Bounded Cover Pebbling Ratios

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Abstract

A pebbling move on a graph consists of taking two pebbles off one vertex and placing one on an adjacent vertex. $\pi(G)$, the pebbling number of a graph, is the minimum number of pebbles needed so that from any initial arrangement of the pebbles, after a series of pebbling moves, it is possible to move to any vertex in the graph. $\gamma(G)$, the cover pebbling number of a graph is the minimum number of pebbles needed so that from any initial arrangement of the pebbles, after a series of pebbling moves, it is possible to have at least one pebble on every vertex of a graph. $\rho(G)$, the cover pebbling ratio, is defined as $\gamma(G) / \pi(G)$, the cover pebbling number divided by the pebbling number. This paper will discuss bounded cover pebbling ratios for star graphs, wheel graphs, cycles, and complete r-partite graphs.

1 Introduction

1.1 Overview

The game of pebbling was first suggested by Lagarias and Saks. Chung [1] introduced the concept in the, literature and defined the pebbling number, π (G). A pebbling move on a graph consists of taking two pebbles off one vertex and placing one on an adjacent vertex. $\pi(G)$, the pebbling number of a graph is the minimum number of pebbles needed so that from any initial arrangement of the pebbles, after a series of pebbling moves, it is possible to move to any root vertex in the graph. Hurlbert [2] and others have published results for general formulas concerning pebbling numbers for families of graphs. Details will be discussed later in the paper.

Crull et all. [3] introduced the concept of cover pebbling. $\gamma(G)$, the cover pebbling number of a graph is the minimum number of pebbles needed so that from any initial arrangement of the pebbles, after a series of pebbling moves, it is possible to have at least one pebble on every vertex of a graph. Yerger and Watson [4] published some general formulas for cover pebbling for families of graphs. Details will be discussed later in the paper as well as some results the authors of this paper believe to be new concerning general formulas for the cover pebbling number of specific families of graphs.

Crull et all. [3] were also the first to introduce $\rho(G)$, the cover pebbling ratio. It is defined as $\gamma(G) / \pi(G)$, the cover pebbling number divided by the pebbling number. They also found $\rho(K_n)$ and $\rho(P_n)$ approach 2. In the closing of that paper they posed the open question concerning other families of graphs who's cover pebbling ratios are bounded. None of the authors of this paper have seen results to this question published. This paper will find bounded cover pebbling ratios for star graphs, wheel graphs, cycles, and complete r-partitie graphs.

1.2 Definitions

All graphs, G, mentioned in this paper are simple connected graphs. "n" will refer to the number of vertices in a graph unless otherwise stated. "v" will refer to a vertex in a graph. K_n is the complete graph on n vertices. P_n is the path on n vertices. S_n is the star graph with a total of n vertices. W_n is the wheel graph with a total of n vertices. C_n is the cycle on n vertices. $K_{s1,s2,\dots,sr}$ is the complete r-partite graph with $s_1, s_2, \dots, s_r$ vertices in vertex classes $c_1, c_2, \dots, c_r$ respectively.

1.3 Known Results

The general formula for the pebbling number of a graph is known for many families of graphs. This paper will review several of these formulas.

Hurlbert [2] shows that $\pi(K_n) = n$. This result can be shown by taking the case of n-1 pebbles on a complete graph. If one were to arrange the pebbles so there was one pebble on every vertex except one, then there would be no way to reach that vertex. By adding the n^{th} pebble it becomes possible to reach any vertex since it's either added to the vertex without a pebble or it's added to a vertex that already has one pebble which is only a distance of one away from the empty vertex.

It is known that if a vertex *v* is distance *d* from vertex *w,* then 2^d pebbles are required on *v* to be able to reach *w* after a series of pebbling moves. Following this result Hurlbert [2] shows $\pi(P_n) = 2^{n-1}$.

Pachter et all. [6] found general formulas for the pebbling number of cycles. We have changed the notation here for later purposes, but the formula is equivalent. For even cycles they define the pebbling number as $\pi(C_n) = 2^{n/2}$ and for odd cycles it is defined as $\pi(C_n) = 2\lfloor(2^{(n+1)/2})/3\rfloor + 1$.

Watson and Yerger [4] found a general formula for the cover pebbling number of wheel graphs. $\gamma(W_n) = 4n - 5$. In the same paper they also found a general formula for the cover pebbling number of complete r-partitie graphs. $\gamma(K_{s1, s2, s3, \ldots, sr}) = 4s_1 + 2s_2 + 2s_3 \ldots + 2s_r - 3$. Where $s_1 \geq s_2 \geq \ldots \geq s_r$.

Vuong and Wyckoff [5] proved a theorem known as the stacking theorem which essentially finds the cover pebbling number for any graph. It states the configuration of pebbles that requires the most pebbles to be cover solved happens when all pebbles are placed on a single vertex. From there they state:

$$s(v) = \sum_{u \in V(G)} 2^{d(u,v)}$$

Do this for every vertex v in G. d(u,v) denotes the distance from u to v. Then the cover pebbling number is the largest s(v) that results.

Crull et all. [3] introduced $\gamma(G)$, the cover pebbling number of a graph. In the same paper they prove general formulas for the cover pebbling number of complete graphs and paths. $\gamma(K_n) = 2n - 1$. This can easily be verified with the stacking theorem. For any v in G there will be n-1 vertices distance one from v. $2(n - 1)$ pebbles will be needed to place a pebble on every other vertex, and one pebble to remain on v, or $2n - 1$ pebbles. They also show $\gamma(P_n) = 2^n - 1$.

In the same paper Crull er all. [3] introduce $\rho(G)$, the cover pebbling ratio. It is defined as $\gamma(G) / \pi(G)$. They also find the only published bounded cover pebbling ratios, $\rho(K_n)$ and $\rho(P_n)$ which both approach 2.

2 Preliminary Results

Theorem 1 $\pi(W_n) = n$ for $n > 3$

Proof. If n-1 pebbles are placed on G at every vertex except one then the vertex without a pebble can never be reached. If one pebble is added to G then it either covers the empty vertex or joins another pebble on a vertex. If the empty vertex is the center vertex then which ever spoke has 2 pebbles can now move a pebble to the center. If the empty vertex is one of the spokes then either the n^{th} pebble was placed on the center vertex which would allow for a direct pebbling move to the empty spoke or it was placed on a spoke. Regardless of which spoke it was placed on there will now be a path, of trivial length, between the spoke with 2 pebbles and the spoke with no pebbles. Each vertex on this path has one pebble on it. This will allow for a pebbling move from one end of the path to the other. Any other configuration of pebbles will have at least 2 pebbles on at least 2 spokes, which after 2 pebbling moves will leave 2 pebbles on the center vertex which will allow for a pebbling move to any vertex in G.

Theorem 2 . $\pi(K_{s1, s2, s3, \ldots, sr}) = n$

Proof. If one pebble is placed one every vertex except one then the empty vertex can never be reached. If the n^{th} pebble is placed on G it is either placed on the open vertex or on a vertex that already has a pebble on it. Regardless of what vertex it is placed on there will be a path between the vertex with 2 pebbles and the empty vertex that has a pebble on every vertex. This will allow for a series of pebbling moves from one end of the path to another. Any other configuration of pebbles will have at least 2 vertices with at least2 pebbles on each. This will allow for a pebbling move either directly to the open vertex or to a intermediary vertex first.

Theorem 3 $\pi(S_n) = n + 1$

Proof. If one pebble is placed one every vertex except one then the empty vertex can never be reached. When the n^{th} pebble is placed on the graph there are configurations that do not allow for a pebbling move to all vertices. For example take S_5 and place 3 pebbles on a spoke and one pebble on two other spokes. That configuration will not allow for a pebbling move to the 4^{th} spoke. When the $n+1^{th}$ pebble is added there are 4 options. It can be added to the center, which would allow for a pebbling move to the empty spoke. It can be added to the empty spoke, which would leave only the center without a pebble which can be reached. Finally, it can be added to either one of the spokes with only one pebble on it, which would allow for a pebble to be moved to the center vertex which when combined with the pebble from the spoke with 3 pebbles would allow for a pebbling move to the empty vertex.

Those results are also true for similar configurations on larger star graphs. The only difference would be the number of vertices with one pebble on them. There is one final possibility left to talk about; that is multiple spokes with 3 or more pebbles on them. Any such configuration would allow for 2 pebbling moves which would put 2 pebbles on the center vertex which would allow for a pebbling move to any vertex in S_n.

Theorem 4 $\gamma(S_n) = 4n – 5$

Proof. The stacking theorem produces two sums, when all the pebbles are places on the center vertex and when they are placed on one of the n-1 spokes. Placing them on the center vertex yields a sum of $2n + 1$. Placing them on any of the spokes will yield $4n – 5$, regardless of which spoke they are placed on. A configuration of 2n + 1 pebbles on a single spoke will not allow for a cover pebbling solution.

Theorem 5 $\gamma(C_n) = 3(2^{n/2} - 1)$ for even cycles and $\gamma(C_n) = 2^{(n+3)/2} - 3$ for odd cycles.

Proof. Regardless of which vertex is picked in a cycle the stacking theorem will give identical results, which means there is only one case for odd cycles and one for even cycles. Even cycles will produce a sum of the following form: $1 + 2 + 2 + 4 + 4 + \ldots + 2^{n/2}$. Where there is only 1 of the last term. This can be simplified to the above equation. The same logic follows for odd cycles. All sums will be of the form $1 + 2 + 2 + 4 + 4 \ldots + 2^{n/2} + 2^{n/2}$ which can be simplified to the above equation.

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Theorem 6 $\rho(S_n) = 4$

Proof. $\rho(S_n) = \gamma(S_n) / \pi(S_n)$. $\rho(S_n) = (4n - 5)/(n+1)$ The limit as n approaches infinity for this equation is 4.

Theorem 7 $\rho(W_n) = 4$

Proof. $\rho(W_n) = \gamma(W_n) / \pi(W_n)$. $\rho(W_n) = (4n - 5)/n$ The limit as n approaches infinity for this equation is 4.

Theorem 8 $\rho(C_n) = 3$

Proof. For even cycles: $\rho(C_n) = \gamma(C_n) / \pi(C_n)$. $\rho(C_n) = 3(2^{n/2} - 1) / 2^{n/2}$ The limit as n approaches infinity for this equation is 3.

For odd cycles: $\rho(C_n) = \gamma(C_n) / \pi(C_n)$. $\rho(C_n) = [2^{(n+3)/2} - 3] / [2\lfloor(2^{(n+1)/2})/3\rfloor + 1]$ The limit as n approaches infinity for this equation is 3.

Theorem 9 $\rho(K_{s1,\, s2,\, s3,\, \ldots,\, sr})$ where $s_1 \geq s_2 \geq \ldots \geq s_r$ =
4 as s_1 approaches infinity and $(n - s_1)$ does not.
2 as $(n - s_1)$ approaches infinity and s_1 does not.

Note: Yerger and Watson's [4] result concerning the cover pebbling number for complete r-partitie graphs is $\gamma(K_{s1,\, s2,\, s3,\, \ldots,\, sr}) = 4s_1 + 2s_2 + 2s_3 \ldots + 2s_r - 3$ where $s_1 \geq s_2 \geq \ldots \geq s_r$,. We have slightly modified their form to meet our needs: $\gamma(K_{s1,\, s2,\, s3,\, \ldots,\, sr}) = 4s_1 + 2(n-s_1) - 3$, where $s_1 \geq s_2 \geq \ldots \geq s_r$.

Proof. $(K_{s1,\ldots}) = \gamma(K_{s1,\ldots}) / \pi(K_{s1,\ldots})$. $[4s_1 + 2(n-s_1) - 3]/n$. The limit as s_1 approaches infinity is 4. The limit as n approaches infinity is 2. This intuitively makes sense. It takes 4 pebbles to get from one vertex in a class to another vertex in the same class, since the distance is 2. As the largest class of vertices gets larger it will take closer and closer to 4 times as many pebbles to cover the graph then its pebbling number it. By similar logic if the biggest class of vertices was a smaller and smaller percentage of the graph the cover pebbling number would be twice as big as the pebbling number, since it the most common distance in the graph would be one.

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