
y[n] &= x[n^2] \quad \text{(time-scaling system)}
\end{aligned}$$

Example 1.7:

Do the following systems have memory?

$y[n] = (x[n])^2$ only depends upon the input at time n so the system is memoryless.

$y[n] = x[n] - x[n-1]$ depends upon the input at both time n and $n-1$ so the system is not memoryless.

Example 1.8:

Is the system $y[n] = \frac{1}{2}(x[n] + x[n-1])$ linear? Check that superposition and homogeneity holds:

1. Superposition:

$$y_1[n] = \frac{1}{2}(x_1[n] + x_1[n-1])$$
$$y_2[n] = \frac{1}{2}(x_2[n] + x_2[n-1])$$

Now, apply $x_3[n] = x_1[n] + x_2[n]$ to the system S and check if $y_3[n] = y_1[n] + y_2[n]$.

$$y_3[n] = \frac{1}{2}(x_1[n] + x_2[n] + x_1[n-1] + x_2[n-1])$$
$$y_3[n] = \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1])$$
$$y_3[n] = y_1[n] + y_2[n]$$

Thus, superposition holds.

2. Homogeneity: Apply $x_4[n] = \alpha x[n]$ and see if $y_4[n]$ is equal to $\alpha y[n]$.

$$y_4[n] = \frac{1}{2}(\alpha x[n] + \alpha x[n-1])$$
$$y_4[n] = \frac{\alpha}{2}(x[n] + x[n-1])$$
$$y_4[n] = \alpha y[n]$$

Thus, homogeneity also holds. Therefore the system is linear.

Example 1.9:

Is the system $y[n] = x^2[n]$ linear?

1. Superposition:

$$y_1[n] = x_1^2[n]$$
$$y_2[n] = x_2^2[n]$$

   Now, apply $x_3[n] = x_1[n] + x_2[n]$ to the system and check if $y_3[n] = y_1[n] + y_2[n]$.

$$y_3[n] = [x_1[n] + x_2[n]]^2 = x_1^2[n] + 2x_1[n]x_2[n] + x_2^2[n] \neq x_1^2[n] + x_2^2[n]$$

   Thus, superposition fails and the system is **not** linear; however, let's also check homogeneity.

2. Homogeneity: Apply $x_4[n] = \alpha x[n]$ and see if $y_4[n]$ is equal to $\alpha y[n]$.

$$y_4[n] = (\alpha x[n])^2$$
$$y_4[n] = \alpha^2 x^2[n]$$
$$y_4[n] = \alpha^2 y[n] \neq \alpha y[n]$$

   Thus, homogeneity also fails and again the system is not linear. Showing either property does not hold is sufficient to show the system is not linear.

Example 1.10:

Is the system $y[n] = x[n] - x[n-1]$ time-invariant?

By our steps outlined above, $y_2[n] = x[(n - n_o)] - x[(n - n_o) - 1]$ and

$$\begin{aligned} y_3[n] &= x[(n - n_o)] - x[(n-1) - n_o] \\ &= x[n - n_o] - x[n - n_o - 1]. \end{aligned}$$

Thus, $y_2[n] = y_3[n]$ and the system is time invariant.

Example 1.11:

Is the system $y[n] = nx[n]$ time-invariant?

Here we have $y_2[n] = (n - n_o)x[(n - n_0)]$ and $y_3[n] = nx[n - n_o]$. Thus, $y_2[n] \neq y_3[n]$ and the system is **not** time invariant.

Example 1.12:

Are the following systems causal?

$y[n] = x[n-1] + x[n]$ is a causal system since all input indices are less than the output index, i.e., $n - 1 \le n$ and $n \le n$.

$y[n] = x[n+1] + x[n]$ is a non-causal system since one of the input indices is greater than the output index, i.e., $n + 1 > n$.

Example 1.13:

Is $y[n] = x[n] - x[n-1]$ stable?

We start by assuming we have a bounded input $x[n]$ and then try to show that the output, $y[n]$, is bounded.
So, suppose that $|x[n]| \le B$. Let's check if $|y[n]| \le B_2$?

$$|y[n]| = |x[n] - x[n-1]| \le |x[n]| + |x[n-1]| \le B + B = 2B$$

Thus, $|y[n]| \le 2B$ and hence $y[n]$ is bounded and the system is BIBO stable.

Example 1.14:

Recall that we showed the system $y[n] = nx[n]$ was not time invariant. Let's show that it is also unstable.

Start with a bounded input $x[n]$ such that $|x[n]| \le B$. Let's try to show that $|y[n]| \le B_2$:

$$|y[n]| = |nx[n]| = |n||x[n]|$$

Pick a specific input, say $x[n] = B$.

$$|y[n]| = |n|B,$$

which is clearly unbounded. Thus, the system is unstable.

Note: To show instability, it is sufficient to come up with a single bounded input for which the corresponding output is unbounded.