Compendium MAT260

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1 Preliminaries

1.1 Norms

Definition 1. *Let V be a linear spave over $\mathbb{R}$. A function $\|\cdot\| : V \to \mathbb{R}$ is a norm on V if it satisfies the following:*

(i) $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in \mathbb{R}$

(ii) $x + y \leq \|x\| + \|y\| \ \ \forall x, y \in V$

(iii) $\|x\| = 0 \Leftrightarrow x = 0$

Example 1. *p - norm:* $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$

p = 2: $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$

$p = \infty$: $\|x\|_\infty = \max\limits_{i \in \{1,\ldots,n\}} |x_i|$

1.1.1 Matrix norms

A matrix norm has the following properties:

(i) $\|\lambda A\| = |\lambda| \|A\| \ \ \forall \lambda \in \mathbb{R}, \ \forall A \in \mathbb{R}^{\ltimes,\ltimes}$

(ii) $\|A + B\| \leq \|A\| + \|B\| \quad \forall A, B \in \mathbb{R}^{\ltimes,\ltimes}$

(iii) $\|A\| = 0 \to A = 0_n$

(iv) $\|AB\| \leq \|A\| \|B\| \quad \forall A, B \in \mathbb{R}^{\ltimes,\ltimes}$

Example 2.

1. $\|A\|_1 = \sup\limits_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max\limits_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}|$. *Max column sum*

2. $\|A\|_\infty = \sup\limits_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max\limits_{i=1,\ldots,n} \sum_{j=1}^{n} |a_{ij}|$. *Max row sum*

3. $\|A\|_2 = \sup\limits_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$. *Also:* $\sqrt{\rho(A^T A)}$, *ρ is the spectrum of A. Euclidean norm*

4. $\|A\|_F = (\sum_i^n \sum_j^n |a_{ij}|^2)^{1/2}$. *Frobenius norm*

1.1.2 Function norm

Function norms we have used in this course are among others:

1. $\|f\|_p = (\int_a^b |f(x)|^2 \omega(x)dx)^{1/p}$ where $\omega(x)$ is some weight function.
2. $\|f\|_\infty = \max_{a\leq x\leq b} |f(x)|$

1.2 Banach stuff

Definition 2 (Banach space)**.** *A Banach space is a normed space that is complete. I.e every Cauchy sequence is convergent (to an element of the space).*

Theorem 1. *Let $(X,\|\cdot\|)$ be a Banach space and let $U \subseteq X$ be a subset of X and $f : U \to X$ be a function. If*

(i) *U is closed*

(ii) *f is a contraction*

(iii) $f(U) \subseteq U$

then f has a unique fixed point $x^* \in U$, i.e $x^* = f(x^*)$. Moreover, the sequence $x_n = f(x_{n-1})$, with $x_0 \in U$ arbitrary, converges to x^*.

1.3 Inner product

Definition 3. *Let V be a vector space. $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}(\mathbb{C})$ is called an inner product on V over $\mathbb{R}(\mathbb{C})$ if:*

(i) $\langle x, x\rangle \geq 0 \, \forall x \in \mathbb{R}(\mathbb{C})$

(ii) $\langle x, x\rangle = 0 \Rightarrow x = 0$

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall \alpha\beta \in \mathbb{R}(\mathbb{C}), \quad \forall x, y, z \in V$

(iv) $\langle x, y\rangle = \overline{\langle y, x\rangle}$