PEBBLING CERTAIN PRODUCT DISTRIBUTIONS

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Abstract. A pebbling step on a graph consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. Given some subset R of the vertices of G, the pebbling number of G for R is the minimal number of pebbles such that there is a sequence of pebbling steps at the end of which each vertex of R has a pebble on it, no matter how the pebbles are initially placed on the vertices of the graph. If G and H are two graphs, some goal distributions on $G \Box H$ have the property that they can be expressed as a product of goal distributions on G and H; these are called product goal distributions. In this paper we study the pebbling number of a particular kind of product goal distributions and express the pebbling number of the product distribution in terms of the pebbling numbers of the factors. We do this by introducing *color pebbling* which may be of interest in its own right.

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1. Introduction and definitions

The game of pebbling was first suggested by Lagarias and Saks as a tool for solving a number-theoretical conjecture of Erdös. Chung successfully used this tool to prove the conjecture and established other results concerning pebbling numbers. In doing so she introduced pebbling to the literature [**?**]. Begin with a graph G and a certain number of pebbles placed on its vertices. A pebbling step consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. In (regular) pebbling, a target vertex is selected, and the goal is to move a pebble to the target vertex. The minimum number of pebbles such that, regardless of their initial placement and regardless of the target vertex, we can pebble that vertex is called *the pebbling number of* G.

A generalization of pebbling is to allow the "goal" of the game to be to place a pebble (or possibly a number of pebbles) on a set of vertices R_G^P that satisfy a particular property P_G. In regular pebbling the property is that R_G^P contains only one vertex. Another well studied version is for R_G^P to contain all vertices of G (this is known as cover pebbling) or for R_G^P to contain a set of vertices so that each vertex of G is at most distance 1 from a vertex in R_G^P (dominance pebbling). We will call a subset R_G^P a *goal distribution with the property* P_G and denote by $\gamma(P_G)$ the minimum number of pebbles to guarantee that regardless of their initial distribution we can achieve every goal distribution with property P_G after a sequence of pebbling steps. In the particular case when P_G is the property that R_G^P contains all vertices of G, there is clearly a unique goal distribution and in this case we will abreviate $\gamma(P_G)$ by $\gamma(G)$ (i.e. $\gamma(G)$ is the cover pebbling number of G). In the case that P_G is the property that the goal distribution contains only one vertex (as in regular pebbling) we will denote $\gamma(P_G)$ by $\rho(G)$.

One of the main open problems of pebbling is studying how pebbling numbers behave under graph products.

Graham's Conjecture. *If $G\Box H$ is the product of G and H (Definition 1.1), then $\rho(G\Box H) \leq \rho(G) \times \rho(H)$.*

In this paper we discuss certain kinds of goal distributions on products of graphs and determine their pebbling numbers. We do this by "translating" a distribution on a product of graphs to a distribution on one of the factors by introducing colors.

Let $G = (V, E)$ be any graph. A *distribution* D of pebbles to the vertices of G is any initial arrangement of pebbles on some subset of V. Given a goal property P_G, we call D *solvable for* P_G if for any R^P_G there is a sequence of pebbling steps at the end of which each vertex in R^P_G has at least one pebble on it. For any positive integer n we call D *n-solvable for* P_G if there is a sequence of steps at the end of which each vertex in any R^P_G has at least n pebbles placed on it.

Definition 1.1. *Say graphs G and H have vertex sets $V(G) = \{w_1, \ldots, w_g\}$ and $V(H) = \{v_1, \ldots, v_h\}$, respectively. The product of G and H, $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ (Cartesian product) and with edge set*

$$\begin{aligned} E(G\Box H) =& \{\big((w_1, v_1), (w_2, v_2)\big) |\, w_1 = w_2 \text{ and } (v_1, v_2) \in V(H)\} \\ & \cup \{\big((w_1, v_1), (w_2, v_2)\big) |\, v_1 = v_2 \text{ and } (w_1, w_2) \in V(G)\}. \end{aligned}$$

Suppose P_G and P_H are goal properties on G and H respectively. Then there is an induced goal property $P_{G\Box H}$ on $G\Box H$ where for every $R^P_{G\Box H}$ there exist R^P_G and R^P_H such that$(v_i, w_j) \in R_{G\Box H}$ if and only if $v_i \in R_G$ and $w_j \in R_H$. A distibution $R^P_{G\Box H}$ on $G\Box H$ will be called a *product goal distribution* if it is induced by goal distributions on R^P_G and R^P_H on G and H respecitvely. We will refer to R^P_G and R^P_H as the *factors* of $R^P_{G\Box H}$. A property $P_{G\Box H}$ will be called a product goal property if there exist properties P_G and P_H such that every goal distribution $R^P_{G\Box H}$ is a product goal distribution with factors R^P_G and R^P_H.

Most natural goal properties on a product graph are in fact product goal distributions. Examples are regular pebbling and cover pebbling. Another example is the property that each copy of G in the product $G\Box H$ has a pebble on some subset of vertices where the subset is the same for all copies. This is the product property that we will explore in this paper. Namely we will prove the following.

Theorem 1.2. *Suppose $P_{G\Box H}$ is a goal property on $G\Box H$ and $P_H = V(H)$. Then $\gamma(P_{G\Box H}) = \gamma(P_G) \times max_j\{\sum_{i=1}^{h} 2^{dist(w_j, w_i)}\} = \gamma(P_G) \times \gamma(H)$.*

2. Color Pebbling

Let G and H be two graphs with vertices $w_1, .., w_g$ and $v_1, \ldots, v_h$ respectively. Recall that their product has vertex set $\{(w_i, v_j) | i = 1, \ldots, g;\ j = 1, \ldots, h\}$. We will associate to each distribution on $G\Box H$ a certain distribution of colored pebbles on H. We will call a distribution t-colored (or a t-distribution) if each pebble in the distribution has been assigned one of t possible colors. A color-respecting pebbling step for a colored distribution consists of taking two pebbles of the same color from some vertex and placing one of these pebbles on an adjacent vertex. When considering colored distributions we allow only color-respecting steps. A colored distibution D on a graph G is n-solvable for some goal distribution if there is a sequence of color preserving pebbling steps at the end of which each vertex in the goal distribution has at least n pebbles regardless of their color. If the goal distribution is clear from the context, we will not specify it.

To each distribution D on $G\Box H$ we associate a color distribution $\tilde{D}$ on H in the following way: use colors $c_1, c_2, \ldots, c_g$ to assign color c_i to each pebble that D places on vertices (w_i, v_j) (for any j). Collapse $G\Box H$ to a single copy of H, which we call $\tilde{H}$ for clarity, by identifying $G\Box\{v_i\}$ in $G\Box H$ with vertex V_i in $\tilde{H}$. We place all pebbles from $G\Box\{v_i\}$ on V_i.

Fix a goal distribution $R_{G\Box H}$ with factors R_G and R_H.

Lemma 2.1. *Let G and H be graphs and D be a distribution on $G\Box H$. If the associated g-distribution $\tilde{D}$ on $\tilde{H}$ is $\gamma(G)$-solvable for R_H, then D is solvable for $R_{G\Box H}$.*

Proof. By hypothesis there is a sequence of color-respecting pebbling steps beginning with $\tilde{D}$ at the end of which there are $\gamma(G)$ pebbles on each vertex of $\tilde{R}_H$. Because the steps respect color we could have performed them in $G\Box H$: taking two pebbles of color c_i from V_j, discarding one and placing the other one on V_k in $\tilde{R}_H$ corresponds to taking two pebbles from vertex (w_i, v_j) and placing one of them on (w_i, v_k). So there is a sequence of steps on $G\Box H$, consisting only of moving pebbles from one copy of G to another, at the end of which if $v_i \in R_H$, then G_i has $\gamma(G)$ pebbles. Now each vertex in $R_{G\Box H}$ is in some copy of G and may be reached using the $\gamma(G)$ pebbles on it, so D is solvable for $R_{G\Box H}$.

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Within the usual concept of pebbling (t=1), given M pebbles on a vertex v we can always move $\lfloor \frac{M}{2} \rfloor$ pebbles to an adjacent vertex, possibly having to leave one pebble on v in the case when M is odd. The analogous statement holds for colored distributions.

Lemma 2.2. *Suppose a vertex v in the support of a t-colored distribution has $M > t$ pebbles. Given any integer $E \leq M - t$, at least $\lfloor E/2 \rfloor$ pebbles initially on v can be placed on an adjacent vertex using color-respecting steps.*

Proof. Consider the set T of all pebbles on the given vertex v. We will construct a subset S of size at least $M - t$ consisting of pebbles all of which can be placed in same-color pairs. If a color has an odd number of representatives in T remove one pebble of that color. As there are only t colors, at most t pebbles are removed. Let S be the subset of all remaining pebbles, $|S| \geq M - t$. Now by removing pebbles in pairs of the same color we can obtain a smaller set, also containing even numbers of pebbles of each color, of size E if E is even or of size $E-1$ if E is odd. Half of all pebbles in a given color can be moved to an adjacent vertex while discarding the other half. Thus we can move at least $\lfloor E/2 \rfloor$ pebbles to an adjacent vertex. $\square$

Proof of Theorem 1.2: To show that $\gamma(P_{G\square H}) \leq \gamma(P_G) \times \gamma(H)$, by Lemma 2.1, it suffices to show that every g-colored distribution on H with at least $\gamma(H)\gamma(P_G)$ pebbles is $\gamma(P_G)$-solvable, i. e. there is a sequence of pebbling steps at the end of which every vertex of H has at least $\gamma(P_G)$ pebbles. Note that $g < \gamma(G)$. Let D_H be such a distribution and assume that it is not $\gamma(P_G)$-solvable. We model the proof of this result on the proof of the main theorem in [**?**] but we need to acertain that the proof indeed carries over when colored distributions are considered.

The *value* of a pebble is the number of pebbles that have gone into it, as defined in [**?**]. In other words instead of thinking of a pebbling steps as movind one pebble and discarding one, we will think of a pebbling step as picking up two pebbles, joining them together by adding their values and then placing the new, bigger pebble on an adjacent vertex. Thus the total value of all pebbles remains the same throughout the pebbling game. As we only allow color preserving steps, all pebbles remain a single color.

Call a vertex in H *fat* if there are more than $\gamma(P_G)$ pebbles on it (regardless of their color), *thin* if there are fewer than $\gamma(R_G)$ pebbles on it and *perfect* if there are exactly $\gamma(P_G)$ pebbles on it. Let (f, t) be a pair of a fat and a thin vertex that are a minimum distance from each other amongst all such pairs and let $p_1, p_2, ..., p_n$ be the shortest path between them. Each vertex p_i has exactly $\gamma(P_G)$ pebbles on it. By Lemma 2.2, as $g < \gamma(P_G)$ we can move one pebble from f to p_1. Now p_1 is fat so we can move one pebble from p_1 to p_2 etc until finally t has an additional pebble and each of $f, p_1, .., p_n$ has one fewer pebbles. Then pick a new pair of a fat and thin vertex and repeat the precedure untill no fat vertices are left. This process always satisfies the condition that the value of each pebble never exceeds 2^d where d is the distance of the pebble from the nearest fat vertex.

Let v_j be the last fat vertex that survives the above recursive procedure. Then if a pebble p is on vertex v_i, $value(p) \leq 2^{dist(v_j,v_i)}$. As at the end of the game there are at most $\gamma(G)$ pebbles on each vertex of H and at least one vertex has strictly fewer than $\gamma(P_G)$ pebbles (as D was assumed to not be solvable), the initial number of pebbles in D must have been less than $\gamma(P_G) \times max_j\{\sum_{i=1}^{h} 2^{dist(v_j,v_i)}\}$ contradicting the hypothesis. In fact this proof shows that as in cover pebbling, to compute the color pebbling number of H it suffices to find a vertex v_j such that $\sum_{i=1}^{h} 2^{dist(w_j,w_i)}$ is maximal.

To prove the equality

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