On Graph Pebbling, Threshold Functions, and Supernormal Posets *

Glenn Hurlbert

Department of Mathematics

Arizona State University

Tempe, Arizona 85287-1804

email: hurlbert@math.la.asu.edu

September 4, 2000

*Dedicated to the Memory of Paul Erdős

Abstract

We survey results on the pebbling numbers of graphs as well as their historical connection with a number theoretic question of Erdős and Lemke. We also present new results on pebbling threshold functions of various natural graph sequences and relate the question of their existence to a strengthening of the normal property of posets. In particular we show that the multiset lattice is not supernormal.

1991 AMS Subject Classification: 05C99 (05D05, 06A07, 11B75, 90D43)

Key words: pebbling number, threshold function, LYM inequality, normal poset, supernormal poset.

1 Introduction

Suppose t pebbles are distributed onto the vertices of a graph G. A pebbling step $[u, v]$ consists of removing two pebbles from one vertex u and then placing one pebble at an adjacent vertex v. We say a pebble can be *moved* to a vertex r, the *root* vertex, if we can repeatedly apply pebbling steps so that in the resulting distribution r has one pebble. The pebbling step $[u, v]$ is *greedy* if $dist(v, r) < dist(u, r)$, and *semigreedy* if $dist(v, r) \leq dist(u, r)$. Here the function *dist* denotes distance.

For a graph G, we define the *pebbling number*, $f(G)$, to be the smallest integer t such that for any distribution of t pebbles to the vertices of G, one pebble can be moved to any specified root vertex r. If D is a distribution of pebbles on the vertices of G and it is possible to move a pebble to the root vertex r, then we say that D is *r-solvable*. Otherwise, D is *r-unsolvable*. Then D is *solvable* if it is r-solvable for all r, and *unsolvable* otherwise. We denote by $D(v)$ the number of pebbles on vertex v in D and let the *size*, $|D|$, of D be the total number of pebbles in D, that is $|D| = \sum_v D(v)$. This yields another way to define $f(G)$, as one more than the maximum t such that there exists an unsolvable pebbling distribution D of size t.

Throughout this paper G will denote a simple connected graph, where $n(G) = |V(G)|$, and $f(G)$ will denote the pebbling number of G. For any two graphs G_1 and G_2, we define the *cartesian product* $G_1 \Box G_2$ to be the graph with vertex set $V(G_1 \Box G_2) = \{(v_1, v_2) | v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1 \Box G_2) = \{((v_1, v_2), (w_1, w_2)) | (v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2)) \text{ or } (v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1))\}$. Thus the m-dimensional cube Q^m can be written as the cartesian product of an edge with itself m times.

In this paper we survey the current knowledge regarding the many questions surrounding this simple graph game. Our aim is to tie together several of Paul Erdős's favorite subjects, namely, graph theory, number theory, probability, and extremal set theory. It is interesting that graph pebbling arose out of attempts to answer a question of Erdős.

We begin in Section 2 by reviewing the obvious general upper and lower bounds, the known results for the pebbling numbers of various classes of graphs, conjectures involving products, diameter, and connectivity, and alternative definitions of pebbling. Section 3 details the origins of graph pebbling and its connections with number theory. In Section 4 we define threshold functions for graph sequences, analogous to threshold functions for random graph properties, and discuss known results and open problems. Finally, section 5 deals with the underlying set theory on which the existence of pebbling threshold functions rely, in contrast to random graph threshold functions.

2 Pebbling Numbers

If one pebble is placed at each vertex other than the root vertex, r, then no pebble can be moved to r. Also, if w is at distance l from r, and $2^l - 1$ pebbles are placed at w, then no pebble can be moved to r. On the other hand, if more than $(2^d - 1)(n - 1)$ pebbles are placed on the vertices of a graph of diameter d then either every vertex has at least one pebble on it or some vertex w has at least 2^d pebbles on it. In either case one can immediately pebble from w to any vertex r. We record these observations as

Fact 2.1 *Let $d = diam(G)$ and $n = n(G)$. Then $max\{n, 2^d\} \leq f(G) \leq (2^d-1)(n-1)+1$.*

Of course this means that $f(K_n) = n$, where K_n is the complete graph on n vertices. Let P_n denote the path on $n+1$ vertices. A simple weight function method shows that $f(P_n) = 2^n$. For a given distribution D and leaf root r define the weight $w(D) = \sum_v w(v)$, where $w(v) = D(v)/2^{dist(v,r)}$. Because the weight of a distribution is preserved under greedy pebbling steps, D is an r-unsolvable distribution if and only if $w(D) < 1$. Because pebbling reduces the size of a distribution, if D has maximum size with respect to r-unsolvable distributions then all its pebbles lie on the leaf opposite from r, implying $|D| = 2^n - 1$. Finally, for any other choice of root r, one applies the above argument to both sides of r and notices that $(2^a-1)+(2^b-1) < 2^{a+b}-1$.

Let C_n be the cycle on n vertices. It is easy to see that $f(C_5) = 5$ and $f(C_6) = 8$ and so each of the two lower bounds are relevant. The pebbling numbers of cycles is derived in [29]. In the case of larger odd cycles, the pebbling number is larger than both lower bounds.

Theorem 2.2 *[29] For $k \geq 1$, $f(C_{2k}) = 2^k$ and $f(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$.*

We can use this result to prove that the pebbling number of the Peterson graph P is 10. A common drawing of P displays inner and outer 5-cycles. Consider a distribution D of size 10 with root r, having no pebble on it. If a neighbor s of r has a pebble on it then by symmetry we may draw P so that r is on the outer cycle and s is on the inner. Because $f(C_5) = 5$ we may assume that there are less than 5 pebbles on the outer cycle, and thus more than 5 on the inner. We may ignore one of the pebbles on s and use 5 of the other pebbles to move a second pebble to s, then one to r. On the other hand we consider the case that the neighbors of r are also void of pebbles, so all 10 pebbles are on the 6-cycle formed by the nonneighbors. From there one can show that 2 pebbles can be moved to a neighbor of r, then one to r.

The pebbling number of a tree T on n vertices is more complicated. Consider a partition $Q = (Q_1, \ldots, Q_m)$ of the edges of T into paths $Q_1, \ldots, Q_m$, written so that $q_i \geq q_{i+1}$, where $q_i = |Q_i|$. Any choice of root vertex r in T induces an orientation of the edges of T and thus also on each path Q_i. The orientation on Q_i determines a root r_i of Q_i, which may or may not be an endpoint of Q_i. If r_i is an endpoint of Q_i then we say that Q_i is *well r-directed.* We call Q an *r-path partition* of T if each path Q_i is well r-directed, and a *path partition* if it is an r-path partition for some r. The path partition Q *majorizes* another, Q', if its sequence of path lengths majorizes that of the other. That is, if $q_j > q'_j$, where $j = \min\{i : q_i \neq q'_i\}$. A path (resp. r-path) partition of T is *maximum* (resp. *r-maximum*) if no other path (resp. r-path) partition majorizes it.

Theorem 2.3 *[27] Let $(q_1, q_2, \ldots, q_m)$ be the nonincreasing sequence of path lengths of a maximum path partition $Q = (Q_1, \ldots, Q_m)$ of a tree T. Then $f(T) = \left(\sum_{i=1}^{m} 2^{q_i}\right) - m + 1$.*

The crucial idea in the argument is to find the right generalization to use for an induction proof. Say that a distribution is *k-fold r-solvable* if it is possible to move k pebbles to the vertex r after a sequence of pebbling steps. Define $f(G, r; k)$ to be the minimum t so that every distribution of size at least t is k-fold r-solvable. Moews [27] proves that $f(T,r;k) = k2^{q_1} + \left(\sum_{i=2}^{m} 2^{q_i}\right) - m + 1$, where $(q_1, q_2, \ldots, q_m)$ is the nonincreasing sequence of path lengths of a maximum r-path partition $Q = (Q_1, \ldots, Q_m)$ of the tree T. If $T - r$ is the union of trees $T_1, \ldots, T_s$, with each T_j rooted at a neighbor r_i of r, then one uses as the inductive step the equality

$$f(T,r;k) = \max\left\{ \sum_{i=1}^{s} f(T_i, r_i; k_i + 1) \right\},$$

where the maximum is over all $k_1, \ldots, k_s$ which satisfy $\sum_{i=1}^{s} \lfloor k_i/2 \rfloor < k$.

It is natural to ask what pebbling solutions can look like. We say a graph G is *greedy* (*semi-greedy*) if every distribution of size at least $f(G)$ has a solution in which every pebbling step is greedy (semi-greedy). We say a graph G is *tree-solvable* if every distribution of size at least $f(G)$ has a solution in which the edges traversed by pebbling steps form an acyclic subgraph. The 5-cycle $abcde$ is not greedy, as witnessed by the distribution $D(c,d) = (3,2)$, with root a. Worse yet, let H be the graph formed from the 6-cycle $abcdef$ by adjoining new vertices g to a and c, and h to a and e. It is not difficult to show that $f(H) = 9$. Then the distribution $D(a,b,f,g,h) = (1,3,3,1,1)$ is not semi-greedily d-solvable, so H is not semi-greedy. Also, H is not tree-solvable. Indeed, the distribution $D(b,c,d,g,h) = (1,5,1,1,1)$ has no tree-solution for the root f.

2.1 Diameter and Connectivity

Another natural interest is to find necessary and sufficient conditions for a graph G to satisfy $f(G) = n(G)$. Although this seems very difficult, several results are relevant.

Fact 2.4 *If G has a cut vertex then $f(G) > n(G)$.*

Indeed, let x be a cut vertex and G_1 and G_2 be two components of $G - x$. Choose $r \in V(G_1)$ and $y \in V(G_2)$. The r-unsolvable distribution D, defined by $D(r, x, y) = (0, 0, 3)$ and $D(v) = 1$ for all other v, witnesses this fact.

Since a graph G of girth g satisfies $f(G) \geq f(C_g)$, one can easily show the following.

Fact 2.5 *If $girth(G) > 2 \log n$ then $f(G) > n(G)$.*

Thus it is natural to ask if there is a constant g so that $girth(G) > g$ implies $f(G) > n$, where $n = n(G)$. More generally, is it true that for all m there is a constant $g = g(m)$ so that $girth(G) > g$ implies $f(G) > mn$?

Now consider upper bounds.

Theorem 2.6 *[29] If G has diameter 2 then $f(G) \leq n(G) + 1$.*

Graphs G for which $f(G) = n(G) + i$ are called *Class i*. Consider the graph H, the union of the 6-cycle $abcdef$ and the clique ace. G has diameter two and no cut vertex, and yet $f(H) = 7$ (the distribution $D(b, d) = (3, 3)$ is f-unsolvable). Clarke, Hochberg and Hurlbert [7] characterized precisely which diameter two graphs are Class 1. The characterization is based on the structure of H. They developed an $O(n^5)$ algorithm which tests for Class 1 membership of diameter 2 graphs, and thus also serves as a test for Class 0 membership. A key ingredient in the classification proof is showing that in any size $n(G)$ unsolvable distribution D of a 2-connected, diameter 2, Class 1 graph G, there is no vertex with two pebbles on it and there are exactly two vertices with three pebbles on them. Since $maxD < 4$ it follows that there are at most four vertices which are void of pebbles, one of which must be the root r. Hence, G is not 4-connected, since otherwise if $D(v) = 3$ then we can solve D by pebbling along one of the vr-paths having no void vertex. However, the full characterization goes further and yields the following as a corollary.

Theorem 2.7 *[7] If G is 3-connected and has diameter 2 then $f(G) = n(G)$.*

Now, for fixed p the probability that the random graph $G(n,p)$ is 3-connected and has diameter 2 tends to 1 as n tends to infinity. Thus we have

Corollary 2.8 *[7] Almost all graphs are Class 0.*

Since the Petersen graph P is 3-connected and has diameter 2, we obtain our second proof that $f(P) = 10$. To extend this relationship between connectivity and diameter, Clarke, et. al. [7], had conjectured the following, which was recently proved.

Theorem 2.9 *[9] There is a function $k(d)$ such that if G is a $k(d)$-connected, diameter d graph then G is Class 0.*

The proof yielded the value of $k(d) = 2^{2d+3}$. It is shown in [7] that the function $k(d)$ must be at least $2^d/d$, and we believe it is less than 2^d. It is worth mentioning that Czygrinow, et. al., use this result to obtain the following improvement on Corollary 2.8.

Corollary 2.10 *[9] Consider the random graph $G(n,p)$ on n vertices in which each edge is included independently with probability p. Let Q be the event that $G(n,p)$ is Class 0. Then for any $d > 0$ we have*

(a) $p \gg (n \log n)^{1/d}/n$ *implies that* $Pr(Q) \to 1$ *as* $n \to \infty$*, and*

(b) $p \ll \log n/n$ *implies that* $Pr(Q) \to 0$ *as* $n \to \infty$.

Since being Class 0 is a monotone increasing graph property, its threshold function exists (see [3]) and is between the two functions above. It would be quite interesting to find the actual value of the Class 0 threshold function.

A nice family of graphs in relation to Theorem 2.9 is the following. For $n \geq 2t+1$, the *Kneser graph*, $K(n,t)$, is the graph with vertices $\binom{[n]}{t}$ and edges $\{A, B\}$ whenever $A \cap B = \emptyset$. The case $t = 1$ yields the complete graph K_n and the case $n = 5$ and $t = 2$ yields the Petersen graph P, both of which are Class 0. When $t \geq 2$ and $n \geq 3t-1$ we have $diam(K(n,t)) = 2$. Also, it is not difficult to show that $\kappa(K(n,t)) \geq 3$ in this range. Indeed, $\kappa = \kappa(K(n,t)) \geq \binom{n-2t+1}{t}$, the minimum size of a common neighborhood of two nonadjacent vertices. When $n \geq 3t$, $\kappa \geq t+1 \geq 3$. In the case $n = 3t-1$, it is easy to explicitly find 3 pairwise internally disjoint paths between any pair of vertices. (Using the techniques of [13], one can improve this to $\kappa(K(n,t)) \geq min\left\{t\binom{n-t}{t-1}, \binom{n-t}{t} - (t-1)\binom{n-2t+1}{t-1}\right\}$. It would be interesting in its own right to find $\kappa(K(n,t))$ more accurately.) Thus, we know by Theorem 2.7 that such graphs are of Class 0. The family of Kneser graphs is interesting precisely because the graphs become more sparse as n decreases toward $2t+1$, so the diameter increases and yet the connectivity decreases.

Question 2.11 *For $2t+1 \le n < 3t-1$, is $K(n,t)$ of Class 0?*

Because Pachter and Snevily [29] also proved that diameter two graphs have the 2-pebbling property (see below), it is interesting as well to ask whether $K(n,t)$ has the 2-pebbling property when $n < 3t-1$.

2.2 Products and 2-Pebbling

Chung proved that $f(Q^m) = 2^m$. What she proved is, in fact, more general. Let $\overline{d} = \langle d_1, \ldots, d_m \rangle$ and denote by $P_{\overline{d}}$ the graph $P_{d_1} \Box \cdots \Box P_{d_m}$.

Theorem 2.12 *[5] For all nonnegative $\overline{d} = \langle d_1, \ldots, d_m \rangle$, $f(P_{\overline{d}}) = 2^{d_1+\cdots+d_m}$.*

The following, more general conjecture, has generated a great deal of interest.

Conjecture 2.13 *(Graham) For all G_1 and G_2, $f(G_1 \Box G_2) \le f(G_1)f(G_2)$.*

There are few results which support Graham's conjecture. Among these, the conjecture holds for a tree by a tree [27], a cycle by a cycle (with possibly some small exceptions: it holds for $C_5 \Box C_5$ [15], and otherwise for $C_m \Box C_n$, provided m and n are not both from the set $\{5, 7, 9, 11, 13\}$ [29], and a clique by a graph with the 2-pebbling property [5].

A graph G has the *2-pebbling property* if, for any distribution D of size at least $2f(G) - q(D) + 1$, it is possible to move two pebbles to any specified root r after a sequence of pebbling steps. Here, $q(D)$ is the size of the *support* of D, the number of vertices v with $D(v) > 0$. Among the graphs known to have the

2-pebbling property are cliques, trees [5], cycles [29], and diameter two graphs [29]. The only graph known not to have the 2-pebbling property is the *Lemke graph* L, whose vertex set is $\{a, b, c, d, w, x, y, z\}$ and whose edge set consists of the union of the complete bipartite graphs $\{a\} \times \{b, c, d\}$ and $\{b, c, d\} \times \{w, z\}$ with the path w, x, y, z, a. As a witness consider that $f(L) = 8$ and let $D(a, b, c, d, w, x, y, z) = (8, 1, 1, 1, 0, 0, 0, 1)$. It is impossible to move two pebbles to the root x. In [12] one can find a family of graphs $L_0, L_1, L_2, \ldots$, each of which is conjectured not to have the 2-pebbling property. $L_0 = L$ and L_k is formed from L_{k-1} by subdividing each of the four edges incident with a exactly once.

The importance of the 2-pebbling property arises from its use in Chung's proof of Theorem 2.12. Clarke and Hurlbert [6, 18] generalize Chung's technique to cover a larger class of graphs than cartesian products. Given two graphs G_1 and G_2, denote by $B(G_1, G_2)$ the set of all bipartite graphs F such that $E(F) \subseteq V(G_1) \times V(G_2)$ and such that F has no isolated vertices. We let $\mathcal{M}(G_1, G_2)$ be the set of graphs $\{H | H = (G_1 + G_2) \cup F$ for some $F \in B(G_1, G_2)\}$, where $+$ denotes the vertex disjoint graph union.

Theorem 2.14 *[6, 18] Let G_1 and G_2 have the 2-pebbling property and $H \in \mathcal{M}(G_1, G_2)$. Then $f(H) \leq f(G_1) + f(G_2)$. Furthermore, if $f(H) = f(G_1) + f(G_2)$ then H has the 2-pebbling property.*

The proof follows essentially the same argument found in [5]. The useful corollary is that, whenever the hypothesis of Theorem 2.14 holds, if also $f(G_i) = n(G_i)$ for each i then $f(H) = n(H)$ and H has the 2-pebbling property. A pretty instance of this stronger result is our third proof that the Petersen graph P has pebbling number 10. Indeed, $P \in \mathcal{M}(C_5, C_5)$. However, we also obtain that P has the 2-pebbling property because C_5 does.

The power of Theorem 2.14 is shown in that it is easily used to prove its own generalization. Denote by $\mathcal{M}(G_1, \ldots, G_t)$ the set of all graphs H such that $H[V_i \cup V_j] \in \mathcal{M}(G_i, G_j)$ for all $i \neq j$, where $V_i = V(G_i)$. Here, $H[X]$ denotes the subgraph of H induced by the vertex set X.

Theorem 2.15 *[6, 18] Let G_i have the 2-pebbling property for $1 \leq i \leq t$ and let $H \in \mathcal{M}(G_1, \ldots, G_t)$. Then $f(H) \leq \sum_{i=1}^{t} f(G_i)$. Furthermore, if $f(H) = \sum_{i=1}^{t} f(G_i)$ then H has the 2-pebbling property.*

The analogous corollary is that, whenever the hypothesis of Theorem 2.15 holds, if also $f(G_i) = n(G_i)$ for each i then $f(H) = n(H)$ and H has the 2-pebbling property. This corollary yields a simple proof of another result of Chung, verifying another instance of Graham's conjecture. Notice that $G \square K_m \in \mathcal{M}(G, \ldots, G)$.

Theorem 2.16 *[5] If G has the 2-pebbling property then $f(G \square K_m) \leq m f(G)$.*

A similar result is proved by Moews.

Theorem 2.17 *[27] If G has the 2-pebbling property and T is a tree then $f(G \square T) \leq f(G) f(T)$.*

Hence, since trees have the 2-pebbling property, the cartesian product of two trees also satisfies Graham's conjecture.

Finally, we remark that the pebbling number of a product can sometimes fall well inside the range $n(G_1 \Box G_2) < f(G_1 \Box G_2) < f(G_1)f(G_2)$. Consider the graph $H = P_3 \Box S_4$. (S_n is the *star* with n vertices, also denoted $K_{1,n-1}$). Easily, $f(P_3) = 4$ and $f(S_4) = 5$. Although each pebbling number is only one more than the number of vertices, $f(H) = 18$ is far greater than $n(H) = 12$; it would be interesting to discover how much greater this gap can be for other graphs. Also, notice that $18 < f(P_3)f(S_4)$, a strict inequality. More importantly, as observed by Moews [28], H is not semi-greedy. Indeed, think of H as three pages of a book, let r be the corner vertex of one of the pages, x the farthest corner vertex of a second page, u, v and w the three vertices of the third page, and let $D(u, v, w, x) = (1, 1, 1, 15)$. This shows that even semi-greediness is not preserved by the product of two greedy graphs.

3 Number Theory

The concept of pebbling in graphs arose from an attempt by Lagarias and Saks to give an alternative proof of a theorem of Kleitman and Lemke. An elementary result in number theory which follows from the pigeonhole principle is as follows.

Fact 3.1 *For any set $N = \{n_1, \ldots, n_q\}$ of q natural numbers, there is a nonempty index set $I \subset \{1, \ldots, q\}$ such that $q \,\Big|\, \sum_{i \in I} n_i$.*

Erdős and Lemke conjectured in 1987 that the extra condition $\sum_{i \in I} n_i \leq \mathrm{lcm}(q, n_1, \ldots, n_q)$ could also be guaranteed. In 1989 Lemke and Kleitman proved

Theorem 3.2 *[22] For any any set $N = \{n_1, \ldots, n_q\}$ of q natural numbers, there is a nonempty index set $I \subset \{1, \ldots, q\}$ such that $q \,\Big|\, \sum_{i \in I} n_i$ and so that $\sum_{i \in I} \gcd(q, n_i) \le q$.*

This proves the Erdős-Lemke conjecture because of the string of inequalities

$$\sum_{i \in I} n_i = \frac{1}{q} \sum_{i \in I} q n_i = \frac{1}{q} \sum_{i \in I} \operatorname{lcm}(q, n_i) \gcd(q, n_i)$$
$$\le \frac{1}{q} \operatorname{lcm}(q, n_1, \ldots, n_q) \sum_{i \in I} \gcd(q, n_i) \le \operatorname{lcm}(q, n_1, \ldots, n_q).$$

The proof offered by Kleitman and Lemke had many cases and did not seem to be the most natural proof. It was the intention of Lagarias and Saks to introduce graph pebbling as a more intuitive vehicle for proving the theorem. If the formula for the general pebbling number of a cartesian product of paths is as was believed then the number theory result would follow easily. It was Chung [5] who finally pinned down such a formula. Recently, pebbling has been used to extend the result further. Denley [11] proved that if each $n_i | q$ (with $n_i \le n_{i+1}$) and $\sum_{p \text{ prime},\ p|q} 1/p \le 1$, then there is a nonempty I such that $q = \sum_{i \in I} n_i$ and $n_i | n_j$ for all $i < j$.

Kleitman and Lemke went on to make more general conjectures on groups. First, let G be a finite group of order q with identity e, and let $|g|$ denote the order of the element g in G. Then for any multisubset $N = \{g_1, \ldots, g_q\}$ of G there is a nonempty I such that $\prod_{i \in I} g_i = e$ and $\sum_{i \in I} 1/|g_i| \le 1$. Their prior theorem is merely the case $G = \boldsymbol{Z}_q$, and they verified this conjecture for $G = \boldsymbol{Z}_p^n$, for dihedral G, and also for all $q \le 15$. Second, let H be a subgroup of a group G with $|G/H| = q$ and let $N = \{g_1, \ldots, g_q\}$ be any multisubset of G. Then there is a nonempty I such that $\prod_{i \in I} g_i \in H$ and

$\sum_{i\in I} 1/|g_i| \leq 1\Big/|\prod_{i=1}^{q} g_i|$. The first conjecture is the case $H = \{e\}$ here, and this second conjecture they verified for all $|G| \leq 11$. It would be very interesting to see what pebbling could say about these two conjectures.

In order to describe Chung's proof (see [5, 7]) of Theorem 3.2 we need to define a more general pebbling operation on a product of paths.

A *p-pebbling step* in G consists of removing p pebbles from a vertex u, and placing one pebble on a neighbor v of u. The definitions for r-solvability, and so on, carry over to p-pebbling. Recall the definition of the graph $P_{\overline{d}}$ from section 2.2. Each vertex $v \in V(P_{\overline{d}})$ can be represented by a vector $\overline{v} = \langle v_1, \ldots, v_m \rangle$, with $0 \leq v_i \leq d_i$ for each $i \leq m$. Let $\overline{e}_i = \langle 0, \ldots, 1, \ldots, 0 \rangle$ be the i^{th} standard basis vector and $\overline{0} = \langle 0, \ldots, 0 \rangle$. Then two vertices u and v are adjacent in $P_{\overline{d}}$ if and only if $\overline{u} - \overline{v} = \pm\overline{e}_i$ for some integer $1 \leq i \leq m$. If $\boldsymbol{p} = (p_1, \ldots, p_m)$, then we may define $\boldsymbol{p}$*-pebbling* in $P_{\overline{d}}$ to be such that each pebbling step from $\overline{u}$ to $\overline{v}$ is a p_i-pebbling step whenever $\overline{u} - \overline{v} = \pm\overline{e}_i$. We denote the $\boldsymbol{p}$-pebbling number of $P_{\overline{d}}$ by $f_{\boldsymbol{p}}(P_{\overline{d}})$.

For integers $p_i, d_i \geq 1$, $1 \leq i \leq m$, we use $\boldsymbol{p}^{\overline{d}}$ as shorthand for the product $p_1^{d_1} \cdots p_m^{d_m}$. Chungs's proof uses the following result.

Theorem 3.3 *[5] Every distribution of size at least $\boldsymbol{p}^{\overline{d}}$ is $\overline{0}$-solvable via greedy $\boldsymbol{p}$-pebbling.*

One can actually prove more.

Theorem 3.4 *[7] The $\boldsymbol{p}$-pebbling number $f_{\boldsymbol{p}}(P_{\overline{d}}) = \boldsymbol{p}^{\overline{d}}$. Morover, $P_{\overline{d}}$ is greedy.*

In order to prove Theorem 3.2 from Theorem 3.3 one first defines a pebbling distribution D in $P_{\overline{d}}$ which depends on the set of integers $\{x_1, \ldots, x_d\}$. Here, $|D| = \boldsymbol{p}^{\overline{d}}$, where $\boldsymbol{p}^{\overline{d}} = \prod_{i=1}^{m} p_i^{d_i}$ is the prime factorization of d and $\overline{d} = \langle d_1, \ldots, d_m \rangle$. In what follows, each pebble will be named by a set, and $\overline{c}(B)$ will denote the vertex (coordinates) on which the pebble B sits. We let x_j correspond to the pebble $A_j = \{x_j\}$, which we place on the vertex $\overline{c}(A_j) = \langle c_1, \ldots, c_m \rangle$ of $P_{\overline{d}}$, where $d/\gcd(x_j, d) = \boldsymbol{p}^{\overline{c}}$. For each vertex $\overline{u} = \langle u_1, \ldots, u_m \rangle$ define the set $X(\overline{u}) = \{A | \overline{c}(A) = \overline{u}\}$ to denote those pebbles currently sitting on $\overline{u}$, and let $\overline{u}^{(i)} = \langle u_1, \ldots, u_i - 1, \ldots, u_m \rangle$.

For a set B we make the following recursive definitions. The *value* of B is defined as $val(B) = \sum_{A \in B} val(A)$, with $val(\{A_j\}) = x_j$. The function GCD is defined as $GCD(B) = \sum_{A \in B} GCD(A)$, where $GCD(\{A_j\}) = \gcd(x_j, d)$. Finally, $Set(B) = \bigcup_{A \in B} Set(A)$, where $Set(A_j) = A_j$.

We say that B is *well placed* at $\overline{c}(B) = \langle c_1, \ldots, c_m \rangle$ when

$$\boldsymbol{p}^{\overline{d} - \overline{c}(B)} | val(B) \tag{1}$$

and

$$GCD(B) \leq \boldsymbol{p}^{\overline{d} - \overline{c}(B)} \ . \tag{2}$$

It is important to maintain a numerical interpretation of $\boldsymbol{p}$-pebbling so that moving a pebble to $\overline{0}$ corresponds to finding a set J which satisfies the conclusion of Theorem 3.2. For this reason we introduce the following operation, which corresponds to a greedy p_i-pebbling step in which a numerical condition must hold in order to move a pebble. It is shown that this condition holds originally for D (Lemma 3.5) and is maintained throughout (Lemma 3.6).

Numerical Pebbling Operation. If W is a set of p_i pebbles such that every pebble $A \in W$ sits on the vertex $\overline{c}(A) = \overline{u}$, and there is some $B \subseteq W$ such that $p_i^{b_i} | val(B)$, where $b_i = d_i - c_i + 1$, then replace $X(\overline{c})$ by $X(\overline{c}) \setminus W$, and replace $X(\overline{c}^{(i)})$ by $X(\overline{c}^{(i)}) \cup B$.

Lemma 3.5 *A_j is well placed for $1 \leq j \leq d$.*

Lemma 3.6 *Suppose $B \subseteq X(\overline{u})$, $|B| \leq p_i$, and $p_i^{b_i} | val(B)$ for $b_i = d_i - u_i + 1$. Suppose further that for every $A \in B$, A is well placed at $\overline{u}$. Then B is well placed at $\overline{u}^{(i)}$.*

Lemma 3.7 *Suppose $|X(\overline{u})| \geq p_i$, and for all $A \in X(\overline{u})$, A is well placed at $\overline{u}$. Then there exists some $B \subseteq X(\overline{u})$ such that $|B| \leq p_i$ and $p_i^{b_i} | val(B)$ where $b_i = d_i - u_i + 1$.*

By Lemma 3.5 the pebbles corresponding to each of the numbers are initially well placed. Lemma 3.6 guarantees that applying the Numerical Pebblng Operation maintains the well placement of the pebbles. Lemma 3.7 establishes that every graphical pebbling operation can be converted to a numerical pebbling operation. Then by Theorem 3.3 we can repeatedly apply the numerical pebbling operation to move a pebble to $\overline{0}$. This pebble B is then well placed at $\overline{0}$. Thus, for $J = \{j | x_j \in Set(B)\}$, we have $d = \boldsymbol{p}^{\overline{d}} \mid val(B) = \sum_{j \in J} x_j$ by (1), and $\sum_{j \in J} gcd(x_j, d) = \sum_{j \in J} x_j = GCD(B) \leq \boldsymbol{p}^{\overline{d}} = d$ by (2). This proves Theorem 3.2. Interestingly, it is Fact 3.1 which is used to prove Lemma 3.7.

Naturally, one can generalize Graham's Conjecture 2.13 in the following way.

Conjecture 3.8 *[7] For all* G_1, G_2, p_1, p_2, $f_{(p_1,p_2)}(G_1 \square G_2) \leq f_{p_1}(G_1) f_{p_2}(G_2)$.

4 Thresholds

For this section we will fix notation as follows. The vertex set for any graph on n vertices will be taken to be $\{v_i | i \in [n]\}$, where $[n] = \{0, \ldots, n-1\}$. That way, any distribution $D : V(G_n) \rightarrow \boldsymbol{N}$ is independent of G_n. Let

$\mathcal{G} = (G_1, \ldots, G_m, \ldots)$, G_n a generic graph on n vertices;

$\mathcal{H} = (H_1, \ldots, H_m, \ldots)$, H_n a generic graph on n vertices;

$\mathcal{T} = (T_1, \ldots, T_m, \ldots)$, T_n a generic tree on n vertices;

$\mathcal{K} = (K_1, \ldots, K_m, \ldots)$, K_n the complete graph on n vertices;

$\mathcal{P} = (P_1, \ldots, P_m, \ldots)$, P_n the path on n vertices;

$\mathcal{C} = (C_1, \ldots, C_m, \ldots)$, C_n the cycle on n vertices;

$\mathcal{S} = (S_1, \ldots, S_m, \ldots)$, S_n the star on n vertices;

$\mathcal{W} = (W_1, \ldots, W_m, \ldots)$, W_n the wheel on n vertices;

$\mathcal{Q} = (Q^1, \ldots, Q^m, \ldots)$, Q^m the cube on $n = 2^m$ vertices;

$D_n : [n] \rightarrow \boldsymbol{N}$, a distribution on n vertices.

Let $h : \boldsymbol{N} \to \boldsymbol{N}$ and for fixed n consider the probability space X_n of all distributions D_n of size $h = h(n)$. We denote by P_n^+ the probability that D_n is G_n-solvable and let $t : \boldsymbol{N} \to \boldsymbol{N}$. We say that t is a *threshold function* for $\mathcal{G}$, and write $th(\mathcal{G}) = \Theta(t)$, if $P_n^+ \to 0$ whenever $h(n) \ll t(n)$ and $P_n^+ \to 1$ whenever $h(n) \gg t(n)$. Although the existence of such a function $th(\mathcal{G})$ has yet to be established and may be impossible for some graph sequences $\mathcal{G}$, the family of cliques is not one of them. (If $th(\mathcal{G}) \ll th(\mathcal{H})$ then the alternating graph sequence $(G_1, H_2, G_3, H_4, \ldots)$ may seem to have no threshold function — see Theorem 4.1 and Theorem 4.2(d) — its threshold function is merely the alternation of the corresponding two threshold functions. There is no demand on the continuity of such a function.)

Theorem 4.1 *[6] The threshold function $th(\mathcal{K}) = \Theta(\sqrt{n})$.*

This result is merely a reformulation of the so-called "Birthday problem" in which one finds the probability that 2 of t people share the same birthday, assuming n days in a year.

Among the results of Czygrinow, Eaton, Hurlbert and Kayll are the following.

Theorem 4.2 *[10]*

(a) *For all $\mathcal{G}$, if $th(\mathcal{G})$ exists then, for all $\epsilon > 0$, $th(\mathcal{G}) = o(n^{1+\epsilon})$.*

(b) *If $\mathcal{G}$ is a sequence of graphs of bounded diameter and $th(\mathcal{G})$ exists then $th(\mathcal{G}) = O(n)$.*

(c) *If $th(\mathcal{Q})$ exists then $th(\mathcal{Q}) = O(n)$.*

(d) *If* $th(\mathcal{C})$ *exists then* $th(\mathcal{C}) = \Omega(n)$.

(e) *If* $th(\mathcal{P})$ *exists then* $th(\mathcal{P}) = \Omega(n)$.

(f) $th(\mathcal{S}) = \Theta(\sqrt{n})$.

(g) $th(\mathcal{W}) = \Theta(\sqrt{n})$.

It is interesting that even within the family of trees, the threshold functions can vary so dramatically, as in the case for paths and stars. Diameter seems to be a critical parameter.

Two important theorems are currently lacking in the theory to date. One is an existence theorem for threshold functions of certain families of graph sequences. Another is a monotonicity theorem for threshold functions.

Conjecture 4.3 *For every graph sequence* $\mathcal{G}$ *there is a threshold function* $th(\mathcal{G})$.

Conjecture 4.4 *If* $f(G_n) \leq f(H_n)$ *for all* n *and both* $th(\mathcal{G})$ *and* $th(\mathcal{H})$ *exist then* $th(\mathcal{G}) \leq th(\mathcal{H})$.

For any integer t and graph G denote by $p(G, t)$ the probability that a randomly chosen distribution D of size t on G solves G. If both $th(\mathcal{G})$ and $th(\mathcal{H})$ exist then Conjecture 4.4 would follow from the statement that, if $f(G_n) \leq f(H_n)$ then for all t we have $p(G_n, t) \geq p(H_n, t)$. Unfortunately, although seemingly intuitive, this statement is false. Using the Class 0 pebbling characterization theorem of [7], we discovered in [10] a family of pairs of graphs (G_n, H_n), one pair for each n divisible by 3, for which such a statement fails.

5 Set Theory

In order to describe the connection between extremal set theory and threshold functions we need to define some notation.

Let $[n] = \{1, 2, \ldots, n\}$, $Set\{n\}$ be the poset of subsets of $[n]$, $MSet\{n\}$ be the poset of submultisets of $[n]$, and $BMSet\{n, b\}$ be the poset of b-bounded submultisets of $[n]$ (that is, no element appears more than b times). Ordered by the relation of inclusion, we call $Set\{n\}$ the *subset lattice* (also called the Boolean algebra) and $BMSet\{n, b\}$ the *multiset lattice* (the product of n chains C_{b+1} on $b + 1$ elements each; isomorphic to the lattice of divisors of an integer which is the b^{th} power of a product of n distinct primes).

Let $Set[n, w]$ be the set of weight-w subsets of $[n]$, $MSet[n, w]$ be the set of weight-w submultisets of $[n]$, and $BMSet[n, w, b]$ be the set of b-bounded weight-w submultisets of $[n]$. Also let

$bin[n, w] = |Set[n, w]| = \binom{n}{w}$,

$mul[n, w] = |MSet[n, w]| = \binom{n+w-1}{w}$, and

$bmul[n, w, b] = |BMSet[n, w, b]| = \sum_{i=0}^{\lfloor w/(b+1) \rfloor} (-1)^i bin[n, i] mul[n, w - i(b+1)]$.

Next, let $\boldsymbol{N} = \{1, 2, \ldots\}$, $S[w]$ be the set of weight-w subsets of $\boldsymbol{N}$, $MS[w]$ be the set of weight-w submultisets of $\boldsymbol{N}$, and $BMS[w, b]$ be the set of b-bounded weight-w submultisets of $\boldsymbol{N}$. Since $BMS[w, b]$ is the most general setting ($S[w] = BMS[w, 1]$ and $MS[w] = BMS[w, w]$), we will define the well known *Colex* order L on $BMS[w, b]$.

Let $M \in BMS[w, b]$ with multiplicities $(m_1, m_2, \ldots, m_l)$ for some l so that $m_j = 0$ for all $j > l$. Define the function $C : BMS[w, b] \to \boldsymbol{N}$ by

$C(M) = \sum_{i=1}^{l} m_i(b+1)^{i-1}$. Then for $A, B \in BMS[w,b]$ we have $L(A,B)$ if and only if $C(A) < C(B)$. It is easy to see that, for any f and n, the first f multisets of $BMSet[n,w,b]$ are precisely the first f members of $BMS[w,b]$.

To define *shadows*, again let M be an element of $BMS[w,b]$ with multiplicities $(m_1, m_2, \ldots, m_l)$ for some l so that $m_j = 0$ for all $j > l$. Now denote by $Shad[M]$ the set of all $A \in BMS[w-1,b]$ with multiplicities $(a_1, a_2, \ldots, a_l)$ so that $a_i \leq m_i$ for all $i \leq l$. When $\mathcal{F}$ is a family of multisets, each of which is in $BMS[w,b]$, denote by $Shad[\mathcal{F}]$ the family of all $A \in Shad[M]$ for some $M \in \mathcal{F}$, and let $shad[\mathcal{F}] = |Shad[\mathcal{F}]|$. For the vector $\overline{v} = \langle v_1, v_2, \ldots, v_l \rangle$ with each $v_i \geq v_{i+1}$ define $col[\overline{v}, w, b] = \sum_{j=1}^{l} bmul[v_j, w+1-j, b]$ and let $Col[\overline{v}, w, b]$ be the first $col[\overline{v}, w, b]$ multisets in the Colex order on $BMS[w,b]$. Finally, define $Shad[\overline{v}, w, b] = Shad[Col[\overline{v}, w, b]] = Col[\overline{v}, w-1, b]$ and let $shad[\overline{v}, w, b] = |Shad[\overline{v}, w, b]| = col[\overline{v}, w-1, b]$. It is not difficult to show that for any natural numbers f, w and b, there exist natural numbers $l, v_1, v_2, \ldots, v_l$ such that $f = col[\overline{v}, w, b]$. In 1969 Clements and Lindström proved the following.

Theorem 5.1 *[8] Let $\overline{v}$, w and b be given with $\mathcal{F} \in BMS[w,b]$ and $|\mathcal{F}| = col[\overline{v}, w, b]$. Then $shad[\mathcal{F}] \geq shad[\overline{v}, w, b] = col[\overline{v}, w-1, b]$.*

The important special cases when $b = 1$ and $b \geq w$ had been proven earlier by Kruskal [21] and Katona [20] and by Macaulay [25], respectively. The Kruskal-Katona theorem plays a crucial role in many combinatorial contexts, including the dimension theory of partially ordered sets ([19]). It is of fundamental importance in the probability estimates below.

Since $bmul[n,w,b]$ is a polynomial in the variable n, it is well-defined when n takes on the value of some real number x. Of course, for any natural numbers f, w and b, there is a real number x for which $f = bmul[x, w, b]$. In 1979 Lovász proved the following version of the Kruskal-Katona theorem.

Theorem 5.2 *[23] Let w be given with $\mathcal{F} \in BMS[w, 1] = S[w]$ and $|\mathcal{F}| = bmul[x, w, 1] = bin[x, w]$. Then $|Shad[\mathcal{F}]| \geq bmul[x, w-1, 1] = bin[x, w-1]$.*

For a general ranked poset $\mathcal{P}$ and family $\mathcal{F}$ of elements of $\mathcal{P}$, let $\mathcal{F}_w$ be those elements of $\mathcal{F}$ of rank w. Then define the probability $p(\mathcal{F}_w) = |\mathcal{F}_w|/|P_w|$. A family $\mathcal{F}$ is *monotone decreasing* if $A \subset B \in \mathcal{F}$ implies $A \in \mathcal{F}$. Also, $\mathcal{F}$ is an *antichain* if no pair of elements of $\mathcal{F}$ are related in $\mathcal{P}$. The poset $\mathcal{P}$ is *LYM* (has the *LYM property*) if, for any antichain $\mathcal{F}$ in $\mathcal{P}$, $\sum_w p(\mathcal{F}_w) \leq 1$. Also, $\mathcal{P}$ is *normal* (has the *normalized matching property*) if, for any monotone decreasing family $\mathcal{F}$, $p(\mathcal{F}_u) \geq p(\mathcal{F}_w)$ whenever $0 < u < w$. Finally, we define $\mathcal{P}$ to be *supernormal* if $p(\mathcal{F}_u)^w \geq p(\mathcal{F}_w)^u$ whenever $0 < u < w$.

It is known that $\mathcal{P}$ is LYM if and only if $\mathcal{P}$ is normal, and that $Set\{n\}$ is LYM, and hence normal, for all n [2, 24, 26, 30]. Canfield's and Harper's product theorems [4, 14] show that $BMSet\{n, b\}$ is LYM, and hence normal, for all $b \leq n$. An important consequence of Theorem 5.2 is that $Set\{n\}$ is supernormal. It is precisely this inequality which allows one to prove the following.

Theorem 5.3 *[3] Let $\mathcal{F}$ be any monotone decreasing family of subsets of $[n]$. Then there exists a threshold function $th(\mathcal{F})$ for $\mathcal{F}$. That is, $p(\mathcal{F}_w) \rightarrow 1$ when $w \ll th(\mathcal{F})$ and $p(\mathcal{F}_w) \rightarrow 0$ when $w \gg th(\mathcal{F})$.*

Of course, the analogous theorem holds for monotone increasing families. Most notably, as a corollary one obtains the existence for threshold functions of monotone properties of graphs (such as for connectedness, hamiltonicity, or subgraph containment, but not for induced subgraph containment).

In our case we would like to mimic these results for multisubsets in order to prove existence for pebbling threshold functions of graph sequences. Unfortunately, not all of the results generalize well. We have discovered that the Lovász-type version (Theorem 5.2) fails in both the Macauley and Clements-Lindström contexts. For example, let $\mathcal{F} = Col[\langle 4\rangle, 7, b]$ for any $b > 3$. It may be that $Col[\langle n\rangle, b+c, b]$ fails the test for all n and c, provided b is large enough. On the other hand, it may be that $Col[\langle n,\rangle, w, b]$ passes the test for all n and b, provided w is large enough.

Our purpose in trying to generalize Theorem 5.2 is in trying to prove that $BMSet\{n,b\}$ is supernormal for all n and b. However,

Theorem 5.4 *[17] There exist n and b so that $BMSet\{n,b\}$ is not supernormal.*

Our purpose in investigating the supernormality of $BMSet\{n,b\}$ is in trying to generalize Theorem 5.3, which would yield the existence of pebbling threshold functions for arbitrary graph sequences as a corollary. However, such a generalization may be true even in the absence of supernormality. There is some evidence that, in the Clements-Lindström setting, for fixed u and w, $|p(\mathcal{F}_u)^w - p(\mathcal{F}_w)^u| \to 0$ as $n \to \infty$.

References

[1] S. Bezrukov and K. Engel, Properties of graded posets preserved by some operations, The mathematics of Paul Erdős, II, *Algorithms Combin.* **14** (1997), 79–85.

[2] B. Bollobás, *Combinatorics*, Cambridge Univ. Press, Cambridge, New York (1986).

[3] B. Bollobás and A. Thomason, Threshold functions, *Combinatorica* **7** (1987), 35–38.

[4] E. R. Canfield, A sperner property preserved by products, *Linear and Multilinear Alg.* **9** (1980), 151–157.

[5] F. R. K. Chung, Pebbling in hypercubes, *SIAM J. Disc. Math.* **2** (1989), 467–472.

[6] T. A. Clarke, Pebbling on graphs, *Master's Thesis, Arizona St. Univ.* (1996).

[7] T. A. Clarke, R. A. Hochberg and G. H. Hurlbert, Pebbling in diameter two graphs and products of paths, *J. Graph Th.* **25** (1997), 119–128.

[8] G. F. Clements and Lindström, A generalization of a combinatorial theorem of Macaulay, *J. Combin. Theory* **7** (1969), 230–238.

[9] A. Czygrinow, G. H. Hurlbert, H. A. Kierstead, and W. T. Trotter, A note on graph pebbling, *preprint* (1999).

[10] A. Czygrinow, N. Eaton, G. H. Hurlbert, and P. M. Kayll, On pebbling threshold functions for graph sequences, *preprint* (1999).

[11] T. Denley, On a result of Lemke and Kleitman, *Comb., Prob. and Comput.* **6** (1997), 39–43.

[12] J. A. Foster and H. S. Snevily, The 2-pebbling property and a conjecture of Graham's, *preprint* (1995).

[13] C. D. Godsil, J. Nešetřil and R. Nowakowski, The chromatic connectivity of graphs, *Graphs and Combin.* **4** (1988), 229–233.

[14] L. H. Harper, The morphology of partially ordered sets, *J. Combin. Theory (A)* **17** (1974), 44–58.

[15] D. Herscovici and A. Higgins, The pebbling number of $C_5 \square C_5$, *preprint* (1996).

[16] W. N. Hsieh and D. J. Kleitman, Normalized matching in direct products of partial orders, *Stud. Appl. Math.* **52** (1973), 285–289.

[17] G. H. Hurlbert, *unpublished* (1997).

[18] G. H. Hurlbert, Two pebbling theorems, *Congress. Numer. (to appear).*

[19] G. H. Hurlbert, A. V. Kostochka and L. A. Talysheva, The dimension of interior levels of the boolean lattice, *Order* **11** (1994), 29–40.

[20] G. O. H. Katona, A theorem on finite sets, in *Theory of Graphs (P. Erdős and G. O. H. Katona, eds.)*, Akadémiai Kiadó, Budapest (1968), 187–207.

[21] J. B. Kruskal, The number of simplices in a complex, in *Mathematical Optimization Techniques*, Univ. California Press, Berkeley (1963), 251–278.

[22] P. Lemke and D. J. Kleitman, An addition theorem on the integers modulo n, *J. Number Th.* **31** (1989), 335–345.

[23] L. Lovász, *Combinatorial Problems and Exercises*, North Holland Pub., Amsterdam, New York, Oxford (1979).

[24] D. Lubell, A short proof of Sperner's lemma, *J. Combin. Theory* **1** (1966), 299.

[25] F. S. Macaulay, Some properties of enumeration in the theory of modular systems, *Proc. Lond. Math. Soc.* **26** (1927), 531–555.

[26] L. D. Meshalkin, A generalisation of Sperner's theorem on the number of subsets of a finite set (in Russian), *Teor. Probab. Ver. Primen.* **8** (1963), 219–220; English translation in *Theory of Probab. and its Apps.* **8** (1964), 204–205.

[27] D. Moews, Pebbling graphs, *J. Combin. Theory (B)* **55** (1992), 244–252.

[28] D. Moews, *personal communication* (1997).

[29] L. Pachter, H. S. Snevily, and B. Voxman, On pebbling graphs, *Congressus Numerantium* **107** (1995), 65–80.

[30] K. Yamamoto, Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343–353.