Erdős–Ko–Rado theorems for chordal graphs and trees

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One of the more recent generalizations of the Erdős–Ko–Rado theorem, formulated by Holroyd, Spencer and Talbot, defines the Erdős–Ko–Rado property for graphs in the following manner: for a graph G, vertex $v \in G$ and some integer $r \geqslant 1$ denote the family of independent r-sets of $V(G)$ by $\mathcal{J}^{(r)}(G)$ and the subfamily $\{A \in \mathcal{J}^{(r)}(G): v \in A\}$ by $\mathcal{J}_v^{(r)}(G)$, called a star. Then G is said to be r-EKR if no intersecting subfamily of $\mathcal{J}^{(r)}(G)$ is larger than the largest star in $\mathcal{J}^{(r)}(G)$. In this paper, we prove that if G is a disjoint union of chordal graphs, including at least one singleton, then G is r-EKR if $r \leqslant \frac{\mu(G)}{2}$, where $\mu(G)$ is the minimum size of a maximal independent set.

We also prove Erdős–Ko–Rado results for chains of complete graphs, which are special chordal graphs obtained by blowing up edges of a path into complete graphs, as well as prove preliminary results for trees.

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1. Introduction

Let $X = [n] = \{1, \dots, n\}$ be a set of size n. We denote the power set of X by $\mathcal{P} = \mathcal{P}(X) = \{A \mid A \subseteq X\}$. A family $\mathcal{A}$ is a collection of sets in $\mathcal{P}$. $\mathcal{A}$ is said to be an intersecting family if $A, B \in \mathcal{A}$ imply $A \cap B \neq \emptyset$. An intersecting r-uniform hypergraph is an intersecting family where all sets have cardinality r. The problem of finding how large an intersecting family can be is trivial: an intersecting family can have size at most 2^{n-1} with $\mathcal{P}(X_x) = \{A: A \subset X, \ x \in A\}$ being one of the extremal families.

If we consider this problem for intersecting r-uniform hypergraphs, we see that the problem is trivial for $n \leqslant 2r$ because the set of all r-sets in X, denoted by $X^{(r)}$, is intersecting for $n < 2r$, and if $n = 2r$, every family contains exactly one of any two complimentary sets, so the maximum size is at most $\frac{1}{2}\binom{n}{r} = \binom{n-1}{r-1}$.

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If $n > 2r$, then the problem is solved by the Erdős–Ko–Rado theorem [6], one of the seminal results in extremal set theory.

Theorem 1.1 *(Erdős–Ko–Rado theorem [6]). Let* $2 \leqslant r < n/2$ *and let* $\mathcal{A} \subset X^{(r)}$ *be an intersecting hypergraph. Then*

$$|\mathcal{A}| \leqslant \binom{n-1}{r-1},$$

with equality iff $\mathcal{A} = X_x^{(r)} = \{A \mid A \in X^{(r)}, \ x \in A\}$ *for some* $x \in X$.

There have been generalizations of the theorem in different directions. Deza and Frankl [4] give a very nice survey of the EKR-type results proved in the 1960s, 70s and 80s. In this paper, we concern ourselves with the generalization for graphs, formulated by Holroyd, Spencer and Talbot in [8].

1.1. Erdős–Ko–Rado property for graphs

The Erdős–Ko–Rado property for graphs is defined in the following manner.

For a graph G, vertex $v \in V(G)$ and some integer $r \geqslant 1$ denote the family of independent r-sets of $V(G)$ by $\mathcal{J}^{(r)}(G)$ and the subfamily $\{A \in \mathcal{J}^{(r)}(G): v \in A\}$ by $\mathcal{J}_v^{(r)}(G)$, called a star. Then G is said to be r-EKR if no intersecting subfamily of $\mathcal{J}^{(r)}(G)$ is larger than the largest star in $\mathcal{J}^{(r)}(G)$. If every maximum sized intersecting subfamily of $\mathcal{J}^{(r)}(G)$ is a star, then G is said to be strictly r-EKR. This can be viewed as the Erdős–Ko–Rado property on a ground set, but with additional structure on this ground set. In fact, the Erdős–Ko–Rado theorem can be restated in these terms as follows.

Theorem 1.2 *(Erdős–Ko–Rado theorem [6]). The graph on* n *vertices with no edges is* r*-EKR if* $n \geqslant 2r$ *and strictly* r*-EKR if* $n > 2r$.

There are some results giving EKR-type theorems for different types of graphs. The following theorem was originally proved by Berge [1], with Livingston [10] characterizing the extremal case.

Theorem 1.3. *(See Berge [1], Livingston [10].) If* $r \geqslant 1$, $t \geqslant 2$ *and* G *is the disjoint union of* r *copies of* K_t, *then* G *is* r*-EKR and strictly so unless* $t = 2$.

Other proofs of this result were given by Gronau [7] and Moon [12]. Berge [1] proved a stronger result.

Theorem 1.4. *(See Berge [1].) If* G *is the disjoint union of* r *complete graphs each of order at least 2, then* G *is* r*-EKR.*

A generalization of Theorem 1.3 was first stated by Meyer [11] and proved by Deza and Frankl [4].

Theorem 1.5. *(See Meyer [11], Deza and Frankl [4].) If* $r \geqslant 1$, $t \geqslant 2$ *and* G *is a disjoint union of* $n \geqslant r$ *copies of* K_t, *then* G *is* r*-EKR and strictly so unless* $t = 2$ *and* $r = n$.

In the paper which introduced the notion of the r-EKR property for graphs, Holroyd, Spencer and Talbot [8] prove a generalization of Theorems 1.4 and 1.5.

Theorem 1.6. *(See Holroyd et al. [8].) If* G *is a disjoint union of* $n \geqslant r$ *complete graphs each of order at least 2, then* G *is* r*-EKR.*

The compression technique used in [8], which is equivalent to contracting an edge in a graph, was employed by Talbot [14] to prove a theorem for the kth power of a cycle.

Definition 1.7. The *kth power of a cycle* C_n^k is a graph with vertex set $[n]$ and edges between $a, b \in [n]$ iff $1 \leqslant |a - b \bmod n| \leqslant k$.

Theorem 1.8. *(See Talbot [14].) If $r, k, n \geqslant 1$, then C_n^k is r-EKR and strictly so unless $n = 2r + 2$ and $k = 1$.*

An analogous theorem for the kth power of a path is also proved in [8].

Definition 1.9. The *kth power of a path* P_n^k is a graph with vertex set $[n]$ and edges between $a, b \in [n]$ iff $1 \leqslant |a - b| \leqslant k$.

Theorem 1.10. *(See Holroyd et al. [8].) If $r, k, n \geqslant 1$, then P_n^k is r-EKR.*

Observe that the condition $r \leqslant n/2$ is not required for the graphs C_n^k and P_n^k because for each of the two graphs, there is no independent set of size greater than $n/2$, so the r-EKR property holds vacuously if $r > n/2$.

The compression proof technique is also employed to prove a result for a larger class of graphs.

Theorem 1.11. *(See Holroyd et al. [8].) If G is a disjoint union of $n \geqslant 2r$ complete graphs, cycles and paths, including an isolated singleton, then G is r-EKR.*

The problem of characterizing when a graph G is 2-EKR is addressed by Holroyd and Talbot in [9].

Theorem 1.12. *(See Holroyd and Talbot [9].) Let G be a non-complete graph of order n with minimum degree δ and independence number α.*

1. *If $\alpha = 2$, then G is strictly 2-EKR.*
2. *If $\alpha \geqslant 3$, then G is 2-EKR if and only if $\delta \leqslant n - 4$ and strictly so if and only if $\delta \leqslant n - 5$, the star centers being the vertices of minimum degree.*

Holroyd and Talbot also present an interesting conjecture in [9].

Definition 1.13. The minimum size of a maximal independent vertex set of a graph G is the *minimax independent number*, denoted by $\mu(G)$.

We note that $\mu(G)$ is sometimes referred to as the *independent domination number*.

Conjecture 1.14. *Let G be any graph and let $1 \leqslant r \leqslant \frac{1}{2}\mu$; then G is r-EKR (and is strictly so if $2 < r < \frac{1}{2}\mu$).*

This conjecture seems hard to prove or disprove; however, restricting attention to certain classes of graphs makes the problem easier to tackle. Borg and Holroyd [2] prove the conjecture for a large class of graphs, which contain a singleton as a component.

Definition 1.15. (See Borg and Holroyd [2].) For a monotonic non-decreasing (mnd) sequence $\mathbf{d} = \{d_i\}_{i \in \mathbb{N}}$ of non-negative integers, let $M = M(\mathbf{d})$ be the graph such that $V(M) = \{x_i \colon i \in \mathbb{N}\}$ and for $x_a, x_b \in V(M)$ with $a < b$, $x_a x_b \in E(M)$ iff $b \leqslant a + d_a$. Let $M_n = M_n(\mathbf{d})$ be the subgraph of M induced by the subset $\{x_i \colon i \in [n]\}$ of $V(M)$. Call M_n an mnd graph.

Definition 1.16. (See Borg and Holroyd [2].) For $n > 2$, $1 \leqslant k < n - 1$, $0 \leqslant q < n$, let $C_{q,n}^{k,k+1}$ be the graph with vertex set $\{v_i \colon i \in [n]\}$ and edge set $E(C_n^k) \cup \{v_i v_{i+k+1 \bmod n} \colon 1 \leqslant i \leqslant q\}$. If $q > 0$, call $C_{q,n}^{k,k+1}$ a *modified kth power of a cycle*.

Borg and Holroyd [2] proved the following theorem.

Theorem 1.17. *Conjecture* 1.14 *is true if G is a disjoint union of complete multipartite graphs, copies of* mnd *graphs, powers of cycles, modified powers of cycles, trees, and at least one singleton.*

Our main result in this paper extends the class of graphs which satisfy Conjecture 1.14 by proving the conjecture for all chordal graphs which contain a singleton. Note that the mnd graphs in Theorem 1.17 are chordal.

We also define a special class of chordal graphs, and prove a stronger EKR result for these graphs. Finally, we consider similar problems for trees.

1.2. Main results

Definition 1.18. A graph G is *chordal* if every cycle of length at least 4 has a chord.

It is easy to observe that if G is chordal, then every induced subgraph of G is also chordal.

Definition 1.19. A vertex v is called *simplicial* in a graph G if its neighborhood is a clique in G.

Consider a graph G on n vertices, and let $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices of G. Let the graph G_i be the subgraph obtained by removing the vertex set $\{v_1, \ldots, v_{i-1}\}$ from G. Then σ is called a *simplicial elimination ordering* if v_i is simplicial in the graph G_i, for each $1 \leqslant i \leqslant n$. We state a well-known characterization for chordal graphs, due to Dirac [5].

Theorem 1.20. *A graph G is a chordal graph if and only if it has a simplicial elimination ordering.*

It is easy to see, using this characterization of chordal graphs, that the mnd graphs of Definition 1.15 are chordal.

Proposition 1.21. *Every* mnd *graph is chordal.*

Proof. It can be seen that ordering the vertices of M_n according to the corresponding degree sequence **d**, as stated in Definition 1.15, gives a simplicial elimination ordering. □

Note that, with or without the non-decreasing condition on the sequence **d**, the resulting graph is an interval graph — use the interval $[a, a + d_a]$ for vertex x_a — which is chordal regardless.

We prove the non-strict part of Conjecture 1.14 for disjoint unions of chordal graphs, containing at least one singleton.

Theorem 1.22. *If G is a disjoint union of chordal graphs, including at least one singleton, and if $r \leqslant \frac{1}{2}\mu(G)$, then G is r-EKR.*

We also consider graphs which do not have singletons. Consider a class of chordal graphs constructed as follows.

Let P_{n+1} be a path on n edges with $V(P_{n+1}) = \{v_1, \ldots, v_{n+1}\}$. Label the edge $v_i v_{i+1}$ as i, for each $1 \leqslant i \leqslant n$. A *chain of complete graphs*, of length n, is obtained from P_{n+1} by replacing each edge of P_{n+1} by a complete graph of order at least 2 in the following manner: to convert edge i of P_{n+1} into K_s, introduce a complete graph K_{s-2} and connect v_i and v_{i+1} to each of the $s - 2$ vertices of the complete graph. Call the resulting complete graph G_i, and call each G_i a *link* of the chain. We call v_i and v_{i+1} the *connecting vertices* of this complete graph, with the exception of G_1 and G_n, which have only one connecting vertex each (the ones shared with G_2 and G_{n-1} respectively). In general, for each $2 \leqslant i \leqslant n$, call v_i the $(i-1)$th connecting vertex of G. Unless otherwise specified, we will refer to a chain of complete graphs as just a *chain*. We will call an isolated vertex a *trivial chain* (of length 0), while a complete graph is simply a chain of length 1. Call a chain of length n *special* if $n \in \{0, 1\}$ or if $n \geqslant 2$ and the following conditions hold:

1. $|G_i| \geqslant |G_{i-1}| + 1$ for each $2 \leqslant i \leqslant n-1$;
2. $|G_n| \geqslant |G_{n-1}|$.

We prove the following results for special chains.

Theorem 1.23. *If G is a special chain, then G is r-EKR for all $r \geqslant 1$.*

Theorem 1.24. *If G is a disjoint union of 2 special chains, then G is r-EKR for all $r \geqslant 1$.*

We will also consider similar problems for bipartite graphs. A basic observation about complete bipartite graphs, and its obvious generalization for complete multipartite graphs, are mentioned below.

- If $G = K_{m,n}$ and $m \leqslant n$, then G is r-EKR for all $r \leqslant \frac{m}{2}$.
- If $G = K_{m_1,\ldots,m_k}$, with $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_k$, then G is r-EKR for all $r \leqslant \frac{m_1}{2}$.

It is easy to see why these hold. If $\mathscr{B} \subseteq \mathscr{J}^r(G)$ is intersecting, then each $A \in \mathscr{B}$ lies in the same partite set. Clearly, if $2r \leqslant m \leqslant n$, then G is r-EKR by Theorem 1.1. A similar argument works for complete multipartite graphs as well.

Holroyd and Talbot [9] proved Conjecture 1.14 for a disjoint union of two complete multipartite graphs.

If we consider non-complete bipartite graphs with high minimum degree δ, it seems that they usually have low μ (always at most $\min\{n - \delta, n/2\}$). Instead one might consider bipartite graphs with low maximum degree Δ in order to have higher values of μ (always at least $\frac{n}{\Delta+1}$). In particular, sparse graphs are likely to have high μ. Here we look at trees, a particularly sparse class of bipartite graphs.

One of the difficult problems in dealing with graphs without singletons is that of finding centers of maximum stars. We consider this problem for trees, and conjecture that there is a maximum star in a tree that is centered at a leaf.

Conjecture 1.25. *For any tree T on n vertices, there exists a leaf x such that for any $v \in V(T)$, $|\mathcal{J}_v^r(T)| \leqslant |\mathcal{J}_x^r(T)|$.*

We prove this conjecture for $r \leqslant 4$.

Theorem 1.26. *Let $1 \leqslant r \leqslant 4$. Then a maximum sized star of r-independent vertex sets of T is centered at a leaf.*

The rest of the paper is organized as follows: in Section 2, we prove Theorem 1.22; in Section 3, we give proofs of Theorems 1.23 and 1.24; and in Section 4, we prove Theorem 1.26.

2. Chordal graphs

We begin by fixing some notation. For a graph G and a vertex $v \in V(G)$, let $G - v$ be the graph obtained from G by removing vertex v. Also, let $G \downarrow v$ denote the graph obtained by removing v and its set of neighbors from G. We note that if G is a disjoint union of chordal graphs and if $v \in G$, the graphs $G - v$ and $G \downarrow v$ are also disjoint unions of chordal graphs. Denote by $N[v] = N_G[v]$ the closed neighborhood $\{u \in V(G): u = v \text{ or } uv \in E(G)\}$ of v.

2.1. Structural lemmas

We state and prove a series of lemmas, which we will use in the proof of Theorem 1.22.

Lemma 2.1. *Let G be a graph containing an isolated vertex x. Then, for any vertex $v \in V(G)$, $|\mathcal{J}_v^r(G)| \leqslant |\mathcal{J}_x^r(G)|$.*

Proof. Let $v \in V(G)$, $v \neq x$. We define the function $f : \mathcal{J}_v^r(G) \to \mathcal{J}_x^r(G)$ by

$$f(A) = \begin{cases} A & \text{if } x \in A, \text{ and} \\ A \setminus \{v\} \cup \{x\} & \text{otherwise.} \end{cases}$$

It is easy to see that the function is injective, and this completes the proof. □

Lemma 2.2. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then the following inequalities hold*:

1. $\mu(G - v_2) \geqslant \mu(G)$;
2. $\mu(G \downarrow v_2) + 1 \geqslant \mu(G)$.

Proof. We begin by noting that the condition $N[v_1] \subseteq N[v_2]$ implies that $v_1 v_2 \in E(G)$.

1. We will show that if I is a maximal independent set in $G - v_2$, then I is maximally independent in G. Suppose I is not a maximal independent set in G. Then $I \cup \{v_2\}$ is an independent set in G. Thus, for any $u \in N[v_2]$, $u \notin I$. In particular, for any $u \in N[v_1]$, $u \notin I$. Thus, $I \cup \{v_1\}$ is an independent set in $G - v_2$. This is a contradiction. Thus, I is a maximal independent set in G.

Taking I to be the smallest maximal independent set in $G - v_2$, we get $\mu(G - v_2) = |I| \geqslant \mu(G)$.

2. We will show that if I is a maximal independent set in $G \downarrow v_2$, then $I \cup \{v_2\}$ is a maximal independent set in G. Of course, $I \cup \{v_2\}$ is independent, so suppose it is not maximal. Then, for some vertex $u \in G \downarrow v_2$ and $u \notin I \cup \{v_2\}$, $I \cup \{u, v_2\}$ is an independent set. Thus, $I \cup \{u\}$ is an independent set in $G \downarrow v_2$, a contradiction.

Taking I to be the smallest maximal independent set in $G \downarrow v_2$, we get $\mu(G \downarrow v_2) + 1 = |I| + 1 \geqslant \mu(G)$. □

Corollary 2.3. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then the following statements hold*:

1. *If $r \leqslant \frac{1}{2}\mu(G)$, then $r \leqslant \frac{1}{2}\mu(G - v_2)$;*
2. *If $r \leqslant \frac{1}{2}\mu(G)$, then $r - 1 \leqslant \frac{1}{2}\mu(G \downarrow v_2)$.*

Proof. 1. This follows trivially from the first part of Lemma 2.2.

2. To prove this part, we use the second part of Lemma 2.2 to show

$$r - 1 \leqslant \frac{1}{2}\mu(G) - 1 = \frac{\mu(G) - 2}{2} \leqslant \frac{\mu(G \downarrow v_2)}{2} - \frac{1}{2}. \qquad \square$$

Suppose G is a non-empty graph and H is a non-empty component of G. Let $|V(H)| = m \geqslant 2$. Let $\{v_1, \ldots, v_m\}$ be a simplicial elimination ordering of H and let $v_1 v_i \in E(H)$ for some $i \geqslant 2$. Let $\mathcal{A} \subseteq \mathcal{J}^r(G)$ be an intersecting family. We define a compression operation $f_{1,i}$ for the family $\mathcal{A}$. Before we give the definition, we note that if A is an independent set and if $v_i \in A$, then $A \setminus \{v_i\} \cup \{v_1\}$ is also independent. Now let

$$f_{1,i}(A) = \begin{cases} A \setminus \{v_i\} \cup \{v_1\} & \text{if } v_i \in A, v_1 \notin A, A \setminus \{v_i\} \cup \{v_1\} \notin \mathcal{A}, \text{ and} \\ A & \text{otherwise.} \end{cases}$$

Then we define the family $\mathcal{A}'$ by

$$\mathcal{A}' = f_{1,i}(\mathcal{A}) = \{f_{1,i}(A) : A \in \mathcal{A}\}.$$

It is not hard to see that $|\mathcal{A}'| = |\mathcal{A}|$. Next, we define the families

$$\mathcal{A}'_i = \{A \in \mathcal{A}': v_i \in A\},$$
$$\bar{\mathcal{A}}'_i = \mathcal{A}' \setminus \mathcal{A}'_i, \quad \text{and}$$
$$\mathcal{B}' = \{A \setminus \{v_i\}: A \in \mathcal{A}'_i\}.$$

Then we have

$$\begin{aligned}
|\mathcal{A}| &= |\mathcal{A}'| \\
&= |\mathcal{A}'_i| + |\bar{\mathcal{A}}'_i| \\
&= |\mathcal{B}'| + |\bar{\mathcal{A}}'_i|.
\end{aligned} \tag{1}$$

We prove the following lemma about these families.

Lemma 2.4.

1. $\bar{\mathcal{A}}'_i \subseteq \mathcal{J}^r(G - v_i)$.
2. $\mathcal{B}' \subseteq \mathcal{J}^{(r-1)}(G \downarrow v_i)$.
3. $\bar{\mathcal{A}}'_i$ is intersecting.
4. $\mathcal{B}'$ is intersecting.

Proof. It follows from the definitions of the families that $\bar{\mathcal{A}}'_i \subseteq \mathcal{J}^r(G - v_i)$ and $\mathcal{B}' \subseteq \mathcal{J}^{(r-1)}(G \downarrow v_i)$. We need only prove that the two families are intersecting. Consider $A, B \in \bar{\mathcal{A}}'_i$. If $v_1 \in A$ and $v_1 \in B$, we are done. If $v_1 \notin A$ and $v_1 \notin B$, then $A, B \in \mathcal{A}$ and hence $A \cap B \neq \emptyset$. So suppose $v_1 \notin A$ and $v_1 \in B$. Then $A \in \mathcal{A}$. Also, either $B \in \mathcal{A}$, in which case we are done, or $B_1 = B \setminus \{v_1\} \cup \{v_i\} \in \mathcal{A}$. Then $|A \cap B| = |A \cap B \setminus \{v_1\} \cup \{v_i\}| = |A \cap B_1| > 0$.

Finally, consider $A, B \in \mathcal{B}'$. Since $A \cup \{v_i\} \in \mathcal{A}'_{v_i}$, $A \cup \{v_1\} \in \mathcal{A}$ and $A \cup \{v_i\} \in \mathcal{A}$. A similar argument works for B. Thus, $|(A \cup \{v_1\}) \cap (B \cup \{v_i\})| > 0$ and hence, $|A \cap B| > 0$. □

The final lemma we prove is regarding the star family $\mathcal{J}^r_x(G)$, where x is an isolated vertex.

Lemma 2.5. *Let G be a graph containing an isolated vertex x and let $v \in V(G)$, $v \neq x$. Then we have*

$$\left| \mathcal{J}^r_x(G) \right| = \left| \mathcal{J}^r_x(G - v) \right| + \left| \mathcal{J}^{(r-1)}_x(G \downarrow v) \right|.$$

Proof. Partition the family $\mathcal{J}^r_x(G)$ into two parts. Let the first part contain all sets containing v, say $\mathcal{F}_v$, and let the second part contain all sets which do not contain v, say $\bar{\mathcal{F}}_v$. Then $\mathcal{F}_v = \mathcal{J}^{(r-1)}_x(G \downarrow v)$ and $\bar{\mathcal{F}}_v = \mathcal{J}^r_x(G - v)$. □

We proceed to a proof of Theorem 1.22.

2.2. Proof of Theorem 1.22

Proof. The theorem trivially holds for $r = 1$, so suppose $r \geqslant 2$. Let G be a disjoint union of chordal graphs, including at least one singleton, and let $\mu(G) \geqslant 2r$. We do induction on $|G|$. If $|G| = \mu(G)$, then $G = E_{|G|}$, and we are done by the Erdős–Ko–Rado theorem. So suppose $|G| > \mu(G)$, and there is one component, say H, which is a chordal graph having m vertices, $m \geqslant 2$. Let $\{v_1, \ldots, v_m\}$ be a simplicial ordering of H and suppose $v_1 v_i \in E(H)$ for some $i \geqslant 2$. Since the neighborhood of v_1 is a clique, we have $N[v_1] \subseteq N[v_i]$. Also, let x be an isolated vertex in G. Let $\mathcal{A} \subseteq \mathcal{J}^r(G)$ be intersecting.

Define the compression operation $f_{1,i}$ and the families $\bar{\mathcal{A}}'_i$ and $\mathcal{B}'$ as before. Using Eq. (1), Lemmas 2.1, 2.2, 2.4, 2.5, Corollary 2.3 and the induction hypothesis, we have

$$
\begin{aligned}
|\mathcal{A}| &= |\bar{\mathcal{A}}'_i| + |\mathcal{B}'| \\
&\leqslant |\mathcal{J}^r_x(G - v_i)| + |\mathcal{J}^{(r-1)}_x(G \downarrow v_i)| \\
&= |\mathcal{J}^r_x(G)|. \quad \square
\end{aligned}
$$

3. Special chains

The main technique we use to prove Theorem 1.23 is a compression operation that is equivalent to compressing a clique to a single vertex. In a sense, it is a more general version of the technique used in [8]. We begin by stating and proving a technical lemma, similar to the one proved in [8]. We will then use it to prove Theorem 1.23 by induction.

3.1. A technical lemma

Let $H \subseteq G$ with $V(H) = \{v_1, \ldots, v_s\}$. Let G/H be the graph obtained by contracting the subgraph H to a single vertex. The contraction function c is defined by

$$
c(x) = \begin{cases} v_1 & \text{if } x \in H, \text{ and} \\ x & \text{if } x \notin H. \end{cases}
$$

When we contract H to v_1, the edges which have both endpoints in H are lost and if there is an edge $xv_i \in E(G)$ such that $x \in V(G) \setminus V(H)$, then there is an edge $xv_1 \in E(G/H)$. Duplicate edges are disregarded.

Also, let $G - H$ be the (possibly disconnected) graph obtained from G by removing all vertices in H.

Lemma 3.1. *Let $G = (V, E)$ be a graph and let $\mathcal{A} \subseteq \mathcal{J}^r(G)$ be an intersecting family of maximum size. If H is a subgraph of G with vertex set $\{v_1, \ldots, v_s\}$, and if H is isomorphic to K_s, then there exist families $\mathcal{B}$, $\{\mathcal{C}_i\}_{i=2}^s$, $\{\mathcal{D}_i\}_{i=2}^s$, $\{\mathcal{E}_i\}_{i=2}^s$ satisfying:*

1. $|\mathcal{A}| = |\mathcal{B}| + \sum_{i=2}^s |\mathcal{C}_i| + |\bigcup_{i=2}^s \mathcal{D}_i| + \sum_{i=2}^s |\mathcal{E}_i|$;
2. $\mathcal{B} \subseteq \mathcal{J}^r(G/H)$ *is intersecting*; *and*
3. *for each $2 \leqslant i \leqslant s$,*
   (a) $\mathcal{C}_i \subseteq \mathcal{J}^{r-1}(G - H)$ *is intersecting,*
   (b) $\mathcal{D}_i = \{A \in \mathcal{A}: v_1 \in A \text{ and } N(v_i) \cap (A \setminus \{v_1\}) \neq \emptyset\}$, *and*
   (c) $\mathcal{E}_i = \{A \in \mathcal{A}: v_i \in A \text{ and } N(v_1) \cap (A \setminus \{v_i\}) \neq \emptyset\}$.

To prove Lemma 3.1, we will need a claim, which we state and prove below.

Claim 3.2. *Let $H \subseteq G$ be isomorphic to K_s, $s \geqslant 3$. Let $\mathcal{A} \subseteq \mathcal{J}^r(G)$ be an intersecting family of maximum size. Suppose $A \cup \{v_i\}, A \cup \{v_j\} \in \mathcal{A}$ for some $i, j \neq 1$ and $c(A \cup \{v_i\}) = A \cup \{v_1\} \in \mathcal{J}^r(G/H)$. Then $A \cup \{v_1\} \in \mathcal{A}$.*

Proof. Since we have $c(A \cup \{v_i\}) \in \mathcal{J}^r(G/H)$, $B = A \cup \{v_1\} \in \mathcal{J}^r(G)$. Suppose $B \notin \mathcal{A}$. Since $\mathcal{A}$ is an intersecting family of maximum size, $\mathcal{A} \cup \{B\}$ is not an intersecting family. Thus there exists a $C \in \mathcal{A}$ such that $B \cap C = \emptyset$. So we have $C \cap (A \cup \{v_i\}) = v_i$ and $C \cap (A \cup \{v_j\}) = v_j$. Hence $v_i, v_j \in C$. This is a contradiction since v_i and v_j are adjacent to each other. $\square$

Proof of Lemma 3.1. Define the following families:

1. $\mathcal{B} = \{c(A): A \in \mathcal{A} \text{ and } c(A) \in \mathcal{J}^r(G/H)\}$; and
2. for each $2 \leqslant i \leqslant s$:

(a) $\mathcal{C}_i = \{A \setminus \{v_1\}: \ v_1 \in A$ and $A \setminus \{v_1\} \cup \{v_i\} \in \mathcal{A}\}$,
(b) $\mathcal{D}_i = \{A \in \mathcal{A}: \ v_1 \in A$ and $N(v_i) \cap (A \setminus \{v_1\}) \neq \emptyset\}$, and
(c) $\mathcal{E}_i = \{A \in \mathcal{A}: \ v_i \in A$ and $N(v_1) \cap (A \setminus \{v_i\}) \neq \emptyset\}$.

If $A, B \in \mathcal{A}$ and $A \neq B$, then $c(A) = c(B)$ iff $A \triangle B = \{v_i, v_j\}$ for some $1 \leqslant i, j \leqslant s$. Using this and Claim 3.2 (if $s \geqslant 3$), we have

$$\left| \{A \in \mathcal{A}: c(A) \in \mathcal{J}^r(G/H)\} \right| = |\mathcal{B}| + \sum_{i=2}^{s} |\mathcal{C}_i|.$$

Also, if $A \in \mathcal{A}$, then $c(A) \notin \mathcal{J}^r(G/H)$ iff $A \in \bigcup_{i=2}^{s} \mathcal{D}_i \cup \bigcup_{i=2}^{s} \mathcal{E}_i$. Thus, we have $|\mathcal{A}| = |\mathcal{B}| + \sum_{i=2}^{s} |\mathcal{C}_i| + |\bigcup_{i=2}^{s} \mathcal{D}_i| + |\bigcup_{i=2}^{s} \mathcal{E}_i|$. By the definition of the $\mathcal{E}_i$'s, $\bigcup_{i=2}^{s} \mathcal{E}_i$ is a disjoint union, so we have

$$|\mathcal{A}| = |\mathcal{B}| + \sum_{i=2}^{s} |\mathcal{C}_i| + \left| \overset{s}{\underset{i=2}{\mathring{\bigcup}}} \mathcal{D}_i \right| + \sum_{i=2}^{s} |\mathcal{E}_i|.$$

It is obvious to show that $\mathcal{B}$ is intersecting since $\mathcal{A}$ is.

Let $2 \leqslant i \leqslant s$. To see that $\mathcal{C}_i$ is intersecting, suppose $C, D \in \mathcal{C}_i$ and $C \cap D = \emptyset$. But $C \cup \{v_1\}$ and $D \cup \{v_i\}$ are in $\mathcal{A}$ and hence, are intersecting. This is a contradiction. □

3.2. Proof of Theorem 1.23

Before we move to the proof of Theorem 1.23, we will prove one final claim regarding maximum sized star families in G.

Claim 3.3. *If G is special chain of length n, then a maximum sized star is centered at an internal vertex of G_1.*

Proof. First note that for any i, there is a trivial injection from a star centered at a connecting vertex of G_i to a star centered at an internal vertex of G_i, which replaces the star center by that internal vertex in every set of the family. So suppose $\mathcal{Q}$ is a star centered at an internal vertex u of any of the graphs G_i, $i \neq 1$. Let $G_1 = K_m$. Consider the following cases.

1. Suppose u is in G_2. In this case, define an arbitrary bijection between the $m - 1$ internal vertices of G_1 and any $m - 1$ internal vertices of G_2 containing u, such that u corresponds to an internal vertex of G_1, say v (note that this can always be done, since if $n = 2$, then $|G_2| \geqslant m$, with one connecting vertex, while if $n \geqslant 3$, then $|G_2| \geqslant m + 1$, with two connecting vertices).
2. Suppose u is in some G_i such that $i \geqslant 3$. Then define an arbitrary bijection between the m vertices of G_1 and any m internal vertices of G_i including u such that u corresponds to an internal vertex of G_1, say v.

Next, consider any set in $\mathcal{Q}$. If it contains a vertex w in G_1, replace that vertex by b and replace u by the vertex in G_i corresponding to w. If it does not contain a vertex in G_1, replace u by v. This defines the injection from $\mathcal{Q}$ to a star centered at v. □

We now give a proof of Theorem 1.23.

Proof of Theorem 1.23. Let $\mathcal{J}_1^r(G)$ be a maximum sized star family in G, where 1 is an internal vertex of G_1.

We do induction on r. The result is trivial for $r = 1$. Let $r \geqslant 2$. We do induction on n (n is the number of links). For $n = 1$, result is vacuously true. If $n = 2$, then for $r = 2$, we use Theorem 1.12 to conclude that G is 2-EKR while the result is vacuously true for $r \geqslant 3$. So let $n \geqslant 3$. Let $\mathcal{A} \subseteq \mathcal{J}^r(G)$ be an intersecting family of maximum cardinality. Let the vertices of $G_n = K_s$ be labeled from n_1

to n_s (let n_1 be the connecting vertex which also belongs to G_{n-1}). Define the compression opera-tion c on G and the clique K_s as before. Let the families $\mathcal{B}$, $\{\mathcal{C}_i\}_{i=2}^s$, $\{\mathcal{D}_i\}_{i=2}^s$, $\{\mathcal{E}_i\}_{i=2}^s$ be defined as in Lemma 3.1.

Clearly, for G, $\mathcal{D}_i = \emptyset$ for each $2 \leqslant i \leqslant s$. So, by Lemma 3.1,

$$\mathcal{A} = \mathcal{B} + \sum_{i=2}^{s} |\mathcal{C}_i| + \sum_{i=2}^{s} |\mathcal{E}_i|.$$

Let $G_{n-1} = K_t$. Let the vertices of G_{n-1} be labeled from m_1 to m_t ($t \leqslant s$), with $m_t = n_1$. For every $1 \leqslant i \leqslant t-1$ and $2 \leqslant j \leqslant s$ define a set $\mathcal{H}_{ij}$ of families by

$$\mathcal{H}_{ij} = \{A \in \mathcal{A} : m_i \in A, \ n_j \in A\}.$$

We note that $\bigcup_{i=1}^{t-1} H_{ij} = \mathcal{E}_j$ for each $2 \leqslant j \leqslant s$, and since each of the $\mathcal{H}_{ij}$'s are also disjoint, we have

$$\sum_{i=2}^{s} |\mathcal{E}_i| = \sum_{1 \leqslant i \leqslant t-1, 2 \leqslant j \leqslant s} |\mathcal{H}_{ij}|.$$

Now, consider a complete bipartite graph $K_{t-1,s-1}$. Label the vertices in part 1 from m_1 to m_{t-1} and vertices in part 2 from n_2 to n_s.

Partition the edges of the bipartite graph $K_{t-1,s-1}$ into $s-1$ matchings, each of size $t-1$. For each matching M_k ($1 \leqslant k \leqslant s-1$), define the family

$$\mathcal{F}_{M_k} = \bigcup_{i,j,m_in_j \in M_k} \left(\mathcal{H}_{ij} - \{n_j\}\right),$$

where a family $\mathcal{H} - \{a\}$ is obtained from $\mathcal{H}$ by removing a from all its sets. Then of course

$$\sum_{1 \leqslant i \leqslant t-1, 2 \leqslant j \leqslant s} |\mathcal{H}_{ij}| = \sum_{1 \leqslant i \leqslant s-1} |\mathcal{F}_{M_i}|.$$

For each $1 \leqslant k \leqslant s-1$, F_{M_k} is a disjoint union and is intersecting. The intersecting property is obvious if both sets are in the same $\mathcal{H}_{ij} - \{n_j\}$ since they contain m_i. If in different such sets, adding distinct elements which were removed (during the above operation) gives sets in the original family which are intersecting.

Finally, if we consider families $C_{n_i} \cup F_{M_{i-1}} \subseteq \mathcal{J}^{(r-1)}(G - G_n)$ for $2 \leqslant i \leqslant s$, each such family is a disjoint union. It is also intersecting since for $C \in C_{n_i}$ and $F \in F_{M_{i-1}}$, $C \cup \{n_1\}$ and $F \cup \{n_j\}$ for some $j \neq 1$ gives us sets in $\mathcal{A}$. So we get

$$|\mathcal{A}| = |\mathcal{B}| + \sum_{i=2}^{s} |C_{n_i}| + \sum_{1 \leqslant i \leqslant s-1, 2 \leqslant j \leqslant s} |\mathcal{H}_{ij}|$$

$$= |\mathcal{B}| + \sum_{i=2}^{s} |C_{n_i}| + \sum_{1 \leqslant i \leqslant s-1} |\mathcal{F}_{M_i}|$$

$$= |\mathcal{B}| + \sum_{i=2}^{s} \left|(C_{n_i} \cup \mathcal{F}_{M_{i-1}})\right|$$

$$\leqslant \mathcal{J}_1^r(G/G_n) + (s-1)\mathcal{J}_1^{(r-1)}(G - G_n)$$

$$= \mathcal{J}_1^r(G).$$

The last inequality is obtained by partitioning the star based on whether or not it contains one of $\{n_2, \ldots, n_s\}$. $\quad\square$

3.3. Proof of Theorem 1.24

Proof. We do induction on r. Since the case $r = 1$ is trivial, let $r \geqslant 2$. Let G be a disjoint union of 2 *special* chains G' and G'', with lengths n_1 and n_2 respectively. We will do induction on $n = n_1 + n_2$. If $n = 0$, the result holds trivially if $r = 2$ and vacuously if $r \geqslant 3$. So let $n \geqslant 1$. If $n = 1$ or if $n_1 = n_2 = 1$, then $\alpha(G) = 2$. In this case, G is vacuously r-EKR for $r \geqslant 3$. Also, if $r = 2$, then we are done by Theorem 1.12. So, without loss of generality, we assume that G_1 has length at least 2. We can now proceed as in the proof of Theorem 1.23. $\square$

4. Trees

In this section, we give a proof of Theorem 1.26, which states that for a given tree T and $r \leqslant 4$, there is a maximum star family centered at a leaf of T.

Proof of Theorem 1.26. The statement is trivial for $r = 1$. If $r = 2$, we use the fact that for any vertex v, $|\mathscr{J}_v^2(T)| = n - 1 - d(v)$, where $d(v)$ is the degree of vertex v, and thus it will be maximum when v is a leaf.

Let $3 \leqslant r \leqslant 4$. Let v be an internal vertex ($d(v) \geqslant 2$) and let $\mathscr{A} = \mathscr{J}_v^r(T)$ be the star centered at v. Consider T as a tree rooted at v. We find an injection f from $\mathscr{A}$ to a star centered at some leaf. Let v_1 and v_2 be any two neighbors of v and let u be a leaf with neighbor w. Let $A \in \mathscr{A}$.

1. If $u \in A$, then let $f(A) = A$.
2. If $u \notin A$, then we consider two cases.
   (a) If $w \notin A$, let $f(A) = A \setminus \{v\} \cup \{u\}$.
   (b) If $w \in A$, then $B = A \setminus \{w\} \cup \{u\} \in \mathscr{A}$. We consider the following two cases separately.
      - $r = 3$.
        Let $A = \{v, w, x\}$. We know that x cannot be connected to both v_1 and v_2 since that would result in a cycle. Without loss of generality, suppose that $xv_1 \notin E(T)$. Then let $f(A) = A \setminus \{v, w\} \cup \{u, v_1\}$.
      - $r = 4$.
        Let $A = \{v, w, w_1, w_2\}$. We first note that if there is a leaf at distance two from v, then by using 1 and 2(a) above, we can show that the size of the star at this leaf is at least as much as the given star. We again consider two cases.
        – Suppose that $\{v_1, v_2\} \nsubseteq N(w_1) \cup N(w_2)$. By symmetry, suppose $v_1 \notin N(w_1) \cup N(w_2)$. In this case, let $f(A) = A \setminus \{w, v\} \cup \{u, v_1\}$.
        – Suppose that $\{v_1, v_2\} \subseteq N(w_1) \cup N(w_2)$. Label so that $v_i \in N(w_i)$ for $1 \leqslant i \leqslant 2$ (in particular, v_i is the parent of w_i). Since neither w_1 nor w_2 is a leaf, they have at least one child, say x_1 and x_2, respectively. In this case, let $f(A) = \{u, x_1, x_2, v_1\}$. For this case, injection is less obvious. We show it by contradiction as follows. Let $f(\{v, w, w_1, w_2\}) = f(\{v, w, y_1, y_2\}) = \{u, x_1, x_2, v_1\}$. We may assume that $y_1 \neq w_1$ and let y_i be the child of v_i and x_i be the child of y_i; then certainly $v_1 w_1 x_1 y_1 v_1$ gives a cycle in T, a contradiction. $\square$

We believe that Conjecture 1.25 holds true for all r. However, it is harder to prove because it is not true that every leaf centered star is bigger than every non-leaf centered star; an example is illustrated in Fig. 1.

For each vertex, the first number denotes the label, while the second number denotes the size of the star centered at that vertex. We note that $\mathcal{J}_8^5(T) = 9$, while $\mathcal{J}_1^5(T) = 10$. However, we note that the maximum sized stars are still centered at leaves 9 and 10.

We also point out that this example satisfies an interesting property, first observed by Colbourn [3].

Property 4.1. *Let G be a bipartite graph with bipartition $V = \{V_1, V_2\}$ and let $r \geqslant 1$. We say that G has the* bipartite degree sort *property if for all $x, y \in V_i$ with $d(x) \leqslant d(y)$, $\mathcal{J}_x^r(T) \geqslant \mathcal{J}_y^r(T)$.*
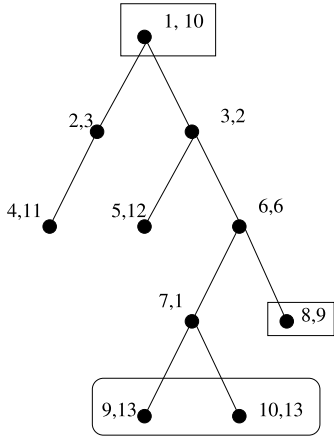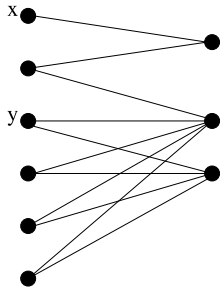
Fig. 1. Tree T on 10 vertices, $r = 5$.


Fig. 2. $G_{4,2}$.

Not all bipartite graphs satisfy this property. Neiman [13] constructed the following counterexample, with $r = 3$.

Fix positive integers t and k with $t \geqslant 2k \geqslant 4$. Let $G = G_{t,k}$ be the graph obtained from the complete bipartite graph $K_{2,t}$ and P_{2k} by identifying one endpoint of P_{2k} to be a vertex in $K_{2,t}$ lying in the bipartition of size 2. Let x be the other endpoint of the path, and let y be a vertex in $K_{2,t}$ lying in the bipartition of size t, of degree 2. An example is shown in Fig. 2.

Let $Y = \mathcal{J}_y^3(G)$ and let $X = \mathcal{J}_x^3(G)$. We have, for $t \geqslant 2k$,

$$
\begin{aligned}
Y - X &= \mathcal{J}^2(G \downarrow y) - \mathcal{J}^2(G \downarrow x) \\
&= \binom{t + 2k - 2}{2} - \left| E(G \downarrow y) \right| - \binom{t + 2k - 1}{2} + \left| E(G \downarrow x) \right| \\
&= \binom{t + 2k - 2}{2} - \binom{t + 2k - 1}{2} + 2t - 1 \\
&= (t + 2k - 2)(-1) + 2t - 1 \\
&= t - 2k + 1 \\
&> 0.
\end{aligned}
$$

We show that a similar construction acts as a counterexample for all $r > 3$. Given $r > 3$, consider the graph $G = G_{t,2}$, $t > r$. Let x and y be as defined before, with $d(x) = 1$ and $d(y) = 2$. Let $Y = \mathcal{J}_y^r(G)$ and $X = \mathcal{J}_x^r(G)$. We have $X = \binom{t+1}{r-1}$ and $Y = \binom{t+1}{r-1} + \binom{t-1}{r-2}$. It follows that, for $t > r$, $Y > X$.
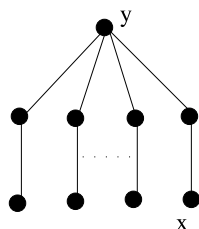
Fig. 3. Tree T on $2n + 1$ vertices which satisfies Property 4.1.
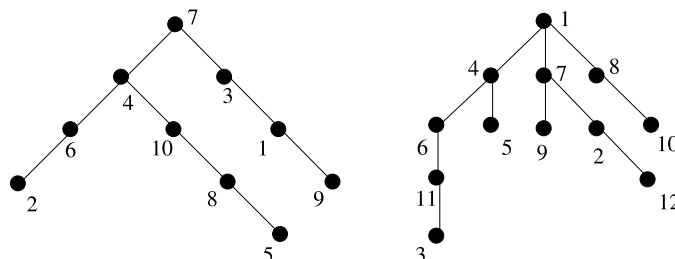
Fig. 4. Trees T_1 (left) and T_2 (right) which do not satisfy Property 4.1.

If one considers trees, we see that the tree in Fig. 1 satisfies this property. It is also not hard to show that the path P_n satisfies this property, since for all $r \geqslant 1$, $\mathcal{J}_{v_1}^r(P_n) = \mathcal{J}_{v_n}^r(P_n) \geqslant \mathcal{J}_{v_i}^r(P_n)$ holds for each $2 \leqslant i \leqslant n - 1$.

Another infinite family of trees that satisfy the property is the depth-two stars shown in Fig. 3. Let $Y = \mathcal{J}_y^r(T)$ and let $X = \mathcal{J}_x^r(T)$. Then we have $Y = \mathcal{J}^{r-1}(T \downarrow y) = \binom{n}{r-1}$ and $X = \binom{n-1}{r-2} + 2^{r-1}\binom{n-1}{r-1}$. It is then easy to note that when $r \geqslant 1$, $X - Y \geqslant 0$.

However, it turns out that not all trees satisfy this property. A counterexample, for $n = 10$ and $r = 5$, is shown as T_1 in Fig. 4. Observe that the vertex labeled 8, with degree 2, and the vertex labeled 4, with degree 3, lie in the same partite set, but we have $\mathcal{J}_4^5(T_1) = \{\{2, 3, 4, 8, 9\}, \{2, 3, 4, 5, 9\}\}$ and $\mathcal{J}_8^5(T_1) = \{\{2, 3, 4, 8, 9\}\}$. Note that, in this example, $r = \frac{n}{2}$. Another counterexample, with $n = 12$ and $r = 5$, is shown as T_2 in Fig. 4.

We see that the vertices labeled 1 and 2, with degrees 3 and 2 respectively, lie in the same partite set. It can be checked that $|\mathcal{J}_1^5(T_2)| = 32$ and $|\mathcal{J}_2^5(T_2)| = 28$.

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