Pebbling in Diameter Two Graphs and Products of Paths

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Abstract: Results regarding the pebbling number of various graphs are presented. We say a graph is of Class 0 if its pebbling number equals the number of its vertices. For diameter d we conjecture that every graph of sufficient connectivity is of Class 0. We verify the conjecture for $d = 2$ by characterizing those diameter two graphs of Class 0, extending results of Pachter, Snevily and Voxman. In fact we use this characterization to show that almost all graphs have Class 0. We also present a technical correction to Chung's alternate proof of a number theoretic result of Lemke and Kleitman via pebbling.
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1. INTRODUCTION

Suppose p pebbles are distributed onto the vertices of a graph G. A pebbling step consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be *moved* to a vertex r, the *root* vertex, if we can repeatedly apply pebbling steps so that in the resulting distribution r has one pebble. For a graph G, we define the *pebbling number,* $f(G)$, to be the smallest integer m such that for any distribution of m pebbles to the vertices of G, one pebble can be moved to any specified root vertex r. If D is a distribution of pebbles on the vertices of G and there is some choice of a root r such that it is impossible to move a pebble to r, then we say that D is a *bad* pebbling distribution. We denote by $D(v)$ the

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number of pebbles on vertex v in D and let the size, $|D|$, of D be the total number of pebbles in D, that is $|D| = \sum_{v \in V(G)} D(v)$. This yields another way to define $f(G)$, as one more than the maximum k such that there exists a bad pebbling distribution D of size k. Suppose that after several pebbling steps from D we arrive at a pebbling distribution P. In this case we say that P is *derived* from D and use the notation $P(v)$ to refer to the number of pebbles of P on v. In this way $D(v)$ always refers to the initial distribution.

Throughout this paper G will denote a simple connected graph, where $n(G) = |V(G)|$, and $f(G)$ will denote the pebbling number of G. For any two graphs G_1 and G_2, we define the *cartesian product* $G_1 \Box G_2$ to be the graph with vertex set $V(G_1 \Box G_2) = \{(v_1, v_2) | v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1 \Box G_2) = \{((v_1, v_2), (w_1, w_2)) | (v_1 = w_1$ and $(v_2, w_2) \in E(G_2))$ or $(v_2 = w_2$ and $(v_1, w_1) \in E(G_1))\}$.

There are many known results regarding $f(G)$. If one pebble is placed at each vertex other than the root vertex, r, then no pebble can be moved to r. Also, if w is at distance d from r, and $2^d - 1$ pebbles are placed at w, then no pebble can be moved to r. We record this as

Fact 1.1. $f(G) \geq \max\{n(G), 2^{diam(G)}\}$.

Let C_n be the cycle on n vertices. It is easy to see that $f(C_5) = 5$ and $f(C_6) = 8$ and so each of the two lower bounds are important. The pebbling numbers of cycles is derived in [7]. In the case of odd cycles, the pebbling number is larger than both lower bounds.

Result 1.2 *[7].* *For $k \geq 1$, $f(C_{2k}) = 2^k$ and $f(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$.*

The following conjecture has generated a great deal of interest.

Conjecture 1.3 *(Graham).* $f(G_1 \Box G_2) \leq f(G_1) f(G_2)$.

For the interested reader it is worth mentioning that there are few results which verify Graham's conjecture. Among these, the conjecture holds for a tree by a tree [6], a cycle by a cycle [7, 4], (with possibly some small exceptions), and a clique by a graph with the 2-pebbling property [1] (see Section 4 for the definition). It is also proven in [1] that the conjecture holds when each G_i is a path. Let P_n be a path with n vertices and for $\boldsymbol{d} = \langle d_1, \ldots, d_m \rangle$ let $P_{\boldsymbol{d}}$ denote the graph $P_{d_1+1} \Box \cdots \Box P_{d_m+1}$.

Result 1.4 *[1].* *For nonnegative integers $d_1, \ldots, d_m$, $f(P_{\boldsymbol{d}}) = 2^{d_1 + \cdots + d_m}$.*

Chung [1] uses a more general version of Result 1.4 to give an alternate proof of the following result of Lemke and Kleitman [5].

Theorem 1.5 *[5].* *For any given integers $x_1, \ldots, x_d$ there is a nonempty subset $J \subseteq \{1, \ldots, d\}$ such that $d | \sum_{j \in J} x_j$ and $\sum_{j \in J} gcd(x_j, d) \leq d$.*

In Section 3 we correct a technical error which appears in Chung's proof of Theorem 1.5. Our main concern in this paper regards the following.

Result 1.6 *[7].* *If $diam(G) = 2$ then $f(G) = n(G)$ or $n(G) + 1$.*

We say that G is of *Class 0* if $f(G) = n(G)$ and of *Class 1* if $f(G) = n(G) + 1$. In Section 2 we are able to characterize Class 0 graphs of diameter two (Theorem 2.5). As a corollary to this characterization we obtain the following (let $\kappa(G)$ denote the connectivity of G).

Theorem 1.7. *(Section 2). If $diam(G) = 2$, and $\kappa(G) \geq 3$ then G is of Class 0.*

From this it follows that almost all graphs (in the probabilistic sense) are of Class 0, since almost every graph is 3-connected with diameter 2. (3-connectedness follows most easily from the fact that almost all graphs are hamiltonian. Once a hamilton cycle C is drawn in the plane and vertices u and v are chosen, one can almost surely find two chords (u,x) and (v,y) which intersect if drawn in the region interior to C. This structure yields three pairwise vertex-disjoint uv-paths.) We believe that the following is true.

Conjecture 1.8. *There is a function $k(d)$ such that if G is a graph with diam$(G) = d$ and $\kappa(G) \geq k(d)$ then G is of Class 0.*

If this conjecture is true then the function $k(d)$ must be very large, greater than $2^d/d$. Indeed, the following graph G has diameter d and connectivity $\lfloor \frac{2^d-3}{d-1} \rfloor$ and yet is not of Class 0. Let $V = V(G)$ be the disjoint union of $V_0, \ldots, V_d$ with $r \in V_0$, $x \in V_d$, and $|V_j| = \lfloor \frac{2^d-3}{d-1} \rfloor$ or $\lceil \frac{2^d-3}{d-1} \rceil$ for $0 < j < d$, with $\sum_{j=1}^{d-1} |V_j| = 2^d - 3$. Let there be an edge uv whenever $u \in V_i$, $v \in V_j$ and $|i-j| \leq 1$. (G is a *blow-up* of the path P_{d+1}, each vertex of P_{d+1} being replaced by a clique. In general, G is a blow-up of a graph H if G is formed from H by replacing each vertex v of H by a clique $K(v)$—clique sizes may vary—and connecting vertices x and y of G by an edge if and only if $x \in K(u)$, $y \in K(v)$ and $(u,v) \in E(H)$. One may speak of ''the'' blow-up when all clique sizes are equal.) Finally, let D be the pebbling distribution $D(v) = 0$ for all $v \in \{r\} \cup V_1 \cup \cdots \cup V_{d-1}$, $D(x) = 2^d - 1$, and $D(v) = 1$ for all other $v \in V$. Then D is a bad distribution for G of size $n(G)$.

In fact, the blow-up of the cycle C_{2d+1} does a little better, about $\frac{4}{3}\, 2^d/d$. We believe that $k(d)$ is at most 2^d.

2. CHARACTERIZATION

We begin by developing a few lemmas which will help characterize diameter two Class 0 graphs.

Lemma 2.1. *If G is of Class 0 then $\kappa(G) \geq 2$. In particular, if diam$(G) = 2$ and $\kappa(G) = 1$ then G is of Class 1.*

Proof. Let $\kappa(G) = 1$. We show that G is not of Class 0 by presenting a bad pebbling distribution D of size $n(G)$. Since $\kappa(G) = 1$, G has a cut vertex x. Let H_1 and H_2 be two different components of $G - x$ and let r and y be vertices in H_1 and H_2, respectively. Then we define D by $D(v) = 0$ for $v \in \{r, x\}$, $D(y) = 3$, and $D(v) = 1$ for every other vertex v. If $diam(G) = 2$ then Theorem 1.6 implies that G is of Class 1. ∎

Before we proceed with the next lemma we will need to introduce the notation used in its proof. Given a pebbling distribution D, let $S = S_D = \{v \in V(G) | D(v) = 2\}$, $s = |S|$, $T = T_D = \{v \in V(G) | D(v) = 3\}$, $t = |T|$, $Z = Z_D = \{v \in V(G) | D(v) = 0\}$, and $z = |Z|$. For two sets A, $B \subseteq V(G)$ (not necessarily distinct) we denote by AB the set of vertices which are adjacent to some $a \in A$ and $b \in B$ ($b \neq a$). We use similar notation in the case of three sets and write ABx instead of $AB\{x\}$ when one of the sets is a singleton. Furthermore, for $A \subseteq V(G)$, let $N(A)$ be the *neighborhood* of A, that is, the set of vertices which are adjacent to some vertex of A.

Lemma 2.2. *If diam$(G) = 2$, $\kappa(G) \geq 2$, and G is of Class 1 then for any bad distribution D of size $n(G)$ we have $|S_D| = 0$, $|T_D| = 2$.*

Proof. Let D be a bad distribution of size $n(G)$, let r be specified as the root, and let S, T, etc., be defined as above. Certainly, $N(r) \cap (S \cup T) = \emptyset$, and so for all $v \in S \cup T$, $dist(v,r) = 2$.

Because $diam(G) = 2$, we know that $P(v) < 4$ for every vertex v and every P derived from D (and of course $P(r) = 0$). Also, $STr \cup TTr = \emptyset$. Indeed, if $v \in STr \cup TTr$ then one can move one pebble from each of its nonroot neighbors to v, then one from v to r, contradicting $P(r) = 0$. Likewise, if $v \in T \cap N(S \cup T)$ then one can move a fourth pebble from $S \cup T$ to v, contradicting $P(v) < 4$. Thus, for all $u \in S \cup T$, $v \in T$, $dist(u, v) = 2$. From these considerations, we see that $\{r\}$, rS, rT, ST and TT are disjoint subsets of Z, and so

$$1 + s + t + st + \binom{t}{2} \leq z = s + 2t. \tag{1}$$

The equality in (1) follows from artificially redistributing the pebbles of D: since $|D| = n(G)$, we can redistribute the pebbles from $S \cup T$ to Z so that there is exactly one pebble on each vertex (there are $s + 2t$ ''extra''pebbles in $S \cup T$). From (1) we derive the inequality

$$t^2 - (3 - 2s)t + 2 \leq 0. \tag{2}$$

Thus $3 \geq 2s$, which means $s \leq 1$.

If $s = 1$ then $t^2 - t + 2 = (t - \frac{1}{2})^2 + \frac{7}{4} > 0$, contradicting (2). Hence $s = 0$, $t^2 - 3t + 2 = (t - 1)(t - 2)$ and $t = 1$ or $t = 2$. If $t = 1$ then $|Z| = 2$. Let $Z = \{r, v\}$. Then all paths from the vertex in T to r must go through v, making v a cut vertex, a contradiction. Therefore $t = 2$. ∎

Next we define a family $\mathcal{F}$ of 2-connected, diameter 2, Class 1 graphs. We will soon show that every 2-connected, diameter 2, Class 1 graph is in $\mathcal{F}$. The smallest graphs in $\mathcal{F}$ are formed from a 6-cycle $C = rapcqbr$ (in order) by including at least two of the edges between a, b and c. In addition, given any graph $G \in \mathcal{F}$ and any other graph $H = H_p$ (resp. H_q), we can add $V(H)$ to $V(G)$, including also $E(H)$, to obtain a new graph in $\mathcal{F}$, provided that each component of H has some vertex adjacent to p (resp. q), and that each vertex of H is adjacent to both a and c (resp. b and c) and to no other vertex of G. Also, given any graph $G \in \mathcal{F}$ and any other graph $H = H_c$, we can add $V(H)$ to $V(G)$ to obtain a new graph in $\mathcal{F}$, provided that each vertex of H is adjacent to c, to either a or b (or both), and to no other vertex of G. Finally, given any graph $G \in \mathcal{F}$ and any other graph $H = H_r$, we can add $V(H)$ to $V(G)$ to obtain a new graph in $\mathcal{F}$, provided that each vertex of H is adjacent to both a and b, and to no other vertex of G, except possibly r. See Figure 1.

An alternative way of defining the family $\mathcal{F}$ is as follows (see Fig. 2). Simply replace each $H_x \cup \{x\}$ by H'_x for $x \in \{p, q, c, r\}$, with the realization that H'_x is now nonempty and connected for $x \in \{p, q\}$. We chose the former definition in order to make the proof of Theorem 2.5 easier.

Proposition 2.3. *If $G \in \mathcal{F}$ then $diam(G) = 2$, $\kappa(G) = 2$ and G is of Class 1.*

Proof. Clearly $diam(G) = 2$ and $\kappa(G) = 2$. Theorem 1.6 says that G is either of Class 0 or Class 1. We show that G is not of Class 0 by presenting a bad pebbling distribution D of size $n(G)$. We define D by $D(v) = 0$ for $v \in \{a, b, c, r\}$, $D(v) = 3$ for $v \in \{p, q\}$, and $D(v) = 1$ for every other vertex v. ∎

Theorem 2.4. *If $diam(G) = 2$, $\kappa(G) \geq 2$ and G is of Class 1, then $G \in \mathcal{F}$.*

Proof. Let $diam(G) = 2$, $\kappa(G) \geq 2$, let D be a bad pebbling distribution of size $n(G)$ and let r be the root. Using the result and notation from Lemma 2.2, let $T = \{p, q\}$, $rp = \{a\}$, $rq = \{b\}$, and $TT = \{c\}$. Then $Z = \{a, b, c, r\}$. As before, for all distributions P derived from D, $P(r) = 0$ and $P(v) < 4$ for all $v \in V(G)$.

We claim that $\{a, b, c\}$ induces at least two edges. If not, we will show that it is possible to move a pebble to r. Suppose that $c \not\sim a$ and $c \not\sim b$. Because $dist(c, r) = 2$ there is some

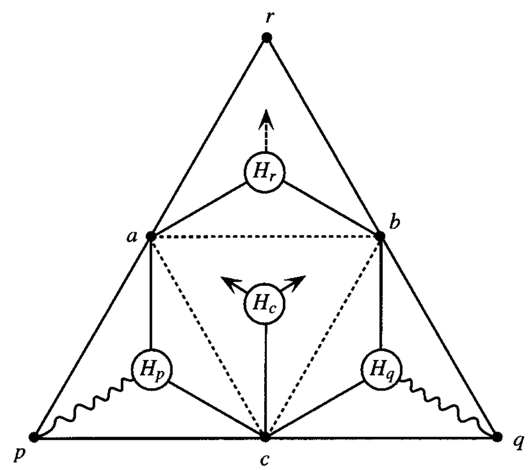

FIGURE 1.

$v \notin Z \cup T$ such that $v \sim c$ and $v \sim r$. But since $D(v) = 1$, it is possible to move one pebble from each of p and q to c, then one pebble from c to r through v. If instead we suppose that $b \not\sim a$ and $b \not\sim c$, then any common neighbor u of p and b can be used to move a pebble from p to b through u. Then we can move a pebble from q to r through b. The case at a is symmetric to the case at b, proving the claim.

Now let V_p be the set of vertices in the same component of $G - \{a, c\}$ as p, and let V_q be the set of vertices in the same component of $G - \{b, c\}$ as q. Either $V_p = V_q$ or $V_p \cap V_q = \emptyset$. If $V_p = V_q$ then since every $v \in V_p$ has $D(v) = 1$ it would be possible to move one pebble along a path from p to q, contradicting $P(q) < 4$. Thus $V_p \cap V_q = \emptyset$. Moreover, if $b \in N(V_p)$ then we could move one pebble from each of p and q to b, and then move one pebble from b to r. Hence

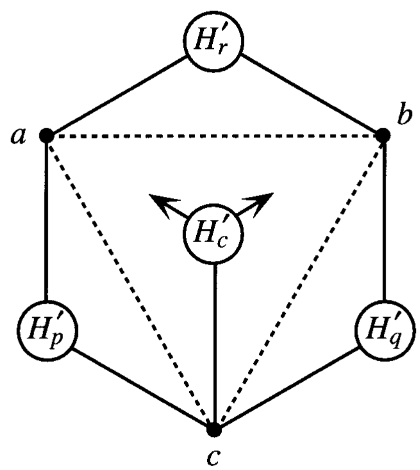

FIGURE 2.

$b \notin N(V_p)$. Of course, $r \notin N(V_p)$. Because we have $dist(p,r) = dist(p,q) = 2$, it must be that $v \sim a$ and $v \sim c$ for all $v \in V_p$. Similarly, $a, r \notin N(V_q)$, and $v \sim b$ and $v \sim c$ for all $v \in V_q$.

Next define V_c to be the set of vertices not yet mentioned which are adjacent to c. We claim that $N(v) \cap \{a,b\} \neq \emptyset$ for all $v \in V_c$. Otherwise $(diam(G) = 2)$ there must exist some vertex w, also not yet mentioned, adjacent to both v and r. As before, we then can move one pebble from each of p and q to c, and then move one pebble from c to r through v and w. Therefore, every $v \in V_c$ must be adjacent to either a or b (or both).

Finally, any vertex not yet mentioned must be adjacent to both a and c in order to have distance 2 from both p and q. Whether it is a neighbor of r is irrelevant. Hence $G \in \mathcal{F}$. ∎

Theorem 2.5. *Let $diam(G) = 2$. Then G is of Class 0 if and only if $\kappa(G) \geq 2$ and $G \notin \mathcal{F}$.*

Proof. Follows immediately from Proposition 2.3 and Theorem 2.4. ∎

Proof of Theorem 1.7. Follows immediately from Theorem 2.5. ∎

3. CORRECTION

The method employed in Chung's proof of Theorem 1.5 is essentially correct, though we are able to present a counterexample to one piece of the argument. We overcome this hurdle by developing more precise notation as technical devices, although we stress that this is still the same proof at heart. We begin by providing the context (see p. 471 of [1]) for Chung's proof (and ours).

A *p-pebbling step* in G consists of removing p pebbles from a vertex u, and placing one pebble on a neighbor v of u. We say that a pebbling step from u to v is *greedy* if $dist(v,r) < dist(u,r)$, and that a graph G is *greedy* if for any distribution of $f(G)$ pebbles on the vertices of G we can move a pebble to any specified root vertex r, in such a way that each pebbling step is greedy.

Let $P_{\boldsymbol{d}} = P_{d_1+1} \square \cdots \square P_{d_m+1}$ be a product of paths, where $\boldsymbol{d} = (d_1, \ldots, d_m)$. Then each vertex $v \in V(P_{\boldsymbol{d}})$ can be represented by a vector $\boldsymbol{v} = \langle v_1, \ldots, v_m \rangle$, with $0 \leq v_i \leq d_i$ for each $i < m$. Let $\boldsymbol{e}_i = \langle 0, \ldots, 1, \ldots, 0 \rangle$, be the i^{th} standard basis vector. $\mathbf{0}$ denotes the vector $\langle 0, \ldots, 0 \rangle$. Then two vertices u and v are adjacent in $P_{\boldsymbol{d}}$ if and only if $|\boldsymbol{u} - \boldsymbol{v}| = \boldsymbol{e}_i$ for some integer $1 \leq i \leq m$. If $\boldsymbol{p} = (p_1, \ldots, p_m)$, then we may define $\mathbf{p}$-*pebbling* in $P_{\boldsymbol{d}}$ to be such that each pebbling step from $\mathbf{u}$ to $\mathbf{v}$ is a p_i-pebbling step whenever $|\boldsymbol{u} - \boldsymbol{v}| = \boldsymbol{e}_i$. We denote the $\mathbf{p}$-pebbling number of $P_{\boldsymbol{d}}$ by $f_{\boldsymbol{p}}(P_{\boldsymbol{d}})$.

Chung's proof uses the following result, which we have phrased in our new terminology. For integers p_i, $d_i \geq 1$, $1 \leq i \leq m$, we use $\boldsymbol{p}^{\boldsymbol{d}}$ as shorthand for the product $p_1^{d_1} \cdots p_m^{d_m}$.

Result 3.1 *[1]. Suppose that $\mathbf{p}^{\mathbf{d}}$ pebbles are assigned to the vertices of $P_{\boldsymbol{d}}$ and that the root $r = \mathbf{0}$. Then it is possible to move one pebble to r via greedy $\mathbf{p}$-pebbling.*

As an aside, we can derive from this a generalization of Result 1.4.

Corollary 3.2. *$f_{\boldsymbol{p}}(P_{\boldsymbol{d}}) = \boldsymbol{p}^{\boldsymbol{d}}$. Moreover, $P_{\boldsymbol{d}}$ is greedy.*

Proof. Suppose the root $r = \langle r_1, \ldots, r_m \rangle \neq \mathbf{0}$. Then $P_{\boldsymbol{d}}$ naturally splits into 2^m smaller graphs, each of which is a product of paths. We show that if D is a pebbling distribution of size $\boldsymbol{p}^{\boldsymbol{d}}$ then one of these graphs contains enough pebbles to apply Chung's result.

Let $\boldsymbol{b} = \langle b_1, \ldots, b_m \rangle = \boldsymbol{d} - \boldsymbol{r}$. For $\delta \in \{0,1\}^m$ let $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\delta) = \langle \sigma_1(\delta), \ldots, \sigma_m(\delta) \rangle$, where $\sigma_i(\delta) = r_i$ if $\delta_i = 0$, and $\sigma_i(\delta) = b_i$ if $\delta_i = 1$. Let G_δ be the subgraph of $P_{\boldsymbol{d}}$ whose vertex set consists of those vertices $\mathbf{v}$ with $v_i \leq \sigma_i$ whenever $\delta_i = 0$ and $v_i \geq \sigma_i$ whenever $\delta_i = 1$. Then $G_\delta \cong P_{\boldsymbol{\sigma}}$. Let D_δ be the subdistribution of D on G_δ. If we suppose that $|D| = \boldsymbol{p}^{\boldsymbol{d}}$ then we show by the Pigeonhole principle that for some δ we have $|D_\delta| \geq \boldsymbol{p}^{\boldsymbol{\sigma}}$. Then we relabel G_δ so that r is labelled by **0** and apply Result 3.1 to finish the proof. The inequality

$$\sum_{\delta\in\{0,1\}^m} \boldsymbol{p}^{\sigma} = \sum_{\delta\in\{0,1\}^m} \prod_{i=1}^{m} p_i^{\sigma_i(\delta)} = \prod_{i=1}^{m}(p_i^{r_i} + p_i^{b_i}) \leq \prod_{i=1}^{m} p_i^{d_i} = \boldsymbol{p}^{\boldsymbol{d}}$$

holds since each $p_i \geq 2$. Thus if each $|D_\delta| < \boldsymbol{p}^{\boldsymbol{\sigma}}$ then $|D| < \boldsymbol{p}^{\boldsymbol{d}}$. ∎

In order to prove Theorem 1.5 from Result 3.1 we first define a pebbling distribution D in $P_{\boldsymbol{d}}$ which depends on the set of integers $\{x_1, \ldots, x_d\}$. Here, $|D| = \boldsymbol{p}^{\boldsymbol{d}}$, where $\boldsymbol{p}^{\boldsymbol{d}} = \prod_{i=1}^{m} p_i^{d_i}$ is the prime factorization of d. In what follows, each pebble will be named by a set, and $\boldsymbol{c}(B)$ will denote the vertex (coordinates) on which the pebble B sits. We let x_j correspond to the pebble $A_j = \{x_j\}$, which we place on the vertex $\boldsymbol{c}(A_j) = \langle c_1, \ldots, c_m\rangle$ of $P_{\boldsymbol{d}}$, where $d/gcd(x_j, d) = \boldsymbol{p}^{\boldsymbol{c}}$. For each vertex $\boldsymbol{u} = \langle u_1, \ldots, u_m\rangle$ define the set $X(\boldsymbol{u}) = \{A|\boldsymbol{c}(A) = \boldsymbol{u}\}$ to denote those pebbles currently sitting on $\mathbf{u}$, and let $\boldsymbol{u}^{(i)} = \langle u_1, \ldots, u_i - 1, \ldots u_m\rangle$.

We are now ready to present a counterexample. In Chung's proof it is stated that, for the prime decomposition of a positive integer $d = p_1^{d_1} \cdots p_m^{d_m}$, for any $1 \leq i \leq m$, and for any set of integers $X = \{x_1, \ldots, x_{p_i}\}$, there is a subset $S \subseteq \{1, \ldots, p_i\}$ such that $p_i | \sum_{k\in S} x_k = y$. This much is true. The error is in the statements that follow, namely,

$$\sum_{k\in S} gcd(x_k, d) \leq p_i \cdot gcd(x_1, d), \tag{3}$$

$$p_i \cdot gcd(x_1, d) = gcd(y, d). \tag{4}$$

The purpose of making the statements is to model the pebbling step numerically, removing X from $\boldsymbol{c}$ and placing y at $\boldsymbol{c}^{(i)}$. It may seem that (4) could be relaxed to an inequality, but we will show that in that case we run the risk of failing (3) at a later stage, a serious problem.

Take, for example, $d = 2 \cdot 5^2$, so that $P_{\boldsymbol{d}} = P_2 \Box P_3$ (see Fig. 3). Let $X = \{1, 7, 13, 17, 23\}$. Then for each $x_k \in X$ we have that $gcd(x_k, d) = 1$, and so $\boldsymbol{c}(A_k) = \langle 1, 2\rangle$ for each k. Consider $i = 2$ ($p_2 = 5$). If $S = \{1, 7, 17\}$ then $y = 25$ and (4) fails. In fact, (4) fails for all choices of S. In this case, since $gcd(y, d) = 25$, we would prefer that y be placed at the vertex $\langle 1, 0\rangle$ ($\langle 0, 1\rangle$ for the other choices of S), although since this represents a pebbling step, we are forced to place y at $\langle 1, 1\rangle$.

Suppose we find another set $X = \{1, 6, 11, 16, 21\}$ of 5 pebbles at $\langle 1, 2\rangle$. Then for $i = 2$ again, our only choice is $S = X$, so $y = 55$. Since $gcd(y, d) = 5$ in this case, we feel quite comfortable placing y at $\langle 1, 1\rangle$. But now consider the next pebbling step, with $i = 1$ ($p_1 = 2$) and $X = \{25, 55\}$ at vertex $\langle 1, 1\rangle$. At this point (3) fails.

Our Claim 3.4 will take the place of statements (3) and (4), after we introduce some technical devices which will maintain the uniformity of the gcd amongst the members of X. For a set B we make the following recursive definitions. The *value* of B is defined as $val(B) = \sum_{A\in B} val(A)$, with $val(\{A_j\}) = x_j$. The function GCD is defined as $GCD(B) = \sum_{A\in B} GCD(A)$, where $GCD(\{A_j\}) = gcd(x_j, d)$. Finally, $Set(B) = \cup_{A\in B} Set(A)$, where $Set(A_j) = A_j$.

We say that B is *well placed* at $\boldsymbol{c}(B) = \langle c_1, \ldots, c_m\rangle$ when

$$\boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{c}(B)} | val(B) \tag{5}$$

and

$$GCD(B) \leq \boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{c}(B)}. \tag{6}$$

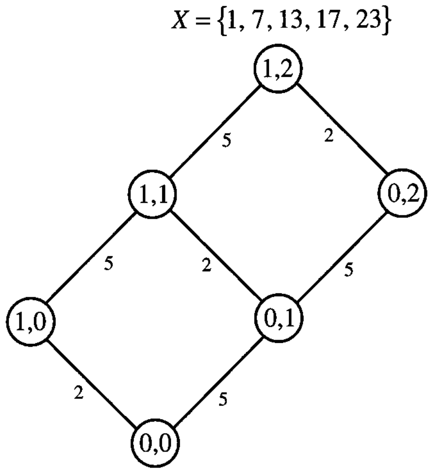

FIGURE 3. The graph $P_{\langle 1,2\rangle} = P_2 \square P_3$.

It is important to maintain a numerical interpretation of $\mathbf{p}$-pebbling so that moving a pebble to r corresponds to finding a set J which satisfies the conclusion of Theorem 1.5. For this reason we introduce the following operation, which corresponds to a greedy p_i-pebbling step in which a numerical condition must hold in order to move a pebble. We will show that this condition holds originally for D (Claim 3.3) and is maintained throughout (Claim 3.4).

Numerical Pebbling Operation. If W is a set of p_i pebbles such that every pebble $A \in W$ sits on the vertex $\boldsymbol{c}(A) = \boldsymbol{u}$, and there is some $B \subseteq W$ such that $p_i^{b_i} | val(B)$, where $b_i = d_i - c_i + 1$, then replace $X(\boldsymbol{c})$ by $X(\boldsymbol{c})\backslash W$, and replace $X(\boldsymbol{c}^{(i)})$ by $X(\boldsymbol{c}^{(i)}) \cup B$.

We are now ready to proceed.

Claim 3.3. *A_j is well placed for $1 \le j \le d$.*

Proof. Condition (5) holds since $\boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{c}(A_j)} = gcd(x_j, d) | x_j = val(A_j)$. Condition (6) holds since $GCD(A_j) = gcd(x_j, d) = \boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{c}(A_j)}$. ∎

Claim 3.4. *Suppose $B \subseteq X(\boldsymbol{u}), |B| \le p_i$, and $p_i^{b_i} | val(B)$ for $b_i = d_i - u_i + 1$. Suppose further that for every $A \in B$, A is well placed at $\mathbf{u}$. Then B is well placed at $\boldsymbol{u}^{(i)}$.*

Proof. Let $\alpha_i = b_i, \alpha_k = d_k - u_k$ for $k \ne i$, and $\alpha = \langle \alpha_1, \ldots, \alpha_m \rangle$. Then $\boldsymbol{p}^{\boldsymbol{\alpha}} = \boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{u}} p_i$. We need to show that $\boldsymbol{p}^{\boldsymbol{\alpha}} | val(B)$ and $GCD(B) \le \boldsymbol{p}^{\boldsymbol{\alpha}}$. First, since every $A \in B$ is well placed we have $\boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{u}} | val(A)$. Thus $\boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{u}} | \sum_{A \in B} val(A) = val(B)$. In addition, $p_i^{b_i} | val(B)$ and so $\boldsymbol{p}^{\boldsymbol{\alpha}} | val(B)$. Second, $GCD(B) = \sum_{A \in B} GCD(A) \le |B| \boldsymbol{p}^{\boldsymbol{d}-\boldsymbol{u}} \le \boldsymbol{p}^{\boldsymbol{\alpha}}$. ∎

The following well-known lemma is proved easily from the Pigeonhole principle.

Lemma 3.5. *If $N = \{n_1, \ldots, n_q\}$ is any set of q integers then there is some subset M of N such that $\sum_{n_i \in M} n_i \equiv 0 \,(mod\ q)$.*

Claim 3.6. *Suppose $|X(\boldsymbol{u})| \ge p_i$, and for all $A \in X(\boldsymbol{u})$, A is well placed at $\mathbf{u}$. Then there exists some $B \subseteq X(\boldsymbol{u})$ such that $|B| \le p_i$ and $p_i^{b_i} | val(B)$ where $b_i = d_i - u_i + 1$.*

Proof. Let $X(\boldsymbol{u}) \supseteq \{B_1, \ldots, B_{p_i}\}$. Since each B_k is well placed, we know that $val(B_k)/(p_i^{d_i-u_i})$ is an integer for $0 \leq k \leq p_i$. Now let $n_k = val(B_k)/(p_i^{d_i-u_i})$ and let $N = \{n_1, \ldots, n_{p_i}\}$. With $q = p_i$, Lemma 3.5 produces a subset $M \subseteq N$ such that $\sum_{n_k \in M} n_k \equiv 0 \,(\mathrm{mod}\, p_i)$. Thus $p_i | \sum_{n_k \in M} n_k$, and so $p_i^{d_i-u_i+1} | \sum_{n_k \in M}(p^{d_i-u_i})n_k = \sum_{n_k \in M} val(B_k)$. In other words, for $B = \{B_k | n_k \in M\}$, we have $p_i^{b_i} | val(B)$. ∎

Finally, we can prove Theorem 1.5.

Proof of Theorem 1.5. By Claim 3.3 the pebbles corresponding to each of the numbers are initially well placed. Claim 3.4 guarantees that applying the Numerical Pebblng Operation maintains the well placement of the pebbles. Claim 3.6 establishes that every graphical pebbling operation can be converted to a numerical pebbling operation. Then by Chung's Result 3.1 we can repeatedly apply the numerical pebbling operation to move a pebble to **0**. This pebble B is then well placed at **0**. Thus, for $J = \{j | x_j \in Set(B)\}$, we have $d = \boldsymbol{p}^{\boldsymbol{d}} | val(B) = \sum_{j \in J} x_j$ by (5), and $\sum_{j \in J} gcd(x_j, d) = GCD(B) \leq \boldsymbol{p}^{\boldsymbol{d}} = d$ by (6). ∎

4. QUESTIONS

For $n \geq 2t+1$, the Kneser graph, $K(n,t)$, is the graph with vertices $\binom{[n]}{t}$ and edges $\{A, B\}$ whenever $A \cap B = \varnothing$. The case $t = 1$ yields the complete graph K_n, which is of Class 0, so consider $t \geq 2$. For $n \geq 3t-1$ we have $diam(K(n,t)) = 2$. Also, it is not difficult to show that $\kappa(K(n,t)) \geq 3$ in this range. Indeed, $\kappa(K(n,t)) \geq \binom{n-2t+1}{t}$, the minimum size of a common neighborhood of two nonadjacent vertices. When $n \geq 3t$ this value is at least $t+1 \geq 3$. In the case $n = 3t-1$, it is easy to explicitly find 3 pairwise internally disjoint paths between any pair of vertices. (Using the techniques of [3], one can improve this to $\kappa(K(n,t)) \geq min\{t\binom{n-t}{t-1}, \binom{n-t}{t} - (t-1)\binom{n-2t+1}{t-1}\}$—it would be interesting in its own right to find $\kappa(K(n,t))$ more accurately.) Thus, we know by Theorem 1.7 that such graphs are of Class 0. The family of Kneser graphs is interesting precisely because it becomes more sparse as n decreases toward $2t+1$, so the diameter increases and yet the connectivity decreases.

Question 4.1. *Is it true that the graphs $K(n,t)$ are of Class 0 when $n < 3t-1$?*

We say a graph G satisfies the *2-pebbling property* if two pebbles can be moved to a specified vertex when the total starting number of pebbles is $2f(G) - q + 1$, where q is the number of vertices with at least one pebble. Pachter and Snevily [7] proved that diameter two graphs have the 2-pebbling property. The 2-pebbling property is important because Conjecture 1.3 holds true when G_1 has the 2-pebbling property and G_2 is either a clique [1], a tree [6], or an even cycle [7].

Question 4.2. *Is it true that the graphs $K(n,t)$ have the 2-pebbling property when $n < 3t-1$?*

Regarding greedy graphs, there are graphs which are not greedy, namely odd cycles. It is conjectured in [7] that bipartite graphs have the 2-pebbling property. We ask

Question 4.3. *Is every bipartite graph greedy?*

Another natural question is whether Conjecture 1.3 can be proved in the case that G_1 and G_2 are both greedy and/or of Class 0. More importantly, one can generalize the conjecture to **p**-pebbling, where $\boldsymbol{p} = \langle p_1, p_2 \rangle$.

Conjecture 4.4. $f_{\boldsymbol{p}}(G_1 \square G_2) \leq f_{p_1}(G_1) f_{p_2}(G_2)$.

Finally, it follows immediately from the Pigeonhole principle that a graph G on n vertices with diameter d has pebbling number $f(G) \leq (n-1)(2^d-1)+1$. It would be interesting to find better general bounds on $f(G)$, especially not involving n. For example, there is no function g such that every graph G of independence number α and diameter d has pebbling number $f(G) \leq g(\alpha)2^d$. Indeed, we define a family of graphs G_m which satisfy $diam(G) = d$ and $\alpha(G) = 2^{d-1}+1$, but which have pebbling number $f(G_m) \to \infty$ as $m \to \infty$. Let Q_n be the n-dimensional cube and let $x \in V(Q_n)$. Then define $G_m = Q_n \cup K_m \cup E$, where the edge set $E = \{xv | v \in V(K_m)\}$. Since $\kappa(G_m) = 1$ we know from Lemma 2.1 that $f(G_m) > 2^d + m$.

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