Another way of thinking about this is through a calculus lens (the argument is not too dissimilar from above). For simplicity assume that S is bounded. Pick any line through S and parametrize it by the variable t. Then the objective function becomes a linear function of t as well, and because S is bounded we maximize the function $z = z(t)$ over the line segment L defined by the interval $t \in [\alpha, \beta]$. Bounded functions have maxima only at critical points and endpoints, and because continuous linear functions rule out nondifferentiable points, the maximum of z on L occurs at an endpoint (even in the case of stationary points). Such an endpoint is a boundary point of S. Thus we may assume that an optimal point is never an interior point of any line segment in S; i.e. an extreme point. We leave it to the reader to fill in more careful and general details.

From Figure 2.1, then, it is clear that $z^* = 24$ at $\mathbf{x}^* = (15, 9)^\mathsf{T}$. We will use the notation $\mathbf{x}^*$ in the general case to indicate the point (or one of the points) where z^* occurs. What we have discussed implicitly is a method which reduces the search for the "best" point from an (uncountably) infinite set to a search from a finite set of extreme points.

Workout 2.1.5 *Why is the set of extreme points of a polytope finite?*

If we have in hand the list of all extreme points of S, then we simply can check through them all to see which yields a maximum. Unfortunately, that list might be rather large (typically exponentially large, in terms of n), and so we may not have time to check them all. (In practical terms, even with 50 variables, never mind the tens of thousands or more encountered in common applications, we don't have time!) In addition, how is one to compute all the extreme points of S? (See Section 3.1.) Thus the geometric discussion doesn't as yet give us a real method for solving a linear problem, although one hopes you'll agree that it gives us plenty of insight into the nature of its solutions. The interplay between the algebraic Simplex algorithm (hang on, it's coming) and its geometric underpinnings would make Descartes both excited and proud.

Problem 2.1.6

$$
\begin{array}{llcrcrcr}
\text{Max.} & z & = & 226x_1 & + & 219x_2 & & \\
\\
\text{s.t.} & & & 197x_1 & + & 185x_2 & \leq & 9,650 \\
 & & & 202x_1 & + & 178x_2 & \leq & 9,595 \\
 & & & 186x_1 & + & 190x_2 & \leq & 9,502 \\
 & & & 191x_1 & + & 196x_2 & \leq & 9,781 \\
 & & & 177x_1 & + & 205x_2 & \leq & 9,661 \\
\\
\& & & & x_1 & , & x_2 & \geq & 0
\end{array}
$$

Workout 2.1.7 *Consider Problem 2.1.6. Draw its corresponding feasible region and use your drawing to find its maximum. [HINT: Would a* MAPLE *plot help?]*

Problem 2.1.8

$$
\begin{array}{llcrcrcrcrcr}
\text{Max.} & z & = & 2x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & & \\
\\
\text{s.t.} & & & x_1 & + & 6x_2 & + & 5x_3 & + & 3x_4 & \leq & 85 \\
 & & & 4x_1 & + & 2x_2 & + & 6x_3 & + & x_4 & \leq & 72 \\
 & & & 7x_1 & + & 4x_2 & + & x_3 & + & 4x_4 & \leq & 91 \\
 & & & 3x_1 & + & x_2 & + & 5x_3 & + & 6x_4 & \leq & 83 \\
\\
\& & & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Workout 2.1.9 *Consider Problem 2.1.8. Draw its corresponding feasible region and use your drawing to find its maximum.*

[MORAL: We need algebraic methods to solve linear problems.]

2.2 Algebraic Lens

We begin by rewriting the problem constraints of Problem 1.3.1 as equalities by introducing a new slack variable for each constraint. We call them slack variables because they pick up the slack, so to speak; that is, they take on whatever values are necessary to create equalities (notice that each slack variable must be nonnegative). In fact, it is a bit handier if we solve for each slack variable. Then Problem 1.3.1 can be written as follows.

Dictionary 2.2.1

$$\begin{array}{llrrrrrrl}
\text{Max.} & z & = & 0 & + & x_1 & + & x_2 & \\
\\
\text{s.t.} & x_3 & = & 90 & - & 3x_1 & - & 5x_2 & \\
 & x_4 & = & 180 & - & 9x_1 & - & 5x_2 & \\
 & x_5 & = & 15 & & & - & x_2 & \\
\\
\& & x_j & \geq & 0 & & & & & 1 \leq j \leq 5
\end{array}$$

dictionary

We call this formulation a **dictionary**, and it has the property that the set of all variables is split in two, those on the left side (ignoring the nonnegativity constraints) appearing exactly once. Other than the objective variable z, the variables on the left hand side of a given dictionary will be called **basic** (the set of which is called the **basis**), and those on the right will be called **nonbasic** (also referred to as **parameters**). One should notice that the initial basis of a LOP in standard form is always the set of slack variables. As we will see, further bases will be some mixture of problem and slack variables.

(non)basic variable

basis

parameter

tableau

Another way of recording the same information is in a **tableau**, which is obtained from a dictionary by first rewriting each equation with all its variables on the left hand side, constants on the right, and then writing the augmented matrix of coefficients of that system. By convention, the row for the objective function is written last. Thus, Problem 1.3.1 has the following tableau, corresponding to the above dictionary.

Tableau 2.2.2

$$\left[\begin{array}{rr|rrrr|r}
3 & 5 & 1 & 0 & 0 & 0 & 90 \\
9 & 5 & 0 & 1 & 0 & 0 & 180 \\
0 & 1 & 0 & 0 & 1 & 0 & 15 \\
\hline
-1 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

Notice that we have ordered the columns in increasing fashion according to subscripts, with z coming last. We have included the dividing lines only as a visual aid to separate problem variables from slack, constraints from objective function, and left sides of the equations from right. The far right column will be referred to as the **b-column**, and the bottom row is called the **objective row**. One can typically spot the basic variables easily in a tableau by finding the simple columns (there are degenerate examples): the columns of the basis form a permutation of the columns of an identity matrix (or a positive multiple of one, as we will see later).

b-column

objective row

Workout 2.2.3 *Consider the linear problem P from Exercise 1.5.10.*

(a) *Write the initial dictionary for P.*

(b) *Write the initial tableau for P.*

By a **solution** we will mean any set of values of the x_j that satisfy the problem constraints when written as equalities. Thus we can refer to either feasible or infeasible solutions, feasible being the case in which the nonnegativity constraints also hold. A solution is **basic** if it corresponds to the values one gets from a dictionary by setting all the parameters to zero. For example, Dictionary 2.2.1 yields the basic (feasible) solution $\mathbf{x} = (0, 0 \mid 90, 180, 15)^{\mathsf{T}}$, with value $z = 0$ (again, the divider distinguishes decision variables from slack). (basic) solution

Workout 2.2.4 *Consider the linear problem P from Exercise 1.5.10.*

(a) *Find an infeasible basic solution to P.*

(b) *Find a feasible nonbasic solution to P.*

When we need to discuss the distinction between basic and nonbasic variables, we will use the shorthand notations β for the set of subscripts of basic variables and π for the set of subscripts of parameters. For example, Dictionary 2.2.1 has $\beta = \{3, 4, 5\}$ and $\pi = \{1, 2\}$. We call a dictionary or tableau **feasible** if its corresponding basic solution is feasible (and **infeasible** otherwise). We say a linear problem is **feasible** if it has a feasible tableau (and **infeasible** otherwise). Whenever a tableau is infeasible we will say that we are in **Phase I** of the Simplex algorithm; **Phase II** if the tableau is feasible. A tableau or dictionary is **optimal** if the corresponding basic solution is optimal. Obviously, the Simplex algorithm will halt at this stage (except for the chance of degeneracy — see Exercise 1.5.4 and Section 2.8).

(in)feasible dictionary/ tableau/problem

Phase I/II

optimal dictionary/ tableau

One might notice from Dictionary 2.2.1 that an increase in x_1 from its basic value of zero would bring about a corresponding increase in the value of z. But since changes in x_3 and x_4 also would occur, we must be careful not to increase x_1 too much. If x_2 is held at zero, then because x_3 and x_4 must remain nonnegative we obtain the following restrictions on x_1.

$$90 - 3x_1 \geq 0 \quad \text{and} \quad 180 - 9x_1 \geq 0\ .$$

The second restriction is the strongest, requiring $x_1 \leq 20$.

Thus, we might increase x_1 all the way up to 20, thereby decreasing x_4 all the way to 0. That has the ring of making x_1 basic and x_4 nonbasic, so we may as well solve the second equation for x_1 and substitute the result into the remaining equations. This produces the new basis $\beta^{(1)} = \{1, 3, 5\}$, parameter set $\pi^{(1)} = \{2, 4\}$, and dictionary below. (The superscripted (1), in general (k), serves to indicate the values after the first, in general k^{th}, modification; thus superscript (0) will denote original information.)

Dictionary 2.2.5

$$\begin{array}{llcrcrcrl}
\text{Max.} & z & = & 20 & + & .444x_2 & - & .111x_4 & \\
\\
\text{s.t.} & x_3 & = & 30 & - & 3.333x_2 & + & .333x_4 & \\
 & x_1 & = & 20 & - & .555x_2 & - & .111x_4 & \\
 & x_5 & = & 15 & - & x_2 & & & \\
\\
\& & x_j & \geq & 0 & & & & & 1 \leq j \leq 5
\end{array}$$

Notice that we have rounded the fractions with denominator 9, so this dictionary is really only an approximation. We might rather maintain exactness by clearing the denominator and writing the following.

Dictionary 2.2.6

$$
\begin{array}{lrcrcrcr}
\text{Max.} & 9z & = & 180 & + & 4x_2 & - & x_4 \\
\\
\text{s.t.} & 9x_3 & = & 270 & - & 30x_2 & + & 3x_4 \\
 & 9x_1 & = & 180 & - & 5x_2 & - & x_4 \\
 & 9x_5 & = & 135 & - & 9x_2 & & \\
\\
\& & x_j & \geq & 0 & & & & \qquad 1 \leq j \leq 5
\end{array}
$$

pivot operation

basic coefficient

Here we have the basic solution $\mathbf{x}^{(1)} = (20, 0 \mid 30, 0, 15)^{\mathsf{T}}$ with $z^{(1)} = 20$. All of this, of course, corresponds to performing a *pivot operation* [2] (suitably modified to clear fractions) on the entry of 9 in row 2, column 1, of Tableau 2.2.2, resulting in Tableau 2.2.7. The notation we use to denote this is $1 \mapsto 4$, since x_1 replaces x_4 in the basis. The 9 is referred to as the **basic coefficient** since it is the coefficient of all the basic variables and z in the dictionary.

Tableau 2.2.7

$$
\left[\begin{array}{rr|rrrr|r}
0 & 30 & 9 & -3 & 0 & 0 & 270 \\
9 & 5 & 0 & 1 & 0 & 0 & 180 \\
0 & 9 & 0 & 0 & 9 & 0 & 135 \\
\hline
0 & -4 & 0 & 1 & 0 & 9 & 180
\end{array}\right]
$$

Workout 2.2.8 *Write the row operations that transformed Tableau 2.2.2 to Tableau 2.2.7.*

By similar analysis on Tableau 2.2.7, an increase in z is incurred by an increase in x_2, but only so far as $x_2 \leq \min\{270/30 = 9, 180/5 = 36, 135/9 = 15\}$, in order to maintain the feasibility of the next basic solution. Thus we arrive at Dictionary 2.2.9 and Tableau 2.2.10, with $\beta^{(2)} = \{1, 2, 5\}$, $\pi^{(2)} = \{3, 4\}$, $\mathbf{x}^{(2)} = (15, 9 \mid 0, 0, 6)^{\mathsf{T}}$, and $z^{(2)} = 24$.

Dictionary 2.2.9

$$
\begin{array}{lrcrcrcr}
\text{Max.} & 30z & = & 720 & - & 4x_3 & - & 2x_4 \\
\\
\text{s.t.} & 30x_3 & = & 270 & - & 9x_3 & + & 3x_4 \\
 & 30x_1 & = & 450 & + & 5x_3 & - & 5x_4 \\
 & 30x_5 & = & 135 & + & 9x_3 & - & 3x_4 \\
\\
\& & x_j & \geq & 0 & & & & \qquad 1 \leq j \leq 5
\end{array}
$$

Tableau 2.2.10

$$
\left[\begin{array}{rr|rrrr|r}
0 & 30 & 9 & -3 & 0 & 0 & 270 \\
30 & 0 & -5 & 5 & 0 & 0 & 450 \\
0 & 0 & -9 & 3 & 30 & 0 & 180 \\
\hline
0 & 0 & 4 & 2 & 0 & 30 & 720
\end{array}\right]
$$

[2] See Appendix A.