

# The Differential

The **differential** of a function  $f(x, y, z)$  (similarly for functions of two variables) is an expression that uses the partial derivatives of the function:

$$df = f_x dx + f_y dy + f_z dz \quad (1)$$

It has some applications, for example, as a convenient way of applying the chain rule in some examples, and it is used for some theoretical developments that we are not going to do.

## Misuse of the Differential

Most mathematics books also use the differential for linear approximation by using finite values for  $dx$ ,  $dy$ , and  $dz$  and then getting a finite approximation for  $df$ . This is a very unfortunate use of the differential because it prevents us from being able to use the differential in the purpose for which it was intended.

Differentials appear in integrals; for example, we will eventually write an integral for the volume of a region in space as

$$V = \iint_R f(x, y) dA,$$

where  $R$  is the (2D) projection of the solid onto the  $xy$ -plane,  $f(x, y)$  is the height of the solid over the point  $(x, y)$ , and  $dA$  is a differential in 2D. The usual way integrals are presented is as limits of Riemann sums, with the differential as part of the notation but having no independent meaning. It is much easier to do applications with definite integrals if we interpret  $dA$  as an infinitesimal bit of area in the  $xy$ -plane located at the point  $(x, y)$ . With this interpretation, we can identify  $f(x, y) dA$  as the infinitesimal bit of volume located over the point  $(x, y)$  and interpret the integral as an infinite sum of these infinitesimal volumes.

If we use Equation (1) for finite linear approximation, say with  $dx = 0.1$ , then we are employing the notation of the differential in a way that is inconsistent with its use in definite integrals. Inconsistent use of notation in mathematics makes it much harder for students of mathematics to learn the concepts. Mathematicians need to be very careful not to do this, and for this reason, we will NEVER use the differential in my class for anything that is large enough to be assigned a numerical value. It will ALWAYS refer only to quantities that are arbitrarily small, meaning that they are individually 0 but collectively finite when infinitely many of them are added together.

## The Right Way to Do Linear Approximation

The notation of the differential contributes nothing to linear approximation. Just use finite differences, being careful to use “approximately equal” rather than “equal.” For example, to calculate the change in a function  $f(x, y) = x^2y + y^2$  when moving away from the point  $(2, 1)$ , we calculate the partial derivatives,

$$f_x = 2xy, \quad f_y = x^2 + 2y,$$

evaluate them at the point,

$$f_x(2, 1) = 4, \quad f_y(2, 1) = 6,$$

and write the linear approximation formula

$$\Delta f \approx 4\Delta x + 6\Delta y$$

using the values of the partial derivatives. In a specific instance, we are given finite values of  $\Delta x$  and  $\Delta y$  and use them to calculate an approximation for  $\Delta f$ . For example, at the point  $(2.1, 0.8)$ , we have

$$\Delta x = 0.1, \quad \Delta y = -0.2, \quad \Delta f \approx 4(0.1) + 6(-0.2) = -0.8.$$