Compendium MAT260

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1 Preliminaries

1.1 Norms

Definition 1. *Let V be a linear spave over $\mathbb{R}$. A function $\|\cdot\| : V \to \mathbb{R}$ is a norm on V if it satisfies the following:*

(i) $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in \mathbb{R}$

(ii) $x + y \leq \|x\| + \|y\| \;\; \forall x, y \in V$

(iii) $\|x\| = 0 \Leftrightarrow x = 0$

Example 1. *p - norm:* $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$

$p = 2$: $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$

$p = \infty$: $\|x\|_\infty = \max\limits_{i \in \{1,\dots,n\}} |x_i|$

1.1.1 Matrix norms

A matrix norm has the following properties:

(i) $\|\lambda A\| = |\lambda| \|A\| \;\; \forall \lambda \in \mathbb{R}, \;\; \forall A \in \mathbb{R}^{\ltimes,\ltimes}$

(ii) $\|A + B\| \leq \|A\| + \|B\| \quad \forall A, B \in \mathbb{R}^{\ltimes,\ltimes}$

(iii) $\|A\| = 0 \to A = 0_n$

(iv) $\|AB\| \leq \|A\| \|B\| \quad \forall A, B \in \mathbb{R}^{\ltimes,\ltimes}$

Example 2.

1. $\|A\|_1 = \sup\limits_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max\limits_{j=1,\dots,n} \sum_{i=1}^n |a_{ij}|$. *Max column sum*

2. $\|A\|_\infty = \sup\limits_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max\limits_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$. *Max row sum*

3. $\|A\|_2 = \sup\limits_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$. *Also:* $\sqrt{\rho(A^T A)}$, *ρ is the spectrum of A. Euclidean norm*

4. $\|A\|_F = (\sum_i^n \sum_j^n |a_{ij}|^2)^{1/2}$. *Frobenius norm*

1.1.2 Function norm

Function norms we have used in this course are among others:

1. $\|f\|_p = (\int_a^b \left|f(x)\right|^2 \omega(x)dx)^{1/p}$ where $\omega(x)$ is some weight function.
2. $\|f\|_\infty = \max\limits_{a\leq x\leq b}\left|f(x)\right|$

1.2 Banach stuff

Definition 2 (Banach space)**.** *A Banach space is a normed space that is complete. I.e every Cauchy sequence is convergent (to an element of the space).*

Theorem 1. *Let $(X,\|\cdot\|)$ be a Banach space and let $U \subseteq X$ be a subset of X and $f : U \to X$ be a function. If*

(i) U is closed

(ii) f is a contraction

(iii) $f(U) \subseteq U$

then f has a unique fixed point $x^* \in U$, i.e $x^* = f(x^*)$. Moreover, the sequence $x_n = f(x_{n-1})$, with $x_0 \in U$ arbitrary, converges to x^*.

1.3 Inner product

Definition 3. *Let V be a vector space. $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}(\mathbb{C})$ is called an inner product on V over $\mathbb{R}(\mathbb{C})$ if:*

(i) $\langle x, x\rangle \geq 0\, \forall x \in \mathbb{R}(\mathbb{C})$

(ii) $\langle x, x\rangle = 0 \Rightarrow x = 0$

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall \alpha\beta \in \mathbb{R}(\mathbb{C}), \quad \forall x, y, z \in V$

(iv) $\langle x, y\rangle = \overline{\langle y, x\rangle} \ \ \forall x, y \in V$

Example 3. *1.* $\mathbb{R}$*:* $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$

2. $\mathbb{C}$*:* $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y_i}$

3. $\langle f, g\rangle = \int_a^b f(x)g(x)dx$*. Inner product of functions*

1.4 Properties of a matrix

1.4.1 Positive definite

A matrix is positive definite if $\langle Ax, x\rangle \geq 0$ ($\langle Ax, x\rangle = 0 \Rightarrow x = 0$).
A matrix that is symmetric is positive definite if all its eigenvalues are positive.

To check if a symmetric (Hermitian in the complex case) matrix is positive definite, we can do two things:

1. Check if all eigenvalues are greater than 0
2. Check if all main minors are greater than 0 (i.e. $\det(A_{ii}) > 0$)

1.4.2 Diagonally dominant

I A is diagonally dominant, then A is regular, i.e. A is non-singular, i.e. det(A) $\neq 0$.

2 Condition of a problem

2.1 Condition analysis by solving linear systems

We have the system $A\overrightarrow{x} = \overrightarrow{b}$ where $A \in \mathbb{R}^{\ltimes,\ltimes}, b \in \mathbb{R}^{\ltimes}$ and det(A) $\neq 0$

2.1.1 Condition number of a matrix A

1. $K_{abs} = \left\|A^{-1}\right\|$
2. $K_{rel} = \|A\|\left\|A^{-1}\right\|$

2.2 Preconditioning

Having a system $A\overrightarrow{x} = \overrightarrow{b}$, preconditioning means to multiply our system with a matrix $B \in \mathbb{R}^{\ltimes,\ltimes}$ such that K(BA)<K(A). We are solving $BA\overrightarrow{x} = B\overrightarrow{b}$, det(B) $\neq 0$.

The easiest preconditioning is to take B = diag(d_i)