Designing Undergraduate Research Projects in Mathematical Modeling for an ODE Course (or research)

Glenn Ledder

Department of Mathematics
University of Nebraska-Lincoln
gledder@unl.edu

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Why Choose **Modeling** Projects?

Practical Reasons:

1. Most students in a DE course are not going to be mathematicians. If they use DEs at all, it will be in modeling.
2. Modeling appeals broadly to intellectual curiosity.
3. Most textbook "applications" have little or no modeling.

Pedagogical Reasons:

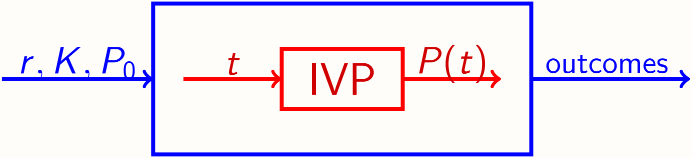
4. Modeling emphasizes the function concept in a more sophisticated way than can be done elsewhere.
5. Modeling focuses attention on the interpretation of algebraic symbols, a largely unrecognized weakness for most students.

Models as Functions of Parameters

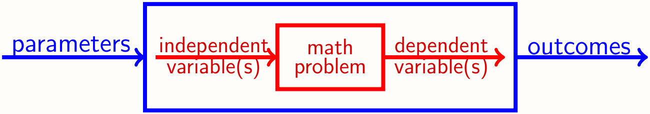
How do we view the model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0 > 0, \qquad r, K > 0?$$

- **Narrow** view:
  Initial value problem for $P(t)$, with parameters r, K, and P_0.
- **Broad** view:
  Function that maps parameters r, K, and P_0 to outcomes.



Models as Functions of Parameters



- **Narrow** view: Math problem with **fixed parameters**.
  - The narrow view is used to determine the outcomes.
    - Narrow view questions are trivial: "Given $K = 10$, $R = 1$, and $P_0 = 1$, when does the population reach $P = 5$?"
- **Broad** view: Outcomes as **functions of parameters**.
  - The important questions are in the broad view.
    - Do solutions with any initial condition always approach K?
    - At what point is the population growth the fastest?

A Textbook-Style **Modeling** Problem

The susceptible and recovered population fractions in the SIR disease model (cite worldbank.org article) are related by the initial value problem

$$\frac{dS}{dR} = -\mathcal{R}_0 S, \qquad S(1 - s_0) = s_0, \quad S(1 - s_\infty) = s_\infty,$$

where s_0 and s_∞ are the initial and final susceptible population fractions, R is the recovered population fraction, and $\mathcal{R}_0$ is the *basic reproduction number* of the disease.

1. Explain why the initial and terminal conditions are evaluated at $1 - s_0$ and $1 - s_\infty$.
2. Plot a graph showing how the final susceptible population depends on $\mathcal{R}_0 \in (1, 8]$ for the case where the entire population is initially susceptible.

Project Design: Modeling Tasks

- **B**uild
  - Use mechanistic assumptions to construct a model of a scenario or experiment.
- **A**nalyze
  - Use *qualitative methods* to obtain outcomes such as equilibrium solutions and stability.
- **S**imulate
  - Use software such as MATLAB or R to obtain numerical solutions.
- **I**nterpret
  - Address general questions in the context of the model setting.
- **C**ritique
  - Discuss possible model limitations.

Project Design: Choosing Outcomes

Epidemiology Example

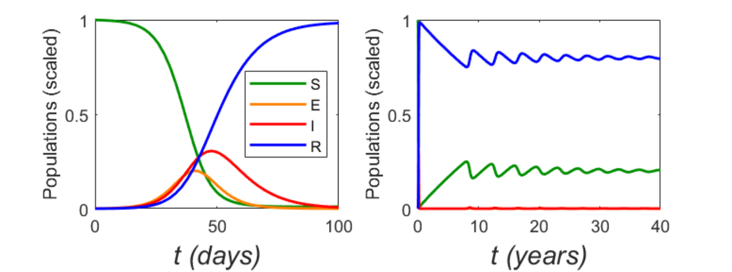
- Maximum number of new infections?
- Maximum number of hospitalizations per million?
  (compared to an average of 2800 hospital beds per million)
  - Serves as a measure of the stress on the health care system
- Percent deaths? (0.06% is 200,000 people)
  - Serves as a measure of the human cost
- Final fraction of susceptibles?
  - Serves as a measure of the risk of a new outbreak
- Times for any of these events?

Project Design: Asking and Addressing Questions

- Models must be designed to answer specific questions.
  - If we want to know the impact of COVID-19 on health care resources, we need to modify a standard disease model to track hospitalizations and/or ICU patients.
- Some common question types:
  - Is a specific claim supported by modeling or not?
  - What effect does parameter x have on outcome y?
- Strategies for addressing questions
  - Run simulations for several values of a parameter and compare simulation plots.
  - Calculate an outcome y for a large set of values of parameter x and plot y vs x.

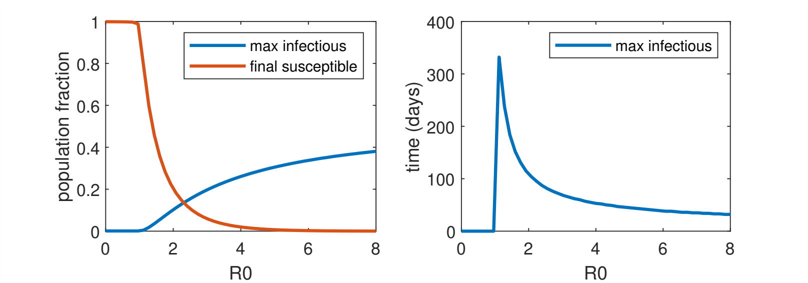
Simulations

- Simulations require known parameter values.
  - Plots show model behavior.
  - Results only guaranteed to apply to the specific parameter set.
  - Multiple simulations can *help* us understand the generality of the results and the effects of the parameters.



Parameter Studies

- Parameter studies systematically explore the impact of a parameter on one or more outcomes.
  - They show outcome vs parameter, not individual simulation results.



Structuring Projects to Get Better Results

- Break projects into a cumulative sequence of assignments.
  - Multiple submissions allow you to start projects before students have all necessary background.
  - Multiple submissions allow you to make sure everyone has the correct model before starting the analysis, ...
  - Cumulative submissions allow students a chance to improve their writing.
- Assume that your students do not know how to write about mathematics, even if they are good writers in general.
  - Provide short examples of A and C+ student writing.
    - Don't identify which is which, but assign the task of identifying which is better and why to the students.

Immune System: Origin and Background

Origin:

- I heard a research talk on the subject and found the associated research paper.
- I created a simplified version by decreasing the number of components.
- I included the simplified version in *Mathematical Modeling for Epidemiology and Ecology* as an extended exercise set.

Background:

- general: dynamical system construction, scaling
- DE background: equilibria, stability, nullcline plots
- matrix theory: Jacobian, eigenvalues, Routh-Hurwitz conditions

Immune System: Model Construction

- Three populations: a (p)athogen, a (g)eneralist macrophage, and a (s)pecialist macrophage
- p grows logistically, g are produced at a constant rate, s are produced at a rate that increases as p increases
- gp encounters kill g as well as p, but sp encounters only kill p.

After scaling[1], we get the model

$$p' = p(1 - p - rg - ks)$$

$$g' = b(1 - g - mgp)$$

$$s' = a\left(\frac{p}{p+h} - s\right)$$

[1] This took several tries to get right.

Immune System: Model Analysis

$$p' = p(1 - p - rg - ks)$$

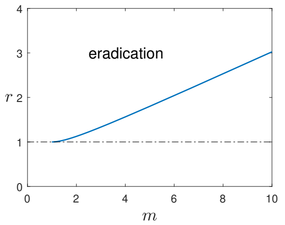
$$g' = b(1 - g - mgp)$$

$$s' = a\left(\frac{p}{p+h} - s\right)$$

- Do pg system.
  1. Use algebra to obtain a relationship between equilibrium p and the parameters.
  2. Use nullcline analysis and/or the Routh-Hurwitz conditions to determine stability for different cases based on inequalities in the parameter space.
  3. Determine the relationship between the 'good' parameters r and k and the 'bad' parameters m and h for eradication.
- Repeat for ps system and pgs systems.

pg Immune System: Results and Interpretation

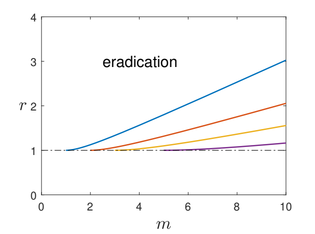
- Eradication requires $r > 1$ if $m < 1$; as m increases, the requirement for r increases, eventually becoming linear.



- The generalist can defeat some pathogens, but not all.
- *All results are for the model, not the real system.*
  - To assess the model, we need to ask experts if the model predictions match the real world.

ps and *pgs* Immune Systems: Results and Interpretation

- Eradication is impossible in the *ps* system. In the *pgs* system, eradication is easier than the *pg* system.



1. The specialists cannot defeat any pathogens on their own because they stop being produced as the pathogen level decreases.
2. With both present, the specialist significantly reduces the required generalist effectiveness.

Project Resources (write to gledder@unl.edu for more)

- *SIMIODE*, at qubeshub.org
- *Using Modelling to Motivate and Teach Differential Equations*, *IJMEST* special issue v55-02, ed. Winkel (2023)
  - A large collection of papers containing project ideas on a variety of subjects.
  - doi: 10.1080/0020739X.2024.2310875
- *CODEE* (at https://www.scholarship.claremont.edu)
  - A journal of DE case studies, including
    - Ledder, Qualitative Analysis of a Resource Management Model, (2021)
- *UMAP Journal* (at https://www.comap.com)
  - A journal of case studies, including
    - Ledder, Mathematical Model for a Mission to Mars, (2002)

- Ledder, *Mathematical Modeling for Epidemiology and Ecology*, 2nd ed., Springer (2023)
    - Contains a large number of case studies, projects, and extended projects distibuted over multiple sections.

Textbook

- Bryan, *Differential Equations: A Toolbox for Modeling the World*, 2nd ed., SIMIODE ($39 online)
    - The only ODE textbook with my personal seal of approval.
        - I'm not being paid, and I don't know the author personally (until we met at the AMS session this morning).

Undergraduate Research Mentoring

- Ledder, Mentoring Undergraduate Research in Mathematical Modeling, *Bull. Math. Biol.*, 84:77, (2022)
    - doi 10.1007/s11538-022-01040-4

Last Words

As a self-funded post-doc (*aka Professor Emeritus*), I enjoy (*expenses-paid*) visits to institutions to give talks and advice.

Thanks for your attention!

Write to gledder@unl.edu for a copy of this talk.