t-Pebbling in k-connected diameter two graphs

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Abstract

Graph pebbling models the transportation of consumable resources. As two pebbles move across an edge, one reaches its destination while the other is consumed. The t-pebbling number is the smallest integer m so that any initially distributed supply of m pebbles can place t pebbles on any target vertex via pebbling moves. The 1-pebbling number of diameter two graphs is well-studied. Here we investigate the t-pebbling number of diameter two graphs under the lense of connectivity.

1 Introduction

Graph pebbling has an interesting history, with many challenging open problems. Calculating pebbling numbers of graphs is a well known computationally difficult problem. See [4, 5] for more background.

A configuration C of pebbles on the vertices of a connected graph G is a function $C:V(G)\to\mathbb{N}$ (the nonnegative integers), so that C(v) counts the number of pebbles placed on the vertex v. We write |C| for the $size \sum_v C(v)$ of C; i.e. the number of pebbles in the configuration. A pebbling step from a vertex u to one of its neighbors v reduces C(u) by two and increases C(v) by one. Given a specified root vertex r we say that C is t-fold r-solvable if some sequence of pebbling steps places t pebbles on r. We are concerned with determining $\pi_t(G,r)$, the minimum positive integer m such that every configuration of size m on the vertices of G is t-fold r-solvable. The t-pebbling number of G is defined to be $\pi_t(G) = \max_{r \in V(G)} \pi(G,r)$. We avoid writing t when t = 1.

Pebbling number of diameter 2 graphs was solved and characterized by the following theorem. For the purpose of the present work, it is enough to know that a pyramidal graph has no *universal* vertex (a vertex adjacent to every other vertex) and has connectivity 2.

Theorem 1 [2, 6] For a diameter 2 graph G with connectivity k and n vertices, $\pi(G) = n + 1$ if and only if k = 1 or G is pyramidal. Otherwise (i.e. k = 2 and G is not pyramidal, or $k \ge 3$), $\pi(G) = n$.

In contrast, other than the following bound, little is known about the t-pebbling number of diameter 2 graphs.

^{*2000} AMS Subject Classification: 05C40, 05C75, 05C87 and 05C99. Keywords: graph pebbling, pebbling number, connectivity.

Theorem 2 [3] If G is a diameter 2 graph on n vertices then $\pi_t(G) \leq \pi(G) + 4t - 4$. Moreover, $\liminf_{t \to \infty} \pi_t(G)/t = 4$.

The goal of the present paper is to determine the exact t-pebbling number of a large subfamily of diameter 2 graphs by considering their connectivity. Define $\mathcal{G}(n,k)$ to be the set of all k-connected graphs on n vertices having a universal vertex. Set $f_t(n,k) = n + 4t - k - 2$ and $h_t(n) = n + 2t - 2$. Notice that $h_t(n) \geq f_t(n,k)$ if and only if $k \geq 2t$. Define $p_t(n,k) = \max\{f_t(n,k), h_t(n)\}$. The main result is the following theorem which is proved in Section 3.

Theorem 3 If $G \in \mathcal{G}(n,k)$ then $\pi_t(G) = p_t(n,k)$.

We observe from our result that, for any fixed t, in the family of graphs with universal vertex, there are graphs whose t-pebbling number is much lower than the bound given by Theorem 2, and also that there are graphs reaching that bound: when $k \geq 2t$ we have $\pi_t(n,k) = (n+4t-4)-2(t-1)$; when k < 2t $\pi_t(n,k) = (n+4t-4)-(k-2)$.

It will be useful to take advantage of Menger's Theorem. The version of Menger's theorem that we use is the following (exercise 4.2.28 in [7]).

Theorem 4 (Menger's Theorem) [7] Let G be a k-connected graph and $S = \{v_1, \ldots, v_k\}$ be a multiset of vertices of G. For any $r \notin S$ there are k pairwise-internally-disjoint paths, one from each v_i to r.

2 Technical Lemmas

We begin with a lemma that is used to prove lower bounds on the pebbling number of a graph by helping to show that certain configurations are unsolvable.

For a vertex v, define its open neighborhood N(v) to be the set of vertices adjacent to v, and its closed neighborhood $N[v] = N(v) \cup \{v\}$. We say that a vertex y is a junior sibling of a vertex x (or, more simply, junior to x) if $N(y) \subseteq N[x]$, and that y is a junior if it is junior to some vertex x.

Lemma 5 (Junior Removal Lemma) [1] Given the graph G with root r and t-fold r-solvable configuration C, suppose that y is a junior with C(y) = 0. Then C (restricted to G - y) is t-fold r-solvable in G - y.

Given a configuration C of pebbles, we say that a path $Q=(r,q_1,\ldots,q_j)$ with $j\geq 1$ is a *slide* from q_j to r if no q_i is zero (it has no pebbles on) and q_j has at least two pebbles.

A potential move is a pair of pebbles sitting on the same vertex. To say that C has j potential moves means that the j pairs are pairwise disjoint. For example, any configuration on 5 vertices with values 0, 1, 1, 2, and 7 has 4 potential moves. The potential of C, pot(C), is the maximum j for which C has j potential moves. Because every solution that requires a pebbling move uses a potential move, the following fact is evident.

Fact 6 Let r be an empty vertex in a configuration C with pot(C) < t. Then C is not t-fold r-solvable.

Basic counting yields the following lemma.

Lemma 7 (Potential Lemma) Let G be a graph on n vertices. If C is a configuration on G of size n + y $(y \ge 0)$ having z zeros, then $pot(C) \ge \lceil \frac{y+z}{2} \rceil$.

A nice application of the Potential Lemma is the following, which we will use repeatedly in the arguments that follow.

Lemma 8 (Slide Lemma) Let r be a vertex of a k-connected graph G. Let C be a configuration on G of size n+y $(y \ge 0)$ with z zeros. If $\lceil \frac{y+3z}{2} \rceil \le k$ then C is $\lceil \frac{y+z}{2} \rceil$ -fold r-solvable.

Proof. Set $p = \lceil \frac{y+z}{2} \rceil$. By Lemma 7 we can choose a set P of p potential moves. Note that the hypothesis implies that $p \leq k - z$. Delete all non-root zeros to obtain G'. Since G is k-connected, G' is p-connected. Thus Menger's Theorem 4 implies that there are p pair-wise disjoint slides in G' from P to r, which yield p r-solutions.

3 Proof of Theorem 3

The proof will follow from Lemmas 9 and 10, below. Let u be a universal vertex of a graph $G \in \mathcal{G}(n,k)$. If C is a configuration of size n+2t-3 with u empty and every other vertex odd then $\mathsf{pot}(C) = t-1$, and so C is not t-fold u-solvable. Hence $\pi_t(G,u) \geq n+2t-2$. On the other hand, if $|C| \geq n+2t-2$ then $\mathsf{pot}(C) \geq t$ when u is empty, and $\mathsf{pot}(C) \geq t-1$ when u is not; either way C is t-fold u-solvable because u is universal. Thus $\pi_t(G,u) = n+2t-2$, which is at most $p_t(n,k)$ always.

3.1 Lower bound

Clearly, $\pi_t(G) \geq \pi_t(G, u) = h_t(n)$. Now let r be any non-universal vertex of G, and let s be a vertex at distance two from r. Let X be any (r, s)-cutset of size k (in particular, $u \in X$) and define the configuration $F_t(n, k)$ by placing 0 on r and X, 4t - 1 on s, and 1 on each vertex of $V(G) - (X \cup \{r, s\})$; then $|F_t(n, k)| = (4t - 1) + (n - k - 2) = f_t(n, k) - 1$.

Since the vertices of $X - \{u\}$ have 0 pebbles and all them are juniors to u, Lemma 5 states that if t pebbles can reach r then 2t pebbles can reach u. But, with exactly 2t-1 potential moves in F, by Fact 6, we can place at most 2t-1 pebbles on u. Therefore $\pi_t(G,r) \geq f_t(n,k)$, implying $\pi_t(G) \geq f_t(n,k)$.

We record these results as

Lemma 9 For $G \in \mathcal{G}(n,k)$ we have $\pi_t(G) \geq p_t(n,k)$.

| k^t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|-----|----|-----|------|----|----|
| 2 | 0 | 4 | 8 | 12 | 16 | 20 | 24 | 28 |
| 3 | 0 | 3 | 7 | 11 | 15 | 19 | 23 | 27 |
| 4 | 0 | 2 | 6 | 10 | 14 | 18 | 22 | 26 |
| 5 | 0 | 2 | (5) | 9 | 13 | 17 | 21 | 26 |
| 6 | 0 | 2 | 4 | 8 | 12 | 16 | 20 | 24 |
| 7 | 0 | 2 | 4 | 7 | 11 | 15 | 19 | 23 |
| 8 | 0 | 2 | 4 | 6 | 10 | 14 | 18 | 22 |
| 9 | 0 | 2 | 4 | 6 | 9 | 13 | 17 | 21 |
| 10 | 0 | 2 | 4 | 6 | 8 | 12 | 16 | 20 |
| 11 | 0 | 2 | 4 | 6 | 8 : | (11) | 15 | 19 |

Figure 1: The values m for which $\pi_t(G) = |V(G)| + m$.

3.2 Upper bound

We will prove that any configuration of size $f_t(n, k)$ when $k \leq 2t$, and of size $h_t(n)$ when $k \geq 2t$, is t-fold r-solvable for any $r \in V(G)$.

Lemma 10 For $k \geq 2$, let $G \in \mathcal{G}(n,k)$ with a universal vertex u, and let r be any root vertex. Then $\pi_t(G,r) \leq p_t(n,k)$.

Proof. First note that the lemma is true when t = 1. Indeed, in this case we have $k \geq 2t$, and so $p_t(n,k) = h_t(n) = n + 2t - 2 = n$. On the other hand, because no pyramidal graph has a universal vertex, we have from Theorem 1 that $\pi(G) = n$, hence $\pi(G, r) \leq n$.

In addition, the lemma holds for k=2. Indeed, in this case we have $k \leq 2t$, and so $p_t(n,k) = f_t(n,k) = n+4t-k-2 = n-4t-4$. Also, we have by Theorem 2 that $\pi_t(G,r) \leq n+4t-4$.

Hence, we may assume that $t \geq 2$ and $k \geq 3$. Figure 1 shows the structure of this proof. As was noted above, the grey section has been proven before. We continue by proving the dashed-bordered, lower left section and diagonal circled entries together, and then the solid-bordered, upper right section by induction.

Base case.

We will simultaneously address the case k = 2t - 1 (the circled entries), for which $|C| = f_t(n, k) = n + 2t - 1$, and the case $k \ge 2t$ (the dashed-bordered section), for which $|C| = h_t(n) = n + 2t - 2$, by writing $k \ge 2t - 1$ and considering a configuration of size $|C| = n + 2t - 2 + \phi$, where $\phi = 1$ if 2t - 1 = k and 0 otherwise. The natural idea we leverage here is repeating the argument that zeros force potential which, combined with connectivity, yields either more solutions or more zeros.

Let $x \ge 0$ such that k = 2t - 1 + x. By Lemma 7, since we may assume that C(r) = 0 (otherwise induct on t), we have at least $\lceil (2t - 2 + 1)/2 \rceil = t$ potential

moves. Therefore, we have at least t solutions if there are at least t different slides from them to r.

Thus we consider the case in which there are at most t-1 slides; that is, from some of the vertices in which a potential move is sitting, say v, there is no path to r without an internal zero after considering the remaining t-1 slides. Since G is k-connected, that implies that C has at least k-(t-1) zeros between v and r and so, because of r, C has at least k - (t - 1) + 1 = t + 1 + x zeros.

Assume that there are exactly z = t + 1 + j zeros, for some $j \ge x$. Then, by Lemma 7, C has at least

$$\left\lceil \frac{(2t-2) + (t+1+j)}{2} \right\rceil = t + \left\lceil \frac{t-1+j}{2} \right\rceil$$

potential moves. If there are at least $t - \left\lceil \frac{t-1+j}{2} \right\rceil$ slides from them to r, then we can use those slides for that many solutions. Then, the other $\left\lceil \frac{t-1+j}{2} \right\rceil$ solutions can be obtained from the remaining $2 \left\lceil \frac{t-1+j}{2} \right\rceil$ potential moves, putting $2\lceil \frac{t-1+j}{2}\rceil$ pebbles on the universal vertex u and then $\lceil \frac{t-1+j}{2}\rceil$ on r. Otherwise, there are at most $t-\lceil \frac{t-1+j}{2}\rceil-1$ slides, from which we find, using

k = 2t - 1 + x, at least

$$k - \left(t - \left\lceil \frac{t - 1 + j}{2} \right\rceil - 1\right) + 1 = t + x + \left\lceil \frac{t - 1 + j}{2} \right\rceil + 1$$

zeros. Clearly, this number cannot exceed the total number of zeros z=t+1+j; therefore $j\geq x+\left\lceil\frac{t-1+j}{2}\right\rceil\geq x+\frac{t-1+j}{2},$ and so $j\geq t-1+2x.$

2t + 2x + i. Applying Lemma 7 again, there are at least

$$\left\lceil \frac{(2t-2) + (2t+2x+i)}{2} \right\rceil = 2t + x - 1 + \lceil i/2 \rceil$$

potential moves.

If either $x \geq 1$ or $i \geq 1$, then we can move 2t pebbles to the universal vertex u, and then t to r.

Hence, we consider the case for which x = i = 0; i.e. k = 2t - 1, z = 2t, and |C| = n + 2t - 1 (because $\phi = 1$ in such a case). We let T be the star centered on u, having leaves r and the nonzero vertices of G. Clearly, T is a subgraph of G with n + 2t - 1 pebbles on it and with either 2 + (n - z) or 1 + (n - z) vertices, depending on whether u is empty or not. In either case $n(T) \le 2 + n - z = 2 + n - 2t$. Therefore, since

$$\pi_t(T,r) = n(T) + 4t - 3 < (2 + n - 2t) + 4t - 3 = n + 2t - 1 = |C(T)|,$$

we see that C is r-solvable.

Induction step.

Finally, we consider the cases when k < 2t - 1 (the solid-bordered section); so $|C| = f_t(n,k) = n + 4t - k - 2$. Since $2(t-1) = 2t - 1 - 1 \ge k$, we have $\pi_{t-1}(G,r) = f_{t-1}(n,k) = n+4(t-1)-k-2 = n+4t-k-2-4 = |C|-4$. Hence, if C has a solution of cost at most 4, we are done. Otherwise, there is at most one vertex v having two or more pebbles, and on such a vertex there are at most 3 pebbles. This implies the contradiction $|C| \leq 3 + (n-2)$, which completes the proof.

In future work we intend to study k-connected diameter two graphs without a universal vertex, and use that work as a base step toward studying graphs of larger diameter.

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