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The pebbling number of $C_5 \times C_5$

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Abstract

Chung has defined a pebbling move on a graph G to be the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The pebbling number $f(G)$ of a connected graph is the least number of pebbles such that any distribution of $f(G)$ pebbles on G allows one pebble to be moved to any specified, but arbitrary vertex. Graham conjectured that for any connected graphs G and H, $f(G \times H) \leqslant f(G)f(H)$. We show that Graham's conjecture holds when $G = H = C_5$.

Keywords: Pebbling; Graham's conjecture; Cartesian products, $C_5 \times C_5$

1. Introduction

Pebbling in graphs was first considered by Chung [2]. Consider a graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex, and the placement of one of those pebbles on an adjacent vertex.

Chung [2] defined the *pebbling number* of a connected graph, which we denote $f(G)$, as follows: $f(G)$ is the minimum number of pebbles such that from any distribution of $f(G)$ pebbles on the vertices of G, any designated vertex can receive one pebble after a finite number of pebbling moves.

This paper explores the pebbling number of the Cartesian product $C_5 \times C_5$. The idea for Cartesian products comes from a conjecture of Graham's [2]. The conjecture states that for any graphs, G and H, $f(G \times H) \leqslant f(G)f(H)$. Moews [5] confirms this conjecture for trees. Snevily and Foster [7] generalize Moews's result to the case when G is a tree, and H satisfies an additional property called the *two-pebbling property*. They also prove that it holds whenever G is an even cycle and H satisfies the

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two-pebbling property, and whenever G and H are both odd cycles, and one of them has at least 15 vertices. Chung also mentions that the pebbling number of $C_5 \times C_5 \times \cdots \times C_5$ is not known.

We were particularly intrigued by C_5 since it is the largest cycle whose pebbling number is equal to its number of vertices. A graph whose pebbling number equals the number of its vertices was termed *demonic* by Jessup [4]. In this paper, we show that $C_5 \times C_5$ is demonic. That is, the pebbling number of $C_5 \times C_5$ is 25, satisfying Graham's conjecture.

This paper is organized as follows. In Section 2, we present some of the basic notions and results of the pebbling number, and of Cartesian products of graphs. In Section 3, we prove our main result that $C_5 \times C_5$ is demonic. Finally, in Section 4, we briefly mention possibilities for further research.

2. Preliminaries

2.1. Pebbling and C_5

Definitions. A *pebbling* of a connected graph is a placement of pebbles on the vertices of the graph. A *pebbling move* consists of removing two pebbles from a vertex, throwing one pebble away, and moving the other pebble to an adjacent vertex. The *pebbling number of a vertex v in a graph G* is the smallest number $f(G,v)$ with the property that from every placement of $f(G,v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The *pebbling number of the graph G*, denoted $f(G)$, is the maximum $f(G,v)$ over all the vertices in G.

We state some immediate facts from [2] about $f(G)$.

1. $f(G) \geqslant |V(G)|$, where $V(G)$ is the number of vertices of G.
2. $f(G) \geqslant 2^D$, where D is the diameter of the graph G.
3. $f(K_n) = n$, where K_n is the complete graph on n vertices.
4. $f(P_n) = 2^{n-1}$, where P_n is the path on n vertices.

We say a graph is *demonic* if $f(G) = |V(G)|$. Two more useful concepts are the *t-pebbling number* of a graph, and the *two-pebbling property*. We now define these terms.

Definition. The *t-pebbling number of a vertex v in a graph G* is the smallest number $f_t(G,v)$ with the property that from every placement of $f_t(G,v)$ pebbles on G, it is possible to move t pebbles to v by a sequence of pebbling moves. The *t-pebbling number of the graph G*, denoted $f_t(G)$, is the maximum $f_t(G,v)$ over all the vertices in G.

Definition. Suppose p pebbles are placed on a graph G in such a way that q vertices of G are occupied, i.e. there are exactly q vertices which have one pebble or more. We say the graph G *satisfies the two-pebbling property* if we can put two pebbles on any specified vertex of G starting from every configuration in which

$$p \geqslant 2f(G) - q + 1,$$

or equivalently,

$$\frac{p+q}{2} > f(G).$$

Relating these ideas to C_5, we have the following propositions:

Proposition 2.1. *The pebbling number of the 5-cycle C_5 is $f(C_5)=5$.*

Proposition 2.2. *The t-pebbling number of C_5 is $f_t(C_5)=4t+1$.*

Proposition 2.3. *C_5 satisfies the two-pebbling property.*

We prove Proposition 2.2. Proving Propositions 2.1 and 2.3 involves examining various cases. We leave this to the reader.

Proof of Proposition 2.2. The proof is by induction on t, where the case $t=1$ is Proposition 2.1. For $t \geqslant 2$, if there are nine or more pebbles, we have at least four pebbles on one of the paths (x_1,x_2,x_3) or (x_5,x_4,x_3). Using four of the pebbles from one of these paths, we can put one pebble on x_3. Then, using the remaining $4(t-1)+1$ pebbles, we can put $t-1$ more pebbles on x_3. To show that $4t$ pebbles are not sufficient, put three pebbles on x_1 and $4t-3$ pebbles on x_5. It is then impossible to move t pebbles to x_3. □

2.2. Direct products

We now define the direct product of two graphs, and discuss some results on the pebbling number of such graphs.

Definition. If $G=(V_G,E_G)$ and $H=(V_H,E_H)$ are two graphs, the direct product of G and H is the graph, $G \times H$, whose vertex set is the Cartesian product

$$V_{G\times H} = V_G \times V_H = \{(x,y) : x \in V_G, y \in V_H\},$$

and whose edges are given by

$$E_{G\times H} = \{((x,y),(x',y')) : x=x' \text{ and } (y,y') \in E_H, \text{ or } (x,x') \in E_G \text{ and } y=y'\}.$$

We can depict $G \times H$ pictorially by drawing a copy of H at every vertex of G and connecting each vertex in one copy of H to the corresponding vertex in an adjacent copy of H. We write $\{x\} \times H$ (respectively, $G \times \{y\}$) for the subgraph of vertices whose projection onto V_G is the vertex x, (respectively, whose projection onto V_H is y). If the vertices of G are labeled x_i, then for any distribution of pebbles on $G \times H$,

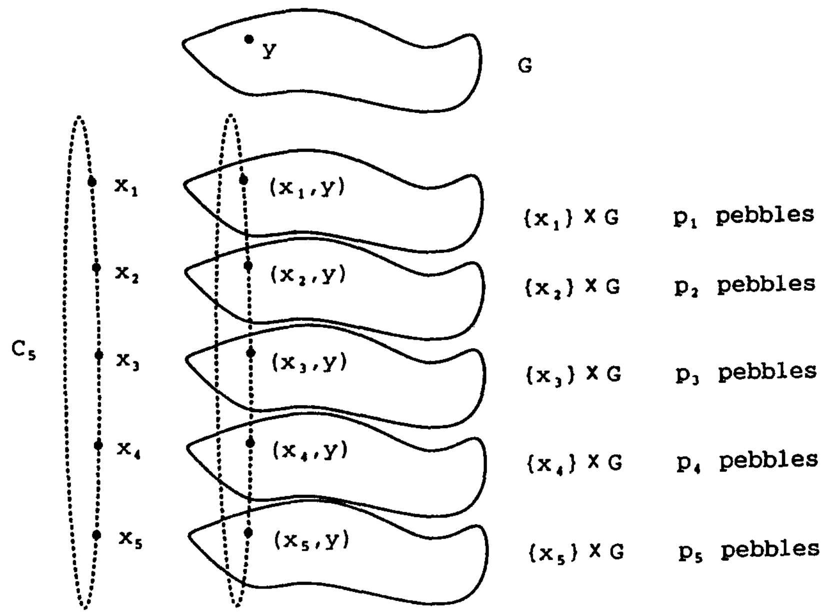

Fig. 1. $C_5 \times G$.

we write p_i for the number of pebbles on $\{x_i\} \times H$ and q_i for the number of occupied vertices of $\{x_i\} \times H$. For example, Fig. 1 shows the graph $C_5 \times G$ for an arbitrary graph G. Conjecture 2.4, by Graham, suggests a constraint on $f(G \times H)$. It would follow from our more general Conjecture 2.5.

Conjecture 2.4 (Graham). *The pebbling number of $G \times H$ satisfies $f(G \times H) \leqslant f(G)f(H)$.*

Conjecture 2.5. *The pebbling number of every vertex (v,w) in $G \times H$ satisfies $f(G \times H, (v,w)) \leqslant f(G,v)f(H,w)$.*

Lemma 2.6 describes how many pebbles we can transfer from one copy of H to an adjacent copy of H in $G \times H$. The proof is straightforward, and left to the reader. We use it to prove Propositions 2.7 and 2.8 — some simple, but useful cases of Conjectures 2.4 and 2.5. Proposition 2.7 is a special case of a result of Chung's [2], and Snevily and Foster [7] independently proved a more general version of Proposition 2.8.

Lemma 2.6. *Let (x_i,x_j) be an edge in G. Suppose that in $G \times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\} \times H$. If $q_i - 1 \leqslant k \leqslant p_i$, and if k and p_i have the same parity, then k pebbles can be retained on $\{x_i\} \times H$, while moving*

$$\frac{p_i - k}{2}$$

pebbles onto $\{x_j\} \times H$. If k and p_i have opposite parity, we must leave $k+1$ pebbles on $\{x_i\} \times H$, so we can only move

$$\frac{p_i - (k+1)}{2}$$

pebbles onto $\{x_j\} \times H$. In particular, we can always move at least

$$\frac{p_i - q_i}{2}$$

pebbles onto $\{x_j\} \times H$.

Proposition 2.7 (Chung [2]). *Let K_2 be the complete graph on two vertices x_1 and x_2, and suppose G satisfies the two-pebbling property. Then $f(K_2 \times G) \leqslant 2f(G)$. In particular, $f(K_2 \times C_5) \leqslant 10$.*

Proof. Without loss of generality, assume the target is (x_1, y) for some y. If

$$p_1 + \frac{p_2 - q_2}{2} \geqslant f(G),$$

we can use Lemma 2.6 to put $f(G)$ pebbles on $\{x_1\} \times G$. Since this subgraph is isomorphic to G, we can then put a pebble on (x_1, y). Also, since G satisfies the two-pebbling property, if

$$\frac{p_2 + q_2}{2} > f(G),$$

we can put two pebbles on (x_2, y), and then use a pebbling move to pebble (x_1, y). Hence, the only distributions from which we cannot pebble the target satisfy the inequalities

$$p_1 + \frac{p_2 - q_2}{2} < f(G),$$
$$\frac{p_2 + q_2}{2} \leqslant f(G).$$

But adding these together shows that $p_1 + p_2 < 2f(G)$ in any configuration from which we cannot pebble some target. □

Proposition 2.8. *Suppose G satisfies the two pebbling property, and consider the graph $P_3 \times G$. To pebble a target vertex on the middle copy of G, it suffices to start with $3f(G)$ pebbles on $P_3 \times G$. In particular, 15 pebbles are sufficient to pebble any vertex on the middle cycle of $P_3 \times C_5$.*

Proof. Label the vertices of P_3 by x_1, x_2 and x_3 in order. The target in $P_3 \times G$ is then (x_2, y). Since G has the two pebbling property, we can put two pebbles on (x_1, y) unless

$$\frac{p_1 + q_1}{2} \leqslant f(G).$$

By Lemma 2.6 and Proposition 2.7, we can pebble (x_2, y) directly by transferring pebbles from $\{x_1\} \times G$ unless

$$\frac{p_1 - q_1}{2} + p_2 + p_3 < 2f(G).$$

But if both these inequalities hold, then adding them together gives

$$p_1 + p_2 + p_3 < 3f(G).$$

Thus, any distribution of pebbles from which we cannot reach some vertex on the middle copy of G must begin with fewer than $3f(G)$ pebbles. □

Note $f(K_2) = 2$ and $f(P_3, x_2) = 3$, so Propositions 2.7 and 2.8 are consistent with Conjecture 2.5. In Section 3, we show that Conjectures 2.4 and 2.5 hold for $G = H = C_5$.

3. Pebbling $C_5 \times C_5$

In this section, we show that Conjectures 2.4 and 2.5 apply to the graph $C_5 \times C_5$. Fig. 2(a) shows this graph. Fig. 2(b) is another representation of $C_5 \times C_5$, but for clarity, we have eliminated from the drawing edges connecting the leftmost vertices to the rightmost vertices, and edges connecting the topmost and bottommost vertices. Thus, we represent $C_5 \times C_5$ by a 5 by 5 grid. By analogy to Fig. 2(b), we refer to $\{x_i\} \times C_5$ as the ith *row of* $C_5 \times C_5$. Similarly, we call $C_5 \times \{x_i\}$ the ith *column of* $C_5 \times C_5$. We also use a 5-tuple $[r_1, r_2, r_3, r_4, r_5]$ where each r_i denotes the number of pebbles on the ith row of $C_5 \times C_5$. In this 5-tuple, we use r_i+ to signify a configuration with *at least* r_i pebbles on the ith row, and we use an asterisk $(*)$ in place of r_i when we are indifferent as to the number of pebbles on the ith row. Finally, without loss of generality, we assume the target vertex on $C_5 \times C_5$ is (x_3, x_3).

Theorem 3.1. *$C_5 \times C_5$ is demonic, i.e. $f(C_5 \times C_5) = 25$.*

Proof. The proof involves several, progressively harder steps. We break down the possible configurations of 25 pebbles on $C_5 \times C_5$ according to the distribution of pebbles on $C_5 \times \{x_3\}$. The steps are as follows:
1. There are four or more pebbles on $C_5 \times \{x_3\}$.
2. There is a pebble on a vertex adjacent to (x_3, x_3).
3. At least two vertices of $C_5 \times \{x_3\}$ are occupied.
4. There are at least two pebbles on $C_5 \times \{x_3\}$.
5. There is one pebble on $C_5 \times \{x_3\}$.
6. There are no pebbles on $C_5 \times \{x_3\}$.

Step 1: Suppose there are at least four pebbles on $C_5 \times \{x_3\}$. If we can get five pebbles onto this subgraph, we can pebble (x_3, x_3), since $C_5 \times \{x_3\}$ is isomorphic to C_5, and $f(C_5) = 5$. If there are four pebbles on $C_5 \times \{x_3\}$, paint them red, and paint the

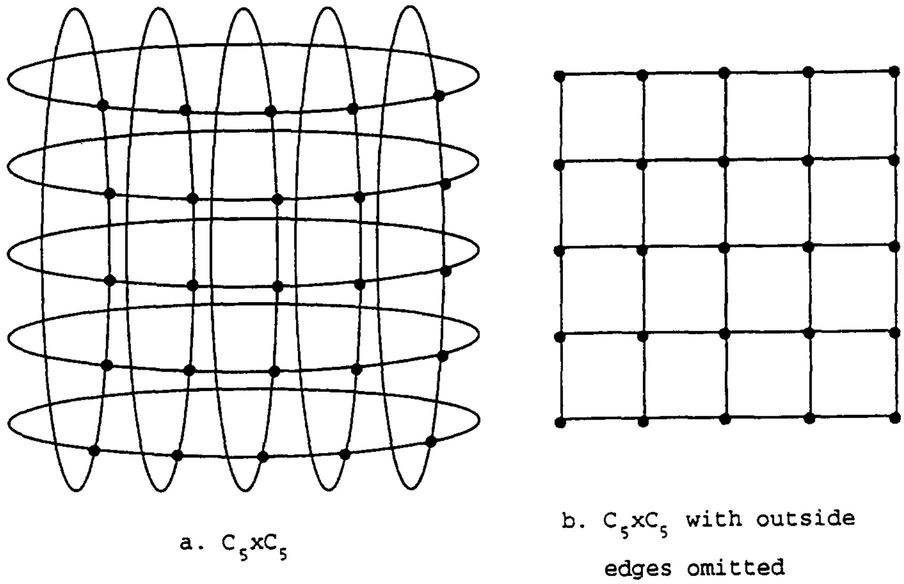

Fig. 2. Two representations of $C_5 \times C_5$.

remaining 21 pebbles black. By the pigeonhole principle, there is some $\{x_i\} \times C_5$ with at least five black pebbles. Hence, we can move a black pebble to (x_i,x_3). We now have four red pebbles and one black pebble on $C_5 \times \{x_3\}$, so we can pebble (x_3,x_3).

Step 2: Suppose a vertex adjacent to (x_3,x_3) is occupied. Without loss of generality, assume that vertex is (x_2,x_3). Now the first two rows of $C_5 \times C_5$ form a subgraph isomorphic to $K_2 \times C_5$. Suppose we can put 10 pebbles on these rows in addition to the one already on (x_2,x_3). Using these pebbles, we could move a second pebble to (x_2,x_3), and then move a pebble to the target. Also, if we can put 10 pebbles on the third and fourth rows, we can pebble the target. If we use the pebbles on the fifth row, then by Lemma 2.6 we can put either

$$p_1 + p_2 + \frac{p_5 - q_5}{2}$$

pebbles on the first two rows, or

$$p_3 + p_4 + \frac{p_5 - q_5}{2}$$

pebbles on the third and fourth rows. Therefore, if we cannot pebble (x_3,x_3), then both the inequalities

$$p_1 + p_2 + \frac{p_5 - q_5}{2} \leqslant 10,$$

$$p_3 + p_4 + \frac{p_5 - q_5}{2} \leqslant 9$$

hold. Adding these together gives

$$p_1 + p_2 + p_3 + p_4 + p_5 - q_5 \leqslant 19, \tag{1}$$

in any configuration from which we cannot pebble the target. Since $q_5 \leqslant 5$, such a configuration has at most 24 pebbles.

Step 3: Suppose we have a configuration of 25 pebbles on $C_5 \times C_5$, and we cannot pebble (x_3,x_3). From Step 2, only (x_1,x_3) and (x_5,x_3) can be occupied on $C_5 \times \{x_3\}$. We now show that they cannot both be occupied. To do this, we use the notion of a *transmitting subgraph*, which we now define.

Definition. Given a pebbling of G, a *transmitting subgraph* of G is a path $x_0,x_1,\dots,x_k$ such that there are at least two pebbles on x_0 and at least one pebble on each of the other vertices in the path, except possibly x_k. In this case, we can *transmit* a pebble from x_0 to x_k.

Now, suppose both (x_1,x_3) and (x_5,x_3) are occupied. By Proposition 2.8, if 15 pebbles are on the middle three rows of $C_5 \times C_5$, we can pebble the target. Otherwise, we have $p_1 + p_5 \geqslant 11$, so without loss of generality, $p_1 \geqslant 6$.

We make two observations. First, if we could put a pebble on (x_2,x_3) and retain six pebbles on the first row, we could use the pebble already on (x_1,x_3) and the remaining five pebbles on the first row to get two pebbles to (x_1,x_3). Then the vertices $\{(x_1,x_3),(x_2,x_3),(x_3,x_3)\}$ would form a transmitting subgraph. Alternatively, we could try to move a pebble on (x_4,x_3) while retaining six pebbles on the first row and the pebble on (x_5,x_3). In this case, we could again move a second pebble onto (x_1,x_3), and now the vertices $\{(x_1,x_3),(x_5,x_3),(x_4,x_3),(x_3,x_3)\}$ would form a transmitting subgraph.

From Lemma 2.6, we can keep six pebbles on the first row and still transfer $(p_1 - 7)/2$ to another row. We can achieve the first goal (a pebble on (x_2,x_3)) by putting five pebbles on the second row. We can achieve the second (a pebble on (x_4,x_3)) by putting 11 pebbles on the fourth and fifth rows; for in that case, we have one already on (x_5,x_3) and we can apply Proposition 2.7 to the remaining 10 pebbles with (x_4,x_3) as the target. If we cannot reach either objective, then both of the following inequalities hold.

$$\frac{p_1 - 7}{2} + p_2 \leqslant 4,$$

$$\frac{p_1 - 7}{2} + p_5 + p_4 \leqslant 10.$$

Adding these inequalities together, we have

$$p_1 + p_2 + p_4 + p_5 \leqslant 21,$$

or $p_3 \geqslant 4$. In this case, we apply Step 1 to $\{x_3\} \times C_5$ instead of $C_5 \times \{x_3\}$.

Step 4: Suppose we start with a configuration of 25 pebbles with at least two pebbles on $C_5 \times \{x_3\}$. Using Steps 2 and 3, we can put one pebble on the target unless all these pebbles are on one of (x_1,x_3) or (x_5,x_3). Without loss of generality, we assume (x_5,x_3) is not occupied. We make a pebbling move from (x_1,x_3) to (x_2,x_3). From the logic of Step 2, we can pebble the target unless (1) holds for this new configuration. However, since (x_5,x_3) is not occupied, we have $q_5 \leqslant 4$. Therefore, there could be at most 23 pebbles in this new configuration. But since we made only one pebbling move from a starting configuration with 25 pebbles, this is not possible.

Step 5: Now, suppose there is a pebble on (x_1,x_3), and that $C_5 \times \{x_3\}$ is otherwise unoccupied. We note that if $p_3 \geqslant 2$, we can pebble (x_3,x_3) using Step 4 on $\{x_3\} \times C_5$. Hence, we assume $p_3 \leqslant 1$, so

$$p_1 + p_2 + p_4 + p_5 \geqslant 24. \tag{2}$$

Now, if we can get 15 pebbles on rows 2, 3, and 4 of C_5, we are done by Proposition 2.8. Therefore, we assume that the transfer of pebbles from row 1 to row 2 and from row 5 to row 4 gives

$$\frac{p_1 - 5}{2} + p_2 + p_3 + p_4 + \frac{p_5 - 4}{2} \leqslant 14.$$

Multiplying this inequality by 2 and using the equation

$$p_1 + p_2 + p_3 + p_4 + p_5 = 25,$$

we find

$$p_1 + p_5 \geqslant 13. \tag{3}$$

Case 1: Suppose $p_1 \geqslant 6$. If we could put five pebbles on the second row while retaining six pebbles on the first row, we could create a transmitting subgraph $\{(x_1,x_3),$ $(x_2,x_3),$ $(x_3,x_3)\}$. Toward this end, we can transfer $(p_5 - 4)/2$ pebbles from the fifth row to the first, and then transfer pebbles from the first row to the second. Thus, if we cannot move five pebbles to the second row while retaining six pebbles on the first row, we must have

$$\frac{((p_5 - 4)/2 + p_1) - 7}{2} + p_2 \leqslant 4.$$

Multiplying this inequality by 4 yields

$$p_5 + 2p_1 + 4p_2 \leqslant 34. \tag{4}$$

If we can get a total of nine pebbles to $\{x_4\} \times C_5$, we can put two pebbles on (x_4,x_3) and then make a pebbling move from there to (x_3,x_3). We would attempt to get nine pebbles to $\{x_4\} \times C_5$ by first transferring $(p_1 - 5)/2$ pebbles from the first row to the fifth row, and then transferring as many pebbles as possible from the fifth row to the fourth row. If we cannot end up with nine pebbles on the fourth row, then

$$\frac{((p_1 - 5)/2 + p_5) - 4}{2} + p_4 \leqslant 8.$$

Multiplying this inequality by 4 gives

$$p_1 + 2p_5 + 4p_4 \leqslant 45. \tag{5}$$

Now, suppose $p_4 \geqslant 5$. In this case, we can achieve a distribution of the form $[6+, *, *, 5+, 5+]$ unless

$$\frac{p_1 - 7}{2} + p_5 + \frac{p_4 - 6}{2} \leqslant 4.$$

Simplifying this inequality, we see that we could pebble (x_3, x_3) unless

$$p_1 + 2p_5 + p_4 \leqslant 21. \tag{6}$$

However, adding together (4)–(6) gives

$$4p_1 + 4p_2 + 5p_4 + 5p_5 \leqslant 100.$$

This is impossible if $p_4 \geqslant 5$ and (2) holds.

Thus, assume $p_4 \leqslant 4$. Now, we can achieve the distribution $[6+, *, *, 5+, 5+]$ unless

$$\frac{((p_1 - 7)/2 + p_5) - 6}{2} + p_4 \leqslant 4,$$

which becomes

$$p_1 + 2p_5 + 4p_4 \leqslant 35. \tag{7}$$

But adding together (4) and (7) gives

$$3p_1 + 4p_2 + 4p_4 + 3p_5 \leqslant 69,$$

and this contradicts (2).

Case 2: Suppose $p_1 \leqslant 5$. Then $p_5 \geqslant 8$ from (3). If $p_5 = 8$, we must have $p_1 = 5$ and either $p_2 \geqslant 5$ or $p_4 \geqslant 7$. If $p_2 \geqslant 5$, we can put a pebble on (x_2, x_3). We could then move a pebble from the fifth row to the first row. With six pebbles on the first row, including one already on (x_1, x_3), we can put a second pebble on that vertex, thereby creating a transmitting subgraph $\{(x_1, x_3), (x_2, x_3), (x_3, x_3)\}$. Alternatively, if $p_4 \geqslant 7$ we could move two pebbles from the fifth row to the fourth row. We could then put two pebbles on (x_4, x_3), and from there, we could pebble (x_3, x_3).

If $p_5 \geqslant 9$, we can put two pebbles on (x_5, x_3), by Proposition 2.2. Now, we could create a transmitting subgraph either by putting an additional pebble either on (x_2, x_3) or on (x_4, x_3). If neither of these transmitting subgraphs can be created, we must have $p_2 \leqslant 4$ and $p_4 \leqslant 4$. Therefore, we can pebble the target unless the inequalities

$$\frac{((p_5 - 4)/2 + p_1) - 7}{2} + p_2 \leqslant 4, \tag{8}$$

$$\frac{p_5 - 10}{2} + p_4 \leqslant 4 \tag{9}$$

hold, since if (8) fails, we can arrange a configuration of the form [6+, 5+, *, *, *], including one pebble already on (x_1, x_3), and if (9) fails, we can arrange the configuration [*, *, *, 5+, 9+]. In either case, we can pebble (x_3, x_3).

Multiplying (8) and (9) by 4 and adding them together with our assumption that $p_1 \leqslant 5$ gives

$$3p_1 + 4p_2 + 4p_4 + 3p_5 \leqslant 75.$$

Subtracting (2), we find $p_2 + p_4 \leqslant 3$, and $p_1 + p_5 \geqslant 21$. But $p_1 \leqslant 5$, so $p_5 \geqslant 16$. If $p_5 \geqslant 17 = f_4(C_5)$, we can move four pebbles to (x_5, x_3), and if $p_5 = 16$ and $p_1 = 5$, we can move one pebble from the fifth row to the first row. Then we can put three pebbles on (x_5, x_3) (since $f_3(C_5) = 13$) and two pebbles on (x_1, x_3).

Step 6: Finally, suppose $C_5 \times \{x_3\}$ is unoccupied. Grilliot [3] outlined one argument for this case; we present a different one here.

From Step 5, we assume $p_3 = 0$. By Proposition 2.8, 15 pebbles on rows 2, 3, and 4 would be sufficient to pebble the target. Thus, we assume

$$\frac{p_1 - 4}{2} + p_2 + p_4 + \frac{p_5 - 4}{2} \leqslant 14.$$

Multiplying by two and using the equation

$$p_1 + p_2 + p_4 + p_5 = 25,$$

we find

$$p_1 + p_5 \geqslant 14. \tag{10}$$

We can also pebble the target if we can get nine pebbles on the second (or fourth) row. Thus, we assume

$$\frac{((p_5 - 4)/2 + p_1) - 4}{2} + p_2 \leqslant 8,$$

$$\frac{((p_1 - 4)/2 + p_5) - 4}{2} + p_4 \leqslant 8,$$

or

$$p_5 + 2p_1 + 4p_2 \leqslant 44, \tag{11}$$

$$p_1 + 2p_5 + 4p_4 \leqslant 44. \tag{12}$$

Without loss of generality, we also assume that

$$p_1 + p_2 \geqslant 13. \tag{13}$$

Subtracting (13) and (10) from (11) gives us $3p_2 \leqslant 17$. Hence, $p_2 \leqslant 5$ and $p_1 \geqslant 8$.

Furthermore, if $p_1 = 8$, then $p_2 = 5$ and $p_5 \geqslant 6$ (by virtue of (10)). In this case, we could move a ninth pebble onto the first row from the fifth row, and then create a transmitting subgraph $\{(x_1, x_3), (x_2, x_3), (x_3, x_3)\}$. Hence, we assume $p_1 \geqslant 9$. In this case, we could arrange nine pebbles on the first row and five pebbles on the second row unless

$$\frac{((p_5-4)/2)+p_1-10}{2}+p_2\leqslant 4,$$

or

$$p_5+2p_1+4p_2\leqslant 40. \tag{14}$$

We now aim for the pattern [9+, *, 0, 5+, 5+]. We consider two cases; either $p_4\geqslant 5$ or $p_4\leqslant 4$. If $p_4\geqslant 5$, we can pebble the target unless (12) and both of the following inequalities hold:

$$\frac{p_1-10}{2}+p_2\leqslant 4,$$

$$\frac{p_1-10}{2}+\frac{p_4-6}{2}+p_5\leqslant 4,$$

for if (12) fails, we can get nine pebbles on the fourth row, and if one of the other inequalities fails, we can arrange either the pattern [9+, 5+, 0, *, *] or [9+, *, 0, 5+, 5+]. However, some algebra shows that these inequalities cannot hold simultaneously if $p_4\geqslant 5$; we get

$$p_1+2p_5+4p_4\leqslant 44,$$

$$2p_1+4p_2\leqslant 36,$$

$$p_1+p_4+2p_5\leqslant 24,$$

and adding these together gives

$$4p_1+4p_2+5p_4+4p_5\leqslant 104,$$

or $p_4\leqslant 4$, contradicting our original assumption.

If $p_4\leqslant 4$, we can arrange the pattern [9+, *, 0, 5+, 5+] unless

$$\frac{((p_1-10)/2+p_5)-6}{2}+p_4\leqslant 4,$$

or

$$p_1+2p_5+4p_4\leqslant 38. \tag{15}$$

Now, suppose (14) and (15) both represent equalities, i.e.

$$p_5+2p_1+4p_2=40, \tag{16}$$

$$p_1+2p_5+4p_4=38. \tag{17}$$

Adding these gives

$$3p_1+4p_2+4p_4+3p_5=78,$$

which implies $p_2+p_4=3$, and $p_1+p_5=22$. The only choice for $[p_1, p_2, 0, p_4, p_5]$ which satisfies (16) and (17) and has $p_1<17$ and $p_1+p_2\geqslant 13$ is [14, 1, 0, 2, 8]. In this case, we use the two pebbling property of C_5. If the eight pebbles on the fifth row are confined to one or two vertices of that row, we can transfer three pebbles to the first row, and seventeen pebbles are sufficient to 4-pebble any vertex. Otherwise, we can 2-pebble (x_5,x_3) and 3-pebble (x_1,x_3), which puts five pebbles on the target C_5.

If equality either does not hold in (14) or does not hold in (15), we have

$$3p_1 + 4p_2 + 4p_4 + 3p_5 < 78,$$

so $p_2 + p_4 \leqslant 2$ and $p_1 + p_5 \geqslant 23$. If we relax the restriction that $p_1 + p_2 \geqslant 13$, we may assume that $p_1 \geqslant 12$. If $p_1 = 12$ and $p_5 \geqslant 11$, we can transfer a pebble from the fifth row to the first row, and then we can 3-pebble (x_1,x_3) and 2-pebble (x_5,x_3). Otherwise, we have $p_1 \geqslant 13$, so if we paint 13 pebbles on the first row red, and the remaining ten (or more) pebbles on the first and fifth rows black, we can put three red pebbles and one black pebble on (x_1,x_3) (since the first and fifth rows form a subgraph isomorphic to $K_2 \times C_5$). With four pebbles on (x_1,x_3), we can then pebble (x_3,x_3). $\square$

4. Open problems

Our results apply to $C_5 \times C_5$. Chung's question on the pebbling number of $C_5 \times C_5 \times \cdots \times C_5$ remains unanswered. Even $f(C_5 \times C_5 \times C_5)$ is unknown, and while we have found $f(C_5 \times C_5)$, we have not shown whether it satisfies the two-pebbling property.

There are also variants of pebbling problems. Arquila and Fredricksen [1] have asked about the fewest pebbles for which *some* configuration (instead of *every* configuration) would allow every vertex to be reached. Pachter et al. [6] independently discuss this variant; they call it *optimal pebbling.* Pachter et al. also consider a continuous analog of pebbling, using sand instead of pebbles.

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