Conditions for Weighted Cover Pebbling of Graphs

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Abstract

In a graph G with a distribution of pebbles on its vertices, a pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. A weight function on G is a non-negative integer-valued function on the vertices of G. A distribution of pebbles on G covers a weight function if there exists a sequence of pebbling moves that gives a new distribution in which every vertex has at least as many pebbles as its weight. In this paper we give some necessary and some sufficient conditions for a distribution of pebbles to cover a given weight function on a connected graph G. As a corollary, we give a simple formulation for the 'weighted cover pebbling number' of a weight function W and a connected graph G, defined by Crull et al. to be the smallest number m such that any distribution on G of m pebbles is a cover for W. Also, we prove a cover pebbling variant of Graham's Conjecture for pebbling.

1 Introduction

Suppose k pebbles are placed on the vertices of a graph G. Define a pebbling move on this distribution as the removal of two pebbles from one vertex together with the addition of one pebble to an adjacent vertex. In the game of pebbling, one attempts to place a pebble on a specified vertex in a graph by a sequence of pebbling moves on some starting distribution of pebbles. This game was first suggested by Lagarias and Saks, in an attempt to provide a short alternate proof of a result in additive number theory. Chung [1] later succeeded in creating such a proof and in so doing defined the pebbling number $\pi(G)$ of a connected graph G, the minimum number m such that one can win the pebbling game with any

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target vertex and any starting distribution of m pebbles. See [3] for a more complete treatment of pebbling.

Crull et al. [2] introduced weighted cover pebbling, a variation on the game of pebbling. In weighted cover pebbling, we begin with a connected graph G and a non-negative integer valued 'weight function' on the vertices of G. Starting from an initial distribution of pebbles, a player performs a sequence of pebbling moves in an attempt to create a final distribution on G such that every vertex has at least as many pebbles as its weight. Crull et al. also defined the weighted cover pebbling number of a connected graph G, $\Phi_W(G)$, as the minimum number m such that one can win the cover pebbling game (relative to the weight function W) with any starting distribution of m pebbles.

In this paper we provide some necessary and some sufficient conditions for a given distribution of pebbles to allow a solution for the weighted cover pebbling game, relative to a strictly positive weight function. As a direct corollary of these results, we give a simple formulation of the value $\Phi_W(G)$ for any strictly positive weight function W, which vastly simplifies earlier proofs for particular weighted cover pebbling numbers of hypercubes [4], complete multipartite graphs [5], and trees [2]. This corollary is equivalent to an affirmative answer to question 10 posed in [2].

2 Preliminaries

We now give a list of notations and definitions that characterize the game of cover pebbling. Our characterization attempts to generalize and formalize many of the standard terms previously developed in the literature, both for pebbling and cover pebbling.

Definition 2.1. *Let G be a graph, and let $\mathfrak{I}$ denote the set of all integer-valued functions on $V(G)$. Let $\mathfrak{D}$ denote all functions in $\mathfrak{I}$ that are nonnegative. We call a function D a* distribution *on G if and only if $D \in \mathfrak{D}$.*

We may also call a distribution a *weight function*. Following this, we will refer to a current or previous configuration of pebbles on a graph as a distribution, and reserve use of the term 'weight function' for a desired distribution to be reached after completion of pebbling moves.

Definition 2.2. *Let G be a graph, and let p, q be two adjacent vertices of G. For all $D \in \mathfrak{D}$, define the function $P_{p,q} : \mathfrak{D} \mapsto \mathfrak{I}$ by*

$$[P_{p,q}(D)](s) = \begin{cases} D(s) - 2 & s = p \\ D(s) + 1 & s = q \\ D(s) & \text{otherwise} \end{cases}$$

If $P(D) \in \mathfrak{D}$, then P is called a pebbling move *on D. We also consider the identity function to be a pebbling move on D for all $D \in \mathfrak{D}$.*

Definition 2.3. *Let G be a graph, and let D, D' be distributions on G. If there exists a sequence of functions $P_1, \ldots, P_n$ such that P_i is a pebbling move on $P_{i-1} \circ \ldots \circ P_1(D)$ $\forall i \in [n]$, and*

$$P_n \circ \ldots \circ P_1(D) = D',$$

then we say that D' is derivable from D.

Definition 2.4. *Let G be a graph, and let D, D' be distributions on G. If $D(q) \leq D'(q)$ $\forall q \in V(G)$, then we say that D is contained in D', and if q is a vertex such that $D(q) > D'(q)$, then we say that q is a distribution node of D relative to D'. Also, if there exists a distribution derivable from D that contains D', then we say that D is a cover of D', or D covers D'.*

Definition 2.5. *Let H be an induced subgraph of G, G an induced subgraph of K, and D a distribution on G. By D_H we denote the distribution on H defined by*

$$D_H(q) = D(q) \quad \forall q \in V(H).$$

By D_K we denote the distribution on K defined by

$$D_K(q) = \begin{cases} D(q) & q \in V(G) \\ 0 & q \in V(K) \setminus V(G) \end{cases}.$$

Also, if $S \subseteq V$, by D_S we denote the distribution $D_{G[S]}$ on $G[S]$ and by $D|_S$, the distribution on G such that

$$D|_S(q) = \begin{cases} D(q) & q \in S \\ 0 & q \in S^c \end{cases}.$$

Definition 2.6. *Let S be a nonempty subset of $V(G)$, and let D be a distribution on G. Then we define the* standard value *of D with respect to S as*

$$V_S(D) = \sum_{q \in V(G)} D(q) \cdot 2^{d(q,S)}$$

where

$$d(q, S) = \min_{r \in S} d(q, r).$$

We also define the stacking number *of G relative to W to be*

$$SN_W(G) = \max_{q \in V(G)} V_{\{q\}}(W) = \max_{q \in V(G)} \sum_{u \in V(G)} D(u) \cdot 2^{d(u,q)}.$$

Observation 2.7. *Let S_1, S_2 be two nonempty subsets of $V(G)$, and let D_1, D_2 be two distributions on G. Then the following statements are true:*

1. *If $S_1 \subset S_2$, then $V_{S_1}(D_1) \geq V_{S_2}(D_1)$.*
2. *If D_1 is properly contained in D_2, then $V_{S_1}(D_1) < V_{S_1}(D_2)$.*

3. *If there exists a legal pebbling move P on D_1 s.t. $P(D_1) = D_2$, then $V_{S_1}(D_1) \geq V_{S_1}(D_2)$.*

4. *If D_1 is a cover of D_2, then $V_{S_1}(D_1) \geq V_{S_1}(D_2)$.*

Note that, by property 4, a distribution D covers a weight function W only if

$$V_S(D) \geq V_S(W) \quad \forall\, S \subseteq V(G),\, S \neq \emptyset.$$

3 Principal Result

In this section we present our primary result, a strong sufficient condition for a distribution D to cover a given positive weight function W.

Lemma 3.1. *Let G be a connected graph, W a weight function on G, and D a distribution on G. If D has no distribution nodes relative to W, then D is a cover of W if and only if $D = W$.*

Proof. By assumption, D has no distribution nodes relative to W, thus $D(q) \leq W(q)$ $\forall q \in V(G)$. By Observation 2.7 we have that D covers W only if $V_{V(G)}(D) \geq V_{V(G)}(W)$. Thus D covers W only if $D = W$. The reverse implication is obvious. $\square$

Lemma 3.2. *Let G be a connected graph, W a positive weight function on G, $v_0 \in V(G)$, and D a distribution on G such that $D(s) \leq W(s) \quad \forall s \neq v_0$. If $V_{\{v_0\}}(D) \leq V_{\{v_0\}}(W)$, then there exists a distribution D^* on G contained in W and derivable from D such that $V_{\{v_0\}}(D^*) = V_{\{v_0\}}(D)$.*

Proof. By induction on $D(v_0)$.
If $D(v_0) \leq W(v_0)$, then we simply set $D^* = D$.
Now assume that $D(v_0) > W(v_0)$, and that the lemma holds for any suitable distributions D' with $D'(v_0) < D(v_0)$. Select q in $V(G)$ such that $D(q) < W(q)$ and $d(v_0, q)$ is minimized. Such a point must exist, for otherwise W is properly contained in D and $V_{\{v_0\}}(D) > V_{\{v_0\}}(W)$. Let $v_0, \dots, v_k, q$ be a shortest path from v_0 to q. As q is at minimum distance to v_0, we have that $D(v_i) = W(v_i)\ \forall i \in [k]$. Define the distribution D^I on G by

$$D^I(s) = \begin{cases} D(s) - 2 & s = v_0 \\ D(s) - 1 & s \in \{v_1, \dots, v_k\} \\ D(s) + 1 & s = q \\ D(s) & \text{otherwise} \end{cases}.$$

D^I is derivable from D by $P_{v_k,q} \circ P_{v_{k-1},v_k} \circ \cdots \circ P_{v_0,v_1}(D) = D^I$. Also,

$$V_{\{v_0\}}(D^I) = V_{\{v_0\}}(D) - [2 + 2^1 + \cdots + 2^{d(v_0,q)-1} - 2^{d(v_0,q)}] = V_{\{v_0\}}(D)$$

and $D^I(s) \leq W(s)\ \forall s \neq v_0$, thus D^I satisfies the induction hypothesis for $D(v_0) - 2$. Now we have a distribution D^* on G contained in W and derivable from D^I such that $V_{\{v_0\}}(D^I) = V_{\{v_0\}}(D^*)$. Clearly D^* satisfies for D. $\square$

Note that the proof of Lemma 3.2 implies an algorithm for achieving the desired distribution.

Lemma 3.3. *Let G be a connected graph, W a positive weight function on G, $v_0 \in V(G)$, and D a distribution on G such that $D(s) \leq W(s)$ $\forall s \neq v_0$. Then D covers W if and only if $V_{\{v_0\}}(D) \geq V_{\{v_0\}}(W)$.*

Proof. By Observation 2.7, we have that D covers W only if $V_{\{v_0\}}(D) \geq V_{\{v_0\}}(W)$. Now, assume that $V_{\{v_0\}}(D) \geq V_{\{v_0\}}(W)$. Consider the distribution $D^{\#}$ defined by

$$D^{\#}(q) = \begin{cases} D(v_0) - [V_{\{v_0\}}(D) - V_{\{v_0\}}(W)] & q = v_0 \\ D(q) & \text{otherwise} \end{cases}.$$

$D^{\#}$ satisfies the conditions of Lemma 3.2, thus there exists a distribution D^* on G contained in W and derivable from $D^{\#}$ such that $V_{\{v_0\}}(D^*) = V_{\{v_0\}}(D^{\#})$. As $V_{\{v_0\}}(D^*) = V_{\{v_0\}}(D^{\#}) = V_{\{v_0\}}(D) - [V_{\{v_0\}}(D) - V_{\{v_0\}}(W)] = V_{\{v_0\}}(W)$ and D^* is contained in W, we have that $D^* = W$. D^* is derivable from $D^{\#}$, thus $D^{\#}$ is a cover of W, and as $D^{\#}$ is contained in D, this gives that D is also a cover of W. ∎

Theorem 3.4. *Let G be a connected graph, W a positive weight function on G, and D a distribution on G with a nonempty set of distribution nodes $\{d_1, \ldots, d_n\}$ relative to W. If $V_{\{d_1,\ldots,d_n\}}(D) \geq max_{i \in [n]} V_{\{d_i\}}(W)$, then D covers W.*

Proof. By induction on the number of distribution nodes of D.
Suppose first that D has only distribution node d_1. Then

$$V_{\{d_1\}}(D) \geq V_{\{d_1\}}(W),$$

and by Lemma 3.3 D covers W.
Suppose next that D has distribution nodes $d_1, \ldots, d_n$ with $n \geq 2$, and that the theorem holds for any distribution $\widetilde{D}$ having less than n distribution nodes relative to W. For all $i \in [n]$ define $E(d_i) = \{v \in V(G) \,|\, d(v, d_i) \leq d(v, d_j) \ \forall j \in [n]\}$. Let $G_i = G[E(d_i)], D_i = D_{G_i}, W_i = D_{G_i}$ $\forall i \in [n]$. If D_i covers G_i $\forall i \in [n]$, then it is clear that D covers G. Otherwise $\exists j \in [n]$ such that D_j does not cover G_j. Assume without loss of generality $j = n$. By Lemma 3.2, there exists a distribution D^* on G_n derivable from D_n such that D^* is contained in W_n and

$$V_{\{d_n\}}(D^*) = V_{\{d_n\}}(D_n).$$

Thus,

$$V_{\{d_n\}}(D^*_G) = V_{\{d_n\}}(D|_{E(d_n)}).$$

Consider the distribution

$$D' = D|_{E(d_n)^c} + D^*_G$$

derivable from D. Clearly $\{d_1, \ldots, d_{n-1}\}$ is the set of distribution nodes for D'. Also,

$$\begin{aligned}
V_{\{d_1,\ldots,d_{n-1}\}}(D') &\geq V_{\{d_1,\ldots,d_n\}}(D') \\
&= V_{\{d_1,\ldots,d_n\}}(D|_{E(d_n)^c}) + V_{\{d_1,\ldots,d_n\}}(D^*_G) \\
&= V_{\{d_1,\ldots,d_n\}}(D|_{E(d_n)^c}) + V_{\{d_n\}}(D^*_G) \\
&= V_{\{d_1,\ldots,d_n\}}(D|_{E(d_n)^c}) + V_{\{d_n\}}(D|_{E(d_n)}) \\
&= V_{\{d_1,\ldots,d_n\}}(D|_{E(d_n)^c}) + V_{\{d_1,\ldots,d_n\}}(D|_{E(d_n)}) \\
&= V_{\{d_1,\ldots,d_n\}}(D) \\
&\geq \max_{i\in[n]} V_{\{d_i\}}(W) \\
&\geq \max_{i\in[n-1]} V_{\{d_i\}}(W).
\end{aligned}$$

Thus D' satisfies the induction hypothesis for $n-1$ and D' covers W. Since D' is derivable from D, D also covers W. □

4 Corollaries

Corollary 4.1. *Stacking Theorem*
If G is a connected graph and W is a positive weight function on G, then $\Phi_W(G) = SN_W(G)$.

Proof. Let D be a distribution on G such that $|D| \geq SN_W(G)$, where we define $|D| = \sum_{q\in V(G)} D(q)$. If D has no distribution nodes relative to W, then we have that $|D| \geq SN_W(G) \geq |W|$, thus by Lemma 3.1 we have that D covers W. Otherwise, let $\{d_1, \ldots, d_n\}$ be the set of distribution nodes of D relative to W. Now we have $V_{\{d_1,\ldots,d_n\}}(D) \geq |D| \geq SN_W(G) \geq \max_{i\in[n]} V_{\{d_i\}}(W)$, thus by Theorem 3.4 we have that D covers W.

Let q be a vertex in G such that $V_{\{q\}}(W)$ is maximum. Then the distribution D on G having $V_{\{q\}}(W) - 1 = SN_W(G) - 1$ pebbles on q and 0 pebbles on all other vertices fails to cover W, as we have $V_{\{q\}}(D) = V_{\{q\}}(W) - 1 < V_{\{q\}}(W)$. □

As discussed before, this reduces the problem of finding $\Phi_W(G)$ to a matter of computing

$$SN_W(G) = \max_{v\in V(G)} \sum_{u\in V(G)} W(u) \cdot 2^{d(u,v)}.$$

The corollary that follows further simplifies the computation of $\Phi_W(G)$ in certain cases. Also, this result provides encouragement for those working on a proof for Graham's Conjecture in standard pebbling.

Definition 4.2. *If W_1 is a weight function on a graph G and W_2 is a weight function on a graph H, then we define the weight function $W_1 \times W_2$ on the graph $G \times H$ by*

$$[W_1 \times W_2](g, h) = W_1(g)W_2(h) \quad \forall\, (g, h) \in G \times H.$$

Corollary 4.3. *Let G and H be connected graphs. If W_1 and W_2 are positive weight functions on G and H respectively, then $\Phi_{W_1 \times W_2}(G \times H) = \Phi_{W_1}(G)\Phi_{W_2}(H)$.*

Proof. For all $(g, h) \in G \times H$ we have

$$
\begin{aligned}
SN_{W_1 \times W_2}(g,h) &= \sum_{\substack{g^* \in G \\ h^* \in H}} [W_1 \times W_2](g^*, h^*) 2^{d((g,h),(g^*,h^*))} \\
&= \sum_{h^* \in H} \sum_{g^* \in G} W_1(g^*) W_2(h^*) 2^{d(g,g^*)} 2^{d(h,h^*)} \\
&= \sum_{h^* \in H} W_2(h^*) 2^{d(h,h^*)} \sum_{g^* \in G} W_1(g^*) 2^{d(g,g^*)} \\
&= SN_{W_1}(g) \sum_{h^* \in G} W_2(h^*) 2^{d(h,h^*)} \\
&= SN_{W_1}(g) SN_{W_2}(h).
\end{aligned}
$$

The result now follows easily from the stacking theorem. □

5 Conjectures

Our results rely heavily upon the standard value function, and the relationship between its values both on W and D with respect to particular subsets of $V(G)$. By Observation 2.7 we have that D is a cover of W only if

$$V_S(D) \geq V_S(W) \quad \forall S \subset V(G).$$

This necessary condition can be proven using only properties 2 and 3 of V_S, as given in Observation 2.7. Any function on the set of distributions having both of these properties we call a *general value function*, or simply a value function. It can easily be proven that if A is a value function, then D covers W only if

$$A(D) \geq A(W).$$

We conjecture that the converse is true: that if $A(D) \geq A(W)$ for all value functions A, then D covers W. We leave as an open question whether a stronger condition is true: whether if $V_S(D) \geq V_S(W)$ $\forall S \subset V(G)$, then D covers W.

If G is a connected graph, then $\pi(G)$ can be thought of as the maximum of a finite set of values $\Phi_{W_1}(G), \ldots, \Phi_{W_n}(G)$. Our results do not apply to the computation of these values, as the weight functions involved are not strictly positive. However, it is clear that a good extension of our results to all weight functions would provide a determination of the pebbling numbers of graphs.

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