In the case of Game 9.2.1, suppose that Barbie knows that Ken's strategy is to play his columns in the proportion $\mathbf{x}^* = (12, 0, 16, 0, 5)^\mathsf{T}/33$. Because $\mathbf{A}\mathbf{x}^* = (36, -1, -1, -1)^\mathsf{T}/33$, Barbie would be unwise to ever risk a big payout by playing her first row. However, she could play her remaining rows without caution since they share a common expected value. For example she could play solely row 3. Because any stochastic combination of these values is at least $-1/33$, Ken's strategy guarantees himself at least that much of a win per round, on average.

Similarly, if Ken knows that Barbie will play her rows in the proportion $\mathbf{y}^* = (0, 13, 14, 6)^\mathsf{T}/33$, then since $\mathbf{y}^{*\mathsf{T}}\mathbf{A} = (-1, -10, -1, -16, -1)/33$, Ken would be wise to avoid his second and fourth columns, playing only his columns 1, 3 and 5 in any stochastic manner. For example he could play row 5 only. Barbie's strategy gurantees her losses to be at most $-1/33$ per round, on average. Hence the value of Game 9.2.1 is exactly $-1/33$.

Let's move on to discuss how one might find such strategies $\mathbf{x}^*$ and $\mathbf{y}^*$. We say that a strategy $\mathbf{x}$ (or $\mathbf{y}$) is **pure** if one of its entries is a 1 and the rest are 0; it is **mixed** otherwise. The key observation is the following lemma. pure/mixed strategy

Lemma 9.2.2 *Given any strategy* $\mathbf{x}$ *there exists a pure strategy* $\mathbf{y}_\mathbf{x}$ *that minimizes the value of* $\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x}$ *over all stochastic* $\mathbf{y}$.

Proof. As we have seen above, the notion is to pick the smallest value out of $\mathbf{b} = \mathbf{A}\mathbf{x}$. Formally, let the vector $\mathbf{u}_i$ be the pure strategy with its 1 in position i. Then $\mathbf{u}_i^\mathsf{T}\mathbf{A}\mathbf{x} = b_i$. Now define $b_\mathbf{x} = \min_i b_i$.

Workout 9.2.3 *Prove that every stochastic vector* $\mathbf{y}$ *satisfies* $\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} \geq b_\mathbf{x}$.

Thus the pure response is given by $\mathbf{y}_\mathbf{x} = \mathbf{u}_{i^*}$, where $i^* = \text{argmin}\{b_i\}_{i=1}^m$. $\diamond$

The partner to Lemma 9.2.2 the following lemma, which we state without proof.

Lemma 9.2.4 *Given any strategy* $\mathbf{y}$ *there exists a pure strategy* $\mathbf{x}_\mathbf{y}$ *that maximizes the value of* $\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x}$ *over all stochastic* $\mathbf{x}$. $\diamond$

In light of Lemma 9.2.3, Ken will consider all of his possible strategies $\mathbf{x}$ and find each resulting $b_\mathbf{x}$. His best option, then, is to choose that $\mathbf{x}$ for which $b_\mathbf{x}$ is maximum. If we delve into the proof of Lemma 9.2.3 again, we see that there is another way to define $b_\mathbf{x}$. For a finite set S recall that its minimum $\min(S)$ equals its greatest lower bound $\text{glb}(S)$. In the case of $S = \{b_i\}_{i=1}^m$ we have

$$b_\mathbf{x} = \min(S) = \text{glb}(S) = \max\{x_0 \mid x_0 \leq b_i, i = 1, \ldots, m\} \ . \tag{9.1}$$

We can rephrase even this. Let $\mathbf{J}_k$ be the vector of k ones, and say that $\mathbf{A}$ is $m \times n$. Then Equation (9.1) becomes

$$b_\mathbf{x} = \max\{x_0 \mid x_0\mathbf{J}_m \leq \mathbf{A}\mathbf{x}\} \ . \tag{9.2}$$

Now, since Ken seeks $\max b_\mathbf{x}$ over all stochastic $\mathbf{x}$, we derive from Equation (9.2) his LOP

$$\max\{x_0 \mid \mathbf{J}_n\mathbf{x} = 1,\ x_0\mathbf{J}_m - \mathbf{A}\mathbf{x} \leq \mathbf{0},\ \mathbf{x} \geq \mathbf{0}\} \ . \tag{9.3}$$

Workout 9.2.5

(a) *Write Ken's LOP for Game 9.2.1.*

(b) *Use part a to find the value of Game 9.2.1 and find Ken's optimal strategy.*

Workout 9.2.6 *Use Lemma 9.2.4 to show that Barbie's LOP is*

$$\min\{y_0 \mid \mathbf{J}_m\mathbf{y} = 1,\ y_0\mathbf{J}_n - \mathbf{A}^\mathsf{T}\mathbf{y} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0}\}\ . \tag{9.4}$$

Workout 9.2.7

(a) *Write Barbie's LOP for Game 9.2.1.*

(b) *Use part a to find the value of Game 9.2.1 and find Barbie's's optimal strategy.*

Workout 9.2.8 *Let $\mathbf{A} = (a_{i,j})$ be an $m \times n$ payoff matrix, with respect to Ken, for a game against Barbie.*

(a) *Write Ken's LOP for this game using summation notation.*

(b) *Write Barbie's LOP for this game using summation notation.*

(c) *Prove that these LOPs are dual to each other.*

(d) *Use the General Fundamental Theorem 6.2.9 to prove that both LOPs are optimal.*

Now we can etch in stone the realization that each player can give away their strategy without repurcussion.

Minimax Theorem

Theorem 9.2.9 *For every $m \times n$ matrix $\mathbf{A}$ there are stochastic $\mathbf{x}^*$ and $\mathbf{y}^*$ such that $\min_\mathbf{y} \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x}^* = \max_\mathbf{x} \mathbf{y}^{*\mathsf{T}}\mathbf{A}\mathbf{x}$, where* min *and* max *are taken over all stochastic m- and n-vectors, respectively.*

Proof. We claim that x^* and y^* are the optimal strategies for Ken and Barbie, respectively, in the game defined by payoff matrix $\mathbf{A}$. By Equation (9.3) we have $\mathbf{A}\mathbf{x}^* \geq z^*\mathbf{J}_m$ (with equality in at least one coordinate by the maximization of z^*), so that $\min_\mathbf{y} \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x}^* = z^*$, the optimal value of Ken's LOP. Similarly, Equation (9.4) implies $\max_\mathbf{x} \mathbf{y}^{*\mathsf{T}}\mathbf{A}\mathbf{x} = w^*$, the optimal value of Barbie's LOP. By Workout 9.2.8c and the General Duality Theorem 6.4.1, $z^* = w^*$. ◇

Another formulation of this result is that $\max_\mathbf{x} \min_\mathbf{y} \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} = \min_\mathbf{y} \max_\mathbf{x} \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x}$, which can also be proven in similar fashion with the ideas presented here; see Exercise 9.5.18.

Let's reconsider Game 9.2.1 and it optimal strategies. In hindsight, it makes perfect sense that $x_4^* = 0$. Compare column 4 to column 2 — every entry that differs is worse for Ken in column 4. Thus, if Ken plays an optimal strategy that includes playing column 4, the strategy derived from it by replacing every such play by column 2 can only improve his payoffs, and thus is also optimal.

dominated column

We say that the j^{th} column $\mathbf{A}_{\cdot,j}$ is **dominated** by the k^{th} column $\mathbf{A}_{\cdot,k}$ if $a_{i,j} \leq a_{i,k}$ for every $1 \leq i \leq m$.

Workout 9.2.10 *Let column k dominate column j in payoff matrix $\mathbf{A}$, and let $\mathbf{A}'$ be the matrix $\mathbf{A}$ with its j^{th} column removed. Compare the duals of their respective LOPs to conclude that the value of the game has been preserved.*

dominated row

Likewise, we say that the i^{th} row $\mathbf{A}_{i,\cdot}$ is **dominated** by the k^{th} row $\mathbf{A}_{k,\cdot}$ if $a_{i,j} \geq a_{k,j}$ for every $1 \leq j \leq n$. Here the inequality is reversed because it is Barbie who is trying to avoid paying Ken more than is necessary.

Workout 9.2.11 *Let row k dominate row i in payoff matrix $\mathbf{A}$, and let $\mathbf{A}'$ be the matrix $\mathbf{A}$ with its i^{th} row removed. Compare their respective LOPs to conclude that the value of the game has been preserved.*