Assignment – Parameter Estimation

Solution 1 :-

For normal population :-

$$pmf :- \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

Likelihood function :- $h(\theta_1, \theta_2)$

$$h(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x_i-\theta_1}{\theta_2}\right)^2}$$

$$\ln h(\theta_1, \theta_2) = \sum_{i=1}^{n} \left[\ln \frac{1}{\sqrt{2\pi}} + \ln \frac{1}{\sqrt{\theta_2}} - \frac{1}{2} \frac{(x_i-\theta_1)^2}{\theta_2} \right]$$

$$\ln h(\theta_1, \theta_2) = \sum_{i=1}^{n} \left[-\ln\sqrt{2\pi} - \ln\sqrt{\theta_2} - \frac{(x_i-\theta_1)^2}{2\theta_2} \right]$$

$$\ln h(\theta_1, \theta_2) = -n \ln\sqrt{2\pi} - n \ln\sqrt{\theta_2} - \frac{\sum_{i=1}^{n}(x_i-\theta_1)^2}{2\theta_2}$$

Taking partial derivative w.r.t. θ_1

$$\frac{\partial}{\partial\theta_1} \ln h(\theta_1, \theta_2) = \frac{\sum_{i=1}^{n} 2(x_i-\theta_1)}{2\theta_2}$$

$$\frac{\partial}{\partial\theta_1} \ln h(\theta_1, \theta_2) = \sum_{i=1}^{n} \frac{(x_i-\theta_1)}{\theta_2}$$

For getting the maximum value :-

$$\frac{\partial}{\partial \theta_1} \ln h(\theta_1, \theta_2) = 0$$

$$\Rightarrow \Sigma x_i - \Sigma \theta_1 = 0$$

$$\Sigma x_i = n\theta_1$$

$$\theta_1 = \frac{\Sigma x_i}{n}$$

$$\hat{\theta}_1 = \bar{x} \quad \text{[Mean]}$$

Taking double derivative :-

$$\frac{\partial^2}{\partial \theta_1^2} \ln (\theta_1, \theta_2) = \frac{\Sigma(-1)}{\theta_2} = \frac{-n}{\theta_2} < 0$$

Since, it is < 0, So, the obtained value is maximum.

Now, taking the partial derivative w.r.t. $\sqrt{\theta_2}$

$$\frac{\partial}{\partial \theta} \ln h(\theta_1, \sigma) = \frac{-n}{\sigma} + \frac{\Sigma (x_i - \theta_1)^2}{\sigma^3}$$

To get max value, $\frac{\partial}{\partial \sigma} \ln h(\theta_1, \sigma^2) = 0$

$$\Rightarrow \frac{n}{\sigma} = \frac{\Sigma (x_i - \theta_1)^2}{\sigma^3}$$

$$\Rightarrow \theta_2 = \frac{\Sigma (x_i - \theta_1)^2}{n}$$

Puttin $\theta_1 = \bar{x}$,

$$\boxed{\theta_2 = \frac{\sum (x_i - \bar{x})^2}{n}}$$ (variance)

Now taking the double derivative :-

$$\frac{\partial^2}{\partial \theta_2} \ln h(\theta_1, \theta_2) = \frac{n}{\theta_2} - \frac{3}{\theta_2^2} \sum (x_i - \theta_1)^2$$

$$= \frac{n}{\theta_2} - \frac{3}{\theta_2^2} n \theta_2$$

$$= -\frac{2n}{\theta_2} < 0$$

Hence, maximum value.

Overall, $\hat{\theta}_1 = \bar{x}$

$$\hat{\theta}_2 = \frac{\sum (x_i - \theta_1)^2}{n}$$ ans

Solution - 2 :-

For binomial distribution:-

$$pmf = {}^mC_x \, \theta^x (1-\theta)^{m-x}$$

Likelihood function:-

$$L(\theta) = \prod_{i=1}^{n} \left[{}^mC_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i} \right]$$

$$L(\theta) = \prod_{i=1}^{n} \left[{}^mC_{x_i} \right] . \theta^{\sum_{i=1}^{n} x_i} . (1-\theta)^{\sum_{i=1}^{n}(m - x_i)}$$

$$L(\theta) = \left(\prod {}^mC_{x_i} \right) . \theta^{\sum x_i} . (1-\theta)^{nm - \sum x_i}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln {}^mC_{x_i} + \sum x_i \ln\theta + (nm - \sum x_i) . \ln(1-\theta)$$

Taking derivative w.r.t. θ :-

$$\frac{\partial}{\partial\theta} \ln L(\theta) = \frac{\sum x_i}{\theta} + \frac{nm - \sum x_i}{1-\theta} . (-1)$$

$$\frac{\partial}{\partial\theta} \ln L(\theta) = \frac{\sum x_i - \theta \sum x_i - nm\theta + \theta \sum x_i}{\theta(1-\theta)}$$

$$\frac{\partial}{\partial\theta} \ln L(\theta) = \frac{\sum x_i - nm\theta}{\theta(1-\theta)}$$

To get max. value, put :-

$$\frac{\partial}{\partial\theta} \ln h(\theta) = 0$$

$$\Rightarrow \quad \theta = \frac{\sum x_i}{nm} = \frac{\bar{X}}{m}$$

$$\Rightarrow \hat{\theta} = \frac{\bar{X}}{m}$$

Taking double derivative :-

$$\frac{\partial^2}{\partial\theta^2} \ln h(\theta) = -\frac{\sum x_i}{\theta^2} - \frac{(nm - \sum x_i)}{(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2} \ln h(\theta) = \frac{-\sum x_i - \theta^2 \sum x_i + 2\theta \sum x_i - \theta^2 nm + \sum x_i \theta^2}{\theta^2 (1-\theta)^2}$$

$$= -\left[\frac{\sum x_i + \theta^2 nm - 2\theta \sum x_i}{\theta^2 (1-\theta)^2}\right]$$

$$= \frac{-nm (1-\theta)\theta}{\theta(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2} \ln h(\theta) = \frac{-nm}{\theta(1-\theta)} < 0 \quad [\theta \in (0,1)]$$

Since double derivative is <0, so, obtained value is maximum.

$$\therefore \boxed{\hat{\theta} = \frac{\bar{X}}{m}} \quad \text{ans.}$$