

Solving Linear Partial Differential Equations on Finite Spatial Domains¹

This procedure assumes that the problem is “mostly homogeneous” (a homogeneous equation is one for which 0 is a solution);² that is,

- The differential equation is homogeneous.
 - If time is a variable, all boundary conditions must be homogeneous and all but one initial condition homogeneous.
 - If time is not a variable, there must be exactly one nonhomogeneous boundary condition.
1. Find a complete set of component solutions $u_n = h_n \phi_n$, where the complement function h_n depends only on the variable associated with the nonhomogeneous condition and the eigenfunction ϕ_n depends on all of the variables that are associated only with homogeneous conditions.
 - (a) Substitute $u = h\phi$ into the differential equation and separate variables. Define the eigenvalues so that they are normally positive (so $\phi'' = -\lambda\phi$, not $\phi'' = +\lambda\phi$).
 - (b) Determine the appropriate boundary/initial conditions for ϕ and h . Note that ϕ should have as many boundary conditions as the order of its differential equation.
 - (c) Make appropriate assumptions about the sign of the eigenvalues (take $\lambda = \omega^2$, $\lambda = 0$ if needed, and rarely $\lambda = -\omega^2$). Solve the corresponding differential equations with boundary conditions to get the eigenvalues and a 1-parameter family of eigenfunctions for each eigenvalue. Arbitrarily take the parameter to be 1.
 - (d) Solve the corresponding problem for h_n to obtain the component solutions $u_n = h_n \phi_n$
 2. Construct a general solution $u = \sum C_n u_n$, where C_n are generalized Fourier coefficients to be determined.
 3. Determine the generalized Fourier coefficients for the nonhomogeneous condition.
 - (a) Write down the algebra problem that the coefficients must satisfy for the nonhomogeneous condition with general function.
 - (b) Derive the coefficient formula by multiplying the equation by ϕ_m and the appropriate weight function σ , integrating over the appropriate interval, and using orthogonality.
 - (c) Evaluate the coefficient formula for the specific nonhomogeneous function.

¹Problems with infinite spatial domains require completely different methods.

²There are modifications that can be used if there is a nonhomogeneity in the PDE or in a BC for a time-dependent problem.