STARK-HEEGNER CYCLES OVER ARBITRARY NUMBER FIELDS

LENNART GEHRMANN

ABSTRACT. TODO

Contents

Introduction

TODO

Notations. All rings are assumed to be commutative and unital. The group of invertible elements of a ring R will be denoted by R^*. Given an R-module M we put $M^\vee = \mathrm{Hom}_R(M,R)$. If S is an R-algebra and M an R-module, we put $M_S = M \otimes_R S$. If R is a ring and G a group, we will denote the group ring of G over R by $R[G]$. Let H be an open subgroup of a locally profinite group G and M an R-linear representation M of H. The *compact induction* $\text{c-ind}_H^G M$ of M from H to G is the space of all functions $f\colon G \to M$ such that:

- f has finite support modulo H and
- $f(hg) = h.f(g)$ for all $h \in H, g \in G$.

Compact induction $\text{c-ind}_H^G M$ is an R-module on which G acts R-linearly via the right regular representation. Let $\chi\colon G \to R^*$ be a character. We write $R[\chi]$ for the G-representation, which underlying R-module is R itself and on which G acts via the character χ. More generally, if M is any $R[G]$-module, we put $M(\chi) = M \otimes_R R(\chi)$. The trivial character will be denoted by $\mathbb{1}$.

Acknowledgements. TODO

1. The setup

We fix an algebraic number field F with ring of integers $\mathcal{O}$. In addition, we fix a finite place $\mathfrak{p}$ of F lying above the rational prime p and choose embeddings

$$\mathbb{C} \xleftarrow{\iota_\infty} \overline{\mathbb{Q}} \xhookrightarrow{\iota_p} \overline{\mathbb{Q}_p}.$$

We let Σ denote the set of all embeddings $\sigma\colon F \hookrightarrow \mathbb{C}$ and for a prime v lying above p we let Σ_v be the set of all continuous embeddings Σ_v. The two chosen embeddings ι_∞ and ι_p yield a decomposition

$$\Sigma = \bigcup_{v|p} \Sigma_v.$$

We denote the number of real places of F by $r_{\mathbb{R}}$ and the number of complex places by $r_{\mathbb{C}}$. If v is a place of F, we denote by F_v the completion of F at v. If v is a finite place, we let $\mathcal{O}_v$ denote the valuation ring of F_v and ord_v the additive valuation such that $\mathrm{ord}_v(\varpi) = 1$ for any local uniformizer $\varpi \in \mathcal{O}_v$. We write $\mathcal{N}(v)$ for the cardinality of the residue field of $\mathcal{O}_v$.

Let $\mathbb{A}$ be the adele ring of F, i.e the restricted product over all completions F_v of F. We write $\mathbb{A}^\infty$ (respectively $\mathbb{A}^{\mathfrak{p},\infty}$) for the restricted product over all completions of F at finite places (respectively finite places different from $\mathfrak{p}$). More generally, if S is a finite set of places of F we denote by $\mathbb{A}^S$ the restricted product of all completions F_v with $v \notin S$.

If H is an algebraic group over F and v is a place of F, we write $H_v = H(F_v)$. If l is a (possible infinite) rational place we put $H_l = \prod_{v|l} H_v$. Further, we put $H_p^{\mathfrak{p}} = \prod_{v|p,\ v\neq\mathfrak{p}} H_v$.

Throughout the article, we fix an inner form $\widetilde{G}$ of the algebraic group GL_2/F, which is split at the prime $\mathfrak{p}$. We denote the centre of $\widetilde{G}$ by Z and put $G = \widetilde{G}/Z$. If G is split, we always identify it with PGL_2. Similarly, if v is a place of F at which G is split, we choose an isomorphism of G_v with $PGL_2(F_v)$. We write q for the number of Archimedean places at which G is split.

At last, we fix a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ with the following properties:

- π is cohomological with respect to an algebraic coefficient system $V_{\mathrm{al},\mathbb{C}}$ (see Section 2.2 for more details) and
- $\pi_{\mathfrak{p}}$ is the (smooth) Steinberg representation $\mathrm{St}^\infty_{\mathfrak{p}}(\mathbb{C})$ of $G_{\mathfrak{p}} = PGL_2(F_{\mathfrak{p}})$.

2. Automorphic L-invariants

The aim of this section is to define automorphic $\mathcal{L}$-invariants. We follow broadly the same steps as Spieß in the Hilbert modular parallel weight 2 setting (see [Spi14]), though our arguments are more involved since our representation is not necessarily ordinary anymore.

2.1. Cohomology of $\mathfrak{p}$-arithmetic groups.

Throughout this section we fix a ring R.

Let $\mathrm{Div}(\mathbb{P}^1(F))$ denote the free abelian group on $\mathbb{P}^1(F)$ and $\mathrm{Div}_0(\mathbb{P}^1(F))$ the kernel of the map

$$\mathrm{Div}(\mathbb{P}^1(F)) \to \mathbb{Z},\ \sum_P m_P P \mapsto \sum_P m_P.$$

The $PGL_2(F)$-action on $\mathbb{P}^1(F)$ induces an action on $\mathrm{Div}_0(\mathbb{P}^1(F))$. If G is non-split, we put $\mathrm{H}^i_c(G(F), A) = \mathrm{H}^i(G(F), A)$. If G is split, we define $\mathrm{H}^i_c(G(F), A) = \mathrm{H}^{i-1}_c(G(F), \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}_0(\mathbb{P}^1(F)), A))$. In this case the boundary map associated to the short exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}(\mathbb{P}^1(F)), A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}_0(\mathbb{P}^1(F)), A) \longrightarrow 0$$

yields a map

$$\delta\colon\ \mathrm{H}^i_c(G(F), A) \longrightarrow \mathrm{H}^i(G(F), A). \tag{2.1}$$

Given a compact, open subgroup $K^{\mathfrak{p}} \subseteq G(\mathbb{A}^{\mathfrak{p},\infty})$, an $R[K^{\mathfrak{p}}]$-module $N^{\mathfrak{p}}$, an $R[G_{\mathfrak{p}}]$-module $M_{\mathfrak{p}}$ and an $R[G(F)]$-module N we define $\mathcal{A}_R(K^{\mathfrak{p}}, N^{\mathfrak{p}}, M_{\mathfrak{p}}; N)$ as the space of all R-bilinear maps $\Phi\colon G(\mathbb{A}^{\mathfrak{p},\infty}) \times N^{\mathfrak{p}} \times M_{\mathfrak{p}} \to N$ such that $\Phi(gk, kn, m) = k\Phi(g, n, m)$ for all $g \in G(\mathbb{A}^{\mathfrak{p},\infty})$, $k \in K^{\mathfrak{p}}$, $n \in \mathbb{N}^{\mathfrak{p}}$ and $m \in M_{\mathfrak{p}}$. The R-module $\mathcal{A}_R(K^{\mathfrak{p}}, N^{\mathfrak{p}}, M_{\mathfrak{p}}; N)$ carries a natural $G(F)$-action given by

$$(\gamma.\Phi)(g, n, m) = \gamma.(\Phi(\gamma^{-1}g, n, \gamma^{-1}.m)).$$

Most of the times the module $N^{\mathfrak{p}}$ is equal to R. In this case we put

$$\mathcal{A}_R(K^{\mathfrak{p}}, M_{\mathfrak{p}}; N) = \mathcal{A}_R(K^{\mathfrak{p}}, R, M_{\mathfrak{p}}; N).$$

Example 2.1. *If $M_{\mathfrak{p}}$ is of the form $\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} R$ for some compact, open subgroup $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$, we put*

$$\mathcal{A}(K^{\mathfrak{p}}K_{\mathfrak{p}}; N) = \mathcal{A}_R(K^{\mathfrak{p}}, M_{\mathfrak{p}}; N)$$

where $? \in \{\emptyset, c\}$. By definition we have a natural $G(F)$-equivariant isomorphism

$$\mathcal{A}(K^{\mathfrak{p}}K_{\mathfrak{p}}; N) \xrightarrow{\cong} C(G(\mathbb{A}^{\infty})/K^{\mathfrak{p}}K_{\mathfrak{p}}, N).$$

More generally, suppose $M_{\mathfrak{p}}$ is of the form $\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} N_{\mathfrak{p}}$ for some compact, open subgroup $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$ and some $R[G_{\mathfrak{p}}]$-module $N_{\mathfrak{p}}$ and that $N^{\mathfrak{p}}$ is a $G(\mathbb{A}^{\mathfrak{p},\infty})$-module. Then the map

$$\left(\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} R\right) \otimes_R N_p \longrightarrow \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} N_{\mathfrak{p}},\ (f,n) \longmapsto [g \mapsto f(g)\cdot g.n]$$

is an isomorphism of $R[G_{\mathfrak{p}}]$-modules. Hence, its inverse (and a similar map for the $N^{\mathfrak{p}}$-part) induces an isomorphism of $R[G(F)]$-modules

$$\mathcal{A}_R(K^{\mathfrak{p}}, N^{\mathfrak{p}}, M_{\mathfrak{p}}; N) \xrightarrow{\cong} C(G(\mathbb{A}^{\infty})/K^{\mathfrak{p}}K_{\mathfrak{p}}, \mathrm{Hom}_R(N^{\mathfrak{p}} \otimes_R N_{\mathfrak{p}}, N)).$$

Definition 2.2. *An $R[G_{\mathfrak{p}}]$-module M is called flawless if*

- *M is projective as an R-module and*
- *there exists a finite length exact resolution*

  $$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

  of $R[G_{\mathfrak{p}}]$-modules, where each P_i is a finite direct sum of modules of the form

  $$\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L$$

  with $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$ a compact, open subgroup and L an $R[K_{\mathfrak{p}}]$-module which is finitely generated projective over R.

Proposition 2.3. *Suppose that M is a flawless $R[G_{\mathfrak{p}}]$-module and that $N^{\mathfrak{p}}$ if finitely generated projective as an R-module. For $? \in \{\emptyset, c\}$ we have:*

(a) The R-module $\mathrm{H}^d_?(G(F), \mathcal{A}_R(K^{\mathfrak{p}}, N^{\mathfrak{p}}, M_{\mathfrak{p}}; N))$ is finitely generated for all d if R is Noetherian and N is finitely generated as an R-module.

(b) If S is a flat R-algebra, then the canonical map

$$\mathrm{H}^d_?(G(F), \mathcal{A}_R(K^{\mathfrak{p}}, N^{\mathfrak{p}}, M_{\mathfrak{p}}; N)) \otimes_R S \longrightarrow \mathrm{H}^d_?(G(F), \mathcal{A}_S(K^{\mathfrak{p}}, N^{\mathfrak{p}}_S, M_{\mathfrak{p},S}; N_S))$$

is an isomorphism for all $d \in \mathbb{Z}$.

Proof. This is essentially Proposition 4.9 of [Geh18]. $\square$

Example 2.4. *Let N be an $R[G(F)]$-module and $K^{\mathfrak{p}}K_{\mathfrak{p}} \subseteq G(\mathbb{A}^{\infty})$ a compact, open subgroup. In light of Example 2.1 we put*

$$\mathrm{H}^d(X_{K^{\mathfrak{p}}K_{\mathfrak{p}}}, N) = \mathrm{H}^d(G(F), \mathcal{A}(K^{\mathfrak{p}}K_{\mathfrak{p}}; N))$$

respectively

$$\mathrm{H}^d_c(X_{K^{\mathfrak{p}}K_{\mathfrak{p}}}, N) = \mathrm{H}^d_c(G(F), \mathcal{A}(K^{\mathfrak{p}}K_{\mathfrak{p}}; N)).$$

If $K^{\mathfrak{p}}K_{\mathfrak{p}}$ is neat or R is a field of characteristic 0, we can identify these groups with the N-valued singular cohomology (respectively singular cohomology with compact support) of the locally symmetric space of level $K^{\mathfrak{p}}K_{\mathfrak{p}}$ associated to G.

Let Ω be a finite extension of $\mathbb{Q}_p$ with ring of integers R, $V_{\mathfrak{p}}$ an Ω-Banach representation of $G_{\mathfrak{p}}$ and $V^{\mathfrak{p}}$ a finite dimensional continuous Ω-representation of $G_p^{\mathfrak{p}}$. We view $V^{\mathfrak{p}}$ as a $G(F)$-representation via the embedding $G(F) \hookrightarrow G_p^{\mathfrak{p}}$. Let $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ be a sign character. We define

$$\mathcal{A}_\Omega^{\mathrm{ct}}(K^{\mathfrak{p}}, V_{\mathfrak{p}}; V^{\mathfrak{p}}(\epsilon)) = C(G(\mathbb{A}^{\mathfrak{p},\infty})/K^{\mathfrak{p}}, \mathrm{Hom}_{\Omega,\mathrm{ct}}(V_{\mathfrak{p}}, V^{\mathfrak{p}}(\epsilon))).$$

Now let $V_{\mathfrak{p}}$ merely be a be a locally convex topological Ω-vector space equipped with a continuous $G_{\mathfrak{p}}$-action. Suppose that $V_{\mathfrak{p}}$ admits an open $R[G_{\mathfrak{p}}]$-lattice $M_{\mathfrak{p}}$ that is flawless. Since $M_{\mathfrak{p}}$ is finitely generated, it follows that the completion of $V_{\mathfrak{p}}$ with respect to $M_{\mathfrak{p}}$ is the universal unitary completion $V_{\mathfrak{p}}^{\mathrm{univ}}$ of $V_{\mathfrak{p}}$. We have the following automatic continuity statement.

Corollary 2.5. *Let $V_{\mathfrak{p}}$ be a finite length, locally $\mathbb{Q}_p$-algebraic representation of $G_{\mathfrak{p}}$ that admits a $G_{\mathfrak{p}}$-stable separated R-lattice and let $V^{\mathfrak{p}}$ be a finite dimensional Ω-representation of $G_p^{\mathfrak{p}}$. Then the canonical map*

$$\mathrm{H}^d_?(G(F), \mathcal{A}_\Omega^{\mathrm{ct}}(K^{\mathfrak{p}}, V_{\mathfrak{p}}^{\mathrm{univ}}; V^{\mathfrak{p}}(\epsilon))) \longrightarrow \mathrm{H}^d_?(G(F), \mathcal{A}_\Omega(K^{\mathfrak{p}}, V_{\mathfrak{p}}; V^{\mathfrak{p}}(\epsilon)))$$

is an isomorphism for all characters ϵ and $? \in \{\emptyset, c\}$.

Proof. By [Vig08], Proposition 0.4, the representation $V_{\mathfrak{p}}$ admits a flawless R-lattice $M_{\mathfrak{p}}$. Since $V^{\mathfrak{p}}$ is finite dimensional, Example 2.1 implies that

$$\mathcal{A}_\Omega(K^{\mathfrak{p}}, V_{\mathfrak{p}}; V^{\mathfrak{p}}(\epsilon)) = \mathcal{A}_\Omega(K^{\mathfrak{p}}, V^{\mathfrak{p},\vee}, V_{\mathfrak{p}}; \Omega(\epsilon)).$$

Again, by finite-dimensionality of $V^{\mathfrak{p},\vee}$ we see that it admits a $K_{\mathfrak{p}}$-stable lattice $N_{\mathfrak{p}}$. Therefore, Proposition 2.3 (b) implies that the canonical map

$$\mathrm{H}^d_?(G(F), \mathcal{A}_R(K^{\mathfrak{p}}, N^{\mathfrak{p}}, M_{\mathfrak{p}}; R(\epsilon))) \otimes_R \Omega \longrightarrow \mathrm{H}^d_?(\mathcal{A}_\Omega(K^{\mathfrak{p}}, V_{\mathfrak{p}}; V^{\mathfrak{p}}(\epsilon)))$$

is an isomorphism. But the former can be identified with the cohomology group $\mathrm{H}^d_?(G(F), \mathcal{A}_\Omega^{\mathrm{ct}}(K^{\mathfrak{p}}, V_{\mathfrak{p}}^{\mathrm{univ}}; V^{\mathfrak{p}}(\epsilon)))$ and, thus, the claim follows. $\square$

2.2. **The π-isotypical component.** We determine the π-isotypical component of various cohomology groups.

By assumption π is cohomological with respect to an algebraic coefficient system $V_{\mathrm{al},\mathbb{C}}$, i.e. there exists an irreducible algebraic $\mathbb{C}$-representation $V_{\sigma,\mathbb{C}}$ of $G_{\mathbb{C}}$ for every embedding $\sigma \in \Sigma$ such that

$$V_{\mathrm{al},\mathbb{C}} = \bigotimes_{\sigma\in\Sigma} V_{\sigma,\mathbb{C}}$$

and

$$\mathrm{Hom}_{\mathbb{C}[G(\mathbb{A}^\infty)]}(\pi^\infty, \varinjlim_{K^{\mathfrak{p}} K_{\mathfrak{p}}} \mathrm{H}^*(X_{K^{\mathfrak{p}} K_{\mathfrak{p}}}, V_{\mathrm{al},\mathbb{C}}^\vee)) \neq 0.$$

Here we let $G(F)$ act on $V_{\sigma,\mathbb{C}}^\vee$ via the embedding σ.

For the remainder of the article we fix a finite extension $\mathbb{Q}_\pi \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}$ such that

- $|\mathrm{Hom}(F, \mathbb{Q}_\pi)| = |\mathrm{Hom}(F, \overline{\mathbb{Q}})|$ and
- the finite part $\pi^{\mathfrak{p},\infty}$ away from $\mathfrak{p}$ of π has a model over $\mathbb{Q}_\pi$, i.e. $\pi^{\mathfrak{p},\infty} = \pi^{\mathfrak{p},\infty}_{\mathbb{Q}_\pi} \otimes_{\mathbb{Q}_\pi} \mathbb{C}$.

By the first assumption on $\mathbb{Q}_\pi$ each $V_{\sigma,\mathbb{C}}$ (viewed as an representation of $G(F)$) has a model $V_{\sigma,\mathbb{Q}_\pi}$ over $\mathbb{Q}_\pi$ and we put $V_{\mathrm{al},\mathbb{Q}_p i} = \otimes_\sigma V_{\sigma,\mathbb{Q}_\pi}$.

Let Ω be a field extension of $\mathbb{Q}_\pi$ and $K^{\mathfrak{p}} \subseteq G(\mathbb{A}^{\mathfrak{p},\infty})$ a compact, open subgroup such that $(\pi^{\mathfrak{p},\infty}_{\mathbb{Q}_\pi})^{K^{\mathfrak{p}}} \neq 0$. We denote the Ω-valued Hecke algebra of level $K^{\mathfrak{p}}$ away from $\mathfrak{p}$ by

$$\mathbb{T} = \mathbb{T}(K^{\mathfrak{p}})_\Omega = C_c(K^{\mathfrak{p}} \backslash G(\mathbb{A}^{\mathfrak{p},\infty})/K^{\mathfrak{p}}, \Omega).$$

If V is a $\mathbb{T}(K^{\mathfrak{p}})_\Omega$-module, we write

$$V[\pi] = \mathrm{Hom}_{\mathbb{T}}((\pi_\Omega^{\mathfrak{p},\infty})^{K^{\mathfrak{p}}}, V).$$

The Ω-valued smooth Steinberg representation $\mathrm{St}_{\mathfrak{p},\Omega}$ of $G_{\mathfrak{p}}$ is the space of all locally constant Ω-valued functions on $\mathbb{P}^1(F_{\mathfrak{p}})$ modulo constant function. The invariants of $\mathrm{St}_{\mathfrak{p},\Omega}$ under the Iwahori subgroup $\mathbb{I}_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$ are one-dimensional. Thus, by Frobenius reciprocity there exists a unique (up to scalar) non-zero $G_{\mathfrak{p}}$-equivariant map

$$\text{c-ind}_{I_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega \longrightarrow \mathrm{St}_{\mathfrak{p},\Omega},$$

which in turn induces a Hecke-equivariant map

$$\mathrm{ev}^{(d)}\colon \mathrm{H}^d_?(G(F), \mathcal{A}(K^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p},\Omega}; N)) \longrightarrow \mathrm{H}^d_?(X_{K^{\mathfrak{p}} I_{\mathfrak{p}}}, N) \tag{2.2}$$

for every $\Omega[G(F)]$-module N.

Proposition 2.6. *The following holds:*

(a) For every character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ and $? \in \{\emptyset, c\}$ we have

$$\dim_\Omega \mathrm{H}^d_?(X_{K^{\mathfrak{p}} I_{\mathfrak{p}}}, V^\vee_{\mathrm{al},\Omega}(\epsilon))[\pi] = \binom{r_{\mathbb{C}}}{d-q}.$$

(b) The map $\mathrm{ev}^{(d)}$ induces an isomorphism

$$\mathrm{H}^d_?(G(F), \mathcal{A}_\Omega(K^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p},\Omega}; V^\vee_{\mathrm{al},\Omega}(\epsilon)))[\pi] \xrightarrow{\mathrm{ev}^{(d)}} \mathrm{H}^d_?(X_{K^{\mathfrak{p}} I_{\mathfrak{p}}}, V^\vee_{\mathrm{al},\Omega}(\epsilon))[\pi]$$

for every character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ and all d.

(c) For every character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ we have

$$\dim_\Omega \mathrm{H}^d_?(G(F), \mathcal{A}_\Omega(K^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p},\Omega}; V^\vee_{\mathrm{al},\Omega}(\epsilon)))[\pi] = \binom{r_{\mathbb{C}}}{d-q}.$$

Proof. The proof of [Geh19b], Proposition 3.7, also works in this more general setup. □

It is well known that the space of smooth extensions of the trivial representation Ω with the Steinberg representation is one-dimensional (see for example [Cas74], Theorem 2 (b) for the case $\Omega = \mathbb{C}$). We fix a smooth non-split extension

$$0 \longrightarrow \mathrm{St}_{\mathfrak{p},\Omega} \longrightarrow \mathcal{E} \longrightarrow \Omega \longrightarrow 0.$$

This induces a short exact sequence

$$0 \longrightarrow \mathcal{A}(K^{\mathfrak{p}}, \Omega; V^\vee_{\mathrm{al},\Omega}(\epsilon)) \longrightarrow \mathcal{A}_{\mathbb{Q}}(K, \mathcal{E}; V^\vee_{\mathrm{al},\Omega}(\epsilon)) \longrightarrow \mathcal{A}(K^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p},\Omega}; V^\vee_{\mathrm{al},\Omega}(\epsilon)) \to 0.$$

The boundary map of the associated the long exact cohomology sequence induces the map

$$\mathrm{H}^d_?(G(F), \mathcal{A}(K^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p},\Omega}; V^\vee_{\mathrm{al},\Omega}(\epsilon)))[\pi] \xrightarrow{c^{(d)}_?[\pi]^\epsilon} \mathrm{H}^{d+1}_?(G(F), \mathcal{A}(K^{\mathfrak{p}}, \Omega; V^\vee_{\mathrm{al},\Omega}(\epsilon)))[\pi]$$

on π-isotypical components.

Lemma 2.7. *The map $c^{(d)}_?[\pi]^\epsilon$ is an isomorphism for every sign character ϵ and every degree d.*

Proof. The proof of [Geh19b], Lemma 3.8, also works in this more general setup. □

This together with Proposition 2.6 (c) implies:

Corollary 2.8. *For every character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ we have*

$$\dim_\Omega \mathrm{H}^{d+1}_?(G(F), \mathcal{A}_\Omega(K^{\mathfrak{p}}, \Omega; V^\vee_{\mathrm{al},\Omega}(\epsilon)))[\pi] = \binom{r_{\mathbb{C}}}{d-q}.$$

2.3. **P-adic special series.** Throughout this section we fix a finite extension $\Omega \subseteq \overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ such that the image of every continuous embedding $\sigma \in \Sigma_\mathfrak{p}$ is contained in Ω. We write R for its ring of integers.

Given an even integer $l \geq 0$ we let

$$V(l)_\Omega = \operatorname{Sym}^l \Omega^2 \otimes \det^{-l/2}$$

be the algebraic representation of $PGL_{2,\Omega}$ of highest weight l. We fix a tuple $k_\mathfrak{p} = (k_\sigma)_{\sigma\in\Sigma_\mathfrak{p}}$ of even integers $k_\sigma \geq 0$ and put $V(k_\mathfrak{p})_\Omega = \bigotimes_{\sigma\in\Sigma_\mathfrak{p}} V(k_\sigma)_\Omega$. We view $V(k_\mathfrak{p})_\Omega$ as a $G_\mathfrak{p}$-representation by letting it act on the k_σ-factor via the embedding $\sigma\colon G_\mathfrak{p} \hookrightarrow PGL_2(\Omega)$. Note that every irreducible $\mathbb{Q}_p$-rational Ω-representation of $G_{F_\mathfrak{p}}$ arises in this way. We put $\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})_\Omega = \mathrm{St}_{\mathfrak{p},\Omega} \otimes_\Omega V(k_\mathfrak{p})_\Omega$.

Proposition 2.9. *The locally $\mathbb{Q}_p$-algebraic representation $\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})_\Omega$ admits a flawless R-lattice.*

Proof. TODO: the case of Ω-rational representations is [Vig08], Proposition 0.9. Same proof should work here. But it should be somewhere in the literature. □

Let $B \subset G_\mathfrak{p}$ the subgroup of upper triangular matrices. Given a subset $J \subseteq \Sigma_\mathfrak{p}$ and a tuple $l = (l_\sigma)_{\sigma\in\Sigma_\mathfrak{p}}$ of integers we define the J-analytic character

$$\chi_l^J \colon B \longrightarrow \Omega^*,\ \begin{pmatrix} a & u \\ 0 & d \end{pmatrix} \longmapsto \prod_{\sigma\in J} \sigma(a/d)^{l_\sigma}.$$

We let $I'(k_\mathfrak{p})_\Omega^J = \left(\operatorname{Ind}_B^{G_\mathfrak{p}} \chi_{-k/2}^J\right)^{J-\mathrm{an}}$ be the locally J-analytic induction of the character $\chi_{-k/2}^J$ from B to $G_\mathfrak{p}$ and put

$$I(k_\mathfrak{p})_\Omega^J = \bigotimes_{\sigma\notin J} V(k_\sigma)_\Omega \otimes I'(k_\mathfrak{p})_\Omega^J.$$

Its subspace of (globally) algebraic vectors can be identified with $V(k_\mathfrak{p})_\Omega$. We define

$$\mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p}, \Omega) = I(k_\mathfrak{p})_\Omega^J / V(k_\mathfrak{p})_\Omega.$$

We have a canonical embedding

$$\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})_\Omega \longrightarrow \mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p})_\Omega.$$

Proposition 2.10. *Suppose that that for all $\sigma \in J$ the following bound holds:*

$$(*) \qquad \sum_{\tau\in\Sigma_\mathfrak{p},\ \tau\neq\sigma} k_\tau \leq k_\sigma.$$

Then the embedding $\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})_\Omega \longrightarrow \mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p})_\Omega$ induces an isomorphism of Ω-Banach representations

$$\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})_\Omega^{\mathrm{univ}} \longrightarrow \mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p})_\Omega^{\mathrm{univ}}.$$

Proof. TODO: should be somewhere in the literature (Breuil, de Ieso, Kidwell). REMARK: this is essentially Teitelbaum's extension of Amice-Velu theory. □

Remark 2.11. *A standard non-criticality assumption often used in control theorems for overconvergent cohomology (see for example [BSW19], theorem 8.7) is that equation $(*)$ holds for all $\sigma \in \Sigma_\mathfrak{p}$. This forces the prime $\mathfrak{p}$ to be of degree one or two and the weight $(k_\sigma)_{\sigma\in\Sigma_\mathfrak{p}}$ to be parallel.*

The following construction of extensions is due to Breuil (see [Bre04], Section 2.1). Let $\lambda\colon F_\mathfrak{p}^*\colon \Omega$ be a J-analytic homomorphism. We define $\tau(\lambda)$ to be the two dimensional Ω-representation given by

$$\begin{pmatrix} a & u \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \lambda(a/d) \\ 0 & 1 \end{pmatrix}$$

and put $\tau^J(k_\mathfrak{p},\lambda)=\tau\otimes\chi^J_{-k/2}$. The short exact sequence

$$0\longrightarrow \chi^J_{-k/2}\longrightarrow \tau^J(k_\mathfrak{p},\lambda)\longrightarrow \chi^J_{-k/2}\longrightarrow 0$$

induces the short exact sequence

$$0\longrightarrow I\prime(k_\mathfrak{p})^J_\Omega\longrightarrow \left(\mathrm{Ind}_B^{G_\mathfrak{p}}\,\tau(k_\mathfrak{p},\lambda)\right)^{J-\mathrm{an}}\longrightarrow I'(k_\mathfrak{p})^J_\Omega\longrightarrow 0$$

of locally J-analytic representations. Tensoring with $\otimes_{\sigma\notin J}V(k_\sigma)_\Omega$ yields a self-extension of $I(k_\mathfrak{p})^J_\Omega$. Finally, pullback via $V(k_\mathfrak{p})_\Omega\hookrightarrow I(k_\mathfrak{p})^J_\Omega$ and pushforward along $I(k_\mathfrak{p})^J_\Omega\twoheadrightarrow \mathrm{St}_\mathfrak{p}(k_\mathfrak{p})^{J-\mathrm{an}}_\Omega$ yields an exact sequence

$$0\longrightarrow \mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p})_\Omega\longrightarrow \mathcal{E}^J(k_\mathfrak{p},\lambda)_\Omega\longrightarrow V(k_\mathfrak{p})_\Omega\longrightarrow 0. \tag{2.3}$$

Remark 2.12. *Given two locally $\mathbb{Q}_p$-analytic Ω-representations W_1 and W_2 we denote by $\mathrm{Ext}^1_{\mathrm{an}}(W_1,W_2)$ the space of locally $\mathbb{Q}_p$-analytic extensions of W_2 by W_1. The map*

$$\mathrm{Hom}_{J-\mathrm{an}}(F_\mathfrak{p}^*,\Omega)\longrightarrow \mathrm{Ext}^1_{\mathrm{an}}(\mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p})_\Omega,V(k_\mathfrak{p})_\Omega),\ \lambda\longmapsto \mathcal{E}^J(k_\mathfrak{p},\lambda)_\Omega$$

is an isomorphism. In the case $F_\mathfrak{p}=\mathbb{Q}_p$ this is due to Breuil. In fact, an analogous statement is true for higher rank groups as well (see [Din19], Theorem 1, and [Geh19a], Theorem 2.13).

2.4. **Automorphic L-invariants.** Let $\Omega\subseteq\overline{\mathbb{Q}_p}$ be a finite extension of $\mathbb{Q}_p$ that contains $\mathbb{Q}_\pi$. We define

$$V_{\mathrm{al},\mathfrak{p},\Omega}=\bigotimes_{\sigma\in\Sigma_\mathfrak{p}}V_{\sigma,\Omega}$$

and

$$V^{\mathfrak{p}}_{\mathrm{al},\Omega}=\bigotimes_{\sigma\notin\Sigma_\mathfrak{p}}V_{\sigma,\Omega}.$$

We can extend the action of $G(F)$ on $V_{\mathrm{al},\mathfrak{p},\Omega}$ (resp. on $V^{\mathfrak{p}}_{\mathrm{al},\Omega}$) to an action of $G_\mathfrak{p}$ (resp. an action of $G_p^{\mathfrak{p}}$). Since $V_{\mathrm{al},\mathfrak{p},\Omega}$ is an irreducible $\mathbb{Q}_p$-rational representation of $G_{F_\mathfrak{p}}$ there exists a unique tuple $k_\mathfrak{p}=(k_\sigma)_{\sigma\in\Sigma_\mathfrak{p}}$ of even integers and an isomorphism $V_{\mathrm{al},\mathfrak{p},\Omega}\cong V(k_\mathfrak{p})$, which is unique up to multiplication with a scalar. We have the following chain of isomorphisms

$$\begin{aligned}\mathrm{H}^d_?(G(F),\mathcal{A}_\Omega(K^\mathfrak{p},\mathrm{St}_{\mathfrak{p},\Omega};V^\vee_{\mathrm{al},\Omega}(\epsilon)))&\xrightarrow{2.1}\mathrm{H}^d_?(G(F),\mathcal{A}_\Omega(K^\mathfrak{p},\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})_\Omega;(V^{\mathfrak{p}}_{\mathrm{al},\Omega})^\vee(\epsilon)))\\&\xrightarrow{2.5,\ 2.9}\mathrm{H}^d_?(G(F),\mathcal{A}^{\mathrm{ct}}_\Omega(K^\mathfrak{p},\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})^{\mathrm{univ}}_\Omega;(V^{\mathfrak{p}}_{\mathrm{al},\Omega})^\vee(\epsilon))).\end{aligned}$$

Let $J=J_{\max}\subseteq\Sigma_\mathfrak{p}$ be the maximal set of embeddings such that equation $(*)$ holds for all $\sigma\in J_{\max}$. Given a J-analytic homomorphism $\lambda\colon F_\mathfrak{p}^*\to\Omega$ we denote by

$$\mathcal{E}^J(k_\mathfrak{p},\lambda)_\Omega\in\mathrm{Ext}^1_{\mathrm{an}}(\mathrm{St}_\mathfrak{p}^{J-\mathrm{an}}(k_\mathfrak{p})_\Omega,V(k_\mathfrak{p})_\Omega)$$

be the extension associated to λ at the end of Section 2.3. By Proposition 2.10 we may form the cup product

$$\begin{aligned}&\mathrm{H}^d_?(G(F),\mathcal{A}^{\mathrm{ct}}_\Omega(K^\mathfrak{p},\mathrm{St}_\mathfrak{p}(k_\mathfrak{p})^{\mathrm{univ}}_\Omega;(V^{\mathfrak{p}}_{\mathrm{al},\Omega})^\vee(\epsilon)))\\&\xrightarrow{\cup\mathcal{E}^J(k_\mathfrak{p},\lambda)_\Omega}\mathrm{H}^{d+1}_?(G(F),\mathcal{A}_\Omega(K^\mathfrak{p},V_{\mathrm{al},\mathfrak{p},\Omega};(V^{\mathfrak{p}}_{\mathrm{al},\Omega})^\vee(\epsilon)))\\&\cong\mathrm{H}^{d+1}_?(G(F),\mathcal{A}_\Omega(K^\mathfrak{p},\Omega;(V_{\mathrm{al},\Omega})^\vee(\epsilon))).\end{aligned}$$

Let $c^{(d)}_?(\lambda)[\pi]^\epsilon$ denote the restriction of this map to the π-isotypical component.

Definition 2.13. *We define the $\mathcal{L}$-invariant*

$$\mathcal{L}_?(\pi,\mathfrak{p})^\epsilon \subseteq \mathrm{Hom}_{J_{\max}-\mathrm{an}}(F_\mathfrak{p}^*,\Omega)$$

of π at $\mathfrak{p}$ of sign ϵ as the kernel of the map $\lambda \mapsto c_?^{(q)}(\lambda)[\pi]^\epsilon$.

Note that the $\mathcal{L}$-invariant $\mathcal{L}_?(\pi,\mathfrak{p})^\epsilon$ really depends on the choice of embeddings ι_∞ and $\iota_\mathfrak{p}$ we made at the beginning.

Proposition 2.14. *The following holds for every sign character ϵ:*
(a) $\mathcal{L}_c(\pi,\mathfrak{p})^\epsilon = \mathcal{L}(\pi,\mathfrak{p})^\epsilon$ and
(b) $\mathcal{L}(\pi,\mathfrak{p})^\epsilon \subseteq \mathrm{Hom}_{J_{\max}-\mathrm{an}}(F_\mathfrak{p}^*,\Omega)$ is a subspace of codimension one that does not contain the subspace of locally constant homomorphisms.

Proof. The first claim follows from the fact that all maps considered in the construction commute with the map δ defined in (2.1). The second claim is a direct consequence of Proposition 2.6 and Lemma 2.7. ☐

3. P-adic Hodge theory

4. Stark-Heegner cycles

References

[Bre04] C. Breuil. Invariant $\mathcal{L}$ et série spéciale p-adique. *Annales scientifiques de l'École Normale Supérieure*, 37(4):559–610, 2004.

[BSW19] D. Barrera Salazar and C. Williams. p -adic l -functions for GL_2. *Canadian Journal of Mathematics*, 71(5):1019–1059, 2019.

[Cas74] W. Casselman. On a p-adic vanishing theorem of Garland. *Bull. Amer. Math. Soc.*, 80:1001–1004, 1974.

[Din19] Y. Ding. Simple $\mathcal{L}$-invariants for GL_n. *Transactions of the American Mathematical Society*, to appear, 2019.

[Geh18] L. Gehrmann. On Shalika models and p-adic L-functions. *Israel Journal of Mathematics*, 226(1):237–294, Jun 2018.

[Geh19a] L. Gehrmann. Automorphic L-invariants for reducitve groups. *preprint*, 2019.

[Geh19b] L. Gehrmann. Derived Hecke algebra and automorphic L-invariants. *Trans. Amer. Math. Soc.*, 372(11):7767–7784, 2019.

[Spi14] M. Spieß. On special zeros of p-adic L-functions of Hilbert modular forms. *Inventiones mathematicae*, 196(1):69–138, 2014.

[Vig08] M.-F. Vignéras. A criterion for integral structures and coefficient systems on the tree of $\mathrm{PGL}(2,F)$. *Pure Appl. Math. Q.*, 4(4, Special Issue: In honor of Jean-Pierre Serre. Part 1):1291–1316, 2008.

L. Gehrmann, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany

E-mail address: `lennart.gehrmann@uni-due.de`