

Erdős-Ko-Rado Theorems for Paths in Graphs

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Abstract

A family of sets is s -intersecting if every pair of its sets has at least s elements in common. It is an s -star if all its members have some s elements in common. A family of sets is called s -EKR if all its s -intersecting subfamilies have size at most that of some s -star. For example, the classic 1961 Erdős-Ko-Rado theorem states essentially that the family of r -sized subsets of $\{1, 2, \dots, n\}$ is s -EKR when n is a large enough function of r and s , and the 1967 Hilton-Milner theorem provides the near-star structure of the largest non-star intersecting family of such sets. Two important conjectures along these lines followed: by Chvátal in 1974, that every family of sets that all subsets of its members is 1-EKR, and by Holroyd and Talbot in 2005, that, for every graph, the family of all its r -sized independent sets is 1-EKR when every maximal independent set has size at least $2r$.

In this paper we present similar 1-EKR results for families of length- r paths in graphs, specifically for sun graphs, which are cycles with pendant edges attached in a uniform way, and theta graphs, which are collections of pairwise internally disjoint paths sharing the same two endpoints. We also prove s -EKR results for such paths in suns, and give a Hilton-Milner type result for them as well. A set is a transversal of a family of sets if it intersects each member of the family, and the transversal number of the family is the size of its smallest transversal. For example, stars have transversal number 1, and the Hilton-Milner family has transversal number 2. We conclude the paper with some transversal results involving what we call triangular families, including a few results for projective planes.

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1 Introduction

Let \mathcal{F} be a family of sets, and use the shorthand notations $\cup\mathcal{F} = \cup_{A \in \mathcal{F}} A$ and $\cap\mathcal{F} = \cap_{A \in \mathcal{F}} A$. We say that \mathcal{F} is *s-intersecting* if $|A \cap B| \geq s$ for all $A, B \in \mathcal{F}$, and \mathcal{F} is an *s-star* if $|\cap\mathcal{F}| \geq s$. By *intersecting* we mean 1-intersecting, and by *star* we mean 1-star. We say that \mathcal{F} is *exactly s-intersecting* if $|A \cap B| = s$ for all $A, B \in \mathcal{F}$. For $X \subseteq \cup\mathcal{F}$, we define the *full s-star of \mathcal{F} at X* to be $\mathcal{F}_X = \{A \mid X \subseteq A \in \mathcal{F}\}$. We also write $\mathcal{F}^r = \{A \in \mathcal{F} \mid |A| = r\}$. For each of these, note that the definition is trivial in the case $s = 0$, and so throughout we will implicitly assume that $s > 0$. We say that \mathcal{F} is **EKR** if some star \mathcal{F}_x satisfies $|\mathcal{F}_x| \geq |\mathcal{H}|$ for all intersecting $\mathcal{H} \subseteq \mathcal{F}$, and *strictly EKR* if every maximum intersecting family is a star. We extend this language to graphs as follows. Let G be a graph and \mathcal{F} be a family of subgraphs of G . We say that G is \mathcal{F} -EKR if \mathcal{F} is EKR.

One of the cornerstones of extremal set theory is the following result of Erdős, Ko, and Rado.

Theorem 1. [8] *There is a function $n_0(r, s)$ such that, if $\mathcal{F} \subseteq \binom{[n]}{r}$ is s-intersecting and $n \geq n_0(r, s)$, then $|\mathcal{F}| \leq \binom{n-s}{r-s}$. Moreover, equality holds if and only if \mathcal{F} is an s-star.*

For $s = 1$ we have $n_0(r, s) = 2r + 1$, with the statement for $n = 2r$ being true without the uniqueness of the star extremum. For $s > 1$ and $n < n_0(r, s)$, the sizes and structures of maximum s -intersecting families were determined in the Complete Intersection Theorem of [1].

It is natural to ask which families of sets \mathcal{F} give a theorem similar to Theorem 1; that is, which \mathcal{F} are EKR? Chvátal [6] made the following conjecture.

Conjecture 2. [6] *Every subset-closed family of sets is EKR.*

One of the more well-studied families along these lines is the family of independent sets of a graph G , which we denote by $\mathcal{I}(G)$. This study was formally initiated by Holroyd, Spencer and Talbot in [15], although earlier traces of the idea are found in [2, 4, 7]. In [16] we find the following conjecture. Let $\mu(G)$ denote the *independent domination number* of G ; i.e. the minimum size of a maximal independent set of G .

Conjecture 3. [16] *Let G be any graph and suppose that $1 \leq r \leq \mu(G)/2$. Then G is \mathcal{I}^r -EKR, and is strictly so if $2 < r < \mu(G)/2$.*

There is a growing literature affirming this conjecture — see [5, 10, 12, 17] for recent examples and a survey containing history and past references.

Simonovits and Sós [24, 25] introduced the problem of finding the size of the largest family of graphs on n vertices such that the intersection of every pair is in a given set L of graphs. Recently, Frankl, et al. [13]

proved results when the family is restricted to trees on n vertices and L is the set of all edges. In this paper we restrict our attention to the case that $\mathcal{F} \in \{\mathcal{P}(G), \mathcal{P}^r(G)\}$; i.e. \mathcal{F} is either the family of all paths in a graph G , or the family of all paths on r vertices in G , with L being the set of all vertices of G . Observe that, unlike $\mathcal{I}(G)$, $\mathcal{P}(G)$ is not subset-closed, as not all subgraphs of paths are paths.

In fact, we will tacitly assume that G is connected throughout, since an intersecting subfamily must always live inside a component of G . For $x \in \cup \mathcal{F}$ define $\deg(x) = |\mathcal{F}_x|$, and denote $\Delta(\mathcal{F}) = \max_{x \in \cup \mathcal{F}} \deg(x)$. More generally, we write $\Delta_s(\mathcal{F}) = \max_{|X|=s} |\mathcal{F}_X|$. We say that \mathcal{F} is r -uniform if $|A| = r$ for all $A \in \mathcal{F}$ (i.e. $\mathcal{F} = \mathcal{F}^r$), and write $\mathcal{F} - x = \{A - \{x\} \mid A \in \mathcal{F}\}$. A *transversal* of a family \mathcal{F} is a set X such that $X \cap F \neq \emptyset$ for each $F \in \mathcal{F}$; we denote by $\tau(\mathcal{F})$ the minimum size of a transversal of \mathcal{F} , called the *transversal number* of \mathcal{F} . The set $\cup \mathcal{F}$ is the *trivial* transversal of \mathcal{F} ; all others are *nontrivial*.

Theorem 1 can be expressed using this transversal language; it says that, when n is large enough in terms of r , among the maximum size intersecting families is a family with transversal number one. In general, an EKR-type theorem gives a condition under which there is guaranteed to be a maximum size intersecting family with transversal number one.

Hilton and Milner considered the problem of determining the size of the largest non-star intersecting family under the same conditions as the EKR Theorem. Let $X = \{2, \dots, r+1\}$ and define $\mathcal{M} = \{H \in \binom{[n]}{r}_1 \mid H \cap X \neq \emptyset\} \cup \{X\}$. Furthermore, define $\mathcal{K} = \{H \in \binom{[n]}{r} \mid |H \cap [3]| \geq 2\}$. In [14] they proved the following theorem.

Theorem 4. [14] *If $n \geq 2r$, $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting, and \mathcal{F} is not a star then $|\mathcal{F}| \leq |\mathcal{M}| = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. Additionally, equality holds if and only if $\mathcal{F} \cong \mathcal{M}$ or $r = 3$ and $\mathcal{F} \cong \mathcal{K}$.*

Since \mathcal{M} is not a star, the Hilton-Milner Theorem states that, for large enough n , among the intersecting families with transversal number at least two, \mathcal{M} has maximum size.

In Section 2 we prove some EKR results for paths in graphs, beginning with a few elementary results, followed by our main EKR theorems for paths in *sun* graphs and *theta* graphs. A sun graph is formed from a cycle by adding a fixed number of pendants to each vertex, and theta graphs are unions of internally vertex disjoint paths between two fixed vertices (see their formal definitions in Subsections 2.2 and 2.3). We prove in Theorems 9 and 14, respectively, that each of these classes are EKR, for the appropriate range of path lengths, and also that suns are s -EKR similarly. Then we present in Section 3 some transversal results involving what we call triangular families, including a few results for projective planes. We finish with several interesting questions in Section 4.

2 Path EKR Results

2.1 Elementary Results

For a graph G , recall that $\mathcal{P} = \mathcal{P}(G)$ is the set of all paths in G , and $\mathcal{P}^r = \mathcal{P}^r(G)$ is the set of all paths in G with exactly r vertices.

Theorem 5. *Let T be a tree. Then every intersecting family of paths in T is a star.*

Proof. Let \mathcal{F} be an intersecting family of paths on the tree T . Suppose that \mathcal{F} is not a star. Then there must be 3 paths in \mathcal{F} which do not form a star. That is, there exist $P_1, P_2, P_3 \in \mathcal{F}$ such that $P_1 \cap P_2 \cap P_3 = \emptyset$. Given that each $P_i \cap P_j \neq \emptyset$, choose $x \in P_1 \cap P_2$, $y \in P_1 \cap P_3$, and $z \in P_2 \cap P_3$. Then we can traverse from x to y along P_1 (avoiding z), then y to z along P_3 (avoiding x), then back to x along P_2 (avoiding y), which defines a closed trail. Since every closed trail contains a cycle, this contradicts the definition of a tree. \square

Theorem 6. *Let G be a graph with girth $\text{gir}(G) > 3l$. Then every intersecting family of paths of length at most l in G is a star.*

Proof. Let \mathcal{F} be an intersecting family of paths of length at most l on a graph G with girth greater than $3l$. Suppose that this family is not a star. Then there must be 3 paths in \mathcal{F} which are not a star. That is, there exist $P_1, P_2, P_3 \in \mathcal{F}$ such that $P_1 \cap P_2 \cap P_3 = \emptyset$. Given that each $P_i \cap P_j \neq \emptyset$, choose $x \in P_1 \cap P_2$, $y \in P_1 \cap P_3$, and $z \in P_2 \cap P_3$. Then we can traverse from x to y along P_1 (avoiding z), then y to z along P_3 (avoiding x), then back to x along P_2 (avoiding y), which defines a closed trail of length at most $3l$. Since every closed trail contains a cycle, this contradicts the definition of G . \square

Corollary 7. *If T is a tree then T is \mathcal{P} -EKR. If $\text{gir}(G) > 3r - 3$ then G is \mathcal{P}^r -EKR.*

Observe that the complete graph K_3 is not \mathcal{P}^2 -EKR.

Fact 8. *If $G \neq K_3$ then G is strictly \mathcal{P}^2 -EKR.*

Proof. We prove the contrapositive. Suppose that G is not \mathcal{P}^2 -EKR. Because every intersecting family of edges of G is either a triangle or a star, this implies that some triangle is bigger than every star — i.e., $\Delta(G) \leq 2$ — and so G is a path or cycle. By Theorem 5, G cannot be a path of any length. By Theorem 6, G is not a cycle of length longer than three. Hence $G = C_3 = K_3$. \square

Because of Fact 8 we will assume throughout the remainder of this manuscript that $r \geq 3$.

2.2 Suns

Define the *sun* with t rays S_n^t to have vertex set $\{v_i^j \mid 0 \leq i < n, 0 \leq j \leq t\}$, with edges $v_i^0 v_{i+1}^0$, and $v_i^j v_i^0$ for all $i \in [n]$, and $0 < j \leq t$. We imagine the vertices $v_0^0, v_1^0, \dots, v_{n-1}^0$ to be written clockwise around a circle.

Theorem 9. *Let $3 \leq s+2 \leq r \leq \lfloor \frac{n+s-1}{2} \rfloor$ and $t \geq 0$. Then S_n^t is \mathcal{P}^r -EKR, and strictly so when $r < (n+s-1)/2$. If $\mathcal{F} \subseteq \mathcal{P}^r(S_n^t)$ is an s -intersecting family then $|\mathcal{F}| \leq (r-s+1) + 2t(r-s) + \binom{t}{2}(r-s-1) = 3 + 4t + \binom{t}{2}$ when $r = s+2$ and $|\mathcal{F}| \leq (r-s+1) + 2t(r-s) + t^2(r-s-1)$ when $r \geq s+3$.*

Proof. When $t = 0$, the theorem is equivalent to Katona's Cycle Lemma [21], so we will assume throughout that $t > 0$.

We begin by displaying an s -intersecting family of maximum size, after which we will prove the upper bound. Finally we will show that only a s -star can achieve that bound. First, we introduce some notation.

Let $\vec{P}_i^{(j,k)}$ be the clockwise path of length $r-1$, beginning at v_i^j and ending at $v_{i'}^k$, for the appropriate i' , and $\overleftarrow{P}_i^{(j,k)}$ be the counter-clockwise path of length $r-1$, beginning at v_{i+s-2}^j and ending at $v_{i'}^k$, for the appropriate i' . Observe that for $r = 3$ the definitions require that $i' = i$ and $j \neq k$ whenever both $j > 0$ and $k > 0$. Note that this yields four types of paths, depending on whether or not each of j and k is zero. More precisely, define the following four families that partition the set of all paths of length $r-1$ in S_n^t :

- $\mathcal{P}^{(0,0)} = \{\overleftarrow{P}_i^{(0,0)}, \vec{P}_i^{(0,0)} \mid i \in [n]\},$
- $\mathcal{P}^{(0,1)} = \{\overleftarrow{P}_i^{(0,k)}, \vec{P}_i^{(0,k)} \mid i \in [n], 0 < k \leq t\},$
- $\mathcal{P}^{(1,0)} = \{\overleftarrow{P}_i^{(j,0)}, \vec{P}_i^{(j,0)} \mid i \in [n], 0 < j \leq t\},$ and
- $\mathcal{P}^{(1,1)} = \{\overleftarrow{P}_i^{(j,k)}, \vec{P}_i^{(j,k)} \mid i \in [n], 0 < j, k \leq t\}.$

Notice that, for each i, j , and k , $\vec{P}_i^{(j,k)} = \overleftarrow{P}_{i'}^{(k,j)}$ for some i' , and that, since $r \leq \lfloor \frac{n+s-1}{2} \rfloor$, $|\vec{P}_i^{(j,k)} \cap \overleftarrow{P}_i^{(j',k')}| < s$ for each i, j, j', k and k' .

Here we define an example of an s -intersecting family of maximum size. For $a, b \in \{0, 1\}$, define $\vec{\mathcal{F}}^{(a,b)}$ to be the family of all paths $\vec{P}_y^{(j,k)}$ with $v_y^0 \in \vec{P}_{a+b}^{(a,b)}$. Note that every such path contains $\{v_{r-s}^0, \dots, v_{r-1}^0\}$. Now set $\mathcal{F}^* = \vec{\mathcal{F}}^{(0,0)} \cup \vec{\mathcal{F}}^{(0,1)} \cup \vec{\mathcal{F}}^{(1,0)} \cup \vec{\mathcal{F}}^{(1,1)}$. Then \mathcal{F}^* is an s -star at $\{v_{r-s}^0, \dots, v_{r-1}^0\}$ of size $(r-s+1) + 2t(r-s) + t^2(r-s-1)$ when $r \geq 4$, or $(r-s+1) + 2t(r-s) + \binom{t}{2}(r-s-1)$ when $r = s+2$.

Now we prove the upper bound. Let \mathcal{F} be an s -intersecting family, and define $\mathcal{F}^{(a,b)} = \mathcal{F} \cap \mathcal{P}^{(a,b)}$, for each $a, b \in \{0, 1\}$. Then $|\mathcal{F}| = |\mathcal{F}^{(0,0)}| + |\mathcal{F}^{(0,1)}| + |\mathcal{F}^{(1,0)}| + |\mathcal{F}^{(1,1)}|$. We calculate an upper bound on $|\mathcal{F}|$ by providing upper bounds on each of these four subfamilies. That is, given $a, b \in \{0, 1\}$, we count the maximum number of paths in $\mathcal{F}^{(a,b)}$.

We calculate $|\mathcal{F}^{(a,b)}|$ as follows. Select some $\vec{P}_x^{(j,k)} \in \mathcal{F}^{(a,b)}$ and consider the maximum number of paths that intersect $\vec{P}_x^{(j,k)}$. Any such path can be written as either $\vec{P}_y^{(j',k')}$ or $\overleftarrow{P}_y^{(j',k')}$ for some $v_y^0 \in \{v_x^0, \dots, v_{x+r-s}^0\}$. Note for each such v_y^0 that, since $\mathcal{F}^{(a,b)}$ is s -intersecting, $\mathcal{F}^{(a,b)}$ has no two paths of opposite type $\vec{P}_y^{(j',k')}$ or $\overleftarrow{P}_y^{(j',k')}$; thus $\mathcal{F}^{(a,b)}$ contains paths of only one of the types $\vec{P}_y^{(j',k')}$ or $\overleftarrow{P}_y^{(j',k')}$.

Define $m_{a,b}(X)$ to be the number of paths of type X ; by symmetry, we have that $m_{a,b}(\vec{P}_y^{(j',k')}) = m_{a,b}(\overleftarrow{P}_y^{(j',k')})$. Then we have

$$|\mathcal{F}^{(a,b)}| \leq \sum_{v_y^0 \in \{v_x^0, \dots, v_{x+r-s}^0\}} m_{a,b}(\vec{P}_y^{(j',k')}) = (r-a-b-s+1)t(a,b), \quad (1)$$

where $t(a,b) = \binom{t}{2}$ when $r = s+2$, and $t(a,b) = t^{a+b}$ otherwise, since the number of non-pendant vertices $v_y^0 \in \{v_x^0, \dots, v_{x+r-s}^0\}$ equals $r-a-b-s+1$, and the number of choices for pendant vertices at each end of a path in $\vec{P}_y^{(j',k')}$ equals $t(a,b)$. Hence

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}^{(0,0)}| + |\mathcal{F}^{(0,1)}| + |\mathcal{F}^{(1,0)}| + |\mathcal{F}^{(1,1)}| \\ &\leq (r-s+1) + 2t(r-s) + t^2(r-s-1) \end{aligned} \quad (2)$$

when $r \geq s+3$, with $|\mathcal{F}| \leq (r-s+1) + 2t(r-s) + \binom{t}{2}(r-s-1) = 3 + 4t + \binom{t}{2}$ when $r = s+2$, as noted above. Hence S_n^t is \mathcal{P}^r -EKR.

Next we show that if $r \leq (n+s-1)/2$ then only a star achieves the upper bound. We will do this by proving that each of the families $\mathcal{F}^{(a,b)}$ is a star, except in the case of $\mathcal{F}^{(0,0)}$ when $s=1$ and $r=n/2$. In addition, each of these families will have the same star center, which will force a star even in the case when $s=1$ and $r=n/2$.

Let $r \leq (n+s-1)/2$ and consider $\mathcal{F}^{(a,b)}$ with $a, b \in \{0,1\}$; if $r = (n+s-1)/2$ and $s=1$ then suppose that $(a,b) \neq (0,0)$. Suppose that all paths in $\mathcal{F}^{(a,b)}$ are of type $\overleftarrow{P}^{(a,b)}$. Then we have a star at $\{v_{x-1}^0, \dots, v_{x+s-2}^0\}$. Similarly, if all paths in $\mathcal{F}^{(a,b)}$ are of type $\vec{P}^{(a,b)}$. Then we have a star at $\{v_{x+r-a-b-s-1}^0, \dots, v_{x+r-a-b-1}^0\}$.

When $(r-a-b) \leq (n+s-1)/2$, we know that $|\overleftarrow{P}_i^{(j,k)} \cap \vec{P}_{i+1}^{(j',k')}| < s$ for all i, j, k, j' , and k' . Thus, in the case that $\overleftarrow{P}_x^{(j,k)} \in \mathcal{F}^{(a,b)}$ for some j and k , $\mathcal{F}^{(a,b)}$ must be of type $\overleftarrow{P}^{(a,b)}$. Otherwise, we have that $\vec{P}_x^{(j,k)} \in \mathcal{F}^{(a,b)}$ for all j and k . Because we similarly may assume that $\mathcal{F}^{(a,b)}$ is not of type $\vec{P}^{(a,b)}$ (since otherwise we are finished), there must be a first h with $\overleftarrow{P}_h^{(j,k)} \in \mathcal{F}$ for some j and k — and hence for all j and k . Thus $\overleftarrow{P}_{h'}^{(j,k)} \in \mathcal{F}^{(a,b)}$ for all $h' > h$ and all j and k . Each of these paths contains vertex $\{v_{h-1}^0, \dots, v_{h+s-2}^0\}$, so $\mathcal{F}^{(a,b)}$ is a star.

To show that \mathcal{F} is a star, all $\mathcal{F}^{(a,b)}$ must share a common star center. Suppose that $\mathcal{F}^{(a,b)}$ is a star at X , and $\mathcal{F}^{(c,d)}$ is a star at $Y \neq X$. By symmetry and relabeling of indices, we may assume that $X = \{v_0^0, \dots, v_{s-1}^0\}$ and $Y = \{v_y^0, \dots, v_{y+s-1}^0\}$, for some $y \leq n/2$. Define z to be the maximum index i (allowing index 0 to be renamed n for this purpose) such that $\mathcal{F}^{(a,b)} \supseteq \vec{P}_i^{(j,k)} \not\supseteq v_{y+s-1}^0$ for some j and k . Because $r \leq n/2$ we have $\vec{P}_y^{(j',k')} \cap \vec{P}_i^{(j,k)} \subseteq Y - \{v_{y+s-1}^0\}$ for some j' and k' , and so $|\vec{P}_y^{(j',k')} \cap \vec{P}_i^{(j,k)}| < s$, a contradiction. Hence $Y = X$.

Now suppose that $\mathcal{F}^{(a,b)}$ is a star at X , and $\mathcal{F}^{(0,0)}$ is not star (and therefore $s = 1$ and $r = n/2$). Again, we may assume that $X = \{v_0^0\}$. Thus there must be some i such that $v_0^0 \notin \vec{P}_i^{(0,0)} \in \mathcal{F}^{(0,0)}$. Now let $\vec{P}_h^{(j,k)} \subseteq \vec{P}_i^{(0,0)}$ for some $\vec{P}_h^{(j,k)} \in \mathcal{F}^{(j,k)}$ with $(j,k) \neq (0,0)$. Then $\vec{P}_0^{(j,k)} \cap \vec{P}_i^{(0,0)} = \emptyset$, a contradiction. Hence $\mathcal{F}^{(0,0)}$ is a star. \square

Corollary 10. *Let $r \leq \lfloor \frac{n}{2} \rfloor$. Then C_n is \mathcal{P}^r -EKR, and strictly so when $r < n/2$.*

Proof. Because $C_n = S_n^0$, the result follows from Theorem 9. \square

Additionally, a stronger result holds for bounded-length paths.

Corollary 11. *Suppose that $S_n^t \neq K_3$. Then S_n^t is strictly $\mathcal{P}^{\leq k}$ -EKR for all $1 \leq k \leq n/2$ and $t \geq 0$.*

Proof. For $k \leq \lfloor n/2 \rfloor$, let $\mathcal{P}^{\leq k}$ be the family of all paths of length at most $k-1$ in S_n^t . Let $\mathcal{F}^{\leq k}$ be a maximum intersecting family of $\mathcal{P}^{\leq k}$. Then let \mathcal{P}^r and \mathcal{F}^r be the corresponding subfamilies of r -paths. From Fact 8 and Theorem 9, we know that S_n^t is strictly \mathcal{P}^r -EKR for all $r < \lfloor n/2 \rfloor$. By symmetry, we may choose $v = v_0^0$ for the center of each maximum star \mathcal{P}_v^r of \mathcal{F}^r in that range. Then we have

$$|\mathcal{F}^{\leq k}| \leq \sum_{1 \leq r \leq k} |\mathcal{F}^r| \leq \sum_{1 \leq r \leq k} |\mathcal{P}_v^r| = |\mathcal{P}_v^{\leq k}|.$$

If $|\mathcal{F}^{\leq k}| = |\mathcal{P}_v^r|$ then, since $v \in \mathcal{F}^1$, $v \in F$ for all $F \in \mathcal{F}^{\leq k}$. \square

The following theorem shows that the EKR property for the collection of all of its paths (rather than just those of uniform length) does not hold for suns other than the cycle.

Theorem 12. *The sun S_n^t is \mathcal{P} -EKR if and only if $t = 0$, and not strictly so when $t = 0$.*

Proof. Let C be the cycle in S_n^t , and for any path P on C denote by \bar{P} the *path complement* of P on C ; i.e. the path induced by $V(C) - V(P)$. Let \mathcal{F} be an intersecting family of maximum size.

Suppose that $t = 0$. Then $|\mathcal{P}| = n^2$. Given that \mathcal{F} is of maximum size, for every $P \in \mathcal{P}$ at most one of P or \bar{P} is in \mathcal{F} . Note that if P has length $n - 1$ then \bar{P} does not exist. Let \mathcal{S}_0 be the star of paths that contain v_0^0 . Then \mathcal{S}_0 consists of exactly one path from each pair $\{P, \bar{P}\}$, and so $|\mathcal{S}_0| = \frac{n^2-n}{2} + n = \frac{n^2+n}{2}$. Thus when $t = 0$, the intersecting family of paths is EKR, and the maximum size is $\frac{n^2+n}{2}$.

Interestingly, the star is not the only intersecting family of maximum size. One can define a Hilton-Milner-type family \mathcal{H} as follows. Set P_0 to be the path on the singleton v_0^0 , with \bar{P}_0 defined as above. Then $\mathcal{H} := \mathcal{S}_0 - \{P_0\} \cup \{\bar{P}_0\}$ is an intersecting non-star with $|\mathcal{H}| = |\mathcal{S}|$. Hence \mathcal{S}_n^t is EKR but not strictly so. In fact, for each pair $\{P, \bar{P}\}$ of complementary paths of \mathcal{S}_n , we can build an intersecting family of maximum size by choosing one of the two paths of size at least $n/2$.

Now suppose that $t > 0$. We proceed to show that \mathcal{S}_n^t is not EKR.

For a path $Q \in \mathcal{P}(\mathcal{S}_n^t)$ we define its *image* $\text{im}(Q)$ to be the graph intersection $Q \cap C$, and denote $\mathcal{P}(P) = \{Q \in \mathcal{P} \mid \text{im}(Q) = P\}$, with $\mathcal{F}(P) = \mathcal{P}(P) \cap \mathcal{F}$. Also define the *image* of \mathcal{F} as $\text{im}(\mathcal{F}) = \{\text{im}(Q) \mid Q \in \mathcal{F}\}$. Clearly $|\mathcal{P}(P)| = (t+1)^2$ for all $P \in C$ having at least one edge. If the path P is a single vertex, however, we have $|\mathcal{P}(P)| = \binom{t+1}{2} + 1$. Thus we obtain

$$|\mathcal{P}(P)| \leq (t+1)^2 \quad (3)$$

for all $P \in C$, with equality if and only if P is not a singleton. Of course, if \mathcal{F} is intersecting of maximum size, then we have that $|\mathcal{F}(P)| = |\mathcal{P}(P)|$ for all $P \in \mathcal{F}$. In particular,

$$|\mathcal{F}| = \sum_{P \in \text{im}(\mathcal{F})} |\mathcal{P}(P)|. \quad (4)$$

Because \mathcal{F} is intersecting, so is $\mathcal{F}' := \text{im}(\mathcal{F})$. From case $t = 0$ above, we know that $|\mathcal{F}'| \leq \frac{n^2+n}{2}$, and so

$$|\mathcal{F}| \leq \left(\frac{n^2+n}{2} \right) (t+1)^2. \quad (5)$$

In the case that \mathcal{F} is a star, we have that \mathcal{F}' is also a star, say \mathcal{S} , and so (3) and (4) imply that

$$\begin{aligned} |\mathcal{F}| &= \sum_{P \in \mathcal{S}} |\mathcal{P}(P)| \\ &= \binom{t+1}{2} + 1 + \left(\frac{n^2+n}{2} - 1 \right) (t+1)^2 \\ &< \left(\frac{n^2+n}{2} \right) (t+1)^2. \end{aligned}$$

On the other hand, we can define $\mathcal{H}^* = \cup_{P \in \mathcal{H}} \mathcal{P}(P)$. Since \mathcal{H} is not a star, neither is \mathcal{H}^* . In addition, by (4) we have

$$|\mathcal{H}^*| = \sum_{P \in \mathcal{H}} |\mathcal{P}(P)| = \left(\frac{n^2 + n}{2} \right) (t + 1)^2.$$

Hence S_n^t is not \mathcal{P} -EKR when $t > 0$. □

Observe that, while sharing a similar notion with Conjecture 2, the $t > 0$ case of Theorem 12 does not refute it. This is because not every vertex-subset of a path is a path; that is, \mathcal{P} is not subset-closed.

We finish this section with the following Hilton-Milner [14] type result, determining both the size and structure of the maximum size non-star intersecting families on the cycle in the relevant range for r .

Theorem 13. *Let $(n + 3)/3 \leq r \leq n/2$ and suppose that $\mathcal{F} \subseteq \mathcal{P}^r(C_n)$ is an intersecting non-star family of maximum-size. Then there exists $S = \{v_a, v_b, v_c\} \subset V(C_n)$ such that $\mathcal{F} = \{F \in \mathcal{P}^r(C_n) \mid |F \cap S| = 2\}$; moreover $|\mathcal{F}| = 3r - n$.*

Proof. Because \mathcal{F} is not a star, there exists $\{A, B, C\} \subseteq \mathcal{F}$ such that $A \cap B \cap C = \emptyset$, and so $A \cup B \cup C = V(S_n)$.

Fixing such A, B , and C , we first define $X = A \cap B$, $Y = A \cap C$, and $Z = B \cap C$, and then define

- $\mathcal{A} = \{D \in \mathcal{F} \mid D \cap Z = \emptyset\}$ and $A' = \cap \mathcal{A}$,
- $\mathcal{B} = \{D \in \mathcal{F} \mid D \cap Y = \emptyset\}$ and $B' = \cap \mathcal{B}$, and
- $\mathcal{C} = \{D \in \mathcal{F} \mid D \cap X = \emptyset\}$ and $C' = \cap \mathcal{C}$.

Note that \mathcal{A}, \mathcal{B} , and \mathcal{C} partition \mathcal{F} since no set of \mathcal{F} intersects all three of X, Y , and Z (since $r \leq n/2$) and no set of \mathcal{F} misses two of X, Y , and Z (e.g. if $D \cap X = \emptyset$ and $D \cap Y = \emptyset$ then $D \cap Z = \emptyset$).

For any set $D = \{v_d, v_{d+1}, \dots, v_{d+r-1}\} \in \mathcal{P}^r(C_n)$, where subscript arithmetic is performed modulo n , define for any i the set $D + i = \{v_{d+i}, v_{d+1+i}, \dots, v_{d+r-1+i}\}$. Then, by maximality of \mathcal{F} , we have, for some A_0, B_0 , and C_0 , $\mathcal{A} = \{A_0 + i\}_{i=0}^j$, $\mathcal{B} = \{B_0 + i\}_{i=0}^k$, and $\mathcal{C} = \{C_0 + i\}_{i=0}^l$, for some j, k , and l .

If $|A' \cap B'| = 0$ then $A_0 \cap B_k \cap C = \emptyset$. But $A_0 \cap X = \emptyset$, and so there is some $f \leq j$ and some $g \leq k$ such that $A_f \cap B_g \cap C \neq \emptyset$ and $A_f \cap B_{g-1} \cap C = \emptyset$, which implies that $n = 2r - 1 \leq n - 1$, a contradiction. Hence $A' \cap B' \neq \emptyset$; similarly, $A' \cap C' \neq \emptyset$ and $B' \cap C' \neq \emptyset$.

If $|A' \cap B'| > 1$ then consider B_{k+1} . We have $B_{k+1} \cap C \neq \emptyset$ since $B_{k+1} \cap C' \neq \emptyset$. But also $B_{k+1} \cap A' \neq \emptyset$, which implies the contradiction that \mathcal{F} is not maximum.

Hence $|A' \cap B'| = 1$; similarly, $|A' \cap C'| = 1$ and $|B' \cap C'| = 1$. Therefore, we have $A' \cap B' = \{v_c\}$, $A' \cap C' = \{v_b\}$, and $B' \cap C' = \{v_a\}$ for some $S = \{a, b, c\} \subseteq V(C_n)$, and so $\mathcal{F} = \{D \mid |D \cap D| = 2\}$.

Finally, since \mathcal{F} is maximum, $\mathcal{A} = \{D \in \mathcal{P}^r \mid \{b, c\} \subseteq D\}$, $\mathcal{B} = \{D \in \mathcal{P}^r \mid \{a, c\} \subseteq D\}$, and $\mathcal{C} = \{D \in \mathcal{P}^r \mid \{a, b\} \subseteq D\}$. Now let $x = (b - a) \bmod n$, $y = (c - b) \bmod n$, and $z = a - c \bmod n$. Then $|\mathcal{F}| = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = (r - z) + (r - y) + (r - x) = 3r - n$. \square

2.3 Theta Graphs

Let $k \geq 2$ and $a_1 \leq \dots \leq a_k$ be given, with $a_2 \geq 2$. For each i let $P_{(i)}$ be the path $P_{a_i} = w_{i,0}w_{i,1} \dots w_{i,a_i-1}$, w_{i,a_i} , with $u = w_{i,0}$ and $v = w_{i,a_i}$. Then define the *generalized theta graph* $\Theta(a_1, \dots, a_k) = \cup_{i=1}^k P_{(i)}$. Each path $P_{(i)}$ is called a *strand* of Θ , while u and v are called its *hubs*. Furthermore define the function

$$f_k(r) = \begin{cases} k + \binom{k}{2}(r-2) & \text{when } 3 \leq r \leq a_1 + 1, \text{ and} \\ (r - a_1 - 2)(k-1)^2 + (a_1 + 2)(k-1) + (r-2)\binom{k-1}{2} & \text{when } a_1 + 2 \leq r \leq a_2 - 1. \end{cases}$$

Theorem 14. *Let $k \geq 2$ and $1 \leq r \leq (a_1 + a_2 + 1)/2$, and suppose that $\Theta = \Theta(a_1, \dots, a_k) \neq \Theta(1, 2) = K_3$. Then Θ is strictly \mathcal{P}^r -EKR. In particular, if \mathcal{F} is an intersecting subfamily of $\mathcal{P}^r(\Theta)$ then $|\mathcal{F}| \leq f_k(r)$, with equality if and only if \mathcal{F} is a star centered on one of the hubs of Θ .*

Proof. Note that the condition on r ensures that the intersection of two paths in \mathcal{F} is a path (i.e., their union does not contain a cycle). The result is true if $k = 2$ by Corollary 10 because $\Theta(a_1, a_2) = C_{a_1+a_2}$. Hence we will assume that $k \geq 3$. The result is also true for $r < (a_1 + a_2 + 3)/3$ by Corollary 7. Thus we will assume that $r \geq (a_1 + a_2 + 3)/3$, which is at least 3 since $a_2 \geq 2$ and $\Theta(a_1, \dots, a_k) \neq \Theta(1, 2)$. Set $\Theta = \Theta(a_1, \dots, a_k)$.

First we calculate the sizes of stars in Θ to find which is the largest; i.e., write $\mathcal{P} = \mathcal{P}(\Theta)$ and find $|\mathcal{P}_x^r|$ for all $x \in V(\Theta)$. We begin by counting $|\mathcal{P}_u^r|$, which equals $|\mathcal{P}_v^r|$ by symmetry.

- If $3 \leq r \leq a_1 + 1$ then $|\mathcal{P}_u^r| = k + \binom{k}{2}(r-2)$. Indeed, there are k paths in \mathcal{P}_u^r with endpoint u . For any other $P \in \mathcal{P}_u^r$, u is one of the $r-2$ interior points of P , and the two subpaths of P on opposite sides of u lie in two different strands $P_{(i)}$ and $P_{(j)}$ for some $\{i, j\} \in \binom{[k]}{2}$.
- If $a_1 + 1 < r \leq (a_1 + a_2 + 1)/2$ then $a_1 < r < a_2$. If $v \in P$ then $P_{(1)} \subset P$ and there are two cases. If one endpoint of P is u or v then the other endpoint is in one of the $k-1$ other strands, and so there are $2(k-1)$ such paths. Otherwise, u is one of $r - a_1 - 2$ vertices on the interior of P and the two endpoints of P are on any of the $k-1$ other strands independently, and so there are $(r - a_1 - 2)(k-1)^2$ such paths. If $v \notin P$ then there are two cases as well. If u is one endpoint of P then the other endpoint is

in one of the other strands, and so there are $k - 1$ such paths. Otherwise, u is an interior vertex of P . Here, we have two subcases. If one endpoint of P is in $P_{(1)}$ then it is one of the $a_1 - 1$ interior vertices of $P_{(1)}$ and the other endpoint is on any of the other $k - 1$ strands, and so there are $(a_1 - 1)(k - 1)$ such paths. Otherwise, the two endpoints x_1 and x_2 are on any pair of paths $P_{(i)}$ and $P_{(j)}$ (respectively), with $1 < i < j$, and u is any of the $r - 2$ interior vertices of P , and so there are $(r - 2)\binom{k-1}{2}$ such paths. Hence $|\mathcal{P}_u^r| = 2(k - 1) + (r - a_1 - 2)(k - 1)^2 + (k - 1) + (a_1 - 1)(k - 1) + (r - 2)\binom{k-1}{2}$.

Next we calculate $|\mathcal{P}_{w_{i,j}}^r|$, with $j \notin \{0, a_i\}$, and show that in each case it is strictly smaller than $|\mathcal{P}_u^r|$. By symmetry we may assume that $j \leq (a_i + 1)/2$.

- If $3 \leq r \leq j + 1$ then $|\mathcal{P}_{w_{i,j}}^r| = r$ because $w_{i,j}$ could be any of the r vertices of P , and $P \subseteq P_{(i)}$. Because $k, r \geq 3$, we have $|\mathcal{P}_u^r| = k + \binom{k}{2}(r - 2) \geq 3 + 3(r - 2) > r = |\mathcal{P}_{w_{i,j}}^r|$, which handles the $r \leq a_1 + 1$ case. Similarly, when $r > a_1 + 1$, we have $|\mathcal{P}_u^r| = (r - a_1 - 2)(k - 1)^2 + (a_1 + 2)(k - 1) + (r - 2)\binom{k-1}{2} \geq 4(r - a_1 - 2) + 2(a_1 + 2) + (r - 2) = 5r - 2a_1 - 2 > r$.
- If $j + 1 < r \leq a_i - j + 1$ then $|\mathcal{P}_{w_{i,j}}^r| = (r - j - 1)(k - 1) + (j + 1)$ because $r - j - 1$ of the paths P extend beyond u to included interior vertices of some other strand $P_{(h)}$, while the remaining paths are all contained in the strand $P_{(i)}$. Note that $(r - j - 1)(k - 1) + (j + 1) = r(k - 1) - (j + 1)(k - 2) < r(k - 1) \leq k + \binom{k}{2}(r - 2)$ because $k \geq 3$. Hence, in this case, $|\mathcal{P}_{w_{i,j}}^r| < |\mathcal{P}_u^r|$ when $r \leq a_1 + 1$. Additionally, $r \geq 3$ implies that $r(k - 1) < 2(k - 1) + \binom{k-1}{2}(r - 2)$, and so $|\mathcal{P}_{w_{i,j}}^r| < |\mathcal{P}_u^r|$ when $r > a_1 + 1$ as well.
- If $a_i - j + 1 < r \leq a_1 + 1$ then $|\mathcal{P}_{w_{i,j}}^r| = (2r - a_i - 2)(k - 1) + (a_i + 2 - r)$ because $a_i + 2 - r$ paths are contained entirely in the strand $P_{(i)}$, while the remaining $2r - a_i - 2$ paths extend beyond u or v to included interior vertices of some other strand.

Since $r, k \geq 3$, we have

$$\begin{aligned} (k - 3)[r(k - 2) - 2k] + (2rk - 8) &= (k - 3)[(r - 2)(k - 2) - 4] + (2rk - 8) \\ &\geq (k - 3)(k - 6) + 6k - 8 = (k - 1)(k - 2) + 8 > 0, \end{aligned}$$

which implies that $2(k - 3)k + 8 < r[(k - 2)(k - 3) + 2k]$. From this and $a_i \geq r - 1$ we obtain

$$[2(k - 1) - 2a_i - 6]k + 8 \leq [2(k - 1) - 2r - 4]k + 8 < r(k - 2)(k - 3),$$

which implies that

$$2(2r - a_i - 2)(k - 2) + 2r < 2k + k(k - 1)(r - 2),$$

and hence that $|\mathcal{P}_{w_{i,j}}^r| < k + \binom{k}{2}(r-2) = f_r(k)$ in this case.

- If $a_1 + 2 \leq r \leq (a_1 + a_2 + 1)/2$, then $|\mathcal{P}_{w_{1,j}}^r| = (r - a_1 - 2)(k - 1)^2 + (a_1 + 2)(k - 1)$ because $r - a_1 - 2$ paths extend beyond both u and v to included interior vertices of two other strands $P_{(h)}$ and $P_{(h')}$ (possibly $h = h'$), while the remaining $a_1 + 2$ paths extend beyond either u or v , exclusively. Therefore $|\mathcal{P}_{w_{1,j}}^r| = (r - a_1 - 2)(k - 1)^2 + (a_1 + 2)(k - 1) < (r - a_1 - 2)(k - 1)^2 + (a_1 + 2)(k - 1) + \binom{k-1}{2}(r - 2) = |\mathcal{P}_u^r|$ for such r because $k \geq 3$.

Second, let $\mathcal{F} \subset \mathcal{P}^r$ be a maximum intersecting family, and suppose that \mathcal{F} is not a star. We proceed to prove that $|\mathcal{F}| < f_k(r)$.

1. Suppose that no path in \mathcal{F} contains both u and v .

- (a) Suppose that some path $P \in \mathcal{F}$ lies entirely in the strand $P_{(i)} - \{u, v\}$ for some i . Because \mathcal{F} is not a star there must be paths $P', P'' \in \mathcal{F}$ such that $P \cap P' \cap P'' = \emptyset$, and so $P \cup P' \cup P'' = P_{(i)} \cup P_{(j)}$, for some j . We may suppose that $P' \in \mathcal{F}_u$ and $P'' \in \mathcal{F}_v$ and that P' and P'' are chosen so that $|P' \cap P''|$ is minimum among such pairs of paths with these properties. From this it follows that $|P' \cap P''| = 1$. Indeed, otherwise define Q by shifting P' by one around the cycle $P_{(i)} \cup P_{(j)}$, toward P and away from P'' . Now the pair Q and P'' have the above properties with $|Q \cap P''| < |P' \cap P''|$, a contradiction.

Let $P' \cap P'' = w_{j,x}$, for some x , and let the other endpoints of P' and P'' be $w_{i,x'}$ and $w_{i,x''}$, respectively. Define $\mathcal{F}_{(i)}$ to be those paths of \mathcal{F} that are contained in the strand $P_{(i)}$. (Note that $|\mathcal{F}_{(i)}| \geq 1$ in this case.) Then define \mathcal{F}_u^j to be the set of paths in $\mathcal{F}_u - \mathcal{F}_{(i)}$ that are contained in the cycle $P_{(i)} \cup P_{(j)}$, and let $\mathcal{F}_u^- = \mathcal{F}_u - \mathcal{F}_u^j$; define \mathcal{F}_v^j and \mathcal{F}_v^- analogously. Thus \mathcal{F} is partitioned into $\mathcal{F}_u \cup \mathcal{F}_v \cup \mathcal{F}_{(i)}$ and, more finely, into $\mathcal{F}_{(i)} \cup \mathcal{F}_u^j \cup \mathcal{F}_u^- \cup \mathcal{F}_v^j \cup \mathcal{F}_v^-$.

Of course, since every path in $\mathcal{F}_{(i)}$ is contained in the strand $P_{(i)}$, we have

$$1 \leq |\mathcal{F}_{(i)}| \leq a_i - r + 1, \quad (6)$$

which implies that $a_i \geq r \geq 3$. Moreover, every path in $\mathcal{F}_{(i)}$ must contain both $w_{i,x'}$ and $w_{i,x''}$ as well, and so $|\mathcal{F}_{(i)}| \leq r - (x'' - x')$. Now notice that

$$\begin{aligned} a_i + a_j &= |P_{(i)} \cup P_{(j)}| \\ &= |P' \cup P''| + (x'' - 1 - x') \\ &= (2r - 1) + (x'' - x) - 1, \end{aligned}$$

and so

$$x'' - x' = (a_i + a_j) - (2r - 2), \quad (7)$$

which yields

$$|\mathcal{F}_{(i)}| \leq 3r - (a_i + a_j) - 2. \quad (8)$$

Next, suppose that $|\mathcal{F}_u^j| > 1$ and let $Q \in \mathcal{F}_u^j$ be such that, among paths in \mathcal{F}_u^j , $|Q \cap P''|$ is maximum. Let $w_{i,y}$ be the endpoint of Q on the strand $P_{(i)}$. Now consider the family \mathcal{S} of $k - 2$ paths with endpoint $w_{i,y+1}$, containing v , and having their other endpoint not in the cycle $P_{(i)} \cup P_{(j)}$. Because each path of \mathcal{S} is disjoint from Q , $\mathcal{S} \cap \mathcal{F} = \emptyset$. Then the family $(\mathcal{F} - \{Q\}) \cup \mathcal{S}$ is intersecting, not a star, and at least as large as \mathcal{F} (because $k \geq 3$). Thus we may assume that $|\mathcal{F}_u^j| = 1$. By the symmetric argument we may also assume that $|\mathcal{F}_v^j| = 1$. At this point, we have that

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_{(i)}| + |\mathcal{F}_u^j| + |\mathcal{F}_u^-| + |\mathcal{F}_v^j| + |\mathcal{F}_v^-| \\ &\leq |\mathcal{F}_{(i)}| + 2 + |\mathcal{F}_u^-| + |\mathcal{F}_v^-|, \end{aligned} \quad (9)$$

and so it remains to estimate $|\mathcal{F}_u^-|$ and $|\mathcal{F}_v^-|$.

Each path in \mathcal{F}_u^- must contain both u and $w_{i,x''}$, so that it intersects both P and P'' . Moreover, if $x'' > r - 2$ then $\mathcal{F}_u^- = \emptyset$, while if $x'' \leq r - 2$ then, for each $x'' \leq h \leq r - 2$, there are exactly $k - 2$ such paths having endpoint $w_{i,h}$. Because $|\mathcal{F}|$ is maximum, all of these would be in \mathcal{F}_u^- . Therefore $|\mathcal{F}_u^-| = (r - 1 - x'')(k - 2)$. Similarly, if $x' < a_i - r$ then $\mathcal{F}_v^- = \emptyset$, while if $x' \geq a_i - r$ then $|\mathcal{F}_v^-| = (r + x' - a_i - 1)(k - 2)$. Hence, in all cases we have

$$\begin{aligned} |\mathcal{F}_u^-| + |\mathcal{F}_v^-| &= [(r - 1 - x'') + (r + x' - a_i - 1)](k - 2) \\ &= [2r - 2 - a_i - (x'' - x')](k - 2) \\ &= [2r - 2 - a_i - (a_i + a_j - 2r + 2)](k - 2) \\ &= (4r - 2a_i - a_j - 4)(k - 2). \end{aligned} \quad (10)$$

Therefore, bounds (9) and (10) yield

$$|\mathcal{F}| \leq |\mathcal{F}_{(i)}| + 2 + (4r - 2a_i - a_j - 4)(k - 2). \quad (11)$$

To show that the upper bound in (11) is at most $f_k(r)$, we consider the two cases for r that define f . First take $3 \leq r \leq a_1 + 1$, which implies that $2r \leq 2a_1 + 2 = (3/2)a_1 + (a_1/2 + 1) + 1 < (3/2)a_i + a_j + 1$ since $a_1 \leq a_i$, $a_1 \leq a_j$, and $a_j \geq 3$. From this, we obtain

$$\begin{aligned} 0 < -2(r-2) + \frac{3}{2}a_i + a_j - 3 &= \left(\frac{2}{2} - \frac{7}{2} + \frac{1}{2}\right)(r-2) + \left(2 - \frac{1}{2}\right)a_i + a_j - 3 \\ &\leq \left(\frac{k-1}{2} - \frac{7}{2} + \frac{1}{k-1}\right)(r-2) + \left(2 - \frac{1}{k-1}\right)a_i + a_j - 3 \\ &= \left(\frac{k}{2} - 4 + \frac{1}{k-1}\right)r + \left(2 - \frac{1}{k-1}\right)a_i + a_j + 5 - \frac{2}{k-1} - k, \end{aligned}$$

because $k \geq 3$. We continue, deriving

$$\frac{a_i - r + 3}{k-1} \leq \left(\frac{k}{2} - 4\right)r + 2a_i + a_j + 4 + \frac{k}{k-1} - k,$$

and so

$$\frac{a_i - r + 3}{k-1} + 4r - 2a_i - a_j - 4 \leq \frac{k}{k-1} + \frac{k}{2}(r-2).$$

Finally, using the bound from (6), we find that

$$\begin{aligned} |\mathcal{F}| &\leq a_i - r + 3 + (4r - 2a_i - a_j - 4)(k-2) \\ &\leq a_i - r + 3 + (4r - 2a_i - a_j - 4)(k-1) < k + \binom{k}{2}(r-2), \end{aligned}$$

achieving the strict bound of $f_k(r)$ for this range of r .

Second, take $a_1 + 2 \leq r \leq a_2 - 1$. In this case we will show that $|\mathcal{F}| \leq f_k(r)$ by splitting each side into two parts and proving that

- i. $|\mathcal{F}_{(i)}| + 2 \leq \binom{k-1}{2}(r-2)$ and
- ii. $(4r - 2a_i - a_j - 4)(k-2) < [(a_1 + 2) + (r - a_1 - 2)(k-1)](k-1)$.

For part (i) we use (8), $r \leq (a_1 + a_2)/2 + 1$, and $k \geq 3$ to write $3r - (a_i + a_j) \leq 3r - (a_1 + a_2) \leq 3r - (2r + 2) = r - 2 \leq \binom{k-1}{2}(r-2)$.

For part (ii) it suffices to prove that $4r - 2a_i - a_j - 4 \leq (a_1 + 2) + (r - a_1 - 2)(k-1)$, which is equivalent to showing that

$$a_1(k-2) \leq (k-5)r + (2a_i + a_j) + 8.$$

Since $3 < 2k + 2(a_i - a_1) + (a_j - a_1)$, we have

$$\begin{aligned} a_1(k-2) &< (k-5)(a_1+2) + 2a_i + a_j + 8 \\ &\leq (k-5)r + 2a_i + a_j + 8, \end{aligned}$$

because $a_1 + 2 \leq r$. Hence $|\mathcal{F}| < f_k(r)$ for this range of r as well.

- (b) Now we know that no path in \mathcal{F} lies entirely in a single strand's interior $P_{(i)} - \{u, v\}$. Thus every path in \mathcal{F} intersects $\{u, v\}$, and because \mathcal{F} is not a star, some path P contains u but not v , and some path P' contains v but not u ; among such pairs, we choose one such that $|P \cap P'|$ is minimum. Note that $|P \cap P'| = 1$. Observe that this property requires $r \geq a_1/2$. Now \mathcal{F} partitions into $\mathcal{F}_u \cup \mathcal{F}_v$. Hence $|\mathcal{F}| = |\mathcal{F}_u| + |\mathcal{F}_v|$.

Having chosen $P \in \mathcal{F}_u$ and $P' \in \mathcal{F}_v$, let $\{i, j\} \in \binom{[k]}{2}$ be such that (without loss of generality) $P \cup P' \subseteq P_{(i)} \cup P_{(j)}$. Notice that \mathcal{F}_v partitions into $\mathcal{F}_v^i \cup \mathcal{F}_v^j$, where $P' \in \mathcal{F}_v^i$ (resp. \mathcal{F}_v^j) if $P' \cap P \subset P_{(i)}$ (resp. $P_{(j)}$). Hence $|\mathcal{F}_v| = |\mathcal{F}_v^i| + |\mathcal{F}_v^j|$.

Let $P'' \in \mathcal{F}_v^j$ such that $|P \cap P''|$ is minimum. Then $|P \cap P''| = 1$; indeed, suppose not. Because \mathcal{F} is maximum we have $P^+ \in \mathcal{F}_u$, where $P^+ \in P_{(i)} \cup P_{(j)}$ has $|P' \cap P^+| = 2$. Let w_{j,h_j} be the endpoint of P in the strand $P_{(j)}$. For $l \neq j$, define P'_l to be the path in the cycle $P_{(j)} \cup P_{(l)}$ beginning at w_{j,h_j} . Now the family $\mathcal{F} - \{P^+\} \cup \{P'_1, \dots, P'_k\} - \{P'_j\}$ is intersecting and larger than \mathcal{F} since $k \geq 3$, a contradiction. Hence $|P \cap P''| = 1$.

Now this implies that $|\mathcal{F}_u| = 1$ since any other path containing u is disjoint from P' or P'' . In the same way that we built P^+ by shifting P toward v along the strand $P_{(i)}$, we can shift P' and P'' toward u along the strands $P_{(i)}$ and $P_{(j)}$ respectively. Among these there are at least two, say Q and Q' , that do not contain u . Then $\mathcal{F} - \{P\} \cup \{Q, Q'\}$ is still intersecting and larger than \mathcal{F} , a contradiction.

2. Now suppose that some path $P \in \mathcal{F}$ contains both u and v . Then there is a unique i such that the strand $P_{(i)} \subset P$. Of course, this implies that $r \geq a_i + 1$, which we know is at least $a_1 + 1$. Similar to the above, we partition $\mathcal{F} = \mathcal{F}_u \cup \mathcal{F}_v \cup \mathcal{F}_{uv}$, where \mathcal{F}_{uv} consists of all the paths of \mathcal{F} that contain both u and v . We will enumerate $|\mathcal{F}| = |\mathcal{F}_u| + |\mathcal{F}_v| + |\mathcal{F}_{uv}|$ in two cases as follows. As above, we choose $P' \in \mathcal{F}_u$ and $P'' \in \mathcal{F}_v$ such that $|P' \cap P''|$ is minimized. Necessarily, $|P' \cap P''| = 1$, and so we write $P' \cap P'' = \{w_{j,x}\}$ for some $j \geq 2$.

- (a) First, let $r = a_1 + 1$, so that $|\mathcal{F}_{uv}| = 1$. Thus we may assume that $i = 1$. (Note that $(a +$

$1 + a_j + 1)/2 \geq r = a_1 + 1$, and so $2a_1 - a_i \leq a_1 - 1$, which we will use below.) Now we have $|\mathcal{F}_u| = (k-1)(r-x-1) + 1$ and $|\mathcal{F}_v| = (k-1)(r-a_j+x-1) + 1$.

Hence

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_u| + |\mathcal{F}_v| + |\mathcal{F}_{uv}| = (k-1)(2r-a_j-2) + 3 \\ &= (k-1)(2a_1-a_j) + 3 < (k-1)(a_1-1) + k \\ &\leq k + \binom{k}{2}(a_1-1) = k + \binom{k}{2}(r-2) = f_r(k). \end{aligned}$$

(b) Second, let $a_1 + 2 \leq r \leq (a_1 + a_2 + 1)/2$. Then $|\mathcal{F}_{uv}| = 2(k-1) + (r-a_i-2)(k-1)^2$ because two paths end at either u or v , while all others have each endpoint on some other strand different from $P_{(i)}$. As in the first case, we have $|\mathcal{F}_u| = (k-1)(r-x-1) + 1$ and $|\mathcal{F}_v| = (k-1)(r-a_j+x-1) + 1$, and so

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_u| + |\mathcal{F}_v| + |\mathcal{F}_{uv}| \\ &= 2(k-1) + (r-a_i-2)(k-1)^2 + (k-1)(2r-a_j-2) + 2. \end{aligned} \quad (12)$$

Now $f_k(r) = 2(k-1) + (r-a_1-1)(k-1)^2 + (a_1+1)(k-1) + \binom{k-1}{2}(r-2)$ in this case, so the right side of (12) is at most $f_k(r)$ when

$$0 \leq (a_i - a_1)(k-1) + (a_1+1) + \frac{1}{2}(k-2)(r-2) + (a_1+a_j-2r) - \frac{2}{k-1}. \quad (13)$$

Now $a_1 + 2 \leq r \leq (a_1 + a_2 + 1)/2$ implies that $a_2 \geq a_1 + 3$, and so $a_i - a_1 \geq a_2 - a_1 \geq 3$. It also implies that $a_1 + a_j - 2r \geq a_j - a_2 - 1 \geq -1$. Hence, the right hand side of Equation (13) is at least $6 + (1/2) - 1 - 1 > 0$, which finishes the proof of part (b), and hence of case (2).

Therefore, \mathcal{F} is a star and $|\mathcal{F}| \leq f_k(r)$, with equality if and only if it is centered on a hub. \square

3 Transversal and Triangular Results

In this section, we collect some related results on transversal numbers which we believe to be of independent interest, as they relate to the following problem of Erdős and Lovász [9].

Problem 15. *Find the value $m(r)$ of the minimum size intersecting, uniform family of r -sets with transversal number r .*

Erdős often said that determining whether $m(r) = O(r)$ was “one of his three favorite combinatorial problems” [20]. In [9] they showed that $m(r) \geq \frac{8}{3}r - 3$, but obtained as an upper bound $m(r) \leq 4r^{3/2} \log r$ – their construction uses random collections of lines in projective planes, and so held when $r - 1$ is the order of a projective plane. This was first improved by Kahn[19] to $m(r) < Cr \log r$, again contingent on the existence of an order $r - 1$ projective plane. Finally, Kahn[20] showed that there is some prime power K so that $m(r) \leq 5(K^2 + k)t$ when $r = Kq + t$, thus confirming the Erdős-Lovász conjecture that $m(r) = O(r)$. It is worth noting that Kahn’s method does not provide any idea how large K might be.

Here, we modify Problem 15 slightly to suit our purposes. First, we think about things in somewhat the opposite direction – rather than fixing r and asking how small a family with transversal number r might be, we instead think of the size of \mathcal{F} as fixed and ask how small its transversal number could be. That is, rather than interpreting the Erdős-Lovász lower bound as saying that $m(r) \geq \frac{8}{3}n - 3$, we think of this as saying that $\tau(\mathcal{F}) \leq \frac{3}{8}(|\mathcal{F}| + 9)$ (subject to the significant restriction that $\tau(\mathcal{F}) = r$).

In Subsection 3.1, we relax the condition that \mathcal{F} must be uniform, proving some basic transversal results involving degree conditions for our intersecting families. In subsection 3.2 we focus on those intersecting families that contain no large stars whatsoever, with $\Delta(\mathcal{F}) = 2$. Finally, in subsection 3.3, we construct some examples in projective planes which demonstrate the sharpness of several of our preceding results in Section 3.

3.1 Elementary Observations

First, we note some straightforward bounds that we will use later. These results are well known, but we include proofs here for completeness.

Fact 16 (See [18], p. 111, Sec. 10.2.). *If \mathcal{F} is an intersecting family of sets then $\tau(\mathcal{F}) \leq \min_{A \in \mathcal{F}} |A|$.*

Proof. Let F be a set in \mathcal{F} of minimum size. Since \mathcal{F} is intersecting, it follows that F is a transversal of \mathcal{F} . Hence $\tau(\mathcal{F}) \leq |F|$. □

Fact 17. *If \mathcal{F} is an intersecting family of sets then $\tau(\mathcal{F}) \leq \lceil |\mathcal{F}|/2 \rceil$.*

Proof. For any intersecting family of sets \mathcal{F} , pair off its sets arbitrarily, with possibly one set remaining. For each pair of sets choose one element from their intersection. Also choose one element from the remaining set if it exists. These choices form a transversal of size $\lceil |\mathcal{F}|/2 \rceil$. Therefore $\tau(\mathcal{F}) \leq \lceil |\mathcal{F}|/2 \rceil$. □

Fact 18 (See [18], p. 111, Sec. 10.2.). *If \mathcal{F} is an intersecting family of sets then $\tau(\mathcal{F}) \geq \lceil |\mathcal{F}|/\Delta(\mathcal{F}) \rceil$.*

This follows from a more general statement in [18], stating that $\tau(\mathcal{F})$ is bounded below by the size of the largest matching in \mathcal{F} . We give a direct proof here.

Proof. Let X be a transversal of \mathcal{F} . For any $x \in X$ we get $\deg(x) \leq \Delta(\mathcal{F})$. Hence X must contain at least $\lceil |\mathcal{F}|/\Delta(\mathcal{F}) \rceil$ elements to be a transversal of \mathcal{F} . \square

3.2 Triangular Families

In the interest of exploring EKR problems, we will be interested in trios of sets with no common intersection. In particular, we say that a trio of sets $\{A, B, C\}$ is *triangular* if $A \cap B \cap C = \emptyset$. We further call a family \mathcal{F} of sets *triangular* if $\{A, B, C\}$ is triangular for all $\{A, B, C\} \subseteq \mathcal{F}$. For example, any family of lines in general position in the plane is both intersecting and triangular. For subsets of a finite set, one may ask how large an intersecting triangular family can be or what is its transversal number. We begin with a characterization.

Fact 19. *An intersecting family \mathcal{F} containing at least two sets is triangular if and only if $\Delta(\mathcal{F}) = 2$.*

Proof. Assume that \mathcal{F} is triangular. That is, for all $\{A, B, C\} \subseteq \mathcal{F}$ we get that $A \cap B \cap C = \emptyset$, which implies $\Delta(\mathcal{F}) < 3$. Furthermore, since \mathcal{F} contains at least two sets and is intersecting $\Delta(\mathcal{F}) > 1$. Hence $\Delta(\mathcal{F}) = 2$.

Suppose that $\Delta(\mathcal{F}) = 2$. Then for any $x \in \cup \mathcal{F}$, we get that x is an element of at most 2 sets in \mathcal{F} . Therefore any three sets from \mathcal{F} are triangular. Hence \mathcal{F} is triangular. \square

Fact 20. *If \mathcal{F} is a triangular intersecting family of sets then $|\mathcal{F}| \leq 1 + \min_{A \in \mathcal{F}} |A|$.*

Proof. Let A be a set in \mathcal{F} of minimum size. Since \mathcal{F} is triangular, it follows from Fact 19 that $\Delta(\mathcal{F}) = 2$. For all $x \in A$, x is contained in at most one other set in \mathcal{F} . Hence $|\mathcal{F}| \leq 1 + |A|$. \square

The following corollary shows that Fact 17 is best-possible.

Corollary 21. *If \mathcal{F} is a triangular intersecting family of sets then $\tau(\mathcal{F}) = \lceil |\mathcal{F}|/2 \rceil$.*

Proof. Since \mathcal{F} is intersecting, it follows from Fact 17 that $\tau(\mathcal{F}) \leq \lceil |\mathcal{F}|/2 \rceil$. As for the lower bound, from Fact 18 we get that $\tau(\mathcal{F}) \geq \lceil |\mathcal{F}|/\Delta(\mathcal{F}) \rceil$. Given that \mathcal{F} is triangular, Fact 19 gives us $\Delta(\mathcal{F}) = 2$. Thus $\tau(\mathcal{F}) = \lceil |\mathcal{F}|/2 \rceil$. \square

It is worth asking whether or not triangular families characterize equality in Fact 17. Curiously, this is true for odd $|\mathcal{F}|$ but not necessarily for all even $|\mathcal{F}|$.

Theorem 22. *If \mathcal{F} is an intersecting family of $2k - 1$ sets with $\tau(\mathcal{F}) = k$ then \mathcal{F} is triangular.*

Proof. Suppose that \mathcal{F} is intersecting, $|\mathcal{F}| = m = 2k - 1$, and $\tau(\mathcal{F}) = k$. If $m \leq 2$ then \mathcal{F} is triangular by definition, so we assume that $m \geq 3$, and let $X = \{x_1, \dots, x_k\}$ be a minimum transversal.

If some $a \in \cup \mathcal{F}$ has $\deg(a) > 2$ then, for $\mathcal{F}' = \mathcal{F} - \mathcal{F}_a$, we have $|\mathcal{F}'| \leq m - 3 = 2k - 4$, and so $\tau(\mathcal{F}') \leq k - 2$ by Fact 17. Now let Y be a minimum transversal of \mathcal{F}' ; then $Y \cup \{a\}$ is a transversal of \mathcal{F} of size at most $k - 1$, a contradiction. Hence $\Delta(\mathcal{F}) = 2$. \square

For $m \in \{4, 6\}$, there are examples of intersecting families \mathcal{F} with $|\mathcal{F}| = m$, $\tau(\mathcal{F}) = m/2$, and $\Delta(\mathcal{F}) > 2$. Indeed, for a given integer $h > 1$, subset $S \subset \{0, \dots, h - 1\} = [h]$, and positive integer i , define the set $S^i = \{s + i \bmod h \mid s \in S\}$. Also define the family $\mathcal{R}_h(S) = \{S^i \mid i \in [h]\}$. Now set $S_2 = \{0, 1, 2\}$ and $S_3 = \{0, 1, 3\}$. It is easy to verify that, for $k \in \{2, 3\}$, the family $\mathcal{R}_{2k}(S_k)$ is intersecting and has size $2k$. Additionally, $\tau(\mathcal{F}_{2k}) = k$ because, for any $T \in \binom{[2k]}{k-1}$, there is some $S^i \in \mathcal{F}_{2k}$ with $S^i \cap T = \emptyset$. Moreover, $\mathcal{R}_{2k}(S_k)$ is not triangular because it is 3-regular.

This simple rotational construction illustrates the conflicting objectives in Problem 15. When $|S|$ is large compared to $2k$, $\mathcal{R}_{2k}(S)$ tends to be intersecting with small transversal number, while if $|S|$ is small, $\mathcal{R}_{2k}(S)$ tends to have large transversal number but not be intersecting.

3.3 Projective Planes

The finite projective plane $\mathbb{P}(q)$, where q is any prime power, is a rich example in extremal combinatorics. Frankl and Graham [11] gave a sufficient condition for a uniform family to be EKR that considers its generalized degrees. They used a construction based on projective planes to show that the condition couldn't be weakened.

Recall the notation $\Delta_s(\mathcal{F}) = \max_{|X|=s} |\mathcal{F}_X|$.

Theorem 23 ([11]). *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting and $\Delta_1(\mathcal{F}) \geq (r^2 - r + 1)\Delta_2(\mathcal{F})$ then \mathcal{F} is EKR. When $r - 1$ is a prime power, there exists a non-EKR $\mathcal{H} \subseteq \binom{[n]}{r}$ with $\Delta_1(\mathcal{H}) = r^2 - r$ and $\Delta_2(\mathcal{H}) = 1$.* \square

The construction of \mathcal{H} in Theorem 23 has $n = m(r - 1)(r^2 - r + 1)$ for some large enough m , and is built from a collection of mutually orthogonal copies of $\mathbb{P}(r - 1)$.

Here we use projective planes to investigate the tightness of Fact 20. Observe that Fact 20 implies that every triangular subfamily of $\mathbb{P}(q)$ has size at most $q + 2$. In fact, this is tight for even q but can be improved to $q + 1$ for odd q , as we show below.

We first introduce some notation for $\mathbb{P}(q)$, as follows. Let \mathbb{F}_q denote the finite field of order q , and set $V = V_q = ((\mathbb{F}_q \cup \{\omega\}) \times \mathbb{F}_q) \cup \{(\omega, \omega)\}$. The set of $q^2 + q + 1$ elements of V serve as both the points of $\mathbb{P}(q)$

and the indices of the lines of $\mathbb{P}(q)$; that is, $\mathbb{P}(q) = \{L_\alpha \mid \alpha \in V\}$, where

- $L_{(m,b)} = \{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q \mid y = mx + b\} \cup \{(\omega, m)\}$, for $(m, b) \in \mathbb{F}_q \times \mathbb{F}_q$,
- $L_{(\omega,b)} = \{(b, y) \in \mathbb{F}_q \times \mathbb{F}_q\} \cup \{(\omega, \omega)\}$, for $b \in \mathbb{F}_q$, and
- $L_{(\omega,\omega)} = \{(\omega, y) \mid y \in \mathbb{F}_q\} \cup \{(\omega, \omega)\}$,

and all arithmetic, unless specified otherwise, is performed in \mathbb{F}_q throughout the remainder of this section.

Of course, $\deg(\alpha) = |L_\alpha| = q + 1$ for all $\alpha \in V$. It is well known that $\mathbb{P}(q)$ is exactly 1-intersecting.

Lemma 24. *For any prime power q , $\tau(\mathbb{P}(q)) = q + 1$.*

Proof. The upper bound follows from Fact 16. The lower bound follows from Fact 18. \square

Theorem 25. *Let q be an odd prime power. If \mathcal{F} is a triangular subfamily of $\mathbb{P}(q)$ of maximum size then $|\mathcal{F}| = q + 1$.*

Proof. We first construct an example to show that triangular families of size $q + 1$ exist. For each $b \in \mathbb{F}_q - \{0, 1\}$, define $m_b = \frac{-b}{1-b}$. Then, we define our family of $q + 1$ lines to be

$$\mathcal{F} = \{L_{(\omega,\omega)}, L_{(\omega,0)}, L_{(0,0)}, L_{(m_2,2)} \cdots, L_{(m_{q-1},q-1)}\}.$$

Note that every line in \mathcal{F} has a distinct slope; hence we just need to show that every point (x, y) has $\deg((x, y)) \leq 2$. Let $(x, y) \in L_{(m_b,b)}$. Then $y = xm_b + b$, and so

$$y = -x \frac{b}{1-b} + b,$$

$$(1-b)(y-b) = -xb, \text{ and}$$

$$b^2 + (x-y-1)b + y = 0.$$

The final equation is quadratic in b and thus has at most two solutions in \mathbb{F}_q , and so $\deg((x, y)) \leq 2$.

Therefore \mathcal{F} is a triangular family of size $q + 1$, showing the lower bound.

For the upper bound, consider a triangular subfamily $\mathcal{F} \subseteq \mathbb{P}(q)$ with $|\mathcal{F}| > q + 1$. By Fact 20, $|\mathcal{F}| = |L| + 1 = q + 2$. By symmetry, we may assume that \mathcal{F} contains the lines $L_{(\omega,\omega)}, L_{(\omega,0)}, L_{(0,0)}$, so we define

$$\mathcal{F}' = \mathcal{F} - \{L_{(\omega,\omega)}, L_{(\omega,0)}, L_{(0,0)}\}.$$

Because \mathcal{F} is 1-intersecting and triangular, and $|\mathcal{F}'| = q - 1$, we know that for every $b \in \mathbb{F}_q^* = \mathbb{F}_q - \{0\}$ there exists a unique $m_b \in \mathbb{F}_q^*$ such that $L_{(m_b, b)} \in \mathcal{F}'$. Moreover, there exists a unique $a_b \in \mathbb{F}_q^*$ such that $(a_b, 0) \in L_{(m_b, b)}$. Hence $m_b a_b + b = 0$; i.e. $b = -m_b a_b$.

Let γ be a generator for $\mathbb{F}_q^* \cong \mathbb{Z}_{q-1}$, so that every $z \in \mathbb{F}_q^*$ satisfies $z = \gamma^e$ for some $e \in \mathbb{Z}_{q-1} = \{0, \dots, q-2\}$. Write $\mathbb{F}_q^* = \{b_1, \dots, b_{q-1}\}$, and relabel m_{b_i} and a_{b_i} as m_i and a_i , respectively. For each $1 \leq i \leq q-1$ define d_i , e_i , and f_i uniquely by $b_i = \gamma^{d_i}$, $m_i = \gamma^{e_i}$, and $a_i = \gamma^{f_i}$, so that $d_i = e_i + f_i \pmod{q-1}$. Hence $\{b_i\} = \{m_i\} = \{a_i\} = \mathbb{F}_q^*$, so that $\{d_i\} = \{e_i\} = \{f_i\} = \mathbb{Z}_{q-1}$. That is, there exist permutations $(d_1 \cdots d_{q-1})$, $(e_1 \cdots e_{q-1})$, and $(f_1 \cdots f_{q-1})$ of \mathbb{Z}_{q-1} such that $d_i = e_i + f_i \pmod{q-1}$ for all i . But, since q is odd, this yields the contradiction that

$$\frac{q-1}{2} = \sum_{i=1}^{q-1} d_i = \sum_{i=1}^{q-1} e_i + \sum_{j=1}^{q-1} f_j = \frac{q-1}{2} + \frac{q-1}{2} = 0 \pmod{q-1}$$

Hence $|\mathcal{F}'| < q - 1$, and so $|\mathcal{F}| \leq q + 1$. □

Theorem 26. *Let $q = 2^t$ for $t \in \mathbb{N}$. If \mathcal{F} is a triangular subfamily of $\mathbb{P}(q)$ of maximum size then $|\mathcal{F}| = q + 2$.*

Proof. Given that all lines in $\mathbb{P}(q)$ contain $q + 1$ points, from Fact 20 we get $|\mathcal{F}| \leq q + 2$. Now we need only show the lower bound by displaying a triangular family of size $q + 2$.

Using the lower bound \mathcal{F}' constructed in Theorem 25, let $\mathcal{F} = \mathcal{F}' \cup \{L_{(1,1)}\}$. Here we must show for any $(x, y) \in L_{(1,1)}$ there is exactly one $L_{(m_b, b)} \in \mathcal{F}$ containing (x, y) . That is, over \mathbb{F}_2 we have

$$\begin{aligned} -x + 1 &= -x \frac{b}{1-b} + b, \\ 0 &= b^2 + (2x-2)b - x + 1 = b^2 + x + 1, \text{ and} \\ b &= \sqrt{x+1}, \end{aligned}$$

which is unique in characteristic 2. Thus there is exactly one line $L_{(m_b, b)} \in \mathcal{F}$ containing (x, y) . Therefore \mathcal{F} is a triangular family of size $q + 2$, showing the lower bound, and thus $|\mathcal{F}| = q + 2$. □

4 More Questions

Define the d^{th} power of a graph G , written $G^{(d)}$, to have $V(G^{(d)}) = V(G)$ and an edge xy for every pair of vertices with $\text{dist}_G(x, y) \leq d$.

Problem 27. Study the \mathcal{P}^r -EKR properties of other graphs such as $C_n^{(d)}$ or subdivisions of complete graphs or complete bipartite graphs. (Note that C_n is a subdivision of $K_{2,2}$ when $n \geq 4$ and the theta graph $\Theta(a_1, \dots, a_k)$ is a subdivision of $K_{2,k}$ when each $a_i \geq 2$.) Also consider graphs C_n^f for $f : V(C_n) \rightarrow \mathbb{N}$, or the more general class of trees plus an edge; i.e. unicyclic graphs.

We note that Frankl, et al. [13] addresses the EKR problem for spanning trees of K_n . Along these lines, it would be equally interesting to study the case of spanning paths of K_n (i.e. $r = n$ in Problem 27).

A family \mathcal{F} is called *Sperner* if there are no two sets $A, B \in \mathcal{F}$ with $A \subseteq B$. Bollobas' inequality [3] gives a very strong condition on the sizes of Sperner families. There is a somewhat complicated history here; Bollobás' inequality is a general version of a sharper-in-some-cases result of Lubell, which had been earlier discovered post-Bollobás but pre-Lubell by Meshalkin, and which can in hindsight be read out of an earlier result of Yamamoto; in order to better reflect all four author's various contributions in chronological order, this is perhaps best called the YBLM inequality. Proving YBLM-type theorem in this \mathcal{H} -EKR context would be a significant step forward; we provide a starting question here.

Question 28. Assume $\mathcal{F} \subseteq \mathcal{P}(C_n)$ is a maximum size family that is both intersecting and Sperner. Is it true that \mathcal{F} must be uniform?

It may be both interesting and necessary to consider cross-intersecting families of paths in pursuit of answering the questions above; bounding the sum or product of the sizes of such cross-intersecting families is also useful and relevant in its own right.

Theorem 9 states that, among the largest intersecting families of $\mathcal{P}^r(C_n)$ having transversal number at least 2 is one having transversal number exactly 2.

Question 29. Given a graph G , is it true that among the largest intersecting families in $\mathcal{P}^r(G)$ (or $\mathcal{P}(G)$) with transversal number at least k is one with transversal number exactly k ?

Finally, we ask the question left open by Theorem 22. There, we show that for odd sized intersecting families of $2k - 1$ sets, that the families achieving $\tau(\mathcal{F}) = k$ are the triangular families. We also showed that this is not necessarily the case for all even n , giving examples when $|\mathcal{F}| \in \{4, 6\}$.

Question 30. For which k does there exist an intersecting family \mathcal{F} of size $2k$ with $\tau(\mathcal{F}) = k$ and $\Delta(\mathcal{F}) > 2$? How can we characterize such families?

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