

THE METHOD OF UNDETERMINED COEFFICIENTS

In standard mathematical terminology, “the” method of undetermined coefficients is a method used to solve relatively simple nonhomogeneous linear differential equations with constant coefficients. However, the method of undetermined coefficients for differential equations is just one example of a family of methods used when you know the structure of a solution to a problem but not the specific solution. In partial fraction decomposition, we write a general form of terms in a sum, using an as yet undetermined coefficient for each. Then we do some work to reduce the original problem to a system of simple equations for the undetermined coefficients. In differential equations, the method of undetermined coefficients involves identifying the structure of a particular solution to a differential equation and then using that structure to reduce the original problem to a simple algebra problem.

The method of undetermined coefficients can be described using a large number of examples, or it can be described in general terms using an abstract formulation. The advantage of the abstract formulation is that once you understand it, there is very little to remember. A smaller number of examples will suffice to illustrate the method.

Linear Differential Operators

- Any linear differential equation of order n has the general form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x)y = g(x).$$

All such equations can be written in the abstract form

$$Ly = g,$$

where the notation “ Ly ” simply refers to the left side of the differential equation.

The symbol L taken by itself represents a *linear differential operator*. These are like functions, except that the input and output of functions are numbers or vectors, while the input and output of linear differential operators are functions of a common variable. The notation $f(x) = x \cos x$ says that the function f consists of instructions to multiply the argument by its cosine. Similarly, the notation $Ly = y'' + y$ says that the operator L consists of instructions to add the argument function to its second derivative. The name “ f ” is often used to refer to an unspecified function and only takes a particular form in examples. We’ll use the name “ L ” the same way. It will be a generic function in theoretical statements and have particular forms in examples.

Operator notation has two important advantages. As noted above, it allows all linear equations to be expressed in the same abstract notation $Ly = g$. The other advantage is that solving an equation $y'' + y = e^x$ (for example) is equivalent to identifying all the functions y that yield e^x as the output of the operator L , defined in this context as instructions to add the function and its second derivative.

The method of undetermined coefficients requires that the operator L have constant coefficients and that the function g be of a particular class. It works for $y'' + y = 1$, but not for $y'' + xy = 1$.

- From here on, we’ll assume that L is a linear differential operator of order n with constant coefficients.

Some Preliminary Examples

Before we develop the general method of undetermined coefficients, it is helpful to start with some simple examples.

- Suppose y_p is any one particular solution of an equation $Ly = g$ and that the general solution of the associated homogeneous equation is an n -parameter family y_h . Then the general solution of $Ly = g$ is $y = y_p + y_h$.

This result follows immediately from the property that a linear operator can be distributed among sums. Thus,

$$L[y_p + y_h] = Ly_p + Ly_h = g + 0 = g.$$

All of the examples below will use the operator L defined by $Ly = y' + y$. This means that each example will have the same homogeneous solution: $y_h = c_1 e^{-x}$.

Example 1

Suppose we want to find a particular solution for $y' + y = 8e^{3x}$. Since L simply adds the derivative and the function, it makes sense that the function we want should be a multiple of e^{3x} . Since we don't know which one, we can use the generic form $y = Ae^{3x}$. Then $y' = 3Ae^{3x}$ and $Ly = 4Ae^{3x}$. We want Ly to be $8e^{3x}$, which is accomplished by taking $A = 2$. The particular solution is $y_p = 2e^{3x}$ and the general solution is $y = 2e^{3x} + c_1 e^{-x}$. \diamond

Example 2

Suppose we want to find a particular solution for $y' + y = x$. This time it makes sense that we should try a first degree polynomial, but we don't know which one. Using the generic form $y = A + Bx$, we get $y' = B$ and $Ly = A + B + Bx$. We want Ly to be x , which requires $A + B = 0$ and $B = 1$. Thus, $A = -1$, $y_p = -1 + x$, and the general solution is $y = -1 + x + c_1 e^{-x}$. \diamond

Example 3

To find a particular solution for $y' + y = e^{-x}$, we can try $y = Ae^{-x}$, as we did in Example 1. However, we then get $Ly = 0$ and we can't choose A to get $Ly = e^{-x}$. The correct trial solution turns out to be Axe^{-x} . Then $y' = Ae^{-x} - Axe^{-x}$, so $Ly = Ae^{-x}$. This works, because $A = 1$ gives us the desired g . So the particular solution is $y_p = xe^{-x}$ and the general solution is $y = xe^{-x} + c_1 e^{-x} = (c_1 + x)e^{-x}$. \diamond

Example 4

To find a particular solution for $y' + y = 2\cos x$, our first thought is that we could use $y = A\cos x$. But then $y' = -A\sin x$ and $Ly = A\cos x - A\sin x$. We can get the right cosine coefficient by choosing $A = 2$, but this does not work because $Ly = 2\cos x - 2\sin x$ and $g = 2\cos x$ are not the same function. Instead, the correct trial solution turns out to be $A\cos x + B\sin x$. Then $y' = -A\sin x + B\cos x$ and $Ly = (A+B)\cos x + (B-A)\sin x$. We can get $\cos x$ from this family by choosing $A+B = 2$ and $B-A = 0$. The result is the particular solution $y_p = \cos x + \sin x$ and the general solution is $y = \cos x + \sin x + c_1 e^{-x}$. \diamond

From these examples, we have the following observations:

1. In many cases, the correct trial solution is merely a version of the function g with specific coefficients replaced by yet undetermined coefficients.

2. Example 3 is different because our first trial solution is actually part of the homogeneous solution; that is, $Ly = 0$. The correct trial solution turns out to be x times the first guess.
3. Example 4 suggests that when g is a cosine or a sine function, the trial solution has to have both cosine and sine terms.
- Note that the equation we get when we set Ly equal to g has one or more undetermined coefficients along with the independent variable x . In this context, the goal is to determine the values of the coefficients that make the equation true *for all* x . This is accomplished by matching the coefficients of different pieces, not by solving for x .

Examples and observations are useful for building intuition, but what we really need is a general method that could be applied to all four examples and a large class of similar problems. This will require some additional notation and terminology.

Generalized Exponential Functions

- *Generalized exponential functions* are all of the functions that can be solutions of some equation $Ly = 0$ (where L is a linear differential operator with constant coefficients).

Example 5

1. $3e^{2x}$ is a generalized exponential function because it solves $y' - 2y = 0$.
2. x^2e^x is a generalized exponential function because it solves $y''' - 3y'' + 3y' - y = 0$. We know this because the characteristic polynomial $r^3 - 3r^2 + 3r - 1$ factors as $(r - 1)^3$. Thus, there is a single characteristic value $r = 1$ of multiplicity 3, which has basic solutions e^x , xe^x , and x^2e^x .
3. $xe^{2x} \cos 3x$ is a generalized exponential function because we would get it as part of the solution of any equation for which the characteristic polynomial has a factor $(r^2 - 4r + 13)^2$ and the roots of the polynomial $r^2 - 4r + 13$ are $2 \pm 3i$.

◇

It is important to determine a general form for generalized exponential functions rather than a mere collection of examples. The easiest way to do this is to consider the solutions from cases where the characteristic polynomial has only one real root or one complex pair, possibly with multiplicity larger than 1.

1. Suppose the characteristic polynomial for L has one real root r of multiplicity s and no complex roots. Then the solution of $Ly = 0$ is given by

$$y_h(x) = (c_1 + c_2x + c_3x^2 + \cdots + c_sx^{s-1})e^{rx} = P_{s-1}(x)e^{rx},$$

where P_d is a name we use to indicate an arbitrary polynomial of degree d .

2. Suppose the characteristic polynomial for L has no real roots and one complex pair $\alpha \pm \beta i$ of multiplicity s . Then the solution of $Ly = 0$ is given by

$$y_h(x) = P_{s-1}(x)e^{\alpha x} \cos \beta x + Q_{s-1}(x)e^{\alpha x} \sin \beta x,$$

where P_d and Q_d are names we use to indicate arbitrary polynomials of degree d .

These results are enough to identify a general form for generalized exponential functions.

1. The family

$$y(x) = P_d(x)e^{rx}, \quad (1)$$

where P_d is an arbitrary polynomial of degree d with nonzero x^d coefficient, includes all of the generalized exponential functions having characteristic value r and degree d .

2. The family

$$y(x) = P_d(x)e^{\alpha x} \cos \beta x + Q_d(x)e^{\alpha x} \sin \beta x, \quad (2)$$

where P_d and Q_d are arbitrary polynomials of degree d (but with different coefficients), includes all of the generalized exponential functions having characteristic value r and degree d . Only one of P_d and Q_d has to have a nonzero x^d coefficient.

Example 6

The family of generalized exponential functions with characteristic value pair $1 \pm 2i$ and degree 1 can be written as

$$y_{1 \pm 2i, 1} = (Ax + B)e^x \cos 2x + (Cx + D)e^x \sin 2x,$$

where A and C are not both zero, in which case the degree would be 0 instead of 1. \diamond

Solving $Ly = g$

Now that we know about generalized exponential functions, it is easy to describe the method of undetermined coefficients. We assume that L has constant coefficients and g is a generalized exponential function.

1. Identify the (possibly complex) characteristic value(s) c and degree d for $g(x)$.
2. Find the family of functions $y_{c,d}$ using the appropriate formula (1) or (2).
3. Identify whether c is a characteristic value (or pair) for the equation $Ly = 0$. If so, let s be the multiplicity. If not, just take $s = 0$.
4. The particular solution is one of the members of the trial solution $y = x^s y_{c,d}(x)$.
5. Determine the coefficients by substituting the trial solution into the left side of the differential equation and choosing the coefficients so that the result of Ly is the function g .

Example 7

Let $Ly = y' + y$. We now apply our method to Examples 1–4.

1. The function $g(x) = 8e^{3x}$ is a generalized exponential function with characteristic value $c = 3$ and degree $d = 0$. This means that $y_{c,d} = Ae^{3x}$. The characteristic value of g is not a characteristic value of L , so $s = 0$. The trial solution is $y = Ae^{3x}$.
2. The function $g(x) = x$ is a generalized exponential function with characteristic value $c = 0$ and degree $d = 1$. This means that $y_{c,d} = (A + Bx)e^{0x}$. The characteristic value of g is not a characteristic value of L , so $s = 0$. The trial solution is $y = A + Bx$.
3. The function $g(x) = e^{-x}$ is a generalized exponential function with characteristic value $c = -1$ and degree $d = 0$. This means that $y_{c,d} = Ae^{-x}$. The characteristic value of g is a characteristic value of L , with multiplicity $s = 1$. The trial solution is $y = x^1 Ae^{-x} = Axe^{-x}$.

4. The function $g(x) = \cos x$ is a generalized exponential function with characteristic value pair $c = 0 \pm i$ and degree $d = 0$. This means that $y_{c,d} = A \cos x + B \sin x$. The characteristic value of g is not a characteristic value of L , so $s = 0$. The trial solution is $y = A \cos x + B \sin x$.

◇

Example 8

To solve $y'' - 4y' + 13y = xe^{2x} \cos 3x + 3e^{2x} \sin x$, we first identify g as a generalized exponential function with $c = 2 \pm 3i$ and $d = 1$. The family of such functions is $y_{c,d} = (A + Bx)e^{2x} \cos 3x + (C + Dx)e^{2x} \sin 3x$. The operator L has one complex pair of characteristic values, which happens to be the same as c . Thus, $s = 1$ and the trial solution is $y = (Ax + Bx^2)e^{2x} \cos 3x + (Cx + Dx^2)e^{2x} \sin 3x$.

◇