Extrema for Functions of Two Variables

All extremum problems have a common structure:

1. Find candidate points.

2. Evaluate the candidates.

The details are different for different classes of problems.

Local Extrema for a 2D Region

1. We define critical points as points that have no tangent plane or a horizontal tangent plane. Only critical points can be local extrema. For the typical case where the partial derivatives are continuous functions, these are points that satisfy

$$\vec{\nabla} f = \vec{\mathbf{0}}.$$

2. The classification of local extrema depends on the shape of the quadratic terms in the Taylor series approximation. This comes down to the sign of the discriminant, which is defined by

$$D = f_{xx} f_{yy} - f_{xy}^2.$$

If $D > 0$ at a critical point, then the critical point is a local extremum (a minimum if f_{xx} and f_{yy} are positive and a maximum if they are negative); while $D < 0$ at a critical point indicates a saddle. If $D = 0$, then the second derivative test is inconclusive.

Example:
Find and classify the critical points of the function $f(x, y) = x^2 y + 4xy - 2y^2$.

The derivatives are

$$f_x = 2xy + 4y, \qquad f_y = x^2 + 4x - 4y,$$

$$f_{xx} = 2y, \qquad f_{xy} = 2x + 4, \qquad f_{yy} = -4.$$

Note that we can immediately draw one conclusion with no further calculations: This function does not have any local minima. We know that because f_{yy} is always negative. Every constant x cross section is concave down everywhere.

The algebra problem of finding the critical points is tricky. The best approach is to observe that the equation $f_x = 0$ can be factored to yield

$$2y(x + 2) = 0.$$

There are two different ways this equation can be satisfied: either $y = 0$ or $x = -2$. We must therefore consider two different cases for $f_y = 0$.

1. In case 1, we satisfy $f_x = 0$ by taking $y = 0$. This reduces the second equation to

$$x^2 + 4x = 0.$$

Factoring this equation yields two results: $x = 0$ and $x = -4$.

2. In case 2, we satisfy $f_x = 0$ by taking $x = -2$. This reduces the second equation to

$$4 - 8 - 4y = 0,$$

from which we obtain $y = -1$.

Thus, we have three critical points: (0,0), (-4,0), and (-2,-1).

The table below shows the computations needed to classify the critical points:

x	y	f_{xx}	f_{yy}	f_{xy}	D	classification
0	0	0	-4	4	-16	saddle
-4	0	0	-4	-4	-16	saddle
-2	-1	-2	-4	0	8	local maximum

Note that $D > 0$ establishes the third point as an extremum; it is a maximum because both f_{xx} and f_{yy} are negative.

Global Extrema on a 2D Region

1. A continuous function on a closed and bounded 2D region must have global extrema, but these could occur in the interior, on a boundary, or on an intersection of boundaries. We must find candidate points in each of these subregions.

   (a) The interior of a region in 2D is of course 2D itself. Global extremum candidates in the interior are the same as local extremum candidates, points where $\vec{\nabla} f = \vec{\mathbf{0}}$.

   (b) The boundary of a 2D region is itself 1-dimensional. Boundary candidates for global extrema must therefore be found using 1D methods. For example, if a boundary is a horizontal line $y = c$, then the function values on that boundary are determined from the 1D function $g(x) = f(x, c)$; hence, we need to look for places where $g'(x) = 0$. Similarly, function values on a boundary $x = c$ are places where $g'(y) = 0$, where $g(y) = f(c, y)$. Other boundaries can be treated by reduction to a function of one variable as well; for example, the boundary $y = x$ can be used to define a 1-variable function $g(x) = f(x, x)$. This method may yield very complicated calculations, so it is often better to use a specialized method for finding extrema on a curve (see below).

   (c) Intersections of boundaries are 0D regions. All such points are candidates for global extrema.

2. There is no point in doing a second derivative test for a global extremum problem. Global extrema are determined by comparing the function values for the candidate points.

Extrema on a Curve in 2D Space

Equations in 2D represent curves in the plane; hence, the requirement that a point be on a curve is given by an equation $g(x, y) = c$ for some function g and constant c. If the curve is simple enough, we can substitute the equation $g = c$ into the function f to get a 1D problem and then solve that problem using one-variable calculus. More generally, critical points on a curve in 2D must satisfy the Lagrange multiplier rule:

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

for some unknown constant λ. In 2D space, the Lagrange multiplier rule provides 2 equations in 3 unknowns. The third equation is the constraint

$$g(x, y) = c.$$

There is no convenient second derivative test for local extrema, but one can usually classify the critical points by context. To identify global extrema, it is necessary (as always) to compare the function values achieved at the candidate points.

The Lagrange multiplier rule determines candidates for local or global extrema on a curve in the plane; hence it applies to the problem of finding 1D candidates for global extrema of a 2D region as well as applying to the problem of finding candidates on a region consisting only of a curve.

Extrema On a Surface

The problem of finding extrema of $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$ is geometrically that of finding extrema on a surface in 3D space; this is fundamentally similar to extrema on a curve in 2D space because there is one constraint. The Lagrange multiplier rule gives 3 equations in 4 unknowns, and the constraint adds the missing equation.

Example:
Find the point(s) on the surface $z = 32/x^2y$ that are closest to the origin.

The point(s) closes to the origin will have the smallest value of $f(x, y, z) = x^2 + y^2 + z^2$. Since this function is never negative, there must be a global minimum. We can think of the problem as minimizing f on the surface $g(x, y, z) = x^2yz = 32$. With

$$\vec{\nabla} f = \langle 2x, 2y, 2z \rangle, \qquad \vec{\nabla} g = \left\langle 2xyz, x^2z, x^2y \right\rangle,$$

we have the equations

$$2x = 2\lambda xyz, \quad 2y = \lambda x^2 z, \quad 2z = \lambda x^2 y, \quad x^2yz = 32.$$

Note that points on the surface $g = 32$ cannot have $x = 0$, $y = 0$, or $z = 0$, so we can solve the first equation to get $\lambda = 1/yz$. Substituting into the second and third and solving each for x^2 gives

$$x^2 = 2y^2, \quad x^2 = 2z^2, \quad x^2yz = 32.$$

Next, we can eliminate x^2 using the first equation to get

$$z^2 = y^2, \quad y^3 z = 16.$$

The first of these equations gives either $z = y$ or $z = -y$. The first case reduces the last equation to

$$y^4 = 16,$$

which has solutions $y = 2$ and $y = -2$, while the second case reduces the last equation to $y^4 = -16$, which has no solutions. For $y = 2$, we have $z = 2$ and $x^2 = 8$, while $y = -2$ yields $z = -2$ and $x^2 = 8$. So there are four points that satisfy all equations:

$$(2\sqrt{2}, 2, 2), \quad (-2\sqrt{2}, 2, 2), \quad (2\sqrt{2}, -2, -2), \quad (-2\sqrt{2}, -2, -2).$$

All of these points yield the same function value $f = 16$, corresponding to a distance of 4 from the origin. Since there must be a global minimum and it must satisfy the Lagrange multiplier rule, no further work is needed.