The Target Pebbling Conjecture on 2-paths

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November 20, 2020

Abstract

Graph pebbling is a network optimization model for satisfying vertex demands with vertex supplies (called pebbles), with partial loss of pebbles in transit. The pebbling number of a demand in a graph is the smallest number for which every placement of that many supply pebbles satisfies the demand. The Target Conjecture (Herscovici-Hester-Hurlbert, 2009) posits that the largest pebbling number of a demand of fixed size t occurs when the demand is entirely stacked on one vertex. This truth of this conjecture could be useful for attacking many open problems in graph pebbling, including the famous conjecture of Graham (1989) involving graph products. It has been verified for complete graphs, cycles, cubes, and trees. In this paper we prove the conjecture for 2-paths.

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1 Introduction

Graph pebbling of the type we study here began as a method for proving a number-theoretic conjecture of Erdős and Lemke ([5]), which was further applied to prove a group-theoretic conjecture of Kleitman and Lemke (see [6]), as well as to prove a result in 2-adic analysis (see [14]). It has since grown into a network optimization model for satisfying vertex demands with vertex supplies (called pebbles), with partial loss of pebbles in transit. While there are a number of different pebbling games on graphs, with a wide range of rules and applications (see [7], [8], [11], [12], [13], [15], [16], [17], [19], [20]), this version is defined as follows.

For a finite, connected graph $G = (V, E)$, a *configuration* C is a non-negative integer-valued function on V, with *size* $|C| = \sum_{v \in V} C(v)$. That is, $C(v)$ represents the number of pebbles on the vertex v, while $|C|$ denotes the total number of pebbles on V. Similarly, a *demand* (or *target distribution*) D is a non-negative integer-valued function on V, with *size* $|D| = \sum_{v \in V} D(v)$. That is, $D(v)$ represents the number of pebbles required to eventually place on the vertex v, while $|D|$ denotes the total demand of pebbles on V. For a configuration C *pebbling step* from u to v removes two pebbles from u and places one of those pebbles on v; the resulting configuration C' is defined by $C'(u) = C(u) - 2$, $C'(v) = C(v) + 1$, and $C'(w) = C(w)$ otherwise. We say that C is *D-solvable* (or that *C solves D*) if C can be converted via pebbling steps to a configuration C^* such that $C^*(v) \geq D(v)$ for all $v \in V$; C is *D-unsolvable* otherwise. The *pebbling number of a demand D in a graph G* is denoted $\pi(G, D)$ and defined to be the smallest m such that every configuration of size m is D-solvable. In the case that $|D| = 1 = D(r)$, we simply write $\pi(G, r)$. When $|D| = t = D(r)$ we say that D is *stacked* (on r); in this case we may write that C *t-fold solves* r instead of that C solves D, with $\pi_t(G, r) = \pi(G, D)$. The *t-fold pebbling number of* G is defined as $\pi_t(G) = \max_{r \in V} \pi_t(G, r)$; if $t = 1$ we omit the subscript and avoid writing "1-fold". As with many fractional analogues of graph theoretical invariants (chromatic number, clique number, matching number, etc. — see [18]), the *fractional pebbling number* is defined to be $\hat{\pi}(G) = \liminf_{n \to \infty} \pi_t(G)/t$. It was proved in [9],[10] that $\hat{\pi}(G) = 2^{\mathsf{diam}(G)}$ for every graph G.

The original application of graph pebbling only involved the case $t = 1$. However, in [5] the problem was immediately generalized so that the parameter t could be used in an inductive manner to prove results on trees. Similarly, the problem was further expanded to more general D in [9], wherein is found the following Target Conjecture.

Conjecture 1. (Target Conjecture) *[9] Every graph G satisfies $\pi(G, D) \leq \pi_{|D|}(G)$ for every target distribution D.*

The authors of [9] verified this conjecture for trees, cycles, complete graphs, and cubes. The generalization to D has proved useful in obtaining results inductively on powers of paths (see [4]). The hope is that it may be a powerful tool more generally, for example on chordal graphs, for which it has been conjectured that the pebbling numbers of chordal graphs of a certain type can be calculated in polynomial time (see [1]). Furthermore, one might suspect that the use of general targets could be helpful in attacking the famous conjecture of Graham (see [5]) that $\pi(G\Box H) \le \pi(G)\pi(H)$, where $\Box$ denotes the cartesian product of graphs.

In this paper we verify Conjecture 1 for the family of 2-paths, defined in Section 2.

1.1 Preliminaries

Before beginning, we introduce a few key concepts used in the proofs. For a fixed vertex r denote by $V_i(r)$ the set of all vertices at distance i from r. For a configuration C we define a vertex v to be a *zero* of C if $C(v) = 0$, and denote the number of zeros of C by $z(C)$, noting that $|V| = s(C) + z(C)$. In addition, we define the *support* of C ($\mathsf{supp}(C)$) to be the set of vertices v with $C(v) > 0$, and denote $s(C) = |\mathsf{supp}(C)|$. Furthermore, we define the *potential* of C ($\mathsf{pot}(C)$) to be the number of pairwise disjoint pairs of pebbles with each pair sitting on the same vertex; this is the total number of initial pebbling steps that can be made from C, and equals $\sum_{v\in V} \lfloor C(v)/2\rfloor$. With more refinement, for a vertex r we define $\mathsf{pot}_i(C, r)$ to be the amount of potential on the vertices of $V_i(r)$. For two vertices r and v, denote by $[r, v]$ the union of all minimum rv-paths, and set $[r, v) = [r, v] - \{v\}$. We say that C is *r-clear* if u is empty for every big vertex v and every $u \in [r, v)$.

Lemma 2. (Potential Lemma) *Let C be a configuration on a graph G with potential $\mathsf{pot}(C)$. Then the following hold.*

1. $\mathsf{pot}(C) \ge \left\lceil \frac{|C|-|V|+z(C)}{2} \right\rceil$.
2. *If C solves a distribution D and* $\mathsf{supp}(C) \cap \mathsf{supp}(D) = \emptyset$ *then* $\mathsf{pot}(C) \ge |D|$.
3. *If C is r-clear and t-fold solves r then* $\sum_{i=1}^{\mathsf{ecc}(r)} 2\mathsf{pot}_i(C, r)/2^i \ge t$.

Proof. Part (1) is a simple consequence of the relation $|C| \le 2\mathsf{pot}(C) + s(C)$. Part (2) holds since placing a pebble on a target requires a potential to make a pebbling step. Part (3) follows from the fact that $2\mathsf{pot}_i(C, r)/2^i$ is the maximum number of pebbles that can reach r from V_i. $\square$

Finally, for a configuration C and single target r, we define the *cost*, $\mathsf{cost}(\sigma)$, of an r-solution σ to be the number of pebbles discarded by σ, including the pebble placed on r; thus it equals one more than the number of pebbling steps of σ. Consequently, the resulting configuration C' of pebbles remaining to use to solve other targets has size $|C'| = |C| - \mathsf{cost}(\sigma)$. We define an r-solution to be cheap if its cost is at most $2^{\mathsf{ecc}(r)}$. We say that a graph G is r-(semi)greedy if every configuration of size at least $\pi(G, r)$ has a (semi)greedy r-solution; that is, every pebbling step in the solution decreases (does not increase) the distance

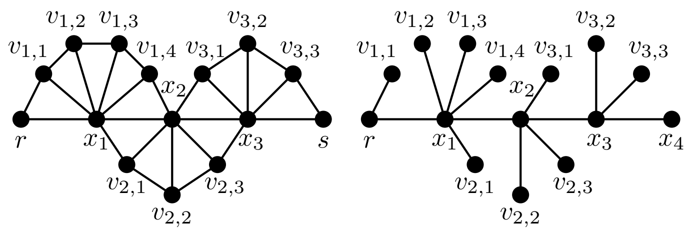

Figure 1: An overlapping fan graph of diameter 4 on the left. The BFS tree rooted at r on the right.

of the moved pebble to r. It is known, for example (see [4]), that trees are greedy and that chordal graphs are semi-greedy.

Lemma 3. **(Cheap Lemma)** *[3] Given the graph G with root r let H be an r-greedy spanning subgraph of G preserving distances to r. Then any configuration of G of size at least $\pi(H, r)$ is cheap.*

In particular, a breadth-first-search spanning tree can play the role of H in the Cheap Lemma. In the case of 2-paths, below, we use a caterpillar as the best choice of such a tree. In the case of split graphs, we find cheap solutions in a more ad hoc manner.

2 2-Paths

2.1 Definition and notation

A *simplicial* vertex is a vertex that forms a complete graph with its neighbors. A *chordal* graph is a graph with no induced cycle of length at least 4. A *k-path* is either the complete graph K_k or a graph G with exactly two simplicial vertices u and v such that the neighborhood of v is K_k and $G - v$ is a k-path.

An *overlapping fan* graph G has two simplicial vertices, we denote r and s, with the shortest path $P = (x_0, x_1, \ldots, x_n)$ between them ($r = x_0, s = x_n$). We call this path the *spine* of G. We denote F_i to be the subgraph of G which consists of the vertex x_i and the $x_{i-1}x_{i+1}$-path $Q_i = (x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1})$, where each $v_{i,j}$ is adjacent to x_i. Every vertex of G is in $P \cup F_i$. An example of a fan graph is seen in Figure 1, where $k_1 = 4$, $k_2 = 3$ and $k_3 = 3$.

By Lemma 2.1 of [2], every 2-path is an overlapping fan graph (and vice-versa). In [2] we find the following result.

Theorem 4. *If G is a 2-path with $\mathsf{diam}(G) = d$ then $\pi_t(G) = t2^d + n - 2d$.*

2.2 Preliminary facts and lemmas

We present some important preliminary facts and lemmas necessary to build our proof for the Target Conjecture. These first two facts are used to determine the pebbling numbers of specially constructed trees which are essential to our proof.

Fact 5. *[5] For any tree, T, rooted at r, $\pi(T,r) = 2^{a_1} + \sum_{i=2}^{k} 2^{a_i} - k + 1$ where a_i is the length of the path, P_i, in a maximum r-path partition, $P = \{P_1, ..., P_k\}$ such that $a_i \geq a_{i+1}$. Note that $a_1 = \mathsf{ecc}(r)$.*

Corollary 6. *If T is a caterpillar, and r is a leaf with maximum eccentricity $d = \mathsf{diam}(G)$, then $\pi(T,r) = 2^d + n - d - 1$.*

Fact 7. *[3] Let G be a graph with cut vertex r. Denote the components of $G - r$ by $G_1, \ldots, G_m$, and define H_i to be the induced subgraph of G on the vertex set $V(G_i) \cup \{r\}$. Then $\pi(G,r) = \left(\sum_{i=1}^{m} \pi(H_i, r)\right) - m + 1$.*

Let T be any Breadth First Search (BFS) spanning tree of G, rooted at r. Then T preserves all distances from r; that is, $\mathsf{dist}_T(v,r) = \mathsf{dist}_G(v,r)$ for all vertices v. For a simplicial vertex r, we define T_r (the *spinal* tree of r) to be such a BFS tree chosen in a specific manner, namely, with the priority to choose vertices of the spine P_1 (from the maximum r-path partition) before other vertices whenever possible. As we see from Figure 1, this produces a caterpillar. Let S be the two simplicial vertices of G.

For an internal, spinal root r, let F_i be the fan centered on r, and let A_r be the set of vertices of F_i that are in no other fan. For each $v \in A_r$ we have that the edge rv is a 2-path, which we denote by G^v. Then $G - A_r$ is the edge union of two 2-paths G^s, for $s \in S$, that share only the vertex r. Note that, for each $v \in S \cup A_r$, we have that r and v are the simplicial vertices of G^v; we denote a spinal tree of r in G^v by $T_r(G^v)$. Now define $T_r = \cup_{v \in S \cup A_r} T_r(G^v)$.

For an internal, non-spinal root r, with r in two fans, let F_i and F_{i+1} be the two fans that share r, centered on x_i and x_{i+1} respectively. Then $G - x_i x_{i+1}$ is the edge union of two 2-paths G^s, for $s \in S$, that share only the vertex r. Note that, for each $s \in S$, we have that r and s are the simplicial vertices of G^s. We denote a spinal tree of r in G^s by $T_r(G^s)$, and so define $T_r = \cup_{s \in S} T_r(G^s)$.

For an internal, non-spinal root r, with r unique to one fan, let F_i be the fan, centered on x_i, that contains r. Then G is the edge union of two 2-paths G^s, for $s \in S$, that shares the edge rx_i.

Note that, for each $s \in S$, we have that r and s are the simplicial vertices of G^s. We denote a spinal tree of r in G^s by $T_r(G^s)$, and so define $T_r = \cup_{s \in S} T_r(G^s)$.

Because every vertex of a 2-path that is not on P_1 is adjacent to some vertex on P_1, this makes T_r an edge-disjoint union of $\deg(r)$ caterpillars, all sharing only the vertex r.

Corollary 8. *Let G be a 2-path of diameter d on n vertices, and T_r be a spinal tree rooted at vertex r. Then* $\pi(T_r, r) =$

1. $2^{\mathsf{ecc}(r)} + 2^{d-\mathsf{ecc}(r)} + n - d - 2$ *if r is spinal for some representation, and*

2. $2^{\mathsf{ecc}(r)} + 2^{d+1-\mathsf{ecc}(r)} + n - d - 3$ *if r is non-spinal in every representation.*

In particular, $\pi(T_r, r) \leq 2^d + n - d - 1$ for all r.

Proof. If r is a spinal root then, for each $T_r(G^v)$, $v \in S \cup A_r$, r is a simplicial vertex, where n_1 is the number of vertices of the first tree with diameter $\mathsf{ecc}(r)$, n_2 is the number of vertices of the second tree with diameter $d - \mathsf{ecc}(r)$, and the subsequent trees are single edges. Note that $n_1 + n_2 + |A_r| = n + 1$ since n_1 and n_2 each contain r. Then, by Fact 7, we have

$$\begin{aligned}\pi(T_r, r) &= (2^{\mathsf{ecc}(r)} + n_1 - \mathsf{ecc}(r) - 1) + (2^{d-\mathsf{ecc}(r)} + n_2 - (d - \mathsf{ecc}(r)) - 1) + |A_r| - 2 + 1 \\ &= 2^{\mathsf{ecc}(r)} + 2^{d-\mathsf{ecc}(r)} + n - d - 2.\end{aligned}$$

If r is non-spinal then, for each $T_r(G^s)$, $s \in S$, r is a simplicial vertex. Note that $\mathsf{diam}(T_r) = d + 1$, since we always gain one more edge in the spine of T_r than the spine of G. Then the first tree has n_1 vertices and diameter $\mathsf{ecc}(r)$, and the second tree has n_2 vertices and diameter $d + 1 - \mathsf{ecc}(r)$. Then, by Fact 7, we have

$$\begin{aligned}\pi(T_r, r) &= (2^{\mathsf{ecc}(r)} + n_1 - \mathsf{ecc}(r) - 1) + (2^{d+1-\mathsf{ecc}(r)} + n_2 - (d + 1 - \mathsf{ecc}(r)) - 1) - 2 + 1 \\ &= 2^{\mathsf{ecc}(r)} + 2^{d+1-\mathsf{ecc}(r)} + n - d - 3.\end{aligned}$$

□

Lemma 9. *If r^* is a simplicial root of a 2-path, G, with the rooted spinal tree, $T^* = T_{r^*}$, then $\pi(T_r, r) \leq \pi(T^*, r^*)$.*

Proof. From corollary 8, we can compare the pebbling numbers for each type of r in G to the pebbling number of r^*.

If r is a spinal root, then we have

$$\begin{aligned}\pi(T_r, r) &= 2^{\mathsf{ecc}(r)} + 2^{d-\mathsf{ecc}(r)} + n - d - 2\\ &\leq 2^d + 1 + n - d - 2\\ &= 2^d + n - d - 1\\ &= \pi(T^*, r^*).\end{aligned}$$

If r is a non-spinal root, then we have

$$\begin{aligned}\pi(T_r, r) &= 2^{\mathsf{ecc}(r)} + 2^{d+1-\mathsf{ecc}(r)} + n - d - 3.\\ &\leq 2^d + 2 + n - d - 3\\ &= 2^d + n - d - 1\\ &= \pi(T^*, r^*).\end{aligned}$$

□

Lemma 10. *Let G be a 2-path and r be any vertex. Then $\pi_2(G) \geq \pi(T^*, r^*)$. Thus, any configuration of size at least $\pi_2(G)$ has a cheap r-solution.*

Proof. Note that for any d, $2^d > d$. Then by Theorem 4 we have:

$$\begin{aligned}\pi(T^*, r^*) &= 2^d + n - d - 1\\ &\leq 2^d + n - d - 1 + (2^d - d + 1)\\ &= 2^{d+1} + n - 2d\\ &= \pi_2(G).\end{aligned}$$

The existence of a cheap solution follows from the Cheap Lemma 3. □

2.3 Verification of the Target Conjecture

Theorem 11. *Let G be a 2-path and D be a target distribution of size t. Then $\pi_t(G) \geq \pi(G, D)$.*

Proof. Let $|C| = \pi_t(G)$. Note from Theorem 4 that $\pi_t(G) = \pi(G) + (t-1)2^d = \pi_{t-1}(G) + 2^d$, which we will use in our induction proof. If $t = 1$, then $\pi_1(G) \geq \pi(G, r)$ for any r, so C solves any D of size 1.

Thus let $t \geq 2$. By the induction hypothesis, we assume that $\pi(G, D') \leq \pi_{t-1}(G)$ for all $|D'| = t - 1$. Let $D = \{r_1, \ldots, r_t\}$ be given (note that the roots r_i are not necessarily distinct). Choose any $r = r_t$ and define $D' = \{r_1, r_2, ..., r_{t-1}\}$. Let T_r denote a spinal tree of r. Then by the Cheap Lemma 10 we have $|C| \geq \pi_2(G) \geq \pi(T_r, r)$, and so C has a cheap solution σ. Thus $|C'| = |C| - \mathsf{cost}(\sigma) \geq \pi_t(G) - 2^d = \pi_{t-1}(G)$. Then $|C'|$ solves $|D'|$ by induction, and so are the remaining roots in the distribution by induction on t. $\square$

3 Remarks

A natural next step for verifying the Target Conjecture would be to consider k-paths for $k \geq 3$. However, the t-fold pebbling numbers for this family are not presently known. In [4] the subfamily of k^{th} powers of paths is studied. Additional interesting families to investigate include diameter two graphs, Class 0 graphs, and chordal graphs, among others. Unfortunately, t-fold pebbling numbers are not known for such broad classes, so more specific sub-classes need to be examined.

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