On the pebbling numbers of some snarks

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Abstract

Graph pebbling is a game played on graphs with pebbles on their vertices. A pebbling move removes two pebbles from one vertex and places one pebble on an adjacent vertex. The pebbling number $\pi(G)$ is the smallest t so that from any initial configuration of t pebbles it is possible, after a sequence of pebbling moves, to place a pebble on any given target vertex. In this paper, we provide the first results on the pebbling numbers of snarks.

Keywords: Graph Pebbling, Snarks.

2020 Mathematics Subject Classification: 05C35, 05C57.

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1 Introduction

Graph pebbling is a mathematical game or puzzle that involves moving pebbles around a connected graph, subject to certain rules. The objective of the game is to place a certain number of pebbles on specific vertices of the graph, typically with the aim of reaching a particular configuration of pebbles or minimizing the number of moves required to achieve a given configuration. Various forms of graph pebbling have applications in number theory, computer science, physics, and combinatorial optimization, and have been studied extensively in mathematics (see [9]).

Throughout this paper, let $G = (V, E)$ be a simple connected graph. The number of vertices and the diameter of G are denoted by $n(G)$ and $D(G)$, respectively. For a vertex w and positive integer k, denote by $N_k[w]$ the set of all vertices that are distance at most k from w.

1.1 Pebbling number

A *configuration* C on a graph G is a function $C : V(G) \to \mathbb{N}$. The value $C(v)$ signifies the number of pebbles at vertex v. The *size* $|C|$ of a configuration C is the total number of pebbles on G. A *pebbling move* consists of removing two pebbles from a vertex and placing one pebble on an adjacent vertex. For a *target* vertex r, C is r-*solvable* if one can place a pebble on r after a sequence of pebbling moves, and is r-*unsolvable* otherwise. Also, C is *solvable* if it is r-solvable for all r. The *pebbling number* $\pi(G, r)$ is the minimum number

t such that every configuration of size t is r-solvable. The *pebbling number of G* equals $\pi(G) = \max_r \pi(G, r)$. A vertex with zero, one, or at least two pebbles on it is called *empty*, *a singleton*, or *big*, respectively.

The basic lower and upper bounds for every graph are $\max\{n(G), 2^{D(G)}\} \leq \pi(G) \leq (n(G) - D(G))(2^{D(G)} - 1) + 1$ [4, 7]. A graph is called *Class 0* if $\pi(G) = n(G)$. It is not yet known whether or not there exist necessary and sufficient conditions for a graph to be Class 0.

1.2 Snarks

We define the important family of *snark* graphs which are cubic, bridgeless, 4-edge-chromatic graphs. (In particular, we allow for arbitrary girth.) They are important for being related to the Four Color Theorem, which holds if and only if no snark is planar [14]. In [1] we find the origins of the study of the pebbling numbers of chordal graphs. Here we begin the systematic study of the pebbling numbers of snarks.

The Petersen graph is the smallest snark, having 10 vertices, and was discovered in 1898 [11]. Since then, many others have been discovered (see [12] for a thorough history and Table 1 for the complete list).

For odd $m = 2k + 1$, we define the m^{th} *flower snark* J_m as follows (see Figure 1) [2]. For each $i \in \pm\{0, 1, \ldots, k\}$ we have vertices v_i, x_i, and y_i all adjacent to z_i. Thus the number of vertices of the m^{th} flower snark is $n(J_m) = 4m$. The vertices $\{u_i\}$ form the natural cycle, with adjacencies given by consecutive indices modulo m. The vertices $\{x_k\}$ (resp. $\{y_i\}$) form

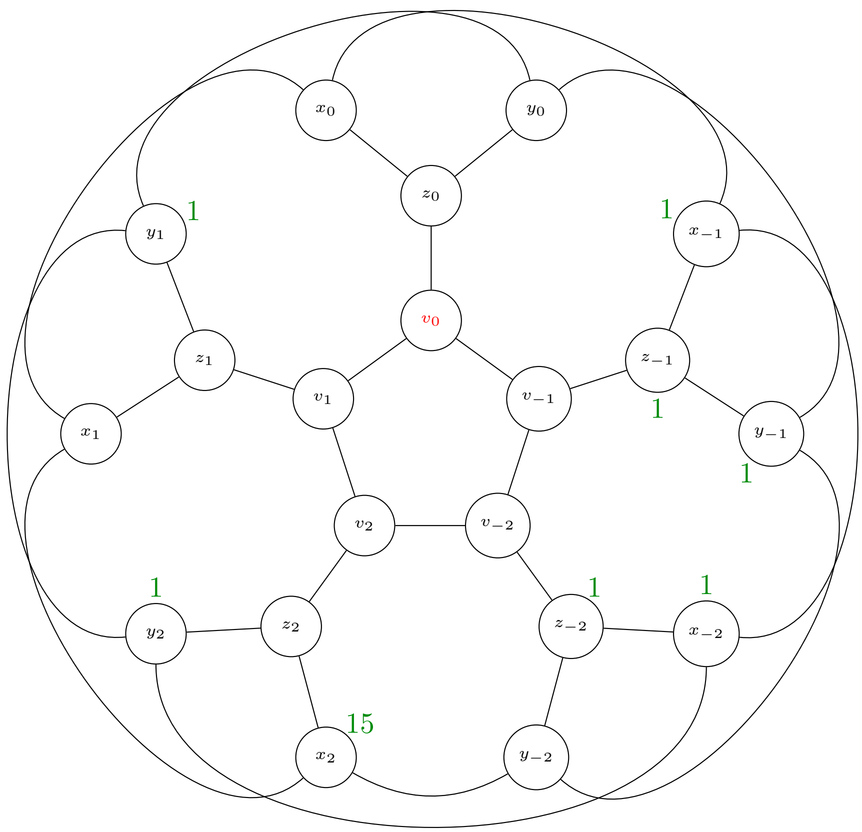

Figure 1: The graph J_5 and its (green) v_0-unsolvable configuration C of size 22, which equals the configuration C^* with an extra pebble on z_{-1}.

a path given by the natural cycle without the edge $x_k x_{-k}$ (resp. $y_k y_{-k}$). Finally, we add the edges $x_k y_{-k}$ and $y_k x_{-k}$. It is easy to see that J_m has a rotational symmetry, with a necessary twist; that is, the automorphisms of J_m yield three vertex orbits. Thus the only targets necessary to contemplate are, without loss of generality, v_0, z_0, and x_0.

1.3 Results

It is known that the Petersen graph is Class 0 [7]. It is the smallest snark and was the only one whose pebbling number is known (we now also know $\pi(J_3)$.) We use the Small Neighborhood Lemma below to prove that the Petersen graph is the only Class 0 snark with at least 23 vertices or girth at least 5.

Theorem 1. *If a snark G is* Class 0 *then either it is the Petersen graph, or $n(G) \leq 22$ and $\mathsf{girth}(G) \leq 4$.*

We also prove the following bounds on the pebbling numbers of snarks.

Theorem 2. *We have $\pi(J_3) = 13$, $23 \leq \pi(J_5) \leq 45$, $40 \leq \pi(J_7) \leq 84$, and for all $k \geq 4$ with $m = 2k+1$, we have $2^{k+2} + 8 \leq \pi(J_m) \leq (93 \cdot 2^{k-1} + 2)/5 + 2k - 3$.*

This corrects a claim of [10] that $\pi(J_m) = 4m + 1$.

2 Techniques

The following lemma (SNL) is used to provide a lower bound on $\pi(G)$.

Lemma 3 (Small Neighborhood Lemma [5]). *Let G be a graph and $u, v \in V(G)$. If $N_a[u] \cap N_b[v] = \emptyset$ and $|N_a[u] \cup N_b[v]| < 2^{a+b+1}$, then G is not Class 0.*

Given a graph G that satisfies the hypothesis of Lemma 3, define the configuration $C^* = C^*_{u,v}$ by $C^*(v) = 2^{a+b+1} - 1$, $C^*(x) = 0$ for all $x \in (N_x[u] \cup N_b[v]) - \{v\}$, and $C^*(z) = 1$ otherwise. The authors of [5] use the first hypothesis to prove that C^* is u-unsolvable and the second hypothesis to show that $|C^*| \geq n(G)$. In fact, we will use this configuration to get even larger lower bounds below.

One can see how SNL is, in some sense, a sharpening of the basic exponential lower bound. One consequence we will use here is the following corollary.

Corollary 4 ([5]). *If G is an n-vertex Class 0 graph with diameter at least 3, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.*

Let T be a subtree of a graph G rooted at vertex r, with at least two vertices. For a vertex $v \in V(T)$ let v^+ denote the *parent* of v; i.e. the T-neighbor of v that is one step closer to r (we also say that v is a *child* of v^+). We call T an *r-strategy* when we associate with it a non-negative *weight function* w with the property that $w(r) = 0$ and $w(v^+) \geq 2w(v)$ for every other vertex that is not a neighbor of r (and $w(v) = 0$ for vertices not in T). Let $\boldsymbol{T}$ be the configuration with $\boldsymbol{T}(r) = 0, \boldsymbol{T}(v) = 1$ for all $v \in V(T)$, and $\boldsymbol{T}(v) = 0$ everywhere else. We now define the *weight* of any configuration C (including $\boldsymbol{T}$) by $w(C) = \sum_{v \in V} w(v)C(v)$. The following lemma (WFL) is used to provide an upper bound on $\pi(G)$.

Lemma 5 (Weight Function Lemma [8]). *Let T be an r-strategy of G with associated weight function w. Suppose that C is an r-unsolvable configuration of pebbles on $V(G)$. Then $w(C) \leq w(\boldsymbol{T})$.*

3 Proof of Theorem 1

Note that every snark has $3n/2$ edges and diameter at least 3. Then, by Corollary 4, if $n(G) > 22$ we get $3n/2 < (5n - 11)/3$. Therefore, every snark with $n(G) > 22$ is not Class 0. The remaining non-Petersen snarks with fewer vertices and girth at least 5 (the flower J_5, the Blanušas, and the Loupekines) all have diameter 4, so for any vertices u and v at distance 4 from each other we have $|N_2[u]| = 10$ and $|N_1[v]| = 4$. Thus $|N_2[u] \cup N_1[v]| < 2^{2+1+1} = 14 < 16$, and so none of these graphs are Class 0 by SNL. $\square$

4 Proof of Theorem 2

First we prove the lower bounds. For these we need only display a configuration, of size one less than the lower bound, that cannot reach some target.

For J_3, define the size 12 configuration C by $C(x_1) = 7$, $C(u) = 1$ for $u \in \{y_0, y_1, z_{-1}, x_{-1}, y_{-1}\}$, and $C(u) = 0$ otherwise. We claim that C cannot reach v_0. Indeed, 7 pebbles cannot reach v_0 at distance 3 without the assistance of other pebbles. If we move a pebble from x_1 to its only nonempty neighbor y_{-1} it can continue through other nonempty vertices until it reaches z_0, z_1, or v_{-1}, none of which can help the remaining 5 pebbles on x_1 to reach v_0. Otherwise, we do not move to y_{-1}, and the analysis is simpler.

For J_5, the v_0-unsolvable configuration $C^*_{v_0,x_k}$ that is provided by SNL has size 21. Notice that we can add a pebble to z_{-1} to obtain the configuration C in Figure 1. It is not difficult to argue that C is also v_0-unsolvable, since any supposed solution would need to use the pebble at z_{-1}.

For $m \geq 7$ (i.e. $k \geq 3$), we will use $C^* = C^*_{v_0,x_k}$ only. One can verify that, for any vertex u of J_m and integer $2 \leq i \leq k$, we have $|N_i[u]| = 2(4i-5)+4 = 8i-6$. From this we can compute $|C^*| = (2^{a+b+1}-1)+(n-|N_a[v_0]|-|N_b[x_k]|)$. In the case that k is even we use $a = k/2$ and $b = a+1$, while if k is odd we use $a = b = (k+1)/2$. In either case we obtain $|C^*| = 2^{k+2} + 8$.

Now we prove the upper bounds, using WFL. For J_3, we define three v_0-strategies $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_{-1}$ by

- $\boldsymbol{T}_0(z_0, x_0, y_0, x_1, y_1, x_{-1}, y_{-1}, z_1, z_{-1}) = (8, 4, 4, 2, 2, 2, 2, 1, 1)$,
- $\boldsymbol{T}_1(v_1, z_1, x_1, y_1, x_0, x_{-1}, y_{-1}) = (8, 4, 2, 2, 1, 1, 1)$ and
- $\boldsymbol{T}_{-1}(v_{-1}, z_{-1}, x_{-1}, y_{-1}, y_0, x_1, y_1) = (8, 4, 2, 2, 1, 1, 1)$,

giving rise to the inequality $|C| \leq \frac{1}{5}(\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}) \leq 64/5$ whenever C

is v_0-unsolvable. Hence $\pi(J_3, v_0) \le 13$. Similar strategies can be found for targets z_0 and x_0 as well, and so $\pi(J_3) \le 13$.

For $m \in \{5, 7\}$, we define v_0-strategies similarly. For $m \ge 9$ (i.e. $k \ge 4$), we instead define three z_0-strategies (see Figure 2) by

- $\boldsymbol{T}_0(v_0, v_1, v_{-1}, v_2, v_{-2}, \ldots, v_k, v_{-k}, z_k, z_{-k}, x_k, y_k, x_{-k}, y_{-k})$
  $= (2^{k+2}, 2^{k+1}, 2^{k+1}, 2^k, 2^k, \ldots, 2, 2, 1, 1, 1, 1)$ and
  $\boldsymbol{T}_0(z_1, z_{-1}, \ldots, z_{k-2}, z_{2-k}, z_{k-1}, z_{1-k}) = (5, 5, \ldots, 5, 5, 4, 4)$;

- $\boldsymbol{T}_1(x_0, x_1, x_{-1}, \ldots, x_k, x_{-k}, z_k, v_k)$
  $= (2^{k+2}, 2^{k+1}, 2^{k+1}, \ldots, 4, 4, 2, 1)$ and
  $\boldsymbol{T}_1(z_{2-k}, z_{1-k}) = (1, 1)$; and

- $\boldsymbol{T}_{-1}(y_0, y_1, y_{-1}, \ldots, y_k, y_{-k}, z_{-k}, v_{-k})$
  $= (2^{k+2}, 2^{k+1}, 2^{k+1}, \ldots, 4, 4, 2, 1)$ and
  $\boldsymbol{T}_{-1}(z_{k-2}, z_{k-1}) = (1, 1)$.

The sum $\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}$ has 3 vertices with coefficient 2^{k+2}, 6 with 2^i (for each $3 \le i \le k+1$), and $2k+6$ with coefficient 5, giving rise to the inequality

$$\begin{aligned} 5|C| &\le \boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1} \\ &= 6(2^3 + \cdots + 2^{k+2}) - 3(2^{k-1}) + 5(2k+6) \\ &= 48(2^k - 1) - 3(2^{k-1}) + 10k + 30 \\ &= 93(2^{k-1}) + 10k - 18, \end{aligned}$$

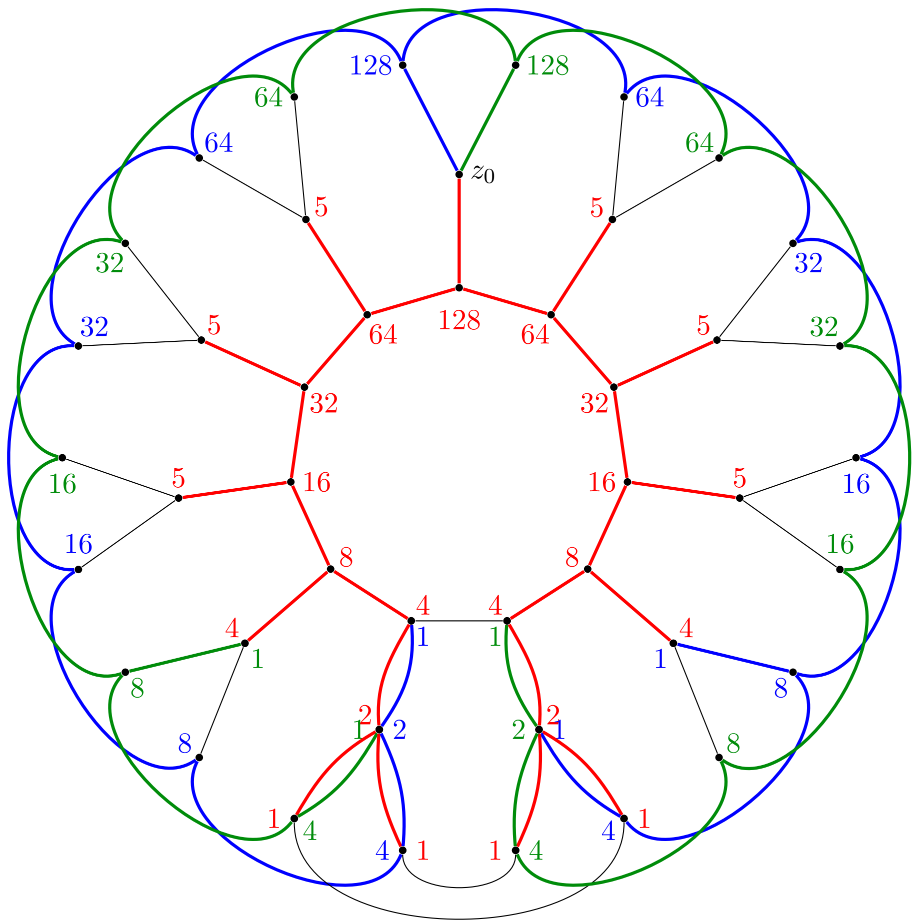

Figure 2: The graph J_{11} and its three z_0-strategies $\boldsymbol{T}_0$ (in red), $\boldsymbol{T}_1$ (in blue), and $\boldsymbol{T}_{-1}$ (in green).

whenever C is v_0-unsolvable. Hence $|C| \leq (93 \cdot 2^{k-1} + 2)/5 + 2k - 4$, and so $\pi(J_m, z_0) \leq (93 \cdot 2^{k-1} + 2)/5 + 2k - 3$. Similar strategies can also be found for targets v_0 and x_0, and so $\pi(J_m) \leq (93 \cdot 2^{k-1} + 2)/5 + 2k - 3$. $\square$

5 Final remarks

Table 1 shows the current knowledge of the pebbling numbers of several well known snarks, using the basic bounds mentioned in the introduction, as well as Theorems 1 and 2. We also note that the Watkins lower bound comes from a more complicated argument that will be included in a follow-up article, correcting a claim of [13] that $\pi(W_{50}) = 166$.

Acknowledgements

This research was partially supported by CAPES Finance Code 001, FAPERJ (grant no. E-26/202.793/2017, E-26/010.002239/201), and CNPq (grant no. 407635/2018-1, 305356/2021-6).

Snark	$n(G)$	$D(G)$	$\pi(G)$
Petersen	10	3	10
Flower J_3	12	3	13
Flower J_5	20	4	$23 \leq \pi(J_5) \leq 45$
Flower J_7	28	5	$40 \leq \pi(J_5) \leq 84$
Flower J_m $(m \geq 9)$	$4m$	$(m-1)/2$	$2^{k+2} + 8 \leq \pi(J_m) \leq (93 \cdot 2^{k-1} + 2)/5 + 2k - 3$
Blanuša (1 and 2)	18	4	$19 \leq \pi(G) \leq 211$
Loupekine (1 and 2)	22	4	$23 \leq \pi(G) \leq 271$
Double-Star	30	4	$31 \leq \pi(G) \leq 391$
Szekeres	50	7	$128 \leq \pi(G) \leq 5462$
Watkins	50	7	$169 \leq \pi(G) \leq 5462$

Table 1: Bounds on the pebbling numbers of several well known snarks.

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