A New Dominating Set Game on Graphs

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Abstract

We introduce a new two-player game on graphs, in which players alternate choosing vertices until the set of chosen vertices forms a dominating set. The last player to choose a vertex is the winner. The game fits into the scheme of several other known games on graphs. We characterize the paths and cycles for which the first player has the winning strategy. We also create tools for combining graphs in various ways (via graph powers, Cartesian products, graph joins, and other methods) for building a variety of graphs whose games are won by the second player, including cubes, multidimensional grids with an odd number of vertices, most multidimensional toroidal grids, various trees such as specialized caterpillars, the Petersen graph, and others. Finally, we extend the game to groups and show that the second player wins the game on abelian groups of even order with canonical generating set, among others.

Keywords: dominating set, games on graphs, graph power, Cartesian product, graph join, Cayley graph

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1 Introduction

In this work we introduce a new combinatorial game, which we call the *graph domination* (**GD**) *game*. On a given graph, two players take turns choosing vertices until the set of chosen vertices forms a dominating set. The winner is the player who was last to choose a vertex. This game is related to a wide range of other games played on graphs, and generalizes the following natural game on groups. For a group Γ with generating set Σ, players alternate choosing elements of Γ until every unchosen element h equals $h'g$ for some chosen h' and some $g \in \Sigma$. In this instance, the game played on a group is equivalent to the game played on the Cayley graph of that group.

All graphs $G = (V, E)$ are assumed to be finite and connected. We denote the path on n vertices by P_n and the cycle on n vertices by C_n, and refer to the degree 1 vertices of a path as endpoints, and the degree 2 vertices as interior vertices. The open (closed) neighborhoods of a vertex u and of a set of vertices U are denoted $N(u)$ (resp. $N[u] = N(u) \cup \{u\}$) and $N(U) = \cup_{u \in U} N(u)$, respectively. We also define the set notation $N_X(u) = N(u) \cap X$. For $u, v \in V(G)$, the function $\mathsf{dist}_G(u, v)$ measures the distance between them; i.e., the number of edges in a shortest uv-path. This allows us to define the *closed distance-d neighborhood* of u: $N^d[u] = \{v \in V \mid \mathsf{dist}(u, v) \leq d\}$. A set of vertices $D \subseteq V$ is called a *dominating set* if $V = N[D]$. More generally, D is a *distance-d dominating set* if $V = N^d[D]$.

1.1 History

There is a large body of work in graph theory that studies, not the graphs themselves, but the activities or games involving the placement of objects on the vertices of a graph and movements of them along edges under certain rules for various purposes. Some versions of these model the spread of information or disease, supply-demand optimization, network search, and other applications, while others are studied purely for game-theoretic research. Autonoma such as Conway's Game of Life [12] and bootstrap percolation [6] can be thought of as 0-person games, while others such as chip firing [5], zero forcing [1], and graph burning [7] can be thought of as 1-person games, or puzzles. The objects of some 2-person games like cops & robbers [19] and graph pebbling [10, 18] are moved around by players, while some like Go [3] and Hex [14] are not. Our game is of this last category. An interesting and extremely popular game is the k-person board game RISK$^{\circledR}$ [16]. Its initial setup of shares some of the above characteristics, as pieces are placed on vertices of an implicit graph in order to eventually dominate it.

Several games like ours have been studied. For example, one can play a game in which two players alternate choosing vertices so that the set of chosen vertices is independent. Phillips and Slater [21] initiated

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the study of the case in which the last player to chose a vertex wins, while Phillips and Slater [20] and Goddard and Henning [13] have studied the case of one player trying to maximize and the other player trying to minimize the size of the final chosen set. Huggan, Huntemann, and Stevens [17] studied the game in which players alternate choosing vertices of a hypergraph H (in their case a block design) so that the chosen set does not contain any edge of H, with the winner being last player able to choose a vertex.

As mentioned, in the game we study here, the players alternate choosing vertices until the chosen vertices form a dominating set. In this context, Brešar, et al. [8, 9] have studied the "maximizer-minimizer" version of the game, and Henning, Klavžar, and Rall [15] did the same for total dominating sets. (Another interesting maximize-minimize variation is studied by Alon, et al. [2], in which players alternate orienting the edges of a graph until the entire graph is oriented, observing the size of the resulting oriented domination number.)

Here we study the winner-loser version, the winner being the player who first creates a dominating set from the chosen vertices.

1.2 Game Theory Basics

We follow the traditional language and notation of [4, 11]. Because a **GD** game is a finite, perfect information game with no ties, we know that one of the players has a winning strategy. At every stage of the game, the player who is next to play is denoted Next with the other player denoted Prev (thought of as "previous"). At the beginning of the game, we say that Player 1 is next, and write P1 = Next, with Player 2 being P2 = Prev. For clarity, notice that after the first move we have Next = P2 and Prev = P1; That is, P1 and P2 are the permanent names of the players, while Next and Prev describe which of them is about to play. Then the set N is defined to be the set of all positions of the game for which Next has the winning strategy, while P is defined to be the set of all positions of the game for which Prev has the winning strategy. By identifying a game with its initial position, we can say that the game is in N (or P) if its initial position is in N (or P).

Every position $\mathcal{Q}$ corresponds to a set B of chosen vertices, so we define the *size* of $\mathcal{Q}$ to be $|B|$. Because Next must always choose a vertex that has not yet been chosen, we will abuse notation slightly by writing $u \in \mathcal{Q}$ to mean that u is a move available for N to play from position $\mathcal{Q}$. The position created from $\mathcal{Q}$ by Next choosing the vertex $u \in \mathcal{Q}$ is denoted $\mathcal{Q}(u)$, and corresponds to the chosen set $B \cup \{u\}$. Thus, P and N can be calculated recursively by first placing the position corresponding to $V(G)$ in P. Then we repeatedly consider all positions $\mathcal{Q}$ of size one less than prior: if $\mathcal{Q}$ corresponds to a dominating set then place it in P; otherwise, if $\mathcal{Q}(u) \in$ P for some u then place it in N; otherwise, place it in P. A common technique in game theory for proving that a position is in P is to display a *pairing strategy*: a function ϕ from the possible moves for N to those for P, along with a condition C such that all winning positions satisfy C and, for every

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position $\mathcal{Q}$ that satisfies C and every move u from $\mathcal{Q}$, $\mathcal{Q}(u)$ does not satisfy condition C and $\mathcal{Q}(u, \phi(u))$ does satisfy C. We make use of this idea below.

Let **GD** to refer to domination games on general graphs; for games on paths and cycles, we refer to them by **PD** and **CD**, respectively.

1.3 GD and PD Game Notations

To assist with the analysis of game positions and player strategies, it will be useful to keep track of more than just the chosen vertices B; in particular, we will also distinguish vertices that are or are not dominated by B. When the graph G is understood (in our case $G = P_n$ or C_n), we will leave it out of the positional notation. With this finer perspective, a **GD** *position* $\mathcal{Q}$ on G is a partition $\{B, S, W\}$ of $V(G)$ (for *black, shaded,* and *white*) such that no white vertex is adjacent to a black vertex. The interpretation here is that B is the set of chosen vertices, $S = N(B) - B$ is the set of non-chosen vertices dominated by B, and B is a dominating set of G if and only if $W = \emptyset$. **GD** positions with $W = \emptyset$ are called *trivial*. Thus all trivial positions are in P. We will use the notations $B_\mathcal{Q}$, (resp. $S_\mathcal{Q}, W_\mathcal{Q}$) when necessary to indicate the black (resp. shaded, white) vertices of position $\mathcal{Q}$.

Define two **GD** positions $\mathcal{Q}_1 = \{B_1, S_1, W_1\}$ and $\mathcal{Q}_2 = \{B_2, S_2, W_2\}$ on graphs G_1 and G_2 to be *isomorphic*, written $\mathcal{Q}_1 \cong \mathcal{Q}_2$, if there is a graph isomophism $\varphi : G_1 - B_1 \to G_2 - B_2$ such that $\varphi(S_1) = S_2$ and $\varphi(W_1) = W_2$. Clearly, if $\mathcal{Q}_1 \cong \mathcal{Q}_2$ then $\mathcal{Q}_1 \in \mathsf{P}$ if and only if $\mathcal{Q}_2 \in \mathsf{P}$.

Now define the game $\mathbb{D} = \mathbb{D}(G)$ with initial position $\mathcal{Q}_0$ having $W = V(G)$. At any stage of the game with position $\mathcal{Q}$, if $W_\mathcal{Q} \neq \emptyset$ then Next chooses some vertex $u \in S_\mathcal{Q} \cup W_\mathcal{Q}$, resulting in the position $\mathcal{Q}(u)$ having $B_{\mathcal{Q}(u)} = B_\mathcal{Q} \cup \{u\}$, $S_{\mathcal{Q}(u)} = (S_\mathcal{Q} - \{u\}) \cup N_W(u)$, and $W_{\mathcal{Q}(u)} = W_\mathcal{Q} - (\{u\} \cup N(u))$.

1.4 Involutions

For a graph G, an *involution* of G is an automorphism of G of order two. We say that an involution ϕ is *d-involutionary* if, for all $v \in V(G)$, $\mathsf{dist}(v, \phi(v)) \geq d$. Define a graph to be *d-involutionary* if it has d-involution. Observe that an involution ϕ is 1-involutionary if and only if it has no fixed points, and is 3-involutionary if and only if $N[v] \cap N[\phi(v)] = \emptyset$ for all $v \in V(G)$. The definition can be extended to a position $\mathcal{Q}$ by requiring that ϕ be an automorphism of $\mathcal{Q}$. The importance of a 3-involution is that it gives rise to a pairing strategy, as we show in Lemma 1.

Suppose that Γ is a group with elements Λ and generating set Σ (which we consider as closed under inverses). For the purposes of this paper, it is not important what set of relations Γ has; consequently, we will use the notation $\Gamma = (\Lambda, \Sigma)$ to denote any such group. The Cayley graph $\mathfrak{C}(\Gamma, \Sigma)$ is defined to be the

undirected graph $G = (V, E)$ with $V = \Lambda$ and $E = \{\{g, h\} \in \binom{\Lambda}{2} \mid gh^{-1} \in \Sigma\}$. We say that (Γ, Σ) is *involutionary* if there is some $g \in \Lambda - \Sigma$ of order 2, and *3-involutionary* if g also cannot be expressed as the product of two generators. Thus, if (Γ, Σ) is 3-involutionary then so is $\mathfrak{C}(\Gamma, \Sigma)$. Additionally, we define the game $\mathbb{D}(\Gamma, \Sigma)$ to be the game described above, in which players alternate choosing elements of Γ until every unchosen element h equals $h'g$ for some chosen h' and some $g \in \Sigma$. The last person able to choose an element is the winner. It is easy to see that $\mathbb{D}(\Gamma, \Sigma) = \mathbb{D}(G)$, where $G = \mathfrak{C}(\Gamma, \Sigma)$.

1.5   Graph constructions

Among our results are statements involving the following graph constructions.

Graph power. Given a graph $G = (V, E)$ and positive integer d, the dth *power of G* is denoted $G^{(d)} = (V, F)$, where $F = \{\{x, y\} \in \binom{V}{2} \mid \mathsf{dist}_G(x, y) \leq d\}$. Observe that $N_G^d[u] = N_{G^{(d)}}[u]$. For example, if M is a perfect matching in K_{2t}, then $C_{2k}^{(k-1)} \cong K_{2k} - M$.

Cartesian product. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *Cartesian product of G_1 and G_2* is denoted $G_1 \square G_2 = (V, E)$, where $V = V_1 \times V_2$ and $E = \{\{(u, v_1), (u, v_2)\} \mid \{v_1, v_2\} \in G_2\} \cup \{\{(u_1, v), (u_2, v)\} \mid \{u_1, u_2\} \in G_1\}$. For example, the 2-dimensional $m \times k$ grid on mk vertices is isomorphic to $P_m \square P_k$. Also, the d-dimensional cube equals $P_2^d = P_2 \square P_2 \square \cdots \square P_2$ (d copies of P_2).

Join. Given graphs $G = (V, E)$ and $H = (W, F)$, each with distinguished vertices $x \in V$ and $y \in F$, the *join of (G, x) and (H, y)* is denoted $(G, x) \cdot (H, y) = (V', E')$, where $V' = V \cup W$, setting $x = y$, and $E' = E \cup F$. For example, for any such x and y, the join $(C_n, x) \cdot (C_m, y)$ can be drawn to resemble a figure eight or infinity symbol.

Dangling. Given a graph $G = (V, E)$ with a sequence of distinct, distinguished vertices $\mathbf{x} = (x_1, \ldots, x_k)$ and a sequence of graphs $\mathbf{H} = (H_1, \ldots, H_k)$ (each $H_i = (W_i, F_i)$) with corresponding sequence of distinguished vertices $\mathbf{y} = (y_1, \ldots, y_k)$ (each $y_i \in W_i$), a *dangling of $\mathbf{H}$ on G* is denoted $(G, \mathbf{x}) \cdot (\mathbf{H}, \mathbf{y}) = (V', E')$, where $V' = V \cup_{i=1}^k W_i$, setting each $x_i = y_i$, and $E' = E \cup_{i=1}^k F_i$. One can think of dangling as a sequence of joins done in parallel. For example, if $\mathbf{x}$ is any ordering of V and each $H_i \cong K_2$ with any choice of y_i, then $(C_n, \mathbf{x}) \cdot (\mathbf{H}, \mathbf{y})$ is known as the *k-sunlet* graph S_k, shown in Figure 1, below.

Bridging. Given a graph $G = (V, E)$ with a distinguished vertex x and a sequence of graphs $\mathbf{H} = (H_1, \ldots, H_k)$ (each $H_i = (W_i, F_i)$) with corresponding sequence of distinguished vertices $\mathbf{y} = (y_1, \ldots, y_k)$ (each $y_i \in W_i$), a *bridging of (G, x) and $(\mathbf{H}, \mathbf{y})$* is denoted $(G, x) \cdot (\mathbf{H}, \mathbf{y}) = (V', E')$, where $V' = V \cup_{i=1}^k W_i$ and $E' = E \cup_{i=1}^k (F_i \cup \{\{x, y_i\}\})$. If each $(H_i, y_i) = (H, y)$ we denote this operation by $(G, x)\text{–}(H, y)^k$, with
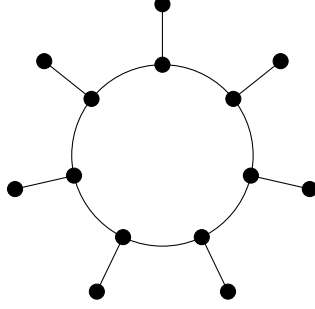
Figure 1: The 7-sunlet graph S_7.

the exponent suppressed when $k = 1$. For example, for degree 2 vertices x and y, the bridging $(P_3, x) \cdot (P_3, y)$ yields the internally 3-regular caterpillar shown in Figure 2.
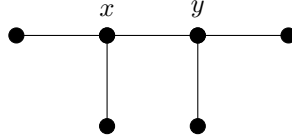

Figure 2: The internally 3-regular caterpillar $(P_3, x) \cdot (P_3, y)$.

1.6 Our results

The following lemma is fundamental to carving out many results.

Lemma 1 (Involution Lemma). *If $\mathcal{Q}$ is a 3-involutionary position, then $\mathcal{Q} \in \mathsf{P}$. In particular, if the graph G is 3-involutionary then $\mathbb{D}(G) \in \mathsf{P}$.*

Proof. We show that $\mathcal{Q} \notin \mathsf{N}$ by proving that if $\mathcal{Q}$ is a 3-involutionary position then, for all $u \in \mathcal{Q}$ we have (1) $\mathcal{Q}(u)$ is not trivial and (2) $\mathcal{Q}(u, v)$ is 3-involutionary for some $v \in \mathcal{Q}(u)$. Thus P1 can never end the game.

Indeed, let ϕ be 3-involution of $\mathcal{Q}$, suppose that Next chooses $u \in \mathcal{Q}$, and define $v = \phi(u)$. Because ϕ is an involution, it partitions the vertices into pairs $\{z, \phi(z)\}$ such that the elements of each pair are of the same "color" (white or shaded). Thus, if $u \in W_{\mathcal{Q}}$ then $v \in W_{\mathcal{Q}}$. Because ϕ is 3-involutionary, $v \in W_{\mathcal{Q}(u)}$. Also, if $u \in S_{\mathcal{Q}}$ then $v \in S_{\mathcal{Q}}$. So, if $\mathcal{Q}(u)$ is trivial, then there is some $x \in N_{W_{\mathcal{Q}}}(u)$, which implies that $\phi(x) \in N_{W_{\mathcal{Q}}}(v)$. But because ϕ is 3-involutionary, $\phi(x) \notin N[u]$, and so $f(x) \in W_{\mathcal{Q}(u)}$, a contradiction. This proves part (1). For part (2), we observe that ϕ is a 3-involution of $\mathcal{Q}(u, v)$. □

The Involution Lemma yields several immediate results.

Corollary 2. *For all $d \geq 2$ we have $\mathbb{D}(P_2^d) \in \mathsf{P}$, and $\mathbb{D}(P_2) \in \mathsf{N}$.*

Proof. Label the two vertices of P_2 by 0 and 1; then this labeling naturally extends to the vertices of $G = P_2^d$ by labels in $\{0, 1\}^d$. Define the *antipodal* mapping ϕ on G by $\phi(x_1, \ldots, x_d) = (1 - x_1, \ldots, 1 - x_d)$. For $d = 1$, the statement is obvious. For $d = 2$, if P1 plays u then P2 wins by playing $\phi(u)$. For $d \geq 3$, ϕ is 3-involutionary, so $\mathbb{D}(G) \in \mathsf{P}$ by Lemma 1. $\qquad\square$

Corollary 3. *For every 3-involutionary group Γ with generating set Σ, we have $\mathbb{D}(\Gamma, \Sigma) \in \mathsf{P}$.*

Proof. Let (Γ, Σ) be a 3-involutionary group, and g be an order 2 element of $\Lambda - \Sigma$ that cannot be expressed as the product of two generators. Define $G = \mathfrak{C}(\Gamma, \Sigma) = (V, E)$, and let $\phi : G \to G$ be an automorphism given by $\phi(v) = gv$. Then ϕ is also an involution on G. To see that ϕ is 3-involutionary, first notice that $\phi(u) = gu \notin N[u]$ since $g \notin \Sigma$. Then supposing, for contradiction, that there is an element $v \in N(u) \cap N[\phi(u)]$, we see that there are generators $a, b \in \Sigma$ such that $abu = \phi(u) = gu$. Canceling u yields $ab = g$, contradicting the fact that g cannot be written as a product of at most two generators. This shows that φ is 3-involutionary.

$\square$

Example 4. *Let $n \geq 6$ be even and let $k < n/4$ be a positive integer. Consider the game in which two players alternate choosing numbers from $\{0, 1, 2, \ldots, n - 1\}$ until every unchosen number is at distance at most k from some chosen number, where the distance between two numbers a and b equals $|a - b| \mod n$, and the winner is the last player to move. This game can be modeled by the game on the group $\Gamma = \mathbb{Z}_n$, with generating set $\Sigma = \{\pm 1, \ldots, \pm k\}$, where $\mathbb{Z}_n$ denotes the cyclic group of order n. Then $n/2 \notin \Sigma$ and cannot be expressed as the sum of two elements in the generating set, which implies that the automorphism $\phi(x) = x + n/2 \mod n$ is 3-involutionary, and so, by Corollary 3, $\mathbb{D}(\Gamma, \Sigma) \in \mathsf{P}$. We can also observe that $\mathfrak{C}(\Gamma, \Sigma) \cong C_n^{(k)}$, and thus this also shows that $C_n^{(k)} \in \mathsf{P}$.*

Example 5. *Let $n_1, \ldots, n_d$ be positive integers, each at least 3, and suppose that n_1 is even and greater than or equal to 6. Consider the group $\Gamma = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$. Let $e_j \in \Gamma$ be the element with 1 in coordinate j and 0 in every other coordinate, and let the canonical generating set of Γ be $\Sigma = \{\pm e_j \mid 1 \leq j \leq d\}$. Then, since $(n_1/2, 0, \ldots, 0) \notin \Sigma$ and cannot be expressed as $e_j + e_\ell$ for any $1 \leq j, \ell \leq d$, by Corollary 3 we conclude that $\mathbb{D}(\Gamma, \Sigma) \in \mathsf{P}$. Furthermore, we can observe that $\mathfrak{C}(\Gamma, \Sigma) \cong C_{n_1} \square \cdots \square C_{n_d}$, which shows that $C_{n_1} \square \cdots \square C_{n_d} \in \mathsf{P}$.*

For an abelian group Γ, we say that the representation $\Gamma \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ is *good* if $n_1 \geq 6$ or if $\Gamma \cong \mathbb{Z}_2^k \times \mathbb{Z}_4^m$ for some k and m. Notice that every even-order abelian group has a good representation. Then Example 5 proves the following proposition unless $\Gamma \cong \mathbb{Z}_2^k \times \mathbb{Z}_4^m$, for which case Corollary 2 applies because $\mathfrak{C}(\mathbb{Z}_2^k \times \mathbb{Z}_4^m, \Sigma) = P_2^{k+2m}$, where Σ is the canonical generating set.

Proposition 6. *Let* $\Gamma \not\cong \mathbb{Z}_2$ *be an even-order abelian group with good representation and corresponding canonical generating set* Σ. *Then* $\mathbb{D}(\Gamma, \Sigma) \in \mathsf{P}$. $\qquad\qquad\qquad\square$

We will defer the proof of the following Theorem to Section 3.1.

Theorem 7. *The game* $\mathbb{D}(G) \in \mathsf{P}$ *for the following graphs* G:

1. $C_n^{(k)}$, *for even* $n \geq 6$ *and* $1 \leq k < n/4$.

2. $G_0 \,\square\, H$, *whenever (a)* G_0 *is 3-involutionary and* H *is any graph, or (b)* G_0 *is 2-involutionary and* H *is 1-involutionary.*

3. $(G_0, (x, \phi(x))) \cdot ((H, H), (y, y))$, *such that* ϕ *is a 3-involution of* G_0, $H = (W, F)$ *is any graph, and* $y \in F$.

4. $(G_0, \mathbf{x}) \cdot (\mathbf{H}, \mathbf{y})$, $|V(G_0)| \geq 2$, $\mathbf{x} = V(G_0)$, *and each* $H_i \cong K_{1,m_i}$ *with odd* $m_i \geq 1$ *and* y_i *is a dominating vertex of* H_i.

5. $(G_0, x)\text{--}(G_0, x)^k$, *such that* x *is a cut-vertex of* G_0, *some component of* $G_0 - x$ *is a singleton, and* k *is odd.*

Paths and odd cycles are examples of graphs that are not 3-involutionary, so they require additional techniques. We are able to characterize which path and cycle games are won by Player 1, as follows.

Theorem 8. *For* $n \geq 3$, $\mathbb{D}(C_n) \in \mathsf{N}$ *if and only if* n *is odd but not equal to 5.*

Theorem 9. *For* $n \geq 2$, $\mathbb{D}(P_n) \in \mathsf{N}$ *if and only if* n *is odd or* $n \in \{2, 6, 8, 10, 12\}$.

In Section 2 we develop the lemmas necessary to prove Theorems 7–9, which we do in Section 3. Section 4 contains theorems about games on specific graphs such as caterpillars, Cartesian products of paths and cycles, and the Petersen graph. We offer several interesting questions, open problems, and a conjecture in Section 5.

2 Key Lemmas

2.1 GD games

For a set of **GD** positions $\mathcal{Q}_i$ on corresponding graphs G_i ($1 \leq i \leq k$), define the *game sum* $\mathcal{Q} = \oplus_{i=1}^k \mathcal{Q}_i$ to have $B_\mathcal{Q} = \cup_{i=1}^k B_{\mathcal{Q}_i}$, $S_\mathcal{Q} = \cup_{i=1}^k S_{\mathcal{Q}_i}$, $W_\mathcal{Q} = \cup_{i=1}^k W_{\mathcal{Q}_i}$ on the disjoint graph union $+_{i=1}^k G_i$ so that, for each $i \neq j$, no move in $\mathcal{Q}_i$ affects $\mathcal{Q}_j$. Each $\mathcal{Q}_i$ is called a *component* of $\mathcal{Q}$. While game sums are normally fairly

simple to interpret, the complexity in this case is that players can continue to play on the shaded vertices of a component that has been "won" (is trivial).

Fact 10. *For any position $\mathcal{Q}$ and any vertex $u \in \mathcal{Q}$, we have $|S_{\mathcal{Q}(u)}| + |W_{\mathcal{Q}(u)}| = |S_{\mathcal{Q}}| + |W_{\mathcal{Q}}| - 1$.*

Proof. This holds because $|B_{\mathcal{Q}(u)}| = |B_{\mathcal{Q}}| + 1$. $\qquad\square$

Lemma 11. *For* **GD** *games $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \cong \mathcal{B}$ then $\mathcal{A} \oplus \mathcal{B} \in \mathsf{P}$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be **GD** games, $f : \mathcal{A} \to \mathcal{B}$ be an isomorphism. Define $\phi : \mathcal{A} \oplus \mathcal{B} \to \mathcal{A} \oplus \mathcal{B}$ by setting $\phi(a) = f(a)$ for all $a \in \mathcal{A}$ and $\phi(b) = f^{-1}(b)$ for all $b \in \mathcal{B}$. Then ϕ is 3-involution of $\mathcal{A} \oplus \mathcal{B}$ and so, by Lemma 1, $\mathcal{A} \oplus \mathcal{B} \in \mathsf{P}$. $\qquad\square$

2.2 PD games

To describe **PD** games, we first make an observation that will yield simpler notation that will facilitate our analysis. The observation is that every interior vertex of a path is a cut vertex, so when Next chooses an interior vertex u, the game on the original path P becomes a sum of games on two paths defined by $P - u$. Thus we define the position $\mathcal{P}_k^i$ to be the path on k vertices having i shaded endpoints. Then, for example, if u has distance at least two from each endpoint of $\mathcal{P}_k^i$ then $\mathcal{P}_k^i(u) \cong \mathcal{P}_a^h \oplus \mathcal{P}_b^j$, for some $a, b \geq 2$, $a + b = k - 1$, $1 \leq \{h, j\} \leq 2$, and $h + j = 2 + i$. When u is an endpoint or neighbor of one, it is slightly trickier to write a general formula, but simple to calculate a particular instance; for example, if u is a neighbor of the shaded endpoint of $\mathcal{P}_7^1$ then $\mathcal{P}_7^1(u) \cong \mathcal{P}_1^1 \oplus \mathcal{P}_5^1$. Notice that, with this notation, there is no mention of B since it no longer exists — the chosen vertices are not colored black but instead are removed — so the original definition of S no longer applies. Instead we know now that S is a subset of the endpoints of paths, and endpoints next to a chosen vertex become shaded.

We define a **PD** position $\mathcal{Q}$ to be a finite sum of such $\mathcal{P}_k^i$ positions. We say a **PD** position or game $\mathcal{Q} = \oplus_{j=1}^m \mathcal{P}_{k_j}^{i_j}$ is *standard* if $i_j = 2$ when $k_j > 1$, and $i_j = 1$ otherwise; that is, every endpoint is shaded. Thus, if $\mathcal{Q}$ is standard then, for every move u in $\mathcal{Q}$, we have that $\mathcal{Q}(u)$ is standard. To simplify the notation of standard positions for most of this paper, we write $\mathcal{P}_k$ in place of $\mathcal{P}_k^2$ for $k > 1$ and $\mathcal{P}_1$ in place of $\mathcal{P}_1^1$, with $\mathcal{P}_0$ denoting the empty position (having no vertices).

Given standard $\mathcal{Q} = \oplus_{i=1}^m \mathcal{Q}_i$ where $Q_i \cong \mathcal{P}_{n_i}$, we define the functions $\mathsf{one}(\mathcal{Q}) = |\{i \mid n_i = 1\}|$, $\mathsf{four}(\mathcal{Q}) = |\{i \mid n_i = 4\}|$, and $\mathsf{odd}(\mathcal{Q}) = |\{i \mid n_i \text{ is odd}\}|$. We say that $\mathcal{Q}$ is *even* if $|W_{\mathcal{Q}}| + |S_{\mathcal{Q}}|$ is even. Equivalently, $\mathsf{odd}(\mathcal{Q})$ is even. We say that $\mathcal{Q}$ is *totally even* if both $|W_{\mathcal{Q}}|$ and $|S_{\mathcal{Q}}|$ are even and $\mathsf{four}(\mathcal{Q})$ is even. Note that

$|S_\mathcal{Q}|$ is even if and only if $\mathsf{one}(\mathcal{Q})$ is even. Thus, an even $\mathcal{Q}$ is totally even if and only if $\mathsf{one}(\mathcal{Q})$ and $\mathsf{four}(\mathcal{Q})$ are both even.

Denote by x_i the vertex $x \in \mathcal{Q}_i$. Let ϕ denote the automorphism of any path that swaps its endpoints. We say that x_i is the *center* of $\mathcal{Q}_i$ if $\phi(x_i) = x_i$. Note that if x_i is the center of $\mathcal{Q}_i$ then n_i is odd.

Lemma 12. *Let $\mathcal{Q} = \oplus_{i=1}^m \mathcal{P}_{n_i}$ be nontrivial such that $|S_\mathcal{Q}| + |W_\mathcal{Q}|$ is odd. Then Next has a move $u \in \mathcal{Q}$ such that $\mathcal{Q}(u)$ is totally even.*

Proof. By Fact 10, $\mathcal{Q}(u)$ is even for any move u, and we therefore recall that $\mathcal{Q}(u)$ is totally even if and only if $\mathsf{one}(\mathcal{Q}(u))$ and $\mathsf{four}(\mathcal{Q}(u))$ are both even. We consider the following four cases:

- $\mathsf{one}(\mathcal{Q})$ and $\mathsf{four}(\mathcal{Q})$ are both odd: Since $\mathsf{four}(\mathcal{Q})$ is odd we can write $\mathcal{Q} \cong \mathcal{A} \oplus \mathcal{P}_4$ for some game $\mathcal{A}$. Then Next plays an interior vertex $u \in \mathcal{P}_4$, so that $\mathsf{one}(\mathcal{Q}(u)) = \mathsf{one}(\mathcal{Q}) + 1$ and $\mathsf{four}(\mathcal{Q}(u)) = \mathsf{four}(\mathcal{Q}) - 1$.

- $\mathsf{one}(\mathcal{Q})$ is even and $\mathsf{four}(\mathcal{Q})$ is odd: Since $\mathsf{four}(\mathcal{Q})$ is odd we can again write $\mathcal{Q} \cong \mathcal{A} \oplus \mathcal{P}_4$. Then Next plays an endpoint $u \in \mathcal{P}_4$, so that $\mathsf{one}(\mathcal{Q}(u)) = \mathsf{one}(\mathcal{Q})$ and $\mathsf{four}(\mathcal{Q}(u)) = \mathsf{four}(\mathcal{Q}) - 1$.

- $\mathsf{one}(\mathcal{Q})$ is odd and $\mathsf{four}(\mathcal{Q})$ is even: Since $\mathsf{one}(\mathcal{Q})$ is odd we can again write $\mathcal{Q} \cong \mathcal{A} \oplus \mathcal{P}_1$. Then Next plays the unique vertex $u \in \mathcal{P}_1$, so that $\mathsf{one}(\mathcal{Q}(u)) = \mathsf{one}(\mathcal{Q}) - 1$ and $\mathsf{four}(\mathcal{Q}(u)) = \mathsf{four}(\mathcal{Q})$.

- $\mathsf{one}(\mathcal{Q})$ and $\mathsf{four}(\mathcal{Q})$ are both even: Since $\mathsf{one}(\mathcal{Q})$ is even, so is $|S_\mathcal{Q}|$, which makes $|W_\mathcal{Q}|$ odd. Thus there is some i for which $n_i \geq 3$ is odd, and so we write $\mathcal{Q} \cong \mathcal{A} \oplus \mathcal{P}_{n_i}$. Then Next plays the center u of $\mathcal{P}_{n_i}$. By symmetry, $\mathsf{one}(\mathcal{Q}(u))$ and $\mathsf{four}(\mathcal{Q}(u))$ are both even.

Therefore, in all cases there is a move u such that $\mathcal{Q}(u)$ is totally even. $\square$

Fact 13. *Let $\mathcal{Q} = \oplus_{i=1}^m \mathcal{P}_{n_i}$ be nontrivial and totally even. Then for all vertices $u \in \mathcal{Q}$, the position $\mathcal{Q}(u)$ is nontrivial.*

Proof. The only even positions that can be won in one move are of the form $(\oplus_{i=1}^m \mathcal{P}_1) \oplus \mathcal{P}_4$, where m is even. Such positions are not totally even. $\square$

Theorem 14.

Version A: *Let $\mathcal{Q} = \oplus_{i=1}^m \mathcal{P}_{n_i}$ be nontrivial. Then $\mathcal{Q} \in \mathsf{P}$ if and only if $\mathcal{Q}$ is even and $|W_\mathcal{Q}| \geq 4$, or $\mathcal{Q}$ is isomorphic to $\mathcal{Q} \cong (\oplus_{i=1}^m \mathcal{P}_1) \oplus \mathcal{P}_3 \oplus \mathcal{A}$, where m is odd and $\mathcal{A}$ is either $\mathcal{P}_4$ or $\mathcal{P}_1 \oplus \mathcal{P}_3$.*

Version B: *Let $\mathcal{Q} = \oplus_{i=1}^{m}\mathcal{P}_{n_i}$ be nontrivial. Then $\mathcal{Q} \in \mathsf{P}$ if and only if $\mathcal{Q}$ is even and $|W_{\mathcal{Q}}| \geq 4$, or $\mathcal{Q}$ is isomorphic to one of the following:*

- *$(\oplus_{i=1}^{m}\mathcal{P}_1) \oplus \mathcal{P}_3 \oplus \mathcal{P}_3$, where m is even*

- *$(\oplus_{i=1}^{m}\mathcal{P}_1) \oplus \mathcal{P}_4 \oplus \mathcal{P}_3$, where m is odd*

We recall that $\mathcal{Q} = \oplus_{i=1}^{m}\mathcal{P}_{n_i}$ is a finite position and therefore must lead to a trivial position in finitely many steps. Thus, $\mathcal{Q}$ is in exactly one of P and N.

Proof of Theorem 14. Say that $\mathcal{Q}$ has *condition C* if $\mathcal{Q}$ is totally even or $\mathcal{Q}$ is even and $|W_{\mathcal{Q}}| \geq 4$. We argue that if $\mathcal{Q}$ has condition C then $\mathcal{Q}(u)$ is nontrivial for all $u \in \mathcal{Q}$, and then use Lemma 12 to conclude that $\mathcal{Q}(u,v)$ has condition C. Thus $\mathsf{P1}$ can never win $\mathcal{Q}$; i.e. $\mathcal{Q} \in \mathsf{P}$. Indeed, Fact 13 takes care of the totally even case, and $|W_{\mathcal{Q}}| \geq 4$ implies that $|W_{\mathcal{Q}(u)}| \geq 1$ for all $u \in \mathcal{Q}$. This shows that if $\mathcal{Q}$ is even and $|W_{\mathcal{Q}}| \geq 4$, then $\mathcal{Q} \in \mathsf{P}$.

Next, we gh at the two cases listed in the statement of the theorem. First, suppose that $\mathcal{Q} \cong (\oplus_{i=1}^{m}\mathcal{P}_1) \oplus \mathcal{P}_3 \oplus \mathcal{P}_3$, where m is even. Then we can write it as $\mathcal{Q} \cong \mathcal{A} \oplus \mathcal{A}$ where $\mathcal{A} \cong \left(\oplus_{i=1}^{m/2}\mathcal{P}_1\right) \oplus \mathcal{P}_3$. Then $\mathcal{Q} \in \mathsf{P}$ by Lemma 11. Second, suppose that $\mathcal{Q} \cong (\oplus_{i=1}^{m}\mathcal{P}_1) \oplus \mathcal{P}_4 \oplus \mathcal{P}_3$, where m is odd. Without loss of generality, we can assume that $m = 1$. If $\mathsf{P1}$ chooses $u \in \mathcal{P}_3$ or $v \in W_{\mathcal{P}_4}$, then $\mathsf{P2}$ will choose v or u, respectively, to win. If $\mathsf{P1}$ chooses $u \in S_{\mathcal{P}_1}$ or $v \in S_{\mathcal{P}_4}$, then $\mathsf{P2}$ will choose v or u, respectively, to yield $\mathcal{Q}(u,v) \cong \mathcal{P}_3 \oplus \mathcal{P}_3$, which $\mathsf{P2}$ wins by Lemma 11. Hence $\mathcal{Q} \in \mathsf{P}$.

Conversely, by contrapositive, we first suppose that $\mathcal{Q}$ is odd. By Lemma 12, $\mathsf{P1}$ has a move u such that $\mathcal{Q}(u)$ is totally even and thus in P. Therefore, $\mathcal{Q} \in \mathsf{N}$.

The only remaining case is $\mathcal{Q}$ is even, $|W_{\mathcal{Q}}| < 4$, and $\mathcal{Q}$ is not isomorphic to one of the games listed in the statement of the theorem. One can see that $\mathcal{Q} \cong (\oplus_{i=1}^{m}\mathcal{P}_1) \oplus \mathcal{P}_k$, where $k \in \{3,4,5\}$ and $m \equiv k \pmod 2$. In each of the cases, there are at most three white vertices and they are all consecutive. Thus, $\mathsf{P1}$ can win in one move. $\qquad\square$

Remark 15. *In the notation of Theorem 14, suppose that $|W_{\mathcal{Q}}| \geq 10$. Then, after any move u of $\mathsf{P1}$, any response v by $\mathsf{P2}$ keeps the game even by Fact 10. Also, $|W_{\mathcal{Q}(u,v)}| \geq 4$ and so $\mathcal{Q}(u,v) \in \mathsf{P}$ by Theorem 14. Therefore, while there remain at least 10 white vertices, $\mathsf{P2}$'s response to $\mathsf{P1}$ can be arbitrary.*

Corollary 16. *The position $\mathcal{P}_k \in \mathsf{P}$ if and only if k is even and $k \neq 4$.*

Proof. This follows as a special case of Theorem 14. $\qquad\square$

3 Proofs

3.1 Proof of Theorem 7

For the first 3 cases, we show that the graph under consideration has a 3-involution. Then the results follow from Lemma 1.

1. This was proved in Example 4.   ◇

2. For case (a), let $\phi' : G_0 \to G_0$ be a 3-involution, and define the involution ϕ on $G_0 \,\square\, H$ by setting $\phi(g,h) = (\phi'(g), h)$. For case (b), let $\phi' : G_0 \to G_0$ be a 2-involution and $\phi'' : H \to H$ be a 1-involution, and define the involution ϕ on $G_0 \,\square\, H$ by setting $\phi(g,h) = (\phi'(g), \phi''(h))$. In both cases, ϕ is 3-involutionary.   ◇

3. Let $G_0 = (V, E)$ and $H = (W, F)$, and set $G = (G_0, (x, \phi(x))) \cdot ((H, H), (y, y))$. We will say that $W = \{w_1, ..., w_n\}$, and define $W_i = \{w_1^i, ..., w_n^i\}$ for $i = 1, 2$. Then we can write $V(G) = V \cup W_1 \cup W_2$. Let $\phi : G_0 \to G_0$ be a 3-involution. Then define $\overline{\phi} : G \to G$ by setting $\overline{\phi}(u) = \phi(u)$ for $u \in V(G)$, $\overline{\phi}(w_j^1) = w_j^2$ for $w_j^1 \in W_1$, and $\overline{\phi}(w_j^2) = w_j^1$ for $w_j^2 \in W_2$. Then $\overline{\phi}$ is a 3-involution.   ◇

4. We prove this statement for a broader class of graphs and for a broader set of positions on them. Let $\mathcal{G}$ be the set of positions $\mathcal{Q}$ on graphs $G = ((G_0, \mathbf{x}) \cdot (\mathbf{H}, \mathbf{y})) \cup G_1 \cup G_2$, where $G_1 \cup G_2$ is an independent set, with the following properties:

   - $|V(G_0)| \geq 2$;

   - $\mathbf{x} = V(G_0)$;

   - no $x_i \in \mathcal{Q}_B$;

   - each $H_i \cong K_{1, m_i}$, with odd $m_i \geq 1$;

   - each y_i is a dominating vertex of H_i;

   - G_1 is a set of evenly many isolated vertices in $\mathcal{Q}_W$; and

   - G_2 is a set of evenly many isolated vertices in $\mathcal{Q}_S$.

   For such $\mathcal{Q} \in \mathcal{G}$, we prove by induction on $|\mathcal{Q}|$ that $\mathcal{Q} \in \mathsf{P}$. At each stage, when a player plays a vertex v in some graph G, we remove black vertices so that $G(v) = G - v$. By doing so, when isolated vertices occur, they get moved in $G_1 \cup G_2$ according to their color.

   Let $\mathcal{Q} \in \mathcal{G}$ and $k = |V(G_0)|$. We write $N_{H_i}(y_i) = \{z_{i,1}, \ldots, z_{i, m_i}\}$ and suppose that $\mathsf{P1}$ plays $u \in H_i$. Since $k \geq 2$, there is some $j \neq i$ such that $z_{j,1} \in \mathcal{Q}_W(u)$, and so u is not a winning move. We are done

if P2 has a winning move, so we assume otherwise and show that P2 has a move $v \in \mathcal{Q}(u)$ that makes $\mathcal{Q}(u,v) \in \mathcal{G}$, which will complete the induction. For ease of notation, relabel the $\{H_i\}$ so that P1 plays in H_k.

First consider the case that $u = x_k$. If $k = 2$ then P2 has a winning move $v = x_1$, so we must have $k > 2$. Thus P2 plays $v = z_{k,m_k}$, so that $\mathcal{Q}(u,v) \in \mathcal{G}$. This is because an even number of vertices, namely $\{z_{k,1}, \ldots, z_{k,m_k-1}\}$, have been moved into G_2.

Next consider the case that $u = z_{k,m_k}$. If $m_k = 1$ and $k = 2$, then P2 has the winning move $v = x_1$, so we must have either $m_k \geq 2$ or $k \geq 3$. If $m_k \geq 2$, then P2 can play $v = z_{k,m_k-1}$, while if $k \geq 3$ then P2 can play $v = x_k$. In either case, we have $\mathcal{Q}(u,v) \in \mathcal{G}$. In the former case, this is because no new isolated vertices were created and the oddness of leaves was maintained. In the latter case, this is because an even number of shaded leaves were moved to G_2. $\diamond$

5. Rewrite the bridging $G = (G_0, x)$–$(G_0, x)^k$ as $(G_0, x_0) \cdot (\mathbf{G}, \mathbf{x})$, where $\mathbf{G} = (G_1, \ldots, G_k)$, $\mathbf{x} = (x_1, \ldots, x_k)$, and each $G_i \cong G_0$. Since k is odd, we may pair G_j with G_{j+1} for all even $j < k$, which allows us to extend the identity map on G_0 to the automorphism ϕ of G composed of the natural isomorphisms between each pair G_j and G_{j+1}. Now we define the corresponding mirroring strategy for P2 that plays $\phi(u)$ for each play u by P1. We claim that this is a winning strategy.

Write z_i for a singleton in $G_i - x_i$. Suppose that P1 wins and let u be the winning move by P1. That is, there is a position Q on G such that $W_Q \neq \emptyset$ and $W_{Q(u)} = \emptyset$. This means that there is some $w \in W_Q \cap N[u]$, which implies that $\phi(w) \in W_Q \cap N[\phi(u)]$. Because $\phi(w) \notin W_{Q(u)}$, it must be that $\phi(w) \in N(u)$, and so $u = x_i$, for some i. But this implies that $z_i \in W_Q$ since z_i is a leaf, and so $\phi(z_i) \in W_Q$. Since $\phi(z_i) \notin N(x_i)$, we arrive at the contradiction $\phi(z_i) \in W_{Q(u)}$. Hence P2 wins. $\diamond$

This completes the proof of Theorem 7. $\square$

3.2   Proof of Theorem 8

Recall the statement of Theorem 8: For $n \geq 3$, $\mathbb{D}(C_n) \in \mathsf{N}$ if and only if n is odd but not equal to 5. Let $\mathcal{D}_n = \mathbb{D}(C_n)$. Because of the symmetry of C_n, for any u we have $\mathcal{D}_n(u) \cong \mathcal{P}_{n-1}$, and so $\mathcal{D}_n \in \mathsf{N}$ if and only if $\mathcal{P}_{n-1} \in \mathsf{P}$. The result follows from Corollary 16. $\square$

3.3   Proof of Theorem 9

Recall the statement of Theorem 9: For $n \geq 2$, $\mathbb{D}(P_n) \in \mathsf{N}$ if and only if n is odd or $n \in \{2, 6, 8, 10, 12\}$. Let $\mathcal{D}_n = \mathbb{D}(P_n)$ and $\mathsf{P1} = \mathsf{Next}$, and write $\{0, 1, \ldots, n-1\}$ for the vertex labels of $\mathcal{D}_n$.

We first suppose that n is odd. $\mathsf{P1}$ plays the center vertex $u = (n-1)/2 \in \mathcal{D}_n$. Then $\mathcal{D}_n(u) \cong \mathcal{A} \oplus \mathcal{B}$ where $\mathcal{A} \cong \mathcal{B}$, and thus $\mathcal{D}_n(u) \in \mathsf{P}$ by Lemma 11. Thus $\mathcal{D}_n \in \mathsf{N}$.

At this point we recall the more specific notation $\mathcal{P}_k^i$ for any path on k vertices with exactly i shaded endpoints. Thus, $\mathcal{D}_k$ can also be written as $\mathcal{P}_k^0$.

Next we suppose that n is even and consider the following cases:

$n = 2$: Here $\mathsf{P1}$ wins with any move so $\mathcal{D}_2 \in \mathsf{N}$.

$n = 4$: Let $\mathsf{P1}$ play $u \in \mathcal{D}_4$. By symmetry, we can assume that $u = 0$ or $u = 1$. In both cases, $\mathsf{P2}$ wins by playing vertex 3. Thus $\mathcal{D}_4 \in \mathsf{P}$.

$n = 6$: In this case $\mathsf{P1}$ plays an endpoint u, so that $\mathcal{D}_6(u) \cong \mathcal{P}_5^1$. From this, it is easy to see that, for any move v by $\mathsf{P2}$ the at most three white vertices of $\mathcal{D}_6(u, v)$ are consecutive, and subsequently can be dominated on the next move by $\mathsf{P1}$. Hence $\mathcal{D}_6 \in \mathsf{N}$ (and $\mathcal{P}_5^1 \in \mathsf{P}$).

$n = 8$: Now $\mathsf{P1}$ plays a neighbor u of an endpoint. If $\mathsf{P2}$ plays one neighbor v of u then $\mathsf{P1}$ plays the other w, so that $\mathcal{D}_8(u, v, w) \cong \mathcal{P}_5^1$, which is in P by the argument in the case $n = 6$ above. So we assume otherwise. If $\mathsf{P2}$ plays v so that the at most 3 white vertices of $\mathcal{D}_8(u, v)$ are consecutive, then $\mathsf{P1}$ plays to dominate them on the next move. The only remaining case is that $\mathsf{P2}$ plays the vertex v at distance two from the unshaded endpoint of $\mathcal{D}_8(u)$. Now $\mathcal{D}_8(u, v) \cong \mathcal{P}_1^1 \oplus \mathcal{P}_3^2 \oplus \mathcal{P}_2^1$, and $\mathsf{P1}$ plays the unique vertex w in $\mathcal{P}_1^1$. At this point, note that any move in either component $\mathcal{P}_3^2$ or $\mathcal{P}_2^1$ dominates that component, so that, when $\mathsf{P2}$ plays in one of them, $\mathsf{P1}$ plays in the other to win. Therefore $\mathcal{D}_8 \in \mathsf{N}$.

$n \in \{10, 12\}$: For these values of n, the case analysis is more extensive, but not insightful. We verified that $\mathcal{D}_n \in \mathsf{N}$ for each such n by computer.

$n \geq 14$: The following strategy can be used by $\mathsf{P2}$ to win this game. Irrespective of the first two moves of $\mathsf{P1}$, the first two moves of $\mathsf{P2}$ should be the endpoints of $\mathcal{P}_n^i$. If $\mathsf{P2}$ cannot choose an endpoint because $\mathsf{P1}$ has already played it, then $\mathsf{P2}$ can play any other vertex. Thus, after each player has made two moves, we arrive at the even position $\mathcal{Q} = \mathcal{P}_{a_1} \oplus \mathcal{P}_{a_2} \oplus \mathcal{P}_{a_3}$, where $a_1 + a_2 + a_3 = n - 4$ and each $a_i \geq 0$. Because each path has at most 2 shaded endpoints, $|W_\mathcal{Q}| \geq (n - 4) - 6 \geq 4$. Hence $\mathcal{Q} \in \mathsf{P}$ by Theorem 14, and so $\mathcal{D}_n \in \mathsf{P}$.

4   Other specific graphs

The path P_n can be described as the internally 2-regular tree on n vertices. A natural extension of this class of trees is the set of internally r-regular trees. (Internally 3-regular trees can be thought of as rooted binary trees with an additional pendant vertex attached to the root.) One instance in this class is the set of internally r-regular caterpillars. A *caterpillar* is a tree such that every vertex not on a longest path P_k (called the *spine*) is adjacent to some vertex of P_k. Such a caterpillar is called *even* (*odd*) if k is *even* (*odd*). The bridging construction yields the following theorem.

Theorem 17. *If G is an even internally r-regular caterpillar with $r \geq 3$, then $\mathbb{D}(G) \in \mathsf{P}$.*

Proof. Let $e = \{x_1, x_2\}$ be the middle edge of the spine P of G; then $G - e$ is the disjoint union of two isomorphic copies of some tree $T_1 \cong T_2$. Then $G = (T_1, x_1)$–(T_2, x_2) and each $T_i - x_i$ has a singleton component, so $\mathbb{D}(G) \in \mathsf{P}$ by Theorem 7.5. $\square$

Theorem 18. *If G is the k-sunlet graph S_k, with ~~even~~ $k \geq 2$, then $\mathbb{D}(G) \in \mathsf{P}$.*

Proof. As noted after the dangling definition, $S_k = (C_k, \mathbf{x}) \cdot (\mathbf{H}, \mathbf{y})$. Hence $\mathbb{D}(S_k) \in \mathsf{P}$ by Theorem 7.4. $\square$

Another interesting class of graphs to consider are the grids $P_k \,\square\, P_m$. Because of the complex pattern of $\mathbb{D}(P_n)$ winner given in Theorem 9, one might guess that no simple pattern describes the winner of grids in general. However, evidence suggests otherwise.

Theorem 19. *If km is odd then $\mathbb{D}(P_k \,\square\, P_m) \in \mathsf{N}$.*

Proof. Define $s = (k-1)/2$ and $t = (m-1)/2$, and use the coordinate system $\{-s, \ldots, 0, \ldots, s\} \times \{-t, \ldots, 0, \ldots, t\}$ for the vertices of $G = P_k \,\square\, P_m$. Define the involution ϕ on G by $\phi(a, b) = (-a, -b)$. Then ϕ is 3-involutionary on $G - (0, 0)$. Hence, $\mathsf{P1}$ wins $\mathbb{D}(G)$ by playing $u = (0, 0)$ because $G(u) = G - (0, 0) \in \mathsf{P}$ by Lemma 1. $\square$

Notice that the same argument yields that $\mathbb{D}(P_{k_1} \,\square\, P_{k_1} \,\square\, \cdots \,\square\, P_{k_d}) \in \mathsf{N}$ whenever $k_1 k_2 \cdots k_d$ is odd. Similarly, toroidal grids are equally interesting. As pointed out in Example 5, which follows from Theorem 7.2, even higher-dimensional products of cycles are in P, provided that some cycle has length at least 6. We record this below.

Theorem 20. *Suppose that $n_1 \geq 6$ is even, $3 \leq n_2 \leq \cdots \leq n_d$ are integers, and $G = C_{n_1} \,\square\, C_{n_2} \,\square\, \cdots \,\square\, C_{n_d}$. Then $\mathbb{D}(G) \in \mathsf{P}$.* $\square$
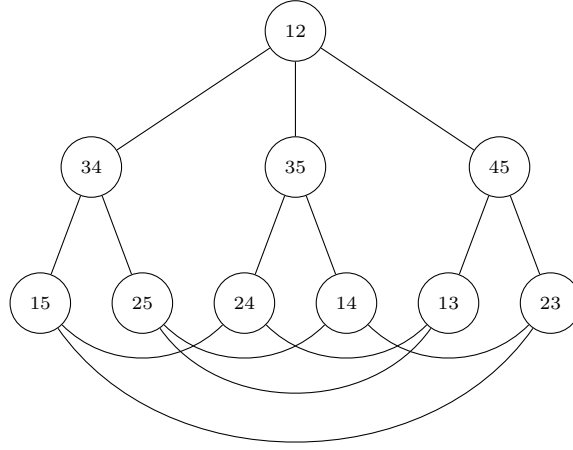
Figure 3: The Petersen graph $P = K(5, 2)$ with its vertex labels shown.

For $m \geq 2t + 1$, the *Kneser graph* $K(m, t)$ has vertex set $\binom{[m]}{t}$, with edges between disjoint pairs of vertices. The *Petersen graph* equals $K(5, 2)$ and is of great importance in many areas of graph theory. It is natural to investigate which player wins the games on these graphs.

Theorem 21. *For the Petersen graph P, we have $\mathbb{D}(P) \in \mathsf{P}$.*

Proof. We simplify the notation of the vertices of P by writing ij instead of $\{i, j\}$ (see Figure 3). By symmetry, we may assume that P1 chooses 12. Because $P - N[12]$ is a 6-cycle C, which is 3-involutionary, P2 will lose by choosing any vertex u of C (P1 responds by playing the vertex of C opposite from u). By symmetry again, we may assume that P2 chooses 35. Similarly, P1 loses by choosing a vertex of C, and so chooses, by symmetry, 34. But now P2 wins by choosing 45. □

5 Final Comments

Given the results of Section 4, we offer the following open problems and conjectures.

Problem 22. *Find the winning player for $\mathbb{D}(G)$ when G is an odd internally regular caterpillar.*

The first several cases of the following conjecture are easy to verify by hand.

Conjecture 23. *For $k, m > 1$ with mk even, we have $\mathbb{D}(P_k \square P_m) \in \mathsf{P}$.*

Problem 24. *For integers $3 \leq n_1 \leq \ldots \leq n_d$, with every even $n_i = 4$, determine the winner of $\mathbb{D}(G)$ for $G = C_{n_1} \square \cdots \square C_{n_d}$.*

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A general statement one might hope to prove about Cartesian products involves the case in which the winner of one of the graphs is known.

Question 25. *Let G, H be graphs such that $\mathbb{D}(G) \in \mathsf{P}$. Under what conditions is it true that $G \square H \in \mathsf{P}$?*

For example, Theorem 7.23 states that $\mathbb{D}(G \square H) \in \mathsf{P}$ if G is 3-involutionary, so the question is open for graphs in P that are not 3-involutionary, such as $\mathbb{D}(P_4)$. If Conjecture 23 is true, though, $\mathbb{D}(P_4 \square P_m) \in \mathsf{P}$ for any $m \geq 2$.

A similar question arises when considering graph powers, along the lines of Theorem 7.1.

Question 26. *For what conditions on a graph G can we determine the winner of $\mathbb{D}(G^{(k)})$?*

We listed five graph constructions in Section 1.5 that produce P games by Theorem 7. There are many other graph constructions one might consider.

Question 27. *What other graph constructions produce games in P?*

Finally, we offer a reachable problem on problem on Kneser graphs.

Problem 28. *Find the winning player for $\mathbb{D}(G)$ when G is the Kneser graph $K(m, 2)$ and $m \geq 6$.*

While solving this game on a general graph appears to be out of reach, the game can be played on a number of other classes of graphs that we believe to be natural directions for future study. Some of the techniques described above may shed light on the case of trees. In particular, spiders, other caterpillars, and other internally 3-regular trees seem to be approachable cases to work on in the future; complete bipartite graphs are also a natural choice, as are non-Cartesian products of graphs and higher order Kneser graphs.

Finally, there are several related variants of the graph domination game. The first is the Misère version in which the first player to create a dominating set loses rather than wins. Another version that can be played is that players take turns choosing vertices until the complement of the set of chosen vertices is not a dominating set. Additionally, all of these variants can be played with "dominating sets" replaced by "total dominating sets", which are subsets $D \subseteq V$ of vertices of a graph with the property that $N(D) = V$.

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Statements and Declarations