The Equivalence of the Auxiliary and Shortcut Methods for the Simplex Algorithm

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Abstract

We describe two standard, different-looking Simplex method pivotting rules and prove their equivalence by framing them as two different shortcuts of the same extended method. The first is the auxiliary method, often found and thoroughly justified in graduate texts, such as by Chvátal. The second is a simpler method, often found but never justified in undergraduate pre-business texts, such as by Mizrahi-Sullivan. This article bridges this gap in explanation.

1 Introduction

Dantzig's Simplex algorithm for solving linear optimization problems (LOPs — here we purposely avoid the term *program*) is well known and well studied. There are many pivotting rules available, each with various advantages and disadvantages, some practical, some pedagogical. As typically described (we assume familiarity with [1] throughout) the algorithm has two *phases*, for handling infeasible and feasible bases, respectively. Here we are concerned with two different approaches to Phase I, as found in [1] (called the *Auxiliary Method*) and [3] (which we call the *Shortcut Method*). While the former is fully justified and proved to work, the latter is left unexplained. Our goal is to prove the equivalence of these methods. These ideas appear also in the forthcoming book [2].

1.1 Set-up

We begin by describing the relevant procedures on the following LOP in *standard maximization form*.

Problem 1

$$
\begin{array}{llll}
Max. \quad z = & \sum_{j=1}^{n} c_j x_j & & \\
s.t. & \sum_{j=1}^{n} a_{i,j} x_j & \leq b_i & (1 \leq i \leq m) \\
 & x_j & \geq 0 & (1 \leq j \leq n)
\end{array}
$$

The initial tableau T_0 for Problem 1 is below. Notice that we differ from the setup in [1] in the objective row only: by placing all variables on the left of the equality, we find the objective coefficients negated.

Tableau 2

$$
\begin{array}{cccc|ccccc|c}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 & 0 & b_1 \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 & 0 & b_2 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & 0 & 0 & \cdots & 1 & 0 & b_m \\
\hline
-c_1 & -c_2 & \cdots & -c_n & 0 & 0 & \cdots & 0 & 1 & 0
\end{array}
$$

The columns before the first separating line hold the coefficients of the *problem variables* x_j $(1 \le j \le n)$. The columns between the two separating lines hold the coefficients of the *slack variables* x_{n+i} $(1 \le i \le m)$ and the *objective variable* z. The final *b-column* holds the *constraint values* b_i $(1 \le i \le m)$ and the *objective value* 0. As these entries change after pivot operations we will ignore the line separating problem and slack variables and work on the general tableau T' below.

Tableau 3

$$
\begin{array}{cccccccc|c}
a'_{1,1} & a'_{1,2} & \cdots & a'_{1,n} & a'_{1,n+1} & a'_{1,n+2} & \cdots & a'_{1,n+m} \quad 0 & b'_1 \\
a'_{2,1} & a'_{2,2} & \cdots & a'_{2,n} & a'_{2,n+1} & a'_{2,n+2} & \cdots & a'_{2,n+m} \quad 0 & b'_2 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \quad \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \quad \cdot & \cdot \\
a'_{m,1} & a'_{m,2} & \cdots & a'_{m,n} & a'_{m,n+1} & a'_{m,n+2} & \cdots & a'_{m,n+m} \quad 0 & b'_m \\
\hline
a'_{m+1,1} & a'_{m+1,2} & \cdots & a'_{m+1,n} & a'_{m+1,n+1} & a'_{m+1,n+2} & \cdots & a'_{m+1,n+m} \quad d' & z'
\end{array}
$$

In each such tableau the columns corresponding to the basic variables typically will form a permutation matrix. In our presentation they will form the multiple d' times a permutation matrix, where the *basic coefficient* d' is the determinant of the original columns of the basic variables (excluding the objective row). This is in order that all tableau entries are integral (a simple consequence of Cramer's Rule). A basic variable x_j whose only nonzero coefficient in T' occurs in row i will have current basic value $x_j = b'_i/d'$, while the current objective value equals $z = z'/d'$. It is not difficult to show that this integral property is preserved by the following *pivot operation*.

$$
\begin{aligned}
\hat{T}_r &= sT'_r \\
\hat{T}_i &= s(a'_{r,c}T'_i - a'_{i,c}T'_r)/d' \qquad (1 \le i \le m+1,\ i \ne r)
\end{aligned}
$$

Here, the pivot is on entry $a'_{r,c}$, $s = \text{sign}(a_{r,c})$, and T'_i (resp. $\hat{T}_i$) denotes the i^{th} row of the tableau before (resp. after) the pivot. In fact, the new basic coefficient $\hat{d} = |a'_{r,c}|$.

We denote by β the set of basic variables. The remaining variables we call *parameters*, denoted by π. In the pivot operation above we have $\hat{\beta} = (\beta' \cup \{x_c\}) - \{x_j\}$ and $\hat{\pi} = (\pi' \cup \{x_j\}) - \{x_c\}$, where x_j is the basic variable whose only nonzero entry (d') in T' occurs in row r. We denote this pivot operation by $c \mapsto j$.

1.2 Example

Consider the following linear optimization problem.

Problem 4

$$
\begin{array}{lrrrrrrrl}
Max. \; z & = & 26x_1 & + & 23x_2 & + & 24x_3 & & \\
s.t. & & -8x_1 & + & 3x_2 & + & 4x_3 & \leq & -231 \\
 & & 4x_1 & - & 9x_2 & + & 2x_3 & \leq & -322 \\
 & & x_1 & + & 5x_2 & - & 7x_3 & \leq & -268 \\
\& & & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

The tableau corresponding to $\beta = \{3, 5, 6\}$ and $\pi = \{1, 2, 4\}$ is below.

Tableau 5

$$
\begin{array}{rrr|rrrr|r}
-7 & 2 & 3 & 1 & 0 & 0 & 0 & -210 \\
22 & -26 & 0 & -1 & 3 & 0 & 0 & -705 \\
-57 & 30 & 0 & 9 & 0 & 3 & 0 & -2640 \\
\hline
-266 & -11 & 0 & 26 & 0 & 0 & 3 & -5460
\end{array}
$$

One can make the pivot $1 \mapsto 6$ on the entry $a'_{3,1} = -57$ by performing the following row operations.

$$
\begin{array}{rcl}
\hat{T}_1 & = & -(-57T'_1 + 7T'_3)/3 \\
\hat{T}_2 & = & -(-57T'_2 - 22T'_3)/3 \\
\hat{T}_3 & = & -T'_3 \\
\hat{T}_4 & = & -(-57T'_4 + 266T'_3)/3
\end{array}
$$

The result is as follows, with $\beta = \{1, 3, 5\}$ and $\pi = \{2, 4, 6\}$.

Tableau 6

$$
\begin{array}{rrr|rrrr|r}
0 & -32 & 57 & -2 & 0 & -7 & 0 & 2170 \\
0 & -274 & 0 & 47 & 57 & 22 & 0 & -32755 \\
57 & -30 & 0 & -9 & 0 & -3 & 0 & 2640 \\
\hline
0 & -2869 & 0 & -304 & 0 & -266 & 57 & 130340
\end{array}
$$

1.3 Pivot Rules

In each of the procedures below we outline the method for finding the pivot entry. We begin with Phase II, signified by having every $b'_i \geq 0$.

Phase II Procedure.

1. Find $c = \min\{i \mid a_{m+1,i} < 0, i \leq m+n\}$.
   - If no such $a_{m+1,i}$ exists then the tablueau is optimal.
2. Find $r = \text{argmin}_i\{b'_i/a'_{i,c} \mid a'_{i,c} > 0\}$.
   - If no such $a_{i,c}$ exists then the problem is unbounded.
   - If there are several i sharing the same minimum ratio $b'_i/a'_{i,c}$ then each corresponds to a unique basic variable x_j, where $j = j(i)$. Let I_c be the set of all such i and choose $r = \text{argmin}_i\{j(i) \mid i \in I_c\}$.
3. Pivot on entry $a_{r,c}$.

While any such c for which $a_{m+1,c} < 0$ can be used as a pivot column, and any such $r \in I_c$ can be used as a pivot row, the choices outlined above all correspond to what is commonly known as the Least Subscript implementation, or Bland's Rule (which guarantees that the Simplex algorithm will halt).

For Phase I, we know that some $b'_i < 0$. Since Step 2 above preserves Phase II, we must have that some $b_{i'} < 0$ in the original Tableau 2. Thus for the instance of Problem 1 having some $b_i < 0$ we consider the following *Auxiliary Problem.*

Problem 7

$$\begin{array}{lllllll} Max. & v = & -x_0 & & & & \\ s.t. & & -x_0 & + & \sum_{j=1}^n a_{i,j}x_j & \leq b_i & (1 \leq i \leq m) \\ & & x_0 & , & x_j & \geq 0 & (1 \leq j \leq n) \end{array}$$

Problem 7 is feasible at $x_0 = -\min\{b_i\}$ with $x_j = 0$ for $1 \leq j \leq n$. The problem is also bounded by $v \leq 0$, and so there exists an optimum value v^* by the Fundamental Theorem. Clearly, $v^* = 0$ if and only if Problem 1 is feasible. We now describe the Auxiliary Method of [1].

Phase I Auxiliary Method.

1. Find $r = \text{argmin}_i\{b_i \mid b_i < 0\}$.

   - If no such b_i exists then use the Phase II procedure.
   - If there are several i sharing the same minimum b_i then each corresponds to a unique basic variable x_j, where $j = j(i)$. Let I_c be the set of all such i and choose $r = \text{argmin}_i\{j(i) \mid i \in I_c\}$.

2. Pivot on the entry $a_{r,0}$ of the initial tableau for the Auxiliary Problem 7.

3. Repeat the Phase II procedure to solve Problem 7.

4. If $v^* < 0$ then Problem 1 is infeasible. Otherwise use the optimal basis β^* as the starting basis for Problem 1.

It is easy to prove that the tableau after pivotting on $a_{r,0}$ is feasible, justifying Step 3, above. Moreover, the final tableau delivered by the Auxiliary Method is in Phase II. As before, the Least Subscript implementation is used throughout.

Now we describe the Phase I procedure found in [3], which for reasons that will soon be apparent we call the Shortcut Method. This procedure is defined for every tableau T'.

Phase I Shortcut Method.

1. Find $r = \text{argmin}_i\{b'_i \mid b'_i < 0\}$.

   - If no such b'_i exists then use the Phase II procedure.
   - If there are several i sharing the same minimum b'_i then each corresponds to a unique basic variable x_j, where $j = j(i)$. Let I be the set of all such i and choose $r = \text{argmin}_i\{j(i) \mid i \in I\}$.

2. Find $c = \min\{j \mid a'_{r,j} < 0\}$.

   - If no such $a'_{r,j}$ exists then Problem 1 is infeasible.

3. Pivot on the entry $a_{r,c}$.

4. Repeat Steps 1–3 until Phase II is required.

The equivalence of these two Phase I methods is best illustrated by example.

2 Comparison of Methods by Example

Consider Problem 4. Its auxiliary is the following LOP.

Problem 8

$$\begin{array}{llrcrcrcrcr}
Max. & v = & -x_0 \\
\\
s.t. & & -x_0 & - & 7x_1 & + & 2x_2 & + & 3x_3 & \leq & -210 \\
 & & -x_0 & + & 5x_1 & - & 8x_2 & + & x_3 & \leq & -305 \\
 & & -x_0 & + & 2x_1 & + & 4x_2 & - & 9x_3 & \leq & -250 \\
\\
\& & & x_0 & , & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

We display below the sequence of tableaux and pivots that solve Problem 8 according to the given pivot rules. Note that we have numbered the tableaux Ti (and pivots Pi) somewhat oddly. We also have left some space where columns may appear or disappear in future tableaux relating to these.

Tableaux 9 (Auxiliary Method)

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	v	
T0 :	−1	−8	3	4	1	0	0	0	−231
	−1	4	−9	2	0	1	0	0	−322
	−1	1	5	−7	0	0	1	0	−268
	1	0	0	0	0	0	0	1	0

P1 : $0 \mapsto 5$

T1 :	0	−12	10	2	1	−1	0	0	95
	1	−5	8	−1	0	−1	0	0	305
	0	−3	12	−10	0	−1	1	0	55
	0	5	−8	1	0	1	0	1	−305

P3 : $2 \mapsto 6$

T3 :	0	−114	0	124	12	−2	−10	0	590
	12	−36	0	68	0	−4	−8	0	3220
	0	−3	12	−10	0	−1	1	0	55
	0	36	0	−68	0	4	8	12	−3220

P5 : $3 \mapsto 4$

T5 :	0	−114	0	124	12	−2	−10	0	590
	124	274	0	0	−68	−30	−26	0	29930
	0	−126	124	0	10	−12	2	0	1060
	0	−274	0	0	68	30	26	124	−29930

$P6 : 1 \mapsto 0$

T6 :	114	0	0	274	−36	−32	−46	0	28820
	124	274	0	0	−68	−30	−26	0	29930
	126	0	274	0	−47	−57	−22	0	32755
	274	0	0	0	0	0	0	274	0

Make note that Pivot *P1* was performed so that Tableau *T1* would be feasible. By pivoting x_0 into the basis, Tableau *T1* will be feasible if and only if the leaving variable is chosen to be that basic variable whose current value is most negative. After that, the Phase II pivoting rules take over.

Take a moment now to explore the effect of these same pivots on the original Problem 4. To do this, we *extend* the Auxiliary Method as follows. Think of each pivot $x_l \mapsto x_k$ as the pair of pivots $x_l \mapsto x_0$ and $x_0 \mapsto x_k$. These two pivots have the same effect of exchanging x_k and x_l, while leaving x_0 basic. But the splitting of each pivot like this will allow us to analyze each individual choice of entering and leaving variables. Now we will include in the tableaux the columns and objective rows of both objective variables z and u, arriving at the following sequence of tableaux.

Tableaux 10 (Extended Method)

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	z	v	
T0 :	−1	−8	3	2	1	0	0	0	0	−231
	−1	4	−9	2	0	1	0	0	0	−322
	−1	1	5	−7	0	0	1	0	0	−268
	0	−26	−23	−24	0	0	0	1	0	0
	1	0	0	0	0	0	0	0	1	0

$P1 : 0 \mapsto 5$

T1 :

0	−12	10	2	1	−1	0	0	0	95
1	−5	8	−1	0	−1	0	0	0	305
0	−3	12	−10	0	−1	1	0	0	55
0	−26	−23	−24	0	0	0	1	0	0
0	5	−8	1	0	1	0	0	1	−305

$P2 : 2 \mapsto 0$

T2 :

−10	−60	0	42	9	3	0	0	0	−3045
1	−4	9	−2	0	−1	0	0	0	322
−12	29	0	−53	0	5	9	0	0	−4022
21	−326	0	−262	0	−23	0	9	0	7406
8	0	0	0	0	0	0	0	9	0

$P3 : 0 \mapsto 6$

T3 :

0	−114	0	124	12	−2	0	0	0	590
0	−3	12	−10	0	−1	0	0	0	55
12	−36	0	68	0	−4	12	0	0	3220
0	−399	0	−522	0	−21	0	12	0	1155
0	36	0	−68	0	4	0	0	12	−3220

$P4 : 3 \mapsto 0$

T4 :

−124	−218	0	0	53	41	42	0	0	−36701
0	−30	53	0	0	−7	−2	0	0	2790
12	−29	0	53	0	−5	−9	0	0	4022
522	−2764	0	0	0	−281	−262	53	0	160698
53	0	0	0	0	0	0	0	53	0

$P5 : 0 \mapsto 4$

T5 :

124	274	0	0	−68	−30	−26	0	0	29930
0	−126	124	0	10	−12	2	0	0	1060
0	−114	0	124	12	−2	−10	0	0	590
0	−9082	0	0	522	−304	−218	124	0	37600
0	−274	0	0	68	30	26	0	124	−29930

$P6 : 1 \mapsto 0$

T6 :	124	218	0	0	-53	-41	-42	0	0	36701
	126	0	218	0	-30	-52	-32	0	0	32250
	114	0	0	218	-29	-43	-60	0	0	36625
	9082	0	0	0	-2764	-3294	-3268	218	0	2574976
	218	0	0	0	0	0	0	0	218	0

Now one can see why the tableaux were numbered in the Auxiliary Method the way they were: Pivot *Pk* in the Auxiliary sequence became Pivots *P(k-1)* and *Pk* in the Extended sequence. Let's now translate the rules we used in the Auxiliary Method into rules for the Extended Method. In both cases we start by choosing the row whose b-column was most negative, in this case row 2.

The entering varialble x_2 was chosen in Auxiliary Pivot *P3* because of the -8 in the objective row of Tableau *T1*, the first negative number we see when reading left-to-right (Least Subscript rule). This corresponds to the 8 in row 2 of Tableau *T1*, or the -8 in row 2 of Tableau *T0*. Neglecting the auxiliary variable x_0, this translates into a rule which chooses the variable with the least subscript whose coefficient in the corresponding row of the prior tableau is negative.

Then x_6 was chosen as the leaving variable in Auxiliary Pivot *P3* because it was the variable which placed the greatest restriction on x_2. That is, its *b-ratio* of $\frac{55}{12}$ was smallest among nonnegative b-ratios. Soon we shall see that $\frac{55}{12}$ was the smallest for the same reason that -3220 was the most negative in Tableau *T2*. In other words, x_6 was chosen as the leaving variable in Extended Pivot *P3* because -3220 was most negative in Tableau *T2*.

Thus we have transformed the rules of pivoting into ones which refer only to even numbered tableaux, suggesting that we can skip the odd numbered tableaux by combining the two pivot operations $x_0 \mapsto x_k$ and $x_l \mapsto x_0$ into the single exchange $x_l \mapsto x_k$. Now the method for finding a Phase I pivot is the reverse of that which finds the pivot in Phase II in the following sense. Instead of choosing an entering variable first, followed by a leaving variable, we now choose a leaving variable first, then an entering variable. The leaving variable rule picks that basic variable whose current basic value is most negative, while the entering variable rule picks that variable with the least subscript whose coefficient in the pivot row is negative. These are the Shortcut Method rules. Of course, now we can delete the column and row corresponding to the auxiliary objective variable u since these are (virtually) unchanged throughout the sequence of even tableaux. Also, x_0 is never basic, so we can delete its column as well.

Tableaux 11 (Shortcut Method)

	x_1	x_2	x_3	x_4	x_5	x_6	z	
T0 :	-8	3	4	1	0	0	0	-231
	4	-9	2	0	1	0	0	-322
	1	5	-7	0	0	1	0	-268
	-26	-23	-24	0	0	0	1	0

P2 : $2 \mapsto 5$

T2 :	-60	0	42	9	3	0	0	-3045
	-4	9	-2	0	-1	0	0	322
	29	0	-53	0	5	9	0	-4022
	-326	0	-262	0	-23	0	9	7406

P4 : $3 \mapsto 6$

T4 :	-218	0	0	53	41	42	0	-36701
	-30	53	0	0	-7	-2	0	2790
	-29	0	53	0	-5	-9	0	4022
	-2764	0	0	0	-281	-262	53	160698

P6 : $1 \mapsto 4$

T6 :	218	0	0	-53	-41	-42	0	36701
	0	218	0	-30	-52	-32	0	32250
	0	0	218	-29	-43	-60	0	36625
	0	0	0	-2764	-3294	-3268	218	2574976

3 Equivalence of Methods

Now we move to justifying in general the claim that achieving the minimum b-ratio of $\frac{55}{12}$ was equivalent to having the b-value -3220 be most negative.

Theorem 12 *The Auxiliary and Shortcut Methods are equivalent in the sense that, ignoring the auxiliary variable x_0, they make the same sequence of decisions for incoming and outgoing variables.*

Proof. In the case that Phase I halts on infeasibility, the two methods are clearly the same. Thus we assume that, at some stage of the Auxiliary Method, we hold a feasible tableau, a portion of which, including the incoming variable and the b-column, is shown below.

$$\begin{array}{cccc|c} a & \cdot & & & b \\ c & & \cdot & & d \\ e & & & \cdot & f \\ \hline -e & & & & -f \end{array}$$

For example, it could be the first three rows of Tableau *T1*. We know that $b, d, f \geq 0$ since the tableau is feasible. Suppose that the Auxiliary Method chooses to pivot in the first row, in particular on the entry $a > 0$. The second row represents any other row such that a pivot on c is also under consideration (i.e., $c > 0$), and the third row includes the basic auxiliary variable x_0.

The first thing to notice is that, other than the auxiliary coefficient, while x_0 is in the basis the objective row is the negative of the auxiliary row (as displayed above). This is caused by the objective function being equal to $-x_0$: the relation clearly holds after the first pivot and is easily seen to be preserved by future pivots until x_0 leaves. Thus, because the Auxiliary Method pivots on a, we must have $e > 0$. For ease of presentation, we momentarily ignore the prospect of ties; thus $0 \leq \frac{b}{a} < \frac{d}{c}$.

Note that the relevant portion above the objective line of the corresponding tableau during the Extended Method is identical (outside of permuting the rows). Since x_0 is basic, $-f$ was most negative in the previous Extended tableau, and so $f > 0$ as well. The Extended Method then chooses the pivot entry e, resulting in the following tableau.

$$\begin{array}{cccc|r} 0 & \cdot & & & be - af \\ 0 & & \cdot & & de - cf \\ e & & & \cdot & f \end{array}$$

Now suppose now $de - cf < 0$. This implies that $\frac{d}{c} < \frac{f}{e}$, and then $\frac{b}{a} < \frac{d}{c}$ implies that $b < \frac{ad}{c}$. Hence $b - d < \frac{d}{c}(a - c) < \frac{f}{e}(a - c)$, which means that $e(b - d) < f(a - c)$, or $be - af < de - cf$. Thus the Extended Method pivots in the first row, just like the Auxilliary Method. The argument is of course reversible: for $b, d \geq 0$, $a, c, e, f > 0$, and $\frac{d}{c} < \frac{f}{e}$, we have $be - af < de - cf$ if and only if $\frac{b}{a} < \frac{d}{c}$. This means that the the Auxiliary and Shortcut Methods make the same choice of leaving variable. We already know that the incoming variable is the same for each, so both methods make the same sequence of pivots.

It is fairly straightforward to include Bland's tiebreaking rule in the above arguments, completing the proof. $\square$

References

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[3] A. Mizrahi and M. Sullivan, *Finite Mathematics* (8th ed.), Wiley, New York, 1999.