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Thresholds for random distributions on graph sequences with applications to pebbling

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Abstract

Let $\mathbf{G} = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a sequence of graphs with G_n having n vertices and having a random distribution of $t(n)$ pebbles to its vertices. If $s \geqslant 2$ is an integer, the event that G_n has s or more vertices with two or more pebbles has threshold $t(n) = \Theta(\sqrt{n})$. If $t(n) = c\sqrt{n}$, then the limiting distribution for the number of vertices with multiple pebbles is Poisson(c^2). The threshold for the event that G_n has at least one vertex with s or more pebbles is $t(n) = \Theta(n^{(s-1)/s})$. These results are used to establish new bounds for thresholds for pebbling on sequences of graphs with bounded diameters. If for some d, diameter $(G_n) \leqslant d$ for all n, and if for some $p \in (0,1]$, maximum degree $(G_n) \subseteq \Omega(n^p)$, then the threshold th($\mathbf{G}$) for the solvability of $\mathbf{G}$ is in $O(n^{1-0.5p})$.

Let G_n be a connected graph with n vertices and a distribution of t pebbles on the vertices of G_n. A pebbling step consists of removing two pebbles from a vertex v and placing one pebble on an adjacent vertex to v. For a given vertex r, we say the distribution is r-solvable if a pebble may be placed on r by a sequence of pebbling steps. In this case, we say that it is possible to "pebble" to r. The distribution is solvable if it is r-solvable for every vertex r in G_n. A central question is for a given G_n and t, what proportion of all distributions of t pebbles to G_n are solvable?

Certain natural families of graphs are best described as sequences of graphs indexed by the number of vertices. The family of complete graphs K_n is a good example. Let $\mathbf{G} = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a sequence of graphs with G_n having n vertices and let

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$t=t(n)$ be a natural number. Let $P_G(n,t)$ be the proportion of all distributions of t pebbles onto G_n that are solvable. If we think of the distribution as selected at random from all possible distributions of t pebbles on G_n then $P_G(n,t)$ is the probability that G_n is solvable. Here, the random selection is made uniformly with all distinguishable distributions having the same probability (the pebbles are indistinguishable, but of course the vertices are distinguishable).

For a given graph sequence, we are interested in finding a threshold function so that the graphs in the sequence will become solvable with high probability if the number of pebbles is essentially greater than the threshold function. In other words, if the number of pebbles $t(n)$ grows more quickly than the threshold function then the probability G_n is solvable tends to one, and if the number of pebbles grows more slowly than the threshold function the probability G_n is solvable tends to zero.

The following definitions and notations are taken from [3]. We write

$$f \ll g \quad \text{or} \quad g \gg f \quad \text{if} \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Define

$$\mathrm{o}(g)=\{f \mid f \ll g\} \quad \text{and} \quad \omega(g)=\{f \mid g \ll f\}.$$

$$\mathrm{O}(g)=\{f \mid \exists c>0,\ \&\ k>0 \text{ such that } f(n)/g(n)<c \text{ for all } n>k\},$$

$$\Omega(g)=\{f \mid \exists c>0,\ \&\ k>0 \text{ such that } g(n)/f(n)<c \text{ for all } n>k\},$$

$$\Theta(g)=\mathrm{O}(g)\cap\Omega(g).$$

Then $f\in\mathrm{o}(g)\Leftrightarrow g\in\omega(f)$ and $\mathrm{o}(g)\subset\mathrm{O}(g)$ and $\omega(g)\subset\Omega(g)$.

Let $\mathbf{G}=\{G_1,G_2,\ldots,G_n,\ldots\}$ be a sequence of graphs. We say a function f is a threshold for $\mathbf{G}$ and write $f\in\mathrm{th}(\mathbf{G})$ if $\lim_{n\to\infty} P_G(n,t)=1$ whenever $t\gg f$ and $\lim_{n\to\infty} P_G(n,t)=0$ whenever $t\ll f$. More generally, let E be some pebble distribution property and let $P_G(n,t;E)$ be the probability that G_n with a random distribution of t pebbles possesses the property E. Then we say the function f is a threshold for property E and write $f\in\mathrm{th}(\mathbf{G};E)$ if $\lim_{n\to\infty} P_G(n,t;E)=1$ whenever $t\gg f$ and $\lim_{n\to\infty} P_G(n,t;E)=0$ whenever $t\ll f$.

For a survey of known results for graph sequences see [3,4]. In [1] it is established that threshold functions always exits for every graph sequence. It is known that the threshold of any sequence belongs to $\Omega(n^{1/2})\cap\mathrm{o}(n^{1+\varepsilon})$. A few graph sequences are known to have the threshold $\Theta(n^{1/2})$. The following theorem and corollary appear in [2]. Theorem 1 and its proof are included here because it is the building block for all of the subsequent results in the paper.

Theorem 1. *Let $\mathbf{G}=\{G_1,G_2,\ldots,G_n,\ldots\}$ be a graph sequence and let E_0 be the event that no vertex of G_n has two or more pebbles. Then $\lim_{n\to\infty} P_G(n,c\sqrt{n};E_0)=\mathrm{e}^{-c^2}$. The event $\bar{E}_0$ that at least one vertex of G_n has two or more pebbles has $\mathrm{th}(\mathbf{G};\bar{E}_0)=\Theta(\sqrt{n})$.*

Proof. The total number of distributions of t pebbles to G_n is

$$N = \binom{n+t-1}{t}.$$

The total number of distributions of t pebbles to G_n with no vertex receiving two or more pebbles is $N_0 = \binom{n}{t}$. The probability that G_n has no vertices with two or more pebbles is

$$P = P_G(n,t;E_0) = \frac{N_0}{N} = \frac{n!}{t!(n-t)!} \div \frac{(n+t-1)!}{t!(n-1)!} = \frac{n\cdots(n-t+1)}{(n-1+t)\cdots n}$$
$$= \left(\frac{n}{n-1+t}\right)\cdots\left(\frac{n-t+1}{n}\right).$$

Notice this last product of fractions is written from largest factor to the smallest. Therefore,

$$\left(\frac{n-t+1}{n}\right)^t < P < \left(\frac{n}{n-1+t}\right)^t.$$

Letting the number of pebbles be $t = c\sqrt{n}$ we have

$$\left(\frac{n-c\sqrt{n}+1}{n}\right)^{c\sqrt{n}} < P < \left(\frac{n}{n-1+c\sqrt{n}}\right)^{c\sqrt{n}}.$$

To simplify these bounds let $x = \sqrt{n}$.

$$\left(\frac{x^2-cx+1}{x^2}\right)^{cx} < P < \left(\frac{x^2}{x^2-1+cx}\right)^{cx}$$

or

$$\left(\left(1-\frac{c}{x}+\frac{1}{x^2}\right)^x\right)^c < P < \left(\left(1-\frac{c}{x}+\mathrm{O}\left(\frac{1}{x^2}\right)\right)^x\right)^c.$$

Letting $n \to \infty$, we have $x \to \infty$ and by an application of l'Hopital's Rule we get that both of these bounds tend to e^{-c^2}. Now let $\bar{E}_0$ be the event that G_n has at least one vertex with two or more pebbles. Then as $n \to \infty$, $P(\bar{E}_0) \to 1 - \mathrm{e}^{-c^2}$. Notice that $\lim_{c\to 0}(1-\mathrm{e}^{-c^2}) = 0$ and $\lim_{c\to\infty}(1-\mathrm{e}^{-c^2}) = 1$. If the number of pebbles is $t \ll \sqrt{n}$ then for any $c > 0$, $t < c\sqrt{n}$ for sufficiently large n. In this case $P(\bar{E}_0) \to 0$. On the other hand, if the number of pebbles is $t \gg \sqrt{n}$ then for any $c > 0$, $t > c\sqrt{n}$ for sufficiently large n. In this case $P(\bar{E}_0) \to 1$. This shows that $\mathrm{th}(\mathbf{G}; \bar{E}_0) = \Theta(\sqrt{n})$. □

Let $\mathbf{K} = \{K_1, K_2, \ldots, K_n, \ldots\}$ be the sequence of complete graphs. If a vertex v of K_n receives two pebbles then K_n is solvable in one pebbling step since every vertex is adjacent to the vertex v. On the other hand, if $t(n) < n$ and no vertex of K_n receives two pebbles then K_n is unsolvable.

Corollary 2. $\mathrm{th}(\mathbf{K}) = \Theta(\sqrt{n})$.

Theorem 3. *Let $G = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a graph sequence and let E be the event that at least s vertices of G_n receive two or more pebbles. Event E has threshold* $\mathrm{th}(G; E) = \Theta(\sqrt{n})$.

Proof. Let E_i be the event that exactly i vertices of G_n receive two pebbles and the rest of the vertices have zero or one pebble. Let F be the event that at least one vertex of G_n has three or more pebbles. Then

$$E = \left(\bigcup_{i=s}^{\lfloor t/2 \rfloor} E_i \right) \cup FE \quad \text{(disjoint union) and}$$

$$\bar{E} = \left(\bigcup_{i=0}^{s-1} E_i \right) \cup F\bar{E} \quad \text{(disjoint union).}$$

As before the total number of distributions of t pebbles to the n vertices of G_n is

$$N = \binom{n+t-1}{t}.$$

Let $|E_i|$ denote the total number of distributions in the event E_i. Each of the distributions in the event E_i is a partition of the vertices of G_n; i vertices receive two pebbles each, $t-2i$ vertices receive one pebble each, and the remaining $n-t+i$ vertices receive no pebbles. Hence

$$|E_i| = \frac{n!}{i!(t-2i)!(n-t+i)!}, \quad P(E_i) = \frac{|E_i|}{N}$$

and

$$\frac{P(E_{i+1})}{P(E_i)} = \frac{n!i!(t-2i)!(n-t+i)!}{(i+1)!(t-2i-2)!(n-t+i+1)!n!} = \frac{(t-2i)(t-2i-1)}{(i+1)(n-t+i+1)}.$$

Let $t(n) = c\sqrt{n}$. Then

$$\frac{P(E_{i+1})}{P(E_i)} = \frac{(c\sqrt{n}-2i)(c\sqrt{n}-2i-1)}{(i+1)(n-c\sqrt{n}+i+1)} \to \frac{c^2}{i+1} \quad \text{as } n \to \infty.$$

By Theorem 1, $P(E_0) \to \mathrm{e}^{-c^2}$. Combining these last two results, a recursive argument shows

$$P(E_i) \to \frac{\mathrm{e}^{-c^2} c^{2i}}{i!}$$

and

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor t(n)/2 \rfloor} P(E_i) = \mathrm{e}^{-c^2} \left(1 + c^2 + \frac{c^4}{2!} + \frac{c^6}{3!} + \cdots \right) = \mathrm{e}^{-c^2} e^{c^2} = 1.$$

This shows that as $n \to \infty$ the E_i's consume all the probability. Consequently, $P(F) \to 0$ and hence, $P(F\bar{E}) \to 0$. Therefore,

$$P(\bar{E}) = \sum_{i=0}^{s-1} P(E_i) + P(F\bar{E}) \to \mathrm{e}^{-c^2}\left(1 + c^2 + \frac{c^4}{2!} + \cdots + \frac{c^{2s-2}}{(s-1)!}\right) + 0 = \mathrm{e}^{-c^2} f(c),$$

where

$$f(c) = \left(1 + c^2 + \frac{c^4}{2!} + \cdots + \frac{c^{2s-2}}{(s-1)!}\right).$$

Finally,

$$P(E) \to 1 - \frac{f(c)}{\mathrm{e}^{c^2}} \quad \text{and} \quad 1 - \frac{f(c)}{\mathrm{e}^{c^2}} \to \begin{cases} 0 & \text{as } c \to 0^+, \\ 1 & \text{as } c \to \infty. \end{cases}$$

This shows event E has threshold $\mathrm{th}(G;E) = \Theta(\sqrt{n})$ as claimed. □

We will show later (Theorem 11) that event F in the above proof has threshold $\mathrm{th}(G;F) = \Theta(n^{2/3})$. Let X be the number of vertices with two or more pebbles. The above proof shows that $P(X = i) \to \mathrm{e}^{-c^2} c^{2i}/i!$, which is a Poisson probability.

Corollary 4. *Let $\mathbf{G} = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a graph sequence with G_n receiving $c\sqrt{n}$ pebbles distributed at random. Let X be the number of vertices with two or more pebbles. Then the limiting distribution for X is Poisson(c^2). In particular, the expected value $E(X) \to c^2$ and the standard deviation $\sigma(X) \to c$.*

Up to this point most of our results are just combinatorial results about random distributions to the vertices of graphs. The next theorem is the main application to pebbling thresholds for the solvability of graph sequences.

Theorem 5. *Let $G = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a sequence of graphs with G_n having n vertices. Suppose there exists constants d, a, and p with $a > 0$, $0 < p \leqslant 1$, and d a positive integer, such that diameter$(G_n) \leqslant d$ for all n, and maximum degree$(G_n) \geqslant an^p$ for all n. Then* $\mathrm{th}(G) \subseteq \mathrm{O}(n^{1-0.5p})$.

Before proving this theorem we will state some corollaries. Define the following sequences of graphs:

$\mathbf{S} = \{S_1, \ldots, S_n, \ldots\}$, where S_n is the star on n vertices.
$\mathbf{W} = \{W_1, \ldots, W_n, \ldots\}$, where W_n is the wheel n vertices.
$\mathbf{B} = \{K_{1,1}, \ldots, K_{m,m}, \ldots\}$, where $K_{m,m}$ is the complete bipartite.
$\mathbf{K} \times \mathbf{K} = \{K_1 \times K_1, \ldots, K_m \times K_m, \ldots\}$, where $K_m \times K_m$ is the cross product of the complete graph K_m on m vertices with itself.

Corollary 6. (i) $\mathrm{th}(\mathbf{S}) = \Theta(\sqrt{n})$.
(ii) $\mathrm{th}(\mathbf{W}) = \Theta(\sqrt{n})$.

(iii) $\mathrm{th}(\mathbf{B}) = \Theta(\sqrt{n})$.
(iv) $\mathrm{th}(\mathbf{K} \times \mathbf{K}) \subseteq \mathrm{O}(n^{3/4})$.

Proof. (i) The maximum degree of S_n is $n - 1$. Therefore, we can take $a = 0.5$ (say), $p = 1$, and $d = 2$.

(ii) The maximum degree of W_n is $n - 1$. Therefore, we can take $a = 0.5$ (say), $p = 1$, and $d = 2$.

(iii) Each vertex of $K_{m,m}$ has degree m. Therefore, $a = 0.5$, $p = 1$, and $d = 2$.

(iv) Each vertex of $K_m \times K_m$ has degree $2m - 2$ and $K_m \times K_m$ has order $n = m^2$. Therefore, $a = 1$ (say), $p = 0.5$, and $d = 2$. □

The following corollary is an improvement to the $\mathrm{O}(n)$ upper bound of Corollary 3.5 of [3]:

Corollary 7. *Let $G = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a sequence of graphs with G_n having n vertices and with diameter$(G_n) \leqslant d$ for all n. Then* $\mathrm{th}(G) \subseteq \mathrm{O}(n^{1-1/2d})$. *In particular, if $d = 2$,* $\mathrm{th}(G) \subseteq \mathrm{O}(n^{3/4})$, *and if $d = 3$,* $\mathrm{th}(G) \subseteq \mathrm{O}(n^{5/6})$.

Proof. The order n, the diameter d and the maximum degree Δ of a graph are related by the inequality

$$n \leqslant 1 + [(\Delta - 1)^d - 1]\frac{\Delta}{\Delta - 2}.$$

This implies that $\Delta > an^{1/d}$ for some $a > 0$. The result follows from Theorem 5 with $p = 1/d$. □

The next task is to prove Theorem 5. The basic idea is as follows. Let v be a vertex of G_n with degree at least an^p. If at least 2^d neighbors of v have two or more pebbles, then we can transfer 2^d pebbles to v. Since G_n has diameter no larger than d, a pebble can now be transferred to any other vertex from v. By Theorem 3, this scenario occurs when the number of pebbles on the neighborhood of v reaches the threshold $\sqrt{an^p}$. We can expect to achieve this threshold when the total number of pebbles on G_n reaches the threshold $n^{1-0.5p}$. The actual number of pebbles Y falling on the neighborhood of v is a random variable. We'll need to show that Y exceeds $c\sqrt{an^p}$ with probability that tends to 1. For this reason we need to compute the expected value and standard deviation of Y.

Let G_n be a graph with vertices numbered 1 through n. Let X_i be the number of pebbles on vertex i if t pebbles are distributed randomly to G_n. The following three lemmas appear in the appendix of the paper [3]:

Lemma 8. $E(X_i) = t/n$.

Lemma 9. $E(X_i^2) = (2t^2 + t(n - 1))/n(n + 1)$.

Lemma 10. $E(X_iX_j) = (t^2 - t)/n(n + 1)$, $i \neq j$.

Using these lemmas we compute the variance and the covariance.

$$V(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{2t^2 + t(n-1)}{n(n+1)} - \frac{t^2}{n^2} = \frac{t(t+n)(n-1)}{n^2(n+1)}.$$

$$\mathrm{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j) = \frac{t^2 - t}{n(n+1)} - \frac{t^2}{n^2} = \frac{-tn - t^2}{n^2(n+1)}.$$

Note than the covariance is negative. Let $Y_m = \sum_{i=1}^m X_i$ be the total number of pebbles falling on a subgraph with m vertices. Then

$$E(Y_m) = \frac{mt}{n}.$$

$$V(Y_m) = \sum_{i=1}^m V(X_i) + 2 \sum_{1 \leqslant i < j \leqslant m} \mathrm{Cov}(X_i, X_j) < \sum_{i=1}^m V(X_i) = \frac{mt(t+n)(n-1)}{n^2(n+1)}.$$

Let σ_{Y_m} be the standard deviation of Y_m. Let $m = an^p$ (the degree) and let $t = cn^{1-0.5p}$ (the total number of pebbles to be distributed). Then

$$\frac{\sigma_{Y_m}}{E(Y_m)} < \frac{\sqrt{\dfrac{mt(t+n)(n-1)}{n^2(n+1)}}}{\dfrac{mt}{n}} = \frac{\sqrt{\dfrac{an^p cn^{1-0.5p}(cn^{1-0.5p} + n)(n-1)}{n^2(n+1)}}}{\dfrac{an^p cn^{1-0.5p}}{n}}$$

$$= \sqrt{\frac{(n-1)(cn^{1-0.5p})}{ac(n+1)n^{1+0.5p}}} \in \Theta\left(\frac{1}{n^{0.25p}}\right).$$

Therefore, as $n \to \infty$, $\sigma_{Y_m}/E(Y_m) \to 0$. It follows from Chebyshev's inequality that $P(Y_m > \frac{1}{2}E(Y_m)) \to 1$ as $n \to \infty$. The bottom line is that if we desire to exceed a certain target number of pebbles in a particular subgraph choose t so that the expected number of pebbles in the subgraph is twice (or any multiple greater than one of) the target. The probability we exceed half the expected number (i.e., exceed the target number) goes to 1 as n goes to infinity.

We can now prove Theorem 5.

Proof of Theorem 5. Let v_n be a vertex of G_n that has degree at least an^p and let H_n be a set of an^p vertices adjacent to v_n. If 2^d or more vertices in H_n receive two or more pebbles, then G_n can be solved as follows. Transfer 2^d pebbles to v_n and then since diameter$(G_n) \leqslant d$, one can pebble to any other vertex.

According to Theorem 3, the above event occurs when the number of pebbles falling on H_n is $\Theta(\sqrt{an^p})$. To exceed any particular target number $c\sqrt{an^p}$ of pebbles on H_n let $t = (2c/\sqrt{a})n^{1-0.5p}$. Then the expected number of pebbles falling on H_n is

$$\frac{mt}{n} = an^p \frac{(2c/\sqrt{a})n^{1-0.5p}}{n} = 2c\sqrt{an^p}.$$

The necessity of the extra factor of “2” is explained in the comments before the proof. We have shown that if $t \geqslant n^{1-0.5p}$ then the probability that G_n is solvable tends to 1 as n goes to infinity. In other words, $\mathrm{th}(G) \subseteq \mathrm{O}(n^{1-0.5p})$. □

The next theorem is another threshold result for random distributions to the vertices of graphs. Its applications to pebbling are more modest than Theorem 5. It duplicates the results of Corollary 7 in one case and in the other cases it is not as good. The proof of the theorem is a lot of work but the theorem may find better applications in the future.

Theorem 11. *Let $\mathbf{G} = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a sequence of graphs with G_n having n vertices and let $s \geqslant 2$ be an integer. Let E be the event that G_n has at least one vertex with s or more pebbles. Event E has threshold* $\mathrm{th}(\mathbf{G}; E) = \Theta(n^{(s-1)/s})$.

Before proving the above theorem, we need the following intuitively obvious lemma. The reader is invited to find a shorter proof.

Lemma 12. *Let G_n be a graph with n vertices numbered 1 through n and with t pebbles distributed randomly. Let E_i be the event that vertex i receives s or more pebbles where s is some fixed positive integer. Then $P(\bar{E}_1 \cdots \bar{E}_k) \leqslant P(\bar{E}_1) \cdots P(\bar{E}_k)$.*

Proof. We will show $P(\bar{E}_1 \cdots \bar{E}_k) < P(\bar{E}_1) \cdots P(\bar{E}_k)$ by induction on k. First, we must show that $P(E_i E_j) < P(E_i)P(E_j)$ for $i \neq j$.

Let

$$A = P(E_i E_j) = \frac{\binom{n+t-2s-1}{t-2s}}{N},$$

where

$$N = \binom{n+t-1}{t}$$

and let

$$B = P(E_i)P(E_j) = \frac{\binom{n+t-s-1}{t-s}^2}{N^2}.$$

We need to show that $A/B < 1$.

$$\frac{A}{B} = \frac{(n+t-2s-1)![(t-s)!(n-1)!]^2(n+t-1)!}{(t-2s)!(n-1)![(n+t-s-1)!]^2 t!(n-1)!}$$

$$= \frac{(n-t-2s-1)!(n+t-1)!(t-s)!(t-s)!}{(n+t-s-1)!(n+t-s-1)!(t-2s)!t!}$$

$$= \prod_{m=0}^{s-1} \frac{(n+t-1-m)(t-s-m)}{(n+t-s-1-m)(t-m)}.$$

$$= \prod_{m=0}^{s-1} \frac{(n+t-1-m)(t-m) - s(t-m) - s(n-1)}{(n+t-1-m)(t-m) - s(t-m)}.$$

Since $s(n-1) > 0$ (unless $n=1$, but in this case Lemma 12 is trivially true), the numerator in every fraction making up the factors of the product is smaller than the denominator, we have $A/B < 1$.

An elementary exercise shows that $P(E_iE_j) < P(E_i)P(E_j)$ for $i \neq j$ implies that $P(\bar{E}_i\bar{E}_j) < P(\bar{E}_i)P(\bar{E}_j)$ for $i \neq j$. This completes the first step in the inductive argument.

Now assume that $P(\bar{E}_1 \cdots \bar{E}_{k-1}) < P(\bar{E}_1) \cdots P(\bar{E}_{k-1})$ for all n and t. Let $A_i = \{X_k = i\}$ be the event that the kth vertex receives i pebbles. Then

$$P(\bar{E}_1 \cdots \bar{E}_{k-1}A_i) = P((\bar{E}_1A_i) \cdots (\bar{E}_{k-1}A_i)) < P(\bar{E}_1A_i) \cdots P(\bar{E}_{k-1}A_i)$$
$$< P(\bar{E}_1) \cdots P(\bar{E}_{k-2})P(\bar{E}_{k-1}A_i).$$

The first inequality above is by the induction hypothesis applied in the case of n-1 vertices and t-i pebbles (deleting vertex k from the graph along with its i pebbles).

Finally,

$$P(\bar{E}_1 \cdots \bar{E}_k) = \sum_{i=0}^{s-1} P(\bar{E}_1 \cdots \bar{E}_{k-1}A_i) < \sum_{i=0}^{s-1} P(\bar{E}_1) \cdots P(\bar{E}_{k-2})P(\bar{E}_{k-1}A_i)$$
$$= P(\bar{E}_1) \cdots P(\bar{E}_{k-2})P(\bar{E}_{k-1}\bar{E}_k)$$
$$< P(\bar{E}_1) \cdots P(\bar{E}_{k-2})P(\bar{E}_{k-1})P(\bar{E}_k). \qquad \square$$

Proof of Theorem 11. Number the vertices of G_n 1 through n and let E_i be the event that vertex i receives s or more pebbles. The number of distributions in the event E_i is

$$|E_i| = \binom{n+t-s-1}{t-s}$$

and the probability of E_i is

$$P(E_i) = \frac{|E_i|}{N} = \frac{(n+t-s-1)!}{(t-s)!(n-1)!} \div \frac{(n+t-1)!}{t!(n-1)!}$$
$$= \frac{t(t-1)\cdots(t-s+1)}{(n+t-1)(n+t-2)\cdots(n+t-s)}.$$

Therefore,

$$\left(\frac{t-s+1}{n+t}\right)^s < P(E_i) < \left(\frac{t}{n+t-1}\right)^s.$$

We will use the upper bound to show $\mathrm{th}(\mathbf{G};E) \subset \Omega(n^{(s-1)/s})$ and the lower bound to show $\mathrm{th}(\mathbf{G};E) \subset \mathrm{O}(n^{(s-1)/s})$

$$P(E) = P\left(\bigcup_1^n E_i\right) < \sum_1^n P(E_i) < n\left(\frac{t}{n+t-1}\right)^s$$

Now let $t = cn^{(s-1)/s}$. Then

$$P(E) < \frac{n(cn^{(s-1)/s})^s}{(n+cn^{(s-1)/s}-1)^s} = \frac{c^s n^s}{(n+cn^{(s-1)/s}-1)^s} \to c^s \quad \text{as } n \to \infty.$$

Since $c^s \to 0$ *as* $c \to 0$ any $t \ll cn^{(s-1)/s}$ will cause $P(E)$ to go to zero an $n \to \infty$. Therefore, $\mathrm{th}(\mathbf{G};E) \subset \Omega(n^{(s-1)/s})$.

Now for the other direction.

$$P(E) = P\left(\bigcup_1^n E_i\right) = 1 - P\left(\bigcap_1^n \bar{E}_i\right) > P(E) > 1 - P(\bar{E}_1)\cdots P(\bar{E}_n)$$

$$= 1 - (1 - P(E_1))^n > 1 - \left(1 - \left(\frac{t-s+1}{n+t}\right)^s\right)^n.$$

Again let $t = cn^{(s-1)/s}$. Then

$$P(E) > 1 - \left(1 - \left(\frac{cn^{(s-1)/s}-s+1}{n+cn^{(s-1)/s}}\right)^s\right)^n = 1 - \left(1 - \frac{c^s}{n} + O\left(\frac{1}{n^{(s+1)/s}}\right)\right)^n$$

$$\to 1 - \mathrm{e}^{-c^s} \quad \text{as } n \to \infty.$$

Note that $1 - \mathrm{e}^{-c^s}$ goes to 1 as c goes to infinity. Therefore, if $t \gg cn^{(s-1)/s}, P(E)$ will go to 1 as $n \to \infty$. Thus $\mathrm{th}(\mathbf{G};E) \subset \mathrm{O}(n^{(s-1)/s})$. $\square$

Let $\mathbf{G} = \{G_1, G_2, \ldots, G_n, \ldots\}$ be a sequence of graphs with G_n having n vertices and suppose each G_n has diameter two. If some vertex of G_n receives four or more pebbles then G_n is solvable. The event that at least one vertex receives four or more pebbles has thresholds $\Theta(n^{3/4})$ by Theorem 11. This implies that $\mathrm{th}(\mathbf{G}) \subseteq \mathrm{O}(n^{3/4})$. This duplicates the first special case of Corollary 7.

Suppose instead that the graphs of $\mathbf{G}$ all have diameter three. If some vertex of G_n receives eight or more pebbles then G_n is solvable. At least one vertex receiving eight or more pebbles has threshold $\Theta(n^{7/8})$. This implies that $\mathrm{th}(\mathbf{G}) \subseteq \mathrm{O}(n^{7/8})$. This is not as good as the second special case sited in Corollary 7.

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