# A RECIPE FOR RESOLVING REAL RIDDLES

## DAY 1 — LOGIC

GLENN SUN
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**By the end of today's class, you will be able to:**

- Understand the statement of the Tarski–Seidenberg theorem.
- Convert an English sentence to the input form expected by the Tarski–Seidenberg algorithm.
- Apply various reductions to simplify logical sentences into a form that is easier to reason about.

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# 1 Historical context

Questions about $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and beyond have puzzled mathematicians for centuries. In 1694, Issac Newton argued with his contemporaries about whether 12 or 13 identical balls could be arranged to simultaneously touch a ball of the same size in the middle, a question about $\mathbb{R}^3$ that was not resolved until 1953. (The answer is 12, this is the *kissing spheres* problem.) And practically all of high school geometry can be formulated as questions about $\mathbb{R}^2$.

In the 1930s, mathematicians like Alan Turing and Kurt Gödel started to develop a theory of the foundations of mathematics: What is a system of logic, and which one have we been using all this time? What does it mean to compute something, or prove something? And most surprising of all, they determined that there exist problems that impossible for computers to solve (i.e. the Halting problem), and that there are mathematical statements about the natural numbers $\mathbb{N}$ for which it is impossible to decide whether they are true or false (Gödel's first incompleteness theorem). Don't worry if you haven't heard about these, we won't need them! The point is that it is hard, and often actually impossible, to decide in general whether a statement is true or false.

It is somewhat surprising then, that Alfred Tarski discovered in 1948 **a computer algorithm to decide and prove whether any statement about $\mathbb{R}$ is true or false.** (You might think that this contradicts Gödel's incompleteness if you think $\mathbb{N} \subseteq \mathbb{R}$ means "$\mathbb{N}$ is easier than $\mathbb{R}$," but it's actually the other way around—the larger set gives you more flexibility, so deciding it is easier.) Today, we call his method the **Tarski–Seidenberg theorem**.

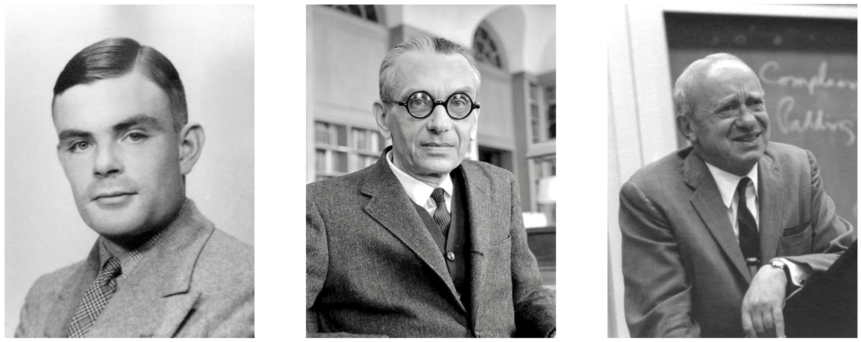

**Figure 1.** From left to right: (1) Alan Turing (1912–1954), a British mathematician best known for the Turing machine, and for breaking the Enigma code during World War II. He died of suicide after being persecuted for homosexuality. (2) Kurt Gödel (1906–1978), an Austrian-American mathematician best known for his two incompleteness theorems, and for showing that the Axiom of Choice is independent of ZFC. He died after struggling with anorexia and mental instability, weighing 65 lbs upon death. (3) Alfred Tarski (1901-1983), a Polish-American mathematician best known for his work in logic and model theory, the Banach–Tarski paradox, and today's topic, the Tarski–Seidenberg theorem. He died from heart problems and lung disease.

# 2 Some basics about logic

Recall that we often use the following symbols as shorthand when talking about logic:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\wedge$ | $\vee$ | $\neg$ | $\rightarrow$ | $\leftrightarrow$ | $\forall$ | $\exists$ |
| and | or | not | implies | if and only if | for all | there exists |

The symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$ are called *logical connectives*, and the symbols $\forall$ and $\exists$ are called *quantifiers*.

For example, the sentence "if $(x-1)(x-2)=0$, then $x=1$ or $x=2$" can be written as:

$$\forall x,\ [(x-1)(x-2)=0] \rightarrow ([x=1] \vee [x=2]).$$

Notice three things:

1. When we say "for all $x$" ($\forall x$) or "there exists $x$" ($\exists x$), technically we need to specify what universe $x$ belongs to. **For this class, the universe will always be $\mathbb{R}$.**

2. This sentence is what we call a *first-order formula*. In particular, a *second-order formula* would allow use to write things like "$\forall A \subseteq \mathbb{R}$", which we do not allow. We only allow $\forall x \in \mathbb{R}$ and $\exists x \in \mathbb{R}$ (and by the above, we typically just write $\forall x$ and $\exists x$).

3. Every variable used in the sentence is quantified (i.e. appears after a $\forall$ or $\exists$). That is, there are no *free variables.* It doesn't make sense to prove sentences with free variables.

We need this symbolic form because we can't write an algorithm that operates on English sentences! (Our theorem was discovered 75 years before ChatGPT.) So it's important to formally define how our sentences are allowed to look.

**Problem 1.** Rewrite the following conventional sentences using logical notation. No English words!

1. For $x, y \in \mathbb{R}$, we have $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$.

2. There is a real number that is strictly bigger than every *other* real number.

3. Every cubic equation (with non-zero leading coefficient) has a real root.

4. Two distinct lines with the same slope have no points of intersection.

5. (♦♦) There exists $S \subseteq [0, 3]$ with $|S| = 4$ and for every pair of distinct elements $x, y \in S$, the distance between $x$ and $y$ is at least 1. (Remember that we want a first-order sentence, which only quantifies over $\mathbb{R}$, not subsets of $\mathbb{R}$.)

# 3 The big theorem becomes a little smaller

The main theorem of this course is the following:

**Theorem 1** (Tarski–Seidenberg)**.** Consider the problem where as input, we are given a first-order sentence involving only the following symbols:

$\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$, $=$, $\neq$, $<$, $>$, $\leq$, $\geq$, $+$, $\cdot$, rational numbers, parentheses, variables.

The desired output is whether the sentence is true or false.

Then there is an algorithm that solves the above problem.

We will prove this over the next week!

**Problem 2.** For each of the first-order sentences in Problem 1, which of them can be proven using the Tarski–Seidenberg algorithm?

The parts of the sentence like $[xy \leq x - 1]$ or $[x^3 = 2]$ that only involve:

$$=, \neq, <, >, \leq, \geq, +, \cdot, \text{rational numbers, parentheses, variables}$$

(no quantifiers or logical connectives) are called *atoms*.

**Problem 3.**

1. Try to describe what an atom can look like. (This is an open-ended question.)

2. I claim that it sufficient to assume all atoms are of the form $[p(x_1, \ldots, x_n) \diamond 0]$, where $\diamond$ is one of $=$, $\neq$, $<$, $>$, $\leq$, and $\geq$, and $p$ is a *multivariate polynomial*. (I haven't defined this term yet!) Explain what a multivariate polynomial should be in order to make the claim true.

The theorem statement allows a ton of different symbols in the formula, which makes things pretty hard. The rest of this worksheet focuses on **reducing the number of symbols** or **restricting the order they appear in**, without changing the meaning of the sentence.

There are a couple of different parts that we can simplify: **the atoms** themselves, **the logical structure**, and **the quantifier structure**. Let's start with the atoms.

**Problem 4.**

1. Rewrite $[xy \leq 3x + 1]$ and $[x^3 \neq -2]$ into equivalent expressions using only:

   $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $=$, $>$, $+$, $\cdot$, rational numbers, parentheses, variables

   (no $\neq$, $<$, $\leq$, and $\geq$).

2. Explain in general how you would rewrite atoms to use only the above.

Now, the remaining symbols are:

$\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $=$, $>$, $+$, $\cdot$, rational numbers, parentheses, variables.

Next, let's simply the logical structure.

**Problem 5.**

1. Suppose I say, "If it rains, I will bring an umbrella." Under what circumstances would this be a lie, vs. technically the truth? Can you rephrase this sentence to avoid conditional words like "if" or "whenever"?

2. Explain in general how to rewrite any sentence involving:

$$\forall,\ \exists,\ \wedge,\ \vee,\ \neg,\ \rightarrow,\ \leftrightarrow,\ =,\ >,\ +,\ \cdot, \text{ rational numbers, parentheses, variables}$$

into an equivalent sentence using only:

$$\forall,\ \exists,\ \wedge,\ \vee,\ \neg,\ =,\ >,\ +,\ \cdot, \text{ rational numbers, parentheses, variables.}$$

The following rules are called *De Morgan's laws*: for any expressions $X$ and $Y$,

- $\neg(X \wedge Y)$ is equivalent to $\neg X \vee \neg Y$.
- $\neg(X \vee Y)$ is equivalent to $\neg X \wedge \neg Y$.

If you're not familiar with these, take a minute to convince yourself why they are true!

**Problem 6.** We will now make the sentence simpler by restricting the order in which $\wedge$, $\vee$, and $\neg$ appear.

1. Let $A$, $B$, and $C$ represent some atoms. Use De Morgan's laws to rewrite

$$\neg((A \vee B) \wedge \neg(A \wedge C))$$

such that $\neg$ only appears attached to an atom. (In other words, $\neg$ should not appear in front of any parentheses.)

2. Rewrite the English sentence "Every day the sun rises" to avoid the words "every", "for all", and all other synonyms. Then rewrite $\neg(\exists x, \forall y, [x < y^2])$ so that $\neg$ only appears attached to an atom.

3. Explain in general how to rewrite any first-order sentence so that $\neg$ only appears attached to an atom. This is called *negation normal form*.

Finally, let's simplify the quantifier structure.

**Problem 7.** We will now make the sentence simpler by restricting the order in which $\forall$ and $\exists$ appear.

1. Without changing the atoms, rewrite $\forall x,\ (\exists y,\ [x < y]) \wedge \neg(\forall y,\ [y \geq x])$ into an equivalent sentence where all of the quantifiers appear at the beginning.

2. Explain in general how to rewrite any first-order sentence so that all of the quantifiers appear at the beginning. This is called *prenex normal form*.

3. (♦♦♦) Give a formal proof that the sentence you described in part 2 is logically equivalent to the original.

**Problem 8.** Explain how to take a sentence in prenex normal form and rewrite it into an equivalent sentence with only $\exists$ (no $\forall$). The resulting sentence may not be in prenex normal form. (Hint: Problem 6.2.)

Actually, there's one more simplification we can do. After Problem 7, we have something that looks like

$$\exists x,\ \neg\exists y,\ \exists z,\ (\text{some quantifier-free formula in } x,\ y, \text{ and } z)$$

(of course, with possibly more or less variables). We showed how to put the quantifier-free part in negation normal form (all $\neg$ attached to atoms). There is an even more restrictive form called *disjunctive normal form* (DNF form), and it looks something like

$$(A \wedge \neg B \wedge C) \vee (B \wedge \neg D) \vee (\neg A \wedge E),$$

(here $A$, $B$, $C$, $D$, and $E$ are atoms). Note that all $\neg$ are attached to atoms, like in negation normal form. Furthermore, the whole formula is an "OR of ANDs". Specifically, the things in parentheses are ANDs of atoms and possibly negations, we call each one a *DNF term*, and the whole formula is just the OR of a bunch of DNF terms.

**Problem 9.** Convince yourself of the distributivity law:

$$X \wedge (Y \vee X) \text{ is equivalent to } (X \wedge Y) \vee (X \wedge Z).$$

Use this rule to help you rewrite $(\neg A \vee B) \wedge (A \vee \neg C)$ in DNF form. (And convince yourself how you could do so in general.)

Phew! That was a lot of work, but we got something very useful out of it. Now, instead of having to deal with all first-order sentences using dozens of different symbols in all sorts of orders, we only have to deal with sentences that look something like

$$\exists x, \neg\exists y, \exists z, \text{(some DNF formula in } x, y, \text{ and } z).$$

We're now in good shape to take a close look at these tomorrow.

# 4 Extra problems

**Problem 10.** (♦♦) Apply the reduction process from Problem 3 through 9 to the first-order sentences you wrote in Problem 1. It's a tedious slog to do all of them, but it's good practice to do at least one!

**Problem 11.** (♦♦♦) Rewrite the kissing spheres problem from the introduction as a first-order sentence: The maximum number of mutually non-intersecting spheres of radius 1, all tangent to a sphere at the origin of radius 1, is 12.

**Problem 12.** (♦♦♦) Problem 8 asked you to give a *syntactic* procedure to convert a formula into DNF form. Syntactic means you worked directly with the symbols in the formula itself. This requires you to pattern-match within the formula, which is how we're used to doing algebra, but it's not the only way.

Can you come up with a *semantic* method to convert to DNF form? This means that instead of changing the symbols in the formula step-by-step, the method is based on evaluating the formula, in other words plugging in values (true/false) for atoms.