ON DIMENSION IN THE CUBE

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Abstract. For integers $0 \leq k < r \leq n$, let $\mathcal{K}$ and $\mathcal{R}$ denote the families of all subsets of $[n] = \{1, 2, \ldots, n\}$ of size k and r, respectively. Denote by $P(k, r; n)$ the containment order on $\mathcal{K} \cup \mathcal{R}$ ($A < B$ whenever $A \subset B$) and let $d(k, r; n)$ be its order dimension. We characterize those linear extensions of $P(k, r; n)$ which reverse the maximum number of critical pairs, and we count how many they reverse. This count yields a lower bound of $d(k, r; n) \geq n - \lfloor (k+1)(n-r+1)/2 \rfloor + 2$ for large enough n in terms of k and r. Also, we prove that the fractional dimension, as defined by Brightwell and Scheinerman, of $P(k, r; n)$ is at most $r - k + 2$. It is interesting that this upper bound is independent of n, unlike the case for standard dimension.

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1. Introduction

For integers $0 \leq k < r \leq n$, let $\mathcal{K}$ and $\mathcal{R}$ denote the families of all subsets of $[n] = \{1, 2, \ldots, n\}$ of size k and r, respectively. Denote by $P(k, r; n)$ the containment order on $\mathcal{K} \cup \mathcal{R}$ ($A < B$ whenever $A \subset B$) and let $d(k, r; n)$ be its dimension, defined as follows. Following the standard terminology of [**15**] we say that L is a *linear extension* of a partially ordered set (poset) P if L is a linear order which contains all the relations of P. A *critical pair* (x, y) of P satisfies the three properties (i) $x \not< y$, (ii) $z < x$ implies $z < y$ and (iii) $z > y$ implies $z > x$ (notice the important lack of symmetry in the definition). Then P has *dimension* d if d is the smallest integer so that there are linear extensions $L_1, \ldots, L_d$ of P with the property that for every critical pair (x, y) in P there is an i having $x > y$ in L_i. We say in this case that the extension L_i *reverses* (x, y) and that the set $\mathcal{L} = \{L_1, \ldots, L_d\}$ is a *realizer* for P. Of primary importance in this paper is to characterize those linear extensions of $P(k, r; n)$ which reverse the maximum number of critical pairs. We prove that these are precisely those extensions which we will soon define as pseudo-lex. We will assume throughout that $0 < k < r < n$, since it is trivial that $d(0, n; n) = 1$ and $d(0, m; n) = d(m, n; n) = 2$ for $0 < m < n$.

We finish this section with some new and relevant definitions. In the next section we review some of the principle results on the problem of finding $d(k, r; n)$. We also state our results there. Section 3 is devoted to presenting the terminology needed to state the Kruskal-Katona theorem and prove our result on pseudo-lex extensions, and in section 4 we count the number of critical pairs they reverse. Our final section contains a proof that the fractional dimension of $P(k, r; n)$ is at most $r - k + 2$. It is interesting that this upper bound is independent of n, unlike the case for standard dimension.

Given any family $\mathcal{F}$ of subsets of $[n]$, let L be a linear extension of the containment order on $\mathcal{F}$ and for $x \in \cup\mathcal{F}$ let $\mathcal{F}_1(x) = \{A \in \mathcal{F} : x \in A\}$ and $\mathcal{F}_0(x) = \{A \in \mathcal{F} : x \notin A\}$. Now define the *center* $\operatorname{cent}(L) = \{x \in \cup\mathcal{F} : A > B \text{ in } L \text{ for all } A, B \in \mathcal{F} \text{ with } x \in A - B\}$. Notice that the center can be empty ($L = \langle 1 < 2 < 3 < 12 \rangle$) or can contain more than one element ($L = \langle 1 < 23 \rangle$). However, the center of any extension of $P(k, r; n)$ contains at most one element because if $x, y \in \operatorname{cent}(L)$ then, say, $\mathcal{F}_1(x) \subseteq \mathcal{F}_1(y)$, which when $\mathcal{F} = \mathcal{R}$ implies $x = y$ since $r < n$. Whenever the center contains only one element, we will avoid the use of set braces.

We now can define recursively a *pseudo-lex* extension on $\mathcal{F}$. First, L is pseudo-lex if $A > B$ in L whenever $|A| > |B|$. Also, L is pseudo-lex if there is an element $x \in \mathrm{cent}\,(L)$ and the restriction of L to each $\mathcal{F}_i(x)$ is both nonempty and pseudo-lex. As a special case, we define L to be *lexicographic* (*lex*) with respect to the permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $[n]$ if $A > B$ in L whenever the first element of σ belonging to exactly one of $\{A, B\}$ belongs to A. L is lex if it is lex with respect to some permutation σ.

Finally, for a linear extension L of the poset P, we let $\mathrm{crit}\,(L, P)$ be the set of critical pairs of P which are reversed by L, and we call L *maximal* if there is no other extension E of P with $\mathrm{crit}\,(L, P) \subset \mathrm{crit}\,(E, P)$. Clearly, we may assume that any realizer we choose for P consists entirely of maximal extensions. If no other extension of P reverses more critical pairs than does L, then we call L a *maximum* extension. It is not difficult to see that (K, R) is critical in $P(k, r; n)$ precisely when $K \in \mathcal{K}$, $R \in \mathcal{R}$ and $K \not\subset R$.

2. Results

Dushnik and Miller [5] first proved that $d(1, n-1; n) = n$. $P(1, n-1; n)$ is known as the standard example, the smallest poset of dimension n. Dushnik [4] then found exact values for $d(1, r; n)$ when $r \geq 2\sqrt{n} - 2$, namely, if

$$2 \leq j \leq \sqrt{n}$$

and

$$\lfloor n/j \rfloor + j - 2 \leq r < \lfloor n/(j-l) \rfloor + j - 3$$

then $d(1, r; n) = n - j + 1$.

For smaller values of r we find several important, but less precise results. Pushing Dushnik's ideas, Füredi and Kahn [8] showed that $d(1, r; n) > r^2/4$ for all $r \leq \sqrt{n}$. By choosing extensions at random they also proved that $d(1, r; n) < r(r+1)\log(n/r)$ for all $r < n$. Together, these imply that

$$(\log n)^2/4 < d(1, \log n; n) \le (\log n)^3 + o((\log n)^3).$$

Recently, Kierstead [11] proved that $d(1, \log n; n) = \Omega((\log n)^3/\log\log n)$.

Spencer [14] also studied the case $k = 1$. Using the Erdős/Szekeres theorem, the Erdős/Ko/Rado theorem, and the concept of scrambling families, he proved that

$$\log\log n \le d(1, 2; n) \le \log\log n + (1/2 + o(1))\log\log n.$$

He then used probabilistic methods to show $d(1, r; n) = \Theta(\log\log n)$ for fixed r. Through connections with shift graphs and interval orders, Fűredi, Hajnal, Rödl and Trotter [7] proved that

$$d(1, 2; n) = \log\log n + (1/2 + o(1))\log\log\log n.$$

The case $k = 2$ was first studied by Hurlbert, Kostochka and Talysheva [9], who proved that $d(2, n-2; n) = n - 1$ for $n \ge 5$ and that $d(2, n-3; n) \le n - 2$ for $n \ge 6$. In fact, they showed that the poset $P(2, n-2; n)$ is $(n-1)$-irreducible, that is, the removal of any one of its elements leaves a subposet of dimension $n - 2$. Their methods rely on a close look at maximal pseudo-lex extensions and on a sharper analysis of the counts we perform in section 4. In addition, much of the work for the case $k \ge 2$ is built upon the following elementary proposition.

Proposition 2.1. *Let $0 < k < r < n$.*

(1) *Let $d(k, k+1, \ldots, r; n)$ be the dimension of the poset $P(k, k+1, \ldots, r; n)$ generated by all m-element subsets of $[n]$ with $k \le m \le r$. Then*

$$d(k, k+1, \ldots, r; n) = d(k, r; n).$$

(2) *If $0 < k_1 \le k_2 < r_2 \le r_1 < n$, then $d(k_2, r_2; n) \le d(k_1, r_1; n)$.*

(3) *If* $0 < m < k$, *then* $d(k-m, r-m; n-m) \le d(k, r; n)$.

(4) $d(k, r; n) = d(n-r, n-k; n)$.

Proof. The proof of part (1) rests on the fact that $P(k, r; n)$ is a subposet of $P(k, k+1, \ldots, r; n)$, each having the same set of critical pairs. Part (2) is an immediate corollary to part (1). Part (3) follows from the observation that $P(k-m, r-m; n-m)$ is isomorphic to the subposet of $P(k, r; n)$ generated by the sets containing $\{1, 2, \ldots, m\}$. Finally, the *dual* P^D of a poset P is obtained by changing every relation $x < y$ to $x > y$ and is realized by $\mathcal{L}^D = \{L_1^D, \ldots, L_q^D\}$, where $\mathcal{L} = \{L_1, \ldots, L_q\}$ realizes P. Thus, part (4) follows from the fact that $P(n-r, n-k; n)$ is the dual of $P(k, r; n)$. ■

Few results are known for larger k, mostly obtained by Brightwell, Kierstead, Kostochka, and Trotter [2], who discovered that

$$d(k, k+1; n) \le (6/\log 3)\log n,$$

and more generally that

$$d(k, k+m; n) = O(m^2 \log n).$$

Combining with the previous results, using Proposition 2.1(3), and setting $n = 2k+1$, this puts $d(k, k+1; n)$ asymptotically between $\log\log n$ and $\log n$ (actually $\log n/\log\log n$ according to a modification by Kostochka [12]).

Regarding complementary levels, Fűredi [6] notices that the lower bound of $d(k, n-k; n) \ge n-k-\sqrt{k}$ for $n > 2k + \sqrt{k}$ follows easily from Dushnik's result and Proposition 2.1(3). More generally, the following result is equally simple.

Theorem 2.2. *If* $2 \le j \le \sqrt{n-k+1}$ *and* $n \ge [(2j-1)/(j-1)]k + j(j-3)/(j-1)$ *then* $d(k, n-k; n) \ge n-k-j+2$. ■

From this we not only get Fűredi's observation when $j = \sqrt{k}$, but also the interesting case when $j = 2$. Namely, $d(k, n-k; n) \ge n-k$ for $n \ge 3k-2$. In [6], Fűredi also uses the concept of cross intersecting families of sets to prove that $d(k, n-k; n) = n-2$ for $k \ge 3$ and $n \ge 250k^3$.

As previously mentioned, we concern ourselves here with maximal extensions of $P(k,r;n)$, in particular $P(k,n-s;n)$. We prove the following characterization.

Theorem 2.3. *E is a maximum extension of $P(k,r;n)$ if and only if E is a maximal pseudo-lex extension.*

Theorem 2.4. *For $0<k<r<n$, let $g(k,r;n)$ be the number of critical pairs reversed by a maximum extension of $P(k,r;n)$. Then*

$$g(k,n-s;n)=\sum_{i=0}^{k-1}\sum_{j=0}^{s-1}\binom{i+j}{i}\binom{n-1-i-j}{k-1-i}\binom{n-1-i-j}{s-1-j}.$$

We also prove the following negative result, which is somewhat surprising. In many cases, for example crowns (see [15, p.34]), it is a particular set of maximum extensions which forms a minimal realizer. Here this may not be the case.

Theorem 2.5. *If $\mathcal{L}=\{L_1,\ldots,L_q\}$ realizes $P(k,r;n)$ and each L_i is lex, then $q\geq n-1$.*

In [3] Brightwell and Scheinerman define a *t-fold realizer* of a poset P to be a set of linear extensions $\mathcal{L}=\{L_1,\ldots,L_q\}$ with the property that, for every critical pair (x,y) there is a set I, $|I|=t$ with $x>y$ in L_i for all $i\in I$. Then the *fractional dimension* $d^f(P)$ of P is $d^f(P)=\inf_t\{|\mathcal{L}|/t:\mathcal{L}\text{ is a }t\text{-fold realizer of }P\}$. We prove the following in section 5.

Theorem 2.6. *Let $d^f(k,r;n)=d^f(P(k,r;n))$ be the fractional dimension of $P(k,r;n)$ for $0<k<r<n$. Then $d^f(k,r;n)\leq r-k+2$.*

3. Shadows and the Colex Order

Given any family $\mathcal{F}$ of m-subsets of $[n]$, let L be a linear order on $\mathcal{F}$. We say that L is *colexicographic* (*colex*) if L is lex with respect to $\sigma=(n,n-1,\ldots,2,1)$. Define the *j-shadow* of $\mathcal{F}$ to be $\partial^j(\mathcal{F})=\{B:|B|=m-j\text{ and }B\subset A,\text{ some }A\in F\}$. Since we will be dealing only with k-subsets and r-subsets of $[n]$, we will write ∂ instead of ∂^{r-k} for convenience.

If L is a linear extension of $P(k, r; n)$ whose restriction to $\mathcal{R}$ is $L(\mathcal{R}) = \langle R_1 < R_2 < \cdots < R_{\binom{n}{r}} \rangle$, then we set $L(\mathcal{R}, t) = \{R_1, R_2, \ldots, R_t\}$ and let $\delta(L, R_t) = \{K \in \mathcal{K} : K < R_t \text{ in } L\}$. We note that (K, R_t) is reversed by L precisely when $K \notin \delta(L, R_t)$. Also, L is maximal if and only if $\delta(L, R_t) = \partial(L(\mathcal{R}, t))$. Now let $g(L)$ be the number of critical pairs reversed by L, and let $g(L, R_t)$ be the number of those pairs (K, R_t) reversed in which t is fixed. It is clear that if E and L are both lex then $g(E) = g(L)$. It is also not difficult to see that if L is lex, then it is maximal. However, we would like to say that it is maximum. For this we will need the following version of the Kruskal-Katona theorem (see [1, 10, 13]), tailored to our needs.

Theorem 3.1. (Kruskal 1963, Katona 1968). *Let $0 < k < r < n$, $1 \le t \le \binom{n}{r}$, and let $\mathcal{R}$ be the family of all r-subsets of $[n]$. Suppose $\mathcal{F} \subseteq \mathcal{R}$, $|F| = t$, and let $\partial(\mathcal{F})$ be its $(r-k)$-shadow. Then, of all such $\mathcal{F}$, $|\partial(\mathcal{F})|$ is minimum if and only if $\mathcal{F}$ is isomorphic to the least t sets in the colex order on $\mathcal{R}$.* ∎

By definition, we can replace the phrase "least t sets in the colex order on $\mathcal{R}$" by "$L(\mathcal{R}, t)$ for some lex extension L."

Lemma 3.2. *Every lex extension of $P(k, r; n)$ is a maximum extension. Moreover, if E and L are extensions of $P(k, r; n)$ and L is lex, then $g(E) = g(L)$ if and only if E is maximal and, for all $t \le \binom{n}{r}$, $E(\mathcal{R}, t)$ is isomorphic to $L(\mathcal{R}, T)$.*

Proof. Let E be any extension of $P(k, r; n)$, and let L be a lex extension of $P(k, r; n)$. We must show that $g(E) \le g(L)$. Let $E(\mathcal{R}) = \langle A_1 < A_2 < \cdots < A_{\binom{n}{r}} \rangle$ and $L(\mathcal{R}) = \langle B_1 < B_2 < \cdots < B_{\binom{n}{r}} \rangle$. Then, by the preceding discussion and Theorem 3.1, we have

$$\begin{aligned} g(E) &= \sum_{t=1}^{\binom{n}{r}} g(E, A_t) \\ &= \sum_{t=1}^{\binom{n}{r}} \left[\binom{n}{r} - |\delta(E, A_t)| \right] \\ &\le \sum_{t=1}^{\binom{n}{r}} \left[\binom{n}{r} - |\partial(E(\mathcal{R}, t))| \right] \qquad (1) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{t=1}^{\binom{n}{r}} \left[\binom{n}{r} - |\partial(L(\mathcal{R},t))| \right] && (2)\\
&= \sum_{t=1}^{\binom{n}{r}} \left[\binom{n}{r} - |\delta(L,B_t)| \right]\\
&= \sum_{t=1}^{\binom{n}{r}} g(L,B_t)\\
&= g(L).
\end{aligned}
$$

We see from the inequalities (1) and (2) that if E, L are extensions of $P(k,r;n)$ and L is lex, then $g(E) = g(L)$ if and only if E is maximal and $E(\mathcal{R},t)$ is isomorphic to $L(\mathcal{R},t)$ for every $t \le \binom{n}{r}$. ∎

Claim 3.3. *Suppose E is a maximal extension of $P(k,r;n)$ and L is a lex extension of $P(k,r;n)$. Then E is pseudo-lex if and only if, for all $t \le \binom{n}{r}$, $E(\mathcal{R},t)$ is isomorphic to $L(\mathcal{R},t)$.*

Proof. We use induction on n to show sufficiency. This is vacuously true when $n = r$ (and when $k = 0$), so suppose $0 < k < r < n$. Since E is maximal pseudo-lex , by definition it must have a center, say x (otherwise it reverses no critical pairs). Let y be the center of L. Then $E_0(x)$ and $L_0(y)$ are each linear extensions of posets isomorphic to $P(k,r;n-1)$. $E_0(x)$ is pseudo-lex by definition, and $L_0(y)$ is lex since any suborder of a lex order is lex. $E_0(x)$ is also maximal since any extension F_0 which reverses more critical pairs than $E_0(x)$ can be used to show that E is not maximal, as follows. Let F be an extension of $P(k,r;n)$ with center x so that $F_1(x) = E_1(x)$ and $F_0(x) = F_0$. Clearly, F would reverse more critical pairs than E, which is impossible. Hence, $E_0(x)$ is maximal, and by induction, for all $t \le \binom{n-1}{r}$, $E(\mathcal{R},t)$ is isomorphic to $L(\mathcal{R},t)$.

For any family of sets $\mathcal{F}$, let $\mathcal{F} - x = \{A - x : A \in \mathcal{F}\}$. Then by the same argument as above, $E_1(x) - x$ and $L_1(y) - y$ are each linear extensions of posets isomorphic to $P(k-1, r-1; n-1)$, $E_1(x) - x$ being maximal pseudo-lex and $L_1(y) - y$ being lex. By induction, for all $s \le \binom{n-1}{r-1}$, $[E_1(x) - x](\mathcal{R},s)$ and $[L_1(y) - y](\mathcal{R},s)$ are isomorphic, and so by mapping x to y, we can extend to an isomorphism from $E_1(x)(\mathcal{R},s)$ to $L_1(y)(\mathcal{R},s)$. Since this isomorphism also extends to $E_0(x)$ and $L_0(y)$ we have that, for all $t \le \binom{n}{r}$, $E(\mathcal{R},t)$ is isomorphic to $L(\mathcal{R},t)$.

As for necessity, we again use induction on n and assume $0 < k < r < n$. We suppose that L is lex with center y and that, for all $t \leq \binom{n}{r}$, $E(\mathcal{R}, t)$ is isomorphic to $L(\mathcal{R}, t)$. When $t = \binom{n-1}{r}$, $L(\mathcal{R}, t) = \mathcal{R} - y$. Thus there is an x so that $E(\mathcal{R}, t) = \mathcal{R} - x$. Since E is maximal, if $R \in \mathcal{R} - x$ and $x \in K \in \mathcal{K}$, then $K > R$ in E. Hence, $x = \mathrm{cent}(E)$. Also, $E_0(x)$ is a maximal extension of a poset isomorphic to $P(k, r; n-1)$, and so by induction is pseudo-lex. Likewise, $E_1(x) - x$ is a maximal extension of a poset isomorphic to $P(k-1, r-1; n-1)$, and so by induction is pseudo-lex. Hence E is pseudo-lex by definition. ∎

Proof of Theorem 2.3. Let L be a lex extension of $P(k, r; n)$. By Claim 3.3, E is a maximal pseudo-lex extension of $P(k, r; n)$ if and only if, for all $t \leq \binom{n}{r}$, $E(\mathcal{R}, t)$ is isomorphic to $L(\mathcal{R}, t)$. By Lemma 3.2, this is true if and only if E is a maximum extension of $P(k, r; n)$. ∎

4. The Maximum Number of Reversals

In order to derive Theorem 2.4, we note that it suffices to count the number of critical pairs reversed by any lex extension. Since we are free to choose the permutation σ, we let L be the lex extension determined by $\sigma = (1, 2, \ldots, n)$. Then $g(k, r; n) = g(L)$.

Proof of Theorem 2.4. Let $A = \{a_1, \ldots, a_r\} \in \mathcal{R}$ and $B = \{b_1, \ldots, b_k\} \in \mathcal{K}$. Then $B > A$ if and only if, for some $0 \leq i < k$, $a_1 = b_1$, …, $a_i = b_i$, and $b_{i+1} < a_{i+1}$. Let $j = b_{i+1} - (i+1)$ and $s = n - r$. Then $0 \leq j < s$ since $b_{i+1} \leq n - r + i$. Likewise, for any such i and j, we can choose A and B with these properties by choosing $\{a_1, \ldots, a_i\}$ in $\binom{i+j}{i}$ ways, b_{i+1} in one way, $\{b_{i+2}, \ldots, b_k\}$ in $\binom{n-b_{i+1}}{k-1-i} = \binom{n-1-i-j}{k-1-i}$ ways, and $\{a_{i+1}, \ldots, a_r\}$ in $\binom{n-b_{i+1}}{r-i} = \binom{n-1-i-j}{s-1-j}$ ways. The result follows. ∎

If no linear extension of a poset P with c critical pairs reverses more than g of them, and $qg < c$, then $\dim(P) > q$. In $P(k, n-s; n)$ there are exactly $\binom{n}{k}[\binom{n}{s} - \binom{n-k}{s}]$ critical pairs. So, by using the formula for $g(k, n-s; n)$, one is able to prove that, for all k and s, there is an N such that, if $n > N$ then $d(k, n-s; n) \geq n - \lfloor (k+1)(s+1)/2 \rfloor + 2$. We do not prove this here because the estimate on the value of N is not as good as the $250k^3$ found in Fűredi's result, and so his lower bound is always sharper. However, when $k = s = 2$, these estimates yield the result $d(2, n-2; n) \geq n - 2$. The result in [9] that $d(2, n-2; n) = n - 1$ was found by analyzing the value of $g(L)$ when L is a maximal extension of $P(2, n-2; n)$ which is not pseudo-lex. Unfortunately, this approach doesn't work for larger k, s. We finish this section with a proof of Theorem 2.5 on how not to construct a minimal realizer of $P(k, r; n)$.

Proof of Theorem 2.5. Because of Proposition 2.1(d), we may assume that $r > n/2$. Suppose that L_i is lex with respect to $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,n})$. Then let E_i be the linear extension of $P(1, r; n)$ which is also lex with respect to σ_i. We claim that $\mathcal{E} = \{E_i, \ldots, E_q\}$ realizes $P(1, r; n)$, because otherwise there is an element a and a set $B = \{b_1, \ldots, b_r\}$ so that $a > B$ in every E_i. This implies that, for each j, $a > b_j$ in every σ_i, and so $A = \{b_1, \ldots, b_{k-1}, a\} > B$ in every L_i, a contradiction. Recalling Dushnik's theorem, we find $q \geq d(1, r; n) = n - 1$. ∎

5. Fractional Dimension

In [3] Brightwell and Scheinerman define a *t-fold realizer* of a poset P to be a set of linear extensions $\mathcal{L} = \{L_1, \ldots, L_q\}$ with the property that, for every critical pair (x, y) there is a set I, $|I| \geq t$, with $x > y$ in L_i for all $i \in I$. Then the *fractional dimension* $d^f(P)$ of P is $d^f(P) = \inf_t\{|\mathcal{L}|/t : \mathcal{L} \text{ is a } t\text{-fold realizer of } P\}$. We always have $d^f(P) \leq \dim(P)$ since ordinary realizers are 1-fold realizers. Also, if P has c critical pairs and none of its linear extensions reverses more than g of them, then any t-fold realizer of size q must satisfy $qg \geq tc$. Thus, $df(p) \geq c/g$. For example, $P(1, n-1; n)$ has n critical pairs and no extension reverses more than one of them, so $d^f(1, n-1; n) = n$ where $d^f(k, r; n) = d^f(P(k, r; n))$ for $0 < k < r < n$. We can now prove that $d^f(k, r; n) \leq r - k + 2$ with the aid of the following claim.

Claim 5.1. *Let (K, R) be a critical pair of $P(k, r; n)$, σ a randomly chosen permutation of $[n]$, and L_σ the lex extension of $P(k, r; n)$ with respect to σ. Let $p_\sigma(K, R)$ be the probability that (K, R) is reversed by L_σ. Then $p_\sigma(K, R) = (k - a)/(r + k - 2a)$, where $a = |K \cap R|$.*

Proof. With regard to distinguishing K and R, none of the elements of $K \cap R$ or $\overline{K \cup R}$ need to be considered. Of the remaining $r + k - 2a$ elements in $(K - R) \cup (R - K)$, whichever element occurs first in σ determines whether $K > R$ or $R > K$. Thus $p_\sigma(K, R) = |K - R|/(r + k - 2a) = (k - a)/(r + k - 2a)$. ■

Proof of Theorem 2.6. Let $\mathcal{L}$ be the set of all $n!$ lex extensions of $P(k, r; n)$. The number of times the critical pair (K, R) is reversed in $\mathcal{L}$ is $n!(k - a)/(r + k - 2a)$, and hence $\mathcal{L}$ is a t-fold realizer for

$$t = \min_{0 \leq a \leq k-1} n!(k - a)/(r + k - 2a) = n!/(r - k + 2),$$

and the theorem follows. ■

The striking corollary of this is that $d^f(k, k+1; n) = 3$ for $n = 2k + 1$, in contrast to $d(k, k+1; n) = \Omega(\log \log n)$ for $n = 2k + 1$.

Corollary 5.2. *For all* $k \geq 1$ *and* $n = 2k + 1$, $d^f(k, k+1; n) = 3$.

Proof. Theorem 2.6 says $d^f(k, k+1; n) \leq 3$, and Proposition 2.1(c) provides $d^f(k, k+1; n) \geq d(1, 2; 3) = 3$. ■

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