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Research problems on Gray codes and universal cycles

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ABSTRACT

Open problems arising from the Workshop on Generalizations of de Bruijn cycles and Gray Codes at the Banff International Research Station in December, 2004.

The Research Problems section presents unsolved problems in discrete mathematics. In special issues from conferences, most problems come from the meeting and are collected by the guest editors. In regular issues, the Research Problems collect problems submitted individually.

Older problems are acceptable if they are not widely known and the exposition features a new partial result. Concise definitions and commentary (such as motivation or known partial results) should be provided to make the problems accessible and interesting to a broad cross-section of the readership. Problems are solicited from all readers; they should be presented in the style below, occupy at most one journal page, and be sent to

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The problems in this special issue of *Discrete Mathematics* arise from a Workshop on "Generalizations of de Bruijn cycles and Gray Codes" at the Banff International Research Station in December, 2004. The problems were posed by participants of this workshop and others doing related research.

Combinatorial mathematicians have long been interested in efficient ways of listing the elements in a set of combinatorial objects. One classic example is the binary reflected Gray code, which lists all of the binary n-tuples so that successive n-tuples differ in exactly one position. More generally, the term "Gray code" has come to mean a listing of a family of combinatorial objects in an order where consecutive objects differ by one of a small set of specified small change operations.

Another classic example is the de Bruijn cycle of order n. A *de Bruijn cycle* of order n is a cyclic arrangement of 2^n binary digits so that every binary n-tuple appears exactly once as a block of n consecutive digits. Chung, Diaconis, and Graham generalized the notion of de Bruijn cycles to universal cycles for other families of combinatorial objects. We present a variety of such problems here. The problems were collected and edited by

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Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Other comments and information about partial or full solutions may be sent to Professor West (for potential later updates).

PROBLEM 476. Universal Cycles of Subsets

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Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the set of all k-subsets of $[n]$. A cycle $(a_1, a_2, \ldots, a_{\binom{n}{k}})$ of elements of $[n]$ is a *universal cycle* for the k-subsets if every $T \in \binom{[n]}{k}$ appears exactly once as k consecutive elements. Chung, Diaconis, and Graham [1] offered a $100 reward for a resolution of the following conjecture.

Conjecture (CDG Conjecture): For $k \geq 1$, there is an integer $n_0(k)$ such that if $\binom{n-1}{k-1} \equiv 0 \pmod{k}$ and $n > n_0(k)$, then there is a universal cycle for the k-subsets of $[n]$.

Comment: Jackson, Hurlbert, Chung, Diaconis, and Graham proved the conjecture for $k \leq 5$ and for $k = 6$ when $(n, k) = 1$ (see [2] and [4]). Jackson [5] constructed universal cycles for 4-subsets when $n \equiv 2 \pmod{8}$ and $n \geq 10$, with the aid of a computer, completing the proof of the conjecture for $k = 4$. In the same paper, universal cycles of 5-subsets for $(n, 5) = 1$ and $n \geq 8$ were constructed, also with the aid of a computer, thus completing the proof for $k = 5$ as well. Stevens et al. [6] showed that universal cycles of the $(n-2)$-subsets of an n-set do not exist when $n > 3$.

Universal cycles of the k-element multisets from $[n]$ have also been considered. They are obtained for $k = 3$ and $(n, 3) = 1$, for $k = 4$ and $(n, 4) = 1$, and for $k = 6$ and $(n, 6) = 1$ in [3] in this issue. Universal cycles of multisets for $k = 3$ and $k = 4$ were also considered earlier in [4].

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PROBLEM 477. Questions and Refinements for Classical Universal Cycles

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A standard *de Bruijn cycle* of order n is a cyclic arrangement of 2^n binary digits such that the subwords of length n are distinct; these exist for all n. The standard proof shows that a de Bruijn cycle of order n corresponds to an Eulerian circuit in the corresponding de Bruijn graph of order n.

A *universal cycle* for a set S of k-tuples from an alphabet is a cyclic arrangement of $|S|$ characters such that the k-element subwords are the elements of S. An easy generalization of the de Bruijn cycle technique can be used to establish the existence of a universal cycle of the k-permutations of an n-set whenever $k < n$ (see [4], for example).

From an algorithmic point of view, this technique is unsatisfying, because constructing the universal cycle by this method requires constructing the graph, and the space needed is typically exponential. The general open problem is to develop space- and time-efficient algorithms for generating universal cycles. For example, the de Bruijn cycle of length 2^n can be generated by an algorithm that uses $O(n)$ space and $O(2^n)$ time by using the Lyndon word generation algorithm of Fredricksen and Kessler [3] that was analyzed in Ruskey, Savage and Wang [6] (see also Berstel and Pocchiola [1]). Ruskey and Williams [7] gave an algorithm for constructing a universal cycle for the $(n-1)$-permutations of an n-set that uses $O(n)$ space and takes constant time between the output of successive characters in the universal cycle.

Also useful is a compact way to find the position of a given object in the cycle. A *ranking function* for S is a bijection from S to the first $|S|$ positive integers; the corresponding *unranking function* is its inverse. The *Cutting-Down Problem* asks for which m with $1 \le m \le |S|$ there exists universal cycles for an m-subset of S. The *Enumeration Problem* asks for the number of distinct universal cycles for S. The *Generation Problem* asks for an efficient algorithm to list all the universal cycles for S.

We ask these questions for two types of sets S. Let $P_{n,k}$ denote the set of k-permutations of the n-set $[n]$. As noted before, Jackson [4] showed that universal cycles exist for $P_{n,k}$.

Let P'_n be the set of permutations with ties on n. These can be written as a permutation of $[n]$ with consecutive symbols separated by $=$ or $<$. Alternatively, we can list for each element i the number of $<$-signs preceding it. For example, $3 = 1 < 2$ and 010 represent the same object, and (0001201101021) is a universal cycle for P'_n when $n = 3$. Diaconis and Graham [2] showed that universal cycles exist for P'_n.

Problem 1: (Explicit Algorithim) Find an efficient algorithm for constructing an explicit universal cycle of $P_{n,k}$ for $2 < k < n - 1$ (or prove that none exists).

Problem 2: (Explicit Algorithim) Find an efficient algorithm for constructing an explicit universal cycle of P'_n.

Question 3: (Enumeration Problem) How many different universal cycles of $P_{n,k}$ and/or P'_n exist?

Problem 4: (Generation Problem) Describe an efficient algorithm for listing all the universal cycles of $P_{n,k}$ or P'_n.

Question 5: (Ranking and Unranking Problem) Are there simple ranking and unranking algorithms that are consistent with explicit universal cycles for $P_{n,k}$ and/or P'_n (For $k = n - 1$ this is solved in [7])?

Question 6: (Cutting-Down Problem) For which integers m do there exist universal cycles for m-subsets of $P_{n,k}$ and/or P'_n?

Comment: The Cutting-Down Problem is solvable for classical de Bruijn cycles. That is, for each integer m with $n \le m \le 2^n$, there is a cyclic binary arrangement of length m in which no n-tuple appears more than once.

As a variant on the Enumeration Problem with $k = n$, Knuth [5] describes a universal cycle of permutations as a cycle of $n!$ digits from $[n]$ such that each permutation occurs exactly once as a block of $n - 1$ consecutive digits, with its redundant final digit suppressed. When $n = 3$, (121323) is such a cycle. He denotes the number of universal cycles of the permutations of $[n]$ by U_n and states that $U_2 = 1$, $U_3 = 3$, $U_4 = 2^7$, $U_5 = 2^{33} \cdot 3^8 \cdot 5^3$, $U_6 = 2^{190} \cdot 3^{49} \cdot 5^{33}$, and $U_7 = 2^{1217} \cdot 3^{123} \cdot 5^{119} \cdot 7^5 \cdot 11^{28} \cdot 43^{35} \cdot 73^{20} \cdot 79^{21} \cdot 109^{35}$.

The problems for P'_n are sensible, because universal cycles exist for P'_n for all n [2]. This holds by a generalization of the standard construction for de Bruijn cycles and universal cycles of permutations. One constructs a transition digraph whose edges correspond to permutations with ties. One can show that this digraph is strongly connected and any circular rotation of a permutation with ties is still a permutation with ties, thus the transition digraph is eulerian. Any eulerian circuit of this digraph corresponds to a universal cycle of the permutations with ties.

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PROBLEM 478. Universal Cycles for Partitions of an n-Set

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Unordered partitions of the n-set $[n]$ can be represented by words of length n. For two symbols i and j, we put i and j in the same block of the partition if the ith and jth entries in the word are equal; otherwise they are in different blocks. A universal cycle for such partitions must use at least n symbols due to the partition using blocks of size 1. When $n = 4$, (122221123424112) is a universal cycle using only 4 symbols. Note that 1222 represents the partition $[\{1\}, \{2, 3, 4\}]$, but that also 3111 would represent the same partition. It is known that universal cycles of the partitions of $[n]$ using only n symbols always exist for all $n \ge 4$ [1].

Question: Although the existence problem is settled, the ranking and unranking problem, cutting down problem, enumeration problem, and generation problem for universal cycles of set partitions all remain open. (See previous problem for definitions.)

Comment: They are likely to be quite difficult since partitions can be represented in multiple ways.

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PROBLEM 479. Universal Cycles for Subspaces

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Fix a finite field $\mathbb{F}$ and let $\mathbb{F}^n$ be an n-dimensional vector space over $\mathbb{F}$. Let $G(k, n)$ denote the set of k-subspaces (that is, k-dimensional subspaces) of $\mathbb{F}^n$. A cycle $(a_1, \ldots, a_{|G(k,n)|})$ of elements of $\mathbb{F}^n$ is a *universal cycle* for $G(k, n)$ if every element of $G(k, n)$ has a basis consisting of k consecutive elements of the cycle. Many problems involving universal cycles were posed by Chung, Diaconis, and Graham [1].

Question 1: Does there exist a universal cycle for $G(k, n)$ whenever $n \geq k + 1$?

Comment: The construction of a universal cycle when $k = 1$ and $n \geq 2$ is trivial. In this issue, Buhler, Jackson, and Mayer [2] give a recursive construction of universal cycles whenever $k = 2$ and $n \geq 3$.

The construction starts with a simultaneous universal cycle of the 1-subspaces and the 2-subspaces of $\mathbb{F}^3$. More generally, $|G(k, n)| = |G(n - k, n)|$ and it is trivial to construct a simultaneous universal cycle of the subspaces of dimensions 1 and $n - 1$ in $\mathbb{F}^n$. These facts suggests the following question.

Question 2: Does there exist a simultaneous universal cycle for $G(k, n)$ and $G(n - k, n)$ whenever $n \geq k + 1$?

Comment: A possible step toward answering Question 2 could be the following, which is easy to do when $k = 1$.

Problem 3: For $1 \leq k < n/2$, find an explicit map $\alpha : G(k, n) \to G(n - k, n)$ such that $S \subset \alpha(S)$ whenever $S \in G(k, n)$ (the existence of such a map can be proved using Hall's Theorem).

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PROBLEM 480. De Bruijn Tori

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We consider a 2-dimensional generalization of de Bruijn cycles. An R-by-S k-ary array A embedded on a torus is a *de Bruijn torus* with window size (m, n) if every m-by-n array occurs once. For short, we say that A is a k-ary (R, S, m, n)-de Bruijn torus.

Question 1: Is it possible to construct a k-ary (R, S, m, n)-de Bruijn torus whenever R, S, m, n, k are positive integers with $k > 1$, $RS = k^{mn}$, $R > m$, and $S > n$?

Comment: This question has been answered affirmatively for "square" tori (the case $R = S$ and $m = n$). The binary case ($k = 2$) was solved by Fan, Fan, Ma, and Siu [1] and the general case by Hurlbert and Isaak [2]. The problem was also solved for $m = n = 2$ by Hurlbert, Mitchell, and Paterson [3].

As with universal cycles, one can also consider de Bruijn tori for many other combinatorial structures.

Question 2: For which positive integers R, S, m, n, k such that $k > mn$, $R \geq m$, $S \geq n$ and $RS = \prod_{i=0}^{k-1}(mn - i)$ do there exist de Bruijn tori of the mn-permutations of a k-set?

Question 3: For which positive integers R, S, m, n, k such that $k > mn, R \geq m, S \geq n$ and $RS = \binom{k}{mn}$ do there exist de Bruijn tori of the mn-subsets of a k-set?

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PROBLEM 481. Generalizations of Alphabet Overlap Graphs

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For positive integers k, d, s such that $k > s$, the *alphabet overlap graph* $G(k, d, s)$ is the graph whose vertex set is the set of all k-letter words from an alphabet of size d, with two words v and w adjacent if and only if if the word formed by the last $k - s$ letters of one is the same as that formed by the first $k - s$ letters of the other.

Godbole, Knisley, and Norwood [1] proved that $G(k, d, s)$ is Hamiltonian for all nontrivial values of the parameters. When $s = 1$, the statement is equivalent to the existence of a de Bruijn cycle on the set of all k-letter words from a d-letter alphabet. Their proof for general s is by induction on the alphabet size. The result can also be proved using generalizations of the standard methods for constructing de Bruijn cycles or an extension of the ingenious greedy algorithmic proof due to Fredricksen and Maiorana [2].

The authors of [1] asked if their technique generalizes to construct universal cycles of other combinatorial objects. Permutations, permutation with ties, and set partitions are other combinatorial objects that can be represented by words. Similar overlap graphs can be constructed for these objects. For example, a k-permutation is a k-letter word consisting of k distinct letters from the alphabet. A permutation overlap graph $P(k, d, s)$ can be defined as above by restricting the vertex set to k-permutations.

Question: Is the permutation overlap graph $P(k, d, s)$ Hamiltonian whenever $k < d$?

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PROBLEM 482. Gray Codes for Weak Orders

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A *weak order* or *ranking* is a relation $\preceq$ that is transitive and complete. We write $x \equiv y$ if $x \preceq y$ and $y \preceq x$, and $x \prec y$ if $x \preceq y$ but $y \not\preceq x$. Let $[n] = \{1, \ldots, n\}$. A weak order on $[n]$ can be written as a permutation of $[n]$ with consecutive symbols separated by $\equiv$ or $\prec$. Alternatively, we can just list the height of each element, obtaining the n-tuple a where a_i is the height of element i. For example, $3 \equiv 1 \prec 2$ and 010 represent the same weak order, and the thirteen weak orders on {1,2,3} are 000, 100, 010, 001, 011, 101, 110, 012, 120, 201, 021, 210, and 102. Diaconis and Graham [1] called these orders "permutations of n symbols with ties". Permutations with ties can also be thought of as ordered partitions of $[n]$.

Knuth [2] showed that for every n, one can list the weak orders on $[n]$ so that two consecutive objects differ by one of the two elementary operations $a_i \leftrightarrow a_j$ or $a_i \leftarrow a_j$. The first operation exchanges two values; the second copies the second value onto the position of the first (The values in the two positions can be any two).

Question 1: (Knuth) For which positive integers n does there exist a Gray code for the weak orders on $[n]$ using elementary operations of the form $a_i \leftrightarrow a_{i+1}$ and $a_i \leftarrow a_{i+1}$?

Question 2: For which positive integers n does there exist a Gray code for the weak orders on $[n]$ where two consecutive objects always differ in exactly one position?

Comment: For example, when $n = 3$ a suitable listing for Problem 1 is 201, 210, 120, 102, 012, 021, 011, 101, 001, 010, 110, 100, 000, and a suitable listing for Problem 2 is 000, 001, 201, 101, 102, 100, 120, 110, 210, 010, 012, 011, 021.

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PROBLEM 483. Felsner–Trotter Gray Codes

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Let $[n] = \{1, \dots, n\}$. Subsets of $[n]$ correspond to binary n-tuples, with $(x_1, \dots, x_n)$ corresponding to $\{i: x_i = 1\}$. A *Gray code* on subsets of $[n]$ lists the n-tuples in order so that consecutive n-tuples differ in exactly one position.

Question: (Felsner–Trotter Gray Code): Does there always exist an n-bit binary Gray code that starts with the empty set and satisfies the following extra condition: For $x, y \in B_n$ with $x \subset y$, either x comes somewhere before y or x immediately follows y in the code?

Comment: For example, when $n = 4$ a suitable listing is 0000, 0001, 0011, 0010, 0110, 0100, 1100, 1000, 1010, 1011, 1001, 1101, 0101, 0111, 1111, 1110.

This question was originally posed by Felsner and Trotter [2]. Their paper also discusses the connection of these Gray codes with colorings of the diagrams of interval orders. Progress could provide insight into the (in)famous "middle two levels" problem of whether or not there is a Hamilton cycle for the middle two levels of an odd-order Boolean lattice. Recently, Biro and Howard [1] proved for all n that the subsets of size at most 3 can be so ordered.

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PROBLEM 484. Archimedes' Stomachion Puzzle

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The Stomachion is a puzzle consisting of 14 pieces that tile a square. The Stomachion is perhaps the world's oldest puzzle and is thought to have been invented by Archimedes, 2200 years ago. Chung, Diaconis, Graham, and Holmes found 268 different solutions to Archimedes' Stomachion puzzle (assuming two solutions that differ by a reflection or rotation are considered to be the same). Independently, Bill Cutler verified by computer that these are the only solutions to this puzzle. For more information about the Archimedes' Stomachion puzzle see [2]. Three pairs of regions must always appear together in the puzzle, leading to the reduced set of 11 pieces in Fig. 1, called STOMACH at [1].

Problem 1: Find a minimal change ordering of all the solutions to Archimedes' Stomachion puzzle.

Comment: The definition of "minimal" is deliberately not given as finding suitable candidate definitions is part of the challenge of the problem. One candidate move is a flip or a rotation of a symmetric sub-region; In [1] this is called a *simple move* and a cycle by these simple moves through 266 of the 268 solutions is given. The remaining two configurations cannot be reached from these by simple moves.

Problem 2: A related question is to find how many ways a large integer-sided square of side length n can be tiled by one of the Stomachion pieces, for example, the right triangle with sides 1 and 2. When n is large (or even 10) there are some rather non-obvious tilings, e.g., such that the vertices don't lie on integer points of the lattice.

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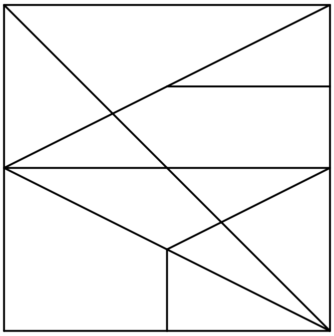

Fig. 1. STOMACH

PROBLEM 485. Linear Extensions of Posets

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A *partially ordered set* (or *poset*) is a set P together with a binary relation $\leq$ on P that is reflexive, transitive, and antisymmetric. A *linear extension* of the poset is an order-preserving permutation of the elements. Given a poset P, let $G(P)$ be the graph whose vertices are the linear extensions of P and whose edges join those extensions that differ by transposing two adjacent elements. For example, if P is a 3-element antichain (no two elements are related), then $G(P)$ is a 6-cycle.

Let $G'(P)$ denote the supergraph of $G(P)$ where edges join extensions differing by any (possibly non-adjacent) transposition. In the same example as above, $G'(P)$ is the complete bipartite graph $K_{3,3}$. The graph $G'(P)$ (and hence $G(P)$) is bipartite.

Conjecture: If the two partite sets of $G(P)$ have the same size, then $G(P)$ is Hamiltonian. A weaker form of the conjecture replaces G by G'.

Comment: This conjecture was posed originally in [2]. The conjecture, or its weaker form, is known to be true in several cases.
The strong form: when P consists of disjoint chains [4].
The strong form: when P is the product of a 2-element and n-element chain [3].
The weak form: when every non-maximal element of P is covered by at least two other elements (this includes the boolean lattice [1]).

The conjecture may be quite difficult, since the problem of determining whether a poset satisfies the hypotheses is #P-complete.

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PROBLEM 486. Directed Cayley Graphs

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In Gray code problems, we want to list a set of combinatorial objects so that consecutive objects differ in a restricted way. The existence of a Gray code is equivalent to finding a Hamiltonian path (or cycle) in an associated graph where the vertices are the objects and vertices are adjacent if they differ in the required way.

The associated graph often is quite symmetric: it may be vertex-transitive, or it may be a Cayley graph. A famous question of Lovász asks whether all (undirected) Cayley graphs are Hamiltonian (see [1]). Schemes for generating combinatorial Gray codes in many cases provide new examples of Cayley graphs with Hamiltonian paths or cycles, from which we might hope to gain insight into Lovász's question.

Non-Hamiltonian examples are known in the case of directed Cayley graphs. Let G_n be the Cayley graph generated by the transposition $(1, 2)$ and the cycle $(1, 2, \ldots, n)$. It can be proved that G_n is not Hamiltonian when n is even using a theorem of Rankin [2] (see also Swan [4]). Nevertheless, Ruskey, Jiang, and Weston [3] showed that G_5 is Hamiltonian and that G_6 has a Hamilton path. Later, McKay and Ruskey showed that G_n is Hamiltonian for $n \in \{7, 9, 11\}$.

Conjecture 1: The graph G_n is Hamiltonian whenever n is odd.

Problem 2: Does the graph G_n have a Hamilton path whenever n is even?

References

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PROBLEM 487. Hamilton Cycles through Matchings

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Recently, Fink [1] proved the conjecture by Kreweras [2] that every perfect matching in the n-dimensional hypercube Q_n extends to (is contained in) a spanning cycle of the hypercube. Earlier, Ruskey and Savage [3] asked a stronger question:

Question: Does every matching (perfect or not) in the hypercube Q_n extend to a spanning cycle?

Comment: The question is also posted at
http://garden.irmacs.sfu.ca/?q=op/matchings_extends_to_hamilton_cycles_in_hypercubes
Note that matchings in Q_n do not necessarily extend to perfect matchings.

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