On the Number of Ones in General Binary Pascal Triangles

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ABSTRACT. This paper answers the question as to whether every natural number n is realizable as the number of ones in the top portion of rows of a general binary Pascal triangle. Moreover, the minimum number $\kappa(n)$ of rows is determined so that n is realizable.

1991 AMS Subject Classification: 05A10, 11B65

Keywords: Pascal triangle, varied initial values, reduced modulo 2, number of ones.

1. Introduction

Consider generating the first k rows of a general binary Pascal triangle. That is, let $L = (l_1 l_2 \dots l_k)$ and $R = (r_2 r_3 \dots r_k)$ be any lists of zeros and ones placed where normally the 1s of the Pascal triangle are, with l_1 at the top, L down the left diagonal and R down the right. Let the remaining entries be filled in by the Pascal recurrence, modulo 2. Denote the resulting triangle $\Delta_k(L,R)$ and let $\delta_k(L,R)$ be the number of its ones. For example, with $L = (11010)$ and $R = (0011)$ we obtain $\delta_5(L,R) = 8$ (see Figure 1).

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 1 & & 1 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 1
\end{array}
$$

Figure 1. $\Delta_5((11010),(0011))$.

Note that δ_k remains the same if any corner is used (by rotation) as its top corner.

We say that n is realizable if it is possible to find k, L, and R so that $\delta_k(L,R) = n$, and we call $\Delta_k(L,R)$ a realizer of n. The first author asked the following question in [4]: Is every natural number n realizable? We

[1]Partially supported by National Security Agency grant #MDA9040210095

answer this affirmatively in Section 2, even with $L = \underline{1}$, the all-ones vector. When the length of the vector is important we will write $\underline{1}^k$ and likewise $\underline{0}^k$ for the all-zeros vector.

It is possible to realize 8 more quickly than in Figure 1. With $L = \underline{1}$ and $R = (110)$ we find $\delta_4(L, R) = 8$. We say that n is k-realizable if it is possible to realize n in k rows, and we denote by $\kappa(n)$ the minimum k such that n is k-realizable. Of course one cannot 3-realize 8 because there are only 6 entries in the first three rows, and thus $\kappa(8) = 4$. We will determine $\kappa(n)$ in Section 4. For this purpose, in Section 3 we will find the maximum number $d(k)$ of ones being possible in a general binary Pascal triangle $\Delta_k(L, R)$.

2. General realizability

Consider $\Delta_k(\underline{1}, \underline{1})$, the standard binary Pascal triangle and write $P^t = \Delta_k(\underline{1}, \underline{1})$ if $k = 2^t$. Then P^t has the well-known recursive structure shown in Figure 2 (see [2,3,5]). Hence, the number of ones in P^t triples when the

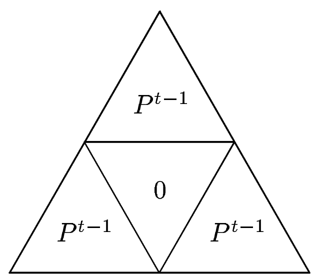


Figure 2. Recursive structure of P^t.

number of rows doubles, and so $\delta(P^t) = 3^t$.

We are now ready to prove that every n is realizable even if $L = \underline{1}$ as in the standard binary Pascal triangle.

Theorem 1. For every natural number $n \leq 3^t$ there are $k \leq 2^t$ and R so that $\Delta_k(\underline{1}, R)$ realizes n.

Proof. An induction base is obvious for $t = 0$. As induction hypothesis every $n \leq 3^{t-1}$ is realizable by $\Delta_k = \Delta_k(\underline{1}, R)$ for some $k = k(n) \leq 2^{t-1}$. Then $\Delta_{2^{t-1}+k(n-3^{t-1})}$ and Δ_{2^t} as in Figures 3 and 4 realize n for $3^{t-1} < n \leq 2 \cdot 3^{t-1}$ and $2 \cdot 3^{t-1} < n \leq 3^t$, respectively. $\square$

3. Maximum number of ones in the first rows

For given k we determine the largest $n = d(k)$ being realizable by $\Delta_k(L, R)$. An earlier completely different proof was given in [1].

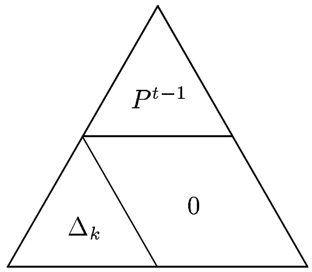


Figure 3. $\Delta_{2^{t-1}+k(n-3^{t-1})}$ for $3^{t-1} < n \leq 2 \cdot 3^{t-1}$.

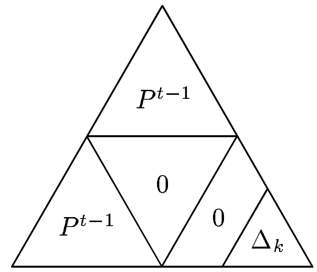


Figure 4. Δ_{2^t} for $2 \cdot 3^{t-1} < n \leq 3^t$.

Theorem 2. The maximum number of ones in $\Delta_k = \Delta_k(L, R)$ is

$$d(k) = \begin{cases} 1 & \text{if } \ k = 1, \\ 2 + \dfrac{1}{3}(k^2 + k + 1) & \text{if } \ k \equiv 1 \pmod 3, \\ 1 + \dfrac{1}{3}(k^2 + k) & \text{if } \ k \equiv 0, 2 \pmod 3, k \neq 8, \\ 27 & \text{if } \ k = 8. \end{cases}$$

Proof. We obtain lower bounds of $d(k)$ using $L = (1\overline{110})$ and $R = (\overline{110})$ where the triples 110 are repeated (see Figure 5), and where a possible zero in the last positions of L and R is substituted by a one. Then there are $2j$ ones in each of the rows $3j - 1$, $3j$, and $3j + 1$, $1 \leq j \leq \lfloor (k+1)/3 \rfloor$. It follows that $d(k) \geq \delta_k(L, R) = \sum_{j=1}^{\lfloor (k+1)/3 \rfloor} 6j$ plus $1 - 4(k+1)/3$, plus $1 - 2(k+1)/3$, and plus 3 for $k \equiv 2, 0$, and 1 (mod 3), respectively. These are the terms of Theorem 2 with the only exception of $k = 8$ where the standard binary Pascal triangle has 2 ones more, that is, $d(8) \geq 3^3$ (see Figure 6).

```
                    1
                  1   1
                1   0   1
              0   1   1   0
            1   1   0   1   1
          1   0   1   1   0   1
        0   1   1   0   1   1   0
      1   1   0   1   1   0   1   1
    1   0   1   1   0   1   1   0   1
  0   1   1   0   1   1   0   1   1   0
1   1   0   1   1   0   1   1   0   1   1
```

Figure 5. $L = (1\overline{110})$ and $R = (\overline{110})$.

```
              1
            1   1
          1   0   1
        1   1   1   1
      1   0   0   0   1
    1   1   0   0   1   1
  1   0   1   0   1   0   1
1   1   1   1   1   1   1   1
```

Figure 6. $d(8) \geq \delta_8(\underline{1}, \underline{1}) = 27$.

To obtain upper bounds of $d(k)$ we notice that in the three corner entries of Δ_k there always have to be ones. Deleting these corner entries from Δ_k leaves Δ'_k, and we have to find upper bounds of the maximum number $f(k) = d(k) - 3$ of ones in Δ'_k.

We use the fact that $x_3 \equiv x_1 + x_2 \pmod 2$ and thus that there are at most 2 ones in each of the sets $\{x_1, x_2, x_3\}$ if they are in one of the positions of Figure 7. Then Δ'_k can be partitioned into

$$\begin{array}{ccc} x_1 & & x_2 \\ & x_3 & \end{array} \qquad \begin{array}{ccccc} x_1 & & \bullet & & x_2 \\ & \bullet & & \bullet & \\ & & x_3 & & \end{array}$$

Figure 7. At most 2 ones.

(1) $\dfrac{k^2-1}{8}$ triangles Δ'_3 and one $\Delta'_{(k+1)/2}$ if $k \equiv 1 \pmod 2$

and into

(2) 3 triangles Δ'_3, 3 triangles $\Delta'_{k/2}$, and one $\Delta'_{(k-2)/2}$ if $k \equiv 0 \pmod 2$

as depicted in Figures 8 and 9.

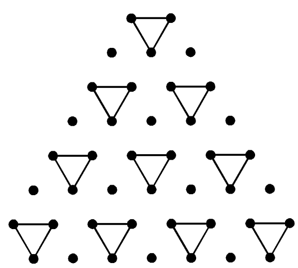


Figure 8. Δ'_k, $k \equiv 1 \pmod 2$.

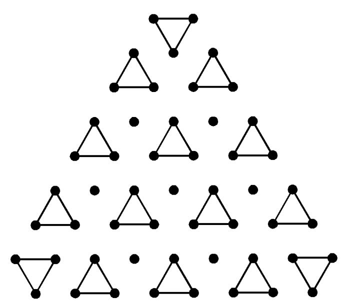
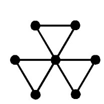


Figure 9. Δ'_k, $k \equiv 0 \pmod 2$.

Denoting by $\varphi_k = \varphi_k(L,R)$ the number of ones in $\Delta'_k(L,R)$ we have the possibilities $\varphi_2 = 0$, $\varphi_3 = 0$ or 2, and $\varphi_4 = 0$, 2, 4, or 6. Thus for small $k \geq 2$ we have the maximum values $f(2) = 0$, $f(3) = 2$, and $f(4) = 6$ where the unique example for $f(4) = 6$ occurs for a zero in the central entry (see Figure 10).



$$\begin{array}{ccccc} & 1 & & 1 & \\ 1 & & 0 & & 1 \\ & 1 & & 1 & \end{array}$$

Figure 10. $f(4) = 6$.

By the partitions (1) and (2) it follows inductively that always

$$\varphi_k \equiv 0 \pmod{2}. \tag{3}$$

To prove $f(k) \le (k^2+k-2)/3$ if $k \equiv 1 \pmod{3}$ and $f(k) \le (k^2+k-6)/3$ if $k \equiv 0, 2 \pmod{3}$ induction steps from k to $k+6$ using (1) and (2) are successful unless in the case of $k \equiv 2 \pmod{6}$. Here $k/2 \equiv 1 \pmod{3}$ and $(k-2)/2 \equiv 0 \pmod{3}$ imply with (2)

$$f(k) \le 6+\left(\frac{k^2}{4}+\frac{k}{2}-2\right)+\frac{1}{3}\left(\left(\frac{k-2}{2}\right)^2+\frac{k-2}{2}-6\right) = \frac{1}{3}(k^2+k-6)+4,$$

that is, we obtain a surplus of 4. To manage this missing case we need two lemmas determining for $k \equiv 1 \pmod{3}$ the designs of Δ_k' with $f(k)$ ones.

Lemma 1. For the last three rows B_k of Δ_k' consisting of $(k-1)/3$ copies of Δ_4' with vertical pairs of entries between these copies, the maximum number $b(k)$ of ones in B_k is $b(k) = 2k-2$, and $b(k)$ is attained only if every copy of Δ_4' has 6 ones in the unique design of Figure 10 and the pairs between the Δ_4's consist of zeros (see Figure 11).

$$\begin{array}{ccccccccccccccccccc}
 & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & \\
1 & & 0 & & 1 & & 1 & & 0 & & 1 & \cdots & 1 & & 0 & & 1 & \cdots & 1 & & 0 & & 1 \\
 & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 &
\end{array}$$

Figure 11. The last three rows B_k of Δ_k'.

Proof of Lemma 1. Consider the vertical pairs of entries between neighboring pairs of Δ_4's.

If both entries of one of the pairs are zero then $b_k = 2k-2$ is attained inductively only if all Δ_4's on both sides have 6 ones. This gives the asserted unique design.

If the upper entries of x pairs, $x > 0$, are ones then always both neighboring Δ_4's have at most 4 ones so that at least $x+1$ of the Δ_4's have 2 ones less than 6 and we get $b_k \le 2k-2-2(x+1)+2x < 2k-2$.

If only all lower entries of all pairs are ones then at least one of the neighboring Δ_4's has at most 4 ones. Any Δ_4' being the reduced Δ_4' for both neighboring lower ones has only zeros. Thus $b(k) \le 2k-2-2((k-1)/3-1) < 2k-2$ since there are at least 2 ones less for all $(k-1)/3-1$ lower ones. $\square$

Lemma 2. The unique Δ_4' with $f(4) = 4$ is shown in Figure 10. All possible Δ_7's with $f(7) = 18$ are shown in Figure 12 where the second Δ_7' can be rotated and where the pair 01 at the corner may be switched to 10. All possible Δ_k's for $k \equiv 1 \pmod{3}$, $k \ge 10$, and with $f(k) = (k^2+k-2)/3$ are shown in Figure 13 where $L' = R' = (l_2 l_3 l_4 \overline{110} 11 l_{k-3} l_{k-2} l_{k-1})$ and

```
    1 1               0 1
   1 0 1             1 1 1
  0 1 1 0           1 0 0 1
 1 1 0 1 1         1 1 0 1 1
1 0 1 1 0 1       1 0 1 1 0 1
 1 1 0 1 1         1 1 0 1 1
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Figure 12. All Δ'_7s with $f(7) = 18$.

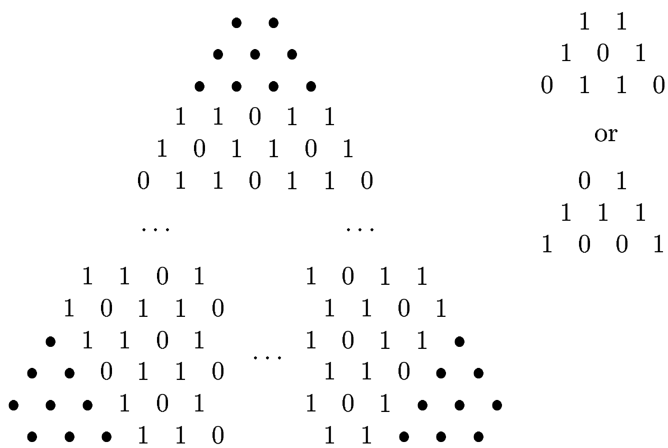

Figure 13. All Δ'_ks for $k \equiv 1 \pmod 3$, $k \geq 10$, and with $f(k) = (k^2 + k - 2)/3$.

where the trapezoids at the corners may be arbitrarily chosen from the depicted two copies with 01 at one corner being switchable.

Proof of Lemma 2. The uniqueness of Δ'_4 in Figure 10 was mentioned above.

For $k \geq 7$ we partition Δ'_k into Δ'_{k-3}, B_k, and a horizontal pair of entries (see Figure 14).

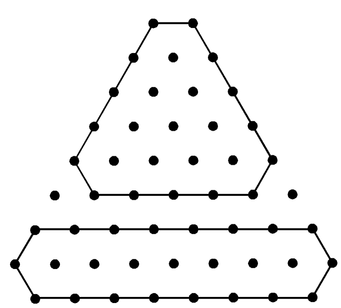

Figure 14. Partition of Δ'_k into Δ'_{k-3}, B_k, and a pair.

For $k \geq 7$ the maximum $f(k) = (k^2+k-2)/3$ cannot be attained if Δ'_{k-3} and B_k both do not have the maximum of ones since (3), Lemma 1, and 2 ones in the horizontal pair imply $\varphi_k \leq f(k-3) - 2 + b(k) - 1 + 2 < f(k)$.

Let $k = 7$. If Δ'_4 has 6 ones then Δ'_7 can be completed to have 18 ones with 00 and with 01 as the horizontal pair and not with 11. If B_7 has the maximum of 12 ones then 18 ones for B'_7 are possible with the pair 00 and with 11 and not with 01. All possibilities are covered by Figure 12.

Let $k = 10$. If Δ'_7 has $f(7) = 18$ ones then we may rotate Δ'_7 such that 11011 is between the horizontal pair of entries. Then 6 ones are determined in the middle part of B_{10}. For the remaining trapezoids at both sides only those of Figure 13 with 6 ones are possible to obtain Δ'_{10} with $f(10) = 18 + 6 + 2 \cdot 6 = 36$ ones.

If B_k for $k \geq 10$ has the maximum of $b(k) = 2k - 2$ ones then 00 for the horizontal pair determines an asserted solution. A one for the pair of entries determines $(k-1)/3$ ones in row $k-3$ and thus $\varphi_k \leq b(k) + (k-1)/3 + f(k-4) + 2 = (k^2+5)/3 < f(k)$ for $k > 7$.

For $k \geq 13$ it remains the case that Δ'_{k-3} has the maximum of $f(k-3)$ ones. Then the last row of Δ'_{k-3} starts at both ends with 110, 011, or 111. The main part $11\overline{011}$ of this row determines $6((k-1)/3 - 4) = 2k - 26$ ones in B_k. At both ends of B_k together with an entry of the horizontal pair there can occur only the six designs of Figure 15. The sixth entry in

$$\begin{array}{c} 0\ \ \underline{1\ \ 1\ \ 0}\ \ 1\ \ 1 \\ \bullet\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0 \\ \bullet\ \ \bullet\ \ 1\ \ 1\ \ 0\ \ 1 \\ \bullet\ \ \bullet\ \ 0\ \ 1\ \ 1\ \ \bullet \end{array} \qquad \begin{array}{c} 1\ \ \underline{1\ \ 1\ \ 0}\ \ 1\ \ 1 \\ \bullet\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0 \\ \bullet\ \ \bullet\ \ 0\ \ 1\ \ 0\ \ 1 \\ \bullet\ \ \bullet\ \ 1\ \ 1\ \ 1\ \ \bullet \end{array} \qquad \begin{array}{c} 0\ \ \underline{1\ \ 1\ \ 1}\ \ 1\ \ 1 \\ \bullet\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 1\ \ 0\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 1\ \ 0\ \ 0\ \ \bullet \end{array}$$

$$\begin{array}{c} 1\ \ \underline{1\ \ 1\ \ 1}\ \ 1\ \ 1 \\ \bullet\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 0\ \ 0\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 0\ \ 0\ \ 0\ \ \bullet \end{array} \qquad \begin{array}{c} 0\ \ \underline{0\ \ 1\ \ 1}\ \ 1\ \ 1 \\ \bullet\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 1\ \ 1\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 0\ \ 0\ \ 0\ \ \bullet \end{array} \qquad \begin{array}{c} 1\ \ \underline{0\ \ 1\ \ 1}\ \ 1\ \ 1 \\ \bullet\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 0\ \ 1\ \ 0\ \ 0 \\ \bullet\ \ \bullet\ \ 1\ \ 1\ \ 0\ \ \bullet \end{array}$$

Figure 15. Possible designs at the ends of B_k, $k \equiv 1 \pmod 3$.

the last row of B_k can be a one only for $k = 13$ if there is a pair 01 or 10 in the middle of the second last row. The 5 entries at the ends contain at most 4 ones each. To attain $f(k)$ ones, two designs of Figure 15 have to have $x = f(k) - f(k-3) - (2k - 26) = 24$ ones. This is possible for the first two designs only since each of the other designs has at most 11 ones. All resulting possibilities are covered by Figure 13. □

Now we are ready to handle the missing case $k \equiv 2 \pmod 6$ in the proof of Theorem 2.

For $k = 8$ we have $f(8) \leq 3f(3) + 3f(4) + f(3)$ by (2). If at least 2 of the unique Δ'_4s with 6 ones occur in Δ'_8 then the central Δ'_3 is forced to be without ones and thus $f(8) \leq 3 \cdot 2 + 3 \cdot 6 = 24$ as asserted. If at most one of the Δ_4's has 6 ones and the other 2 have at most 4 then $f(8) \leq 3 \cdot 2 + 6 + 2 \cdot 4 + 2 = 22$. Thus there is the unique solution (see the part of Figure 6) when all Δ_4's have $f(4) = 6$ ones.

For $k = 14$ we consider (2). If at least 2 of the Δ_7's have $f(7) = 18$ ones then by Lemma 2 (see Figure 12) for Δ_{14}' at most 8 ones occur in Δ_6' (see Figure 16) so that $f(14) \leq 3f(3) + 3f(7) + 8 = 68$. If at most one Δ_7' has 18

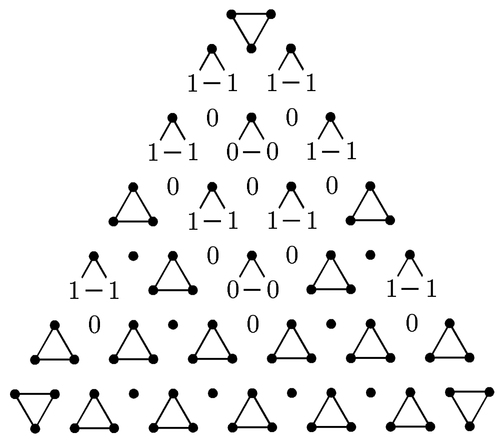

Figure 16. Δ_{14}' with 2 maximum Δ_7's.

ones then by (3) we obtain $f(14) \leq 3f(3) + f(7) + 2(f(7) - 2) + f(6) = 68$.

For $k \equiv 2 \pmod 6$, $k \geq 20$, at least 2 of the $\Delta_{k/2}'$s with the maximum of ones force most entries of $\Delta_{(k-2)/2}'$ to be zero. Using Lemma 2, there are at most 6 ones at each corner. Since $18 \leq f((k-2)/2) - 4$ for $k \geq 20$, we obtain from (2) that $f(k) \leq 3f(3) + 3f(k/2) + f((k-2)/2) - 4 = f(k)$. If at most one $\Delta_{k/2}'$ has $f(k/2)$ ones then, because of (3), we also can subtract 4 on the right part of (2).

Thus the induction step now works for all residue classes modulo 6. However, due to the exceptional case $k = 8$ the induction bases remain unsolved for $k = 15$, 16, and 18.

If $k = 15$ then from (1) we have $f(15) \leq 28f(3) + f(8)$. From another interpretation of the partition as in Figure 8 we get $f(15) \leq 3f(3) + 3f(7) + f(8)$. If Δ_8' has at most 22 ones it follows $f(15) \leq 56 + 22 = 78$ as asserted. Otherwise, there exists the unique Δ_8' with 24 ones. If all three Δ_7's have at most $f(7) - 2$ ones then $f(15) \leq 6 + 3 \cdot 16 + 24 = 78$. It remains that at least one Δ_7' has 18 ones. Because of Lemma 2 we can rotate Δ_{15}' such that Δ_8' and Δ_7' determine the 5^{th} row from the bottom of Δ_{15}' to be 11110001111. Then 2 of the Δ_3's are without ones and $f(15) \leq (28 - 2)f(3) + f(8) = 76$. This proves $f(15) = 78$.

If $k = 16$ then from (2) we have $f(16) \leq 3f(3) + 3f(8) + f(7)$. If all Δ'_8s have at most 22 ones then $f(16) \leq 6 + 66 + 18 = 90$. If at least 2 of the Δ'_8s have 24 ones then Δ'_7 has zeros only and we get $f(16) \leq 6 + 3 \cdot 24 = 78$. If exactly one Δ'_8 has 24 ones then with $f(7) - 2$ ones in Δ'_7 it follows $f(16) \leq 6 + 24 + 44 + 16 = 90$. Otherwise, we have $f(7) = 18$ ones in Δ'_7 and because of Lemma 2 and Figure 12 we can insert digits of Δ'_7 into Δ'_{16} as in Figure 17. If the unique Δ'_8 (see the part of Figure 6) is

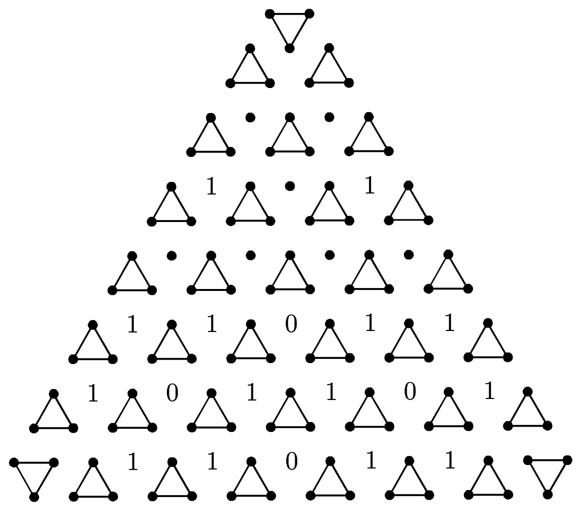

Figure 17. Δ'_{16} with a maximum Δ'_7.

inserted in both possibilities then at least 13 zeros are forced in one of the 2 other Δ'_8s so that one Δ'_8 has at most $33 - 13 = 20$ ones which implies $f(16) \leq 6 + 24 + 22 + 20 + 18 = 90$ which completes the proof of $f(16) = 90$.

If $k = 18$ from (2) we have $f(18) \leq 3f(3) + 3f(9) + f(8)$. If Δ'_8 has at most 22 ones we get $f(18) \leq 112$ as asserted. Thus Δ'_8 is unique with 24 ones. From a variation of (1) (see Figure 8) we obtain $f(9) \leq 3f(3) + 3f(4) + f(5)$ and thus $f(18) \leq 12f(3) + 9f(4) + 3f(5) + f(8) \leq 72 + 9f(4)$. In Figure 18 with the unique Δ'_8 there are 21 small triangles hosting the 9 different Δ'_4s. The 3 triangles in the central region of Δ'_{18} are forced to have only ones or only zeros. All other 18 triangles are forced to have 1 one or 2 ones. At least one of the 3 central triangles has only zeros since otherwise every Δ'_4 has at most 4 ones yielding $f(18) \leq 72 + 9 \cdot 4 = 108$. If at least 2 of the 18 triangles have exactly 1 one then $f(18) \leq 72 + 2 \cdot 3 + 16 \cdot 2 + 2 \cdot 1 = 112$. Otherwise, say the upper 6 triangles of the 18 triangles in Figure 18 have to have 2 ones each. This is possible in the unique way of Figure 18 and it forces the 3 central triangles to have zeros only so that $f(18) \leq 72 + 18 \cdot 2 = 108$. This completes the proof of $f(18) = 112$ and the proof of Theorem 2.

□

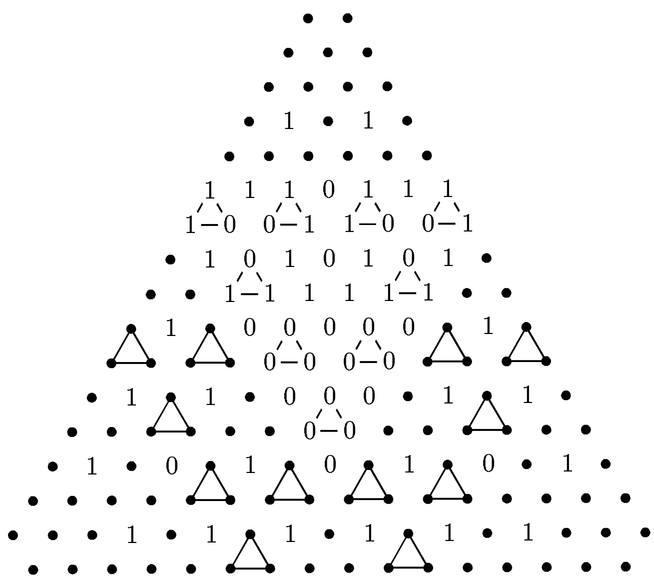

Figure 18. Δ'_{18} with a maximum Δ'_8.

4. Minimum number of rows to realize a number

Of course, the value $n = d(k)$ is k-realizable. The values of n between $d(k-1)$ and $d(k)$ are not $(k-1)$-realizable. However, are all these values k-realizable?

Theorem 3. For the minimum number $\kappa(n)$ of rows of Δ_k such that n is k-realizable we have

$$\kappa(1) = 1,\ \kappa(2) = \kappa(3) = 2,\ \kappa(26) = \kappa(27) = 8,$$
$$\kappa(n) = \left\lceil \sqrt{3n-7} - \frac{1}{2} \right\rceil \quad \text{if } n = \frac{1}{3}(k^2+k+7),\ k \equiv 1 \pmod 3,$$
$$\kappa(n) = \left\lceil \sqrt{3n-4} - \frac{1}{2} \right\rceil \quad \text{otherwise.}$$

Proof. The values of $\kappa(n)$ for $n \le 3$, $n = 26$, and $n = 27$ are obvious (see Figure 6 where a one at a corner can be replaced by a zero). In general, $d(k) - i$ for $i \le 3$ is k-realizable since the ones at the 3 corners of Δ_k can be replaced by zeros.

For $r \ge 4$, $r \equiv 0 \pmod 2$, we consider for $k > r$ the upper triangle $\Delta_r(L, R)$ of $\Delta_k(L, R)$ with $d(k)$ ones and so that $L = (1\overline{110})$ and $R = (\overline{110})$. Then Δ_r has $y(r) = d(r)$ ones for $r \equiv 1 \pmod 3$ and $y(r) = d(r) - 2$ ones otherwise ($k \ne 8$). We will construct an interval of consecutive numbers $y(r+i)$ so that all $n = d(k) - y(r) - i$, $i \ge 0$, are k-realizable. If these intervals overlap one another then $n = d(k) - j$, $0 \le j \le y(r)$ is k-realizable so that because of $d(k) - d(k-1) \leq (2k+1)/3 \leq y(r)$ all values of n between $d(k-1)$ and $d(k)$ are k-realizable for $k \leq 3y(r)/2$.

The construction starts in row r of Δ_k by choosing in row $r-j+1$ a one at the position $r/2-j$ for $1 \leq j \leq r/2+1$. This one together with row $r-j+2$ determines the whole row $r-j+1$. From row $r/2+1$ upwards, zeros are chosen in position 1 of every row (see Figure 19). Then

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&0&&&&&&&&\\
&&&&&&&0&&0&&&&&&&\\
&&&&&&0&&0&&0&&&&&&\\
&&&&&0&&0&&0&&0&&&&&\\
&&&&0&&0&&0&&0&&0&&&&\\
&&&1&&0&&0&&0&&0&&1&&&\\
&&1&&1&&0&&0&&0&&1&&1&&\\
&0&&0&&1&&0&&0&&1&&0&&0&\\
1&&0&&1&&1&&0&&1&&1&&0&&1
\end{array}$$

Figure 19. The construction for $r = 8$.

$y(r) - 2z(r)$ ones are subtracted from $d(k)$ in Δ_k, where $z(r)$ denotes the number $\delta_{(r-2)/2}$ of ones in one of the two identical triangles $\Delta_{(r-2)/2}(L, R)$ at both lower corners of Δ_r and with $L = \underline{1}$, $R = \overline{(001)}$ if the corner inside the row r of Δ_k is interpreted as the top of $\Delta_{(r-2)/2}$.

Now the top triangle $\Delta_{(r+2)/2}$ consists of zeros only and we will use the construction in the proof of Theorem 1 to realize all numbers j for $0 \leq j \leq 3^t$ in the first 2^t rows of $\Delta_{(r+2)/2}$. In the proof of Theorem 1 we have realized all these values of j with $L = \underline{1}$ and now we rotate the triangles so that $L = \underline{1}$ becomes a row of $\Delta_{(r+2)/2}$ having ones only and followed by a row of zeros. Thus we can subtract from $d(k)$ the numbers

$$\text{(4)} \quad \begin{array}{lll} y(r) - 2z(r) - j & \text{for } 0 \leq j \leq 3^t & \text{if } 2^t \leq \frac{r+2}{2} < 2^t + 2^{t-1}, \\ y(r) - 2z(r) - j & \text{for } 0 \leq j \leq 3^t + 2 \cdot 3^{t-1} & \\ & & \text{if } 2^t + 2^{t-1} \leq \frac{r+2}{2} < 2^{t+1}. \end{array}$$

An overlapping of the intervals is guaranteed if

$$\text{(5)} \quad y(r+2) - y(r) - 2(z(r+2) - z(r)) \leq \begin{cases} 3^t & \text{for } r < 2^{t+1} + 2^t - 4, \\ 3^t + 2 \cdot 3^{t-1} & \\ & \text{for } r = 2^{t+1} + 2^t - 4, \\ 3^t + 2 \cdot 3^{t-1} & \\ & \text{for } r < 2^{t+2} - 4, \\ 3^{t+1} & \text{for } r = 2^{t+2} - 4. \end{cases}$$

Using $z(r+2) - z(r) \geq 1$ and $y(r+2) - y(r) \leq (4r+8)/3$ the inequalities (5) are valid for $t \geq 3$ and the last two also for $t = 2$. In the remaining cases for $r \leq 10$ we can check that $y(r) = 53, 37, 25, 15$, and 7 and that $z(r) = 10, 7, 4, 2$, and 1 for $r = 12, 10, 8, 6$, and 4, respectively, and that the corresponding intervals are overlapping one another.

Thus it is proved that all values n between $d(k-1)$ and $d(k)$ are k-realizable.

Now it follows from Theorem 2 that

$$\kappa(n) = \left\lceil \left(\sqrt{12n-11} - 1\right)/2 \right\rceil$$

for $(k^2+k+3)/3 < n \leq (k^2+3k+5)/3$ if $k \equiv 0 \pmod 3$ and also for $(k^2-k+7)/3 < n \leq (k^2+3k+5)/3$ if $k \equiv 2 \pmod 3$ and that

$$\kappa(n) = \left\lceil \left(\sqrt{12n-27} - 1\right)/2 \right\rceil$$

for $(k^2-k+7)/3 < n \leq (k^2+3k+5)/3$ if $k \equiv 1 \pmod 3$. Because of the ceiling function this may be written as in Theorem 3. $\square$

Several further problems remain open. For example, what is the number $\delta_k(L,R)$ of ones in Δ_k for $L = \underline{1}$ and $R = (\overline{101})$? What is the maximum number of ones in $\Delta_k(L,R)$ for $L = \underline{1}$? How many pairs L and R determine a Δ_k with equal numbers of ones and zeros?

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