Discrete Applied Mathematics 161 (2013) 1221–1231

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

General graph pebbling

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ARTICLE INFO

Article history:
Received 29 April 2009
Received in revised form 19 October 2011
Accepted 7 March 2012
Available online 1 April 2012

Keywords:
Pebbling
Configuration
Threshold
Class 0
Zero-sum sequence

ABSTRACT

Graph pebbling is the study of whether pebbles from one set of vertices can be moved to another while pebbles are lost in the process. A number of variations on the theme have been presented over the years. In this paper we provide a common framework for studying them all, and present the main techniques and results. Some new variations are introduced as well and open problems are highlighted.

1. Pebbling framework

We begin by introducing relevant terminology and background on the subject. Here, the term *graph* refers to a simple graph without loops or multiple edges, which we will assume here to be connected. For the definitions of other graph theoretical terms see any standard graph theory text such as [74].

A configuration C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V \to \mathbb{N}$, the nonnegative integers. The value $C(v)$ equals the number of pebbles placed at vertex v, and the *size* of the configuration is the number $|C| = \sum_{v\in V} C(v)$ of pebbles placed in total on G. The *support* of C is defined to be $\sigma(C) = \{v \in V \mid C(v) > 0\}$. When $C(v) \le C'(v)$ for all $v \in V$, we write $C \le C'$. Let $\mathsf{w} : E \to \mathbb{N}$ be a weight function on the edges of G—we denote the resulting weighted graph by G_{w}. A pebbling step along an edge $e = uv$ from u to v reduces by $\mathsf{w}(e)$ the number of pebbles at u and increases by 1 the number of pebbles at v. We say that a configuration D can be *reached* by C if one can repeatedly apply pebbling steps so that, in the resulting configuration C', we have $C'(v) \ge D(v)$ for all $v \in V$. In this case we say that C is *D-solvable* on G_{w}, and the particular sequence S of pebbling steps that witnesses this is called a *D-solution*. One can say that the subject of graph pebbling centers on finding conditions that imply that one configuration D is (or is not) solvable from another C on an edge-weighted graph G_{w}, perhaps because C is (not) large enough in terms of G_{w} and D. This problem was found to be NP-complete in [54] (see also [61,72]), even when w has constant weight 2 and $|D| = 1$. Of course, when considering such problems one may assume that $\sigma(C) \cap \sigma(D) = \emptyset$.

Furthermore, for a set $\mathscr{D}$ of configurations on G, we say that C is *$\mathscr{D}$-solvable* on G_{w} if C can reach every $D \in \mathscr{D}$. In this context one studies worst case, best case, or average case behavior. In the worst case, the *pebbling number* $\pi(G_{\mathsf{w}}, \mathscr{D})$ is defined to be the smallest integer m so that every configuration C of size m is *$\mathscr{D}$-solvable* on G_{w}. That is, the largest configuration that cannot reach some $D \in \mathscr{D}$ has size $\pi(G_{\mathsf{w}}, \mathscr{D}) - 1$. This is the area of most research to date; we discuss it in Section 2. In the best case, the *optimal pebbling number* $\pi^*(G_{\mathsf{w}}, \mathscr{D})$ is defined to be the smallest integer m so that some configuration C of size m is *$\mathscr{D}$-solvable* on G_{w}. Thus, every configuration of size $\pi^*(G_{\mathsf{w}}, \mathscr{D}) - 1$ has some configuration $D \in \mathscr{D}$ that it cannot

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doi:10.1016/j.dam.2012.03.010

reach. We discuss optimal pebbling in Section 3. Average case analysis takes the form of calculating the probability that a randomly chosen configuration of m pebbles reaches $\mathscr{D}$. Here, one aims to quantify the general observation that most large and few small configurations can reach $\mathscr{D}$. Section 4.2 defines the *threshold* τ between what is large and small; the concept depends on an infinite sequence of graphs, rather than on a single graph.

In all cases, when $\mathscr{D} = \{D\}$ we will use the notation D in place of $\mathscr{D}$, and when D is the characteristic function of a vertex r (which often goes by the name *root* in this case)—that is, $D(r) = 1$ and $D(v) = 0$ otherwise—we will use the notation r in place of D. Thus $\pi(G_{\mathsf{w}}, r)$ is the minimum number m so that every configuration of size m can reach r. Of course, $\pi^*(G_{\mathsf{w}}, r) = 1$; more generally, $\pi^*(G_{\mathsf{w}}, D) = |D|$. Similarly, if w is the constant function $\mathsf{w}(e) = q$, we write G_q in place of G_{w}, with the subscript suppressed entirely when $q = 2$. Our consideration of pebbling weights w appears mostly in Sections 2.3 and 2.4.

Here we introduce the new framework of target pebbling. A configuration C on G is *weakly $\mathscr{D}$-solvable* if C can reach some $D \in \mathscr{D}$. The *target pebbling number* $\pi^-(G_{\mathsf{w}}, \mathscr{D})$ is the minimum number m for which every configuration of size m is *weakly $\mathscr{D}$-solvable*. We show in Section 5 how this parameter relates some new questions with some old ones.

Many other results, conjectures, open problems, and variations can be found in [52,53].

2. Pebbling numbers

Denote by $\mathscr{I}$ the set $\{r \mid r \in V\}$ of all root configurations, and for a configuration D on G_{w} define the configuration kD by $(kD)(v) = k(D(v))$ for all $v \in V$, and the set of configurations $k\mathscr{D} = \{kD \mid D \in \mathscr{D}\}$. Then the traditional *$k$-fold pebbling number* $\pi_k(G) = \pi(G, k\mathscr{I})$, with k suppressed when equal to 1.

2.1. Basics

Some of the principles used in pebbling analysis are fairly straightforward but still worth stating. For example, if $C \le C'$ and C is D-solvable then so is C'. This is often applied in contrapositive form. Also, if $E(G) \subseteq E(G')$ and C solves D on G then it does so on G' as well. This is particularly helpful when $D = r$ and G is a spanning tree of G', usually a breadth-first search tree rooted at r. One can think of these as the *Subconfiguration* and *Subgraph Principles*, respectively.

Another useful notion is that of the *cost* of a sequence of pebbling steps, which equals the total number of pebbles moved. The simplest use of it finds that $\pi_k(K_n) = n + 2k - 2$ by induction. Let C_k be the configuration with $C(r) = 0$, $C(u) = 2k - 1$ for some $u \ne r$, and $C(v) = 1$ for all other v. Then since every r-solution has cost at least 2, and removing any such solution yields a configuration $C' \le C_{k-1}$, C_k cannot reach kr. However, the pigeonhole principle finds a cost 2 solution to a given root r from any configuration of size at least n, and so any configuration of size at least $n + 2k - 2$ can reach kr. In general, the minimum cost r-solution from a configuration tv is 2^d, where $d = \mathsf{dist}(v, r)$. These two statements, together with the Subgraph Principle, yield $\pi_k(G) \ge \max\{n + 2k - 2, k2^{\mathsf{diam}(G)}\}$, where $n = n(G)$. If one pushes the argument slightly, for large enough $|C|$ the pigeonhole principle guarantees an r-solution of size at most $2^{\mathsf{diam}(G)}$. This idea was used in [48] to prove the *Fractional Pebbling Theorem* below. We define the fractional pebbling number $\hat{\pi}(G) = \lim_{k\to\infty} \pi_k(G)/k$.

Theorem 2.1.1. *Every graph G satisfies $\hat{\pi}(G) = 2^{\mathsf{diam}(G)}$.*

A tool called the *Weight Argument* appears in [62] for the case $\mathsf{w} = 2$. We descibe it more generally here. For any (not necessarily induced) path $P \subseteq G_{\mathsf{w}}$, define its weight $\mathsf{w}(P) = \prod_{e\in E(P)} \mathsf{w}(e)$. For vertices u and v let $\mathsf{w}(u, v)$ denote the minimum weight of a uv-path. Note that when w is constant the uv-path of minimum weight has minimum length, but in some pathological example it could be arbitrarily long. A necessary condition for the r-solvability of a configuration is given by the following.

Lemma 2.1.2. *If the configuration C is r-solvable on G_{w} then $\sum_{u\in V} C(u)/\mathsf{w}(u, r) \ge 1$.*

The generalized argument for D-solvability takes the following form.

Lemma 2.1.3. *The configuration C is D-solvable on G_{w} if and only if there is a nonnegative integral solution to the system $\{C(u) + \sum_{v\in V} \left(x_{v,u} - \mathsf{w}(u, v)x_{u,v}\right) \ge D(u)$ for all $u \in V\}$.*

Each inequality insures that the process of adding pebbles to and removing pebbles from those at u leaves enough to satisfy D (this generalizes a lemma from [72]). Linear optimization techniques will deliver in polynomial time a rational solution to the system of Lemma 2.1.3 or a certificate of its unsolvability. We can model these constraints by a bipartite graph. Let $Y = \sigma(D)$, $X = V - Y$, and $K = K(D)$ be the complete bipartite graph $X \times Y$. Define the edge weights w' by $\mathsf{w}'(uv) = \mathsf{w}(u, v)$. Then if C reaches D on $K_{\mathsf{w}'}$, it does so on G_{w} as well. Moreover, if C reaches D on G_{w}, it does so fractionally on $K_{\mathsf{w}'}$. Hence $\hat{\pi}(K_{\mathsf{w}'}, D) \le \pi(G_{\mathsf{w}}, D) \le \pi(K_{\mathsf{w}'}, D)$. This certainly motivates the study of generalized pebbling numbers of complete bipartite graphs. For example, can they be calculated in polynomial time?

The fairly intuitive *No-Cycle* Lemma 2.1.4 has proven to be another handy tool. A set of pebbling steps S in a graph G induces the obvious directed subgraph $G(S)$ of G.

Lemma 2.1.4. *If a configuration C is D-solvable then there is a D-solution S for which $G(S)$ is acyclic.*

Note that the undirected subgraph may (and sometimes must) have cycles.

One final trick is introduced in [9], the *Squishing* Lemma 2.1.5. A *thread* in a graph G is a path whose vertices have degree 2 in G.

Lemma 2.1.5. *For every $r \in V(G)$ there is a maximum r-unsolvable configuration such that, on each thread not containing r, all pebbles sit on one vertex or two adjacent vertices.*

The Squishing Lemma provides a simplified proof of the result in [66] that the cycle C_n has pebbling number $\pi(C_{2k}) = 2^k$ and $\pi(C_{2k+1}) = 2\lfloor 2^{k+1}/3 \rfloor + 1$.

In an effort to improve the obvious pigeonhole upper bound $\pi(G) \le (n-1)(2^d - 1) + 1$, where $d = \mathrm{diam}(G)$, Chan and Godbole [11] proved the following.

Theorem 2.1.6. *Let $\mathrm{dom}(G)$ denote the domination number of G. Then*

1. $\pi(G) \le (n-d)(2^d - 1) + 1$,
2. $\pi(G) \le (n + \lfloor \frac{n-1}{d} \rfloor - 1)2^{d-1} - n + 2$, *and*
3. $\pi(G) \le 2^{d-1}(n + 2\mathrm{dom}(G)) - \mathrm{dom}(G) + 1$.

The inequalities in parts 1 and 2 are sharp, and the coefficient of 2 in part 3 can be reduced to 1 in the case of perfect domination.

2.2. Class 0

Graphs G that satisfy $\pi(G) = n(G)$ are known as *Class* 0 graphs, which include the complete graph K_n, the d-dimensional cube Q^d [14], complete bipartite graphs $K_{m,m}$ [15], and many others. Any graph G with a cut vertex x has $\pi(G) > n(G)$. Indeed, let $r \in G_1$ and $u \in G_2$, where G_1 and G_2 are two components of $G - x$. Define the configuration C by $C(r) = C(x) = 0$, $C(u) = 3$ and $C(v) = 1$ for every other vertex v. Then $|C| = n$ and C cannot reach r. On the other hand, in [66] we find the following theorem.

Theorem 2.2.1. *If $\mathrm{diam}(G) = 2$ then $\pi(G) = n(G)$ or $n(G) + 1$.*

Class 0 graphs of diameter 2 are classified in [16]. A particularly crucial graph in the characterization is the graph G, built from the bipartite graph C_6 by connecting all the vertices of one of the parts of the bipartition to each other. G has connectivity 2 and diameter 2 but $\pi(G) > 6 = n(G)$, as witnessed by the configuration $C(a, b) = (3, 3)$, where a, b and the root r are independent. Recently, it was shown in [20] that the diameter two graphs G that are not Class 0 satisfy $\pi_k(G) = n + 4k - 3$ for all $k \ge 1$. Moreover, they use the idea of *cheap* solutions (solutions of cost at most 7) to prove the following.

Theorem 2.2.2. *If $\mathrm{diam}(G) = 2$ then $\pi_k(G) \le n + 7k - 6$.*

Of course, this leaves a substantial gap. In light of Theorem 2.1.1 the truth is surely closer to having coefficient 4 for k than to 7. The following corollary to the diameter two Class 0 characterization appears in [16]. Denote the connectivity of a graph G by $\kappa(G)$.

Theorem 2.2.3. *If $\mathrm{diam}(G) = 2$, and $\kappa(G) \ge 3$ then G is of Class 0.*

From this it follows that almost all graphs (in the probabilistic sense) are of Class 0, since almost every graph is 3-connected with diameter 2. The following result, conjectured in [16], was proved in [25]. This result was used to prove a number of other theorems, including Theorems 2.2.7, 2.4.3 and 4.1.1.

Theorem 2.2.4. *There is a function $k(d) \le 2^{2d+3}$ such that if G is a graph with $\mathrm{diam}(G) = d$ and $\kappa(G) \ge k(d)$ then G is of Class 0. Moreover, $k(d) \ge 2^d/d$.*

An upper bound for diameter 3 graphs was obtained in [8].

Theorem 2.2.5. *If $\mathrm{diam}(G) = 3$ then $\pi(G) \le 3n/2$, which is best possible.*

Theorem 2.2.4 applies well to the family of *Kneser graphs*, $K(m, t)$. For $m \ge 2t + 1$, $K(m, t)$ has vertices $\binom{[m]}{t}$ and edges $\{A, B\}$ whenever $A \cap B = \emptyset$. The case $t = 1$ is the complete graph K_m and the case $m = 5$ and $t = 2$ is the Petersen graph P, both of which are Class 0. When $t \ge 2$ and $m \ge 3t - 1$ we have $\mathrm{diam}(K(m, t)) = 2$. Also, it is not difficult to show that $\kappa(K(m, t)) \ge 3$ in this range, implying that $K(m, t)$ is Class 0 by Theorem 2.2.3. Furthermore, Chen and Lih [13] have shown that $K(m, t)$ is connected, edge transitive, and regular of degree $\binom{m-t}{t}$. A theorem of Lovász [59] states that such a graph has connectivity equal to its degree, and thus $\kappa = \kappa(K(m, t)) = \binom{m-t}{t}$. Therefore, using Theorem 2.2.4, one can obtain the following (see [51]).

Theorem 2.2.6. *For any constant $c > 0$, there is an integer t_0 such that, for $t > t_0$, $s \geq c(t/\log_2 t)^{1/2}$ and $m = 2t + s$, we have that $K(m, t)$ is Class 0.*

What makes the family of Kneser graphs interesting in this context is that the graphs become more sparse as m decreases toward $2t+1$, so the diameter (as well as the girth) increases and yet the connectivity decreases. Thus it is worth discovering whether $K(2t + s, t)$ Class 0 in the range $1 \leq s \ll (t/\log_2 t)^{1/2}$.

Regarding conditions which prohibit Class 0 membership, one can easily show that if $\mathsf{girth}(G) > 2\log n$ then $\pi(G) > n(G)$. The following question was asked in [51]: Is there a constant g so that $\mathsf{girth}(G) > g$ implies $\pi(G) > n(G)$? This question was answered in the negative in [23] by using Theorem 2.2.4, along with a probabilistic (deletion) method analogous to Erdős's construction [30] of graphs of arbitrarily high girth and chromatic number.

Theorem 2.2.7. *Let $g_0(n)$ denote the maximum number g such that there exists a Class 0 graph G on at most n vertices with finite $\mathsf{girth}(G) \geq g$. Then for all $n \geq 3$ we have*

$$\lfloor\sqrt{(\lg n)/2 + 1/4} - 1/2\rfloor \leq g_0(n) \leq 1 + 2\lg n.$$

2.3. Block bounds

In this section we assume that G has at least one cut vertex. Trees play a prominent role here, and their pebbling numbers were worked out in [14]. The work is generalized in [20] as follows. Recall the weight $\mathsf{w}(P) = \prod_{e\in E(P)} \mathsf{w}(e)$ of a path $P \subseteq T_{\mathsf{w}}$. Let r be a vertex of a tree T. An *r-path partition* of T is built by constructing the sequence of pairs (T_i, F_i), where each T_i is a subtree of T and each F_i is a subforest of T, and so that T_i and F_i partition $E(T)$. Starting with $T_0 = r$, at each stage we find a path $P_i \subseteq F_{i-1}$ that shares exactly one vertex, namely an endpoint of P_i, with T_{i-1}, and define $F_i = F_{i-1} - P_i$ and $T_i = T_{i-1} + P_i$. Eventually, $T_t = T$ for some t, and $\mathcal{P} = (P_1, \ldots, P_t)$ is the r-path partition. Then $\mathcal{P}$ is *r-maximal* if each P_i is the heaviest (according to w) such path available. Define $f_k(T, r) = k\mathsf{w}(P_1) \sum_{i=2}^{t} \mathsf{w}(P_i) - t + 1$.

Theorem 2.3.1. *If T is a tree then $\pi_k(T_{\mathsf{w}}, r) = f_k(T, r)$.*

For the case $\mathsf{w} = 2$, the proof in [14] applies induction to the trees in the forest $T - r$, while the proof in [9] uses a weight argument. Note that the condition of heaviest path in the construction of an r-maximal path partition converts to the condition of longest in this case. The proof in [20] for the general case applies induction to the tree T_{t-1}, and uses the No-Cycle Lemma 2.1.4. This result is used to provide an upper bound on the k-fold pebbling numbers of graphs G having connectivity 1. Let $B(G)$ be the block-cutpoint graph of G, and let b_i denote the vertex of $B(G)$ that corresponds to the block B_i in G. For each block B_i, call x_i the cut vertex of G that is closest to r, and denote the edge b_ix_i of $B(G)$ by e_i. Now define the weight $\mathsf{w}(e_i) = \pi(B_i, x_i)$. Finally, if r is not a cut vertex of G, let b be the vertex of $B(G)$ that represents the block B of G that contains r, and define $B'(G)$ to be the graph obtained from $B(G)$ by adding the extra vertex r' and edge $e = br'$ of weight $\mathsf{w}(e) = 1$. All other edges of $B'(G)$ have weight 1. Then we find the following.

Theorem 2.3.2. *Every graph G satisfies $\pi_k(G, r) \leq \pi_k(B'_{\mathsf{w}}(G), r')$.*

We also find in [20] an instance of the Subgraph Principle giving the exact pebbling number.

Theorem 2.3.3. *Let G be a graph for which each of its blocks is a clique. Let $r \in V(G)$ and T be a breadth-first search spanning tree of G rooted at r. Then $\pi_k(G, r) = \pi_k(T, r)$.*

2.4. Products

For two graphs G_1 and G_2, define the *Cartesian product* $G_1\Box G_2$ to be the graph with vertex set $V(G_1\Box G_2) = \{(v_1, v_2)|v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1\Box G_2) = \{\{(v_1, v_2), (w_1, w_2)\}|(v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2)) \text{ or } (v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1))\}$. The cube Q^d can be built recursively from the Cartesian product of paths of length two, and the result [14] that $\pi(Q^d) = 2^d$ (and more generally Theorem 2.4.2) would follow easily from Graham's conjecture, which has generated a great deal of interest.

Conjecture 2.4.1. *Every pair of graphs G_1 and G_2 satisfy $\pi(G_1\Box G_2) \leq \pi(G_1)\pi(G_2)$.*

Graham's conjecture has been verified in a number of instances. Among these, the conjecture holds for a tree by a tree [62], a cycle by a cycle [46,49,66], complete bipartite graphs [33], and wheels and fans [34]. It is also proven in [14] that the conjecture holds when each G_i is a path. Let P_n be a path with n vertices and for $\mathbf{d} = \langle d_1, \ldots, d_m\rangle$ let $P^{\mathbf{d}}$ denote the graph $P_{d_1+1}\Box\cdots\Box P_{d_m+1}$.

Theorem 2.4.2. *For nonnegative integers $d_1, \ldots, d_m$, $\pi(P^{\mathbf{d}}) = 2^{d_1+\cdots+d_m}$.*

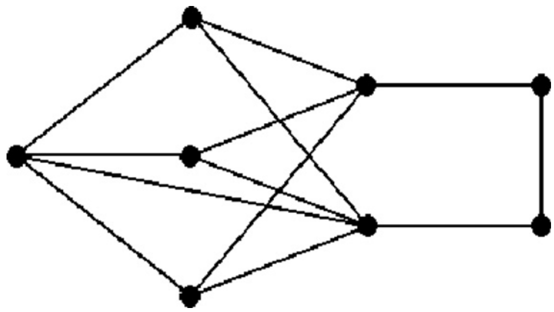

Fig. 1. The Lemke graph.

The conjecture was also verified [23] for graphs of high minimum degree, using Theorem 2.2.4.

Theorem 2.4.3. *If G_1 and G_2 are connected graphs on n vertices that satisfy $\delta(G_i) \geq k$ and $k \geq 2^{12n/k+15}$, then $\pi(G_1 \Box G_2) \leq \pi(G_1)\pi(G_2)$.*

In particular, there is a constant c so that if $k > cn/\lg n$ then $G_1 \Box G_2$ is Class 0. We will present probabilistic versions of Conjecture 2.4.1 in Sections 4.1 and 4.2.

A graph G is said to have the *2-pebbling property* if every configuration of size at least $2\pi(G) - \sigma(C) + 1$ is $2\mathscr{I}$-solvable. This property is crucial in the proof of Theorem 2.4.2. The smallest graph without the 2-pebbling property is the Lemke graph L in Fig. 1. This certainly makes finding $\pi(L^2)$ of great interest with regard to Graham's Conjecture. Given the complexity of calculating π, however, this is a tall order.

We say that a pebbling step from u to v is *greedy* if $\mathsf{dist}(v, r) < \mathsf{dist}(u, r)$, where r is the root vertex, and that a graph G is *greedy* if for any configuration of $\pi(G)$ pebbles on the vertices of G we can move a pebble to any specified root vertex r, in such a way that each pebbling step is greedy.

Returning to $P^{\mathbf{d}}$, each vertex $v \in V(P^{\mathbf{d}})$ can be represented by a vector $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$, with $0 \leq v_i \leq d_i$ for each $i \leq m$. Let $\mathbf{e}_i = \langle 0, \ldots, 1, \ldots, 0 \rangle$ be the *ith* standard basis vector. Denote the vector $\langle 0, \ldots, 0 \rangle$ by $\mathbf{0}$. Then two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent in $P^{\mathbf{d}}$ if and only if $|\mathbf{u} - \mathbf{v}| = \mathbf{e}_i$ for some integer $1 \leq i \leq m$. We define the weight function w on $E(P^{\mathbf{d}})$ by $\mathsf{w}(\mathbf{uv}) = w_i$ whenever $|\mathbf{u} - \mathbf{v}| = \mathbf{e}_i$. The proof of Theorem 2.4.2 uses the following theorem [14]. For integers $w_i, d_i \geq 1,\ 1 \leq i \leq m$, we use $\mathsf{w}^{\mathbf{d}}$ as shorthand for the product $w_1^{d_1} \cdots w_m^{d_m}$.

Theorem 2.4.4. *Every configuration on $P^{\mathbf{d}}_{\mathsf{w}}$ of size at least $\mathsf{w}^{\mathbf{d}}$ is greedily $\mathbf{0}$-solvable.*

This result is actually what gave birth to graph pebbling, since it was invented as a model to prove Theorem 2.5.2.

2.5. Group theory

The origins of graph pebbling reside in combinatorial number theory and group theory. A sequence of elements of a finite group G is called a *zero-sum sequence* if it sums to the identity of G. A simple pigeonhole argument (on the sequence of partial sums) proves the following theorem.

Theorem 2.5.1. *Any sequence of $|\mathsf{G}|$ elements of a finite group G contains a zero-sum subsequence.*

In fact, a subsequence of consecutive terms can be guaranteed by the pigeonhole argument. Furthermore, one can instead stipulate that the zero-sum subsequence has at most N terms, where $N = N(\mathsf{G})$ is the *exponent* of G (the least common multiple of the orders of the elements of G; equals the maximum order of an element of G when G is abelian), and this is best possible.

Initiated in 1956 by Erdős [29], the study of zero-sum sequences has a long history with many important applications in number theory and group theory. In 1961 Erdős et al. [31] proved that every sequence of $2|\mathsf{G}| - 1$ elements of a cyclic group G contains a zero-sum subsequence of length exactly $|\mathsf{G}|$. In 1969 van Emde Boas and Kruyswijk [70] proved that any sequence of $N(1 + \log(|\mathsf{G}|/N))$ elements of a finite abelian group contains a zero-sum sequence. In 1994 Alford et al. [1] used this result and modified Erdős's arguments to prove that there are infinitely many Carmichael numbers. Much of the recent study has involved finding *Davenport's constant* $D(\mathsf{G})$, defined to be the smallest m such that every sequence of m elements contains a zero-sum subsequence [65]. There are a wealth of results on this problem [10,38,39,41,44,68] and its variations [40,64], as well as applications to factorization theory [12] and to graph theory [3].

In 1989 Lemke and Kleitman [58], and independently Chung [14], proved the following theorem (originally stated number-theoretically), strengthening Theorem 2.5.1. Let $\mathbb{Z}_n$ denote the cyclic group of order n, and let $|g|$ denote the order of an element g in the group to which it belongs.

Theorem 2.5.2. *For every sequence $(g_k)_{k=1}^n$ of n elements from $\mathbb{Z}_n$ there is a zero-sum subsequence $(g_k)_{k\in K}$ such that $\sum_{k\in K} 1/|g_k| \le 1$.*

For a sequence S the sum $\sum_{g\in S} 1/|g|$ is known as the *cross number* of S and is an important invariant in factorization theory. As noted above, one can guarantee in Theorem 2.5.1 that $|K| \le N(\mathsf{G})$; here Theorem 2.5.2 strengthens this result to guaranteeing cross number at most 1, and shows also that the equality $|K| = N(\mathsf{G})$ holds if and only if every $|g_k| = N$.

The concept of pebbling in graphs arose from an attempt by Lagarias and Saks to give an alternative (and more natural and structural) proof than that of Kleitman and Lemke; it was Chung who carried out their idea. See also [27] for another extension of this result.

Kleitman and Lemke then conjectured that Theorem 2.5.2 holds for all finite groups. For a subgroup H of G, call a sequence of elements of G an H-*sum sequence* if its elements sum to an element of H. In [28,43] is proved the following theorem (the methods of [28] use graph pebbling).

Theorem 2.5.3. *Let H be a subgroup of a finite abelian group G with $|\mathsf{G}|/|\mathsf{H}| = n$. For every sequence $(g_k)_{k=1}^n$ of n elements from G there is an H-sum subsequence $(g_k)_{k\in K}$ such that $\sum_{k\in K} 1/|g_k| \le 1/|\sum_{k\in K} g_k|$.*

The case $\mathsf{H} = \{e\}$ here gives Theorem 2.5.2 for finite abelian groups, strengthening the van Emde Boas and Kruyswijk result [70]. Kleitman and Lemke also conjectured that Theorem 2.5.3 holds for all finite groups, and verified their conjecture for all dihedral groups (see [58]). For other nonabelian groups, it has been shown recently to hold for the nonabelian solvable group of order 21 (see [28]).

It would be interesting to see whether graph pebbling methods can shed light on the Davenport constant for finite abelian groups. In this regard, one of the most pressing questions is as follows. Write $\mathsf{G} = \prod_{i=1}^r \mathbb{Z}_{n_i}$, where $1 < n_1|n_2|\cdots|n_r$. Then $r = r(\mathsf{G})$ is the *rank* of G. It is natural to guess that $D(\mathsf{G}) = \sum_{i=1}^r n_i - r + 1 = 1 + \sum_{i=1}^r (n_i - 1)$ from pigeonhole intuition. This was conjectured in [65] and is true by Theorem 2.5.2 for rank 1 groups. Moreover, it was proven in [65] for rank 2 groups and p-groups as well. However, it was proven in [70,44] that the conjecture is false for some groups of each rank at least 4. What remains open is the instance of rank 3.

2.6. Cover pebbling

We denote by J the configuration with 1 pebble on each vertex. The paper [19] initiated the study of the *D-cover pebbling number* $\pi(G, D)$, first attempting to find $\pi(G, J)$. We say that the configuration D is *positive* if $D(v) > 0$ for every vertex v. In [19], for positive configurations D the D-cover pebbling number for cliques and trees is found, and it is shown that the ratio between this parameter and the pebbling number can be arbtrarily large, even within the class of trees. Given D define $\min D = \min_v D(v)$. The results are as follows.

Theorem 2.6.1. *For every positive configuration D we have $\pi(K_n, D) = 2|D| - \min D$.*

Given a graph G define $s(G, D) = \max_v \sum_u D(u)2^{\mathrm{dist}(u,v)}$.

Theorem 2.6.2. *For every positive configuration D on any tree T we have $\pi(T, D) = s(T, D)$.*

When $D = J$, a simple instance of T yields the following. The *fuse* $F_l(n)$ is the graph on n vertices composed of a path (or *wick*) on l vertices with $n - l$ independent vertices (*sparks*) incident to one of its endpoints.

Theorem 2.6.3. *For every n and l we have $\pi(F_l(n), J) = (n - l + 1)2^l - 1$. Thus, for $n = 2^l + l$, we have $\pi(F_l(n), J)/\pi(F_l(n)) > (n - \lg n)/2$.*

The statement that $\pi(G, J) = s(G, J)$ was shown true in [55] for cubes ($\pi(Q^d, J) = 3^d$), and for several other graphs in [69,73]. Finally, it was shown to hold for all graphs in [67,71].

Theorem 2.6.4. *For every positive configuration D on any graph G we have $\pi(G, D) = s(G, D)$.*

The main technique in proving Theorems 2.6.2 and 2.6.4 involves showing that the largest J-unsolvable configuration is *simple*; that is, concentrated on a single vertex. For this reason Theorem 2.6.4 is referred to as the *Stacking Theorem*. The same cannot be said in general for configurations that have zeros.

Next, we define the set $\mathscr{C}_k$ of all configurations on G of size k. The pebbling number $\pi(G, \mathscr{C}_k)$ was considered in [52]. In [47] it is conjectured that $\pi(G, \mathscr{C}_k) = \pi_k(G)$ for every graph G and for all $k \ge 1$ (of course, the definitions match for $k = 1$). It is also verified when G is a complete graph, a cycle, or has $\pi(G) = 2^{\mathrm{diam}(G)}$. It is further verified in [50] for trees.

Theorem 2.6.5. *If G is a complete graph, cycle, tree, or has $\pi(G) = 2^{\mathrm{diam}(G)}$ then $\pi(G, \mathscr{C}_k) = \pi_k(G)$.*

3. Optimal pebbling

Recall that the *optimal pebbling number* $\pi^*(G_{\mathrm{w}}, \mathscr{D})$ is defined to be the smallest integer m so that some configuration C of size m is *$\mathscr{D}$-solvable* on G_{w}. In the case that $\mathscr{D} = \mathscr{I}$ and $\mathrm{w} = 2$ we abbreviate to $\pi^*(G)$. The first results in this direction showed that $\pi^*(P_n) = \lceil 2n/3 \rceil$ [66]. An important technique called the *Smoothing* Lemma 3.1 in [9] provides a short proof of this result (see also [35]). In some sense it is analogous to the Squishing Lemma for pebbling since it details structure that one may assume occurs in an extremal configuration. While the Squishing Lemma gathers pebbles together, the Smoothing Lemma spreads them out. Suppose that v has degree 2 in G, and that $C(v) \geq 3$. A *smoothing move* at v removes two pebbles from v and adds one pebble to each of its neighbors. A *smooth* configuration has no smoothing move available. That is, C is smooth if $C(v) \leq 2$ whenever v has degree 2.

Lemma 3.1. *If G has at least 3 vertices then for every vertex r there is a minimum r-solvable configuration that is smooth and has no pebbles on degree 1 vertices.*

In [9] it is proved that $\pi^*(G) \leq \lceil 2n/3 \rceil$ for all G (equality also holds for cycles), and the lower bound $\pi^*(Q^d) \geq (4/3)^d$ for cubes is found in [63]. Caterpillars, cycles and other graphs have been considered in [35–37], and the following interesting analog of Graham's conjecture was proven in [35,36].

Theorem 3.2. *For all graphs G and H we have $\pi^*(G \square H) \leq \pi^*(G)\pi^*(H)$.*

What is most surprising is that, while $\pi(Q^n)$ is known exactly, only $\pi^*(Q^n) = (\frac{4}{3})^{n+O(\log n)}$ is known at present [63]. Some of the newest results consider graphs of high minimum degree. The following result of Czygrinow appears in [9].

Theorem 3.3. *If G is a connected graph with n vertices and $\delta(G) = k$, then $\pi^*(G) \leq \frac{4n}{k+1}$.*

The result is not known to be sharp—the tightest result to date appears in [9].

Theorem 3.4. *For all $t \geq 1$, $k = 3t$ and $n \geq k + 3$, there is a graph G with $\delta(G) = k$ and $\pi^*(G) \geq (2.4 - \frac{24}{5k+15} - o(1))\frac{n}{k+1}$.*

The graphs discovered for this theorem are clever modifications of a blow-up of the vertices of a cycle into cliques. The proof of this and many other theorems relies on the *Collapsing* Lemma 3.5. For $S \subseteq V(G)$, the operation of *collapsing* S forms a new graph H in which S is replaced by a single vertex that is adjacent to all the neighbors of vertices of S that are in $V - S$. It is just like the standard graph contraction but without the requirement that S be connected.

Lemma 3.5. *If H is obtained from G by collapsing sets of vertices then $\pi^*(G) \geq \pi^*(H)$.*

It would be interesting to discover the correct asymptotic coefficient of $\frac{n}{k+1}$, somewhere between 2.4 and 4. The authors also ask if the general upper bound of $\lceil 2n/3 \rceil$ can be improved to $\lceil n/2 \rceil$ for graphs with $\delta(G) \geq 3$.

If girth is also considered then one can say more. Let $c_k(t) = 1 + k\sum_{i=0}^{t-1}(k-1)^i$ and $c'(t) = (2^{2t} - 2^{t+1})\frac{t}{t-1}$. The following theorem of [9] displays an asymptotic bound of $3n/8$.

Theorem 3.6. *Let $k \geq 3$, $t \geq 2$ and $(k, t) \notin (3, 2)$. Then every n-vertex graph G with $\delta(G) = k$ and $\mathsf{girth}(G) \geq 2t + 1$ satisfies $\pi^*(G) \leq 2^{2t}n/(c_k(t) + c'(t))$.*

The optimal pebbling numbers of linear ($P_m \square K_2$), circular ($C_m \square K_2$) and Möbius (circular with a twist) ladders are also determined in [9]: $\pi^* = m$ unless $m \in \{2, 5\}$, with m as a lower bound always.

It is worth noting that most of the optimal counterparts to pebbling parameters have seen little or no study. Optimal cover pebbling is trivial: $\pi^*(G_{\mathrm{w}}, J) = |J|$—in fact, $\pi^*(G_{\mathrm{w}}, D) = |D|$ for every configuration D and weight w. Interesting instances of $\pi^*(G_{\mathrm{w}}, \mathscr{D})$ include $\mathscr{D} = k\mathscr{I}$ (the *k-fold optimal pebbling number* $\pi_k^*(G)$) and $\mathscr{D} = \mathscr{C}_k$. Furthermore, the *fractional optimal pebbling number* $\hat{\pi}^*(G) = \lim_{k\to\infty} \pi_k^*(G)/k$ can be calculated in polynomial time using linear optimization. Finally, finding graphs that satisfy the extreme $\pi^*(G) = \lceil 2n/3 \rceil$ should be worthwhile.

4. Thresholds

4.1. Graph thresholds

The notion that graphs with very few edges tend to have large pebbling number and graphs with very many edges tend to have small pebbling number can be made precise as follows. Let $\mathcal{G}_{n,p}$ be the random graph model in which each of the $\binom{n}{2}$ possible edges of a random graph having n vertices appears independently with probability p. For functions f and g on the natural numbers we write that $f \ll g$ (or $g \gg f$) when $f/g \to 0$ as $n \to \infty$. Let $o(g) = \{f \mid f \ll g\}$ and define $O(g)$ (resp., $\Omega(g)$) to be the set of functions f for which there are constants c, N such that $f(n) \leq cg(n)$ (resp., $f(n) \geq cg(n)$) whenever $n > N$. Finally, let $\Theta(g) = O(g) \cap \Omega(g)$.

Let $\mathscr{P}$ be a property of graphs and consider the probability $\Pr(\mathscr{P})$ that the random graph $\mathcal{G}_{n,p}$ has $\mathscr{P}$. For large p it may be that $\Pr(\mathscr{P}) \to 1$ as $n \to \infty$, and for small p it may be that $\Pr(\mathscr{P}) \to 0$ as $n \to \infty$. More precisely, define the *threshold* of

$\mathcal{P}$, t($\mathcal{P}$), to be the set of functions t for which $p \gg t$ implies that $\Pr(\mathcal{P}) \to 1$ as $n \to \infty$, and $p \ll t$ implies that $\Pr(\mathcal{P}) \to 0$ as $n \to \infty$.

It is not clear that such thresholds exist for arbitrary $\mathcal{P}$. However, we observe that Class 0 is a monotone property (adding edges to a Class 0 graph maintains the property), and a theorem of Bollobás and Thomason [6] states that t($\mathcal{P}$) is nonempty for every monotone $\mathcal{P}$. It is well known [32] that t(connected) $= \Theta(\lg n/n)$, and since connectedness is required for Class 0, we see that t(Class 0) $\subseteq \Omega(\lg n/n)$. In [16] it is noted that t(Class 0) $\subseteq O(1)$. In [25] Theorem 2.2.4 was used to prove the following result.

Theorem 4.1.1. *For all $d > 0$, t(Class 0) $\subseteq O((n \lg n)^{1/d}/n)$.*

4.2. Pebbling thresholds

For this section we will fix notation as follows. The vertex set for any graph on N vertices will be taken to be $\{v_i \mid i \in [N]\}$, where $[N] = \{0, \ldots, N-1\}$. That way, any configuration $C : V(G_n) \to \mathbb{N}$ is independent of G_n. (Here we make the distinction that n is the index of the graph G_n in a sequence $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$, whereas $N = N(G_n)$ denotes its number of vertices.) Let $\mathcal{K} = (K_1, \ldots, K_n, \ldots)$ denote the sequence of complete graphs, $\mathcal{P} = (P_1, \ldots, P_n, \ldots)$ the sequence of paths, and $\mathcal{Q} = (Q^1, \ldots, Q^n, \ldots)$ the sequence of n-dimensional cubes. Let $C_n : [N] \to \mathbb{N}$ denote a configuration on $N = N(G_n)$ vertices.

Let $h : \mathbb{N} \to \mathbb{N}$ and for fixed N consider the probability space X_N of all configurations C_n of size $h = h(N)$, we denote by P_N^+ the probability that C_N is G_n-solvable and let $t : \mathbb{N} \to \mathbb{N}$. We say that t is a *pebbling threshold* for $\mathcal{G}$, and write $\tau(\mathcal{G}) = \Theta(t)$, if $P_N^+ \to 0$ whenever $h(N) \ll t(N)$ and $P_N^+ \to 1$ whenever $h(N) \gg t(N)$. The existence of such thresholds was established in [4] and will be discussed in Section 4.3.

Theorem 4.2.1. *Every graph sequence $\mathcal{G}$ has nonempty $\tau(\mathcal{G})$.*

The first threshold result is found in [15]. The result is merely an unlabeled version of the so-called "Birthday problem", in which one finds the probability that 2 of t people share the same birthday, assuming N days in a year.

Theorem 4.2.2. $\tau(\mathcal{K}) = \Theta(\sqrt{N})$.

The same threshold applies to the sequence of stars $(K_{1,n})$.

It was discovered in [21] that every graph sequence $\mathcal{G}$ satisfies $\tau(\mathcal{K}) \lesssim \tau(\mathcal{G}) \lesssim \tau(\mathcal{P})$, where $A \lesssim B$ is meant to signify that $a \in O(b)$ for every $a \in A$, $b \in B$. The authors also discovered that $\tau(\mathcal{G}) \subseteq O(N)$ when $\mathcal{G}$ is a sequence of graphs of bounded diameter, that $\tau(\mathcal{Q}) \subseteq O(N)$, and that $\tau(\mathcal{P}) \subseteq \Omega(N)$. Surprisingly, the threshold for the sequence of paths has not been determined. The lower bound found in [21] was improved in [4] to $\tau(\mathcal{P}) \subseteq \Omega(N2^{c\sqrt{\lg N}})$ for every $c < 1/\sqrt{2}$, while the upper bound of $\tau(\mathcal{P}) \subseteq O(N2^{2\sqrt{\lg N}})$ found in [4] was improved in [45] to $\tau(\mathcal{P}) \subseteq O(N2^{c\sqrt{\lg N}})$ for every $c > 1$. Finally the lower bound was tightened in [24] to nearly match the upper bound.

Theorem 4.2.3. *For any constant $c < 1$, we have $\tau(\mathcal{P}) \subseteq \Omega(N2^{c\sqrt{\lg N}})$.*

This still leaves room for a wide range of possible threshold functions. It is interesting that even within the family of trees, the pebbling thresholds can vary so dramatically, as in the case for paths and stars. Diameter seems to be a critical parameter. It is quite natural to guess that families of graphs with higher pebbling numbers have a higher threshold, that is, if $\pi(G_n) \le \pi(H_n)$ for all n then $\tau(\mathcal{G}) \le \tau(\mathcal{H})$. But this kind of monotonicity result remains unproven, even for sequences of trees. On the other hand, there is no real reason to believe such a relationship between worst-case behavior and average-case behavior should exist. In fact, consider the following. For a positive integer t and a graph G denote by $p(G, t)$ the probability that a randomly chosen configuration C of size t on G is solvable. Then the monotonicity relationship above would follow from the statement that if $\pi(G_n) \le \pi(H_n)$ then for all t we have $p(G_n, t) \ge p(H_n, t)$. Unfortunately, although seemingly intuitive, this implication is false. Using the Class 0 pebbling characterization theorem of [16], in [21] is found a family of pairs of graphs (G_n, H_n), one pair for each $n = 3k + 4$, for which the implication fails.

What is more reasonable to expect is that, for any functions t_1 and t_2 satisfying $\tau(\mathcal{K}) \lesssim t_1 \ll t_2 \lesssim \tau(\mathcal{P})$, there should be some graph sequence $\mathcal{G}$ such that $t_1 \lesssim \tau(\mathcal{G}) \lesssim t_2$. This was partially proven in [24].

Theorem 4.2.4. *Let t_1 and t_2 be functions satisfying $\tau(\mathcal{K}) \lesssim t_1 \ll t_2 \lesssim \Theta(N)$. Then there is some graph sequence $\mathcal{G}$ such that $t_1 \lesssim \tau(\mathcal{G}) \lesssim t_2$.*

In fact, the family of fuses (defined in Section 3) covers this whole range. What behavior lives above $\Theta(N)$ remains unknown.

It is interesting to consider a pebbling threshold version of Graham's conjecture. Given graph sequences $\mathcal{F}$ and $\mathcal{G}$, define the sequence $\mathcal{H} = \mathcal{F} \square \mathcal{G} = \{F_1 \square G_1, \ldots, F_n \square G_n, \ldots\}$. Suppose that $f(R) \in \tau(\mathcal{F})$, $g(S) \in \tau(\mathcal{G})$, and $h(T) \in \tau(\mathcal{H})$, where $R = N(F_n)$, $S = N(G_n)$, and $T = N(H_n) = RS$.

Question 4.2.5. *Is it true that, for $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ as defined above, we have $h(T) \in O(f(R)g(S))$?*

In particular, one can define the sequence of graphs $\mathcal{G}^k$ in the obvious way. In [23] one finds tight enough bounds on $\tau(\mathcal{P}^k)$ to show that the answer to this question is yes for $\mathcal{F} = \mathcal{P}^i$ and $\mathcal{G} = \mathcal{P}^j$. Another important instance is $\mathcal{H} = \mathcal{K}^2$. Boyle [7] proved that $\tau(\mathcal{K}^2) \in O(N^{3/4})$. This was improved in [5], answering Question 4.2.5 affirmatively for squares of complete graphs.

Theorem 4.2.6. *For $\mathcal{K}^2 = \{K_1^2, \ldots, K_n^2, \ldots\}$ we have $\tau(\mathcal{K}^2) = \Theta(N^{1/2})$.*

This result is interesting because, by squaring, the graphs become fairly sparse, and yet their structure maintains the low pebbling threshold. The proof of the result tied the behavior of pebbling in K_n^2 to the existence of large components in various models of random complete bipartite graphs.

Another interesting related sequence to consider is $\mathcal{P}_l = \{P_l^1, \ldots, P_l^n, \ldots\}$. When $l = 2$ we have $\mathcal{P}_2 = \mathcal{Q}$, and the best result to date is the following theorem of [26] (obtained independently in [2]).

Theorem 4.2.7. *For the sequence of cubes we have $\tau(\mathcal{Q}) \in \Omega(N^{1-\epsilon}) \cap O(N/(\lg\lg N)^{1-\epsilon})$ for all $\epsilon > 0$.*

Let $l = l(n)$ and $d = d(n)$ and denote by $\mathcal{P}_l^d$ the graph sequence $\{P_{l(n)}^{d(n)}\}_{n=1}^{\infty}$, where $P_l^d = (P_l)^d$. Most likely, fixed l yields similar behavior to Theorem 4.2.7.

Conjecture 4.2.8. *For fixed l we have $\tau(\mathcal{P}_l^n) \in o(N)$.*

In contrast, the results of [23] show that $\tau(\mathcal{P}^d) \in \Omega(N)$ for fixed d. Thus it is reasonable to believe that there should be some relationship between the two functions $l = l(n)$ and $d = d(n)$, both of which tend to infinity, for which the sequence $\mathcal{P}_l^d$ has threshold on the order of N.

Problem 4.2.9. Find a function $d = d(n)$ for which $\tau(\mathcal{P}^d) = \Theta(N)$. In particular, how does d compare to n?

Finally one might consider the behavior of graphs of high minimum degree. Define $\mathbf{G}(n, \delta)$ to be the set of all connected graphs on n vertices having minimum degree at least $\delta = \delta(n)$. Let $\mathcal{G}_\delta = \{G_1, \ldots, G_n, \ldots\}$ denote any sequence of graphs with each $G_n \in \mathbf{G}(n, \delta)$. In [22] is proven the following.

Theorem 4.2.10. *For every function $n^{1/2} \ll \delta = \delta(n) \le n - 1$, $\tau(\mathcal{G}_\delta) \subseteq O(n^{3/2}/\delta)$. In particular, if in addition $\delta \in \Omega(n)$ then $\tau(\mathcal{G}_\delta) = \Theta(n^{1/2})$.*

4.3. Shadows

Let $\mathcal{L}$ be a lattice and $\mathcal{L}(t)$ denote the rank t elements of $\mathcal{L}$. (Here, we really only need that $\mathcal{L}$ is a graded poset.) Given any subfamily $\mathcal{A} \subseteq \mathcal{L}_n(t)$, we define its *shadow* $\partial\mathcal{A} = \{C \in \mathcal{L}_n(t) \mid C \subset A \text{ for some } A \in \mathcal{A}\}$, and set $\partial^{i+1}\mathcal{A} = \partial\partial^i\mathcal{A}$.

In the case that $\mathcal{L}$ is the multiset lattice $\mathcal{M}$, for any multiset $A \in \mathcal{M}_n(t)$, and $i \in [n]$, we define $A(i)$ to be the multiplicity of i in A. The *colexicographic (colex) order* on $\mathcal{M}_n(t)$ is defined by setting $A \prec B$ if $A \ne B$ and, for some $i \in [n]$, $A(i) < B(i)$ while $A(j) = B(j)$ for $j > i$. Clements and Lindström [18] proved the following.

Theorem 4.3.1. *Suppose that $\mathcal{F} \subseteq \mathcal{M}_n(t)$, and that $\mathcal{G}$ consists of the first $|\mathcal{F}|$ elements of $\mathcal{M}_n(t)$ in colex order. Then $|\partial^k\mathcal{F}| \ge |\partial^k\mathcal{G}|$ for every $k \ge 1$.*

In other words, the size of the shadow (at any level) of a subset of $\mathcal{M}_n(t)$ is minimized by taking an initial segment of the colex order on $\mathcal{M}_n(t)$. This is a generalization, then, of the Kruskal–Katona Theorem [56,57], which said the same thing for the subset of Boolean lattice $\mathcal{B}_n$. (The colex order in $\mathcal{B}_n(t)$ is the same, since $\mathcal{B}_n(t)$ is just the restriction of $\mathcal{M}_n(t)$ to subsets.) It also extended Macauley's earlier result [60].

In 1979, Lovász proved a version of the Kruskal–Katona theorem which was used by Bollobás and Thomason [6] to prove the existence of threshold functions. An analogous version of the Clements–Lindström theorem was conjectured in [51], and proved in [4]. For x a non-negative real number, let $\left\langle {x \atop t} \right\rangle = (x)(x+1)\cdots(x+t)/t!$. (Note that this counts the number of submultisets of $[x]$ of size t when x is a natural number.)

Theorem 4.3.2. *Suppose that $\mathcal{A} \subseteq \mathcal{M}_n(t)$ and define x by $|\mathcal{A}| = \left\langle {x \atop t} \right\rangle$. Then $|\partial\mathcal{A}| \ge \left\langle {x \atop t-1} \right\rangle$.*

For the case in which $\mathcal{A} = \left\langle {m \atop t} \right\rangle$ for m a natural number, the first $|\mathcal{A}|$ elements of $\mathcal{M}_n(t)$ in colex order are the t-multisets of $\{1, \ldots, m\}$. The shadow of the family consisting of these multisets is the family of $(t-1)$-multisets of $\{1, \ldots, m\}$, of size $\left\langle {m \atop t-1} \right\rangle$, so Theorem 4.3.2 is equivalent to Theorem 4.3.1 in this case. For families of intermediate sizes, Theorem 4.3.2 is a "smoothed" version of Theorem 4.3.1.

Given a family $\mathcal{A} \subseteq \mathcal{M}_n(t)$, and indices i, j with $1 \le i < j \le n$, a *compression* of $\mathcal{A}$ is obtained by taking each member A of $\mathcal{A}$ such that $A(j) \ge 1$ and $A - \{j\} + \{i\} \notin \mathcal{A}$, and replacing it by $A - \{j\} + \{i\}$. A family $\mathcal{A}$ is said to be *compressed* if, for all $1 \le i < j \le n$, we have $A - \{j\} + \{i\} \in \mathcal{A}$ whenever $A \in \mathcal{A}$, i.e., $\mathcal{A}$ is unchanged by any compression. Note that any family can be transformed into a compressed one by a sequence of compressions. Note also that initial segments of the colexicographic order are compressed families, but that these are not the only ones. The key ingredient is the following lemma of Clements [17].

Lemma 4.3.3. *Suppose that $\mathcal{A} \subseteq \mathcal{M}_n(t)$ and let $q(\mathcal{A})$ be a compression of $\mathcal{A}$. Then $|\partial\mathcal{A}| \geq |\partial q(\mathcal{A})|$.*

For $\mathcal{F} \subseteq \mathcal{L}$, we write $\mathcal{F}(t) = \mathcal{F} \cap \mathcal{L}(t)$. $\mathcal{F}$ is said to be *increasing* if $E \supseteq F \in \mathcal{F}$ implies $E \in \mathcal{F}$, and *decreasing* if $E \subseteq F \in \mathcal{F}$ implies $E \in \mathcal{F}$. Now let $\mathcal{L}_1, \ldots, \mathcal{L}_n, \ldots$ be a sequence of lattices of increasing ranks, with subfamilies $\mathcal{F}_n \subseteq \mathcal{L}_n$. The probability that a uniformly randomly chosen element of $\mathcal{L}_n(t)$ is in the family $\mathcal{F}_n$ is $P_t(\mathcal{F}_n(t)) = |\mathcal{F}_n(t)|/|\mathcal{L}_n(t)|$. We say that $t = t(n)$ is a *threshold* for a sequence $\mathscr{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_n, \ldots)$ of increasing families of multisets if, for any function $\omega = \omega(n) \gg 1$, we have $P_{t\omega}(\mathcal{F}_n(t\omega)) \to 1$ and $P_{t/\omega}(\mathcal{F}_n(t/\omega)) \to 0$. We write $\tau(\mathscr{F})$ for the set of all thresholds of $\mathscr{F}$. As noted in Section 4.1, one of the many uses of shadow theorems is in proving threshold existence, such as in Bollobás–Thomason's use of Lovasz's version of the Kruskal–Katona theorem mentioned above. The PI and coauthors [4] proved the following analog of the Bollobás–Thomason Theorem [6], thereby establishing the existence of graph pebbling thresholds (see Section 4.2).

Theorem 4.3.4. *Let $\mathscr{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_n, \ldots)$ be a sequence of increasing families, with $\mathcal{F}_n \subseteq \mathcal{M}_n$ for each n. Define $t = t(n) = \min\{r \mid P_r(\mathcal{F}_n(r)) \geq 1/2\}$. Then $t \in \tau(\mathscr{F})$.*

5. New parameters

5.1. Target pebbling

Another variation is developed in [42], combining ideas from Sections 2 and 3. Let $\mathscr{M}$ denote the set of all dominating sets of G. Then the *domination target pebbling number* (called *domination cover pebbling number* in [42]) equals $\pi^-(G, \mathscr{M})$. The motivation stems from wanting to transport devices from initial positions to eventual positions that allow them to monitor the entire graph. For the complete r-partite graph $K = K_{s_1,\ldots,s_t}$, let $s(K) = 3$ when $\max_i s_i = 2$ and $s(K) = \max_i s_i$ otherwise. Obviously, $\pi^-(K, \mathscr{M}) = s(K)$, since the largest unsolvable configuration sits in one of the parts of K. They find the domination target pebbling number for paths, cycles and complete binary trees.

Theorem 5.1.1. *The domination target pebbling number of the path on n vertices is $\pi^-(P_n, \mathscr{M}) = 2^{n-2} + 2^{n-5} + \cdots + 2^{n \bmod 3} + \lfloor \frac{n \bmod 3}{2} \rfloor$.*

The formula for cycles is similar. The key in both cases is showing that the largest unsolvable configuration is simple, reminiscent of the characterization for maximum unsolvable cover pebbling configurations. However, the same characterization does not hold in the case of domination target pebbling. Let B_h be the complete binary tree of height h, having 2^h leaves. In this case the worst configuration is mostly concentrated on one leaf, with extra single pebbles scattered on half the other leaves, none of which share a common neighbor. The formula for $\pi^-(B_h, \mathscr{M})$ is fairly complicated, but is asymptotically 8^{h-1}, roughly $n^3/64$, where $n = n(B_h)$.

The *radial r-pebbling number* of G is the target pebbling number $\pi^-(G, \mathscr{B}_t(r))$, with $\mathscr{B}_t(r) = \{v \in V \mid \mathrm{dist}(v, r) \leq t\}$ (recall that each vertex v here is interpreted as the root configuration on v). Thus it is the smallest m so that every configuration of size m can reach some vertex at distance at most t from r. Work in [50] relates this number to the free pebbling number below.

5.2. Free pebbling

In *free pebbling*, one is allowed to make some *free moves*—pebbling steps with no loss of pebbles.

The *t-free r-pebbling number* of G, denoted $\phi_t(G, r)$, is defined to be the smallest integer m so that every configuration of size m is r-solvable with at most t free moves. Thus $\phi_0(G, r) = \pi(G, r)$. In [50] we find that $\phi_t(G, r) \leq \pi^-(G, \mathscr{B}_t(r))$: if one can reach $\mathscr{B}_t(r)$ with no free moves, then the t free moves will reach r. It is likely that equality holds always, since it seems advantageous to postpone the free moves until the end.

The *r-pebbling handicap number* of G, denoted $\psi(G, r)$, is defined to be the smallest integer t so that every configuration of size $n = n(G)$ is r-solvable with at most t free moves. Thus a Class 0 graph G has $\psi(G, r) = 0$ for all r. The following is proved in [50]. Let $\psi(G) = \max_r \psi(G, r)$.

Theorem 5.2.1. *Every graph G satisfies $\psi(G) \geq \mathrm{diam}(G) - \lfloor \lg n \rfloor$.*

The bound is tight for paths ($\psi(P_n) = n - \lfloor \lg n \rfloor - 1$) and slightly less so for cycles ($\psi(C_{2k}) = k - \lfloor \lg k \rfloor - 1$).

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