Position auctions with endogenous supply

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Abstract

We consider a multi-object private values setting with quantity externalities. In this setting, a value to a bidder from an object may depend on the total number of objects sold. For example, the likelihood a customer will respond to an advertisement is higher the fewer other advertisements are shown; a spectrum license is more valuable the fewer licenses are being allocated. We raise and solve the problem of finding revenue maximizing and efficiently allocating auctions in such a setting. We show that both optimal and efficient auctions have the property that the quantity of objects sold depends non-trivially on the whole profile of players' valuations. That is, the quantity to sell is determined endogenously, within the auction. We demonstrate that auctions currently used for allocating advertising positions are suboptimal and offer simple designs that can implement (or approximate) optimal and efficient auctions under quantity externalities.

Keywords: quantity externalities, optimal auction, efficient auction, multi-product auction, sponsored search.

1 Introduction

In most models of auctions, the quantity of goods for sale is explicitly specified and treated as given. The players' preferences as well as auction mechanisms considered are described relative to specification of goods. Importantly, the analysis of such models is done under the assumption that the valuation of a given buyer for a given good or a subset of goods is fixed and independent from the quantity of goods sold. Of course, the actual quantity of goods sold can be different; for instance, sellers may choose not to sell at prices below certain levels.

Valuations of buyers do depend on the quantity of goods offered for sale in many circumstances. In spectrum auctions, the value of a license to a given buyer clearly depends on the

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total number of licenses allocated, as future stream of profits depends on the structure of competition after the auction. The more firms are present on the market, the lower are the profits. A piece of art, a collectible coin, a bottle of wine are more valuable the rarer they are. In sponsored search auctions, an advertisement position may be more valuable for an advertiser when fewer competing advertisements are shown.

Whenever a seller can choose quantity to offer and this quantity is not too large, a common practical approach is to separate choices of the quantity and of the auction format, selecting the auction format optimally given the quantity, and choosing quantity either based on the expected performance of the selected format or based on other external reasons. This is how governments have been choosing the number of licenses for spectrum auctions and how search engines have been choosing the number of advertising positions on a page with search results.

In this paper we raise and solve the problem of finding revenue maximizing and efficiently allocating auctions in private values settings with quantity externalities. We offer a simple model in which valuations of buyers for different goods are the product of one-dimensional private values and of goods' qualities. The qualities are common for buyers and depend on the quantity of goods for sale. One-dimensional private information assumption allows us to use conventional methods of auction theory to derive the optimal and efficient mechanisms in a straightforward manner. Both the efficient and optimal mechanisms require computing a maximum of several linear functions of actual and virtual valuations, respectively, and so are relatively simple. In the general case, the resulting allocation functions depend non-trivially on the whole profile of buyers' valuations.

The main application for our model and findings are context advertisement auctions. In these auctions, the objects offered for sale are advertising positions (slots) on the screen that are filled depending on the context of a user's activity, e.g. a search query or a visited page. Advertisers are interested in attracting users to their landing pages. Positions have different values to advertisers, as users are more likely to click on ads in higher positions on the screen no matter what actual ads are shown.[1] The value of a given position to a given advertiser is then a product of the advertiser-specific expected value generated following a click on the ad and the position-specific CTR (click through rate) — a probability that a user clicks on an the ad at that position. We allow these CTRs to vary depending on the total number of positions offered and differently for different positions.[2] Online platforms and especially search engines generate most of their revenue from context advertisement, and so finding the optimal and efficient selling mechanisms are important practical problems.

While maximizing revenue is a natural goal for a for-profit search engine, any search engine faces competition from other platforms for users and advertisers, and therefore has incentives to generate value to its customers. Thus, a likely practical solution lies somewhere in between the optimal and efficient mechanisms.

The industry standard for sponsored search allocative mechanisms has been the gener-

[1] Arkhangelsky, Izmalkov & Khakimova (2013) conduct an experiment at Yandex to evaluate CTRs of the top three advertising positions free from effects of various selection biases and show that the estimated effect for some 300 clusters of keywords is about 80% reduction in the CTR for the second position relative to the first one and for the third position relative to the second one.

[2] Jeziorski & Segal (2015) report that the top ad would receive up to 81% more clicks in a hypothetical world in which it faced no competition.

alized second price auction (GSP), first studied by Varian (2007) and Edelman, Ostrovsky & Schwarz (2007). In the simplest version of this auction, the advertisers submit bids. The advertising positions are filled according to the order of bids: the ad with the highest bid gets the top slot, the ad with the second-highest bid gets the next slot, and so on. Advertisers pay only when a user clicks on their ad, and the price for slot K is the $(K+1)$-st highest bid. Edelman et al. (2007) show that GSP has an equilibrium in which advertisements are allocated efficiently and payments are equal in expectation to those of Vickrey-Clark-Groves (VCG) mechanism. Edelman & Schwarz (2010) show that in a symmetric private values setting without quantity externalities , the optimal mechanism is the GSP auction (or the VCG mechanism, or the generalized English auction) with a reserve price that is independent of the number of bidders and CTRs.

We show that with quantity externalities, the GSP auction and the VCG auction for a fixed quantity of goods for sale are no longer efficient, nor they are revenue-maximizing even with optimally chosen reserve price. One specific reason is that in the efficient and revenue-maximizing mechanisms, the price an advertiser pays for a click at a given position may depend on bids of advertisers at higher positions, whereas this is not the case in GSP or VCG auction for a fixed quantity of goods for sale with or without reserve prices. We show losses in efficiency and revenue for the currently used GSP format by a numerical example.

In practice, search engine companies have considered flexible rules for the number of ads shown, in particular for exclusive display – when only one ad is shown on the page.[3] Theoretical literature explored the designs of auctions that require multi-dimensional bids to let buyers indicate their valuation for exclusive display (Jerath & Sayedi (2011), Deng, Lopomo & Pekeč (2012), Deng & Pekeč (2012), Ghosh & Sayedi (2010)) or the value discount due to possible competitors (Muthukrishnan (2009)). With extra strategic possibilities stemming from multiple bids, such auctions quickly become intractable. In contrast, we show that with one-dimensional private information, both efficient and revenue-maximizing auctions can be found and implemented (or approximated) by asking each buyer to submit a single bid only.

Aseff & Chade (2008) study a problem of finding an optimal auction in the presence of identity specific externalities, when the size of externality bidder i imposes on bidder j depends on pair (i, j). We have a simpler model with object-specific externalities.

There is also a relevant IO literature on online advertising. Jeziorski & Segal (2015) build a dynamic model of utility-maximizing users and find large negative externalities between ads. Athey & Ellison (2011) and Chen & He (2011) model consumers' costly search decisions and provide justification for a top-down search pattern, resulting in lower CTRs for lower positions.

Several papers study the strategic possibility for the seller to vary supply in the uniform-price auctions to affect equilibrium bidding behavior or eliminate some equilibria. Back & Zender (2001) and McAdams (2007) show that the seller will receive higher bids retaining the right to adjust the total quantity sold after bidding. Both papers focus only on uniform price auctions with almost complete information (bidders have constant marginal value v for all goods and v is commonly known to the bidders but not to the seller). Bidders simultaneously submit demand schedules. McAdams (2007) proves that in all equilibria, the outcome is always efficient and the seller always extracts all bidder surplus (price is

[3]See U.S. Patents 20110071908 and 20110071909.

v). In these papers, flexibility in supply is used only as a strategic tool to restrict bidders' behavior, and helps the seller to avoid adverse outcomes. Buyers' preferences do not depend on the total supply. We instead consider environments in which the total supply enters the preferences directly.

Our paper is organized as follows. In Section 2, we present a model of position-specific and quantity externalities. We present the efficient mechanism in Section 3. In Section 4, we find the revenue-maximizing mechanism. In Section 5, we compare the existing GSP auction format with the efficient and optimal mechanisms we derive in terms of efficiency and revenue, respectively. Section 6 concludes.

2 The model

We consider a multi-object independent private values setting. There are K indivisible goods available for sale to N buyers, who demand or can possess at most one good each. The value of a good to a buyer is formed from three sources: (i) a buyer-specific component; (ii) the relative ranking of the good among the other goods; and (iii) the total number of goods offered for sale. To have a simple and tractable model, we assume that the relative ranking and the overall supply effects are common for the buyers. Altogether, the value of buyer i from obtaining good j when $k \leqslant K$ goods are allocated is

$$V_i^{j:k} = \alpha^{j:k} v_i,$$

where $v_i \in \mathbb{R}_+$ is private information of buyer i, independently drawn from cdf $F(v)$ with pdf $f(v) > 0$ on $v \in [0, v^{max}]$; and $\alpha^{j:k}$ are quality multiples commonly known to buyers and the seller, for all $1 \leqslant j \leqslant k \leqslant K$,

$$\alpha^{j:k} \geqslant 0.$$

Goods are naturally ranked: for all $1 \leqslant j < k \leqslant K$,

$$\alpha^{j:k} \geqslant \alpha^{j+1:k},$$

that is, a good with a lower index is ranked higher. Set $\alpha^{\ell:k} = 0$ for $k < \ell \leqslant K$, and denote by $\mathbf{a}_k = \{\alpha^{1:k}, \alpha^{2:k}, \dots, \alpha^{K:k}\}$ the vector of qualities of all the goods when k goods are offered for sale. The following table illustrates our setup and notation for $K = 4$.

	# of goods for sale			
Quality	1	2	3	4
1st good	$\alpha^{1:1}$	$\alpha^{1:2}$	$\alpha^{1:3}$	$\alpha^{1:4}$
2d good	0	$\alpha^{2:2}$	$\alpha^{2:3}$	$\alpha^{2:4}$
3d good	0	0	$\alpha^{3:3}$	$\alpha^{3:4}$
4th good	0	0	0	$\alpha^{4:4}$

Table 1: Qualities of goods depending on the total supply

Note that in the general setting we do not require that $\alpha^{j:k} \geqslant \alpha^{j:k+1}$ for $j \leqslant k$. That is, we allow for the possibility that the quality of good j may actually increase when more goods are sold.

The reservation value to the seller for each good is normalized to 0.

2.1 Context advertisement auctions

In the world of context advertisement, the seller is the company that provides the advertising service, e.g. a search engine; the goods are advertising positions to be shown to a particular user; and buyers are the companies that want to advertise to this user. Private value v_i to advertiser i is the expected value the advertiser receives from the user who responds (*clicks*) on the ad, and $\alpha^{j:k}$ is the *click-through-rate (CTR)*: the probability a user responds to any ad if it is shown in position j out of k ads shown. Without the total supply effects – that is, when the CTR of any position is independent of the number of positions shown – we have $\alpha^{j:k} = \alpha^{j:k'}$ for all $j \leqslant k < k'$, and the model reduces to the classic model of Edelman et al. (2007).

In practice, the main format according to which the ads are allocated is the generalized second price (GSP) auction. In it, the advertisers submit bids for a specific keyword, e.g. "flowers flint michigan." If such a keyword gets associated with an action by some user, for instance when a user from Flint, Michigan searchers for flowers, an auction is run among all advertisers who bid on this keyword. The bids are ranked from high to low, $b_{(1)} \geqslant b_{(2)} \geqslant \ldots$. Advertising slots are ranked according to their CTRs, and if k advertising slots are filled, they are assigned assortatively according to the order of bids. In what follows, bidder (ℓ) denotes the bidder shown in position ℓ for all $\ell \leqslant k$. If a user clicks on an ad in position ℓ, bidder (ℓ) pays the next highest bid, $p_\ell = b_{(\ell+1)}$.[4]

Without quantity effects, Edelman et al. (2007) show that the GSP auction admits an efficient equilibrium equivalent to the dominant strategy efficient equilibrium of the Vickrey-Clark-Groves (VCG) mechanism: the advertisers are allocated according to the order of their private valuations. In a symmetric setup, Edelman & Schwarz (2010) show that the GSP auction with appropriately chosen reserve price is also the optimal mechanism. We will argue that with quantity effects, the GSP auction is neither efficient nor optimal.

In what follows, given that context advertisement auctions are our main application, we will sometimes refer to the seller as a search engine, goods as positions or slots, buyers as advertisers, and quality effects $\alpha^{j:k}$ as CTRs.

Given a vector of reals $y_1, \ldots, y_N$, we denote by $y_{(1)}, \ldots, y_{(N)}$ the decreasing sequence of these numbers with ties broken randomly, so that $y_{(i)} \geqslant y_{(\ell)}$ for any $1 \leqslant i < \ell \leqslant N$.

[4] Clearly, different advertisers may differ in their individual CTRs, as some ads may be more effective or more relevant. To account for individual effects, search engines modify GSP to rank bids based on $b_i w_i$, where w_i is a bidder i's ad quality or relevance: bidder (ℓ) will pay per click the minimal bid it can submit and still be allocated in position ℓ. An immediate example of such a weighting function is $w_i = CTR_i$, where CTR_i is a normalized (say at position 1) click through rate of advertiser i. By considering "flows" of values, that is, by using $v_i w_i$ in place of v_i, the individual effects can be incorporated into our analysis.

3 The efficient auction

In this Section we present an efficient mechanism allocating multiple goods in the presence of quantity externalities, specifically allowing for the quantity of goods to be sold to be determined within the mechanism.

The construction is straightforward. It is the VCG mechanism reflecting the specifics of the environment.

Theorem 1. *The following mechanism has a dominant strategy equilibrium in which all the buyers reports their valuations truthfully, and the resulting allocation is efficient.*

The seller

- *asks each buyer i to submit a bid (report her value) $z_i \geqslant 0$,*
- *based on the submitted vector of bids $\mathbf{z} = (z_1, \ldots, z_n)$, allocates k^* positions to k^* top bidders assortatively, where k^* is chosen to maximize the total buyer surplus*

$$k^* = \underset{k \leqslant K}{\operatorname{argmax}} \, TS(\mathbf{z}, k),$$

$$TS(\mathbf{z}, k) = \sum_{j=1}^{k} \alpha^{j:k} z_{(j)},$$

$$TS(\mathbf{z}) = \max_{k \leqslant K} TS(\mathbf{z}, k),$$

$$z_{(1)} \geqslant \cdots \geqslant z_{(n)}.$$

- *each winner, $(i) \leqslant k^*(\mathbf{z})$, pays the externality she imposes on the other bidders. Thus,*

$$p_i(\mathbf{z}) = TS(\mathbf{z}_{-(i)}) - TS_{-(i)}(\mathbf{z}),$$

  where $\mathbf{z}_{-(i)}$ is an ordered vector of bids excluding $z_{(i)}$, and $TS_{-(i)}$ is a surplus of everyone but buyer (i).[5]

Note that in general, the price for buyer (i) may depend also on the bids of the buyers ranked above him. Indeed, if the optimal quantity of goods allocated is the same when buyer (i) is bidding and when she is not, $k^*(\mathbf{z}_{-(i)}) = k^*(\mathbf{z})$, then $p_{(i)}$ depends only on the bids of the buyers ranked below him,

$$p_i(\mathbf{z}) = \sum_{j=i}^{k} \left(\alpha^{j:k} - \alpha^{j+1:k}\right) z_{(j+1)}.$$

However, if $k^*(\mathbf{z}_{-(i)})$ differs from $k^*(\mathbf{z})$, then, generically, $p_{(i)}$ is affected by changes in qualities of goods allocated to the buyers ranked above (i). For instance, this happens when two goods are sold when buyer (2) is bidding and only one good is sold when buyer (2) is not present. Then,

$$p_2 = (\alpha^{1:1} - \alpha^{1:2}) z_{(1)}.$$

[5]For context advertisement auctions, to obtain prices per click one has to divide these prices by CTRs.

This observation immediately implies that the GSP auction is not efficient under quantity effects, as its prices for each position depend only on the lower bids. In Section 5 ,we present simulations that provide welfare comparison of the GSP auction to the efficient mechanism.

Computing and implementing the efficient mechanism is not difficult. Essentially, the search engine has to compare several linear functions, both to determine the overall supply and the prices. In the context advertisement auctions, the search engine knows position CTRs, $\alpha^{j:k}$, and so an efficient mechanism can be easily implemented.

3.1 An example: two goods

To illustrate the efficient mechanism and later the optimal and other mechanisms, we consider a setting in which two goods for are offered for sale, $K = 2$, in greater detail. It is convenient to use a slightly different notation. Let us normalize the quality of the first good when the two goods are allocated, $\alpha^{1:2} = 1$; denote $\alpha = \alpha^{2:2}$ and $\beta = \alpha^{1:1} - 1$ (so that $\alpha^{1:1} = 1 + \beta$). Thus, β is the premium for exclusivity: the quality of the first good increases by β if it is the only good sold, and $\alpha \leqslant 1$ stands for the relative quality of the second good compared to the first one.

There are three possible allocation outcomes (with special references for context advertisement in brackets):

- "1" – one good is allocated (exclusive display, ED),
- "2" – two goods are allocated (multiple display, MD), and
- "0" – no goods are allocated (no display, ND).

Clearly, given vector of submitted bids $\mathbf{z}$, the efficient allocation rule is:

- "1", if $(1+\beta)z_{(1)} \geqslant z_{(1)} + \alpha z_{(2)}$, or when $\beta z_{(1)} \geqslant \alpha z_{(2)}$;
- "2", otherwise.

Therefore, the highest bidder is the only one obtaining the good if her bid is at least $\frac{\alpha}{\beta}$ times larger than the second highest bid. Conversely, if the two top bids are close to each other, then both goods are allocated. Note that if $\beta < 0$, then both goods are allocated, and if $\beta \geqslant \alpha$, then only one good is allocated, no matter what the bids are. For graphical illustration, see Figure 1.

4 The revenue-maximizing auction

In this Section we characterize the mechanism maximizing the expected revenue to the seller. The benefit of the model we are considering is that we can follow Myerson (1981) approach to characterize incentive constraints and derive the optimal mechanism.

So, by the Revelation Principle, without loss of any generality, we can look for a direct revelation mechanism that induces truth telling. Consider a direct mechanism $(\mathbf{Q}, \mathbf{M})$ where $\mathbf{Q}$ is the allocation rule and $\mathbf{M}$ is the payment rule. An allocation rule $\mathbf{Q} = \{Q_i^{j:k}(\mathbf{z})\}$ is a collection of functions where $Q_i^{j:k}(\mathbf{z}) \in [0,1]$ is the joint probability that bidder i obtains

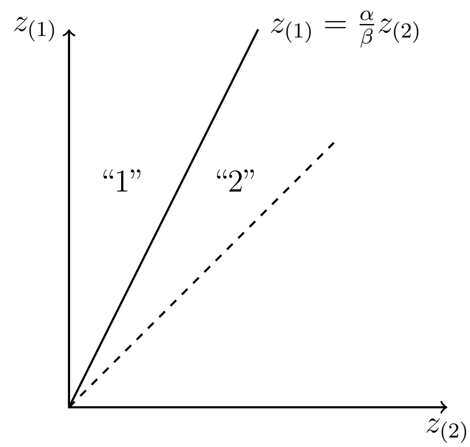

Figure 1: The efficient allocation rule. "1" – one good is allocated (exclusive display), "2" – two goods are allocated (multiple display).

good j and the total number of goods allocated is k. We will refer to each such function as an allocation indicator. Naturally, for all i, j, k, we should have $Q_i^{j:k} \in [0,1]$. The allocation indicators are consistent if they satisfy two ex post allocation requirements:

(i) each buyer i obtains no more than one good, $\sum_{j,k} Q_i^{j:k}(\mathbf{z}) \leqslant 1$ for all i, and

(ii) for each good $j \leqslant K$, $\sum_{i,k} Q_i^{j:k}(\mathbf{z}) \leqslant 1$.

Based on allocation indicators, we can define the *ex post CTR* for bidder i as

$$Q_i(\mathbf{z}) = \sum_{j,k} \alpha^{j:k} Q_i^{j:k}(\mathbf{z}). \tag{1}$$

Then, as in Myerson (1981), we can define the *interim expected CTR* for buyer i as

$$q_i(z_i) = \int Q_i(\mathbf{z}) d\mathbf{F}_{-i}(\mathbf{z}_{-i}). \tag{2}$$

The payment rule is given by the collection of payment functions $\mathbf{M} = \{M_i^{j:k}(\mathbf{z})\}$. Similarly to the allocation functions, we can define the *interim expected payment* of buyer i as

$$m_i(z_i) = \int \sum_{j,k} \alpha^{j:k} M_i^{j:k}(\mathbf{z}) d\mathbf{F}_{-i}(\mathbf{z}_{-i}).$$

Note that with so defined $q_i(z_i)$ and $m_i(z_i)$ we can express and characterize the incentive constraints in the exactly same manner as for the single good auction in Myerson (1981). It is convenient to note immediately that the revenue optimization and individual rationality would imply that $m_i(0) = 0$ for all i. Then, we can state

Lemma 1. $(\mathbf{Q}, \mathbf{M})$ *is incentive compatible if and only if for all buyers i*

(i) interim expected CTRs $q_i(z_i)$ are nondecreasing, and

(ii) interim expected payments satisfy

$$m_i(v_i) = q_i(v_i)v_i - \int_0^{v_i} q_i(t_i)dt_i.$$

The proof is a standard application of the envelope theorem, see e.g. Krishna (2009).

For each i define *virtual valuation function* of buyer i as $\psi(v) = v - \frac{1-F(v)}{f(v)}$. We assume that virtual valuation functions are regular, that is, $\psi(v)$ is non-decreasing, solely for simplicity of exposition.[6]

Then, by integrating over interim expected payments, we obtain

Lemma 2. *The incentive compatible direct revelation mechanism with allocation rule* $\mathbf{Q}$ *generates revenue*

$$R(\mathbf{Q}) = \int \sum_i \psi(z_i)Q_i(\mathbf{z})f(\mathbf{z})d\mathbf{z}. \tag{3}$$

Now we can state

Theorem 2. *The optimal direct revelation mechanism* $(\mathbf{Q}, \mathbf{M})$ *is defined as follows.*

The seller

- *asks each buyer i to submit a bid (report her value) $z_i \geqslant 0$,*
- *for each vector of submitted values $\mathbf{z}$, computes ordered virtual valuations $\psi_j = \psi(z_{(j)})$ for all $j = 1..N$, set $\psi_0 = \infty$ and $\psi_{N+1} = -\infty$.*
- *finds the optimal quantity to sell: determines expected revenues for each $0 \leqslant k \leqslant K$, optimizes over k, selecting minimal number of goods to sell in case of a tie,*

$$R(0) = 0, \quad R(k) = \sum_{j=1}^{k} \psi_j \alpha^{j:k}, \quad k^*(\mathbf{z}) = \min \operatorname*{argmax}_{0 \leqslant k \leqslant K} R(k),$$

- *computes the allocation functions:*

  If $k^*(\mathbf{z}) = 0$, set $Q_i^{j:k}(\mathbf{z}) = 0$ for all i, j, k,

  If $k^*(\mathbf{z}) > 0$, then

  (i) no ties: if $\psi_j > \psi_\ell$ for all $1 \leqslant j < \ell \leqslant k^* + 1$,
  set $Q_{(j)}^{j:k^*}(\mathbf{z}) = 1$ for all $1 \leqslant j \leqslant k^*$ and $Q_i^{j:k}(\mathbf{z}) = 0$ for all other i, j, k,
  (that is, allocate k^* goods to the buyers with k^* highest virtual valuations assortatively)

[6] If for some buyer i $\psi(v)$ is not regular, then an "ironed out" non-decreasing quasi-virtual valuation function $\psi^*(v)$ has to be computed first in the exactly same way as in Myerson (1981). This quasi-virtual valuation function is then to be used in all the statements and derivations below.

(ii) ties correction:[7] *if there exist* j_1, j_2, *such that* $1 \leqslant j_1 \leqslant k^*$, $j_1 < j_2$, $\psi_{j_1 - 1} > \psi_{j_1}$, $\psi_{j_1} = \psi_{j_2}$, *and* $\psi_{j_2} > \psi_{j_2+1}$,

modify $Q_i^{j:k}(\mathbf{z})$ *from (i) as follows: for any* j, ℓ *such that* $j_1 \leqslant j \leqslant j_2$ *and* $j_1 \leqslant \ell \leqslant \min\{k^*, j2\}$, *set* $Q_{(j)}^{\ell:k^*}(\mathbf{z}) = \frac{1}{j_2 - j_1 + 1}$,

(that is, allocate goods from j_1 *to* $\min\{k^*, j_2\}$ *randomly among those whose virtual valuations are tied and are ranked from* j_1 *to* j_2*)*

- *compute the payment functions: for each* i, j, k,

$$M_i^{j:k}(\mathbf{z}) = Q_i^{j:k}(\mathbf{z}) z_i - \int_0^{z_i} Q_i^{j:k}(t_i, \mathbf{z}_{-i}) dt_i.$$

Therefore, the search for the optimal mechanism can be done sequentially: find the optimal allocation of k goods conditionally on all k goods being allocated, and then optimize over k. And the overall optimization problem given the vector of reported bids $\mathbf{z}$ and corresponding virtual values is simple: find a maximum of K linear functions.

Proof. The described mechanism amounts to a point-wise (for each $\mathbf{z}$) optimization of revenue. The only thing that remains to be proven is incentive compatibility of the obtained solution. To show this, by Lemma 1 it suffices to show that interim expected quantities $q_i(z_i)$ are nondecreasing. We will show that effective quantities $Q_i(z_i, \mathbf{z}_{-i})$ are nondecreasing in z_i, from which monotonicity of interim expected quantities trivially follows.

To see what happens when z_i increases, let us trace how the optimal allocation changes when $\psi_i(z_i)$ increases. Two kinds of changes can happen. The first one is that $\psi_i(z_i)$ gets tied and then gets higher in the ranking of virtual values while the optimal quantity k^* remains the same. At a tie, find j_1 and j_2 as in (ii) of the statement of the Theorem, such that $\psi_{j_1} = \psi_{j_2} = \psi_i(\hat{z}_i)$. Then, $Q_i(\hat{z}_i^-, \mathbf{z}_{-i}) = \alpha^{j_2:k^*}$, $Q_i(\hat{z}_i, \mathbf{z}_{-i}) = \sum_{j=j_1}^{j_2} \alpha^{j:k^*} \frac{1}{j_2 - j_1 + 1}$, and $Q_i(\hat{z}_i^+, \mathbf{z}_{-i}) = \alpha^{j_1:k^*}$, where z^- (and z^+) stand for any value lower (higher) than z and such that corresponding virtual valuations are sufficiently close. Clearly $Q_i(z_i, \mathbf{z}_{-i})$ is nondecreasing at around $\hat{z}_i$. (Note that if $j_1 > k^*$ then buyer i is not allocated any goods at z_i below and above $\hat{z}_i$; if $j_2 < k^*$, then buyer i gets allocated a better good, and in the other cases, buyer i passes the threshold at which he gets allocated a good.)

The second possibility is when at some $\hat{z}_i$, the optimal quantity changes, say from k^* to $\hat{k}$, while the rank of the virtual value of buyer i stays the same. Let ℓ be this rank, $\psi_\ell = \psi_i(\hat{z}_i)$. It must be that $R^{k^*}(\hat{z}_i, \mathbf{z}) = R^{\hat{k}}(\hat{z}_i, \mathbf{z})$, and that $R^{\hat{k}}$ increases faster than R^{k^*} in $\psi_i(z_i)$. This means that $Q_i(\hat{z}_i^-, \mathbf{z}_{-i}) = \alpha^{\ell:k^*} < Q_i(\hat{z}_i^+, \mathbf{z}_{-i}) = \alpha^{\ell:\hat{k}}$. If it happens that at some $\hat{z}_i$ both changes occur simultaneously, they can be treated as a combination, and so we obtain that $Q_i(z_i, \mathbf{z}_{-i})$ is non-decreasing in z_i.[8]

[7] The way we deal with ties maintains symmetry (anonymity) among buyers and ensures that when virtual valuations are not regular, each of the types with equal virtual valuations has the same probability of winning each good.

[8] There is a peculiar possibility when at $\hat{z}_i^-$ and $\hat{z}_i^+$ the optimal quantity chosen is k^*, while at $\hat{z}_i$ it is $\hat{k} \neq k^*$. Necessarily virtual values are tied at $(\hat{z}_i, \mathbf{z}_{-i})$. Let j_1 and j_2 be the lowest and the highest tied with i indices of ranked virtual values. As $R^{k^*}(z_i, \mathbf{z}_{-i}) \geqslant R^{\hat{k}}(z_i, \mathbf{z}_{-i})$ at z_i just below and just above $\hat{z}_i$, while they are equal at $\hat{z}_i$, we must have $\alpha^{j_2:k^*} \leqslant \alpha^{j_2:\hat{k}}$ and $\alpha^{j_1:k^*} \geqslant \alpha^{j_1:\hat{k}}$. Since $Q_i(\hat{z}_i^-, \mathbf{z}_{-i}) = \alpha^{j_2:k^*}$, $Q_i(\hat{z}_i^+, \mathbf{z}_{-i}) = \alpha^{j_1:k^*}$, and $\alpha^{j_2:\hat{k}} \leqslant Q_i(\hat{z}_i, \mathbf{z}_{-i}) \leqslant \alpha^{j_1:\hat{k}}$ we have that $Q_i(z_i, \mathbf{z}_{-i})$ is increasing at $\hat{z}_i$.

Clearly, the interim expected payments computed from $\mathbf{M}$ are as in Lemma 1. $\square$

Note that for the optimal multi-object auction with quantity externality in contrast to the single-good optimal auction, it is possible that a buyer with a negative virtual valuation is allocated a good. This may happen if there is a penalty for exclusivity, when quantity effects from an extra good being assigned on the other winners exceed the loss from selling to this buyer exclusively.

If we impose an extra property of *exclusivity* being *valuable*, $\alpha^{j:k} \geqslant \alpha^{j:k+1}$ for all $j \leqslant k$, we can say a little bit more.

Proposition 1. *Suppose exclusivity is valuable. Given a vector of reports $\mathbf{z}_{-i}$, let $z_{\min}$ be the lowest value of z_i (possibly 0), for which a good is allocated to buyer i.*

Then, the number of goods allocated in the optimal mechanism, $k^*(z_i, \mathbf{z}_{-i})$ is non-increasing in z_i on $z_i > z_{\min}$.

Proof. The result easily follows by observing that if $\psi_i(z_i) = \psi_\ell$, then R^{k_1} grows faster in z_i than R^{k_2} for any $\ell \leqslant k_1 < k_2$ as $\alpha^{\ell:k_1} \geqslant \alpha^{\ell:k_2}$. Thus, $k^*(z_i, \mathbf{z}_{-i})$ can only decrease. $\square$

Exclusivity is likely to be valuable in practice. In context advertisement auctions, it is reasonable to expect that the fewer advertising positions shown, the more likely a user is to make a purchase or to act in some other way valuable to the advertiser of a shown ad, since the user is less distracted by other ads. For instance, Jeziorski & Segal (2015) report that the top ad would receive 81% more clicks in a hypothetical world in which it faced no competition.

Proposition 2. *Suppose exclusivity is valuable. The optimal mechanism has a reservation valuation independent of goods' qualities and the number of buyers: for all i, if the report of buyer i is below $\underline{z} = \psi^{-1}(0)$, then no goods are allocated to buyer i no matter what $\mathbf{z}_{-i}$ is.*

For context advertisement auctions, with prices measured per click, this reservation valuation becomes reservation price.

The result readily follows from Eq. (3).

4.1 An example: two goods

Here we maintain the special notation for the two goods that was introduced in Section 3.1.

Given the vector of submitted values $\mathbf{z}$ and corresponding ranked virtual values, $\psi_j = \psi(z_{(j)})$ for all $j = 1..N$, the allocation function of the optimal mechanism is:

- "1" (exclusive display), if $(1+\beta)\psi_1 \geqslant \psi_1 + \alpha\psi_2$ and $\psi_1 > 0$,
- "2" (multiple display), if $(1+\beta)\psi_1 < \psi_1 + \alpha\psi_2$ and $\psi_1 + \alpha\psi_2 > 0$,
- "0" (no display), otherwise.

Note that if $\beta \geqslant \alpha$, the exclusive display dominates multiple display in terms of revenue. Thus, in this case, the optimal mechanism is a second-price auction which sells one good only.

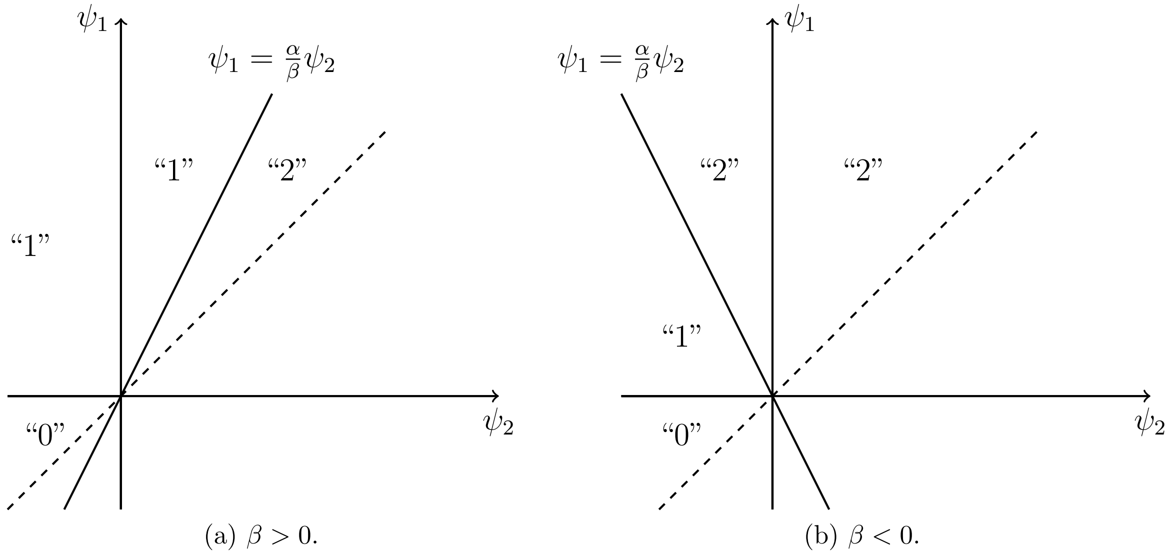

Figure 2: Optimal allocation rule. "1" – one good is allocated (exclusive display), "2" – two goods are allocated (multiple display), "0" – no goods are allocated (no display).

The allocation rule of the optimal mechanism is illustrated in Figure 2 for the two cases of $\beta > 0$ and $-1 < \beta < 0$. Note that when $\beta < 0$, the second good can be allocated to a buyer with negative virtual value.

Now let us describe the optimal mechanism as a function of reported bids. Here we suppose that the setting is symmetric, that is, private signals of the bidders are drawn from the same distribution, and focus on the case when exclusivity is valuable but not too valuable, $\beta \geqslant 0$ and $\beta < \alpha$.

Define the *exclusive sale* function as

$$h(y) := \psi^{-1}\left(\max\{0, \frac{\alpha}{\beta}\psi(y)\}\right). \tag{4}$$

It specifies the lowest value that needs to be reported to obtain the good exclusively, given that the highest of the opponents' reports is y.

The optimal allocation rule in the space of reports is presented in Figure 3. Reservation value $\underline{z} = \psi^{-1}(0)$.

From the perspective of buyer i, the ex post CTR and payment functions of the optimal mechanism (ignoring ties) are:

$$Q_i(z_i, \mathbf{z}_{-i}) = \begin{cases} 1+\beta, & \text{if } z_i = z_{(1)} > h(z_{(2)}), \\ 1, & \text{if } z_i = z_{(1)} < h(z_{(2)}) \text{ and } z_{(2)} > \underline{z}, \\ \alpha, & \text{if } z_i = z_{(2)} > \underline{z} \text{ and } z_{(1)} < h(z_i), \\ 0, & \text{otherwise.} \end{cases}$$

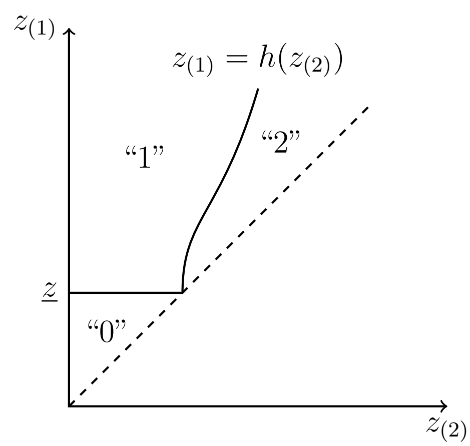

Figure 3: Optimal allocation rule, $\beta > 0$. “1” – one good is allocated (exclusive display), “2” – two goods are allocated (multiple display), “0” – no goods are allocated (no display).

$$M_i(z_i, \mathbf{z}_{-i}) = \begin{cases} \beta h(z_{(2)}) + (1-\alpha) z_{(2)} + \alpha y_2, & \text{if } z_i = z_{(1)},\ z_{(2)} > \underline{z},\ \text{and } z_i > h(z_{(2)}), \\ (1+\beta)\underline{z}, & \text{if } z_i = z_{(1)} > \underline{z} \text{ and } z_{(2)} < \underline{z}, \\ (1-\alpha) z_{(2)} + \alpha y_2, & \text{if } z_i = z_{(1)},\ z_{(2)} > \underline{z},\ \text{and } z_i < h(z_{(2)}), \\ \alpha y_2, & \text{if } z_i = z_{(2)} > \underline{z} \text{ and } z_{(1)} < h(z_i), \\ 0, & \text{otherwise,} \end{cases}$$

where $y_2 = \max\{\underline{z}, z_{(3)}, h^{-1}(z_{(1)})\}$ is the lowest report needed for buyer i to obtain the second good out of the two goods allocated. See Figure 3 for illustration of the optimal allocation rule.

The price per unit, or the price-per-click, is then

$$\frac{M_i(z_i, \mathbf{z}_{-i})}{Q_i(z_i, \mathbf{z}_{-i})} = \begin{cases} \frac{\beta}{1+\beta} h(z_{(2)}) + \frac{(1-\alpha)}{1+\beta} z_{(2)} + \frac{\alpha}{1+\beta} y_2, & \text{if } z_i = z_{(1)},\ z_{(2)} < \underline{z},\ \text{and } z_i > h(z_{(2)}), \\ \underline{z}, & \text{if } z_i = z_{(1)} > \underline{z} \text{ and } z_{(2)} < \underline{z}, \\ (1-\alpha) z_{(2)} + \alpha y_2, & \text{if } z_i = z_{(1)},\ z_{(2)} > \underline{z},\ \text{and } z_i < h(z_{(2)}), \\ y_2, & \text{if } z_i = z_{(2)} > \underline{z} \text{ and } z_{(1)} < h(z_i), \\ 0, & \text{otherwise.} \end{cases}$$

4.2 Practical implementation of the optimal mechanism

When the seller has a single good for sale, the optimal mechanism can be implemented by any common auction form with an appropriately chosen reserve price (Myerson (1981), Riley & Samuelson (1981)). When multiple objects without quantity externalities are offered for sale, the VCG or GSP auctions with the reserve price are optimal (Edelman & Schwarz (2010)). To run these auctions, the seller needs only to determine the optimal reserve price.

With quantity externalities, even with only two goods for sale, neither the VCG auction which ignores quantity externalities, nor the GSP auction with the reserve price is optimal, since, as we have shown, the optimal auction additionally requires the exclusive sale rule (based on function $h(\cdot)$). In addition, $h(\cdot)$ depends on the distributions of valuations in a non-trivial way. Thus, to run the optimal mechanism, the seller would need to recover the distribution of values, the corresponding virtual values, compute $h(\cdot)$ and, generally, have auction rules that depend heavily on these computations. Clearly, this is undesirable and impractical.

The possible practical solution is to approximate $h(\cdot)$, say by a piecewise linear function with two parts, a flat and a strictly increasing one. Then, only a couple of parameters, the level of the first piece and the slope of the second one, need to be evaluated by the seller, similarly to the one parameter (the reserve price) to be determined for settings without quantity externalities.

Examples below provide some ideas of how it can be done. The piecewise linear approximation for $h(\cdot)$ is exact for the uniform distribution of values. Additionally, we find that in the practically important case of log-normal distribution of values (see below), $h(\cdot)$ is approximately linear.

4.2.1 An example: uniformly distributed values

Suppose that $v_i \in [0, 1]$ for all i and $F(x) = x$. Then $\psi(x) = 2x - 1$, $\psi^{-1}(x) = \frac{x+1}{2}$, and the exclusive sale function is:

$$h(y) = \max\left\{\frac{1}{2}, \frac{1}{2}\left(1 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta}y\right\}.$$

Consider only the non-trivial case when there is a competitor above the reserve, $z_{(2)} > \underline{z} = \frac{1}{2}$.

The rule for the exclusive sale becomes

$$2z_{(1)} - 1 > \frac{\alpha}{\beta}(2z_{(2)} - 1), \quad \text{or} \quad \frac{z_{(1)} - \underline{z}}{z_{(2)} - \underline{z}} > \frac{\alpha}{\beta}. \tag{5}$$

Thus the rule for exclusive display says: the top bidder is allocated the exclusive display if his bid exceeds the reservation price more than α/β times the second highest bid exceeds the reservation price. Note that ratio $\frac{\alpha}{\beta}$ does not depend on the distribution of values but only on the quality effects.

For the context advertisement auctions, search engines can compute reserve prices for keyword markets and also estimate CTRs of the individual positions. Therefore, rule (5) is readily implementable.

For an arbitrary distribution, $h(\cdot)$ is not piecewise linear, but can be approximated as such. Alternatively, the exclusive display rule can be chosen so that it is simple and matches the rule based on $h(\cdot)$ empirically well. For instance, the seller may use the threshold for the difference between the highest bids, $z_{(1)} - z_{(2)} > Y$, or the threshold for their proportion, $z_{(1)}/z_{(2)} > Y$, for some $Y \in \mathbb{R}^+$. Note that the latter, the proportion approximation, is a good approximation of (5) when the reserve price $\underline{z}$ is low.

4.2.2 The log-normal distribution

In many applications, where the distribution of valuations is recovered parametrically, the log-normal distribution is selected. In particular, Ostrovsky & Schwarz (2011) find that

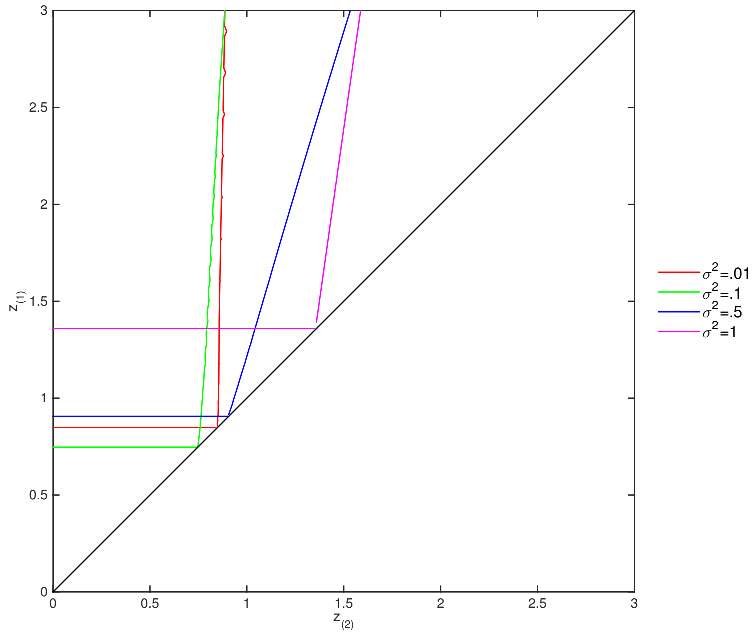

Figure 4: Exclusive sale function $h(\cdot)$ for log-normal distribution for a range of variances σ^2, $\mu = 0$.

the log-normal distribution of values is the good approximation, and use it to find optimal reserve prices for Yahoo! context search auctions.[9]

We find that the piecewise linear approximation for the exclusive sale rule is quite effective. Indeed, the exclusive sale function $h(\cdot)$ is essentially linear in the relevant region, see Figure 4.

Notice also that for low variances σ^2, the exclusive sale function is almost a step function. When h is a step function, either slot is allocated if the valuation is above the reservation price—it is the VCG with a single reservation price. Therefore, for the log-normal distributions with low variance, we expect the the gains from utilizing the optimal mechanism instead of the VCG to be minimal.

5 Revenue and welfare comparisons

In this section, we compare numerically the revenue and welfare that can be achieved by the currently used GSP auction format (and so also by the VCG auction that ignores quantity externalities) with the revenue and welfare that can be obtained by the revenue-maximizing and efficient mechanisms, respectively, in a simple multi-object setting with quantity externalities.

For the setting, we are going to follow the two goods example that we have been con-

[9]The unpleasant feature of the log-normal distribution is that the virtual valuation is not monotone if the variance is above 2.3.

sidering previously. We will set $\alpha = 0.77$,[10] assume that the private signals are distributed uniformly, $F = U[0, 1]$, and vary the number of buyers N and the premium for exclusivity β.

For the welfare comparison, we consider the GSP auction without a reserve price. We account only for the efficiency gains on the advertisers' side of the market, as we cannot measure the surplus of users in this model. If we assume that, in a transaction between a user and an advertiser , following a click on the ad the overall surplus generated is split proportionally, then such a comparison also gives the overall welfare comparison. To compare revenues, we consider the GSP auction with the optimal reserve price computed directly from the distribution of private signals, $R = \psi^{-1}(0) = \frac{1}{2}$, without regard to quantity externalities. Edelman & Schwarz (2010) argue that this is the optimal auction in an environment without quantity externalities.

Table 2 contains combined revenue comparisons between the two formats for different number of participants, N. When the number of bidders is small, $N = 2$, a large share of revenue comes from the exclusive display, which is expected as there is weak competition. When N grows, the difference between the two highest bids shrinks on average, and so does the probability and the share of revenue of the exclusive display. Naturally, when β grows, the exclusive display occurs more often and is more valuable, with the overall difference between the suboptimal GSP and the optimal mechanisms growing as well. Note that interestingly, the effect is not monotone in number of bidders N, with the largest gains for intermediate N. The reason is that with increased competition, the added effects from the optimal reserve price and from the optimal use of exclusive display vanish relative to the GSP (or VCG) auction without the reserve price, and so does the difference between the compared formats (provided β is not too large). When the number of bidders goes up from a small to an intermediate number, GSP with reserve price starts allocating all K slots because a lot of bidders have positive virtual valuations. The optimal auction on the other hand still truncates. When the number of bidders increases further, top bids are dense (no big gaps between bids), therefore the optimal auction does not truncate the list but allocates all K slots.

	$\beta = 0$	$\beta = .1$	$\beta = .2$	$\beta = .5$	$\beta = .8$
$N = 2$	0	0.2%	0.8%	4.3%	10%
$N = 4$	0	0.2%	0.8%	5.8%	17%
$N = 10$	0	0.02%	0.1%	1.5%	11%

Table 2: Percentage increase in revenue of the optimal auction compared to GSP with reserve price for a range of N (number of bidders), iid U[0,1] values per click, 2 slots, $\alpha = .77$, and β is the exclusive display CTR bonus.

The results of the welfare comparison are shown in Table 3. The welfare gain is higher when the exclusivity bonus β is large, and decreases in N.

[10] Arkhangelsky et al. (2013) report that the CTR of the second slot is 77% that of the first slot.

	$\beta = 0$	$\beta = .1$	$\beta = .2$	$\beta = .5$	$\beta = .8$
$N = 2$	0	0.5%	1.8%	11%	29%
$N = 4$	0	0.004%	0.05%	1.9%	13%
$N = 10$	0	0.000%	0.000%	0.04%	4.7%

Table 3: Percentage gain in total welfare (seller+advertisers) of the optimal auction compared to GSP with no reserve price for a range of N (number of bidders), iid U[0,1] values per click, 2 slots, $\alpha = .77$, and β is the exclusive display CTR bonus.

6 Conclusion

In this paper, we have brought attention to issues of auction design in multi-object settings with quantity externalities, when the total supply of goods offered for sale affects buyers' valuations for those goods. We have offered a simple model that allows us to find both the efficient and revenue maximizing auctions in such settings, and derived these mechanisms. For a particular application, we have considered auctions of allocating context advertisements. We have compared our auctions in performance with the existing formats, and offered ways to effectively approximate in practice the optimal auction we have derived.

The key lesson from our analysis is that both the efficient and the optimal auctions have allocation and payment functions that depend non-trivially on the whole profile of buyers' valuations. In particular, whether the buyer with the third-highest bid wins the object and the price she would pay for it may depend on the first-highest and the second-highest bids. Thus, the total number of objects that would be sold is truly determined endogenously within the mechanisms we derive. This stands in contrast to the current practice, where the total quantity to be sold is chosen first, and then the mechanism for sale is selected. At the same time, both the efficient and optimal mechanisms are relatively simple. For a standard environment without quantity externalities, the seller has to find a maximum of one linear function of values for the efficient mechanism and of virtual values for the optimal mechanism, respectively. With quantity externalities, the seller has to compute a maximum of several linear functions, so the overall complexity is similar.

The main application for our model is context advertisement auctions, where quantity externalities are present and can be substantial. Our analysis shows that the generalized second price auction is neither efficient nor optimal under quantity externalities with any reservation price, and, as we demonstrate, significant losses compared to the first best mechanisms may occur. We show that an auction format based on the VCG mechanism can account for the quantity externalities and can be effectively used instead. Recently there seems to be a swing towards truth-telling VCG-based mechanisms in large context advertising platforms (Varian & Harris (2014)). Other markets for which our results can be readily applied are markets in which values of participants are separable in two components: a private one (one-dimensional) and a common one that depends on the overall quantity to be allocated. These can be markets with similar objects offered for sale, e.g. spectrum or other licenses restricting access to the market, rare coins or other collectibles, or shares of a company offered during IPO. Whenever quantity effects are substantial, the effect from using an appropriately constructed efficient or optimal mechanism can be substantial as well.

References

Arkhangelsky, D., Izmalkov, S. & Khakimova, D. (2013), On evaluation of CTRs of different positions in sponsored search auctions. Yandex and New Economics School, mimeo.

Aseff, J. & Chade, H. (2008), 'An optimal auction with identity-dependent externalities', *The RAND Journal of Economics* **39**(3), 731–746.

Athey, S. & Ellison, G. (2011), 'Position auctions with consumer search', *The Quarterly Journal of Economics* **126**(3), 1213–1270.

Back, K. & Zender, J. (2001), 'Auctions of divisible goods with endogenous supply', *Economic Letters* **73**, 29–34.

Chen, Y. & He, C. (2011), 'Paid placement: Advertising and search on the internet', *The Economic Journal* **121**(556), F309–F328.

Deng, C., Lopomo, G. & Pekeč, S. (2012), 'Allocating and pricing exclusivity'. The Fuqua School of Business, Duke University, mimeo.

Deng, C. & Pekeč, S. (2012), 'Optimal allocation of exclusivity contracts'. The Fuqua School of Business, Duke University, mimeo.

Edelman, B., Ostrovsky, M. & Schwarz, M. (2007), 'Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords', *The American Economic Review* **97**(1), 242–259.

Edelman, B. & Schwarz, M. (2010), 'Optimal auction design and equilibrium selection in sponsored search auctions', *American Economic Review* **100**(2), 597–602.

Ghosh, A. & Sayedi, A. (2010), Expressive auctions for externalities in online advertising, *in* 'Proceedings of the 19th international conference on World wide web', WWW '10, ACM, pp. 371–380.

Jerath, K. & Sayedi, A. (2011), 'Exclusive display in sponsored search advertising', *Available at SSRN 1831744* .

Jeziorski, P. & Segal, I. (2015), 'What makes them click: Empirical analysis of consumer demand for search advertising', *American Economic Journal: Microeconomics* **7**(3), 24–53.

Krishna, V. (2009), *Auction theory*, Academic press.

McAdams, D. (2007), 'Adjustable supply in uniform price auctions: Non-commitment as a strategic tool', *Economics Letters* **95**(1), 48–53.

Muthukrishnan, S. (2009), Bidding on configurations in internet ad auctions, *in* 'Proceedings of the 15th Annual International Conference on Computing and Combinatorics', COCOON '09, Springer-Verlag, pp. 1–6.

Myerson, R. (1981), 'Optimal auction design', *Mathematics of Operations Research* **6**(1), 58–73.

Ostrovsky, M. & Schwarz, M. (2011), Reserve prices in internet advertising auctions: A field experiment, *in* 'Proceedings of the 12th ACM conference on Electronic commerce', ACM, pp. 59–60.

Riley, J. & Samuelson, W. (1981), 'Optimal auctions', *The American Economic Review* **71**(3), 381–392.

Varian, H. (2007), 'Position auctions', *International Journal of Industrial Organization* **25**(6), 1163 – 1178.

Varian, H. R. & Harris, C. (2014), 'The VCG auction in theory and practice', *The American Economic Review* **104**(5), 442–445.