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Optimal pebbling of paths and cycles

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Abstract

Distributions of pebbles to the vertices of a graph are said to be solvable when a pebble may be moved to any specified vertex using a sequence of admissible pebbling rules. The optimal pebbling number is the least number of pebbles needed to create a solvable distribution. We provide a simpler proof verifying Pachter, Snevily and Voxman's determination of the optimal pebbling number of paths, and then adapt the ideas in this proof to establish the optimal pebbling number of cycles. Finally, we prove the optimal-pebbling version of Graham's conjecture.

Keywords: optimal pebbling; graph pebbling; Graham's conjecture; cycles.

1 Introduction

The concept of pebbling graphs was introduced by Lagarias and Saks to rephrase a number theoretic conjecture posed by Erdös and Lemke [1]. The idea consists of distributing pebbles to the vertices of a graph, stating a rule for moving pebbles, and asking when at least one pebble may be moved to any vertex. The rule governing moving pebbles states that one may remove two pebbles from a vertex and subsequently place one on an adjacent vertex. We say that a distribution of pebbles to the vertices of a graph is solvable when each vertex can receive a pebble via a sequence of pebbling moves. (This includes sequences of length 0, i.e. when the initial distribution places at least one pebble on the vertex in question.) Two types of pebbling numbers follow immediately:

1. The pebbling number of a graph G, $f(G)$, is the least number such that every distribution of $f(G)$ pebbles to the vertices of G is solvable.
2. The optimal pebbling number of a graph G, $f_{\mathrm{opt}}(G)$, is the least number such that there exists a solvable distribution of $f_{\mathrm{opt}}(G)$ pebbles on G.

Chung established the first results in this area, giving some bounds for pebbling numbers as well as the pebbling numbers for hypercubes, paths, and complete graphs [1]. Others have established the pebbling numbers for various classes of graphs, e.g. cycles [8, 4], complete bipartite graphs [9], and trees [6]. Pebbling numbers of products of graphs have also been studied: we discuss this briefly in Section 4. Optimal pebbling numbers are not as well documented: formulas are known only for paths [7], complete m-ary trees [3], hypercubes [5], and graphs of diameter 2. Interestingly, exact values for optimal pebbling numbers are known only for paths, complete m-ary trees with $m \geq 3$, and diameter-2 graphs. In Section 3 we establish an exact value for the optimal pebbling number of cycles.

To enhance the readability of our proofs, we offer the following notational and linguistic conveniences.

- We often refer to a distribution by name. For the distribution D we use $|D|$ to indicate the size of D, i.e. the number of pebbles in D.
- Suppose the vertices $v_i, v_{i+1}, \ldots, v_{i+j}$ form a path within a graph G. We write $D([v_i, v_{i+1}, \ldots, v_{i+j}]) = [p_i, p_{i+1}, \ldots p_{i+j}]$ to indicate that D places p_k pebbles on vertex v_k for each $k = i, i+1, \ldots, i+j$. Similarly, $D(v) = p$ means that D places p pebbles on the vertex v. We then say that a vertex v is *occupied* if $D(v) > 0$ and *unoccupied* when $D(v) = 0$.
- We often alter a graph by *removing a vertex* of degree 1 or 2. If we remove a vertex v with $\deg(v) = 1$ then it is understood that the edge incident to v is also removed. If $\deg(v) = 2$ then the two edges incident to v are replaced with an edge incident to both of v's original neighbors. Thus a path (respectively, a cycle) on n vertices from which a vertex is removed becomes simply a path (cycle) on $n-1$ vertices.

2 Optimal Pebbling of Paths

Pachter, Snevily and Voxman establish the optimal pebbling number for path graphs [7]. The formula for $f_{\mathrm{opt}}(P_n)$ depends on the value of $n \mod 3$, so we write P_n, the path on n vertices, as $P_n = P_{3t+r}$, where $r \in \{0, 1, 2\}$.

Theorem 1 *[7] The optimal pebbling number of the path on $3t+r$ vertices is $2t+r$, i.e. $f_{opt}(P_{3t+r}) = 2t+r$.*

Proof. We rephrase the distribution given in [7] to show that $f_{\text{opt}}(P_{3t+r}) \leq 2t+r$: Label the vertices of P_{3t+r} sequentially as $v_1, v_2, \ldots v_{3t+r}$. For $1 \leq i \leq 3t$, set $D(v_i) = 2$ if $i \equiv 2 \mod 3$, and $D(v_i) = 0$ otherwise. If $r \geq 1$ set $D(v_{3t+1}) = 1$, and if $r = 2$ then set $D(v_{3t+2}) = 1$ as well. Observe that each vertex is either occupied or adjacent to a vertex with two pebbles. Thus the distribution is solvable.

We ask the reader to check by hand that $f_{\text{opt}}(P_{3t+r}) \geq 2t+r$ when $3t+r$ is 1, 2, and 3. To show $f_{\text{opt}}(P_{3t+r}) \geq 2t+r$ we assume the contrary, and that the smallest index for which the theorem fails is $3t+r$.

Case 1: $r = 0$.

(The logic used here is that of [7].) We rewrite P_{3t-1} as $P_{3(t-1)+2}$. Since this path has a smaller index than the least index for which the theorem fails, we are assured that $f_{\text{opt}}(P_{3(t-1)+2}) = 2(t-1)+2 = 2t$. From our assumption we know that $f_{\text{opt}}(P_{3t}) < 2t$. As $f_{\text{opt}}(P_{3t-1}) \leq f_{\text{opt}}(P_{3t})$, we have

$$2t = f_{\text{opt}}(P_{3t-1}) \leq f_{\text{opt}}(P_{3t}) < 2t,$$

an impossibility.

Case 2: $r = 1$ **or** $r = 2$.

Since we are assuming that $f_{\text{opt}}(P_{3t+r}) \leq 2t+r-1$, we may choose a solvable distribution D of P_{3t+r} such that $|D| = 2t+r-1$. We will modify D to create D^*, a solvable distribution of P_{3t+r-1} with fewer than $f_{\text{opt}}(P_{3t+r-1})$ pebbles, thus producing our desired contradiction.

Subcase 2.1: P_{3t+r} contains a vertex v with $D(v) = 1$.

We modify D and P_{3t+r} by removing v and the pebble on it, thus creating a distribution D^* of $2t+r-2$ pebbles on P_{3t+r-1}. We claim that all the remaining vertices may still be pebbled, in other words, that D^* is solvable. Suppose v was involved in a pebbling step needed to establish the solvability of D. Then v is not an endpoint of the path. Number the vertices of P_{3t+r} sequentially and let $v = v_i$. Without loss of generality we may assume that vertices indexed by j, $j > i$, required pebbles moved from v_{i-1} through v_i to v_{i+1} to pebble them. Assume that, using distribution D, a pebbles could be moved from v_{i-1} to v_i. Then $\lfloor \frac{a+1}{2} \rfloor$ pebbles could be moved from v_{i-1} through v_i to v_{i+1}. Using D^*, the same a pebbles were moved from v_{i-1} to v_i may now be moved directly from v_{i-1} to v_{i+1}. As $a \geq \lfloor \frac{a+1}{2} \rfloor$ for all $a \geq 1$, we see that all vertices initially reachable from D are still reachable from D^*. Of course, any vertex that was reachable from D without using v initially is still reachable from D^*.

Subcase 2.2: D puts at least 2 pebbles on all occupied vertices of P_{3t+r}.

We label the vertices of P_{3t+r} sequentially as $v_1, v_2, \ldots, v_{3t+r}$. Let i be the smallest index for which v_i is occupied and v_{i+1} is unoccupied. (If no such

i exists then renumber the vertices of P_{3t+r} starting from the opposite end.) Create D^* by removing vertex v_{i+1} from the graph and removing two pebbles from vertex v_i. Additionally, if $i \neq 1$, place one additional pebble on v_{i-1}. We now have D^*, a distribution with no more than $2t + r - 2$ pebbles on P_{3t+r-1}.

As the two pebbles removed from v_i could contribute exactly 1 pebble to pebbling steps used to pebble any vertex v_h, $h < i$, the additional pebble placed on v_{i-1} ensures that any vertex v_h, $h < i$, that could be pebbled from D may still be pebbled from D^*.

It remains only to show that any vertices v_{i+j} $(j \geq 2)$ that could be pebbled from D may still be pebbled from D^*. Suppose $D(v_i) = a$; then $D^*(v_i) = a-2$. Suppose that, starting from D, b pebbles were collected on vertex v_{i-1}. Then $\lfloor \frac{b}{2} \rfloor$ pebbles could be moved to v_i, permitting $\lfloor \frac{1}{2}(\lfloor \frac{b}{2} \rfloor + a) \rfloor$ pebbles to be moved to v_{i+1}, and finally $\lfloor \frac{1}{2} \lfloor \frac{1}{2}(\lfloor \frac{b}{2} \rfloor + a) \rfloor \rfloor$ pebbles to be moved to v_{i+2}. Now, starting from D^*, $\lfloor \frac{b+1}{2} \rfloor$ pebbles may be moved from v_{i-1} to v_i, and these $\lfloor \frac{b+1}{2} \rfloor$ pebbles added to v_i's $a-2$ pebbles allowing the placement of $\lfloor \frac{1}{2}(\lfloor \frac{b+1}{2} \rfloor + a - 2) \rfloor$ pebbles on v_{i+2}. It is straightforward to verify that $\lfloor \frac{1}{2}(\lfloor \frac{b+1}{2} \rfloor + a - 2) \rfloor \geq \lfloor \frac{1}{2} \lfloor \frac{1}{2}(\lfloor \frac{b}{2} \rfloor + a) \rfloor \rfloor$ whenever $a \geq 2$, so once again, all v_{i+j} that could be pebbled starting from D can still be pebbled from D^*.

We conclude that our new distribution D^* is a solvable distribution on P_{3t+r-1}. However, $|D^*| \leq 2t+r-2$, thus contradicting the fact that $f_{\text{opt}}(P_{3t+r-1}) = 2t + r - 1$. This discrepancy forces us to conclude that our initial assumption is false, and so the theorem holds for all positive integers.

3 Optimal Pebbling of Cycles

The proof given for path graphs extends nicely to provide the optimal pebbling number for cycle graphs, C_n. Again, we are interested in $n \mod 3$, so we write C_n, the cycle on n vertices, as $C_n = C_{3t+r}$, where $r \in \{0, 1, 2\}$.

Theorem 2 *The optimal pebbling number of the cycle on $3t+r$ vertices is $2t+r$, i.e. $f_{opt}(C_{3t+r}) = 2t + r$.*

Proof. We begin by constructing a solvable distribution of size $2t + r$. Number the vertices of C_{3t+r} sequentially, $C_{3t+r} = (v_1, v_2, \ldots, v_{3t+r})$. For $1 \leq i \leq 3t$, set $D(v_i) = 2$ if $i \equiv 2 \mod 3$, and $D(v_i) = 0$ otherwise. If $r \geq 1$ set $D(v_{3t+1}) = 1$, and if $r = 2$ then set $D(v_{3t+2}) = 1$ as well. Observe that in each case, every vertex is either occupied or is adjacent to a vertex with two pebbles. Thus, the distribution is solvable.

It is straightforward to show that the theorem holds for cycles of lengths 3, 4, and 5. It remains to show that $f_{\text{opt}}(C_{3t+r}) \geq 2t + r$ for $3t + r \geq 6$. Suppose, to the contrary, that $3t + r$ is the least integer for which $f_{\text{opt}}(C_{3t+r}) < 2t + r$.

Case 1: $r = 0$.

Since $C_{3t-1} = C_{3(t-1)+2}$, we have $f_{\text{opt}}(C_{3t-1}) = 2(t-1)+2 = 2t$. However, $f_{\text{opt}}(C_{3t-1}) \leq f_{\text{opt}}(C_{3t})$. Thus,

$$2t = f_{\text{opt}}(C_{3t-1}) \leq f_{\text{opt}}(C_{3t}) < 2t, \quad \text{an impossibility.}$$

Case 2: $r = 1$ or $r = 2$.

Our assumption is that $f_{\text{opt}}(C_{3t+r}) \leq 2t + r - 1$. Therefore, we may choose a solvable distribution D of C_{3t+r} of size $2t + r - 1$. In each case, we modify D to create a solvable distribution on a smaller cycle graph with fewer than the number of pebbles than we know is required.

Subcase 2.1: C_{3t+r} contains a vertex v with $D(v) = 1$.

The proof for Subcase 2.1 of Theorem 1 adapts directly.

Subcase 2.2: D places exactly 2 pebbles on each occupied vertex of C_{3t+r}.

First, since $|D| = 2t+r-1$ and $|D|$ is even, we have $r = 1$. Number the vertices of C_{3t+1} sequentially, $C_{3t+1} = (v_1, v_2, \ldots, v_{3t+1})$, and consider the corresponding sequence of the number of pebbles on the vertices of C_{3t+1}. Note that we may assume that there are at most two consecutive unoccupied vertices in D, since D was assumed to be solvable. Also, there must be a subsequence of vertices, v_i, v_{i+1}, v_{i+2}, with $D([v_i, v_{i+1}, v_{i+2}]) = [2, 0, 2]$ or $D([v_i, v_{i+1}, v_{i+2}]) = [2, 2, 0]$. Otherwise, there would be *exactly* two unoccupied vertices between every pair of occupied vertices, yielding $r = 0$, a contradiction.

In each case, to obtain a new distribution, D^*, remove vertices v_{i+1} and v_{i+2} and their associated pebbles. In the first case, D^* is a solvable distribution since v_{i+3} is either occupied or can be pebbled by v_i, and vertices v_{i-1} and v_{i+4} are unaffected. In the second case, no vertices are affected by the removal of v_{i+1} and v_{i+2}, so D^* is solvable. Also D^* is a distribution on C_{3t-1} with $|D^*| = 2t - 2$. We have reached a contradiction since by hypothesis, $f_{\text{opt}}(C_{3t-1}) = f_{\text{opt}}(C_{3(t-1)+2}) = 2(t-1) + 2 = 2t$.

Subcase 2.3: C_{3t+r} contains some vertex v_1 such that $D(v_1) \geq 3$.

We consider how to construct D^* in each of three possible cases. For the first two cases, we number the vertices of C_{3t+r} sequentially, $C_{3t+r} = (v_1, v_2, \ldots, v_{3t+r})$ in either direction. If either of the vertices adjacent to v_1, say v_2, is unoccupied, D^* may be constructed by removing v_2 and 2 pebbles from v_1, and then adding 1 pebble to v_{3t+r}. D^* permits at least as many pebbles to move to v_3 and v_{3t+r} as does D, therefore D^* is solvable. Say both vertices adjacent to v_1 are occupied. If $D(v_1) = 3$, remove v_1 and its pebbles and place a pebble on each vertex adjacent to v_1 to obtain the distribution D^*. D^* permits at least as many pebbles to move to v_2 and v_{3t+r} as does D, so D^* is solvable.

Finally, if $D(v_1) > 3$, locate the unoccupied vertex closest to v_1 and number the vertices of C_{3t+r} sequentially in that direction. Designate the unoccupied vertex closest to v_1 to be v_j. To obtain D^*, remove 3 pebbles from v_1, place

2 additional pebbles on v_{3t+r} and remove vertex v_j. Observe that D^* permits as least as many pebbles to move to v_{3t+r} as does D. Finally, note that the vertices $v_2, v_3, \ldots, v_{j-1}$ are occupied in both D and D^*. The 3 pebbles that were removed from v_1 could contribute at most 2 pebbles to vertex v_{j-1} and therefore none to vertex v_{j+1} under distribution D. Thus, D^* is indeed solvable.

Thus whenever D places at least three pebbles on some vertex of C_{3t+r}, we can create a solvable distribution, D^*, on C_{3t+r-1} with $|D^*| = 2t + r - 2$. This contradicts our hypothesis.

Having reached a contradiction in all cases, we conclude that our initial assumption is false, and the theorem holds for all positive integers exceeding 2.

4 Graham's conjecture

An important open question in (non-optimal) graph pebbling is Graham's conjecture, a statement about the pebbling number of graph products. We define the product of two graphs G and H to be the graph with vertex set $V(G) \times V(H)$ (Cartesian product) and with edge set

$$\begin{aligned} E(G \times H) = &\{\big((g_1, h_1), (g_2, h_2)\big) \,|\, g_1 = g_2 \text{ and } (h_1, h_2) \in V(H)\} \\ &\cup \{\big((g_1, h_1), (g_2, h_2)\big) \,|\, h_1 = h_2 \text{ and } (g_1, g_2) \in V(G)\}. \end{aligned}$$

Recall that $f(G)$ denotes the pebbling number of the graph G.

Conjecture 1 *[1] For any two graphs G and H, $f(G \times H) \leq f(G)f(H)$.*

While this question is still open, many successful results in support have appeared. (See, for example, [1], [6], and [8].) Most of these take the form of showing that $f(G \times H) \leq f(G)f(H)$ for G from a particular family of graphs and H satisfying the 2-pebbling property[1]. (Herscovici introduces a similar property[2], as there are known to be graphs – the Lemke graphs [8] – that do not satisfy the 2-pebbling property [4].)

The analogous statement for optimal pebbling numbers is easier.

Theorem 3 *For any two graphs G and H, $f_{opt}(G \times H) \leq f_{opt}(G)f_{opt}(H)$*[3].

Proof. Let D_G and D_H be optimally-sized solvable distributions on G and H, respectively. Define the distribution D on $G \times H$ by placing $D_G(v)D_H(w)$

[1]Given a distribution D on a graph G with $|D| = p$ and $D(v) > 0$ for exactly q vertices v, G satisfies the 2-pebbling property if $p + q > 2f(G)$.

[2]G satisfies the path property if for every path P_m, $f(P_m \times G) \leq f(P_m)f(G)$.

[3]Hung-Lin Fu and Chin-Lin Shiue state this result in [3]. The proof has yet to appear; the status of the paper containing their proof is unclear [2].

pebbles on each vertex $(v, w) \in V(G \times H)$. Note that $|D| = f_{\mathrm{opt}}(G) f_{\mathrm{opt}}(H)$, so the theorem is proven once we establish D's solvability.

An arbitrary vertex $(v, w) \in V(G \times H)$ lies in the subgraph $\{v\} \times H$, which is isomorphic to H. Likewise, each vertex (v, w') of $\{v\} \times H$ lies in a subgraph $G \times \{w'\}$ isomorphic to G. Now, consider the distribution D_G on G. Suppose j pebbles can be moved to v under D_G. Note that $j \geq 1$ since D_G is a solvable distribution. Then, starting from the distribution D on $G \times H$, we may move at least $j D_H(w')$ pebbles to the vertex (v, w'). In particular, we may move at least $D_H(w')$ pebbles to each vertex $(v, w') \in V(\{v\} \times H)$. But, since D_H is solvable on H and we have at least $|D_H|$ pebbles arranged appropriately on the vertices of $\{v\} \times H$, we can use edges of the form $\big((v, w'), (v, w'')\big)$ to move a pebble to (v, w). Since the vertex (v, w) was arbitrary, the distribution D is solvable, thus proving the theorem.

Equality in Theorem 3 is achieved by $P_3 \times P_3$, which has optimal pebbling number 4. (Label the vertices of P_3 sequentially as v_1, v_2, v_3 and place all 4 pebbles on $(v_2, v_2) \in V(P_3 \times P_3)$.) The inequality, however, is required: observe that C_4, with optimal pebbling number 3, is isomorphic to $P_2 \times P_2$, and $f_{\mathrm{opt}}(P_2) f_{\mathrm{opt}}(P_2) = 2 \cdot 2 = 4$.

Establishing an upper bound for the optimal pebbling number of a graph requires one only to prove that an appropriately-sized distribution is solvable. Similarly, lower bounds for (non-optimal) pebbling numbers can be proven by demonstrating the non-solvability of a distribution of size one less than the claimed lower bound. It is the constructive nature of demonstrating upper bounds for optimal pebbling numbers that leads to the ease of proving the optimal-pebbling version of Graham's conjecture. This suggests that investigating lower bounds for optimal pebbling numbers may be of interest in future research.

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