MAT 415/512 Combinatorics, Fall 2012

Practice Problems

Problem 1 *At a party of* 10 *people, some pairs of people shake hands and some don't. How many distinct such parties are there, supposing that tracking handshakes is the only way to distinguish them?*

There are 35184372088832 of them. For general n, let H be the set of all possible handshakes and $S = \{\text{shake}, \text{no shake}\}$. The answer is the number of functions $f : H \to S$, which equals $|S|^{|H|} = 2^{\binom{n}{2}}$. In this case, we have 2^{45} of them. (Since a valid interpretation of "*some people shake hands and some don't*" is that in no party does no one or everyone shake hands, one could argue that $2^{45} - 2$ is correct. That interpretation is fine with me.) $\diamond$

NOTE: Since 2^{10} is roughly 10^3, a quick ballpark estimate is $2^5(2^{10})^4 \sim 32 \times 10^{12}$. Extra precision can be garnered from including the extra 10% (nearly 2.5% four times) that had been shaved off. This yields 35.2×10^{12} — pretty close!

Problem 2 *A license plate in the country of Yuforia consists of 12 digits, each of which is from the set* $\{0, 1, 2, \ldots, 9\}$*. How many such license plates have its maximal digit appearing uniquely?*

There are 508371835500 of them. Let P be the set of all such plates, and for $1 \le k \le 9$ let P_k be the set of those whose maximal digit is k. Then $\{P_1, \ldots, P_9\}$ partitions P, and so the Sum Rule implies that $|P| = \sum_{k=1}^{9} |P_k|$. A given plate in P_k has 12 places that the digit k could appear. Every other digit is from the set $\{0, \ldots, k-1\}$, meaning that there are k choices for each of the remaining 11 places. The Product Rule then implies that $|P_k| = 12k^{11}$. Hence $|P| = \sum_{k=1}^{9} 12k^{11}$. $\diamond$

NOTE: The result has the form of an approximation by Riemann sums to the area under the curve of the function $y = 12x^{11}$, over the interval $[0, 9]$, in which the right endpoint of each subinterval is the height of each block. Since the function is increasing, this approximation is an upper bound. In fact, a quick look at their superimposed graphs reveals that this upper bound is at most twice the actual value. Hence $|P| \le 2\int_0^9 12x^{11} = 2(9^{12}) = 564859072962$, a fairly decent approximation. (A MAPLE plot reveals that $2n^d$ is a close approximation when plates have d digits from $\{0, 1, \ldots, n\}$ and $d \sim 3n$. This makes some sense because the polynomial has relatively high degree and the block widths are relatively large. Also, n^d is a close approximation when $n \ge 10d$. This makes sense as well because the block widths are relatively small and the polynomial has relatively small degree.) Interestingly, since there are 10^{12} license plates possible, one chosen randomly will (barely) more likely have the Yuforian property than not.

Problem 3 *From a delegation of 22 Canadians, 25 United Statians and 28 Mexicans, a committee of 15 must be chosen that includes between 2 and 7 delegates from each country, with at least as many United Statians as Canadians, and at least as many Mexicans as United Statians. In how many ways can this be accomplished?*

There are 641556758442600 of them. Let i denote the number of Canadians, j the number of United Statians, and k the number of Mexicans in a given committee C. We say that (i, j, k) is the *characteristic* of C, and that C is *good* if it satisfies the conditions of the problem. If C is good then its characteristic satisfies the constraints $i + j + k = 15$ and $2 \le i \le j \le k \le 7$, with i, j and k all integral. There are exactly six solutions to these constraints: $(i, j, k) \in \{(2, 6, 7), (3, 5, 7), (3, 6, 6), (4, 4, 7), (4, 5, 6), (5, 5, 5)\}$. These can be found by executing the following algorithm, for example.

```
for i from 2 to 7 do
  for j from i to 7 do
    for k from j to 7 do
      if i+j+k=15
        then print (i,j,k)
      end if
    end do
  end do
end do
```

We partition the set of all good committees according to their characteristic. The Product Rule implies there are $\binom{22}{i}\binom{25}{j}\binom{28}{k}$ committees having characteristic (i, j, k). The the Sum Rule implies that there are

$$\binom{22}{2}\binom{25}{6}\binom{28}{7}+\binom{22}{3}\binom{25}{5}\binom{28}{7}+\binom{22}{3}\binom{25}{6}\binom{28}{6}+$$
$$\binom{22}{4}\binom{25}{4}\binom{28}{7}+\binom{22}{4}\binom{25}{5}\binom{28}{6}+\binom{22}{5}\binom{25}{5}\binom{28}{5}$$

good committees in total. $\diamond$

The algorithm can be sped up and made to print out all solutions without testing them, by noting that

$$1 \leq 15 - j - k = i \leq (i + j + k)/3 \leq 5\ ,$$

and that

$$8 - i \leq 15 - i - k = j \leq (j + k)/2 = (15 - i)/2 \leq 13/2\ .$$

Then the algorithm is as follows.

```
for i from max(2,15-7-7) to floor(15/3) do
  for j from max(i,15-i-7) to floor((15-i)/2) do
    print (i,j,15-i-j)
  end do
end do
```

These committees account for about 28% of the total number ($\binom{75}{15} = 2280012686716080$) of committees, without restriction.

Problem 4 *Let n be a nonnegative integer and $x = \sqrt{238747}$. Use the Binomial Theorem to prove that the arithmetic mean of $(1 + x)^n$ and $(1 - x)^n$ is a positive integer.*

Let $y = x^2$ and $a = [(1 + x)^n + (1 - x)^n]$. Then

$$\begin{aligned} a &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k} + \sum_{k=0}^{n}\binom{n}{k}(-x)^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}\left[1 + (-1)^{n-k}\right] \\ &= \sum_{n-k \text{ even}} 2\binom{n}{n-k}y^{(n-k)/2}\ . \end{aligned}$$

Now make the substitution $t = (n - k)/2$. Then the arithmetic mean $a/2 = \sum_{t=0}^{\lfloor n/2 \rfloor}\binom{n}{2t}y^t$. Since every term in the sum is an integer, so is the sum. $\diamond$

Note that the value 238747 could be any integer.

Problem 5 *Use three of the* IMPORTANT IDENTITIES[1] *to prove that*

$$\sum_{k=0}^{38204629939869} \frac{\binom{38204629939869}{k}}{\binom{76409259879737}{k}} = 2 \ .$$

Let $n = 76409259879737$ and $t = 38204629939869$ so that the left hand side can be written $\sum_{k=0}^{t} \binom{t}{k}/\binom{n}{k}$. We will use the following three combinatorial equalities:

$$\binom{n}{t}\binom{t}{k} = \binom{n}{k}\binom{n-k}{t-k} \ ; \tag{1}$$

$$\sum_{r=0}^{t} \binom{m+r}{r} = \binom{m+t+1}{t} \ ; \tag{2}$$

and

$$\binom{n+1}{t}(n+1-t) = (n+1)\binom{n}{t} \ . \tag{3}$$

Then we have

$$\begin{aligned}
\sum_{k=0}^{t} \frac{\binom{t}{k}}{\binom{n}{k}} &= \sum_{k=0}^{t} \frac{\binom{n-k}{t-k}}{\binom{n}{t}} && \text{[by (1)]} \\
&= \frac{\sum_{r=0}^{t} \binom{n-t+r}{r}}{\binom{n}{t}} && \text{[by the substitution } r = t-k\text{]} \\
&= \frac{\binom{n+1}{t}}{\binom{n}{t}} && \text{[by (2) and the evaluation } m = n-t\text{]} \\
&= \frac{n+1}{n+1-t} && \text{[by (3)]} \ .
\end{aligned}$$

Then we use the fact that $n+1 = 2t$ to reveal that $(n+1)/(n+1-t) = 2$. ◇

Of course, this is true for any n and t for which $n+1 = 2t$.

Problem 6 *Find the number of ways to order 14 books onto a bookshelf having 4 shelves.*

There are 59281238016000 of them. Every such arrangement looks like a permutation of some of the books, followed by a divider, followed by a permutation of some of the remaining books, followed by a divider, followed by a permutation of some of the still remaining books, followed by a divider, and followed by a permutation of the final remaining books. That is, it is a permutation of 14 distinguishable books and 3 indistinguishable dividers. In other words, it is a permutation of 17 objects, 3 of which are indistinguishable. Thus the Division Rule yields $17!/3!$ ways. ◇

[1] See Cameron Sec. 3.2 and Exer. 3.13.3.

Problem 7 *Let X be any set of 16 integers and consider all sums of exactly 6 of the numbers from X. Prove that at least 9 of these sums differ from each other by a multiple of 1000.*

Let $f : \binom{X}{6} \to \{0, 1, \ldots, 999\}$ be defined by $f(S) = \left(\sum_{s \in S} s\right) \bmod 1000$. Since $\left|\binom{X}{6}\right| = \binom{16}{6} = 8008$, the Pigeonhole Principle implies that there exist at least $\lceil 8008/1000 \rceil = 9$ subsets $S_1, \ldots, S_9$ such that $f(S_i) = f(S_j)$ for all $1 \le i < j \le 9$. Thus, $\sum_{a \in S_i} a \equiv \sum_{b \in S_j} b \mod 1000$, and so $1000 \,\Big|\, \left(\sum_{a \in S_i} a - \sum_{b \in S_j} b\right)$. ◇

Problem 8 *Let $\mathbb{N}$ be the set of nonnegative integers and suppose that $g : \mathbb{N} \to \mathbb{N}$ is such that $g(1) = 7$ and $g(a + b) = g(a)g(b)$ for all $a, b \in \mathbb{N}$. Prove that $g(n) = 7^n$ for all $n \in \mathbb{N}$.*

We use induction and note that the base case ($n = 0$) is true, as follows. Since $7 = g(1) = g(0 + 1) = g(0)g(1) = 7g(0)$, it must be that $g(0) = 1 = 7^0$.

Now suppose that $g(k) = 7^k$ for some $k \ge 0$. Then $g(k + 1) = g(k)g(1) = 7^k 7^1 = 7^{k+1}$. Hence, by induction, $g(n) = 7^n$ for all $n \in \mathbb{N}$. ◇

Problem 9 *In Jetson City the streets form a 3-dimensional grid. There are East-West streets, North-South streets, and Up-Down streets at each integer mile coordinate of $\mathbb{R}^3$. In how many ways can a thief get from a store at $(12, -5, 6)$ to a hideout at $(21, 5, 18)$ without passing by the police station at $(15, 3, 13)$, and without ever decreasing a coordinate?*

For two locations X and Y let $w(X, Y)$ denote the number of ways to travel from X to Y via shortest paths. We use the result that there are $\binom{e+n+u}{e,n,u}$ ways to travel e steps East, n steps North, and u steps Up, in any order. Let S be the store, H be the hideout, and P be the police station. Thus we have $w(S, H) = \binom{9+10+12}{9,10,12}$, $w(S, P) = \binom{3+8+7}{3,8,7}$, and $w(P, H) = \binom{6+2+5}{6,2,5}$. Now,the Subtraction Rule states that the number of ways to travel from S to H while avoiding P equals $w(S, H)$ minus the number of ways to travel from S to H through P, which equals

$$w(S, H) - w(S, P)w(P, H) = 12847208263890 \ .$$

◇

Problem 10 *Curly, Larry, and Moe are playing* Simplified Jeopardy*, in which the first player to respond to a question receives* \$100*, regardless of the response. In how many ways can 30 questions be responded to without the game ending with all three players tied?*

There are 200340135303309 ways, which we count by complementation. There are 3^{30} ways 30 questions can be responded to by three players, and $\binom{30}{10,10,10}$ ways that the three can end in a tie. Therefore, the Subtraction Rule yields

$$3^{30} - \binom{30}{10, 10, 10}$$

ways to respond to 30 questions without a final tie. ◇

Problem 11 *Prove that $\sum_{k=0}^{n} k(n - k)\binom{n}{k} 2^k = \binom{n}{2} 2^2 3^{n-2}$ for all $n \ge 2$.*

Starting from the Binomial Theorem,

$$(x+y)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \;,$$

we differentiate first with respect to x to obtain

$$n(x+y)^{n-1} \;=\; \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} y^{n-k} \;,$$

and second with respect to y to obtain

$$n(n-1)(x+y)^{n-2} \;=\; \sum_{k=0}^{n} k(n-k) \binom{n}{k} x^{k-1} y^{n-k-1} \;.$$

Next, the substitutions of $x = 2$ and $y = 1$ yield

$$2\binom{n}{2} 3^{n-2} \;=\; \sum_{k=0}^{n} k(n-k) \binom{n}{k} 2^{k-1} \;,$$

and the multiplication of both sides by 2 finishes the proof. $\diamond$

A second proof that is purely algebraic uses the relation $b\binom{a}{b} = a\binom{a-1}{b-1}$ twice, with symmetry used in between, as follows. Make note also of the range of indices and the final index substitution.

$$\begin{aligned}
\sum_{k=0}^{n} k(n-k)\binom{n}{k} 2^k &= \sum_{k=1}^{n-1} n(n-k)\binom{n-1}{k-1} 2^k \\
&= n\sum_{k=1}^{n-1} (n-k)\binom{n-1}{n-k} 2^k \\
&= n\sum_{k=1}^{n-1} (n-1)\binom{n-2}{n-k-1} 2^k \\
&= n(n-1)\sum_{k=1}^{n-1} \binom{n-2}{k-1} 2^k \\
&= 2n(n-1)\sum_{k=1}^{n-1} \binom{n-2}{k-1} 2^{k-1} \\
&= 2^2\binom{n}{2}\sum_{r=0}^{n-2} \binom{n-2}{r} 2^r \\
&= \binom{n}{2} 2^2 3^{n-2} \;.
\end{aligned}$$

The last equality uses the Binomial Theorem. $\diamond$

Here is a purely combinatorial proof. Both sides will count the number of ways in which n people can be elite, regular or non members of a club, with one of the members serving as president and one of the nonmembers serving as moderator of their meetings.

The right hand side counts the number of ways to choose the president and moderator, which of the two is president, and whether the president is an elite or regular member, followed by the number of ways the remaining $n-2$ people can each be elite, regular, or non members.

The left hand side partitions all the possibilities according to the number of club members. With k members, we first choose which k of the n they are, then decide whether each is elite or regular, and finally pick a president among the members and moderator among the nonmembers.

Problem 12 *Find the number of lattice paths from* $(0,0,0)$ *to* $(6,9,4)$ *of length 25. (Notice that these are not minimal paths.)*

There are 30654447623800. Such a lattice path P is a sequence of 25 symbols, each chosen from $\{E,W,N,S,U,D\}$. Let e (resp. w,n,s,u,d) be the number of occurrences of E (resp. W,N,S,U,D), and call (e,w,n,s,u,d) the *type* of P. Then each variable is nonnegative, and we must have

$$e-w=6,\ n-s=9 \text{ and } u-d=4\ , \tag{4}$$

as well as

$$e+w+n+s+u+d=25\ . \tag{5}$$

If we subtract the three equalities in (4) from (5) and divide by 2, we obtain

$$w+s+d=3\ . \tag{6}$$

The solutions to (6) are shown in the following chart.

(w,s,d)	(e,w,n,s,u,d)	$p(e,w,n,s,u,d)$
$(3,0,0)$	$(9,3,9,0,4,0)$	1963217256000
$(0,3,0)$	$(6,0,12,3,4,0)$	41227562376000
$(0,0,3)$	$(6,0,9,0,7,3)$	2248776129600
$(2,1,0)$	$(8,2,10,1,4,0)$	312330018000
$(2,0,1)$	$(8,2,9,0,5,1)$	5889651768000
$(1,2,0)$	$(7,1,11,2,4,0)$	7067582121600
$(1,0,2)$	$(7,1,9,0,6,2)$	1606268664000
$(0,2,1)$	$(6,0,11,2,5,1)$	4417238826000
$(0,1,2)$	$(6,0,10,1,6,2)$	2208619413000
$(1,1,1)$	$(7,1,10,1,5,1)$	818007190000

We reason that there are only these 10 solutions as follows. First note that there are only three solutions if we add the extra constraint that $w \geq s \geq d$: $(3,0,0),(2,1,0),(1,1,1)$. Then realize that there are 3 ways to permute the values of the first solution, 6 ways to permute those of the second, and 1 way the third.

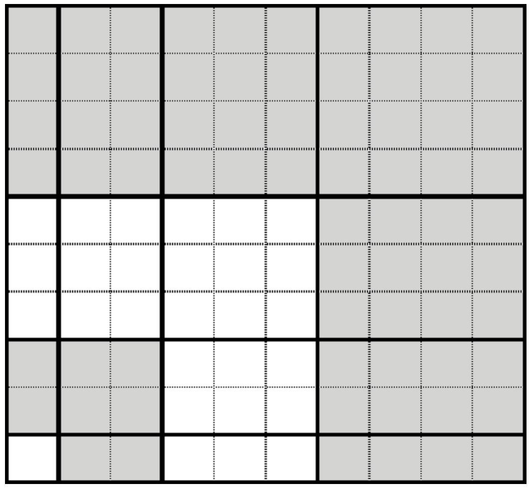
The middle column of the chart shows the resulting solutions (e,w,n,s,u,d) given by (4). Let $p(e,w,n,s,u,d)$ denote the number of paths with type (e,w,n,s,u,d). Then p is determined by the formula

$$p(e,w,n,s,u,d)=\binom{e+w+n+s+u+d}{e,w,n,s,u,d}=\binom{25}{e,w,n,s,u,d}\ , \tag{7}$$

according to the Division Rule, and the corresponding values are listed in the third column of the chart.

The set of lattice paths under consideration is partitioned according to these 10 types. Thus the Sum Rule implies that the number of lattice paths from $(0,0,0)$ to $(6,9,4)$ is the sum of the entries of the third column. ◇

Problem 13 *Consider the figure below. What formula does it suggest regarding the sum of cubes? Use induction to prove your answer.*



This figure F_4 is a square of sidelength $1+2+3+4$. The general figure F_n would be a square of sidelength $\sum_{k=1}^{n} = \binom{n+1}{2}$, having area $A_n = \binom{n+1}{2}^2$.

On the other hand, the alternating gray and white regions partition F_4 into 4 regions $R_1, \ldots, R_4$, with the area of R_i equal to i^3. One sees this by noticing that R_i is further partitioned into a corner $i \times i$ square, $i-1$ $j \times i$ rectangles down the right side ($1 \le j < i$), and their reflections at the top. The rectangles on the side can be combined in reverse with the rectangles at the top to form $i-1$ $i \times i$ squares; along with the corner square these account for i such squares. Therefore, the sum of the areas of the R_i should equal $\sum_{i=1}^{n} i^3$ in general.

Thus, the formula suggested by the figure is $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$ for all $n \ge 1$ ($n \ge 0$ works as well), or

$$\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2 . \tag{8}$$

To prove this we use induction, and note that the base case ($n = 1$) is true: $1 = 1$. Now we suppose that (8) holds when $n = k \ge 1$ and argue that it holds also when $n = k+1$.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \binom{k+1}{2}^2 + (k+1)^3 \\
&= k^2(k+1)^2/4 + (k+1)(k+1)^2 \\
&= [k^2+4k+4](k+1)^2/4 \\
&= (k+1)^2(k+2)^2/4 \\
&= \binom{k+2}{2}^2 .
\end{aligned}$$

◇

Problem 14 *Use induction to prove that*

$$\sum_{k=0}^{t}(-1)^k\binom{n}{k} = (-1)^t\binom{n-1}{t}$$

for all $0 \le t < n$.

We use induction on t. For any $n > 0$, the statement is true when $t = 0$. Indeed, the left-hand side equals $\binom{n}{0} = 1$, while the right-hand side equals $\binom{n-1}{0} = 1$.

Now suppose that the statement is true when $t = r$ for some $0 \le r < n-1$. Then

$$\begin{aligned}
\sum_{k=0}^{r+1}(-1)^k\binom{n}{k} &= \sum_{k=0}^{r}(-1)^k\binom{n}{k} + (-1)^{r+1}\binom{n}{r+1} \\
&= (-1)^r\binom{n-1}{r} + (-1)^{r+1}\binom{n}{r+1} \\
&= (-1)^{r+1}\left[\binom{n}{r+1} - \binom{n-1}{r}\right] \\
&= (-1)^{r+1}\binom{n-1}{r+1} .
\end{aligned}$$

Thus the statement is true when $t = r+1 < n$. $\diamond$

An alternative proof uses induction on n rather than t. Note that the statement is also true when $t \ge n$ because of the Binomial Theorem with $x = 1$ and $y = -1$: the sum on the left truncates to the case $t = n$ because of the extra zeros, while the right is identically zero. When $n = 1$ we have $t = 0$ and $\sum_{k=0}^{0}(-1)^k\binom{1}{k} = \binom{1}{0} = 1 = (-1)^0\binom{0}{0}$. Now we assume that the statement is true when $n = m > 0$, and consider the case $n = m+1$. Then we use Pascal's relation twice to obtain

$$\begin{aligned}
\sum_{k=0}^{t}(-1)^k\binom{m+1}{k} &= \sum_{k=0}^{t}(-1)^k\left[\binom{m}{k-1} + \binom{m}{k}\right] \\
&= \sum_{j=0}^{t-1}(-1)^{j+1}\binom{m}{j} + \sum_{k=0}^{t}(-1)^k\binom{m}{k} \\
&= -(-1)^{t-1}\binom{m-1}{t-1} + (-1)^t\binom{m-1}{t} \\
&= (-1)^t\left[\binom{m-1}{t-1} + \binom{m-1}{t}\right] \\
&= (-1)^t\binom{m}{t} .
\end{aligned}$$

$\diamond$

Note that the statement is true for all $n \ge 0$ and for all t. If $t < 0$ then both sides are zero.

In fact, the statement holds also for all n, including negatives: the numbers $\binom{n}{k} = n^{\underline{k}}/k!$ are well defined for all n and k and satisfy Pascal's relation as polynomials, regardless of their particular inputs. Interestingly, even things like $\binom{-11}{4}$ have combinatorial meaning, since $\binom{-m}{k} = (-1)^k\binom{m+k-1}{k}$.

Problem 15 *A party is held that 174 couples attend. The social protocol prohibits a couple to kiss, but encourages kissing between any other pair of people. Let $k(P)$ denote the number of people kissed by person P, and define $[n] = \{0, 1, \ldots, n-1\}$. Suppose that Q is at the party, and that $\{k(P) \mid P \neq Q\} = [347]$. Use induction to prove that $k(Q) = 173$.*

We first set up a more general scenario to use with induction. Suppose that n couples attend, with person Q among them such that $\{k(P) \mid P \neq Q\} = [2n-1]$, and prove for all $n \geq 1$ the statement $T(n)$ that in such a scenario we have $k(Q) = n-1$. Denote by $s(P)$ the spouse of person P.

To prove the base case $T(1)$ we note that $k(Q) = 0 = 1 - 1$ because there is just one couple and couples don't kiss.

Next consider the case $n = r+1$ (for some $r \geq 1$), where we assume that $T(r)$ holds and that $\{k(P) \mid P \neq Q\} = [2r+1]$. There must be some person A for whom $k(A) = 2r$. Because A did not kiss A or $s(A)$, we know that A kissed everyone else. Also, since there must be some person B for whom $k(B) = 0$, it follows that $B = s(A)$.

Now imagine the subparty without the couple (A, B), and define $k'(P)$ to be the number of people in the subparty kissed by P. Then we have $k'(P) = k(P) - 1$ because every person P in the subparty kissed A and no such P kissed B. Therefore, since $\{k(P) \mid P \notin \{A, B, Q\}\} = \{1, \ldots, 2r-1\}$, we obtain $\{k'(P) \mid P \notin \{A, B, Q\}\} = [2r-1]$.

Thus the hypothesis is satisfied for the case $n = r$. From this we deduce via $T(r)$ that $k'(Q) = r-1$, from which follows $k(Q) = (r-1)+1 = (r+1)-1$. Hence $T(n)$ holds for all $n \geq 1$; in particular, $T(174)$ is true. ◇

Notice that, with almost no extra work, we can show that $k(s(Q)) = k(Q)$. It is certainly true in the base case, and the same algebra makes it true in the inductive step. More generally, we have $k(P) + k(s(P)) = 2n-2$ for all P.

Problem 16 *Define the* Fibonacci numbers $f_0, f_1, \ldots$ *by* $f_0 = f_1 = 1$ *and* $f_n = f_{n-1} + f_{n-2}$ *for* $n \geq 2$. *Prove for every* $k \geq 0$ *and* $j \in \{0, 1\}$ *that*

$$\sum_{i=0}^{k} \binom{k+i+j}{2i+j} = f_{2k+j} \ .$$

We use strong induction on $n = 2k+j$ and, as we will see in the induction step, we will need two base cases. When $n = 0$ we have $k = j = 0$ and $\sum_{i=0}^{0} \binom{i}{2i} = \binom{0}{0} = 1 = f_0$. When $n = 1$ we have $k = 0, j = 1$ and $\sum_{i=0}^{0} \binom{i+1}{2i+1} = \binom{1}{1} = 1 = f_1$. Now we assume that the equality holds for $n \in \{t-1, t\}$ and prove that it holds for $n = t+1$. We will need to break the inductive step into two cases, based on the parity of $t+1$.

Suppose first that $t+1$ is even; i.e. $t+1 = 2(k+1)$ and $k \geq 0$. Then $j = 0$ and Pascal's relation yields

$$\begin{aligned}
\sum_{i=0}^{k+1} \binom{k+1+i}{2i} &= \sum_{i=0}^{k+1} \left[\binom{k+i}{2i-1} + \binom{k+i}{2i} \right] \\
&= \sum_{i=1}^{k+1} \binom{(k+1)+(i-1)}{2(i-1)+1} + \sum_{i=0}^{k} \binom{k+i}{2i}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=0}^{k} \binom{k+r+1}{2r+1} + \sum_{i=0}^{k} \binom{k+i}{2i} \\
&= f_{2k+1} + f_{2k} \\
&= f_{2k+2} \\
&= f_{2(k+1)} \ .
\end{aligned}
$$

Now suppose that $t+1$ is odd; i.e. $t+1 = 2(k+1)+1$ and $k \geq 0$. Then $j=1$ and Pascal's relation yields

$$
\begin{aligned}
\sum_{i=0}^{k+1} \binom{k+1+i+1}{2i+1} &= \sum_{i=0}^{k+1} \left[\binom{k+i+1}{2i} + \binom{k+i+1}{2i+1} \right] \\
&= \sum_{i=0}^{k+1} \binom{(k+1)+i}{2i} + \sum_{i=0}^{k} \binom{k+i+1}{2i+1} \\
&= f_{2(k+1)} + f_{2k+1} \\
&= f_{2k+2} + f_{2k+1} \\
&= f_{2k+3} \\
&= f_{2(k+1)+1} \ .
\end{aligned}
$$

$\diamond$

There is a combinatorial argument that proceeds as follows. For $n \geq 0$ let g_n denote the number of ways to lay n feet of bricks using bricks of length 1 or 2 feet only. Then strong induction shows that $g_n = f_n$ for all $n \geq 0$. Indeed, $g_0 = g_1 = 1$ and $g_n = g_{n-1} + g_{n-2} = f_{n-1} + f_{n-2} = f_n$ for $n \geq 2$, since every brick laying of length n either ends in a 1-foot or 2-foot brick, and there are g_{n-1} of the former and g_{n-2} of the latter. Thus we may use f and g interchangeably.

For $j \in \{0,1\}$ we count the number of brick layings of length $2k+j$, which is f_{2k+j}. Because the only odd length bricks have length 1, the number of bricks of length 1 in such a laying has the same parity as the total length. That is, the number of length 1 bricks equals $2i+j$ for some $0 \leq i \leq k$. Thus we may partition the set of all such brick layings according to i and use the Sum Rule to obtain the result.

If $2i+j$ length 1 bricks are used, then $[(2k+j)-(2i+j)]/2 = k-i$ length 2 bricks are used. This means that there are $(2i+j)+(k-i) = k+i+j$ bricks used in all, and there are $\binom{k+i+j}{2i+j}$ ways to lay them end to end. $\diamond$

Problem 17 *Let X be any finite set and let $f : X \to X$ be any bijection on X. Denote by f^k the function f when $k=1$ and $f \circ f^{k-1}$ when $k > 1$. Define the relation R on X by $(a,b) \in R$ if there is some $k \geq 1$ so that $f^k(a) = b$. Prove that R is an equivalence relation.*

First we prove by induction on k that $f^k = f^i \circ f^{k-1}$ for all $1 \leq i < k$. This is certainly true for $k=2$ by definition. Now we suppose that the statement is true when $k = t \geq 2$ and consider the case when $k = t+1$. The statement is certainly true by definition for $i=1$, and for any $1 < i < t+1$ we have

$$
\begin{aligned}
f^{t+1} &= f \circ f^t && \text{[by definition]} \\
&= f \circ (f^{i-1} \circ f^{t-i+1}) && \text{[by induction]} \\
&= (f \circ f^{i-1}) \circ f^{t-i+1} && \text{[by associativity]} \\
&= f^i \circ f^{t+1-i} && \text{[by induction]} \ .
\end{aligned}
$$

Now we show that R is reflexive. For ease, we denote by f^0 the identity function. Let $x \in X$ and consider the sequence $x, f(x), f^2(x), \ldots, f^{|X|}(x)$. By the Pigeonhole Principle there exist $0 \le i < j \le |X|$ such that $f^i(x) = f^j(x)$, and let j be chosen to be the smallest among all such pairs (i, j). If $i > 0$ then, because f is a bijection (in particular, f is injective), we have $f^{i-1}(x) = f^{j-1}(x)$, which contradicts the minimality of j. Hence $i = 0$, which means that $x = f^j(x)$; i.e. $(x, x) \in R$.

Next we show that R is symmetric. For this we suppose that $(x, y) \in R$, which means that there is a k so that $f^k(x) = y$. Let j be the smallest integer such that $f^j(x) = x$, which exists by reflexivity. Induction shows that $f^{tj}(x) = x$ for all $t \ge 0$. Indeed, this is definitively true for $t = 0$, and $f^{(t+1)j} = f^{tj} \circ f^j$, so that $f^{(t+1)j}(x) = f^{tj}(f^j(x)) = f^{tj}(x) = x$. Thus we may pick any t so that $tj > k$ and define $m = tj - k$. Then we have $f^m(y) = f^{tj-k}(f^k(x)) = f^{tj}(x) = x$, implying that $(y, x) \in R$.

Finally we show that R is transitive. Suppose that $(x, y) \in R$ and $(y, z) \in R$. Then there are i and j such that $f^i(x) = y$ and $f^j(y) = z$. With $k = i + j$ we have $f^k(x) = f^{j+i}(x) = f^j(f^i(x)) = f^j(y) = z$, and so $(x, z) \in R$. $\diamond$

Something we may discuss later is what this has to do with the cycles of a permutation.

Problem 18 *A spice rack tower has 5 levels, with each level holding 4 spice jars, each level able to rotate independently, and the 5 levels stackable in any order. Two displays of spices are considered equivalent if one can be obtained from the other by shuffling and/or rotating the levels. Find the number of inequivalent displays of 20 distinct spices.*

There are 19799007728000 of them. Without regard to equivalences there are 20! displays. But there are 5! ways to shuffle the levels and 4 ways to rotate each of the levels, and so each equivalence class has size $5!4^5$. Thus the Division Rule implies that there are

$$\frac{20!}{5!4^5} = 19799007728000$$

inequivalent displays. $\diamond$

Problem 19 *Consider two arrangements of beads on a necklace as equivalent if one can be obtained from the other by rotating and/or reflecting it. How many inequivalent necklaces contain exactly 3 red beads and exactly* 16192458 *blue beads?*

There are 21849649436377 of them. Let $n = 16192461$ be the number of total beads. Note that n is an odd multiple of 3. For a given necklace, label the read beads R_1, R_2 and R_3. For $\{i, j, k\} = \{1, 2, 3\}$, let x_i be the number of blue beads between R_j and R_k. Then

$$x_1 + x_2 + x_3 = n - 3\ . \tag{9}$$

Of course notice that rotational symmetry makes (x_1, x_2, x_3) equivalent to (x_2, x_3, x_1) and (x_3, x_1, x_2), and then reflectional symmetry makes it equivalent to (x_1, x_3, x_2), (x_2, x_1, x_3) and (x_3, x_2, x_1) as well. In other words, all permutations of a given solution to (9) are equivalent.

There are three types of solutions to (9). The first type has $x_1 = x_2 = x_3 = (n-3)/3$; the second type has $x_1 \ne x_2 = x_3$; the third type has $x_1 < x_2 < x_3$ — note that rotations and reflections may be necessary to put a necklace into the form of the last two types. Let c_i be the number of equivalence classes of type i, and m_i be the size of each equivalence class of type i. Then

$$\binom{n}{3} = \sum_{i=1}^{3} c_i m_i \tag{10}$$

by the Sum and Product Rules, and $\sum_{i=1}^{3} c_i$ is the answer to this problem.

Because $x_1 = (n-3)/3$ is uniquely determined in type 1 (recall that $3|n$), we have $c_1 = 1$, and because all x_i are equal we have $m_1 = n/3$.

For type 2 equivalence classes, we see that any reflection is achieved by a rotation, and because x_1 is distinct from $x_2 = x_3$, we have $m_2 = n$. Also, since $x_1 + 2x_2 = n - 3$ and each $x_i \geq 0$, we have $0 \leq x_2 \leq (n-3)/2$ (which is an integer since n is odd). Thus there are $(n-3)/2 + 1$ values for x_2; however one of those values, namely $(n-3)/3$, is of type 1, and so we have $c_2 = (n-3)/2$.

For type 3 equivalence classes, we note that all rotations and relections yield distinct placements of the red beads, since the x_i are distinct, so that $m_3 = 2n$. Now, by (10), we solve for c_3:

$$\begin{aligned} c_3 &= \frac{1}{m_3}\left[\binom{n}{3} - c_1 m_1 - c_2 m_2\right] \\ &= \frac{1}{2n}\left[\frac{n(n-1)(n-2)}{6} - \frac{n}{3} - \frac{n(n-3)}{2}\right] \\ &= \frac{(n-3)^2}{12} \ . \end{aligned}$$

Hence we have

$$c_1 + c_2 + c_3 \ = \ 1 + \frac{n-3}{2} + \frac{(n-3)^2}{12} \ = \ \frac{n^2+3}{12} \ = \ 21849649436377 \ .$$

◇

Another proof partitions the set N of all necklaces according to the minimum distance between two red beads. Denote by N_d the set of all necklaces whose minimum number of blue beads between a red pair of beads is d. Write $n = 6m+3$, where $m = 2698743$, let $t = \lfloor d/2 \rfloor$, and note that $d \in \{0, 1, \ldots, \frac{n-3}{3}\}$. Then the Sum Rule implies $|N| = \sum_{d=0}^{2m} |N_d|$.

Now consider the necklaces in N_d. For a particular red pair with exactly d blues between them there must be at least d blues outside them as well, on each side. This accounts for $3d+2$ bead positions, so the third red bead is among the remaining $n-(3d+2)$ positions. Because of reflections, in particular the reflection that fixes the red pair, we may assume that it is on the right side, as we look at the necklace, or more carefully, not on the left side (there may be a *middle* position not moved by the reflection).

If $d = 2t$ is even then $n-(3d+2)$ is odd, and so there is a middle position. Thus there are $(n-3d-2-1)/2+1 = 3(m-t)+1$ positions for the third red bead. Note that $0 \leq t \leq m$, and so with $k = m-t$ we have

$$\sum_{t=0}^{m} |N_{2t}| = \sum_{k=0}^{m} (3k+1) \ . \tag{11}$$

If $d = 2t+1$ is odd then $n-(3d+2)$ is even, and so there is no middle position. Thus there are $(n-3d-2)/2 = 3(m-t)-1$ positions for the third red bead. Note that $0 \leq t \leq m-1$, and so with $k = m-t$ we have

$$\sum_{t=0}^{m-1} |N_{2t+1}| = \sum_{k=1}^{m} (3k-1) \ . \tag{12}$$

Now observe that equations (11) and (12) imply

$$\begin{aligned}
|N| &= \sum_{t=0}^{m} |N_{2t}| + \sum_{t=0}^{m-1} |N_{2t+1}| \\
&= 1 + \sum_{k=1}^{m} 6k \\
&= 1 + 6\binom{m+1}{2} \\
&= 1 + 3\left(\frac{n+3}{6}\right)\left(\frac{n-3}{6}\right) \\
&= \frac{12 + (n^2-9)}{12} \\
&= \frac{n^2+3}{12}\,.
\end{aligned}$$

◇

Notice that this answer is very close (the first 6 digits match!) to $\binom{n}{3}/2n = 21849637292032$, which suggests that almost every equivalence class has size $2n$ (indeed, c_1 is constant, c_2 is linear, and c_3 is quadratic). This can be interpreted by saying that almost every necklace has no rotational or reflectional symmetry. For large n, these statements hold for any k (and, in fact, for any number of colors) — any symmetry in a necklace is surely destroyed by swapping adjacent beads of different color, for example.

Problem 20 *How many ways are there to make 4 necklaces of length 10 and 3 necklaces of length 20 from 100 distinct gems?*

There are $100!/[(10!)^4 4!(20!)^3 3!]$ ways to split the 100 gems into 4 groups of size 10 and 3 groups of size 20. Then each group of size 10 makes $10!/(2 \cdot 10)$ different necklaces, while each group of size 20 makes $20!/(2 \cdot 20)$ different necklaces. Thus there are

$$\frac{100!}{(10!)^4 4!(20!)^3 3!} \times \left(\frac{10!}{2\cdot 10}\right)^4 \left(\frac{20!}{2\cdot 20}\right)^3 = \frac{100!}{4!20^4 3!40^3}$$

different necklaces in total (roughly 6.329×10^{145}). ◇

A more direct proof is as follows. Start with any of $100!$ permutations of all the gems. The first 40 gems make the four necklaces of size 10, 10 at a time in order, and the last 60 gems make the three necklaces of size 20, 20 at a time in order. The Division Rule makes the $2k$ necklaces of length k that are rotations and reflections of each other equivalent. Then the Division Rule again makes the $4!$ orderings of the necklaces of length 10 equivalent, as well as the $3!$ orderings of the necklaces of length 20. ◇

Problem 21 *Find the number of inequivalent necklaces consisting of 3 red beads, 5 blue beads and 230 green beads. Necklaces that are reflections and/or rotations of each other are considered equivalent.*

There are 26670887585316 of them. Let $r = 3$ be the number of red beads, $b = 5$ be the number of blue beads, $g = 230$ be the number of green beads, and $n = r + b + g = 238$ be the number of total beads. Since r and n are relatively prime, there are no necklaces with rotational symmetry. Let p be the number of necklaces with reflectional symmetry, and q be the number of necklaces with no symmetry, so that there are $p + q$ such necklaces in total. Then each necklace with reflectional symmetry is in an equivalence class of size n, due to rotations only, while each necklace with no symmetry is in an equivalence class of size $2n$, due to rotations and reflections. Thus the Product and Sum Rules imply that $np + 2nq = \binom{n}{r,b,g}$, the total number of ways to place all the beads without concern for equivalences.

If a necklace has reflectional symmetry then because r and b are odd there must be a red bead opposite a blue bead along the axis of symmetry, which is possible because n is even. Furthermore, the placement of 1 red and 2 blue beads on one side of the axis must be mirrored on the other side. Thus, for $k = g/2$, there are $p = \binom{k+3}{1,2,k}$ ways for these to be placed.

Solving for q we obtain $q = \left[\binom{n}{r,b,g} - np\right]/2n$. Hence

$$p + q = \frac{\binom{n}{r,b,g} + np}{2n} = \frac{\binom{238}{3,5,230} + 238\binom{118}{1,2,115}}{476} .$$

◇

Problem 22 *How many equivalence relations are there on a set of size 6?*

There are 203. Recall that equivalence relations and partitions are different names for the same object: the equivalence classes form a partition. Thus we will count the number N of partitions of the set $S = \{A, B, C, D, E, F\}$, since the particular set of size 6 is irrelevent. We will refer to the following chart in the discussion below.

τ	$\delta(\tau)$	$N(\tau)$
6	1	1
5, 1	1, 1	6
4, 2	1, 1	15
4, 1, 1	2, 1	15
3, 3	2	10
3, 2, 1	1, 1, 1	60
3, 1, 1, 1	3, 1	20
2, 2, 2	3	15
2, 2, 1, 1	2, 2	45
2, 1, 1, 1, 1	4, 1	15
1, 1, 1, 1, 1, 1	6	1

For a particular partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of S denote the *type* of $\mathcal{P}$ to be the multiset $\tau = \tau(\mathcal{P}) = \{|S_1|, \ldots, |S_k|\}$, where we write the members of τ in nonincreasing order. For example, for $\mathcal{P} = \{\{A\}, \{B, F\}, \{C, D\}, \{E\}\}$ we have $\tau(\mathcal{P}) = \{2, 2, 1, 1\}$. Thus the Sum Rule implies that

$$N = \sum_{\tau} N(\tau) ,$$

where $N(\tau)$ denotes the number of partitions of S having type τ.

For a given τ let $\delta_i = \delta_i(\tau)$ be the number of times that i appears in τ. Then every partition of type τ is a partition of S into δ_i parts of size i for each $1 \leq i \leq 6$. Therefore

$$N(\tau) = \binom{6}{\tau} \Big/ \prod_{i=1}^{n} \delta_i! ,$$

where $\binom{6}{\tau}$ is shorthand for $\binom{6}{\tau_1,\ldots,\tau_k}$. In fact, by defining the notation $A! = \prod_{x\in A} x!$ for a set A we can express this more easily as

$$N(\tau) = \frac{6!}{\tau!\delta!} \ ,$$

where $\delta = \{\delta_1, \ldots, \delta_n\}$. We will also write δ in nonincreasing order with suppressed 0s. What remains is to list all possible τ, as shown in the table above, in which we supress set braces for readability.

One can list all τ in this order by constructing the 'highest' τ not already on the list: greedily place the highest number possible in each coordinate without repeating a previous τ. ◇

Problem 23 *Consider two colorings of the vertices of a triangular prism as equivalent if one can be obtained from the other by rotating it. How many inequivalent colorings contain 1 red, 2 blue, and 3 green vertices?*

Wherever the red vertex is originally, rotate the prism so that it is the top vertex of the triangle facing you. This fixes the prism in place so no more rotations are possible. Then there are three cases, based on the number of blue vertices on this triangle. If both vertices are blue then there is one such coloring since all three greens are on the back triangle. If one vertex on the front is blue then there are two choices for which it is, and three choices for which of the back three vertices is the other blue one, for a total of 6 disctinct colorings. If no front vertex is blue then the remaining front vertices are green, and there are three choices for which of the back vertices is the other green. In total, there are $1 + 6 + 3 = 10$ inequivalent colorings. ◇

If we are also allowed to reflect the prism then a similar argument yields $1 + 3 + 2 = 6$ inequivalent colorings. In the second case, both front blue colorings are equivalent, and in the third case, the back green vertex is either adjacent to the front red or not.

Problem 24 *Let* $A = \{(a_1, \ldots, a_k) \mid k \geq 1, \text{ each } a_i \geq 1, \sum_{i=1}^k a_i = 45\}$. *That is, A is the set of all ordered positive compositions of 45 of any length. Find $|A|$.*

We prove that $|A| = 17592186044416 = 2^{44}$. Let $n = 45$ and $A_k = \{(a_1, \ldots, a_k) \mid \text{ each } a_i \geq 1, \sum_{i=1}^k a_i = n\}$, so that $\{A_1, \ldots, A_n\}$ partitions A and the Sum Rule implies that $|A| = \sum_{i=k}^n |A_k|$. With the substitution $b_i = a_i - 1$ for all $1 \leq i \leq k$ we have $\sum_{i=1}^k b_i = n - k$ and every $b_i \geq 0$. By the usual stars-and-bars argument, we obtain $|A_k| = \binom{(n-k)+(k-1)}{k-1} = \binom{n-1}{k-1}$, so that $|A| = \sum_{k=1}^n \binom{n-1}{k-1} = \sum_{t=0}^{n-1} \binom{n-1}{t} = 2^{n-1}$. ◇

A combinatorial proof follows by interpreting each k-composition in A by n stars with $k-1$ bars, such that no two bars are adjacent. That is, for each of the $n-1$ spaces between the n stars, we may place a bar or not, resulting in 2^{n-1} such arrangements. ◇

Problem 25 *How many integral solutions are there to the system* $\sum_{i=1}^{25} x_i = 350$ *with each* $x_i \geq i$*?*

There are 63205303218876 of them. Define $y_i = x_i - i$ for every $1 \leq i \leq 25$. Then

$$\begin{aligned}\sum_{i=1}^{25} y_i &= \sum_{i=1}^{25}(x_i - i) \\ &= \sum_{i=1}^{25} x_i - \sum_{i=1}^{25} i\end{aligned}$$

$$= \; 350 - \binom{26}{2} \; = \; 25 \; ,$$

and every $y_i \geq 0$. Thus there are $\binom{25+24}{24} = \binom{49}{24}$ solutions. $\diamond$

Problem 26 *Find and prove a closed formula for* $\left\{ {m \atop 3} \right\}$ *when* $m \geq 3$. *(Here* $\left\{ {m \atop k} \right\}$ *denotes the number of ways to partition* m *people into* k *nonempty sets.)*

We use the known formula for $\left\{ {m \atop 2} \right\}$ to write

$$\begin{aligned}
\left\{ {m \atop 3} \right\} &= \sum_{j=0}^{m-1} \binom{m-1}{j} \left\{ {j \atop 2} \right\} \\
&= \sum_{j=2}^{m-1} \binom{m-1}{j} \left(2^{j-1} - 1\right) \\
&= \frac{1}{2} \sum_{j=2}^{m-1} \binom{m-1}{j} 2^j - \sum_{j=2}^{m-1} \binom{m-1}{j} \\
&= \frac{1}{2} \left[\sum_{j=0}^{m-1} \binom{m-1}{j} 2^j - \big(1 + 2(m-1)\big) \right] - \left[\sum_{j=2}^{m-1} \binom{m-1}{j} - \big(1 + (m-1)\big) \right] \\
&= \frac{1}{2} 3^{m-1} - 2^{m-1} + \frac{1}{2} \\
&= \frac{3^{m-1} - 2^m + 1}{2} \; .
\end{aligned}$$

$\diamond$

Another proof uses induction and the Pascal-type recurrence relation for Stirling numbers of the second kind. After noticing that the base case holds at $m = 3$, we have, for $m \geq 4$,

$$\begin{aligned}
\left\{ {m \atop 3} \right\} &= \left\{ {m-1 \atop 2} \right\} + 3 \left\{ {m-1 \atop 3} \right\} \\
&= (2^{m-2} - 1) + 3(3^{m-2} - 2^{m-1} + 1)/2 \\
&= (2^{m-1} - 2 + 3^{m-1} - 3(2^{m-1}) + 3)/2 \\
&= (3^{m-1} - 2^m + 1)/2 \; .
\end{aligned}$$

$\diamond$

A third proof is more direct, double-counting the number of surjections $f : [m] \to [3]$. On the one hand the number is $3! \left\{ {m \atop 3} \right\}$. On the other hand, the number can be counted by subtracting from the number of all functions to $[3]$ the number of those that are not onto. The number of functions onto exactly one element of $[3]$ is 3. The number of functions onto exactly two elements of $[3]$ is $3(2^m - 2)$: $3 = \binom{3}{2}$ for the two elements in the image, 2^m for all functions to two elements, and -2 for image exactly one of the two elements. Thus we obtain $\left\{ {m \atop 3} \right\} = [3^m - 3 - 3(2^m - 2)]/3! = (3^{m-1} - 2^m + 1)/2$. $\diamond$

Problem 27 *Use induction and a Pascal-type Stirling relation to prove for all* $1 \leq k \leq n$ *that*

$$\left\{ {n+1 \atop k+1} \right\} = \sum_{i=k}^{n} \left\{ {i \atop k} \right\} (k+1)^{n-i} \; .$$

When $n = k$ the equation reads $\left\{ {n+1 \atop n+1} \right\} = \left\{ {n \atop n} \right\}$, which is true since both sides equal 1. Now suppose that the equation is true for all $n = t - 1 \geq k$ and consider the case $n = t$. We have

$$
\begin{aligned}
\left\{ {t+1 \atop k+1} \right\} &= (k+1)\left\{ {t \atop k+1} \right\} + \left\{ {t \atop k} \right\} \\
&= (k+1)\sum_{i=k}^{t-1} \left\{ {i \atop k} \right\}(k+1)^{t-1-i} + \left\{ {t \atop k} \right\} \\
&= \sum_{i=k}^{t-1} \left\{ {i \atop k} \right\}(k+1)^{t-i} + \left\{ {t \atop k} \right\}(k+1)^0 \\
&= \sum_{i=k}^{t} \left\{ {i \atop k} \right\}(k+1)^{t-i} \ .
\end{aligned}
$$

$\diamond$

A bijective proof is as follows. Define a function $f : \left\{ {[n+1] \atop k+1} \right\} \longrightarrow \binom{[n+1]}{k+1}$ by $f(\{P_1, \ \ldots, P_{k+1}\}) = \{\min(P_1), \ldots, \min(P_{k+1})\}$. Then we partition $\left\{ {[n+1] \atop k+1} \right\}$ according to $\max(f(P))$. The number of partitions $P \in \left\{ {[n+1] \atop k+1} \right\}$ having $\max(f(P)) = i+1$ is $\left\{ {i \atop k} \right\}(k+1)^{n-i}$ because the restriction of P to $[i]$ is a k-partition (since none of the elements of $[i]$ can be in the same part with $i+1$), every integer larger than $i+1$ can be in any of the $k+1$ parts (since none of them can be smallest in its part), and there are $(n+1)-(i+1) = n-i$ of them.

Problem 28 *Use induction to prove that*

$$
\left\{ {n \atop k} \right\} = \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i} i^n
$$

for all $1 \le k \le n$.

We will prove the statement for all $n \ge 0$ and all $k \ge 0$ instead, using induction on n and interpreting $0^0 = 1$. Our induction step will require that $n > 0$ and $k > 0$, so we need to cover the cases $n = 0$ and $k = 0$ in the base step. For $n = k = 0$ we have $\left\{ {0 \atop 0} \right\} = 1 = \frac{1}{1}[(1)\binom{0}{0}0^0]$. For $k = 0$, $n > 0$ we have $\left\{ {n \atop 0} \right\} = 0 = \frac{1}{1}[(1)\binom{0}{0}0^n]$. For $n = 0$, $k > 0$ we have $\left\{ {0 \atop k} \right\} = 0 = \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}$.

Now suppose that the equality holds for all $n \le t-1$ and consider the case $n = t$, $k > 1$. Then

$$
\begin{aligned}
\left\{ {t \atop k} \right\} &= k\left\{ {t-1 \atop k} \right\} + \left\{ {t-1 \atop k-1} \right\} \\
&= k\frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}i^{n-1} + \frac{(-1)^{k-1}}{(k-1)!}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}i^{n-1} \\
&= \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\left(\frac{k}{i}\right)\left[\binom{k}{i} - \binom{k-1}{i}\right] i^n \\
&= \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\left(\frac{k}{i}\right)\binom{k-1}{i-1} i^n \\
&= \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i} i^n \ .
\end{aligned}
$$

$\diamond$

A different induction proof proceeds on k as follows. Again, the base case holds as above, but now we note that the summation can be expanded over all $-\infty < i < \infty$ since the binomial coefficient annihilates terms outside the explicit range. Likewise, we can write the recurrence relation below over all $0 \le j \le n-1$, as the range is similarly implied. Now we assume that the statement holds for all $k \le r-1$, and consider the case $k = r$, $n > 1$.

$$\begin{aligned}
\left\{ n \atop r \right\} &= \sum_{j=0}^{n-1} \binom{n-1}{j} \left\{ j \atop r-1 \right\} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^{r-1}}{(r-1)!} \sum_i (-1)^i \binom{r-1}{i} i^j \\
&= \frac{(-1)^{r-1}}{(r-1)!} \sum_i (-1)^i \binom{r-1}{i} \sum_{j=0}^{n-1} \binom{n-1}{j} i^j \\
&= \frac{(-1)^{r-1}}{(r-1)!} \sum_i (-1)^i \binom{r-1}{i} (i+1)^{n-1} \\
&= \frac{(-1)^{r-1}}{(r-1)!} \sum_s (-1)^{s-1} \binom{r-1}{s-1} s^{n-1} \\
&= \frac{(-1)^r}{r!} \sum_s (-1)^s \left(\frac{r}{s}\right) \binom{r-1}{s-1} s^n \\
&= \frac{(-1)^r}{r!} \sum_s (-1)^s \binom{r}{s} s^n \,.
\end{aligned}$$

◇

A third approach rewrites the equality as $k!\left\{ n \atop k \right\} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$. The left side counts the number of surjections $f : [n] \to [k]$. We will see soon how the technique of inclusion-exclusion counts the same on the right. ◇

Problem 29 *Prove for all nonnegative p, g and b that*

$$\left[p \atop g+b \right] \binom{g+b}{g} = \sum_k \binom{p}{k} \left[k \atop g \right] \left[p-k \atop b \right] .$$

(Here $\left[n \atop k \right]$ denotes the number of ways to seat n people around k nonempty circular tables under rotational equivalence.)

We count the number of ways to arrange p people into g green circular tables and b blue circular tables. On the one hand, we first arrange the p people into $g+b$ circular tables in $\left[p \atop g+b \right]$ ways, and then color g of the $g+b$ tables green in $\binom{g+b}{g}$ ways. The Product Rule establishes the left hand side. On the other hand, we partition the set of all such arrangements according to the number k of people sitting at green tables. To count these arrangements, we first choose the k people in $\binom{p}{k}$ ways, then arrange the k people into g green circular tables in $\left[k \atop g \right]$ ways, and finally arrange the remaining $p-k$ people into b blue circular tables in $\left[p-k \atop b \right]$ ways. The Product Rule determines the count for each k, and the Sum Rule establishes the right hand side. ◇

Notice that the result is trivially true if $g+b < 0$ or $g+b > p$, as both sides equal 0. Thus in the following inductive proof we do not need to be concerned with the value of $g+b$, except only to prove it in the case that $0 \le g+b \le p$. When $p = 0$ we have $g+b = 0$ and so both left and right sides equal 1.

Now suppose that $p = t+1$ and that the result is true when $p = t$. Then

$$\begin{aligned}
\begin{bmatrix} t+1 \\ g+b \end{bmatrix}\binom{g+b}{g} &= \begin{bmatrix} t \\ g+b-1 \end{bmatrix}\binom{g+b}{g} + t\begin{bmatrix} t \\ g+b \end{bmatrix}\binom{g+b}{g} \\
&= \begin{bmatrix} t \\ g+b-1 \end{bmatrix}\binom{g+b-1}{g-1} + \begin{bmatrix} t \\ g+b-1 \end{bmatrix}\binom{g+b-1}{g} + t\sum_k \binom{t}{k}\begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} p-k \\ b \end{bmatrix} \\
&= \sum_k \binom{t}{k}\left\{\begin{bmatrix} k \\ g-1 \end{bmatrix}\begin{bmatrix} t-k \\ b \end{bmatrix} + \begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} t-k \\ b-1 \end{bmatrix} + (t-k)\begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} t-k \\ b \end{bmatrix} + k\begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} t-k \\ b \end{bmatrix}\right\} \\
&= \sum_k \binom{t}{k}\left\{\left(\begin{bmatrix} k \\ g-1 \end{bmatrix} + k\begin{bmatrix} k \\ g \end{bmatrix}\right)\begin{bmatrix} t-k \\ b \end{bmatrix} + \begin{bmatrix} k \\ g \end{bmatrix}\left(\begin{bmatrix} t-k \\ b-1 \end{bmatrix} + (t-k)\begin{bmatrix} t-k \\ b \end{bmatrix}\right)\right\} \\
&= \sum_k \binom{t}{k}\begin{bmatrix} k+1 \\ g \end{bmatrix}\begin{bmatrix} t-k \\ b \end{bmatrix} + \sum_k \binom{t}{k}\begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} t-k+1 \\ b \end{bmatrix} \\
&= \sum_k \left\{\binom{t}{k-1} + \binom{t}{k}\right\}\begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} t-k+1 \\ b \end{bmatrix} \\
&= \sum_k \binom{t+1}{k}\begin{bmatrix} k \\ g \end{bmatrix}\begin{bmatrix} t+1-k \\ b \end{bmatrix}.
\end{aligned}$$

◇

Problem 30 *Prove that* $\begin{bmatrix} 1449 \\ 1032 \end{bmatrix} = \sum_{k=0}^{1032}(416+k)\begin{bmatrix} 416+k \\ k \end{bmatrix}$.

We will prove by induction on m that $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m}(n+k)\begin{bmatrix} n+k \\ k \end{bmatrix}$ for all $m, n \geq 0$. The special case in which $m = 1032$ and $n = 416$ answers the question. For the base case of $m = 0$ we have $\begin{bmatrix} n+1 \\ 0 \end{bmatrix} = 0$ for all $n \geq 0$, as well as $\sum_{k=0}^{0}(n+k)\begin{bmatrix} n+k \\ k \end{bmatrix} = n\begin{bmatrix} n \\ 0 \end{bmatrix} = 0$ for all $n \geq 0$.

Now we assume that the equality holds for all $n \geq 0$ when $m < t$ and consider the case $m = t$ with any $n \geq 0$. Here we use a Pascal-type Stirling relation to derive

$$\begin{aligned}
\begin{bmatrix} t+n+1 \\ t \end{bmatrix} &= \begin{bmatrix} t+n \\ t-1 \end{bmatrix} + (t+n)\begin{bmatrix} t+n \\ t \end{bmatrix} \\
&= \sum_{k=0}^{t-1}(n+k)\begin{bmatrix} n+k \\ k \end{bmatrix} + (t+n)\begin{bmatrix} t+n \\ t \end{bmatrix} \\
&= \sum_{k=0}^{t}(n+k)\begin{bmatrix} n+k \\ k \end{bmatrix}.
\end{aligned}$$

◇

A combinatorial proof proceeds as follows. Let $P = \{p_1, p_2, \ldots, p_{m+n+1}\}$ and $S = \begin{bmatrix} P \\ m \end{bmatrix}$. (We think of P as a set of people.) Then $|P| = m+n+1$ so that $|S| = \begin{bmatrix} m+n+1 \\ m \end{bmatrix}$. We will partition $S = \cup_{k=0}^{m} S_k$ so that $|S| = \sum_{k=0}^{m} |S_k|$, and we will do so in such a way so that $|S_k| = (n+k)\begin{bmatrix} n+k \\ k \end{bmatrix}$. This will prove the (general) result.

Now, for $k = m-j$, define S_k to be those m-cycle permutations for which, in their respective cycles, each of $p_1, \ldots, p_j$ are alone and p_{j+1} is not alone. Then there are $(m+n+1)-(j+1) = n+k$ people remaining to consider, and $m-j = k$ cycles remaining to be filled. Once filled, p_{j+1} could join any of these cycles in $n+k$ ways. Hence $|S_k| = (n+k)\begin{bmatrix} n+k \\ k \end{bmatrix}$. ◇

Problem 31 *Prove for all* $0 \leq k \leq n$ *that*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \sum_{i=k}^{n}\begin{bmatrix} n \\ i \end{bmatrix}\binom{i}{k}.$$

We use induction on n. First, we expand the set of inputs to include $k = -1$ (still $n \geq 0$), and change the summation range to $i \geq 0$. This doesn't change the sum since $\binom{i}{k} = 0$ when $0 \leq i < k$, and $\left[{n \atop i}\right] = 0$ when $i > n$.

The bases cases we will need are (1) $k = -1, n \geq 0$, and (2) $k = n \geq 0$. In case (1) we have $\left[{n+1 \atop 0}\right] = 0$, as well as $\sum_{i=0}^{n} \left[{n \atop i}\right]\binom{i}{-1} = 0$, since each $\binom{i}{-1} = 0$. In case (2) we have $\binom{n+1}{n+1} = 1$, as well as $\sum_{i=0}^{n} \left[{n \atop i}\right]\binom{i}{n} = \left[{n \atop n}\right]\binom{n}{n} = 1$.

Now suppose that the statement is true for the case $n = t-1$ (for all $-1 \leq k \leq t-1$) and consider the case $n = t$ (with $0 \leq k \leq t-1$). Then we write

$$\begin{aligned}
\begin{bmatrix} t+1 \\ k+1 \end{bmatrix} &= \begin{bmatrix} t \\ k \end{bmatrix} + t\begin{bmatrix} t \\ k+1 \end{bmatrix} \\
&= \sum_{i\geq 0} \begin{bmatrix} t-1 \\ i \end{bmatrix}\binom{i}{k-1} + t\sum_{i\geq 0} \begin{bmatrix} t-1 \\ i \end{bmatrix}\binom{i}{k} \\
&= \sum_{i\geq 0} \begin{bmatrix} t-1 \\ i \end{bmatrix}\left\{\binom{i}{k-1} + t\binom{i}{k}\right\} \\
&= \sum_{i\geq 0} \begin{bmatrix} t-1 \\ i \end{bmatrix}\binom{i+1}{k} + \sum_{i\geq 0} \begin{bmatrix} t-1 \\ i \end{bmatrix}(t-1)\binom{i}{k} \\
&= \sum_{i\geq 0} \left\{\begin{bmatrix} t-1 \\ i-1 \end{bmatrix} + (t-1)\begin{bmatrix} t-1 \\ i \end{bmatrix}\right\}\binom{i}{k} \\
&= \sum_{i\geq 0} \begin{bmatrix} t \\ i \end{bmatrix}\binom{i}{k}\,.
\end{aligned}$$

◇

A bijective proof is tricky, but here it is. Suppose that p is a permutation of $[n+1]$ having $k+1$ cycles. We write p in standard cycle form $p = \prod_{j=1}^{k+1} p_j$, where each $p_j = (a_{j,1} \cdots a_{j,m_j})$ is a cycle of length m_j satisfying $a_{j,1} > a_{j,r}$ for every $r > 1$, and where each $a_{j,1} < a_{j+1,1}$. Of course, $a_{k+1,1} = n+1$. We map p to an ordered pair (q, S) where, for some $k \leq i \leq n$, q is a permutation of $[n]$ having i cycles and S is a subset of $[i]$ of size k.

First define $p'_{k+1} = a_{k+1,2} \cdots a_{k+1,m_j}$ and think of it as being written in standard cycle form without its parenthesis. Second, let i' denote the number of cycles of p'_{k+1} and define the permutation q of $[n]$ as $q = p'_{k+1}\prod_{j=1}^{k} p_j$. Notice that q has $i = k + i'$ cycles. Third, write $q = \prod_{j=1}^{i} q_j$ in standard cycle form and define S to be the set of all j such that q_j is one of the cycles of p'_{k+1}.

The map is easily reversible.

Problem 32 *Prove for all $n \geq 1$ that the number of partitions of n into even parts equals the number of partitions of n into parts of even multiplicity.*

Let $E(n)$ be the set of all partitions of n into even parts and $M(n)$ be the set of all partitions of n into parts of even multiplicity. If $p \in E(n)$ with list representation $\{p_1, \ldots, p_r\}$ then $n = \sum_{i=1}^{r} p_i$, which is even since each p_i is even. If $q \in M(n)$ with type representation $(m_1, \ldots, m_n)$ then $n = \sum_{j=1}^{n} jm_j$, which is even since each m_j is even. A bijection between $E(n)$ and $M(n)$ in the case that n is even is given by conjugation. Because conjugation is its own inverse, we need only show that it is well defined in both directions.

Indeed, let $p \in E(n)$ with list representation $\{p_1, \ldots, p_r\}$ and let q be its conjugate with list representation $\{q_1, \ldots, q_s\}$. Then we have $q_{2j-1} = \big|\{i \mid p_i \geq 2j-1\}\big| = \big|\{i \mid p_i \geq 2j\}\big| = q_{2j}$ since there are no odd p_is. Therefore every integer found in q appears in pairs; i.e. $q \in M(n)$.

Inversely, let $q \in M(n)$ with type representation $(m_1, \ldots, m_n)$ and let $p = \{p_1, \ldots, p_r\}$ be its conjugate. Then $p_i = \sum_{j=i}^{n} m_j$, which is even since each m_j is even; i.e. $p \in E(n)$. ◇

Another bijection is defined as follows. Let $p \in E(n)$ have list representation $\{p_1, \ldots, p_r\}$. Since each p_i is even we may define $q_{2i-1} = q_{2i} = p_i/2$. Every integer found in the partition $q = \{q_1, \ldots, q_{2r}\}$ appears in pairs; i.e. $q \in M(n)$. Inversely, any $q \in M(n)$ has list representation $\{q_1, \ldots, q_{2r}\}$, with $q_{2i} = q_{2i-1}$. Hence the partition $p = \{p_1, \ldots, p_r\} \in E(n)$, where $p_i = q_{2i-1} + q_{2i}$. ◇

Problem 33 *Find the number of ways that 12 couples can be split into 5 nonempty committees so that no couple is on the same committee.*

It is 33762010546176. Here we have $n = 12$ and $k = 5$. In general we use PIE, the t^{th} stage being the $\binom{n}{t}$ ways to choose t couples to act like one person, times the number of ways to partition the $2n - t$ "people" into k committees. PIE yields

$$\sum_{t=0}^{n} (-1)^t \binom{n}{t} \left\{ {2n-t \atop k} \right\}$$

ways, in this case

$$\sum_{t=0}^{12} (-1)^t \binom{12}{t} \left\{ {24-t \atop 5} \right\} .$$

◇

An alternative use of PIE first considers the committees as labeled and then divides by $k!$ — here the t^{th} stage puts everyone into at most $k - t$ committees. The couples are kept separate throughout by allowing the women all possible choices, with the men only having choices different from their partner. We obtain

$$\frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{k-t} (k-t)^n (k-t-1)^n ,$$

which equals

$$\frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n (i-1)^n .$$

◇

Problem 34 *Use the Principle of Inclusion-Exclusion to prove, for all* $1 \le k \le n$ *that*

$$n! \binom{n-1}{k-1} = \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^{\overline{n}} .$$

We use the fact that there are $s^{\overline{b}}$ arrangements of b distinct books onto s (possibly empty) bookshelves, labelled $1, \ldots, s$. Let F be the set of all arrangements of n distinct books onto k nonempty bookshelves. Because there are $n^{\underline{k}}$ ways to decide which books will be first on each of the shelves, and the remaining $b = n - k$ books need to be placed to the $s = k$ shelves with shelves allowed to be empty of these remaining books, we have $|F| = n^{\underline{k}} k^{\overline{n-k}}$, which equals the left hand side.

Probably an easier way to express this is to order the books in $n!$ ways, then choose which of the $n - 1$ spaces between the books get the $k - 1$ shelf dividers — putting no two dividers in the same spaces guarantees nonempty shelves.

The right hand side counts the same, as follows. For each $I \subseteq [k]$, let F_I be the set of all arrangements of n distinct books onto the (possibly empty) shelves with labels in I. Since $|F_I| = |F_J|$ whenever $|I| = |J|$, PIE implies that

$$\left|F\right| = \left|F_{[k]}\right| - \binom{k}{k-1}\left|F_{[k-1]}\right| + \binom{k}{k-2}\left|F_{[k-2]}\right| - \cdots = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\left|F_{[k-j]}\right| .$$

Since we know that $|F_I| = |I|^{\overline{n}}$, the result follows from the substitution $i = k - j$. ◇

Problem 35 *A collection of 18 distinct beads consists of striped and solid versions of each of 9 colors. Using all of the beads, find the number of inequivalent necklaces for which beads of the same color are not adjacent.*

There are 59809766768640 of them. Let B be a collection of $2n$ distinct beads consisting of striped and solid versions of each of n colors. Let $[n]$ be the set of colors and denote by $S = S(n)$ the set of all inequivalent necklaces of B for which beads of the same color are not adjacent. Furthermore, for each $I \subseteq [n]$ let S_I the the set of all inequivalent necklaces for which each pair of beads having color in I are adjacent. Because $|S_I| = |S_J|$ for all $|I| = |J|$, PIE implies that

$$\left|S\right| = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left|S_{[k]}\right| .$$

Now consider the necklaces in $S_{[k]}$. For the moment, for each $i \in [k]$, glue the striped and solid beads of color i together and consider them as one bead. Then there are $2n-k$ distinct beads, which make $(2n-k-1)!/2$ inequivalent necklaces. Also, the unglued of the beads of color i can be arranged on the necklace in two ways, and hence $|S_{[k]}| = 2^k(2n-k-1)!/2$. Therefore,

$$\left|S\right| = \frac{1}{2}\sum_{k=0}^{n}(-2)^k\binom{n}{k}(2n-k-1)! .$$

◇

Problem 36 *Prove that* 26585679462804 *is the coefficient of* x^{21} *in*

$$\frac{x^6}{\prod_{t=1}^{6}(1-tx)} .$$

Note that $26585679462804 = \left\{ {21 \atop 6} \right\}$. First we generalize the problem by defining, for all $k \geq 0$,

$$f_k(x) = \frac{x^k}{\prod_{i=1}^{k}(1-ix)} .$$

Then we also write $f_k(x) = \sum_{n\geq 0} a_{k,n}x^n$ and claim that $a_{k,n} = \left\{ {n \atop k} \right\}$ for all $k, n \geq 0$, which we prove by induction on n.

When $n = 0$ we have $\left\{ {0 \atop k} \right\} = 1$ when $k = 0$ and 0 when $k > 0$. Also, since $(1-ix)^{-1} = \sum_{j\geq 0}(ix)^j$, we have that $f_k = x^k\prod_{i=1}^{k}\sum_{j\geq 0}(ix)^j$, in which the smallest exponent of x that appears is k. Hence $a_{k,0} = 1$ when $k = 0$ and 0 when $k > 0$.

Furthermore, when $k=0$ we have $f_0(x)=1$, and so $a_{0,n}=1$ when $n=0$ and 0 when $n>0$. This matches that $\left\{ {n \atop 0} \right\}=1$ when $n=0$ and 0 when $n>0$.

Now suppose that the result holds for $n=m-1>0$ and notice that

$$f_k(x)=\left(\frac{x}{1-kx}\right)f_{k-1}(x)\ .$$

Thus we can write the following sequence of equalities, supposing that $k>0$.

$$\begin{aligned}(1-kx)\sum_{m\geq 0}a_{k,m}x^m &= x\sum_{m\geq 0}a_{k-1,m}x^m\\ \sum_{m\geq 0}a_{k,m}x^m-\sum_{m\geq 0}ka_{k,m}x^{m+1} &= \sum_{m\geq 0}a_{k-1,m}x^{m+1}\\ \sum_{m\geq 1}\left(a_{k,m}-ka_{k,m-1}\right)x^m &= \sum_{m\geq 1}a_{k-1,m-1}x^m\end{aligned}$$

From this we derive $a_{k,m}=a_{k-1,m-1}+ka_{k,m-1}=\left\{ {m-1 \atop k-1} \right\}+k\left\{ {m-1 \atop k} \right\}=\left\{ {m \atop k} \right\}$. ◇

Problem 37 *Find the coefficient of* x^{48} *in* $\left(\frac{1}{1-x^3}\right)^{40}$.

It is $29749251314475=\binom{55}{16}$. We show that the coefficient of y^k in $f(y)=\left(\frac{1}{1-y}\right)^n$ is $\binom{n+k-1}{k}$. The result follows with $y=x^3$.

Since $\frac{1}{1-y}=\sum_{i\geq 0}x^i$, we have that $f(y)=\left(\sum_{i\geq 0}x^i\right)^n=\prod_{j=1}^n\sum_{i\geq 0}x^i=\prod_{j=1}^n\sum_{i_j\geq 0}x^{i_j}$. Every choice of $i_1,\ldots,i_n$ for which $\sum_{j=1}^n i_j=k$ results in an x^k term, so the coefficient of x^k equals the number of solutions to $\sum_{j=1}^n i_j=k$ with each $i_j\geq 0$, which is $\binom{n+k-1}{k}$. ◇

An inductive proof works as well. When $n=1$ we have $\frac{1}{1-y}=\sum_{i\geq 0}x^i$, for which the coefficient of y^k equals $1=\binom{k}{k}$. Now suppose that $\left(\frac{1}{1-y}\right)^t=\sum_{k\geq 0}\binom{t+k-1}{k}y^k$ for some $t\geq 1$. Then

$$\begin{aligned}\left(\frac{1}{1-y}\right)^{t+1} &= \left(\frac{1}{1-y}\right)\left(\frac{1}{1-y}\right)^t\\ &= \left(\sum_{i\geq 0}y^i\right)\left(\sum_{j\geq 0}\binom{t+j-1}{j}y^j\right)\\ &= \sum_{k\geq 0}\left(\sum_{j=0}^k\binom{t+j-1}{j}\right)y^k\\ &= \sum_{k\geq 0}\binom{t+1+k-1}{k}y^k\ .\end{aligned}$$

◇

A different induction proof uses differentiation, the induction step of which follows.

$$\begin{aligned}t(1-y)^{-t-1} &= D_y\left[(1-y)^{-t}\right]\\ &= D_y\left[\sum_k\binom{t+k-1}{k}y^k\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_k k\binom{t+k-1}{k}y^{k-1} \\
&= \sum_k (k+1)\binom{t+k}{k+1}y^k \ .
\end{aligned}
$$

Therefore,

$$
(1-y)^{-(t+1)} = \sum_k \frac{k+1}{t}\binom{t+k}{k+1}y^k = \sum_k \binom{t+k}{k}y^k = \sum_k \binom{(t+1)+k-1}{k}y^k \ .
$$

$\diamond$

Problem 38 *Use generating functions to find a pair of nonstandard (6-sided, none of which are blank) dice having the same distribution of sums as a standard pair of dice.*

Let $S(x) = \sum_{k=1}^{6} x^k$ be the generating function for a standard die. That is, the number of ways to get a sum of k on a single role is the coefficient of x^k in $S(x)$. Thus the generating function for the number of ways to get a sum of k from two dice is $S(x)^2$. If two nonstandard dice, say a red one and a blue one with generating functions $R(x)$ and $B(x)$, respectively, have the same distribution of sums as the standard pair, then $R(x)B(x) = S(x)^2$.

Now $S(x) = x(1+x)(1+x+x^2)(1-x+x^2)$, so

$$
R(x)B(x) = x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2 \ .
$$

We know that neither $R(x)$ nor $B(x)$ has a constant term (there is no way to roll a zero), so each contains x as a factor. Also, since each die is 6-sided, $R(1) = B(1) = 6$, which means that each contains both $(1+x)$ and $(1+x+x^2)$ as factors.

Finally, because they differ from standard dice, one of them, say red, contains $(1-x+x^2)^2$. Hence $R(x) = x(1+x)(1+x+x^2)(1-x+x^2)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8$ and $B(x) = x(1+x)(1+x+x^2) = x + 2x^2 + 2x^3 + x^4$. That is, the red die has sides 1, 3, 4, 5, 6, and 8, while the blue die has sides 1, 2, 2, 3, 3, and 4. $\diamond$

If we allow for blank sides then more solutions arise because the factors $x^2(1-x+x^2)^2$ can be split up in any way other than each die getting $x(1+x)$. The three new possibilities of nonstandard pairs are

$$
\begin{aligned}
&(0,2,3,4,5,7),(2,3,3,4,4,5) \ , \\
&(2,4,5,6,7,9),(0,1,1,2,2,3) \ , \quad \text{and} \\
&(0,1,2,3,4,5),(2,3,4,5,6,7) \ .
\end{aligned}
$$

There are more solutions when the number of sides can vary (of course, the product of the number of sides must be 36); for example, $(1,2,2,3)$ and $(1,3,3,5,5,5,7,7,9)$.

Problem 39 *Use generating functions to prove, for all $m,n,k \geq 0$, that*

$$
\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k} \ .
$$

We use the binomial theorem a few times.

$$
\begin{aligned}
\sum_{k\geq 0}\left[\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}\right]x^k &= \left(\sum_{i\geq 0}\binom{m}{i}x^i\right)\left(\sum_{j\geq 0}\binom{n}{j}x^j\right) \\
&= (1+x)^m(1+x)^n \\
&= (1+x)^{m+n} \\
&= \sum_{k\geq 0}\binom{m+n}{k}x^k \ .
\end{aligned}
$$

Because the two series are identical, the corresponding coefficients of each x^k are equal for all $k \geq 0$. $\diamond$

Problem 40 *Use Newton's Binomial Theorem to prove that the PGF for the sequence* $\{\binom{2n}{n}\}_{n\geq 0}$ *is*

$$\frac{1}{\sqrt{1-4x}} \; .$$

NBT states that $(1-4x)^{-1/2} = \sum_{k\geq 0} \binom{-1/2}{k}(-4x)^k$. We use the relation $\binom{-\alpha}{k} = (-1)^k\binom{\alpha+k-1}{k}$ to show that

$$\begin{aligned}
\binom{-1/2}{k} &= \binom{k-1/2}{k} \\
&= \frac{(-1)^k}{k!}\prod_{i=1}^{k}\frac{2i-1}{2} \\
&= \frac{(-1)^k}{k!}\prod_{i=1}^{k}\left(\frac{2i-1}{2}\right)\left(\frac{2i}{2i}\right) \\
&= \frac{(-1)^k}{k!}\left(\frac{(2k)!}{2^{2k}k!}\right) \\
&= \frac{1}{(-4)^k}\binom{2k}{k} \; .
\end{aligned}$$

Hence $(1-4x)^{1/2} = \sum_{k\geq 0}\binom{2k}{k}x^k$. $\diamond$

The solution can be found combinatorially as well, by using the result of Challenge Problem 1. If one could show that $\sum_k \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$ then we would know that $\left[\sum_k \binom{2k}{k}x^k\right]^2 = \sum_n \left(\sum_k \binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n = \sum_n 4^n x^n = (1-4x)^{-1}$. The result would follow, therefore, by extracting square roots (the positive and negative square roots pairing up accordingly by their evaluation at $x=0$). Now, 4^n is the number of plus-minus sequences of length $2n$. Every such sequence A starts with (empty) partial sum 0, and may or may not contain other partial 0-sums (necessarily of even length). For $0 \leq k \leq n$ let P_k denote those sequences whose longest partial 0-sum is of length $2k$. If A is such a sequence and B is its longest initial 0-sum subsequence, then $A = BC$, where C is positive and of length $2n-2k$. The result of Challenge Problem 1 is that there are exactly $\binom{2n-2k}{n-k}$ such sequences. Since $\{P_k\}$ is a partition, the Sum Rule finishes the proof. $\diamond$

Problem 41 *Find the PGF for the sequence* $\{k^3\}_{k\geq 0}$.

We begin with $(1-x)^{-1} = \sum_{k\geq 0} x^k$, which we differentiate with respect to x to obtain $(1-x)^{-2} = \sum_{k\geq 0} kx^{k-1}$, which becomes $x(1-x)^{-2} = \sum_{k\geq 0} kx^k$ after multiplying by x. Differentiation and multiplication by x again results in

$$2x(1-x)^{-3} + (1-x)^{-2} = \sum_{k\geq 0} k^2 x^{k-1}$$

and

$$\frac{x^2+x}{(1-x)^3} = \sum_{k\geq 0} k^2 x^k \; ,$$

after simplifying the left hand side. Repeating this process one last time yields

$$x\frac{(1-x)^3(2x+1)+(x^2+x)3(1-x)^2}{(1-x)^6} = x\sum_{k\geq 0} k^3x^{k-1} ,$$

which simplifies to

$$\frac{x^3+4x^2+x}{(1-x)^4} = \sum_{k\geq 0} k^3x^k .$$

$\diamond$

One can approach this from another angle by using the identity $k^3 = \binom{k}{3} + 4\binom{k+1}{3} + \binom{k+2}{3}$, so that

$$\begin{aligned}
\sum_{k\geq 0} k^3x^k &= \sum_{k\geq 0}\left[\binom{k}{3} + 4\binom{k+1}{3} + \binom{k+2}{3}\right]x^k \\
&= \sum_{k\geq 0}\binom{k}{3}x^k + 4\sum_{k\geq 0}\binom{k+1}{3}x^k + \sum_{k\geq 0}\binom{k}{3}x^k \\
&= \sum_{k\geq 0}\binom{k+3}{3}x^{k+3} + 4\sum_{k\geq 0}\binom{k+3}{3}x^{k+2} + \sum_{k\geq 0}\binom{k+3}{3}x^{k+1} \\
&= (x^3+4x^2+x)\sum_{k\geq 0}\binom{k+3}{3}x^k \\
&= \frac{x^3+4x^2+x}{(1-x)^4} .
\end{aligned}$$

The identity $k^3 = \binom{k}{3} + 4\binom{k+1}{3} + \binom{k+2}{3}$ can be verified combinatorially by counting the number of functions $f : [3]\to[k]$. A generalization of the argument yields one solution to Challenge Problem 12. $\diamond$

Problem 42 *For $n \geq 0$ let $d(n)$ and $o(n)$ denote the number of ways to partition n into distinct parts and odd parts, respectively (by convention, set $d(0) = o(0) = 1$). Use PGFs to prove that $d(n) = o(n)$ for all n.*

The PGF for $\{d(n)\}_{n\geq 0}$ is $\prod_{k\geq 1}(1+x^k)$, since each part size k can be used at most once. The PGF for $\{o(n)\}_{n\geq 0}$ is $\prod_{t \text{ odd}}\sum_{i\geq 0} x^{it} = \prod_{t \text{ odd}}(1-x^t)^{-1}$, since each odd part size t can be used arbitrarily often. These two PGFs are related by

$$\prod_{k\geq 1}(1+x^k) = \prod_{k\geq 1}\left(\frac{1-x^{2k}}{1-x^k}\right) = \prod_{s \text{ even}}(1-x^s)\Big/\prod_{k\geq 1}(1-x^k) = \prod_{t \text{ odd}}\frac{1}{(1-x^t)} .$$

$\diamond$

There is a combinatorial proof of this result. Suppose n is partitioned into $p = \{m_k \cdot k\}_{k \text{ odd}}$, where m_k denotes the multiplicity of k in the partition p. For each k let $m_k = \sum_{j\geq 0} b_{k,j}2^j$ be the unique binary representation of m_k (each $b_{k,j} \in \{0,1\}$). Then $n = \sum_{k \text{ odd}} m_k k = \sum_{k \text{ odd}}\left(\sum_{j\geq 0} b_{k,j}2^j\right)k = \sum_{j\geq 0}\left(\sum_{k \text{ odd}} b_{k,j}k\right)2^j$. Because every positive integer is uniquely factored into a power of 2 times an odd number, the numbers $\{2^jk\}_{\substack{j\geq 0\\ k \text{ odd}}}$ are distinct. Since the multiplicities $b_{k,j} \in \{0,1\}$, each number 2^jk appears at most once in $q = \{b_{k,j}2^jk\}_{\substack{j\geq 0\\ k \text{ odd}}}$, Thus q is a partition of n into distinct parts. The argument is easily reversible. $\diamond$

Problem 43 *Use PGFs to find a closed formula for the sequence $\{a_n\}_{n\geq 0}$, where $a_0 = 7, a_1 = 105, a_2 = 699$ and $a_n = 7a_{n-1} - 36a_{n-3}$ for $n \geq 3$.*

Problem 44 *Find a particular solution to the recurrence*

$$g_n = 21g_{n-1} - 147g_{n-2} + 343g_{n-3} - 432n + 2592 \quad (n \geq 3)\ , \tag{13}$$

where $g_0 = 12, g_1 = 158$ and $g_2 = 1518$. Then use the characteristic polynomial to find the homogeneous solution to (13). Finally, calculate the general formula for g_n for all $n \geq 0$.

Thus (13) implies that

$$\begin{aligned} 0 &= p_n - 21p_{n-1} + 147p_{n-2} - 343p_{n-3} + 432n - 2592 \\ &= \big(dn+e\big) - 21\big(d(n-1)+e\big) + 147\big(d(n-2)+e\big) - 343\big(d(n-3)+e\big) + 432n - 2592 \\ &= \big[d(1-21+147-343)+432\big]n + \big[d(21-294+1029) + e(1-21+147-343) - 2592\big] \\ &= \big[-216d+432\big] + \big[756d - 216e - 2592\big]\ . \end{aligned}$$

Hence $d = 432/216 = 2$ and $e = (756d - 2592)/216 = -5$.

The characteristic polynomial of (13) is $r^3 - 21r^2 + 147r - 343 = (r-7)^3$, which means that its homogeneous solution is $h_n = (an^2 + bn + c)7^n$, for some a, b, c. This yields $g_n = h_n + p_n = (an^2 + bn + c)7^n + 2n - 5$. The cases $n \in \{0, 1, 2\}$ produce the system

$$\left(\begin{array}{rrr|r} 0 & 0 & 1 & 17 \\ 7 & 7 & 7 & 161 \\ 196 & 98 & 49 & 1519 \end{array}\right) \equiv \left(\begin{array}{rrr|r} 0 & 0 & 1 & 17 \\ 1 & 1 & 1 & 23 \\ 4 & 2 & 1 & 31 \end{array}\right) \equiv \left(\begin{array}{rrr|r} 0 & 0 & 1 & 17 \\ 1 & 1 & 0 & 6 \\ 3 & 1 & 0 & 8 \end{array}\right) \equiv \left(\begin{array}{rrr|r} 0 & 0 & 1 & 17 \\ 1 & 1 & 0 & 6 \\ 2 & 0 & 0 & 2 \end{array}\right)\ ,$$

having the solution $(a, b, c) = (1, 5, 17)$. Consequently, we arrive at the general formula of $g_n = (n^2 + 5n + 17)7^n + (2n - 5)$. $\diamond$