Maximum pebbling number of graphs of diameter three

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Abstract

Given a configuration of pebbles on the vertices of a graph G, a *pebbling move* consists of taking two pebbles off some vertex v and putting one of them back on a vertex adjacent to v. A graph is called *pebbleable* if for each vertex v there is a sequence of pebbling moves that would place at least one pebble on v. The *pebbling number* of a graph G is the smallest integer m such that G is pebbleable for every configuration of m pebbles on G. We prove that the pebbling number of a graph of diameter 3 on n vertices is no more than $\frac{3}{2}n + O(1)$, and, by explicit construction, that the bound is sharp.

Introduction

A *pebbling configuration* on a graph G is a distribution of pebbles on G. A *pebbling move* consists of removing two pebbles lying on the same vertex v, and placing one of these pebbles on some vertex that is adjacent to v. For a given pebbling configuration C, a vertex v is called *pebbleable* if there is a sequence of pebbling moves such that at least one pebble can be placed on v. The *pebbling number* of a graph G is the minimum number m of pebbles that ensure that every vertex of G is pebbleable, no matter what initial configuration of m pebbles we start with.

Let $f(n, d)$ be the maximal pebbling number of a diameter d graph on n vertices. The goal of this paper is to establish asymptotics for $f(n, 3)$, specifically to prove that

$$f(n, 3) = \tfrac{3}{2}n + O(1).$$

Pachter, Snevily and Voxman[3] proved that $f(n, 2) = n + 1$, and Clarke, Hochberg and Hurlbert[2] classified the graphs of diameter two for which the pebbling number equals $n + 1$.

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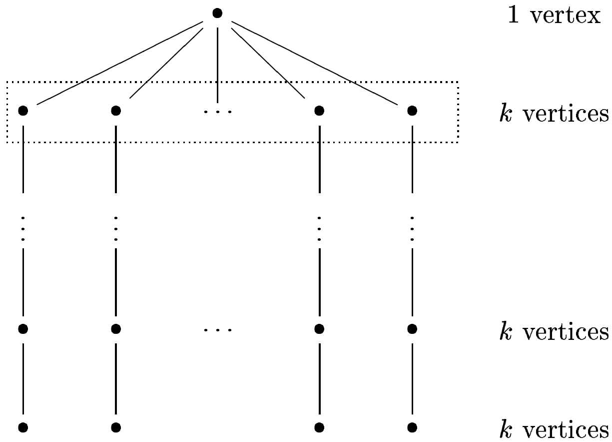
Lower bounds

The lower bound $f(n,d) \geq \frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil} n + O(1)$ can be established by considering the graph



where the enclosed vertices form a clique, and the "height" of the graph is $\lceil \frac{d}{2} \rceil + 1$ levels. If we place $2^{\lceil \frac{d}{2} \rceil} - 1$ pebbles on each vertex of the bottom row, then the top vertex is not pebbleable. Thus, $f(k \lceil \frac{d}{2} \rceil + 1, d) \geq k \left(2^{\lceil \frac{d}{2} \rceil} - 1 \right)$. For even d the vertices in the clique in the graph above do not need to be connected to each other to ensure the correct diameter, and can be replaced by disjoint vertices, obtaining approximately the same lower bound on $f(n,d)$.

Upper bounds

In this section we prove the aforementioned upper bound on $f(n,3)$. In order to illustrate the deficiency of our method for dealing with general d, the results that follow are proven in their greatest possible generality. Suppose that there is some configuration of pebbles on a graph G, so that G is not pebbleable. Fix this configuration, and let A be the set of vertices containing two or more pebbles. Since G has diameter d, for every pair of vertices u and v there is a path $p_{u,v}$ between u and v of length at most d. For each u, v, fix any such $p_{u,v}$, and let $P = \{ p_{u,v} \mid u, v \in A \}$ be the set of such paths between vertices of A.

To prove the upper bound on the number of pebbles on G when $d = 3$ we prove that "almost no" vertices contain more than 3 pebbles, and that "almost all" paths in P are of length 3 and contain no pebbles except at the endpoints. From that we deduce the upper bound on the number of pebbles.

Lemma 1. *There is no vertex containing 2^d pebbles. Moreover, it is impossible to move 2^d pebbles to any single vertex.*

Definition 2. *A vertex v is said to be* accessible *from $u \in A$ at distance r if* *there is path from u to v of length at most r such that some of the pebbles from u can moved to v along this path.*

Lemma 3. *There is constant $m(d,r)$ depending only on d and r such that there is no vertex v that is accessible from $m(d,r)$ different vertices along paths of length at most r.*

Proof. We prove this by induction on r. Since no vertex can be connected to 2^d vertices in A, it follows that $m(d,1) \leq 2^d$. We claim that $m(d,r) \leq (2^d-1)r \cdot m(d,r-1)+1$. If a vertex v was accessible at distance r from more than $(2^d-1)r \cdot m(d,r-1)+1$ vertices, then a pebble could be moved along one of the corresponding paths leaving us with at least $(2^d-2)r \cdot m(d,r-1)+1$ paths of length r emanating from the remaining vertices, namely those vertices whose paths to v do not pass through any of the non-end vertices in the first path. Again a pebble could be moved along one of the remaining paths leaving us with at least $(2^d-3)r \cdot m(d,r-1)+1$ paths. After repeating this process 2^d times we would amass 2^d pebbles at v contradicting Lemma 1. $\square$

Definition 4. *A path $p \in P$ is called* accessible *if there is a vertex on the path that is accessible from both ends of p.*

Lemma 5. *There are less than $\binom{m(d,d)}{2}n$ accessible paths in P.*

Proof. Suppose the contrary, then there is a vertex v through which $\binom{m(d,d)}{2}$ accessible paths go, and which is accessible from both endpoints of each such path. Let $E \subseteq A$ be the set of endvertices of the paths that go through v. Since P contains only one path between any two vertices in A, we have $\binom{|E|}{2} \geq \binom{m(d,d)}{2}$ which violates Lemma 3. $\square$

Lemma 6. *The number of vertices having $2^{\lceil \frac{d}{2} \rceil}$ or more pebbles is less than $c(d)\sqrt{n}$. Moreover, if $d=3$, then the number of such vertices is less than $c(d)\frac{n}{|A|}$.*

Proof. Let m be the number of such vertices. Since paths of length at most d connecting vertices having $2^{\lceil \frac{d}{2} \rceil}$ pebbles at both ends are accessible, and there are $\binom{m}{2}$ of such paths, the first part of theorem follows from Lemma 5. The second part of the theorem follows from Lemma 5 by noting that for $d=3$ every path of length at most 3 with one end having at least four pebbles and the other having at least two is accessible. $\square$

Theorem 7. $f(n,d) \leq (2^{\lceil \frac{d}{2} \rceil}-1)n + O(\sqrt{n})$.

Proof. Since no vertex contains more than 2^d-1 pebbles, by the preceding lemma we get $f(n,d) \leq (2^{\lceil \frac{d}{2} \rceil}-1)n+(2^d-1)c(d)\sqrt{n}$.

$\square$

Theorem 7 improves all the results in Chan and Godbole [1] for fixed values of d and large n. To prove the upper bound for the graphs of diameter three, we need to bound the number of vertices containing exactly one pebble. Let us denote the number of such vertices by α. The key point behind the following Lemma is that $2|A| + \alpha \leq n + O(1)$ under "most" circumstances.

Lemma 8. *If $d = 3$, then there are positive constants c_1, c_2 such that the inequality $|A|(|A| - 1) - c_1 n \geq (|A| - 1)(n - |A| - \alpha) + c_2 |A|$ cannot hold.*

Proof. Set $P_3 = \{ p \in P \mid p$ has length three and contains no pebbles except at the endvertices$\}$. Since a $p \in P$ is not accessible only if $p \in P_3$, by Lemma 5 $|P_3| \geq \binom{|A|}{2} - c_1 n/2$. Let $B = \{ v \in p \mid p \in P_3 \} \backslash A$. By Lemma 3 no vertex in B lies on more than $|A| \, m(d,1)$ paths in P_3. Let C be the set of vertices in B that lie on $|A|$ or more paths. If we count vertices in B with multiplicity according to the number of paths in P_3 they lie on, then the total count $2P_3$ is no more than $(|A| - 1)(n - |A| - \alpha) + |C| \, |A| \, m(d,1)$. Hence, if $2\,|P_3| \geq |A|(|A| - 1) - c_1 n > (|A| - 1)(n - |A| - \alpha) + c_2 \,|A|$ holds, then $|C| \geq c_2/m(d,1)$. Since each vertex in A lies on $|A| - 1$ paths, each vertex $v \in C$ is adjacent to at least two, but no more than $m(d,1)$ vertices in A. Thus, if we move one pebble from each of any two neighbors of v to v, the size of $|P_3|$ is decreased by at most $2m(d,1)\,|A|$. Thus, since $|P_3| > c_2|A|/2$, if $c_2 \geq 2m(d,1)^3$, then we can move two pebbles to each of at least $m(d,1)^2$ vertices of $C' \subset C$. Since every vertex v in C' lies on at least $|A|$ paths, v is adjacent to at least $|A|/m(d,1)$ other vertices in B. By the pigeonhole principle, there is a vertex in A that is adjacent to at least $m(d,1)$ vertices in C'. That violates Lemma 3. $\square$

Theorem 9. *If $d = 3$, then the number of pebbles is less than $\frac{3}{2}n + O(1)$.*

Proof. Since the inequality in Lemma 8 above does not hold, we have $(|A| - 1)(2\,|A| - n + \alpha - c_2) < c_1 n$. If $|A| < n/14$, then we are done because the number of pebbles is no more than $7\,|A| + \alpha < \frac{n}{2} + n$. Else, $2\,|A| - n + \alpha \leq O(1)$. Since α is nonnegative that implies $2\,|A| \leq n + O(1)$. By Lemma 6, the total number of pebbles is no more than $\alpha + 3(|A| - cn/\,|A|) + 7cn/\,|A| \leq n + |A| + O(1) \leq \frac{3}{2}n + O(1)$. $\square$

Further problems

Since the implied constant in $O(1)$ is huge, it is of interest to determine the best possible value of that constant.

It would be interesting to determine whether the lower bounds are sharp for all d. The method in this paper might resolve the case $d = 4$, but the case of general d seems to be beyond the reach of the method since it would involve keeping track of too many different types of paths.

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