The 2-Pebbling Property and a Conjecture of Graham's

Hunter S. Snevily
Department of Mathematics and Statistics
University of Idaho
Moscow, Idaho 83844-0101
snevily@uidaho.edu

James A. Foster
Department of Computer Science
University of Idaho
Moscow, Idaho 83844-0101
foster@cs.uidaho.edu

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Abstract

The *pebbling number* of a graph G, $f(G)$, is the least m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. It is conjectured that for all graphs G and H, $f(G \times H) \leq f(G)f(H)$.

Let C_m and C_n be cycles. We prove that $f(C_m \times C_n) \leq f(C_m)f(C_n)$ for all but a finite number of possible cases. We also prove that $f(G \times T) \leq f(G)f(T)$ when G has the 2-pebbling property and T is any tree.

Keywords: Graphs, Pebbling.

1 Introduction

Throughout this paper, unless stated otherwise, G will denote a simple connected graph on n vertices and $f(G)$ will denote the pebbling number of G (see below).

Suppose p pebbles are distributed onto the vertices of a graph G. A pebbling move (step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the target vertex, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble. In this paper, the letter v will frequently be used to denote the target vertex of the graph under consideration; context should make this clear. For a graph G, we define the *pebbling number* $f(G)$ to be the smallest integer m such that for any distribution of m pebbles to the vertices of G, one pebble can be moved to any specified vertex v.

In this paper, we present evidence that supports the following conjecture:

Conjecture 1 (Graham) $f(G \times H) \leq f(G)f(H)$.

We prove that $f(C_m \times C_n) \leq f(C_m)f(C_n)$ for all but thirty seven cases. Actually, we prove something stronger, but to better understand our results and those of previous researchers, we need the following definition.

Definition 1 *We say a graph G satisfies the* 2-pebbling property *if two pebbles can be moved to any specified vertex when the total starting number of pebbles is $2f(G) - q + 1$, where q is the number of vertices with at least one pebble.*

We now list the results that support Graham's Conjecture.

Theorem 1 (Chung [1]) *Suppose G satisfies the 2-pebbling property; then the following hold:*

1. *$f(G \times K_t) \leq tf(G)$*

2. *If $f(G \times K_t) = tf(G)$, then $G \times K_t$ satisfies the 2-pebbling property.*

Theorem 2 *(Section 2) Suppose G satisfies the 2-pebbling property. Then $f(G \times T) \leq f(G)f(T)$ where T is a tree.*

Theorem 3 *(Section 3) Suppose G satisfies the 2-pebbling property. Then $f(G \times C) \leq f(G)f(C)$, where C is any even cycle with ten or more vertices.*

The reader at this point might be wondering which graphs have the 2-pebbling property. They are: trees [1], all graphs of diameter 2 [6], all cycles (this paper), and probably many more. In fact, we know of only 2 or 3 graphs that fail to have the 2-pebbling property (see Section 4). It is well known that most graphs (in the probabilistic sense) have diameter 2. Therefore, we see that the results of Chung, Theorem 2, and Theorem 3 actually cover a great number of cases of Graham's Conjecture.

In Section 2 we give a proof of Theorem 2. In Section 3 we prove our claim about the Cartesian product of two cycles satisfying Graham's Conjecture, and in Section 4 we describe what is known about graphs that fail to have the 2-pebbling property (Lemke graphs).

2 Trees

Let T be a tree with a specified vertex v. T can be viewed as a directed tree denoted by $\vec{T}_v$ with edges directed toward a specified vertex, also called the root. A path-partition $\mathcal{P} = \{\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_r\}$ is a set of nonoverlapping directed paths the union of which is $\vec{T}$. Throughout this paper, unless stated otherwise, we will always assume that $|E(\vec{P}_i)| \geq |E(\vec{P}_j)|$ whenever $i \leq j$. A path-partition $\mathcal{P} = \{\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_r\}$ is said to majorize another (say $\mathcal{Q} = \{\vec{P}'_1, \vec{P}'_2, \ldots, \vec{P}'_r\}$) if the nonincreasing sequence of its path size majorizes that of the other. That is, if $a_i = |E(\vec{P_i})|$ and $b_i = |E(\vec{P_i'})|$ then $(a_1, a_2, \ldots, a_r) > (b_1, b_2, \ldots, b_t)$ if and only if $a_i > b_i$ where $i = \min\{j : a_j \neq b_j\}$.

A path-partition of a tree T is said to be maximum if it majorizes all other path-partitions. We define the pebbling number $f(G, v)$ to be the smallest integer m such that if m pebbles are distributed on the vertices of G, then one pebble can be moved to v.

Fact 1 (Chung[1]) *The pebbling number $f(T, v)$ for a vertex v in a tree T is $2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} - r + 1$ when $a_1, a_2, \ldots, a_r$ is the sequence of the path sizes in a maximum path-partition $\mathcal{P}$ of $\vec{T_v}$.*

Definition 2 *Let T be a tree and let $v \in V(T)$. If $\deg(v) > 1$ then let T_v^* be the tree obtained from T by deleting all vertices of degree one (leaves) in T. If $\deg(v) = 1$ then let T_v^* be the tree obtained from T by deleting all leaves of T with the exception of v. In either case, we have $v \in V(T_v^*)$.*

Lemma 1 *If $f(T, v) = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} - r + 1$ then $f(T_v^*, v) = 2^{a_1 - 1} + 2^{a_2 - 1} + \cdots + 2^{a_r - 1} - r + 1$, assuming $|V(T)| \geq 2$.*

Proof: We will assume that $a_1 \geq a_2 \geq \cdots a_r \geq 1$. Let $k+1$ be the least integer such that $a_{k+1} = 1$. Hence $2^{a_1-1} + 2^{a_2-1} + \cdots + 2^{a_r-1} - r + 1 = 2^{a_1-1} + 2^{a_2-1} + \cdots + 2^{a_k-1} - k + 1$. Let $\mathcal{P} = \{\vec{P_1}, \vec{P_2}, \ldots, \vec{P_r}\}$ be a maximum path-partition of $\vec{T_v}$. Note that one endpoint (say u_i) of each $\vec{P_i}$ is a leaf of $\vec{T}$ with $u_i \neq v$. Let $\vec{P_i^*} = \vec{P_i} - \{u_i\}$ and note that $\mathcal{P}^* = \{\vec{P_1^*}, \vec{P_2^*}, \ldots, \vec{P_k^*}\}$ is a path-partition of $\vec{T_v^*}$. Let $\mathcal{Q} = \{\vec{P_1'}, \vec{P_2'}, \ldots, \vec{P_t'}\}$ be a maximum path-partition of $\vec{T_v^*}$ and assume that $b_i = |E(\vec{P_i'})|$. Suppose that $(b_1, b_2, \ldots, b_t) > (a_1 - 1, a_2 - 1, \ldots, a_k - 1)$ (i.e. for some j we have $b_j > a_j - 1$). Let j_0 be the least j where this occurs. Then since each path $\vec{P_i'}$ $(1 \leq i \leq j_0)$ in $\mathcal{Q}$ ends at a leaf (say y_i) in $\vec{T_v^*}$ with $y_i \neq v$, we see that the paths $\vec{P_1'}, \vec{P_2'}, \ldots, \vec{P_{j_0}'}$ can all be extended to longer paths in $\vec{T_v}$. This contradicts the fact that $\mathcal{P}$ is a maximum path-partition of $\vec{T_v}$, and we are done. $\square$

In [5], David Moews proved a weaker version of the following theorem (in his definition of 2-pebbling q represented the number of vertices with an *odd* number of pebbles.)

Theorem 2 *Let T be a tree and let G be a graph with the 2-pebbling property. Then $f(G \times T) \leq f(G)f(T)$.*

Proof: Let $z_1, z_2, \ldots, z_n$ be the vertices of T and assume that they have been labeled so that $z_1, z_2, \ldots, z_{r^*}$ are the leaves of T. Let $x_1, x_2, \ldots, x_m$ be the vertices of G, let $G_i \equiv G$ for $1 \leq i \leq n$ and let $V(G_i) = \{x_1^i, x_2^i, \ldots, x_m^i\}$ with $(x_j^i, x_k^i) \in E(G_i)$ if and only if $(x_j, x_k) \in E(G)$. We can construct $G \times T$ by taking our n copies of G (i.e. $G_1, G_2, \ldots, G_n$) and placing edges between x_k^i and x_k^j if and only if $(z_i, z_j) \in E(T)$. Let v be our target vertex in $G \times T$. Assume that $v = x_t^{i_0} \in G_{i_0}$. If $1 \leq i_0 \leq r^*$ then we may assume (after relabeling if necessary) that $v \in G_{i_0} = G_{r^*}$. and we define $r = r^* - 1$. Otherwise we define $r = r^*$. We call $G_1, G_2, \ldots, G_r$ the leaves of $G \times T$. Let $V(T_v^*) = \{x_t^i : r < i \leq n\}$. Then $T_v^* \equiv T_{z_{i_0}}^*$ and we can think of $G \times T_v^*$ as being a subgraph of $G \times T$ (just delete $G_1, G_2, \ldots, G_r$ from $G \times T$).

We will proceed by induction on n. (Actually, we will prove something stronger, we will show that $f(G \times T, v) \leq f(G)f(T, z_{i_0})$—recall that $v = x_t^{i_0}$.) Clearly, our theorem is true if $n = 1$. Now assume that our theorem is true for all trees T_0 with $1 \leq |V(T_0)| = n' < n$ and let T be a tree with n vertices. Assume that $f(T, z_{i_0}) = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} - r + 1$. Suppose that $f(G) \cdot f(T, z_{i_0})$ pebbles have been placed at the vertices of $G \times T$. First, we will show that the leaves of $G \times T$ contain at least $(f(T, z_{i_0}) - 1)f(G) + 1$ pebbles. Suppose they do not, that is suppose that the leaves of $G \times T$ contain at most $(f(T, z_{i_0}) - 1)f(G)$ pebbles. We may assume that every vertex in $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_r)$ contains an odd number of pebbles (this is the worst case scenario). Then we could move at least $\frac{(f(T, z_{i_0}) - 1 - r)f(G)}{2} + f(G)$

pebbles to the vertices of $G \times T_v^*$ (here $T_v^* = T_{z_{i_0}}^*$). But

$$\begin{aligned}
&\frac{(f(T, z_{i_0}) - 1 - r)f(G)}{2} + f(G) \\
&= \frac{(2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} - r + 1)f(G)}{2} - \frac{(r+1)f(G)}{2} + f(G) \\
&= (2^{a_1 - 1} + 2^{a_2 - 1} + \cdots + 2^{a_r - 1} - r + 1)f(G) \geq f(G \times T_v^*, V)
\end{aligned}$$

and we are done.

By our previous remarks we may assume that $G \times T_v^*$ contains at most $\alpha_0 f(G)$ pebbles where $0 \leq \alpha_0 < 1$ and that G_i $(1 \leq i \leq r)$ contains $(j_i + \alpha_i)f(G)$ pebbles where j_i is a nonnegative integer and $0 \leq \alpha_i < 1$. Let q_i be the number of occupied vertices in G_i $(1 \leq i \leq r)$.

Next we claim that $\sum_{i=1}^r q_i > (r - 1 + \alpha_0)f(G)$. For suppose not. Then $\sum_{i=1}^r q_i \leq (r - 1 + \alpha_0)f(G)$. Hence at least

$$\begin{aligned}
&\frac{(f(T, z_{i_0}) - \alpha_0 f(G) - (r - 1 + \alpha_0))f(G)}{2} + \alpha_0 f(G) \\
&= \frac{f(T, z_{i_0})f(G) - 2\alpha_0 f(G) - rf(G) + f(G)}{2} + \alpha_0 f(G) \\
&= \frac{f(T, z_{i_0})f(G)}{2} - \frac{rf(G)}{2} + \frac{f(G)}{2} \\
&= \left(\frac{2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} - r + 1}{2}\right) f(G) - \frac{rf(G)}{2} + \frac{f(G)}{2} \\
&= \left(2^{a_1 - 1} + 2^{a_2 - 1} + \cdots + 2^{a_r - 1} - r + 1\right) f(G) \\
&\geq f(G \times T_v^*, V)
\end{aligned}$$

pebbles can be moved to $G \times T_v^*$ and we are done.

Now let $\sum_{i=0}^r \alpha_i = s \leq r$. Hence $\sum_{i=1}^r j_i = f(T, z_{i_0}) - s$. Note that $\alpha_i f(G) + q_i < 2f(G)$ for $1 \leq i \leq r$. We claim that there exist $i_1, i_2, \ldots, i_s$ such that $i_k \geq 1$ and $\alpha_{i_k} f(G) + q_{i_k} > f(G)$, $k = 1, \ldots, s$.

For suppose not, then

$$\sum_{i=1}^{r}(\alpha_i f(G)+q_i) \le 2(s-1)f(G)+(r-(s-1))f(G) = rf(G)+sf(G)-f(G)$$

But

$$\begin{aligned}
&\sum_{i=1}^{r}(\alpha_i f(G)+q_i) \\
&= \sum_{i=1}^{r}\alpha_i f(G)+\sum_{i=1}^{r} q_i \\
&> \sum_{i=1}^{r}\alpha_i f(G)+(r-1+\alpha_0)f(G) \\
&= \sum_{i=0}^{r}\alpha_i f(G)+(r-1)f(G) \\
&= rf(G)+sf(G)-f(G)
\end{aligned}$$

and this is a contradiction.

Therefore, we may assume (after relabeling if necessary) that $\alpha_i f(G) + q_i > f(G)$ for $1 \le i \le s$. Assume that for some $1 \le i \le s$ we had $j_i = 0$, then G_i contains $\alpha_i f(G) < f(G)$ pebbles. Let $T_{z_i} = T - z_i$ and note that z_i is a leaf in T since G_i is a leaf in $G \times T$. We can think of $G \times T_{z_i}$ as being a subgraph of $G \times T$. Furthermore, we see that $G \times T_{z_i}$ has at least $(f(T, z_{i_0})-1)f(G)+1$ pebbles at its vertices. It is easy to see that $f(T_{z_i}, z_{i_0}) \le f(T, z_{i_0}) - 1$ and we are done.

Therefore we may assume that $j_i \ge 1$ for $1 \le i \le s$. Recall $v = x_t^{i_0}$. Now in G_i $(1 \le i \le s)$ we have $\alpha_i f(G) + q_i > f(G)$ and $j_i \ge 1$. Hence by the 2-pebbling property we can move at least $j_i + 1$ pebbles to x_t^i in G_i $(1 \le i \le s)$. In G_i $(i > s)$ we can move at least j_i pebbles to x_t^i. Now consider the tree $\hat{T} \equiv T$ with $V(\hat{T}) = x_t^i$ $(1 \le i \le n)$. By the above pebbling moves, we see that $\sum_{i=1}^{r} j_i + s = f(T, z_{i_0}) - s + s = f(T, z_{i_0})$ pebbles can be moved to the vertices of $\hat{T}$ and we are done $(v \in V(\hat{T}))$. $\square$

3 Cycles

Before proceeding with this section, one should mention that $f(C_5 \times C_5) = 25$. A proof of this non-trivial result can be found in [3].

Theorem 3 *Let $C \equiv C_{2n}$ $(n \geq 5)$ and let G be a graph with the 2-pebbling property. Then $f(G \times C) \leq f(G)f(C)$.*

Proof: Let $C_{2n} \equiv C = xa_{n-1}a_{n-2} \dots a_2a_1yb_1b_2 \dots b_{n-1}x$, let $z_1, z_2, \dots, z_m$ be the vertices of G, and let $G_u \equiv G$ where $u \in V(C)$ and $V(G_u) = \{z_1^u, z_2^u, \dots, z_m^u\}$ with $(z_j^u, z_k^u) \in E(G_u)$ if and only if $(z_j, z_k) \in E(G)$. We can construct $G \times C$ by taking our $2n$ copies of G (i.e. $G_x, G_{a_{n-1}}, \dots, G_y, \dots, G_{b_{n-1}}$) and placing edges between z_k^u and z_k^w if and only if $(u, w) \in E(C)$. Let $\mathcal{P}_A$ be the induced subgraph of $G \times C$ with vertex set $V(G_y) \cup V(G_{a_1}) \cup \dots \cup V(G_{a_{n-1}})$, let $\mathcal{P}_A^- = \mathcal{P}_A - V(G_{a_{n-1}})$, and let $\mathcal{P}_B$ be the induced subgraph of $G \times C$ with vertex set $V(G_y) \cup V(G_{b_1}) \cup \dots \cup V(G_{b_{n-1}})$, and let $\mathcal{P}_B^- = \mathcal{P}_B - V(G_{b_{n-1}})$. Note that $\mathcal{P}_B \equiv \mathcal{P}_A \equiv G \times P_n$ and that $f(\mathcal{P}_B) \leq 2^{n-1}f(G)$ by Theorem 2. Similarly, we have $\mathcal{P}_B^- \equiv \mathcal{P}_A^- \equiv G \times P_{n-1}$ and $f(\mathcal{P}_B^-) \leq 2^{n-2}f(G)$.

Assume our target vertex is $v = z_t^y$. Define j_u $(u \in V(C))$ to be the number of pebbles at $V(G_u)$ and let $j_B = j_y + \sum_{i=1}^{n-1} j_{b_i}$ and $j_A = j_y + \sum_{i=1}^{n-1} j_{a_i}$. Note that j_B equals the number of pebbles at the vertices of $\mathcal{P}_B$. Define q_u $(u \in V(C))$ to be the number of occupied vertices of G_u. Without loss of generality we will assume that $j_B \geq j_A$ and that $j_y < f(G)$ $(v = z_t^y)$.

First, we establish two simple facts.

Fact 1: $q_x > j_y$.

Proof: Suppose not. Then $j_x \geq 2^n f(G) - 2j_B + j_y$. Hence at least $\frac{j_x - j_y}{2} \geq \frac{2^n f(G) - 2j_B + j_y - j_y}{2} = 2^{n-1}f(G) - j_B$ pebbles could be moved to $V(G_{b_{n-1}})$ from $V(G_x)$ and we would be done since $f(\mathcal{P}_B) \leq 2^{n-1}f(G)$. □

Fact 2: $j_B - j_A < f(G)$.

Proof: Suppose not. Then

$$\begin{aligned} j_x &= 2^n f(G) - j_B - j_A + j_y \geq 2^n f(G) - j_B - (j_B - f(G)) + j_y \\ &= 2^n f(G) - 2j_B + f(G) + j_y \end{aligned}$$

Hence at least $\frac{j_x - f(G)}{2} \geq 2^{n-1}f(G) - j_B + \frac{j_y}{2}$ pebbles can be moved to $V(G_{b_{n-1}})$ from $V(G_x)$ and we are done since $f(\mathcal{P}_B) \leq 2^{n-1}f(G)$. □

Recall that $j_B - j_A < f(G)$ and note that $\frac{j_x - f(G)}{2} \geq \frac{2^n f(G) - 2j_A + j_y - 2f(G)}{2} = 2^{n-1}f(G) - j_A + \frac{j_y}{2} - f(G)$. Hence at least

$$\begin{aligned} & \frac{\left(j_{a_{n-1}} + \frac{j_x - f(G)}{2} - f(G)\right)}{2} \\ \geq \;& \frac{\left(j_{a_{n-1}} + 2^{n-1}f(G) - j_{a_{n-1}} - \left(\sum_{i=1}^{n-2} j_{a_i} + j_y\right) + \frac{j_y}{2} - 2f(G)\right)}{2} \\ = \;& 2^{n-2}f(G) - \frac{\left(\sum_{i=1}^{n-2} j_{a_i} + j_y\right)}{2} + \frac{j_y}{4} - f(G) \end{aligned}$$

pebbles can be moved from $V(G_x)$ and $V(G_{a_{n-1}})$ to $V(G_{a_{n-2}})$. This gives us at least $2^{n-2}f(G) + \frac{\left(\sum_{i=1}^{n-2} j_{a_i} + j_y\right)}{2} + \frac{j_y}{4} - f(G)$ pebbles at the vertices of $\mathcal{P}_A^-$. Thus, if $\frac{\left(\sum_{i=1}^{n-2} j_{a_i} + j_y\right)}{2} + \frac{j_y}{4} \geq f(G)$ we are done, since $f(\mathcal{P}_A^-) \leq 2^{n-2}f(G)$. Therefore we may assume that $\sum_{i=1}^{n-2} j_{a_i} + \frac{3}{2}j_y < 2f(G)$.

Now suppose $j_B \geq 6f(G) - j_y$. Then since $j_B - j_A < f(G)$ and $j_A - j_{a_{n-1}} < 2f(G)$, we must have $j_{a_{n-1}} \geq 3f(G) - j_y$. Hence we can move at least $\frac{j_{a_{n-1}} - f(G)}{2} \geq f(G) - \frac{j_y}{2}$ "extra" pebbles to $V(G_x)$ from $V(G_{a_{n-1}})$. This gives us at least $2^n f(G) - 2j_B + j_y + f(G) - \frac{j_y}{2}$ pebbles on $V(G_x)$. Thus at least $2^{n-1}f(G) - j_B + \frac{j_y}{4}$ pebbles can be moved to $V(G_{b_{n-1}})$ from $V(G_{a_{n-1}})$ and $V(G_x)$ and we are done, since $f(\mathcal{P}_B) \leq 2^{n-1}f(G)$. Therefore we may assume that $j_B = j_{b_{n-1}} + \left(\sum_{i-1}^{n-2} j_{b_i} + j_y\right) < 6f(G) - j_y$.

Now move $2^{n-2}f(G) - (j_B - j_y)$ pebbles to $V(G_{b_{n-1}})$ from $V(G_x)$. Since $n \geq 5$ we are guaranteed that this is a positive number. This leaves at least

$2^n f(G) - 2j_B - 2(2^{n-2} f(G) - (j_B - j_y)) + j_y = 2^{n-1} f(G) - j_y$ pebbles at $V(G_x)$ (and $2^{n-2} f(G)$ pebbles at $\bigcup_{i=1}^{n-1} V(G_{b_i})$). We write this as $(2^{n-1} - 2) f(G) + 2f(G) - j_y$ and recall that $q_x > j_y$. Thus by the 2-pebbling property of G_x we see that 2^{n-1} pebbles can be moved to z_t^x ($v = z_t^y$). Recall that we have $2^{n-2} f(G)$ pebbles at $\mathcal{P}_B - V(G_y) \equiv G \times P_{n-1}$, hence a pebble can be moved to $z_t^{b_1}$. Now $z_t^x, z_t^{b_{n-1}}, z_t^{b_{n-2}}, \ldots, z_t^{b_1}$ is a path P in $G \times C$ of length $n-1$. Thus, utilizing the 2^{n-1} pebbles at z_t^x we see that another pebble can be moved to $z_t^{b_1}$ and we are done. $\square$

The following fact will come in handy.

Fact 2 (see [6]) $f(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1.$

Definition 3 *The* t-pebbling number *of a graph* G *is the smallest number* $f_t(G)$ *with the property that from every placement of* $f_t(G)$ *pebbles on* G *it is possible to move* t *pebbles to any vertex* v *by a sequence of pebbling moves.*

Proposition 1 $f_t(C_{2k+1}) \leq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1 + (t-1)2^k.$

Proof: The proof is by induction on t, where $t = 1$ is Fact 2. For $t \geq 2$, one side of C_{2k+1} or the other must contain 2^k pebbles. We use these 2^k pebbles to place a pebble at our target vertex v and use the remaining $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1 + (t-2)2^k$ pebbles to place $t-1$ more pebbles at v. $\square$

Theorem 4 *Let* $j \geq k$ *(with* $j \geq 5$ *and* $k \geq 1$*); then* $f(C_{2j+1} \times C_{2k+1}) \leq f(C_{2j+1}) f(C_{2k+1})$.

Proof: By Chung's result [1] and Lemma 2 we may assume that $k \geq 2$. We will view $G = C_{2j+1} \times C_{2k+1}$ as a $(2j+1)$ by $(2k+1)$ grid with an extra edge between the first and last vertex of each row, and also between the first and last vertex of each column. Let $C_{2k+1} = y_1 y_2 \ldots y_{2k+1} y_1$. We may assume, without loss of generality, that the center vertex in our grid $(C_{2j+1} \times C_{2k+1})$ is our target vertex v (i.e. $v = \{x_{j+1}\} \times \{y_{k+1}\}$). Now assume $\left(2\left\lfloor\frac{2^{j+1}}{3}\right\rfloor+1\right)\left(2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1\right)$ pebbles have been placed arbitrarily at the vertices of G. We will show that by moving pebbles along the edges of the columns of G (note each column is isomorphic to C_{2k+1}) $2\left\lfloor\frac{2^{j+1}}{3}\right\rfloor+1$ pebbles can be moved to the $k+1$st row of G. Since each row of G is isomorphic to C_{2j+1} we will be done. Let $\mathbf{C}_i$ denote the ith column of G and let p_i equal the number of pebbles at the vertices of $\mathbf{C}_i$. We may assume (after relabeling if necessary) that $p_i \leq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor$ for $1 \leq i \leq r$ and that $p_i \geq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1$ for $i > r$.

Since we will be restricting our pebbling moves along the edges of the $\mathbf{C}_i$s, we want to consider the worst case scenario (i.e. the most wasteful distribution of pebbles possible). Therefore we will assume that $p_i = 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor$ for $1 \leq i \leq r$ and that, for $r+1 \leq i \leq 2j$, $p_i = 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1+t_i2^k+2^k-1$ where t_i is a non-negative integer (here we are making use of Proposition 1).

Note that $p_{2j+1} \leq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1+t_{2j+1}2^k+R$ where t_{2j+1} is a non-negative integer and $0 \leq R \leq 2^k-1$, whence we may assume that at least $r2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+(2j-r)(2^k-1)$ pebbles will be wasted (not moved).

Let

$$\begin{aligned}
\Delta &= \left(2\left\lfloor\frac{2^{j+1}}{3}\right\rfloor+1\right)\left(2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1\right)-2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor r \\
&\quad -(2j-r)(2^k-1)-(2j+1-r)\left(2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1\right) \\
&= \left(2\left\lfloor\frac{2^{j+1}}{3}\right\rfloor+1\right)\left(2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1\right) \\
&\quad +2^kr-(2j)(2^k-1)-(2j+1)\left(2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor+1\right)
\end{aligned}$$

Now, $\Delta = \left(\sum_{i=r+1}^{2j+1} t_i\right)2^k+R$. By Proposition 1, we can move $\sum_{i=r+1}^{2j+1}(t_i+1)$ pebbles to the $k+1$st row of G, and therefore it suffices to show that $\frac{\Delta}{2^k} \geq 2\left\lfloor\frac{2^{j+1}}{3}\right\rfloor+1-(2j+1-r)$. Using the above expression for Δ and the fact that $2\left\lfloor\frac{2^{n+1}}{3}\right\rfloor+1=\frac{2^{n+2}+(-1)^{n+1}}{3}$ we must show that

$$\frac{1}{2^k}\left[\left(\frac{2^{k+2}+(-1)^{k+1}}{3}\right)\left(\frac{2^{j+2}+(-1)^{j+1}}{3}-(2j+1)\right)+2j\right]\geq \frac{2^{j+2}+(-1)^{j+1}}{3}-1 \tag{1}$$

It is easy to show that inequality (1) is true for any $j\geq 6$ and any $k\geq 2$. Direct substitution shows that inequality (1) is true when $j=5$ and $k\in\{2,3,4,5\}$.

This proves our theorem. □

In [6], it is shown that most graphs have the 2-pebbling property. Below, we will show that all cycles have the 2-pebbling property.

Definition 4 *Let $\mathcal{P}=P_{n+1}=v_0v_1\ldots v_n$ be a path of length n. Suppose q_i pebbles are placed at vertex v_i for $0\leq i\leq n$. Then we say that $\mathcal{P}$ has weight $\sum_{i=0}^{n}2^iq_i$ with respect to v_n and this is written as $w_p(v_n)$.*

Proposition 2 *Let $\mathcal{P}=P_{n+1}=v_0v_1\ldots v_n$. If $w_p(v_n)\geq k2^n$ then at least k pebbles may be moved to v_n.*

Proof: We will proceed by induction on n.

Clearly, our lemma is true if $n=0$. Now assume that our lemma is true for all n' with $0\leq n'<n$ and consider the path $\mathcal{P}=P_{n+1}=v_0v_1\ldots v_n$. Assume that $w_p(v_n)=\sum_{i=0}^{n}2^iq_i\geq k2^n$ (here q_i equals the number of pebbles at v_i). Note that if q_0 is odd, then we must have $w_p(v_n)\geq k2^n+1$; hence, if q_0 is odd, then $\frac{q_0-1}{2}+\sum_{i=1}^{n}2^{i-1}q_i\geq k2^{n-1}$ and if q_0 is even then $\frac{q_0}{2}+\sum_{i=1}^{n}2^{i-1}q_i\geq k2^{n-1}$. Both cases tell us that if we move as many pebbles as possible from v_0 to v_1 then the weight of the sub-path $v_1v_2\ldots v_n\equiv P_n$ with respect to v_n is at least $k2^{n-1}$ and we are done. □

Lemma 2 *All cycles have the 2-pebbling property.*

Proof:

Case 1 $C \equiv C_{2n}$. The case $n = 2$ is trivial, hence we will assume that $n \geq 3$. Let q be the number of occupied vertices of C. If $q = 2n$ the proof is trivial. If $q = 2n - 1$ and $n = 3$ then the proof is also very easy to check. Therefore we will assume that $2 \leq q \leq 2n-1$ for $n \geq 4$ and that $2 \leq q \leq 2n - 2$ for $n = 3$. Under these hypotheses, we have $q \leq 2^{n-1}$ for all $n \geq 3$. Let $C_{2n} \equiv C = xa_{n-1}a_{n-2} \ldots a_2a_1vb_1b_2 \ldots b_{n-2}b_{n-1}$ and let v be our target vertex.

Let $\mathcal{P}_A = a_{n-1}a_{n-2} \ldots a_2a_1v$, and let $\mathcal{P}_B = b_{n-1}b_{n-2} \ldots b_2b_1v$, and note that $f(\mathcal{P}_A) = f(\mathcal{P}_B) = 2^{n-1}$.

Let μ_x be the number of pebbles located at x. Define μ_{a_i} and μ_{b_i} similarly. First, we claim that $\mu_x \geq 2^n - q + 2$. For suppose not, then $2 \cdot 2^n - q + 1 - \mu_x \geq 2^n = f(\mathcal{P}_A) + f(\mathcal{P}_B)$. Hence, either $\mathcal{P}_A$ or $\mathcal{P}_B$ has at least 2^{n-1} pebbles. Using 2^{n-1} of these pebbles we can move a pebble to v. This leaves $2 \cdot 2^n - q + 1 - 2^{n-1} \geq 2 \cdot 2^n - 2^{n-1} + 1 - 2^{n-1} = 2^n + 1$ pebbles on the vertices of C, since $q \leq 2^{n-1}$, and we are done.

Now write q as $q = 2r$ if q is even and as $q = 2r + 1$ if q is odd. For now assume that $r \geq 3$. Without loss of generality we may assume that at least r distinct vertices (say $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$) of $\mathcal{P}_A$ contain pebbles. Note that $2^1 + 2^2 + \cdots 2^r \geq 2q$. Using one pebble each from a_{i_j}, $1 \leq j \leq r$, $2^n - 2q$ pebbles from x, and Proposition 2, we see that a pebble can be moved to v at a cost of $2^n - 2q + r$ pebbles. This leaves $2 \cdot 2^n - q + 1 - (2^n - 2q + r) = 2^n + q - r + 1 > 2^n$ pebbles on C and we are done.

There are four remaining cases to consider, $q = 2, 3, 4$ and 5.

Subcase (i) $q = 2$. Without loss of generality we may assume that and that $\mu_{a_i} > 1$ for some i with $1 \leq i \leq n-1$. Using Proposition 2 it is easy to see that two pebbles can be moved to v.

Subcase (ii) $q = 3$. If $\mathcal{P}_A$ (or $\mathcal{P}_B$) contains two distinct vertices each with a pebble on it, we argue as before ($2^1 + 2^2 = 6 = 2 \cdot q$). Thus without loss of generality we may assume that $\mathcal{P}_A$ and $\mathcal{P}_B$ each contain exactly one vertex, say a_i and b_j, with at least one pebble on it. If $\mu_{a_i} \geq 2$ then we can move $2^{n-1} - 2$ pebbles from x to a_{n-1} and hence a pebble to v at a cost of $2^n - 4 + 2 = 2^n - 2$ pebbles. This leaves 2^n pebbles on C and we are done. Therefore we can assume that $\mu_{a_i} = 1$ and $\mu_{b_j} = 1$. This leaves $2 \cdot 2^n - 4$ pebbles at x. Applying Proposition 2 twice we see that 2 pebbles can be moved to v.

The remaining subcases ($q = 4$ and $q = 5$) have similar proofs and will be omitted.

Case 2 $C \equiv C_{2n+1}$. The proof is trivial when $n = 1$ or $n = 2$ and also whenever $q = 2n + 1$. Therefore, we will assume that $n \geq 3$ and $q \leq 2n$. By Proposition 1 we are done whenever the following inequality is satisfied:

$$2\left(2\left\lfloor\frac{2^{n+1}}{3}\right\rfloor + 1\right) - q + 1 \geq 2\left\lfloor\frac{2^{n+1}}{3}\right\rfloor + 1 + 2^n$$

This is easily seen to be true when $n \geq 5$. In fact, the above inequality holds true for both $n = 3$ and $n = 4$ whenever $q \leq 2n - 2$. Therefore there are only four cases to consider ($n = 3$ or $n = 4$ and $q = 2n$ or $q = 2n - 1$). These cases can easily be disposed of by judiciously applying Proposition 2 and will be omitted.

□

Theorem 5 *Let C_m and C_n be cycles. Then $f(C_m \times C_n) \leq f(C_m)f(C_n)$ except (possibly) when both m and n are in S (where $S = \{4, 5, 6, 7, 8, 9\}$) or when $m = 4$ or $m = 6$ and $n = 11$.*

Proof: Combining Chung's result, Theorems 3 and 4, and Lemma 2, we need only consider three cases.

Throughout our proof, we will use the same notation as given in the beginning of Theorem 3—except in our case $G \equiv C_{2k+1}$. The reader might want to review the beginning of Theorem 3 before proceeding.

Case 1. $f(C_4 \times C_{2k+1}) \leq f(C_4)f(C_{2k+1})$ when $k \geq 6$.

Proof:

Recall that $G \equiv C_{2k+1}$ and that our target vertex $v = z_t^y$. From Theorem 2 we know that $f(\mathcal{P}_B) \leq 2f(G)$. If we move as many pebbles as possible from G_x directly to G_{b_1} and also from G_{a_1} to G_y then $\mathcal{P}_B$ will have at least

$$\frac{4f(G) - 2(2k+1) - j_B}{2} + j_B$$

pebbles and this must be less than $2f(G)$ or else we are done.

Hence $j_B \leq 2(2k+1) - 1$. By symmetry, we also have $j_A \leq 2(2k+1) - 1$.

This means that $j_x \geq 4f(G) - (4(2k+1) - 2)$, but $4f(G) - (4(2k+1) - 2) \geq f(G) + 3 \cdot 2^k = f_4(C_{2k+1})$ whenever $k \geq 6$. This means that four pebbles can be moved to z_t^x, hence one pebble can be moved to $z_t^y = v$ and we are done. $\Box$

Case 2. $f(C_6 \times C_{2k+1}) \leq f(C_6)f(C_{2k+1})$ when $k \geq 6$.

Proof: Recall that $G \equiv C_{2k+1}$ and that our target vertex $v = z_t^y$. Let $j^* = j_A + j_B - j_y$ and note that $\frac{j^* - 4(2k+1)}{4}$ pebbles can be moved from the vertices of G_{a_1}, G_{a_2}, G_{b_1}, and G_{b_2} to G_y.

Thus, if $\frac{j^*-4(2k+1)}{4} \geq f(G)$ we are done.

Whence, we may assume that $j^* \leq 4f(G)+4(2k+1)-1$. This tells us that $C_6 \times C_{2k+1} - G_x$ contains at most $4f(G)+4(2k+1)-1$ pebbles. Utilizing the pebbles of G_x we see that $\frac{8f(G)-(2k+1)-j^*}{2} + j^* \leq 4f(G) + 4(2k+1) - 1$ or else we are done. This implies that $j^* \leq 8(2k+1) + (2k+1) - 2 = 9(2k+1) - 2$. Hence $j_x \geq 8f(G) - 9(2k+1) + 2$ but $8f(G) - 9(2k+1) + 2 \geq f(G) + 7 \cdot 2^k = f_8(C_{2k+1})$ when $k \geq 6$. This means that eight pebbles can be moved to z_t^x,hence one pebble can be moved to $z_t^y = v$ and we are done. □

Case 3. $f(C_8 \times C_{2k+1}) \le f(C_8)f(C_{2k+1})$ when $k \ge 5$.

Proof: The proof is very similar to the previous case, and will be omitted. □

□

In view of the fact that $f(C_5 \times C_5) = 25$ [3], and our previous results, we see that $f(C_m \times C_n) \le f(C_m)f(C_n)$ for all but thirty seven possible exceptions.

4 Lemke Graphs

In view of our previous results, it would appear that a better understanding of graphs without the 2-pebbling property might aid us in proving (or disproving) Graham's Conjecture.

Definition 5 *A graph G without the 2-pebbling property is called a* Lemke Graph*.*

This is in honor of Paul Lemke, who was the first to show the existence of such a graph (see Figure 1). With a little effort, one can show that $f(L) = 8$. But if 9 pebbles are placed at vertex x, and 1 pebble each at nodes $a1, a2, a3$ and $b1$, then it is not difficult to show that it is impossible to move 2 pebbles to v. Hence L does not have the 2-pebbling property (since $2f(L)-q+1 = 12$ in this case).

There is no known infinite family $\mathcal{L}$ of Lemke graphs. In Figure 2 we give a possible candidate, where $\mathcal{L} = \{L_1, L_2, \ldots\}$.

Conjecture 2 (Snevily) *L_k is a Lemke graph for each k.*

We will show that $f(L_k) = 2^{k+3}$ implies that L_k is a Lemke graph. Place $2^{k+4} - 7$ pebbles at x and 1 pebble each at $d1, e1, f1$ and $g1$. Then it is

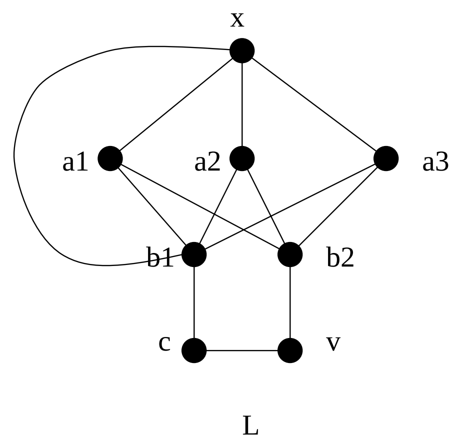

Figure 1: A Lemke Graph

not difficult to show that it is impossible to move 2 pebbles to v. Conjecture 2 appears to be quite difficult—in fact, we do not even know $f(H_k)$ (see Figure 3)—note that H_k is an induced subgraph of L_{k-1}.

Since both trees and cycles have the 2-pebbling property, the following conjecture seems reasonable:

Conjecture 3 (Snevily) *Every bipartite graph has the 2-pebbling property.*

5 Conclusion

At the present time, Graham's conjecture appears to be too difficult. We feel that a solution to the following problem would be noteworthy, since it would imply that $f(C_5 \times C_5) = 25$ (a difficult problem [3]):

Problem 1 *Let $C \equiv C_{2n+1}$ and let G be a graph with the 2-pebbling property. Show that $f(G \times C) \leq f(G)f(C)$.*

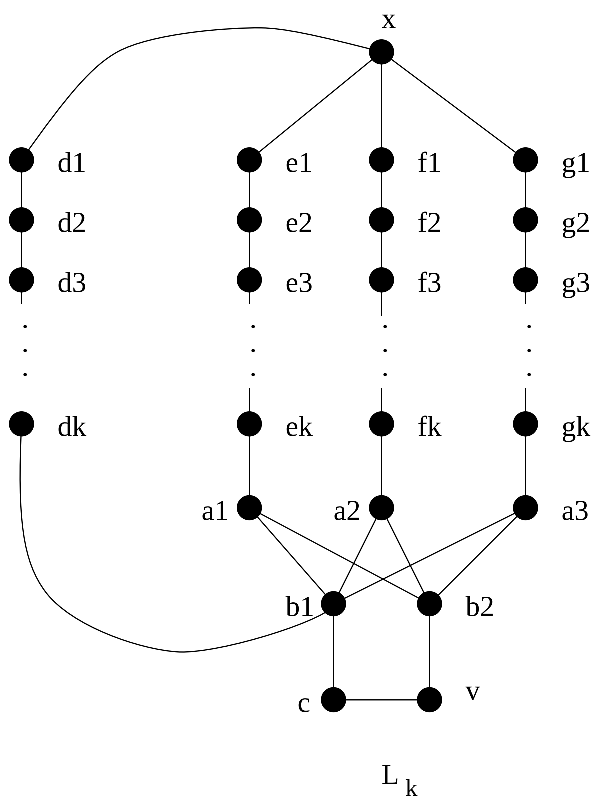

Figure 2: Possible Lemke Graph

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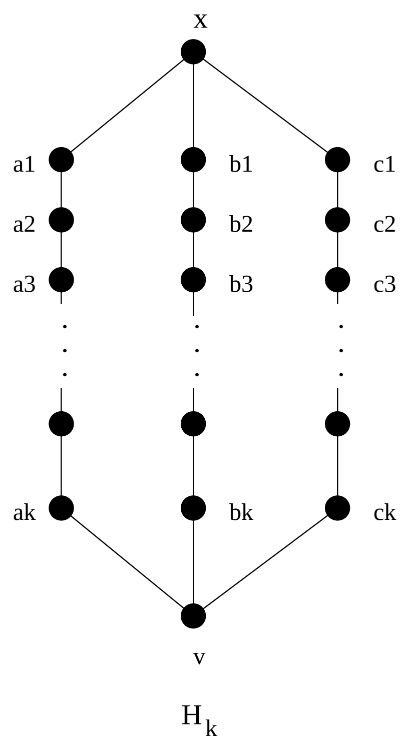

Figure 3: A Subgraph of L_{k-1}

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