Chvátal's conjecture for downsets of small rank

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Abstract

A starting point in the investigation of intersecting systems of subsets of a finite set is the elementary observation that the size of a family of pairwise intersecting subsets of a finite set $[n] = \{1, \ldots, n\}$, denoted by $2^{[n]}$, is at most 2^{n-1}, with one of the extremal structures being family comprised of all subsets of $[n]$ containing a fixed element, called as a *star*. A longstanding conjecture of Chvátal aims to generalize this simple observation for all *downsets* of $2^{[n]}$. In this note, we prove this conjecture for all downsets where every subset contains at most 3 elements.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ and let $2^{[n]}$ (resp. $\binom{[n]}{k}$) denote the family of all subsets (resp. r-sized subsets) of $[n]$. A set system containing sets of size r ($r \geq 1$) is called *r-uniform*. Additionally, let $\binom{[n]}{\leq r}$ be the family of all subsets of size at most r, for any $1 \leq r \leq n$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, call $\mathcal{F}$ a *downset* if $A \in \mathcal{F}$ and $B \subseteq A$ implies $B \in \mathcal{F}$. Denote by $\mathcal{F}^r$ those sets of $\mathcal{F}$ having size r. A family $\mathcal{F} \subseteq 2^{[n]}$ is called *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$. For any $\mathcal{F} \subseteq 2^{[n]}$, let $\mathcal{F}_x = \{A \in \mathcal{F} : x \in A\}$, called the $\mathcal{F}$-star centered at x. Call any $\mathcal{G} \subseteq \mathcal{F}_x$ a partial $\mathcal{F}$-star centered at x, and call x a *center* of such a family. As a family may have more than one center, we call the set of all centers of $\mathcal{G}$ the *head* of $\mathcal{G}$ — it equals the intersection of all the sets of $\mathcal{G}$.

A starting point in the study of intersecting set systems states that any intersecting set system on $[n]$ can contain at most 2^{n-1} subsets, as for any pair $(A, [n] \setminus A)$, where $A \subseteq [n]$, at most one can be in the intersecting family (see [1]). It is clear that the *star* is one of the structures that attains this maximum size. The seminal Erdős–Ko–Rado theorem [4] proves a similar, more non-trivial result for *uniform* set systems.

Theorem 1.1. *[4] Let $r \leq n/2$ and let $\mathcal{F} \subseteq \binom{[n]}{r}$ be intersecting. Then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. Furthermore, if $r < n/2$, then equality holds if and only if $\mathcal{F} = \binom{n}{r}_x$, for some $x \in [n]$.*

In this note, we consider a famous longstanding conjecture of Chvátal (see [3]), which deals with the "Erdős–Ko–Rado" property of downsets. Before we state the conjecture, we formulate the following definitions. For $\mathcal{F} \subseteq 2^{[n]}$ we set $\iota(\mathcal{F})$ to be the size of the largest intersecting subfamily of $\mathcal{F}$ and $\sigma(\mathcal{F}) = max_{x \in [n]} |\mathcal{F}_x|$.

Definition 1.2 (The EKR property)**.** *A set system $\mathcal{F} \subseteq 2^{[n]}$ is* EKR *if $\iota(\mathcal{F}) = \sigma(\mathcal{F})$. Moreover, $\mathcal{F}$ is* strictly *EKR if all of the largest intersecting subfamilies of $\mathcal{F}$ are $\mathcal{F}$-stars.*

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Conjecture 1.3. *[3] If $\mathcal{H} \subseteq 2^{[n]}$ is a downset, then $\mathcal{H}$ is EKR.*

There have been a handful of results confirming this conjecture. For example, the trivial case $\mathcal{H} = 2^{[n]}$ is mentioned in [1], and Theorem 1.1 implies the case for which $\mathcal{H} = \binom{[n]}{\leq k}$. Schonheim [8] solved the case for which the maximal elements of $\mathcal{H}$ share a common element, while Chvátal [3] handled the case for which the maximal sets of $\mathcal{H}$ can be partitioned into two sunflowers (see definition below), each with core size 1. In [3] is also found the case for compressed $\mathcal{H}$; Snevily [9] strengthened this to $\mathcal{H}$ being merely compressed with respect to some element (which also implies [8]). Miklos [7] (and later Wang [11]) verified the conjecture for $\mathcal{H}$ satisfying $\iota(\mathcal{H}) \geq |\mathcal{H}|/2$, and Stein [10] verified it for those $\mathcal{H}$ having m maximal sets, every $m-1$ of which form a sunflower. Most recently, Borg [2] solved a weighted generalization of [9].

In this paper, we prove Conjecture 1.3 for $\mathcal{H} \subseteq \binom{[n]}{\leq 3}$. We also prove a slightly weaker result, one that makes an additional assumption on the size of the maximum intersecting family in $\mathcal{H}$. The advantage of this assumption is that the proof becomes significantly simpler, and the technique, which employs the famous Sunflower Lemma of Erdős and Rado, could potentially be extended for downsets containing larger subsets.

Main Results

We verify Conjecture 1.3 for all downsets consisting of sets of size at most 3.

Theorem 1.4. *Let $\mathcal{H} \subseteq \binom{[n]}{\leq 3}$ be a downset. Then $\mathcal{H}$ is EKR. Moreover $\mathcal{H}$ is strictly EKR, unless one of the following holds.*

(1) *There is a subset $K \in \binom{[n]}{4}$ such that*
  - $\binom{K}{3} \subseteq \mathcal{H}$,
  - *for all $H \in \mathcal{H}$, $H \subseteq K$ or $K \cap H = \emptyset$, and*
  - *the largest star in $\mathcal{H}$ has size 7.*

(2) *There are subsets $K \in \binom{[n]}{3}$ and (possibly empty) $M \subseteq [n] \setminus K$, and a subfamily $\mathcal{Z} = \binom{K}{2} \cup \{Z \in \binom{K \cup M}{3} \mid |Z \cap K| = 2\} \subseteq \mathcal{H}$ such that either*
  - *$K \notin \mathcal{H}$ and the largest star in $\mathcal{H}$ has size $|\mathcal{Z}| = 3|M| + 3$, or*
  - *$K \in \mathcal{H}$ and the largest star in $\mathcal{H}$ has size $|\mathcal{Z}| + 1 = 3|M| + 4$.*

We also prove the following weaker result, which is significantly stronger than the result of [7] for subfamilies of $\binom{[n]}{\leq 3}$.

Theorem 1.5. *Let $\mathcal{H} \subseteq \binom{[n]}{\leq 3}$ be a downset, and let $\mathcal{I} \subseteq \mathcal{H}$ be a maximum intersecting family. If $|\mathcal{I}| \geq 31$, then $\mathcal{I}$ is a star. Hence $\mathcal{H}$ is EKR when $\iota(\mathcal{H}) \geq 31$.*

Of course, some intersecting family (in particular, some star) will be so large if $|\mathcal{H}| > 15n$ or $|\mathcal{H}^3| > 10n$, for example.

Our proofs use the notion of *Sunflowers*, including the famous *Sunflower Lemma* of Erdős and Rado [5], as well as a variant by Håstad, et al [6]. We state both the Sunflower Lemma and the variant below, after the following definitions.

Definition 1.6 (Covering Set and Covering Number)**.** *A set S is a* covering set *for a set system $\mathcal{F}$ if $S \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. The covering number of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the size of the smallest covering set of $\mathcal{F}$.*

Definition 1.7 (Sunflower)**.** *A* sunflower *with k petals and core C is a set system $\{S_1, \ldots, S_k\}$ such that for any $i \neq j$, $S_i \cap S_j = C$. The sets $S_i \setminus C$ are the petals of the sunflower, and must be non-empty. If $k = 1$ then we may choose C to be any proper subset of S_1.*

For a set system $\mathcal{F}$ and set Y, let $\mathcal{F}_Y = \{F \setminus Y : F \in \mathcal{F}, Y \subseteq F\}$.

Definition 1.8 (k-Flower)**.** *A k-flower with core C is a set system $\mathcal{F}$ with $\tau(\mathcal{F}_C) \geq k$.*

Theorem 1.9. *[5] If a family of sets $\mathcal{F}$ is r-uniform and $|\mathcal{F}| > r!(k-1)^r$ sets, then it contains a sunflower with k petals.*

We will use the following variant of Theorem 1.9.

Theorem 1.10. *[6] If $\mathcal{F}$ is r-uniform and $|\mathcal{F}| > (k-1)^r$, then $\mathcal{F}$ contains a k-flower.*

2 Proof of Theorem 1.4

Proof. Let $\mathcal{I}$ be an intersecting subfamily of $\mathcal{H}$ of maximum size. Our goal is to show that either $\mathcal{I}$ must be a star or otherwise that $\mathcal{H}$ contains a star of size equal to that of $\mathcal{I}$, and to characterize the cases for which the latter happens.

If $\mathcal{H}$ does not contain a set of size 3 then $\mathcal{I}$ is a star unless $|\mathcal{I}| = 3$ and $\mathcal{I} = \binom{K}{2}$ for some $K = \{x, y, z\}$. But then $\{\{x\}, \{x, y\}, \{x, z\}\} \subseteq \mathcal{H}_x$, and so $|\mathcal{I}| = |\mathcal{H}_x|$, which is case (2) of the theorem with $M = \emptyset$.

Thus we may assume that $\mathcal{H}$ contains a set of size 3 and, consequently, also contains a star of size 4. Therefore $|\mathcal{I}| \geq 4$. If $\mathcal{I}^1 \neq \emptyset$ or $|\mathcal{I}^2| \geq 4$ then $\mathcal{I}$ is a star and we are done; so we will assume that $\mathcal{I}^1 = \emptyset$ and $|\mathcal{I}^2| \leq 3$ (thus $\mathcal{I}^3 \neq \emptyset$). Our proof splits into cases, based on $|\mathcal{I}^2|$.

We first introduce some notation that we make use of below. Without loss of generality $\bigcup_{I \in \mathcal{I}^2} I = [m]$ for some $m \leq 4$. For $\emptyset \neq J \subset [m]$ define $\bar{J} = [m] \setminus J$, $\mathcal{A}(J) = \{I \in \mathcal{I}^3 \mid I \cap [m] = J\}$, and $C(J) = (\bigcup_{A \in \mathcal{A}(J)} A) \setminus J$. In practice, we relax the notation somewhat to write $\mathcal{A}(2,3)$ instead of $\mathcal{A}(\{2,3\})$, and $C(\bar{2})$ instead of $C(\overline{\{2\}})$, for example. Note that, when $m = 3$, $|C(\bar{i})| = |\mathcal{A}(\bar{i})|$ and $\mathcal{I}^3 \setminus \bigcup_{i \in [3]} \mathcal{A}(\bar{i}) \subseteq \{[3]\}$.

2.1 $|\mathcal{I}^2| = 3$

2.1.1 $\mathcal{I}^2$ is a star

We may assume that $\mathcal{I}^2 = \{\{1,2\}, \{1,3\}, \{1,4\}\}$. If $\mathcal{I}^3 = \mathcal{I}^3_1$, then $\mathcal{I}$ is a star. Otherwise, we must have $\mathcal{I}^3 \setminus \mathcal{I}^3_1 = \{\{2,3,4\}\}$. Therefore $(\mathcal{I} \setminus \{\{2,3,4\}\}) \cup \{\{1\}\} \cup \{\{1,j\} \mid j \in I \in \mathcal{I}^3_1\}$ is a star subfamily of $\mathcal{H}$ that has size at least $|\mathcal{I}|$ and, in fact, is larger unless $I \subseteq [4]$ for every $I \in \mathcal{I}^3_1$. Therefore we must have that $|\mathcal{I}^3_1| \leq 3$.

If $|\mathcal{I}^3_1| < 3$ then, without loss of generality, $\mathcal{I}^3_1 \subseteq \{\{1,2,3\}, \{1,2,4\}\}$, and then $(\mathcal{I} \setminus \{\{1,3\}, \{1,4\}\}) \cup \{\{2\}, \{2,3\}, \{2,4\}\}$ is a larger intersecting subfamily of $\mathcal{H}$, a contradiction. So we are left with the case in which $|\mathcal{I}^3_1| = 3$ and, consequently, $|\mathcal{I}| = 7$ and $\mathcal{H} \supseteq \binom{[4]}{\leq 3}$.

If there is an $H \in \mathcal{H}$ such that both $H \cap [4] \neq \emptyset$ and $H \setminus [4] \neq \emptyset$ then, by taking $h \in H \cap [4]$, we have that $\binom{[4]}{\leq 3}_h \cup \{H\}$ is a star in $\mathcal{H}$ of size $8 > 7$, a contradiction. Hence there is no such H, which is case (1) of the theorem.

2.1.2 $\mathcal{I}^2$ is a triangle

We may assume that $\mathcal{I}^2 = \{\{1,2\}, \{1,3\}, \{2,3\}\}$.

Relabel, if necessary, so that $0 \leq |C(\bar{1})| \leq |C(\bar{2})| \leq |C(\bar{3})|$. Then $(\mathcal{I} \setminus (\mathcal{A}(\bar{1}) \cup \{\{2,3\}\})) \cup \{\{1,s\} \mid s \in C(\bar{3})\} \cup \{\{1\}\}$ is a star subfamily of $\mathcal{H}$ of size $|\mathcal{I}| + |cA(\bar{3})| - |\mathcal{A}(\bar{1})| \geq |\mathcal{I}|$, and so $\mathcal{H}$ is EKR, and strictly so unless $|C(\bar{1})| = |C(\bar{2})| = |C(\bar{3})|$, which we now assume.

If not all the sets $C(\bar{i})$ are the same then, without loss of generality say $C(\bar{1}) \neq C(\bar{2})$, and so $|C(\bar{1}) \cup C(\bar{2})| > |C(\bar{3})|$. Then $(\mathcal{I} \setminus (\mathcal{A}(\bar{3}) \cup \{\{1,2\}\})) \cup \{\{3,s\} \mid s \in C(\bar{1}) \cup C(\bar{2})\} \cup \{\{3\}\}$ is a star subfamily of $\mathcal{H}$ of size $|\mathcal{I}| + |A(\bar{1}) \cup A(\bar{2})| - |\mathcal{A}(\bar{3})| > |\mathcal{I}|$, a contradiction.

Finally, if $C(\bar{1}) = C(\bar{2}) = C(\bar{3})$ then $|\mathcal{H}_1| = |\mathcal{I}|$, so $\mathcal{H}$ is EKR, but not strictly so, giving us case (2) of the theorem.

2.2 $|\mathcal{I}^2| = 2$

We may assume that $\mathcal{I}^2 = \{\{1,2\},\{1,3\}\}$. For each $I \in \mathcal{I}^3$ we must have $1 \in I$ or $\{2,3\} \subset I$. If $I \in \mathcal{I}^3 \setminus (\mathcal{A}(1) \cup \mathcal{A}(2,3))$, then $1 \in I$ and $\{2,3\} \cap I \neq \emptyset$.

If $\mathcal{A}(2,3) = \emptyset$, then $\mathcal{I}$ is a star, so we assume that $\mathcal{A}(2,3) \neq \emptyset$. It must be that $\mathcal{A}(1) \neq \emptyset$, since otherwise $\mathcal{I} \cup \{\{2,3\}\}$ would be a larger intersecting subfamily of $\mathcal{H}$, a contradiction.

Fix an $A \in \mathcal{A}(1)$; then for each $k \in C(2,3)$ we must have that $k \in A$. Thus $|C(2,3)| \leq |A \setminus \{1\}| = 2 \leq |C(1)|$. Hence we have that $(\mathcal{I} \setminus \mathcal{A}(2,3)) \cup \{\{1\}\} \cup \{\{1,i\} \mid i \in C(1)\}$ is a star of size $|\mathcal{I}| + |C(1)| - |\mathcal{A}(2,3)| + 1 > |\mathcal{I}|$, a contradiction.

2.3 $|\mathcal{I}^2| = 1$

We may assume that $\mathcal{I}^2 = \{\{1,2\}\}$. Without loss of generality, both of $\mathcal{A}(1), \mathcal{A}(2)$ are nonempty (otherwise $\mathcal{I}$ is a star and we are done). If, for some $i \in \{1,2\}$, we have that $|\mathcal{A}(i)| \leq |\mathcal{A}(1,2)|$ then $(\mathcal{I} \cup \{A \setminus \{i\} \mid A \in \mathcal{A}(1,2)\} \cup \{\{1,2\} \setminus \{i\}\}) \setminus \mathcal{A}(i)$ is a star-subfamily of $\mathcal{H}$ of size larger than $\mathcal{I}$, which is a contradiction. Thus we know that $|\mathcal{A}(1,2)| < \min(|\mathcal{A}(1)|, |\mathcal{A}(2)|)$.

If $|\mathcal{A}(1)| = |\mathcal{A}(2)| = 1$, then $\mathcal{A}(1,2) = \emptyset$, and $(\mathcal{I} \cup \{\{1,j\} : j \in C(1)\} \cup \{\{1\}\}) \setminus \mathcal{A}(2)$ is a star subfamily of size larger that $\mathcal{I}$, a contradiction., so we may assume without loss of generality that $|\mathcal{A}(1)| \geq 2$.

For $i \in \{1,2\}$, set $\mathcal{A}'(i) = \{A \setminus \{i\} \mid A \in \mathcal{A}(i)\}$; then $|\mathcal{A}'(i)| = |\mathcal{A}(i)|$. Clearly $\mathcal{A}'(1)$ and $\mathcal{A}'(2)$ cross-intersect. If, for some $i \in \{1,2\}$, $\mathcal{A}'(i)$ is an intersecting family then $\mathcal{I} \cup \mathcal{A}'(i) \setminus \mathcal{A}(1,2)$ is an intersecting subfamily of $\mathcal{H}$ that is larger that $\mathcal{I}$, a contradiction, so we have that neither $\mathcal{A}'(1)$ nor $\mathcal{A}'(2)$ is intersecting. Since $\mathcal{A}'(1)$ is not intersecting and $|\mathcal{A}'(1)| \geq 2$, we may assume (by relabeling, if necessary) that $\{\{3,4\},\{5,6\}\} \subseteq \mathcal{A}'(1)$. Because $\mathcal{A}'(2)$ cross-intersects $\mathcal{A}'(1)$ we have $\mathcal{A}'(2) \subseteq \{\{3,5\},\{3,6\},\{4,5\},\{4,6\}\}$. In particular, $|\mathcal{A}(2)| = |\mathcal{A}'(2)| \leq 4$ and, for each $x \in \{3,4,5,6\}$, $\{1,x\}$ is a subset of some set in $\mathcal{A}(1)$. But then $(\mathcal{I} \setminus \mathcal{A}(2)) \cup \{\{1,x\} \mid x \in \{3,4,5,6\}\} \cup \{\{1\}\}$ is an intersecting subfamily of $\mathcal{H}$ that is larger than $\mathcal{I}$, a contradiction.

2.4 $|\mathcal{I}^2| = 0$

Here $\mathcal{I}^2 = \emptyset$ and $\mathcal{I}$ is an intersecting family of 3-sets such that no 2-subset of $[n]$ is contained in every element of $\mathcal{I}$ (otherwise that 2-subset could be added to $\mathcal{I}$).

Let $\mathcal{S}$ be the largest star in $\mathcal{I}$ (clearly $|\mathcal{S}| \geq 2$), and let D be the head of $\mathcal{S}$. If $\mathcal{S} = \mathcal{I}$ then we are done, so define $\mathcal{R} = \mathcal{I} \setminus \mathcal{S}$ and assume that $\mathcal{R} \neq \emptyset$. In particular, for every $R \in \mathcal{R}$ we must have that $R \cap D = \emptyset$; otherwise R could be added to $\mathcal{S}$ to create a larger star. If $|D| \geq 2$ then for any $d \in D$ we have that $\mathcal{I} \cup \{S \setminus \{d\} \mid S \in \mathcal{S}\}$ is a larger intersecting subfamily of $\mathcal{H}$ than $\mathcal{I}$, a contradiction. Therefore $|D| = 1$ and, without loss of generality, $D = \{1\}$.

Let $\mathcal{F}$ be the largest sunflower in $\mathcal{S}$ with core $\{1\}$. Since any $R \in \mathcal{R}$ must intersect every $F \in \mathcal{F}$, we must have that $|\mathcal{F}| \leq 3$. If $|\mathcal{F}| = 1$, then $\{S \setminus \{1\} \mid S \in \mathcal{S}\}$ forms an intersecting family, and from the fact that $|\mathcal{S}| \geq 2$ and $D = \{1\}$, we have that $\mathcal{S} = \{\{1, a, b\}, \{1, a, c\}, \{1, b, c\}\}$ for three different numbers a, b, c. Moreover, we must have $|R \cap \{a, b, c\}| \geq 2$ for every $R \in \mathcal{R}$. This means that $\mathcal{I} \cup \{\{a, b\}\}$ is a larger intersecting subfamily of $\mathcal{H}$ than $\mathcal{I}$, a contradiction. Therefore $2 \leq |\mathcal{F}| \leq 3$.

Let $X = \left(\bigcup_{F \in \mathcal{F}} F\right)$; then $|X| = 2|\mathcal{F}| + 1$. Denote $X^* = X \setminus \{1\}$. Define $Y = \left(\bigcup_{S \in \mathcal{S}} S\right) \setminus X$ and set $\mathcal{S}(Y) = \{S \in \mathcal{S} \mid S \cap Y \neq \emptyset\}$. Then we must have that, for all $y \in Y$, there is an $S \in \mathcal{S}(Y)$ such that $\{1, y\} \subseteq S$ and, for all $x \in X$ (including $x = 1$), there is an $F \in \mathcal{F}$ such that $\{1, x\} \subseteq F$. If $|X \cup Y| = |X| + |Y| = 2|\mathcal{F}| + |Y| + 1 > |\mathcal{R}|$, then $\mathcal{S} \cup \{\{1, k\} \mid k \in X \cup Y\}$ is a star subfamily of $\mathcal{H}$ of size larger than $\mathcal{I}$, a contradiction. So in the rest we assume that $|\mathcal{R}| \geq 2|\mathcal{F}| + |Y| + 1$.

2.4.1 $|\mathcal{F}| = 3$

Without loss of generality, $\mathcal{F} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}\}$. Set $\mathcal{E}$ to be the family of 3-element subsets of X^* that intersect each of $\{2, 3\}, \{4, 5\}, \{6, 7\}$. Then $|\mathcal{E}| = 8$ and $\mathcal{R} \subseteq \mathcal{E}$. However, if $R \in \mathcal{R}$ then $X^* \setminus R \in \mathcal{E} \setminus \mathcal{R}$, and so $|\mathcal{R}| \leq 4 < 7 \leq 2|\mathcal{F}| + |Y| + 1$, a contradiction.

2.4.2 $|\mathcal{F}| = 2$

Without loss of generality, $\mathcal{F} = \{\{1, 2, 3\}, \{1, 4, 5\}\}$. We have $|\mathcal{R}| \geq |Y| + 5$.

Define $\mathcal{S}^* = \mathcal{S} \setminus (\mathcal{F} \cup \mathcal{S}(Y))$. Clearly, $\sum_{x \in X^*} |\mathcal{S}^*_x| = 2|\mathcal{S}^*|$, and $\mathcal{S}^* \subseteq \{\{1, i, j\} \in \mathcal{S} \mid i \in \{2, 3\}, j \in \{4, 5\}\}$. Denote $\mathcal{R}^* = \{R \in \mathcal{R} \mid R \subseteq X^*\}$. Since $|\mathcal{R}^*| \leq 4 < |Y| + 5$, we know that $\mathcal{R} \setminus \mathcal{R}^* \neq \emptyset$.

For each $x \in X^*$ we set $\hat{x}$ to be the integer and C_x to be the 2-set such that $\{\{x, \hat{x}\}, C_x\} = \{\{2, 3\}, \{4, 5\}\}$ (so, in particular, $C_x = C_{\hat{x}}$). Also define $Y_x = \{y \in Y \mid \{1, x, y\} \in \mathcal{S}\}$. For $i \in \{2, 3\}$ and $j \in \{4, 5\}$ let $\mathcal{R}(i, j) = \{R \in \mathcal{R} \mid R \cap X^* = \{i, j\}\}$ and let $R(i, j) = \{y \mid \{i, j, y\} \in \mathcal{R}(i, j)\}$. Note that $R(i, j) \subseteq Y$. The following properties are easy to see.

- **P.1** The collection $\{\mathcal{R}(i, j) \mid i \in \{2, 3\}, j \in \{4, 5\}\}$ partitions $\mathcal{R} \setminus \mathcal{R}^*$; in particular, at least one of these sets is nonempty.
- **P.2** If $\{1, \hat{i}, \hat{j}\} \in \mathcal{S}^*$ then $\mathcal{R}(i, j) = \emptyset$. (Since no element of $\mathcal{R}(i, j)$ intersects $\{1, \hat{i}, \hat{j}\}$.)
- **P.3** $|\mathcal{S}^*| \leq 3$. (This follows from P.1 and P.2.)
- **P.4** If $\min(|R(i, j)|, |R(\hat{i}, \hat{j})|) \geq 1$ then $R(i, j) = R(\hat{i}, \hat{j})$ with $|R(i, j)| = 1$. Therefore if $\min(|\mathcal{R}(i, j)|, |\mathcal{R}(\hat{i}, \hat{j})|) \geq 1$ then $|\mathcal{R}(i, j)| = |\mathcal{R}(\hat{i}, \hat{j})|) = 1$. (Since elements of $\mathcal{R}(i, j)$ and $\mathcal{R}(\hat{i}, \hat{j})$ can intersect in at most one element.)
- **P.5** If $X^* \setminus \{x\} \in \mathcal{R}^*$ then $Y_x = \emptyset$. (Since $X^* \setminus \{x\}$ does not intersect sets of the form $\{1, x, y\}$ for $y \in Y$.)
- **P.6** If $y \in Y_x$ then, for $j \in C_x$, we have $\mathcal{R}(\hat{x}, j) \subseteq \{\{\hat{x}, j, y\}\}$. (Since $\{1, x, y\} \in \mathcal{S}$.)
- **P.7** If $|Y_x| \geq 2$ then $\bigcup_{j \in C_x} \mathcal{R}(\hat{x}, j) = \emptyset$. (This follows from P.6.)
- **P.8** Since $\mathcal{R}^* \neq \mathcal{R}$, we have $\min(|Y_x|, |Y_{\hat{x}}|) \leq 1$ for every $x \in X^*$. (This follows from P.1 and P.7.)
- **P.9** If $|\mathcal{S}^*| = 3$ then $\mathcal{S}^*_x \neq \emptyset$ for all $x \in X^*$.

If, for some $x \in X^*$, we have $|Y_x| \geq 2$, and $|Y_{\hat{x}}| = 1$, then (from P.1, P.6, and P.7) $|\mathcal{R} \setminus \mathcal{R}^*| \leq 2$ and (from P.5) $|\mathcal{R}^*| \leq 2$. But this means that $|\mathcal{R}| \leq 4 < 5 + |Y|$, a contradiction. Therefore we know that if $|Y_x| \geq 2$ then $Y_{\hat{x}} = \emptyset$.

If, for some $x \in X^*$, we have $|Y_x| = |Y_{\hat{x}}| = 1$ (we may assume by relabeling, if necessary, that $x = 2$, so $\hat{x} = 3$), then (from P.1 and P.6) $|\mathcal{R} \setminus \mathcal{R}^*| \leq 4$ and (from P.5) $|\mathcal{R}^*| \leq 2$. Therefore, from $Y \neq \emptyset$, we get $|\mathcal{R}| \leq 6 \leq 5 + |Y|$, therefore $|\mathcal{R}| = 5 + |Y|$ and $|Y| = 1$. Without loss of generality $Y_2 = Y_3 = Y = \{6\}$. Also, $|\mathcal{R}^*| = 2$, and $\mathcal{R}^* = \{\{2, 3, 5\}, \{2, 3, 4\}\}$ and (from P.5) $Y_4 = Y_5 = \emptyset$; consequently $\mathcal{S}(Y) = \{\{1, 2, 6\}, \{1, 3, 6\}\}$. Moreover (using P.6), from $|\mathcal{R} \setminus \mathcal{R}^*| = 4$ we get that, for each $i \in \{2, 3\}$ and $j \in \{4, 5\}$, we have $\mathcal{R}(i, j) = \{\{i, j, 6\}\}$. Thus (from P.2) $\mathcal{S}^* = \emptyset$. But this yields $|\mathcal{I}_2| = 5 > 4 = |\mathcal{S}|$, a contradiction.

Therefore we can now assume, for all $x \in X^*$, that $\min(|Y_x|, |Y_{\hat{x}}|) = 0$. Set $L = \{x \in X^* \mid Y_x \neq \emptyset\}$. Then we have that $|L| \leq 1$ or $L = \{i, j\}$ for some $i \in \{2, 3\}$ and $j \in \{4, 5\}$. For each $x \in X^*$ we have that

$$\Big(|\mathcal{F}_x| + |Y_x| + |\mathcal{S}_x^*|\Big) + \Big(\sum_{j \in C_x} |\mathcal{R}(x, j)| + |\mathcal{R}_x^*|\Big) \;\leq\; |\mathcal{I}_x| \;\leq\; |\mathcal{S}| \;, \tag{1}$$

where we have counted the sets containing 1 before those not containing 1. Of course, $|\mathcal{F}_x| = 1$ and $|S_x^*| \leq 2$. By summing over X^*, we obtain

$$4 + \sum_{x \in X^*} |Y_x| + 2|\mathcal{S}^*| + 2\left(\sum_{i \in \{2,3\}} \sum_{j \in \{4,5\}} |\mathcal{R}(i, j)|\right) + 3|\mathcal{R}^*| \leq 4|\mathcal{S}| \;,$$

which simplifies to

$$4 + \sum_{x \in X^*} |Y_x| + 2|\mathcal{S}^*| + 2|\mathcal{R}| + |\mathcal{R}^*| \leq 4|\mathcal{S}| \;.$$

In particular,

$$|\mathcal{R}| \leq 2|\mathcal{S}| - 2 - \frac{1}{2} \sum_{x \in X^*} |Y_x| - |\mathcal{S}^*| - \frac{1}{2}|\mathcal{R}^*| \;. \tag{2}$$

Now we consider the following three subcases, based on the size of L.

Case $L = \emptyset$

Then each $Y_x = Y = \emptyset$, $\mathcal{S}(Y) = \emptyset$, $|\mathcal{S}| = |\mathcal{S}^*| + 2$, and $|\mathcal{R}| \geq 5 + |Y| = 5$. Using P.3, equation (2) becomes

$$|\mathcal{R}| \leq 2 + |\mathcal{S}^*| - \frac{1}{2}|\mathcal{R}^*| \leq 2 + 3 - 0 = 5.$$

Thus $|\mathcal{R}| = 5$, $|\mathcal{S}^*| = 3$, $\mathcal{R}^* = \emptyset$, $|\mathcal{S}| = 5$, and all inequalities hold with equality in inequality (1).

We may assume, by relabeling, if necessary, that $\mathcal{S}^* = \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}\}$. Then inequality (1) implies that, for $i \in \{2, 4\}$, we have $\sum_{j \in C_i} |\mathcal{R}(i, j)| = 2$, and for $i \in \{3, 5\}$ we have $\sum_{j \in C_i} |\mathcal{R}(i, j)| = 3$. This means that $|\mathcal{R}(3, 5)| - |\mathcal{R}(2, 4)| = 1$ and, therefore, $|R(3, 5)| > |R(2, 4)|$. Hence (using P.4), we must have that $\mathcal{R}(2, 4) = \emptyset$ and $|\mathcal{R}(3, 5)| = 1$. This means that $|\mathcal{R}(2, 5)| = |\mathcal{R}(3, 4)| = 2$, which is a contradiction by P.4.

Case $|L| = 1$

Here we may assume that $L = \{2\}$. Then $Y = Y_2$, $|\mathcal{S}| = 2 + |\mathcal{S}^*| + |Y|$, $\{3, 4, 5\} \notin \mathcal{R}^*$ (from P.5, and from P.6) $|\mathcal{R}(3, j)| \leq 1$ for each $j \in \{4, 5\}$. Using P.3, equation (2) becomes

$$|\mathcal{R}| \leq 2 + |\mathcal{S}^*| + \frac{3}{2}|Y| - \frac{1}{2}|\mathcal{R}^*| \leq 5 + \frac{3}{2}|Y| \;.$$

1. $|Y| = |Y_2| = 1$

   Since $5 + |Y| = 6 \le |\mathcal{R}| \le 6 + \frac{1}{2} - \frac{1}{2}|\mathcal{R}^*|$, we get $|\mathcal{R}| = 6$, $|\mathcal{S}^*| = 3$, and $|\mathcal{R}^*| \le 1$. From P.2 we know that there are $i \in \{2,3\}$ and $j \in \{4,5\}$ such that, if $\{\ell, m\} \ne \{i,j\}$ then $\mathcal{R}(\ell, m) = \emptyset$. Since $\mathcal{R} \setminus \mathcal{R}^* = \mathcal{R}(i,j)$, this implies that $|\mathcal{R}(i,j)| \ge 5$. Moreover, since $|\mathcal{R}(3,k)| \le 1$ for each $k \in \{4,5\}$, we have $i = 2$. Then, using inequality (1), we obtain $7 \le 1 + |Y_2| + |\mathcal{S}_2^*| + \sum_{k \in C_2} |\mathcal{R}(2,k)| + |\mathcal{R}_2^*| \le |\mathcal{S}| = 6$, a contradiction.

2. $|Y| = |Y_2| \ge 2$

   Here we have $\mathcal{R}(3,4) = \mathcal{R}(3,5) = \emptyset$, and so (since $\{3,4,5\} \notin \mathcal{R}^*$) we get that $\sum_{j \in C_2} |\mathcal{R}(2,j)| + |\mathcal{R}_2^*| = |\mathcal{R}|$. Using inequality (1) with $x = 2$ yields $1 + |Y| + |\mathcal{S}_2^*| + |\mathcal{R}| \le 2 + |\mathcal{S}^*| + |Y|$. In other words, $|\mathcal{R}| \le 1 + |\mathcal{S}^*| - |\mathcal{S}_2^*| < 5$, a contradiction.

Case $|L| = 2$

We may assume that $L = \{2,4\}$. Then $|\mathcal{S}| = |\mathcal{S}^*| + 2 + |Y_2| + |Y_4|$ and $Y = Y_2 \cup Y_4$. From (P.5) we have $\mathcal{R}^* \subseteq \{\{2,3,4\},\{2,4,5\}\}$. We need only consider cases for which $|\mathcal{R}| \ge 5 + |Y|$.

1. $|Y_2|, |Y_4| \ge 2$

   From (P.7) we know that $\mathcal{R} \setminus \mathcal{R}^* = \mathcal{R}(2,4)$. Using inequality (1) with $\{i,j\} = \{2,4\}$ and the fact that $5 + |Y| \le |\mathcal{R}|$, we get $|Y_i| + |S_i^*| + 6 + |Y| \le 1 + |Y_i| + |\mathcal{S}_i^*| + |\mathcal{R}(2,4)| + |\mathcal{R}^*| \le |\mathcal{S}| = |\mathcal{S}^*| + 2 + |Y_i| + |Y_j|$, which gives, for each $j \in \{2,4\}$, that $|Y| \le |\mathcal{S}^*| - 4 + |Y_j| < |Y_j|$, a contradiction.

2. $|Y_2| = 1$ and $|Y_4| \ge 2$ (the case $|Y_4| = 1$ and $|Y_2| \ge 2$ is handled symmetrically)

   Without loss of generality, $Y_2 = \{6\}$. From (P.7) we have $\mathcal{R}(2,5) = \mathcal{R}(3,5) = \emptyset$, and from (P.6) we know that $\mathcal{R}(3,4) \subseteq \{\{3,4,6\}\}$ and $\mathcal{R} \setminus \mathcal{R}^* = \mathcal{R}(3,4) \cup \mathcal{R}(2,4)$. Thus, $\mathcal{R}_4 = \mathcal{R}$. Also, $\mathcal{S}^* \subseteq \{\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\}\}$. Set $\mathcal{P} = \mathcal{S}^* \setminus \mathcal{S}_4^*$. Since $\mathcal{R}(2,4) \cup \mathcal{R}(3,4) = \mathcal{R} \setminus \mathcal{R}^* \ne \emptyset$, we get from (P.2) that $|\mathcal{P}| \le 1$. Thus, $\mathcal{I}_4 = \mathcal{I} \setminus (\{\{1,2,6\},\{1,2,3\}\} \cup \mathcal{P})$, so $|\mathcal{I}| \le |\mathcal{I}_4| + 3$. Therefore the family $\mathcal{I}_4 \cup \{\{4\},\{1,4\}\} \cup \{\{4,y\} : y \in Y_4\}$ is a star subfamily of $\mathcal{H}$ of size $|\mathcal{I}_4| + 2 + |Y_4| \ge |\mathcal{I}_4| + 4 > |\mathcal{I}|$, a contradiction.

3. $|Y_2| = |Y_4| = 1$ (so $1 \le |Y| \le 2$)

   In this case $|\mathcal{S}| = 4 + |\mathcal{S}^*|$ and $\mathcal{R} \subseteq \{\{2,3,4\},\{2,4,5\}\}$

   (a) $Y_2 = Y_4 = Y$

   Here we have $|Y| = 1$ so, without loss of generality, $Y = \{6\}$. Then from P.6 we learn that $\mathcal{R}(2,5) \subseteq \{\{2,5,6\}\}$, $\mathcal{R}(3,5) \subseteq \{\{3,5,6\}\}$ and $\mathcal{R}(3,4) \subseteq \{\{3,4,6\}\}$. Therefore $4 = 6 - 2 \le |\mathcal{R}| - |\mathcal{R}^*| = |\mathcal{R} \setminus \mathcal{R}^*| \le 3 + |\mathcal{R}(2,4)|$, and so $|\mathcal{R}(2,4)| \ge 1$. By P.2 we know that $\{1,3,5\} \notin \mathcal{S}^*$, and thus $\mathcal{S}^* \subseteq \mathcal{S}_2^* \cup \{\{1,3,4\}\}$.

i. $|\mathcal{R}(2,4)| > 1$
By P.4 this implies that $\mathcal{R}(3,5) = \emptyset$ and, hence, for $i \in \{2,4\}$, that $\sum_{k\in C_i} |\mathcal{R}(i,k)| \geq |\mathcal{R} \setminus \mathcal{R}^*| - 1$. Using equation (1) we get that $\sum_{k\in C_2} |\mathcal{R}(2,k)| + |\mathcal{R}^*| + 2 + |\mathcal{S}_2^*| \leq 4 + |\mathcal{S}^*|$. Since $7+|\mathcal{S}_2^*| \leq |\mathcal{R}|+1+|\mathcal{S}_2^*| \leq \sum_{k\in C_2} |\mathcal{R}(2,k)|+|\mathcal{R}^*|+2+|\mathcal{S}_2^*|$, we obtain $3+|\mathcal{S}_2^*| \leq |\mathcal{S}^*| \leq 3$. This implies that $S_2^* = \emptyset$ and $|S^*| = 3$, contradicting P.9.

ii. $|\mathcal{R}(2,4)| = 1$

Now we have (from P.4) that $|\mathcal{R}(3,5)| \leq 1$, so we know that $|\mathcal{R}(i,j)| \leq 1$ for every $i \in \{2,3\}, j \in \{4,5\}$. Then $|\mathcal{R} \setminus \mathcal{R}^*| \geq 4$ implies that $|\mathcal{R}(i,j)| = 1$, $|\mathcal{R} \setminus \mathcal{R}^*| = 4$, and $\mathcal{R}^* = \{X_3', X_5'\}$. Using equation (1) with $x \in \{2,4\}$, we get that $7 + |\mathcal{S}_x^*| \leq 4 + |\mathcal{S}^*|$, and so $3 \leq |\mathcal{S}^*| - |\mathcal{S}_x^*| \leq 3$, a contradiction.

(b) $Y_2 \neq Y_4$

Here we have $|Y| = 2$ and, without loss of generality, $Y_2 = \{6\}$ and $Y_4 = \{7\}$. From P.6 we get that $\mathcal{R}(3,j) \subseteq \{\{3,j,6\}\}$ for each $j \in \{4,5\}$ and $\mathcal{R}(i,5) \subseteq \{\{i,5,7\}\}$ for each $i \in \{2,3\}$. This implies, in particular, that $\mathcal{R}(3,5) = \emptyset$. Thus, for $i \in \{2,4\}$, we have $\sum_{k\in C_i} |\mathcal{R}(i,k)| \geq |\mathcal{R} \setminus \mathcal{R}^*| - 1$. In particular,

$$7 \leq |\mathcal{R}| \leq \sum_{k\in C_2} |\mathcal{R}(2,k)| + |\mathcal{R}^*| + 1 \leq \sum_{k\in C_2} |\mathcal{R}(2,k)| + |\mathcal{R}_2^*| + 2 \ .$$

Using inequality (1) with $x = 2$ we get that

$$2 + |\mathcal{S}_2^*| + \sum_{k\in C_2} |\mathcal{R}(2,k)| + |\mathcal{R}_2^*| \leq |\mathcal{S}^*| + 4 \ .$$

Together, these imply that $3+|\mathcal{S}_2^*| \leq |\mathcal{S}^*| \leq 3$, and so $|S^*| = 3$ and $|S_2^*| = 0$, which contradicts P.9.

This completes the proof. $\square$

3 Proof of Theorem 1.5

We now proceed to a proof of Theorem 1.5.

Proof. Let $\mathcal{I}_i = \mathcal{I} \cap \binom{[n]}{i}$, for $i = 1,2,3$. We can assume $\mathcal{I}_1 = \emptyset$, since otherwise, $\mathcal{I}$ is a star. Similarly, we can assume $|\mathcal{I}_2| \leq 3$. Thus, we have $|\mathcal{I}_3| \geq 28$. Since $28 = (4-1)^3 + 1$, we can use Theorem 1.10 to conclude that $\mathcal{I}_3$ contains a 4-flower. Let $k \geq 4$ be maximum such that $\mathcal{S}$ is a k-flower in $\mathcal{I}_3$, and let C be the core of $\mathcal{S}$. As $\mathcal{I}_3$ is 3-uniform and intersecting, every subfamily $\mathcal{G} \subseteq \mathcal{I}$ has $\tau(\mathcal{G}) \leq 3$, which implies that $C \neq \emptyset$. Suppose first that $C = \{a\}$, and suppose $\mathcal{I}$ is not a star centered at a. Let $A \in \mathcal{I}$ be such that $a \notin A$. Consider the family $\mathcal{S}_C$. As $\tau(\mathcal{S}_C) \geq 4$, there exists some $S_1 \in \mathcal{S}_C$ such that $A \cap S_1 = \emptyset$. Consequently, if $S' = S_1 \cup \{a\}$, then $S' \in \mathcal{I}$ and $A \cap S' = \emptyset$, a contradiction. As a result, we may assume that $C = \{a,b\}$. This implies that $\mathcal{S}_C$ is a family of singletons. Consequently, $\mathcal{S}$ is a sunflower with at least 4 petals.[1] Additionally, for every $A \in \mathcal{I}_3$, $A \cap \{a,b\} \neq \emptyset$.

Let $\mathcal{A} = \{A \in \mathcal{I}_3 : A \cap C = \{a\}\}$, and let $\mathcal{B} = \{B \in \mathcal{I}_3 : B \cap C = \{b\}\}$. We have $|\mathcal{I}_3| = |S| + |\mathcal{A}| + |\mathcal{B}|$. Let $\mathcal{A}' = \{A - \{a\} : A \in \mathcal{A}\}$, and $\mathcal{B}' = \{B - \{b\} : B \in \mathcal{B}\}$. If $\mathcal{A}' = \emptyset$ or $\mathcal{B}' = \emptyset$, we can conclude that $\mathcal{I}_3$, and

[1] Note that every sunflower with k petals is a k-flower, but the converse is not always true.

hence, $\mathcal{I}$ is a star (centered at either a or b), so suppose both are non-empty. Since $\mathcal{I}$ is intersecting, $\mathcal{A}'$ and $\mathcal{B}'$ are cross-intersecting families, i.e. for any $A \in \mathcal{A}'$ and $B \in \mathcal{B}'$, $A \cap B \neq \emptyset$. Let $V(\mathcal{A}')$ and $V(\mathcal{B}')$ be the vertex sets of $\mathcal{A}'$ and $\mathcal{B}'$ respectively, and let $n(\mathcal{X}) = |V(\mathcal{X})|$ for $\mathcal{X} \in \{\mathcal{A}', \mathcal{B}'\}$. We first prove the following claims.

Claim 3.1. *If both $\mathcal{A}'$ and $\mathcal{B}'$ are intersecting, or $|\mathcal{A}'| \geq 2$ and $|\mathcal{B}'| \geq 2$, then, $|\mathcal{X}| \leq 2 + n(\mathcal{X})$ for each $\mathcal{X} \in \{\mathcal{A}', \mathcal{B}'\}$.*

Proof. If $\mathcal{A}'$ is intersecting, it is either a triangle, or a star. In either case, the bound follows trivially. A similar argument works for $\mathcal{B}'$, so suppose, without loss of generality that $\mathcal{A}'$ has two disjoint edges, say $\{xy, x'y'\}$. $\mathcal{B}' \subseteq \{xy', y'y, yx', x'x\}$, giving the required bound for $\mathcal{B}'$. Now, if $\mathcal{B}$ has two disjoint edges, we can use a similar argument for $\mathcal{A}'$, so suppose $\mathcal{B}'$ is intersecting. Without loss of generality, suppose $\mathcal{B}' = \{xy', y'y\}$. Then $\mathcal{A}' \subseteq \{xy, x'y'\} \cup \{A \in \binom{[n]}{2} : y' \in A\}$, giving the bound $|n(\mathcal{A}')| \geq |\mathcal{A}'|$. This completes the proof of the claim. ♢

Claim 3.2. *If $\mathcal{A}'$ has a pair of disjoint edges, and $|\mathcal{B}'| = 1$, then $|\mathcal{A}'| \leq n(\mathcal{A}') + (|S| + 1)$.*

Proof. Let $\{xy, x'y'\}$ be a pair of disjoint edges in $\mathcal{A}'$, and, wlog, let $\mathcal{B}' = \{xx'\}$. Let $\mathcal{A}'_x = \{A \in \mathcal{A}' : x \in A\}$, and let $\mathcal{A}'_{x'} = \{A \in \mathcal{A}' : x' \in A\}$. Let $X = \{v \in [n] : v \neq x', xv \in \mathcal{A}_x\}$, $X^* = \{v \in [n] : v \neq x, x'v \in \mathcal{A}_{x'}\}$ and $R = X \cap X^*$. Now, $|\mathcal{A}'| \leq 2|R| + |X \setminus R| + |X^* \setminus R| + 1$, and $n(\mathcal{A}') = 2 + |R| + |X \setminus R| + |X^* \setminus R|$. So, $n(\mathcal{A}') - |\mathcal{A}'| \geq -(|R| + 1)$. Since $|R| \leq |S|$ (otherwise, R would be a bigger sunflower with core $\{a, x\}$ (or $\{a, x'\}$), contradicting the choice of S), we have $n(\mathcal{A}') - |\mathcal{A}'| \geq -(|S| + 1)$. ♢

In the next claim, we give lower bounds on the sizes of $\mathcal{H}_a$ and $\mathcal{H}_b$.

Claim 3.3.

- $|\mathcal{H}_a| \geq 1 + (|S| + n(\mathcal{A}') + 1) + (|S| + |\mathcal{A}'|)$.
- $|\mathcal{H}_b| \geq 1 + (|S| + n(\mathcal{B}') + 1) + (|S| + |\mathcal{B}'|)$.

Proof. We will only give the proof for $\mathcal{H}_a$, as the proof for $\mathcal{H}_b$ follows identically. We know that $|\mathcal{H}_a| = \sum_{i=1}^{3} |\mathcal{H}_a^i|$, where $\mathcal{H}_a^i = \mathcal{H}_a \cap \binom{[n]}{i}$ for $i \in \{1, 2, 3\}$. It is trivial to note that $|\mathcal{H}_a^1| = 1$. Now, consider $\mathcal{H}_a^2$. First, $\{a, b\} \in \mathcal{H}_a^2$. Also, for every $\{a, b, s\} \in S$, $\{a, s\} \in \mathcal{H}_a^2$, as $\mathcal{H}$ is a downset. Similarly, for every $s \in n(\mathcal{A}')$, there exists a $t \in n(\mathcal{A}')$ such that $\{a, s, t\} \in \mathcal{I}_3$, and hence, $\{a, s\} \in \mathcal{H}_a^2$. Thus, $|\mathcal{H}_a^2| \geq |S| + n(\mathcal{A}') + 1$. Also, it is not hard to see that $|\mathcal{H}_a^3| \geq |S| + |\mathcal{A}'|$. This completes the proof of the claim. ♢

We will now prove that either $\mathcal{H}_a$ or $\mathcal{H}_b$ is bigger than $\mathcal{I}$, which will complete the proof of the theorem. It will be sufficient to prove the following claim.

Claim 3.4. $|\mathcal{H}_a| + |\mathcal{H}_b| > 2(|\mathcal{I}_3| + 3)$.

Proof. We will consider two cases, depending on whether or not the hypothesis of Claim 3.1 is true. Suppose the hypothesis of Claim 3.1 holds, so we have $n(\mathcal{X}) - |\mathcal{X}| \geq -2$, for $\mathcal{X} \in \{\mathcal{A}', \mathcal{B}'\}$. Thus, since $|S| > 3$, we have

$$
\begin{aligned}
|\mathcal{H}_a| + |\mathcal{H}_b| &\geq 4 + 4|S| + |\mathcal{A}'| + |\mathcal{B}'| + n(\mathcal{A}') + n(\mathcal{B}') \\
&= (2|S| + 2|\mathcal{A}'| + 2|\mathcal{B}'| + 6) + (n(\mathcal{A}') - |\mathcal{A}'|) + (n(\mathcal{B}') - |\mathcal{B}'|) + 2|S| - 2 \\
&\geq 2(|\mathcal{I}_3| + 3) + (2|S| - 6) \\
&> 2(|\mathcal{I}_3| + 3).
\end{aligned}
$$

Now, assume the hypothesis of Claim 3.1 is false, so, without loss of generality, suppose $\mathcal{A}'$ has a pair of disjoint edges, and $|\mathcal{B}'| = 1$. Clearly, $n(\mathcal{B}') - |\mathcal{B}'| = 1$ and we can use Claim 3.2 to conclude that

$n(\mathcal{A}') - |\mathcal{A}'| \geq -(|S|+1)$. Thus, we have

$$\begin{aligned} |\mathcal{H}_a| + |\mathcal{H}_b| &\geq 4 + 4|S| + |\mathcal{A}'| + |\mathcal{B}'| + n(\mathcal{A}') + n(\mathcal{B}') \\ &\geq (2|S| + 2|\mathcal{A}'| + 2|\mathcal{B}'| + 6) - (|S|+1) + 1 + 2|S| - 2 \\ &\geq 2(|\mathcal{I}_3| + 3) + |S| - 2 \\ &> 2(|\mathcal{I}_3| + 3). \end{aligned}$$

$\diamondsuit$

This proves the theorem. $\square$

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