Laboratory of Plasma Physics, Computational Lab. 1

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March 13, 2025

1 Problem 1: Boris pusher

1.1 How many ion species were accelerated in the spectrometer? Which ions are they? What is the minimum energy that this instrument can measure? What are the maximum energies measured for the species? Can you estimate all of them?

At least 12 ion species were accelerated in the Thomson parabola spectrometer; in Table 1 we have listed only the ion species which have an appreciable superposition between the experimental figure and the results computed with the Boris Pusher algorithm (or computed using the exact analytical solutions shown in the next paragraph), associating to each one an index corresponding to a certain parabola (from 1 to 12).

Element	Isotope	Ion	q[e]/m[m_p]	$E_{kin,min}$[Mev]	$E_{kin,max}$[Mev]	Parabola Index
H	^1H	$^1H^+$	1	0.288	4.2	1
He	^4He	$^4He^+$	0.25	0.094	0.7	7
		$^4He^{2+}$	0.5	0.29	3.25	2
Be	^9Be	$^9Be^+$	0.11	n.w.d	n.w.d	n.w.d
		$^9Be^{2+}$	0.22	0.185	1.5	8
		$^9Be^{3+}$	0.33	0.29	3.5	5
		$^9Be^{4+}$	0.44	0.515	5.85	3
		$^9Be^{5+}$	0.56	0.8	9	2
C	^{12}C	$^{12}C^+$	0.08	0.09	0.6	12
		$^{12}C^{2+}$	0.17	0.185	1.3	10
		$^{12}C^{3+}$	0.25	0.28	2.5	7
		$^{12}C^{4+}$	0.33	0.385	4.5	5
		$^{12}C^{5+}$	0.42	0.6	6.9	3
		$^{12}C^{6+}$	0.5	0.872	10	2
	^{13}C	$^{13}C^{2+}$	0.15	0.18	1.2	11
O	^{16}O	$^{16}O^+$	0.06	n.w.d	n.w.d.	n.w.d.
		$^{16}O^{2+}$	0.13	n.w.d	n.w.d.	n.w.d.
		$^{16}O^{3+}$	0.19	0.28	2.5	9
		$^{16}O^{4+}$	0.25	0.38	3.2	7
		$^{16}O^{5+}$	0.31	0.48	4.8	6
		$^{16}O^{6+}$	0.38	0.65	7.5	4
		$^{16}O^{7+}$	0.44	0.9	10	3
		$^{16}O^{8+}$	0.5	1.2	13	2

Table 1: List of possible ions with the relative charge-to-mass ratio, kinetic energy interval and the index of the associated parabola; n.w.d. stands for "not well defined".

To identify the species associated to each parabola, we faced the problem of redundancy: the same (more or less) charge-to-mass ratio (and so the same parabola) can be obtained from different ions. An exception is obviously given by the most abundant isotope of hydrogen, which is the only ion with a q/m ratio equal to one. Therefore, to simplify the discussion, we considered only the most abundant stable isotopes of each element, selecting just those from the first and the second periods of the periodic table [1, 2].

As predicted from the theoretical models derived in the second paragraph, we observe that less energetic ions along a specific parabola arrive to the detector more deflected from the original beam direction. Moreover, taking into account all the parabolas from the same isotope, a higher curvature is obtained decreasing the charge. On the other hand, when we consider the same charge for different elements, an higher curvature is obtained when the mass is increased.
Proceeding with the analysis, some questions arose:

- Why can a specific energy interval be considered consistent with the experiment that has been carried out? (To say if the guessed ion is right or not).
- Could there be a possibility of reactions in the ion source such that species like Deuterium or even Tritium arrive at the detector?

Trying to understand the physics behind the emission of those ions, we found out that Thomson parabola method is often used as diagnostic method for unconfined high-energy-density laser plasmas, which emits broadband ion spectra in terms of species and their kinetic energies [3, 4]. However, supplementary studies are necessary to get a good understanding of it. To conclude, the presence of specific element lines depends on the target considered in the experiment, so all we can do is to make hypothesis about it; the list of species we reported (^1H, ^9Be, ^{12}C and ^{16}O) covers at least the most visible parabolas, but we cannot be sure that is the effective target composition.
Since only the 11^{th} parabola remained unmatched, we added the carbon isotope $^{13}C^{2+}$, which has an appreciable correspondence to that trajectory (that could be its most probable degree of ionization of that ion).

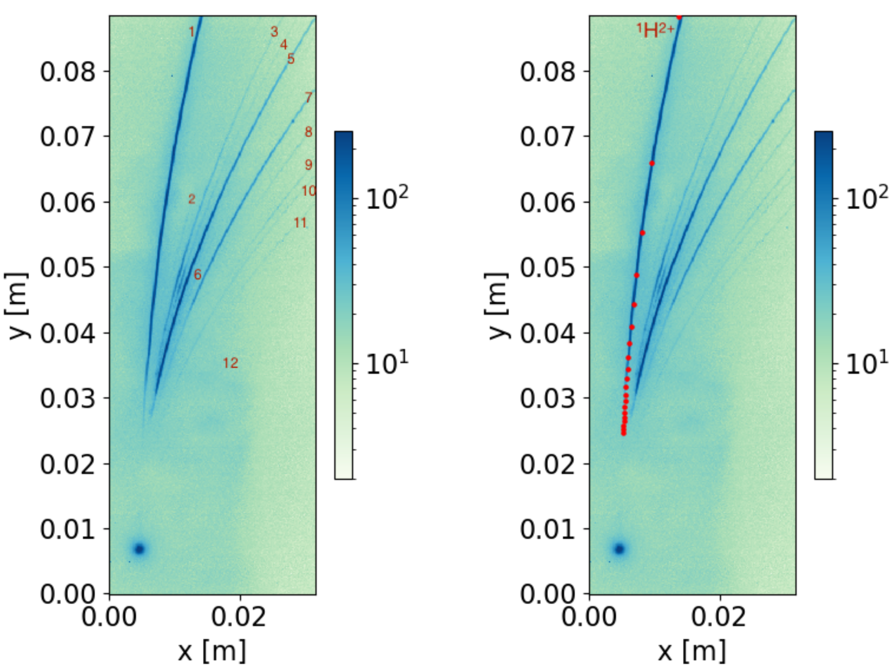


Figure 1: Classification of the detected parabolas (left) and the Hydrogen parabola points computed with the Boris Pusher algorithm (right).

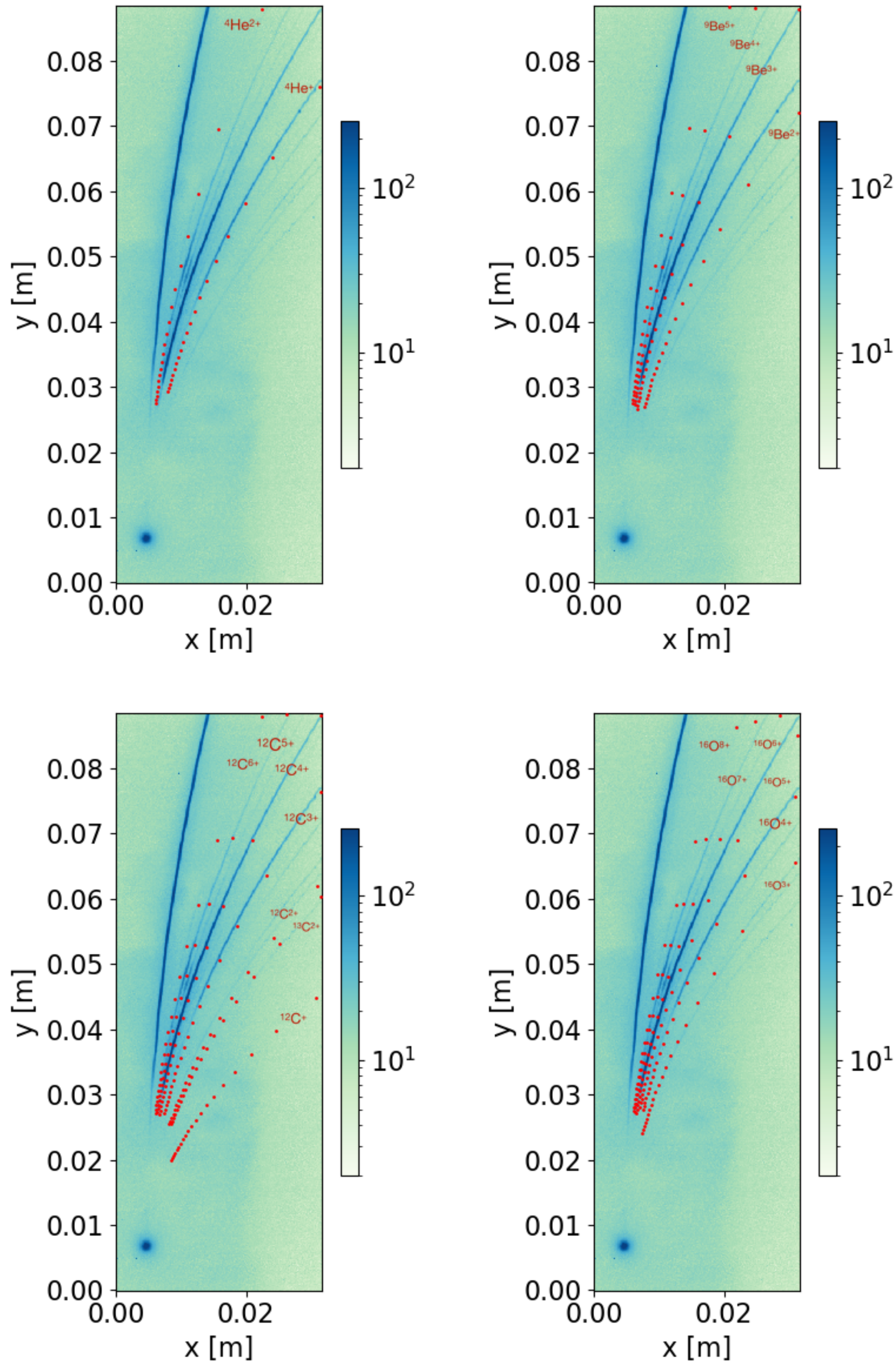


Figure 2: Parabola points of He, Be, C and O ions at various energies. When not possible, the computation of the points has been done with by implementing in Python the analytical solutions, instead of using the BP algorithm, especially at low energies.

1.2 For a given ion species, compare the final positions of the ions obtained with the Boris Pusher and those calculated using an analytical expression (you can find it in the literature or derive yourself). You may plot the deflection with respect to x and y as a function of energy (calculated analytically) in two graphs and overlay some points obtained with the provided code.

In order to get the analytical expression for the position of the ions, we start from Newton-Lorentz equation[5], $m\vec{a} = \vec{F}_L$:

$$\begin{cases} \dfrac{d\vec{v}_x}{dt} = \dfrac{q}{m}\vec{E}_x \\ \dfrac{d\vec{v}_y}{dt} = \dfrac{q}{m}(\vec{v_z} \times \vec{B}_x) = \dfrac{q}{m}B_x v_z \vec{u}_y \\ \dfrac{d\vec{v}_z}{dt} = \dfrac{q}{m}(\vec{v_y} \times \vec{B}_x) = -\dfrac{q}{m}B_x v_y \vec{u}_z \end{cases}$$

Where q is the net charge of the ion, m is the mass, and $B = B_x$ and $E = E_x$ are constant magnetic and electric fields, both applied in the x direction. Relativistic effects have been neglected because, as we can observe from the tables of the previous question, the kinetic energies of the accelerated ions are in the order of a few MeV: $E_k \sim MeV \ll mc^2 \simeq 938MeV$. Solving the system of differential equations we get the trajectories of the ions during their motion inside the plates:

$$\begin{cases} x = x_0 + \dfrac{qE}{2m}t^2 \\ y = y_0 + \dfrac{v_0}{\omega_c}(1 - \cos(\omega_c t)) \\ z = \dfrac{v_0}{\omega_c}\sin(\omega_c t) \end{cases}$$

With the following initial conditions:

$$\begin{cases} x_0 = 0.00472, y_0 = 0.00467, z_0 = 0 \\ v_{x0} = 0, v_{y0} = 0, v_{z0} = v_0 = \sqrt{\dfrac{2E_k}{m}} \end{cases}$$

Where $\omega_c = \frac{qB}{m}$ is the cyclotron frequency. In addition to these displacements, the ions travel across the gap between the exit of the plates and the detector. To simplify the next expressions, we use an approximation for the time spent by the ions inside the plates: $t_p = \frac{1}{\omega_c}\arcsin(\omega_c \frac{z_p}{v_0}) \simeq \frac{z_p}{v_0}$, which holds if $\omega_c \frac{z}{v_0} \to 0$. In our case:

$$\omega_c \frac{z_p}{v_0} = \frac{qBz_p}{\sqrt{2mE_k}} = \frac{N_c \cdot 1.6 \cdot 10^{-19} \cdot 0.4566 \cdot 0.075}{N_A \cdot \sqrt{2 \cdot 1.67 \cdot 10^{-27} \cdot E_k[MeV]/6.242 \cdot 10^{12}}} \simeq 0.237 \frac{N_q}{N_a}\frac{1}{\sqrt{E_k[MeV]}}$$

Where N_q and N_a are respectively the charge number and the mass number. We expect that the approximation will work better when masses and kinetic energies are high. Furthermore, we assume that at the exit of the plates $v_z(t_p) \simeq v_z(0) = v_0$. With this approximation the time spent from the exit of the plates and the detector is just $t_{gap} \simeq \frac{z_{gap}}{v_0}$. The shift of the ions due to the last tract is:

$$\begin{cases} x_{gap} = v_x(t_p)t_{gap} = \dfrac{qEz_p z_{gap}}{m}\dfrac{1}{v_0^2} \\ y_{gap} = v_y(t_p)t_{gap} = (\dfrac{v_0}{\omega_c}\omega_c \sin(\omega_c t))\dfrac{z_{gap}}{v_0} \simeq \omega_c z_p z_{gap}\dfrac{1}{v_0} \\ t_{gap} \simeq \dfrac{z_{gap}}{v_0}, t_p \simeq \dfrac{z_p}{v_0} \end{cases}$$

The approximated expressions for the final position, as function of energy ($v_0 = \sqrt{\frac{2E_k}{m}}$) are:

$$\begin{cases} x_f \simeq x_0 + \dfrac{qEz_p^2}{2m}\dfrac{1}{v_0^2} + \dfrac{qEz_pz_{gap}}{m}\dfrac{1}{v_0^2} = x_0 + \dfrac{qEz_p}{2}(\dfrac{z_p}{2} + z_{gap})\dfrac{1}{E_k} \\ y_f \simeq y_0 + \dfrac{v_0}{w_c}(1 - \cos(\omega_c t_p)) + w_c z_p z_{gap}\dfrac{1}{v_0} \simeq y_0 + \dfrac{v_0}{\omega_c}(\dfrac{w_c^2 z_p^2}{2v_0^2}) + w_c z_p z_{gap}\dfrac{1}{v_0} = y_0 + \dfrac{\omega_c z_p\sqrt{m}}{\sqrt{2}}(\dfrac{z_p}{2} + z_{gap})\dfrac{1}{\sqrt{E_k}} \end{cases}$$

We rewrite for clarity the final expressions grouping the constants into two new values, a and b:

$$\begin{cases} x_f \simeq x_0 + \dfrac{qEz_p}{2}(\dfrac{z_p}{2} + z_{gap})\dfrac{1}{E_k} = x_0 + \dfrac{a}{E_k} \\ y_f \simeq y_0 + \dfrac{\omega_c z_p\sqrt{m}}{\sqrt{2}}(\dfrac{z_p}{2} + z_{gap})\dfrac{1}{\sqrt{E_k}} = y_0 + \dfrac{b}{\sqrt{E_k}} \end{cases}$$

Which are the same expressions used by Treffert et al.[6]. The final positions depend on E_k: $x_f \propto \frac{1}{E_k}$ and $y_f \propto \frac{1}{\sqrt{E_k}}$. If we express x_f as function of y_f, we obtain a parabolic pattern, as observed experimentally:

$$x_f(y_f) = (x_0 - \frac{a}{b^2}y_0) + \frac{a}{b^2}y_f^2$$

We can verify if our approximations hold for the case of the ion of hydrogen by overlapping the obtained expressions with some points calculated with the Boris Pusher algorithm:

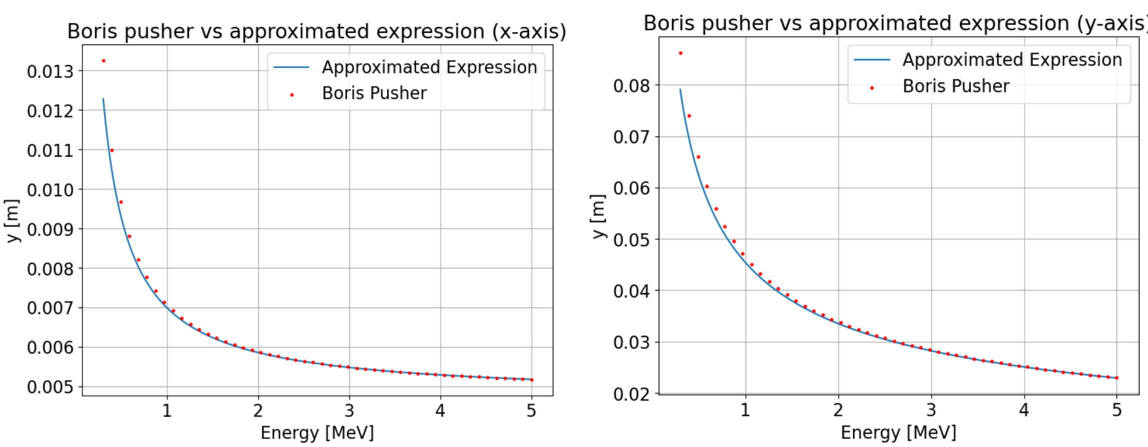


Figure 3: Boris pusher compared to the approximated analytical solutions in the energy interval $(0.3 \div 5)MeV$ of H^+.

The approximation works well for both the x and the y axis at high energies. At low energies the approximation $v_z(t_p) \simeq v_0$ is less accurate, because the ions spend more time inside the plates and the components of the velocity on the x and y axis become more relevant. We can confront the Boris Pusher with the full analytical expression that we have implemented in the script "thomson_analytical.py" [7].

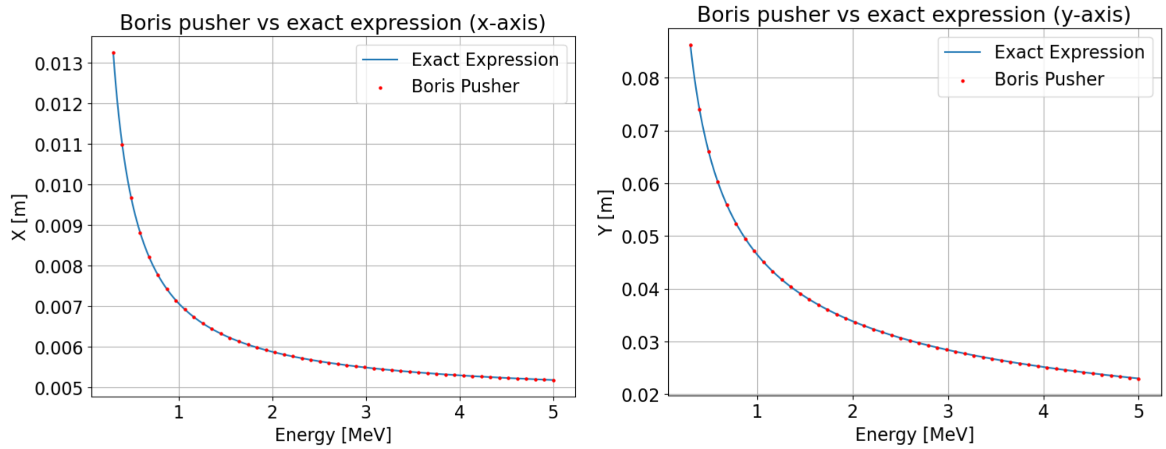


Figure 4: Boris pusher compared to the exact analytical solutions in the energy interval $(0.3 \div 5) MeV$ of H^+.

For hydrogen case the analytical expression and the boris pusher points coincide. However, for low energies and high value of mass, we notice that the Boris Pusher algorithm fails to predict the experimental curve, as for the case of the ${}^4H_e^+$ ion:

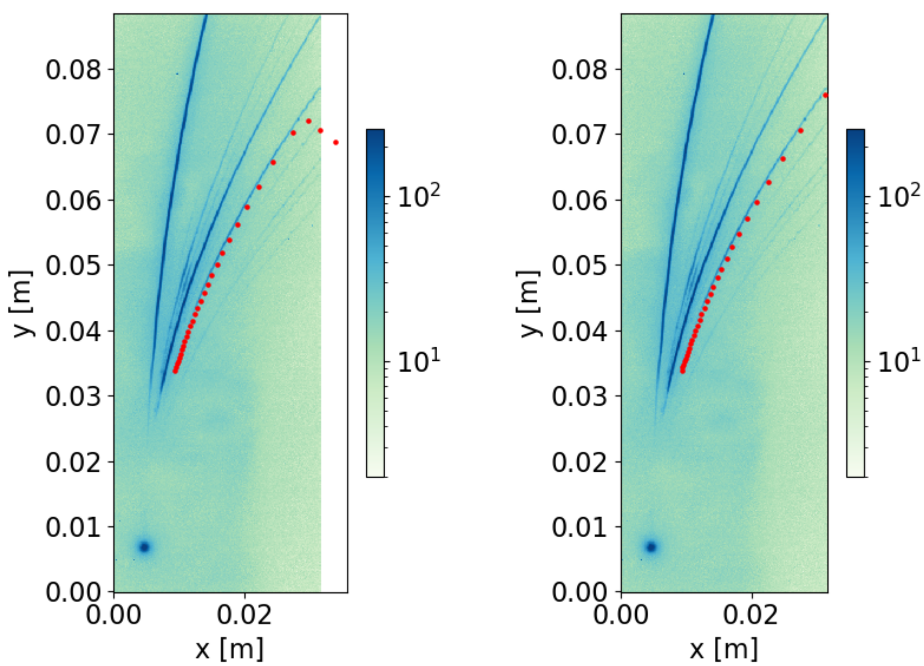


Figure 5: Comparison between BP algorithm (left) and the analytical points (right) for the ${}^4H_e^+$ ion. The BP is not able to correctly compute the lower energies.

The analytical expressions have been used in the first paragraph as an alternative to the Boris Pusher to calculate some of the parabolas.

1.3 Bonus task: Given a certain ion species, how does the energy resolution $(\Delta E/\Delta L)$ vary as a function of the ion energy E? How could it be improved?

The distance of the final points in the x-y plane is given by:

$$l = \sqrt{x_f^2 + y_f^2} = \sqrt{(x_0 + \frac{a}{E_k})^2 + (y_0 + \frac{b}{\sqrt{E_k}})^2}$$

We can take the derivative of l with respect to the kinetic energy and, as first approximation, set it equal to $\Delta l/\Delta E_k$:

$$\frac{dl}{dE_k} = -\frac{\left(by_0\sqrt{E_k} + 2ax_0 + b^2\right)E_k + 2a^2}{2\sqrt{\left(\frac{b}{\sqrt{E_k}} + y_0\right)^2 + \left(\frac{a}{E_k} + x_0\right)^2}\, E_k^3} \simeq \frac{\Delta l}{\Delta E_k}$$

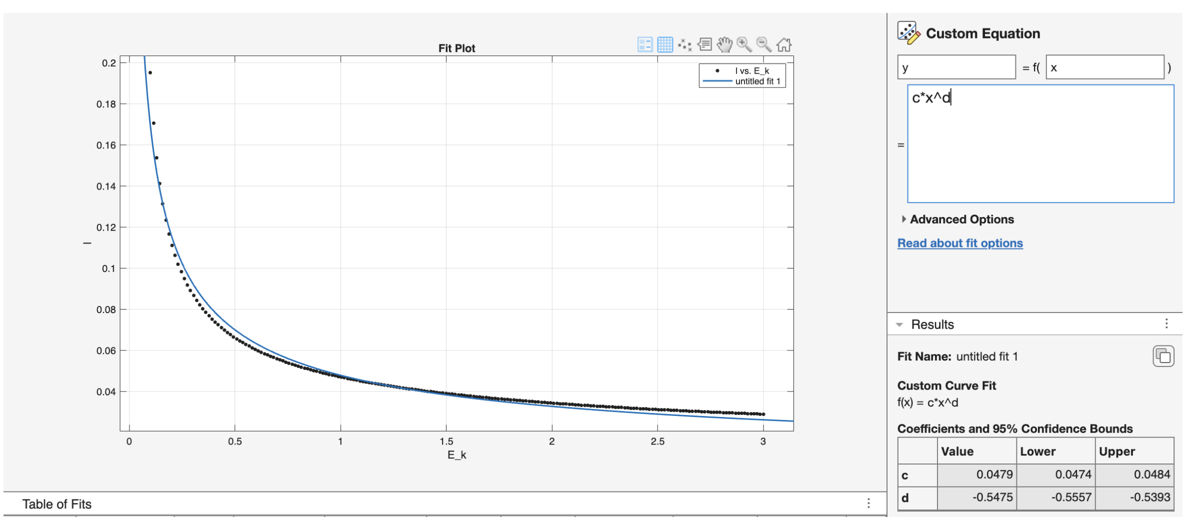
We can analyze the problem in two limit cases:

$$\begin{cases} E_k \to 0 \Rightarrow \dfrac{dl}{dE_k} \sim -\dfrac{a^2}{E_k^2} \Rightarrow |\Delta l| \sim \dfrac{a^2}{E_k^2}|\Delta E_k| \\ E_k \to \infty \Rightarrow \dfrac{dl}{dE_k} \sim -\dfrac{y_0}{2\sqrt{x_0^2+y_0^2}}\dfrac{b}{E_k^{3/2}} \Rightarrow |\Delta l| \sim \dfrac{y_0}{2\sqrt{x_0^2+y_0^2}}\dfrac{b}{E_k^{3/2}}|\Delta E_k| \end{cases}$$

Observing the two limits we can say that a way of increasing Δl is to increment the coefficients a or b, which means to increase the electric field, the magnetic field or the charge, or to decrease the mass. Moreover, since Δl goes as $E^{-1.5\div 2}$, when we are increasing the energy we are also decreasing Δl, so it will be more difficult to distinguish energy points at a fixed ΔE_k at higher energies. In our case, we can just consider the limit $E_k \to \infty$ because $E_k \gg max(b^2, a) \simeq 0.0022$ (for the ion of hydrogen). We are already at the high energy limit, because for this ion we are detecting energies in the interval $(0.3 \div 5)MeV$, as shown in Table 1. We can fit the energy points calculated with the Boris Pusher with a polynomial $l = cE_k^d$ to verify our approximated expression. In the case of the ion of hydrogen we obtain the following fit using the toolbox "Curve Fitter" of MATLAB:

$$l = 0.047E_k^{-0.4634} \Rightarrow dl/dE_k = -0.022E_k^{-1.46} \Rightarrow |\Delta l| \propto E_k^{-1.46}|\Delta E_k|$$

Which is a proportionality near the $\Delta l \propto E_k^{-1.5}\Delta E_k$ that we were expecting, since we are considering energies in the high energy limit.



	Value	Lower	Upper
c	0.0479	0.0474	0.0484
d	-0.5475	-0.5557	-0.5393

Figure 6: Fit for $E_k(0.3 \div 5)MeV$ using the MATLAB Toolbox "Curve Fitter".

References

[1] Table of Isotopic Masses and Natural Abundances, University of Alberta: https://www.chem.ualberta.ca/~massspec/atomic_mass_abund.pdf

[2] IAEA Chart of Nuclides: https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html

[3] J.Riedlinger et al., "Electromagnetic Thomson parabola spectrometer for detection of fs laser-driven keV ions", AIP Advances, vol. 14, 2024.

[4] L. Giuffrida et al., "A Platform for Laser-Driven Ion Sources Generated with Nanosecond Laser Pulses in the Intensity Range of 1013–1015 W/cm2", Quantum Beam Science, 2024, 8(1), 5.

[5] A. Piel, "Plasma Physics: An Introduction to Laboratory, Space and Fusion Plasmas", Springer, 2017.

[6] F. Treffert et al., "Design and Implementation of a Thomson Parabola for Fluence Dependent Energy-Loss Measurements at the Neutralized Drift Compression eXperiment", in Rev. Sci. Instrum. 89, 103302 (2018).

[7] Python codes: https://github.com/glecce3/plasma_physics_computational_lab_1