Chapter 4

The Duality Theorem

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4.1 Primal-Dual Relationship

Consider the following LOP.

Problem 4.1.1

$$\begin{array}{llrcrcrcr} \text{Max.} & z = & 22x_1 & + & 31x_2 & + & 29x_3 & & \\ \text{s.t.} & & x_1 & + & 4x_2 & + & 6x_3 & \leq & 73 \\ & & 5x_1 & - & 2x_2 & + & 3x_3 & \leq & 68 \\ \& & & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

As you are invited to confirm, the simplex algorithm produces the following final tableau.

Tableau 4.1.2

$$\left[\begin{array}{rrr|rrr|r} 0 & 22 & 27 & 5 & -1 & 0 & 297 \\ 22 & 0 & 24 & 2 & 4 & 0 & 418 \\ \hline 0 & 0 & 727 & 199 & 57 & 22 & 18403 \end{array}\right]$$

Tableau 4.1.2 shows the optimum solution $\mathbf{x}^* = (418, 297, 0 \mid 0, 0)^{\mathsf{T}}/22$ with corresponding optimum value $z^* = 18403/22$. Likewise, consider the dual LOP below.

Problem 4.1.3

$$\begin{array}{llrcrcr} \text{Min.} & w = & 73y_1 & + & 68y_2 & & \\ \text{s.t.} & & y_1 & + & 5y_2 & \geq & 22 \\ & & 4y_1 & - & 2y_2 & \geq & 31 \\ & & 6y_1 & + & 3y_2 & \geq & 29 \\ \& & & y_1 & , & y_2 & \geq & 0 \end{array}$$

Problem 4.1.3 has the following final tableau.

Tableau 4.1.4

$$\left[\begin{array}{rr|rrrr|r} 0 & 22 & -4 & 1 & 0 & 0 & 57 \\ 22 & 0 & -2 & -5 & 0 & 0 & 199 \\ 0 & 0 & -24 & -27 & 22 & 0 & 727 \\ \hline 0 & 0 & 418 & 297 & 0 & 22 & -18403 \end{array}\right]$$

Tableau 4.1.4 shows the optimum solution $y^* = (199, 57 \mid 0, 0, 727)/22$ with corresponding optimum value $w^* = 18403/22$.

Curiously, this data shows certain repetitions of values. It looks like the values of $\mathbf{x}^*$ show up in the final dual objective row, but switched around a little. Likewise, the values of $\mathbf{y}^*$ appear in the final primal objective row, with a similar swap of some sort. To be more precise, the pattern seems to be that the *problem* values of $\mathbf{x}^*$ are the final coefficients of the dual *slack* variables, while the *slack* values of $\mathbf{x}^*$ are the final coefficients of the dual *problem* variables.

Problem 4.1.5

$$\begin{array}{llrcrcrcrcrcrcr} \text{Max.} & z = & -22x_1 & - & 18x_2 & - & 27x_3 & - & 23x_4 & + & 16x_5 & - & 12x_6 & & \\ \text{s.t.} & & 4x_1 & + & x_2 & - & 3x_3 & & & + & 2x_5 & + & 7x_6 & \leq & 211 \\ & & 6x_1 & & & + & 2x_3 & + & 5x_4 & - & x_5 & + & 8x_6 & \leq & 189 \\ & & -5x_1 & + & 4x_2 & - & 2x_3 & & & - & x_5 & - & 7x_6 & \leq & -106 \\ & & 3x_1 & + & 9x_2 & & & - & 2x_4 & + & x_5 & + & 4x_6 & \leq & 175 \\ \& & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$

Workout 4.1.6 *Consider Problem 4.1.5.*

(a) *Use the Simplex algorithm to solve it.*

(b) *Without solving the dual linear problem, use the final primal tableau to find the optimal dual variable values* $\mathbf{y}^*$ *(including slacks).*

(c) *Verify that* $\mathbf{y}^*$ *is dual feasible and optimal.*

It will help to articulate this perceived pattern notationally. We return to the general descriptions of primals and duals below.

Problem 4.1.7

$$\begin{array}{llrcll} \text{Max.} & z = & \sum_{j=1}^n c_j x_j & & & (1) \\ \text{s.t.} & & \sum_{j=1}^n a_{ij} x_j \leq & b_i & (1 \leq i \leq m) & (2) \\ \& & & x_j \geq & 0 & (1 \leq j \leq n) & (3) \end{array}$$

Problem 4.1.8

$$\begin{array}{llrcll} \text{Min.} & w = & \sum_{i=1}^m b_i y_i & & & (4) \\ \text{s.t.} & & \sum_{i=1}^m a_{ij} y_i \geq & c_j & (1 \leq j \leq n) & (5) \\ \& & & y_i \geq & 0 & (1 \leq i \leq m) & (6) \end{array}$$

In order to describe the pattern we will need to look at the optimal primal objective row. Just as with all optimal values, let's use c_k^* for the final coefficient of x_k, as shown.

$$\left[\begin{array}{ccccc|ccccc|c} \hline c_1^* & \cdots & c_j^* & \cdots & c_n^* & c_{n+1}^* & \cdots & c_{n+i}^* & \cdots & c_{n+m}^* \; 1 & z^* \end{array}\right]$$

Notice that there is a 1 written instead of the more general d for the coefficient of z. Why did we do that?

It seems as though the pattern we have witnessed is given below.

$$\left[\begin{array}{ccccc|ccccc|c} \hline y_{m+1}^* & \cdots & y_{m+j}^* & \cdots & y_{m+n}^* & y_1^* & \cdots & y_i^* & \cdots & y_m^* \; 1 & z^* \end{array}\right]$$

That is,

$$y_i^* = c_{n+i}^* \quad (1 \leq i \leq m) \qquad \text{and} \qquad y_{m+j}^* = c_j^* \quad (1 \leq j \leq n)\ . \tag{7}$$

Furthermore, we have also noticed time and again that, when the primal has an optimal solution, then so does its dual — in fact, with the same optimal value. Quite possibly, we could take advantage of the detailed pattern above to verify such a phenomenon in general. From *(1)* the value in question is

$$z^* \;=\; \sum_{j=1}^{n} c_j x_j^*\ . \tag{8}$$

From the Weak Duality Theorem (Inequality 1.4.7) we know that if the dual problem is feasible then its optimum is at most this value z^*. In fact we get equality.

Theorem 4.1.9 *If a linear problem P has an optimum z^* then its dual linear problem D has an optimum w^*; moreover, $z^* = w^*$.* Strong Duality Theorem

Proof. Because we already know that every feasible z and w satisfy $z \leq w$, we only need to find a feasible w for which $w = z$. For this we can turn to the y_i^*s defined in *(7)*, and show that they satisfy inequalities *(5)* and *(6)* as well as the equality

$$z^* \;=\; \sum_{i=1}^{m} b_i y_i^*\ . \tag{9}$$

Workout 4.1.10 *Show that the y_i^*s defined in (7) satisfy (6).*

One of the things we can do is write out the optimal objective row, solving for z. With the substitutions from *(7)* we have

$$z \;=\; z^* - \sum_{j=1}^{n} y_{m+j}^* x_j - \sum_{i=1}^{m} y_i^* x_{n+j}\ . \tag{10}$$

Workout 4.1.11 *Use equation (1) and the definition of the slack variables x_{n+j} to derive from equation (10) the equality*

$$\sum_{j=1}^{n} c_j x_j \;=\; \left(z^* - \sum_{i=1}^{m} b_i y_i^*\right) + \sum_{j=1}^{n}\left(\left(\sum_{i=1}^{m} a_{ij} y_i^*\right) - y_{m+j}^*\right) x_j\ . \tag{11}$$

Interestingly, since these are equations that hold for any values of the x_js, we may experiment with various choices. For example, if each $x_j = 1$ we obtain

$$\sum_{j=1}^{n} c_j \;=\; \left(z^* - \sum_{i=1}^{m} b_i y_i^*\right) + \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij} y_i^* - y_{m+j}^*\right) \;.$$

Unfortunately, that experiment tells us nothing to help us show that *(5)* or *(9)* hold.

Workout 4.1.12 *What choice of values for the x_js, plugged into (11), show immediately that (9) holds?*

Now that *(9)* holds, we see that *(11)* reduces to

$$\sum_{j=1}^{n} c_j x_j \;=\; \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij} y_i^* - y_{m+j}^*\right) x_j \;. \tag{12}$$

Of course, we can try similar experiments on equation *(12)*.

Workout 4.1.13 *What choice of values for the x_js, plugged into (12), shows that*

$$c_1 = \sum_{i=1}^{m} a_{i1} y_i^* - y_{m+1}^* \;?$$

Workout 4.1.14 *Do for any c_k what you did for c_1 and use your results to show that (5) holds.*

Now that *(5)*, *(6)* and *(9)* have been verified for the y_i^*s, the Duality Theorem has been proved. ◇

Workout 4.1.15 *Suppose P is a linear problem with 4 variables and 7 constraints.*

(a) *If x_3^* is in the basis, what does that say about some optimal objective coefficient?*
- **[i]** *In turn, what does that say about some optimal dual variable value?*
- **[ii]** *In particular, what does that say about some optimal dual constraint?*

(b) *If x_8^* is in the basis, what does that say about some optimal objective coefficient?*
- **[i]** *In turn, what does that say about some optimal dual variable value?*
- **[ii]** *Also, what does that say about some optimal primal constraint?*

4.2 Complementary Slackness Conditions

Let us return to Problem 4.1.5 and its dual Problem 4.2.1.

Problem 4.2.1

$$\begin{array}{llrcrcrcrcr}
\text{Min.} & w = & 211y_1 & + & 189y_2 & - & 106y_3 & + & 175y_4 & & \\
\\
\text{s.t.} & & 4y_1 & + & 6y_2 & - & 5y_3 & + & 3y_4 & \geq & -22 \\
& & y_1 & & & + & 4y_3 & + & 9y_4 & \geq & -18 \\
& & -3y_1 & + & 2y_2 & - & 2y_3 & & & \geq & -27 \\
& & & & 5y_2 & & & - & 2y_4 & \geq & -23 \\
& & 2y_1 & - & y_2 & - & y_3 & + & y_4 & \geq & 16 \\
& & 7y_1 & + & 8y_2 & - & 7y_3 & + & 4y_4 & \geq & -12 \\
\\
\& & & y_1 & , & y_2 & , & y_3 & , & y_4 & \geq & 0
\end{array}$$