Chapter 12

Economics

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12.1 Shadow Prices

In this section we begin to perform **sensitivity** (or **post-optimality**) **analysis**, which amounts to figuring out how the optimum solution might change under small changes in the LOP. For example, an investment company might need to solve roughly the same problem every day, with only the stock prices changing. Or maybe an airline might consider adding a new route because the current schedule suggests that the resulting income might outweigh its costs. Also, a production company might realize the optimal solution isn't valid because they forgot to include a necessary constraint — could the new problem be solved without starting again from scratch? It may help first to figure out what the dual variables really mean.

sensitivity (post-optimality) analysis

Consider the plight of folk legend Öreg MacDonald.

Problem 12.1.1 *Öreg MacDonald owns* $1,000$ *acres of land and is contemplating conserving, farming, and/or developing it. His annual considerations are as follows. It will only cost him* \$1 *per acre in registration fees to own conservation land, and he will reap* \$30 *per acre in tax savings. Farming will cost him* \$50 *per acre for seeds, from which he can earn* \$190 *per acre by selling vegetables. He can earn* \$290 *per acre by renting developed land, which costs* \$85 *per acre in permits. Öreg has only* \$40,000 *to use, and is also bound by having only* 75 *descendants, each of whom can work at most* 2,000 *hours. How should he apportion his acreage in order to maximize profits, if conservation, farming, and development uses* 12, 240, *and* 180 *hours per acre, respectively?*

The LOP he needs to solve is the following.

Problem 12.1.2

$$\begin{array}{llcrcrcrcr}
\text{Max.} & z & = & 29x_1 & + & 140x_2 & + & 205x_3 & & \\
\\
\text{s.t.} & & & x_1 & + & x_2 & + & x_3 & \leq & 1,000 \\
 & & & x_1 & + & 50x_2 & + & 85x_3 & \leq & 40,000 \\
 & & & 12x_1 & + & 240x_2 & + & 180x_3 & \leq & 150,000 \\
\\
\& & & & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

Here, x_1, x_2, and x_3 are the amounts of land (in acres) that Öreg will use for conservation, farming, and development, respectively. A few Simplex pivots will reveal the optimal solution $z^* = 1251000000/10920$ (about \$114,560.44) at $\mathbf{x}^* = (3750000, 5040000, 2130000)/10920$ (roughly $(343.41, 461.54, 195.05)$). The optimal tableau also displays the optimal dual solution at $\mathbf{y}^* = (286800, 21480, 700)/10920 \approx (26.26, 1.97, .06)$. But, other than providing a certificate of optimality as primal constraint multipliers, what do these dual variables mean?

Take a closer look at the third constraint. The 12 represents the annual number of hours per acre required to work conservation land. The other coefficients represent similar hours-to-acres ratios, meaning that the dimensions of the constraint look something like

$$\left(\frac{\text{hours}}{\text{cons acres}}\right)(\text{cons acres}) + \left(\frac{\text{hours}}{\text{farm acres}}\right)(\text{farm acres}) + \left(\frac{\text{hours}}{\text{dev acres}}\right)(\text{dev acres}) \ \le\ \text{hours}\ ,$$

activity/ product

since x_j is the number of acres put to the j^{th} **activity** (or **product**).

We can analyze the second dual constraint $y_1 + 50y_2 + 240y_3 \ge 140$ similarly. The coefficients, in order, represent the ratio of acres, dollars, and hours to farm acres, with the right-hand result of profit per farm acre. Thus we have constraint

$$\left(\frac{\text{acres}}{\text{farm acres}}\right) y_1 + \left(\frac{\text{dollars}}{\text{farm acres}}\right) y_2 + \left(\frac{\text{hours}}{\text{farm acres}}\right) y_3 \ \ge\ \left(\frac{\text{profit}}{\text{farm acres}}\right)\ ,$$

resource

giving the impression in this case that y_i must be the profit per i^{th} **resource**. Indeed, this interpretation works for the other two constraints as well.

Just as the primal objective function has dimension

$$\left(\frac{\text{profit}}{\text{cons acres}}\right)(\text{cons acres}) + \left(\frac{\text{profit}}{\text{farm acres}}\right)(\text{farm acres}) + \left(\frac{\text{profit}}{\text{dev acres}}\right)(\text{dev acres}) \ =\ \text{profit}\ ,$$

the dual objective function has dimension

$$(\text{acres})\left(\frac{\text{profit}}{\text{acre}}\right) + (\text{dollors})\left(\frac{\text{profit}}{\text{dollar}}\right) + (\text{hours})\left(\frac{\text{profit}}{\text{hour}}\right) \ =\ \text{profit}\ .$$

shadow price (marginal value)

Hence we may view the primal as maximizing the profit of activities, subject to resource constraints, and the dual as minimizing the profit of resources, subject to activity constraints. Each constraint coefficient $a_{i,j}$ measures the amount of resource i used up by one unit of activity j, and each dual variable y_i measures the worth of one unit of resource i. Let us define the **shadow price** (or **marginal value**) of resource i to be the amount that the optimum objective value increases per unit increase in resource i, just to see how this compares with our concept of worth.

Workout 12.1.3 *Re-solve Problem 12.1.2 with each of the following revised resource values.*

(a) *Öreg has* 999, $1,001$, *and* $1,002$ *acres.*

(b) *Öreg has* $39,999$, $40,001$, *and* $40,002$ *dollars.*

(c) *Öreg has* $149,999$, $150,001$, *and* $150,002$ *hours.*

(d) *Öreg has (simultaneously)* 999 *acres,* $40,001$ *dollars, and* $150,002$ *hours.*