Deflection Formalism

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January 18, 2022

$\boldsymbol{\xi}_j$ is the angular vector of the photon with respect to the radial coordinate when it leaves the jth plane,

$$\boldsymbol{\xi} = \hat{k} - (\hat{k} \cdot \hat{r})\,\hat{r} \tag{1}$$

where $\hat{k}$ is the direction of motion of the photon.

1 multiplane approximation

The fundamental equation connecting the angular position on plane i to the angular position on plane $j+1$ is

$$\boldsymbol{\theta}_j = \boldsymbol{\theta}_1 - \sum_{i=1}^{j-1} \frac{D_{j,i}}{D_j} \boldsymbol{\alpha}_i(\boldsymbol{x_i}) \tag{2}$$

$$= \boldsymbol{\theta}_1 - \sum_{i=1}^{j} \beta_{ji}\, \boldsymbol{\alpha}_i(\boldsymbol{x_i}) \tag{3}$$

where D_i is the comoving distance from us to plan i and $D_{j,i}$ is the distance between planes i and j. Multiplying by D_j gives

$$D_j \boldsymbol{\theta}_j = D_j \boldsymbol{\theta}_1 - \sum_{i=1}^{j-1} D_{j,i}\, \boldsymbol{\alpha}_i(\boldsymbol{x_i}) \tag{4}$$

$$D_{j+1} \boldsymbol{\theta}_{j+1} = D_{j+1} \boldsymbol{\theta}_1 - \sum_{i=1}^{j} D_{j+1,i}\, \boldsymbol{\alpha}_i(\boldsymbol{x_i}) \tag{5}$$

Subtracting these gives

$$D_{j+1} \boldsymbol{\theta}_{j+1} - D_j \boldsymbol{\theta}_j = (D_{j+1} - D_j)\, \boldsymbol{\theta}_1 - \sum_{i=1}^{j} \left[D_{j+1,i} - D_{j,i} \right] \boldsymbol{\alpha}_i(\boldsymbol{x_i}) \tag{6}$$

Because $D(z)$ is the comoving angular size distance it can be expressed as an integral over z. Because of this $D_{j+1}-D_j = D_{j+1,j}$ and $D_{j+1,i}-D_{j,i} = D_{j+1,j}$. Using this we can simplify to

$$D_{j+1}\boldsymbol{\theta}_{j+1} - D_j\boldsymbol{\theta}_j = D_{j+1,j}\left[\boldsymbol{\theta}_1 - \sum_{i=1}^{j}\boldsymbol{\alpha}_i(\boldsymbol{x_i})\right] \tag{7}$$

or

$$\boldsymbol{\theta}_{j+1} = \frac{D_j}{D_{j+1}}\boldsymbol{\theta}_j + \frac{D_{j+1,j}}{D_{j+1}}\left[\boldsymbol{\theta}_1 - \sum_{i=1}^{j}\boldsymbol{\alpha}_i(\boldsymbol{x_i})\right] \tag{8}$$

The advantage of this formula over every other one I have seen is that a running sum of the deflections can be maintained and only the information of plane-j is needed to propagate to plane-$j+1$.

The deflections from each plane is $\boldsymbol{\alpha}_i(\boldsymbol{\theta_i})$. The deflections add up so that the deflection at the jth plane is the sum of all the lower redshift planes.

Using the definitions

$$\boldsymbol{A}^i \equiv \frac{\partial\boldsymbol{\theta}_i}{\partial\boldsymbol{\theta}_1} \qquad \boldsymbol{G}^i \equiv \frac{\partial\boldsymbol{\alpha}_i}{\partial\boldsymbol{x}_i} \tag{9}$$

the evolution equation is

$$\boldsymbol{A}^{j+1} = \frac{D_j}{D_{j+1}}\boldsymbol{A}^j + \frac{D_{j+1,j}}{D_{j+1}}\left[\boldsymbol{I} - \sum_{i=1}^{j}\boldsymbol{G}_i\frac{\partial\boldsymbol{x}_i}{\partial\boldsymbol{\theta}_1}\right] \tag{10}$$

$$= \frac{D_j}{D_{j+1}}\boldsymbol{A}^j + \frac{D_{j+1,j}}{D_{j+1}}\left[\boldsymbol{I} - \sum_{i=1}^{j}D_i\boldsymbol{G}_i\boldsymbol{A}^i\right] \tag{11}$$

with the initial condition $\boldsymbol{A}^0 = \boldsymbol{I}$.

2 $\boldsymbol{G}$

3 time-delay

4 references

Seitz, S., Schneider, P. & Ehlers, J., 1994, "Light propagation in arbitrary spacetimes and the gravitational lens approximation", astro-ph/9403056