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A Short Proof that N^3 is Not a Circle Containment Order

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Abstract. A partially ordered set P is called a circle containment order provided one can assign to each $x \in P$ a circle C_x so that $x \leqslant y \Leftrightarrow C_x \subseteq C_y$. We show that the infinite three-dimensional poset N^3 is not a circle containment order and note that it is still unknown whether or not $[n]^3$ is such an order for arbitrarily large n.

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Let

$$N = \{0, 1, 2, \ldots\}, \qquad N^3 = \{(a, b, c) : a, b, c \in N\}.$$

View N^3 as a poset with a partial ordering defined as follows: given $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$,

$$x \leqslant y \Leftrightarrow x_1 \leqslant y_1, x_2 \leqslant y_2 \quad \text{and} \quad x_3 \leqslant y_3.$$

Also define $x < y \Leftrightarrow x \leqslant y$ and $x \neq y$, and $x \sim y \Leftrightarrow x \leqslant y$ or $y \leqslant x$.

A partially ordered set P is called a circle containment order provided one can assign to each $x \in P$ a circle C_x so that $x \leqslant y \Leftrightarrow C_x \subseteq C_y$. In this paper, we use the word circle to mean a circle including its interior. When the tangency of circles is considered, we actually refer to the circles themselves.

Scheinerman and Wierman [1] have proved that $[n] \times [n] \times N$ is not a circle containment order for n large, where $[n] = \{0, 1, 2, \ldots, n\}$. Though slightly weaker, we shall prove the following theorem.

THEOREM 1. *N^3 is not a circle containment order.*

Proof. Suppose there is an assignment of circles to N^3 preserving the partial order. Let $C(i, j, k)$ be the circle assigned to (i, j, k). Consider the following sets:

$$C_1 = \overline{\bigcup_{k=1}^{\infty} C(k, 0, 0)}, \qquad C_{12} = \overline{\bigcup_{k=1}^{\infty} C(k, k, 0)},$$
$$C_2 = \overline{\bigcup_{k=1}^{\infty} C(0, k, 0)}, \qquad C_{13} = \overline{\bigcup_{k=1}^{\infty} C(k, 0, k)},$$

$C_3 = \overline{\bigcup_{k=1}^{\infty} C(0,0,k)}, \qquad C_{23} = \overline{\bigcup_{k=1}^{\infty} C(0,k,k)},$
$C = \overline{\bigcup_{k=1}^{\infty} C(k,k,k)}.$

Clearly, since $\bigcup_{k=1}^{\infty} C(k,k,k) = \lim_{k\to\infty} C(k,k,k)$, there are three possibilities for C:

(1) C is a circle.
(2) C is a half-plane.
(3) C is all of R^2.

In the first case, C_1 must be tangent to C since otherwise there exists n such that $C(n,n,n) \supset C_1 \supset C(k,0,0)$ for all k, which contradicts the fact that $(n,n,n) \not> (k,0,0)$ for $k > n$. Similarly, C_2, C_3, C_{12}, C_{13} and C_{23} are all tangent to C. Since $C_1 \subseteq C_{12}$ and $C_2 \subseteq C_{12}$, it must be that they share a common tangent to C at some point x (and likewise, C_3, C_{13} and C_{23} are tangent at x as well). We can assume that $C_1 \subseteq C_2 \subseteq C_3$. If $x \notin C(0,1,0)$ then there exists n such that $C(0,0,n) \supseteq C(0,1,0)$, contradicting that $(0,0,n) \not> (0,1,0)$. Hence $x \in C(0,1,0)$ and $x \in C(0,k,0)$ for all k, which implies that $C(0,n,0) \supseteq C(1,0,0)$ for some n large enough, while $(0,n,0) \not> (1,0,0)$. Thus, case 1 is impossible.

In cases 2 and 3, consider C_{12}. We already know that C_{12} is not all of R^2, since $(k,k,0) \not> (0,0,1)$ for any k. And C_{12} is not a circle since either $C(n,n,n) \supseteq C_{12} \supseteq C(k,k,0)$ for all k (impossible because $(n,n,n) \not> (k,k,0)$ for $k > n$) or C is a half-plane and C_{12} is tangent to C at some point x. We similarly get both C_1 and C_2 tangent at x, and assume $C_1 \subset C_2$. Then by the same argument from case 1, we have $x \in (0,k,0)$ for all k which implies that $C(0,n,0) \supseteq C(1,0,0)$ for some n, a contradiction. Hence C_{12} is a half-plane.

We argue further that C_1, C_2, C_3, C_{13} and C_{23} are all half-planes, and in fact parallel half-planes ($C_{12} \supseteq C_1, C_2$ so $C_1 \parallel C_2 \parallel C_{12}$, etc.). Now $C_1 \not\subset C_2$ since $(0,k,0) \not> (1,0,0)$ for all k, and similarly $C_{12} \not\subset C_{13}$, $C_1 \not\subset C_{23}$, etc., implying $C_1 = C_2 = \cdots = C_{23}$. Just as in case 1, there must be a common tangent point x shared by $C(k,0,0)$, $C(0,k,0)$ and $C(k,k,0)$, so $C(0,n,0) \supseteq C(1,0,0)$ for n large enough, a contradiction. □

A Few Remarks

This proof uses heavily the notion 'for n large enough', relying on the properties of the limit circles C_j, C_{ij} and C, and thus may suggest that, for arbitrarily large n, $[n]^3$ is a circle containment order, as Scheinermann and Wierman [1] have conjectured. We make the following conjecture.

CONJECTURE 1. *There exists n such that $[n]^3$ is not a circle containment order.*

We believe in fact that n may be very small, say 4 or 5, as we have been unable to construct such an order and have yet to have seen one.

Reference

1. E. R. Scheinerman and J. C. Wierman, 1987, On circle containment orders, preprint.