A Mathematical Model of Opinion Dynamics

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Individual Opinion

- Consider a simple question of opinion with a quantitative answer, such as "Rate your support for childhood vaccination against measles."
- Assume
  1. Opinion can be measured as a value x in a continuous range of real numbers $[-X, X]$ ($X = \infty$ is allowed).
  2. An individual's opinion changes over time according to a mathematical rule
     $$\dot{x} = g(x, t; p),$$
     where p is a vector of parameters used by a mechanistic formula that describes the influences on a person with opinion x at time t.
  3. To complete the model, we need assumptions about the forces that prescribe the quantitative rate function g.

Influences on Individual Opinion

- Possible influences:
  - Opinions of acquaintances
  - Opinions of experts
  - Opinions of "influencers" (non-experts with an audience)
  - News reports on current events
  - General principles of opinion change
- Influences in **our** model:
  - One or more communities of experts, each with a consensus opinion.
  - The broad community of deniers, with one consensus opinion.
  - A weak general tendency for opinions to moderate over time when not reinforced.

Global and Local Influence

- Each influence contributes a term to $\dot{x}$:
  - A **global** influence to an opinion x_j has strength that **increases** as $|x - x_j| \to \infty$:
    - We assume tendency to moderation is a global influence with term $-ax$.
  - A **local** influence to an opinion x_j has strength that **vanishes** as $|x - x_j| \to \infty$:
    - We assume expert opinions and denier opinions are local [1] influences with term

$$b_j \cdot k_j(x - x_j) \cdot (x - x_j),$$

where $\lim_{|h| \to \infty} k_j(h) = 0$.

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[1] I study expert recommendations to inform my opinion and denier recommendations only to get quotes for talks.

Individual Opinion Change Summary

Our model is

$$\dot{x} = g(x,t) = -ax - \sum_{j=1}^{J} b_j \cdot k_j(x - x_j) \cdot (x - x_j),$$

where

- ax (with small a) is a weak global attractor that pulls individuals toward the neutral opinion $x = 0$;
- $b_j \cdot k_j(x - x_j) \cdot (x - x_j)$ is a strong local attractor that pulls individuals toward opinion x_j;
- $k_j(h)$ indicates the relative strength of influence on individuals whose opinion differs from that of the influencer by h.
    - The influence is local because $\lim_{|h|\to\infty} k_j = 0$.

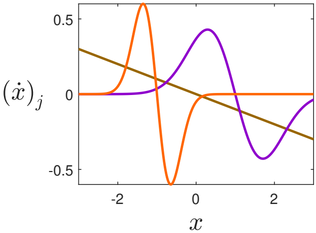
A Relative Strength Function $k(h)$

- We assume k has these properties:
  1. **Influence strength does not depend on direction**: k is even.
  2. **Influencers always attract**: $k \geq 0$.
  3. **Relative strength does not increase with distance**: $k' \leq 0$ for $h > 0$.
  4. **Influence shape is normalized**: $\int_{-\infty}^{\infty} |hk(h)|\, dh = 1$.
     - This requirement guarantees that influence falls to 0 as the subject-influencer opinion difference increases.
- Example: Gaussian form ("width" σ)

$$k(h) = \frac{1}{\sigma^2} e^{-(h/\sigma)^2}.$$

Scenario 1: Status Quo, Individual Influences

- Experts: aggregate influence is strong and broad
  - Local attractor of opinion 1, strength 1, width 1
- Deniers: aggregate influence is moderate and narrow
  - Local attractor of opinion -1, strength 0.7, width 0.5
- Slight tendency toward moderation:
  - Global attractor of strength 0.1



Scenario 1: Status Quo, All Influences Together

- Experts: aggregate influence is strong and broad
  - Local attractor of opinion 1, strength 1, width 1
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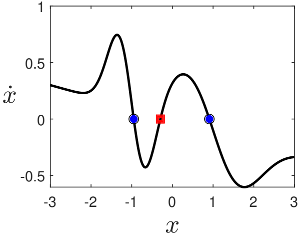


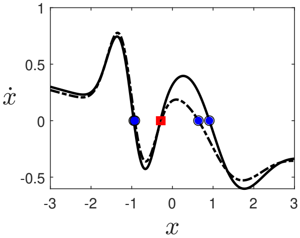
- Two stable and one unstable equilibria.

Changes in 2025

- The Secretary of the US Department of Health and Human Services has said that people should *do their own research* before deciding about vaccination.
  - So we should all use the best scientific methods to collect data and use the best statistical methods to analyze the data.
    - *That's what it means to do research on vaccination.*
  - Most people think "research" is what you are doing when you read a collection of opinions you find on Facebook.
    - *Scientific facts are not determined by popular vote, volume of voice, or belligerence of tone.*
- The FDA announced on 2025/05/20 that they would *require additional studies* before approving more COVID boosters for healthy people under age 65.
  - This sounds good, but the time required for a new study on the latest vaccine iteration is longer than its useful life.
    - *Imagine if it took 2 years to approve each new flu vaccine.*

Scenario Comparison

- Scenario 1 (solid) is as before:
  - Deniers: opinion -1, strength 0.7, width 0.5
  - Experts: **opinion 1**, **strength 1**, width 1
- In scenario 2 (dashed), expert opinion has split into
  - a pro-vaccination medical establishment
    (still opinion 1 and width 1, but with **strength 0.7**),
  - a neutral government position:
    opinion 0, **strength 0.3**, width 1.

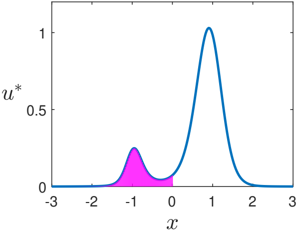


Aggregate Opinion

- What matters is not individual opinion, but aggregate opinion.

1. Aggregate opinion is indicated by an opinion density function $u(x,t)$:

$$P\{x_0 \leq x \leq x_1\}(t) = \int_{x_0}^{x_1} u(x,t)\,dx.$$



- Probability of opinion less than 0 (magenta) is 0.16.
- Must have $u(x,t) \geq 0$ and $\int_{-\infty}^{\infty} u(x,t)\,dx = 1$

The Problem for the Opinion Density Function $u(x,t)$

- Aggregate opinion changes are driven by the gradient of the opinion flux (diffusive is Du_x, convective is $-gu$):

$$u_t = (Fu)_x, \qquad Fu \equiv Du_x - gu,$$

- We assume an initial distribution

$$u(x,0) = u_0(x), \qquad u_0 \geq 0, \quad \int_{-X}^{X} u_0(x)\,dx = 1.$$

- By definition, u must satisfy

$$\int_{-X}^{X} u(x,t)\,dx = 1$$

for some large X. This is equivalent to the no-flux boundary conditions

$$Fu(\pm X) = 0.$$

Equilibrium Opinion Density

If $\lim_{t\to\infty} g(x,t) = \bar{g}(x)$, then the equilibrium density $u^*(x)$ is defined by

$$\bar{F}u^* = Du^*_x - \bar{g}u^* = 0, \qquad \int_{-X}^{X} u^*(x)\,dx = 1.$$

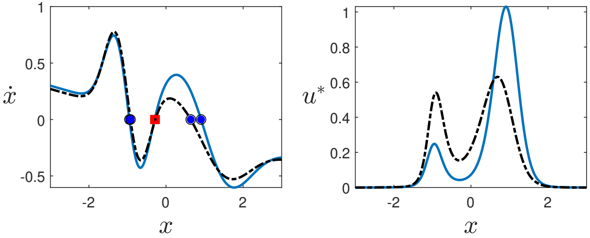
- *This is a problem for a first course in ODEs.*
  - (We should assign some problems with a generic function g rather than a specific function.)

$$u^*(x) = I^{-1}e^{G(x)},$$

where

$$G(x) = D^{-1}\int_0^x g(y)\,dy, \qquad I = \int_{-X}^{X} e^{G(x)}\,dx.$$

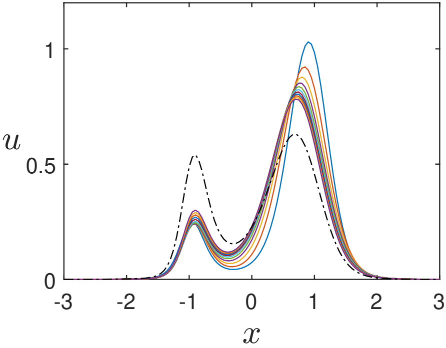
Scenario Comparison



- Changes in individual opinion dynamics (left):
  - Individual pro-vaccination opinion weakens from about 0.9 to about 0.6 because of inconsistent messaging.
- Changes in aggregate opinion dynamics (right):
  - *Much* larger density of vaccine opponents.
  - The peak of vaccine proponents widens and shifts to a more neutral position.
  - Total population fraction with opinion less than 0 goes from 16% to 38% (example result for the given parameter values).

PDE Example

- Initial condition is scenario 1.
- PDE is set to match scenario 2.



- We see a smooth transition, with a quick initial drop of the peak near $x = 1$ and slow growth of the peak at $x = -0.9$.

Theory: Separation of Variables

- Assume $g(x,t) = \bar{g}(x)$ (no time dependence for g).
- One of the key methods taught in an elementary PDE course is separation of variables. We can try that here by looking for solutions of the PDE and BCs of the form

$$u_n(x,t) = \psi_n(t)\phi_n(x)$$

- Critical question, assuming we find some separated solutions:
  - What good is a family of solutions $u_n(x,t) = \psi_n(t)\phi_n(x)$, when these solutions don't satisfy the initial condition?
  - Answer: Some beautiful theory proves that for many problems, an infinite series of these u_n solutions can satisfy ANY initial condition!

Theory: The Eigenvalue Problem

- The assumption of a solution of the form

$$u_n(x,t) = \psi_n(t)\phi_n(x)$$

leads to a boundary value problem for $\phi_n \neq 0$:

$$D\phi_n'' - g\phi_n' + \lambda_n\phi_n = 0, \quad D\phi_n'(\pm X) = g\phi_n(\pm X),$$

where λ_n (if it exists) is an unknown constant.

- By multiplying by u^{*-1}, the ODE can be rewritten as

$$\left[Du^{*-1}\phi_n'\right]' - u^{*-1}g'\phi_n + \lambda_n u^{*-1}\phi_n = 0.$$

  - The ϕ_n' and ϕ_n'' terms have been combined into a single term.

Theory: Existence and Uniqueness

The problem

$$\left[Du^{*-1}\phi_n'\right]' - u^{*-1}g'\phi_n + \lambda_n u^{*-1}\phi_n = 0$$

is a regular Sturm-Liouville problem. These have some amazing properties.

1. There are discrete eigenvalues (smallest, but no largest)

$$\lambda_0 < \lambda_1 < \cdots .$$

2. Any smooth function $u_0(x)$ can be written uniquely as

$$u_0(x) = \sum_{n=0}^{\infty} c_n \phi_n(x).$$

   - This means the full problem for u has a unique solution

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \phi_n(x)$$

Theory: Boundedness and Equilibrium Stability

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \phi_n(x),$$

with

$$\lambda_0 < \lambda_1 < \cdots,$$

but if $\lambda_0 < 0$, then $u \to \infty$ as $t \to \infty$.

3. For a regular Sturm-Liouville problem, ϕ_n has n zeros.
   - The Sturm-Liouville problem with $\lambda = 0$ is the equilibrium problem.
     - So u^* is an eigenfunction with eigenvalue 0.
   - u^* has no zeros.
     - So $\lambda_0 = 0$: u is bounded and u^* is stable.

Approach to Equilibrium

We've learned that u^* is stable. Therefore

$$u(x,t) = u^*(x) + \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\phi_n(x) \sim u^*(x) + c_1 e^{-\lambda_1 t}\phi_1(x).$$

The speed of the approach to equilibrium is determined by λ_1 and the shape by ϕ_1, both unknown.

Define a function

$$Q(x,t;k) \equiv e^{kt}(u - u^*).$$

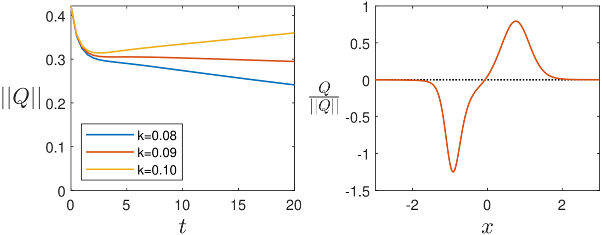
Then

$$Q(x,t;k) \sim c_1 e^{(k-\lambda_1)t}\phi_1(x), \quad t \to \infty.$$

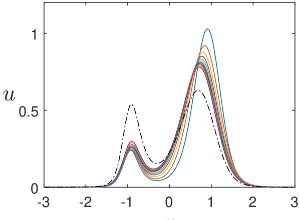
A plot of $||Q||$ vs t should converge to a constant if we choose $k = \lambda_1$. Then $\phi_1 \approx Q/||Q||$.

Approach to Equilibrium

$$Q(x,t;k) \sim c_1 e^{(k-\lambda_1)t}\phi_1(x), \quad t \to \infty.$$



- The eigenvalue is approximately $\lambda_1 = 0.09$.
- Slowest convergence of u is near bimodal peaks of u^*.



Take-Home Messages

1. Reasonable qualitative opinion behavior can come from a model of individual opinion driven by limited influences.
2. Opinion attractors with strong views and local influence lead to a multimodal opinion landscape.
3. Political control of the government medical establishment can have large consequences, for good or ill.
4. **Even a modest weakening of governmental medical establishment support for evidence-based medicine can be expected to greatly increase the numbers of vaccine skeptics and weaken the pro-vaccine views of vaccine supporters.**
5. Standard Sturm-Liouville theory can sometimes be used for theoretical purposes in addition to solving example problems.

Acknowledgements

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