Discrete Mathematics 340 (2017) 1467–1480

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Pebbling in semi-2-trees

Liliana Alcón [a], Marisa Gutierrez [a,b], Glenn Hurlbert [c,*]

[a] Departamento de Matemática, FCE, UNLP, La Plata, Argentina
[b] CONICET, Argentina
[c] Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, USA

ARTICLE INFO

Article history:
Received 27 April 2016
Received in revised form 14 January 2017
Accepted 10 February 2017
Available online 20 March 2017

Keywords:
Pebbling number
k-trees
k-paths
Class 0
Complexity

ABSTRACT

Graph pebbling is a network model for transporting discrete resources that are consumed in transit. Deciding whether a given configuration on a particular graph can reach a specified target is NP-complete, even for diameter two graphs, and deciding whether the pebbling number has a prescribed upper bound is Π_2^{P}-complete. Recently we proved that the pebbling number of a split graph can be computed in polynomial time. This paper advances the program of finding other polynomial classes, moving away from the large tree width, small diameter case (such as split graphs) to small tree width, large diameter, continuing an investigation on the important subfamily of chordal graphs called k-trees. In particular, we provide a formula, that can be calculated in polynomial time, for the pebbling number of any semi-2-tree, falling shy of the result for the full class of 2-trees.

1. Introduction

The fundamental question in graph pebbling is whether a given supply (*configuration*) of discrete pebbles on the vertices of a connected graph can satisfy a particular set of demands on the vertices. The operation of pebble movement across an edge $\{u, v\}$ is called a *pebbling step*: while two pebbles cross the edge, only one arrives at the opposite end, as the other is consumed. We write (u, v) to denote a pebbling step from u to v. The most studied scenario involves the demand of one pebble on a single *root* vertex r. Satisfying this demand is often referred to as *reaching* or *solving* r, and configurations are consequently called either r-*solvable* or r-*unsolvable*.

The *size* $|C|$ of a configuration $C : V \to \mathbb{N} = \{0, 1, \dots\}$ is its total number of pebbles $\sum_{v\in V} C(v)$. The *pebbling number* $\pi(G) = \max_{r\in V} \pi(G, r)$, where $\pi(G, r)$ is defined to be the minimum number s so that every configuration of size at least s is r-solvable. Simple sharp lower bounds like $\pi(G) \geq n$ and $\pi(G) \geq 2^{\mathrm{diam}(G)}$ are easily derived. Graphs satisfying $\pi(G) = n$ are called *Class* 0 and are a topic of much interest. Recent chapters in [13] and [12] include variations on the theme such as k-pebbling, fractional pebbling, optimal pebbling, cover pebbling, and pebbling thresholds, as well as applications to combinatorial number theory, combinatorial group theory, and p-adic diophantine equations, and also contain important open problems in the field.

Computing the pebbling number is difficult in general. The problem of deciding if a given configuration on a graph can reach a particular vertex was shown in [14] and [16] to be NP-complete, even for diameter two graphs [10] or planar graphs [15]. Interestingly, the problem was shown in [15] to be in P for graphs that are both planar and diameter two, as well as for outerplanar graphs (which include 2-trees). The problem of deciding whether a graph G has pebbling number at most k was shown in [16] to be Π_2^{P}-complete.

* Corresponding author.
E-mail address: ghurlbert@vcu.edu (G. Hurlbert).

http://dx.doi.org/10.1016/j.disc.2017.02.011

In contrast, the pebbling number is known for many graphs. For example, in [17] the pebbling number of a diameter 2 graph G was determined to be n or $n+1$. Moreover, [9] and [4] characterized those graphs having $\pi(G) = n+1$, and it was shown in [11] that one can recognize such graphs in quartic time, improving on the order n^3m algorithm of [3]. Beginning a program to study for which graphs their pebbling number can be computed in polynomial time, the authors of [1] produced a formula for the family of split graphs that involves several cases. For a given graph, finding to which case it belongs takes $O(n^{1.41})$ time. The authors also conjectured that the pebbling number of a chordal graph of bounded diameter can be computed in polynomial time.

In opposition to the small diameter, large tree width case of split graphs, we turn here to chordal graphs with large diameter and small tree width.[1] Building on [2], in this paper we study 2-paths, the sub-class of 2-trees whose graphs have exactly two simplicial vertices, as well as what we call semi-2-trees, the sub-class of 2-trees, each of whose blocks are 2-paths, and prove an exact formula that can be computed in linear time.

2. Preliminary definitions and results

In order to simplify notation, for a subgraph $H \subset G$ or subset $H \subset V(G)$ we write $C(H)$ to denote $\sum_{v \in V(H)} C(v)$. We use C_H for the restriction of C to H.

A *simplicial* vertex in a graph is a vertex whose neighbors form a complete graph. It is *k-simplicial* if it also has degree k. A *k-tree* is a graph G that is either a complete graph of size k or has a k-simplicial vertex v for which $G - v$ is a k-tree. A *k-path* is a k-tree with exactly two simplicial vertices. A *semi-2-tree* is a graph in which each of its blocks is a 2-path, with each of its cut-vertices being simplicial in all of its blocks. For the purpose of our work we derive a new characterization of 2-paths that facilitates the analysis of its pebbling number.

Let $P = x_0, x_1, \ldots, x_{d-1}, x_d$ be a shortest rs-path between two vertices $r = x_0$ and $s = x_d$ of G, where $d = \mathrm{dist}(r, s) = \mathrm{diam}(G)$. For $1 \le i \le d-1$, an *$x_{i-1}x_{i+1}$-fan* (*centered on x_i*) is a subgraph F of G consisting of the subpath x_{i-1}, x_i, x_{i+1} of P and a path $Q = x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1}$ with $k_i \ge 1$ such that x_i is adjacent to every vertex of Q. We call F' the set $\{v_{i,1}, \ldots, v_{i,k_i}\}$.

Let F_i be an $x_{i-1}x_{i+1}$-fan and F_{i+1} be an x_ix_{i+2}-fan, centered on x_i and on x_{i+1}, respectively. We say that F_i and F_{i+1} are *opposite-sided* if $F'_i \cap F'_{i+1} = \emptyset$; and that they are *same-sided* when $F'_i \cap F'_{i+1} = \{v_{i,k_i}\}$ and $v_{i,k_i} = v_{i+1,1}$.

The graph G is an *overlapping fan graph* if the following three conditions are satisfied:

- for every $1 \le i \le d-1$, there is a subgraph F_i which is an $x_{i-1}x_{i+1}$-fan centered on x_i,
- for every $1 \le i \le d-2$, F_i and F_{i+1} are either opposite-sided or same-sided, and
- G is the union of the subgraphs F_i for $1 \le i \le d-1$.

If we agree in calling F_1 an *upper* fan, then all further fans of an overlapping fan graph can be classified into upper or *lower* (opposite-sided from upper) — see Fig. 1.

Notice that, in general, the description of a graph as an overlapping fan graph, may be done using different paths P (see the examples in the center and right of Fig. 1). The path P used to describe G as an overlapping fan graph is called the *spine* of G.

In an overlapping fan graph, $|F'_i \cap F'_{i+3}| = 0$; while $|F'_{i-1} \cap F'_{i+1}| \le 1$, with equality if and only if $k_i = 1$. Notice that we can always choose the spine P so that $|F'_{i-1} \cap F'_{i+1}| = 0$ by swapping the names of vertices x_i and $v_{i,1}$, changing the fans F_{i-1}, F_i, and F_{i+1} from being same-sided to F_i being opposite-sided from F_{i-1} and F_{i+1}. Such a choice of path P is called *pleasant* (see Fig. 1).

For an internal vertex x_i of the spine of an overlapping fan graph G, we let A_{x_i} be the set of vertices of F'_i that are in no other fan of G. If $A_{x_i} = \emptyset$ then $k_i = 1$ and $v_{i,1} \in F'_{i-1}$ or F'_{i+1}; or $k_i = 2$ and $v_{i,1} \in F'_{i-1}$ and $v_{i,2} \in F'_{i+1}$. In the former let e_{x_i} be the edge $x_{i-1}v_{i,1}$ or $v_{i,1}x_{i+1}$ respectively, and in the latter let $e_{x_i} = \{v_{i,1}, v_{i,2}\}$. The following fact will be used in Section 5.2.

Claim 1. *If A_{x_i} is empty (non empty) then $G - e_{x_i}$ ($G - A_{x_i}$) is the union of two overlapping fan graphs each one with x_i as simplicial vertex and no other vertex in common.*

A 2-path of diameter 1 is just a path on two vertices. In this case, its *spine* is the graph itself. For larger diameter we have the following lemma.

Lemma 2. *A graph G of $\mathrm{diam}(G) \ge 2$ is a 2-path if and only if it is an overlapping fan graph.*

Proof. An overlapping fan graph is certainly a 2-path.

Let G be a 2-path with simplicial vertices r and s and diameter at least 2. The 2-path on 4 vertices is a fan, and hence an overlapping fan graph, so we assume that G has at least 5 vertices. Let $G' = G - s$, with simplicial vertices r and s'. Since G' is a 2-path, by induction it is also an overlapping fan graph.

If $\mathrm{diam}(G) > \mathrm{diam}(G')$ then the inclusion of s creates a new fan centered on s'. Otherwise, the inclusion of s extends the last fan of G'. In both cases, then, G is an overlapping fan graph. □

[1] One can find the definition of tree-width in [5], but it is not necessary for this paper.

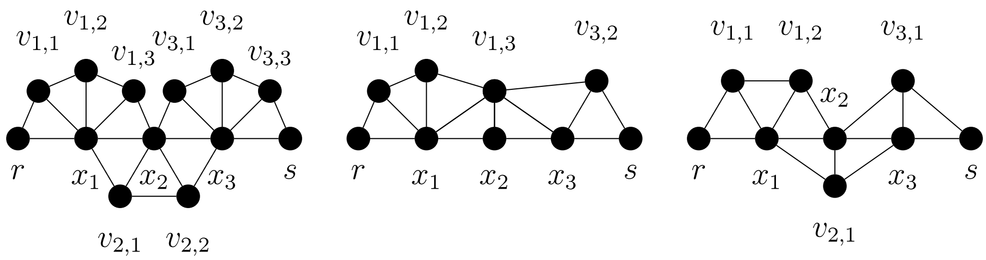

Fig. 1. An overlapping fan graph (left) of diameter 4; fans F_1 and F_3 are same-sided (upper) fans, while F_2 is a lower fan, opposite-sided from F_1 and F_3. An overlapping fan graph with unpleasant (center: $v_{1,3} = v_{2,1} = v_{3,1}$) and pleasant (right: relabeled) shortest *rs*-paths.

Recall that if S is a set of vertices of G then $G - S$ denotes the subgraph of G induced by $V(G) - S$. In an analogous way, if F is a subgraph, we let $G - F$ denote the subgraph of G induced by $V(G) - V(F)$.

With respect to pebbling configurations, we define an *empty vertex* (or *zero*) to be a vertex with no pebbles on it. A *big vertex* has at least two pebbles on it; of course, in an r-unsolvable configuration, every path from a big vertex to the root r must contain at least one zero. A *huge vertex* v has at least $2^{\mathsf{dist}(v,r)}$ pebbles on it; of course, no r-unsolvable configuration has a huge vertex. The *cost* of a pebbling solution σ is the number of pebbles lost during the pebbling steps of σ, plus one for the pebble that reaches r — we denote this by $\mathsf{cost}(\sigma)$. A *cheap* r-solution is an r-solution of cost at most $2^{\mathsf{ecc}(r)}$, where $\mathsf{ecc}(r) = \mathsf{ecc}_G(r)$ is the eccentricity of r in G.

The *t-pebbling number* $\pi_t(G)$ is the minimum number s so that every configuration of size s is t-fold solvable (i.e., can place t pebbles on any root). The t-pebbling number is related to the fractional pebbling number, which measures the limiting average cost of repeated solutions; i.e. $\lim_{t\to\infty}\pi_t(G)/t$. It is also used as a powerful inductive tool for computing the pebbling number. The following theorem was proven in [11].

Theorem 3 (*[11]*)**.** *If G is a graph of diameter 2 then $\pi_t(G) \le \pi(G) + 4t - 4$.*

In what follows we outline the key lemmas and ideas of our proof of the pebbling number for semi-2-trees. In Section 3 we introduce the Cheap Lemma, a powerful mechanism used in tandem with t-pebbling techniques. Section 4 is devoted to 2-paths, which form the base step of our induction argument for semi-2-trees in Section 5. We finish with various remarks for further progress in Section 6.

3. The Cheap Lemma

We begin by introducing the Cheap Lemma, which we believe is a useful tool of independent interest. First we develop a general framework for some key ideas.

Fix a root r in a graph G. We say that a pebbling step from u to v is *greedy* if $\mathsf{dist}(v, r) < \mathsf{dist}(u, r)$. Furthermore, an r-solution σ is *greedy* if each of its pebbling steps is greedy, and a configuration C is *greedy* if it has a greedy r-solution. Finally, G is *greedy* if every configuration of size at least $\pi(G, r)$ is greedy. (If r needs to be specified, we will use the term *r-greedy*.)

Given σ, let G_σ denote the subgraph of edges of G that are traversed by the pebbling steps of σ, oriented by the direction of travel (bi-directed edges are allowed). We say that G_σ is *acyclic* if it contains no directed cycle. The r-solution σ is called *minimal* if no subset of its pebbling steps solves r; it is *minimum* if no r-solution uses fewer steps. A well-known lemma of great use is the No-Cycle Lemma of [6].

Lemma 4 (*No-Cycle Lemma*)**.** *If σ is a minimal r-solution of a configuration on G then G_σ is acyclic.*

Because of the No-Cycle Lemma, we see that every tree is greedy. In particular, if T is a breadth-first-search spanning tree of G, rooted at r, then T is an example of an r-greedy spanning subgraph of G preserving distances to r. Hence any configuration of size at least $\pi(T, r)$ on G has a greedy solution. Indeed, more can be said. Our main point will be that minimal greedy solutions are cheap, which we will show by using weight functions. We say that a configuration is *cheap* if it has a cheap solution.

Lemma 5 (*Cheap Lemma*)**.** *Given a graph G with root r, let G^* be an r-greedy spanning subgraph of G preserving distances to r. Then any configuration on G of size at least $\pi(G^*, r)$ is cheap.*

Proof. For a vertex v define the weight function $w(v) = 2^{-\mathsf{dist}(v,r)}$; let the *weight* of a configuration C be $w(C) = \sum_v C(v)w(v)$. Note that the configuration with a single pebble on r has weight 1.

Suppose that C is a configuration on G of size at least $\pi(G^*, r)$. Let σ be a minimal greedy r-solution from C. Denote by C_σ the configuration on G^* using only the pebbles of C that are used by σ. Then $\mathsf{cost}(\sigma) = |C_\sigma|$.

For any configuration C', let C'' be a configuration that results from making one greedy pebbling step. Then $w(C') = w(C'')$. Applied iteratively to C_σ, this means that $w(C_\sigma) = 1$.

Now $w(C_\sigma) = \sum_v C_\sigma(v)w(v) \geq \sum_v C_\sigma(v)2^{-\mathsf{ecc}(r)}$, and so $\mathsf{cost}(\sigma) = |C_\sigma| = \sum_v C_\sigma(v) \leq 2^{\mathsf{ecc}(r)}w(C_\sigma) = 2^{\mathsf{ecc}(r)}$. □

The pebbling number for a rooted tree (T, r) was first derived in [8], using the notion of its *maximum r-path partition* $\mathcal{P}$. One can compute such a thing iteratively as follows. Beginning with $F = T$, $W = \{r\}$, and $\mathcal{P} = \emptyset$, we choose a longest path P in F having one endpoint in W. Then we add P to $\mathcal{P}$, add its vertices to W, remove its edges from F, and repeat.

Theorem 6 ([8]). *Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a maximum r-path partition of a rooted tree (T, r), with each P_i having length (number of edges) a_i. (By construction, $a_i \geq a_{i+1}$ for $1 \leq i < k$.) Then $\pi_t(T, r) = (t2^{a_1} - 1) + \sum_{i=2}^{k}(2^{a_i} - 1) + 1 = t2^{a_1} + \sum_{i=2}^{k} 2^{a_i} - k + 1$.*

The pebbling number $\pi_t(T)$ is given by choosing r to be a leaf of a longest path of T. We say that a configuration C is *t-extremal* for a rooted tree (T, r) if the following holds. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a maximum r-path partition of (T, r) with each P_i having leaf endpoint v_i. Then $C(v_1) = t2^{a_1} - 1$, $C(v_i) = 2^{a_i} - 1$ for $2 \leq i \leq k$, and $C(v) = 0$ otherwise. The proof of the lower bound in Theorem 6 involves showing (by induction) that such a configuration is not t-fold r-solvable.

For a 2-path G with simplicial root r, we denote by $T^*(G, r)$ any spanning tree of G, rooted at r, that includes the spine of G and all fan vertices as leaves, each one adjacent to its neighbor in the spine closest to r. Notice that $T^*(G, r)$ is an r-greedy spanning subgraph of G preserving distances to r.

For a 2-path G with simplicial vertex r, root eccentricity d, and with n vertices, we define the functions $p_t(G, r) = t2^d + n - 2d$ (suppressing t when $t = 1$) and $q(G, r) = 2^d + n - d - 1$. Note that $p(G, r) < q(G, r) < p_2(G, r)$ when $1 < d$.

Corollary 7. *Let G be a 2-path with simplicial vertex r and diameter d. If C is a configuration of size at least $q(G, r) + (t - 1)2^d$ then C has t distinct cheap r-solutions.*

Proof. For $t = 1$ this follows from the Cheap Lemma 5 and Theorem 6 because for $T^* = T^*(G, r)$ we have $\pi(T^*, r) = q(G, r)$. The general statement follows by induction on t. □

The following two lemmas about pebbling in trees will be used in Section 5.2.

Lemma 8. *Let T be a tree with diameter $d = \mathsf{diam}(T)$, r^* and r be vertices with $\mathsf{ecc}(r) < \mathsf{ecc}(r^*) = d$. Let P^* be a path $v_0v_1 \cdots v_d$ with $v_0 = r^*$ and $v_d = s^*$, labeled so that $\mathsf{dist}(r, s^*) \leq \mathsf{dist}(r, r^*) = \mathsf{ecc}(r)$. Denote by P the path from r to r^*, and set $P^* \cap P = v_0 \cdots v_{h'}$. Define $\overline{h} = d - h'$. Then $\pi_t(T, r) \leq \pi_t(T, r^*) - t(2^d - 2^{\mathsf{ecc}(r)}) + 2^{\overline{h}} - 1 \leq \pi_t(T, r^*) - 2^{d-2}$.*

Proof. Let $\mathcal{P}^*$ be a maximum path partition of T with root r^*. Define $P_0^* = P^*, P_1^*, \ldots, P_k^*$ to be the sequence of paths of $\mathcal{P}^*$ that are used sequentially while traveling from r^* to r in P, and set $d_i^* = \mathsf{length}(P_i^*)$ for each $0 \leq i \leq k$ (so $d_0^* = d$). Next define $P_i' = P \cap P_i^*$, with $h_i' = \mathsf{length}(P_i')$ and $\overline{h}_i = d_i^* - h_i'$ (so $h_0' = h'$ and $\overline{h}_0 = \overline{h}$). Notice that $\mathsf{ecc}(r) = \sum_{i=0}^{k} h_i'$ and $\overline{h} \leq d/2$.

Denote by $\mathcal{P}$ the maximum path partition of T with r as root. We will use the following facts in the calculations below.

- The longest path in $\mathcal{P}$ is P.
- In the component of the tree T–P that contains the path $\hat{P}_i = P_i^*$–P_i', the longest path is $\hat{P}_i$.

From these it follows that each $\hat{P}_i \in \mathcal{P}$ and, subsequently, that $\mathcal{P}^* - \{P_0^*, \ldots, P_k^*\} = \mathcal{P} - \{P, \hat{P}_0, \ldots, \hat{P}_k\}$. Now, by converting $\mathcal{P}^*$ to $\mathcal{P}$, we find that

$$\begin{aligned}
\pi_t(T, r) &= \pi_t(T, r^*) - \left[(t2^d - 1) + \sum_{i=1}^{k}(2^{d_i^*} - 1)\right] + \left[(t2^{\mathsf{ecc}(r)} - 1) + \sum_{i=0}^{k}(2^{\overline{h}_i} - 1)\right] \\
&\leq \pi_t(T, r^*) + t2^{\mathsf{ecc}(r)} - \left(t2^d - 2^{\overline{h}_0}\right) - \left[\sum_{i=1}^{k}\left(2^{d_i^*} - 2^{\overline{h}_i}\right)\right] - 1 \\
&\leq \pi_t(T, r^*) - t(2^d - 2^{\mathsf{ecc}(r)}) + 2^{\overline{h}} - 1 \\
&\leq \pi_t(T, r^*) - t2^{d-1} + 2^{\lfloor d/2 \rfloor} - 1 \\
&\leq \pi_t(T, r^*) - 2^{d-2}. \qquad \square
\end{aligned}$$

Lemma 9. *Let $e = xy$ be a non pendant edge of a tree T and assume that $\mathsf{ecc}(x) \geq \mathsf{ecc}(y)$. If T' is the tree obtained by subdividing the edge e with a new vertex r, then $\pi_t(T', r) = \pi_t(T, x) + 2^a$, where a is the eccentricity of x in the connected component of $T - y$ that contains x (thus, $a + \mathsf{ecc}(x) \leq \mathsf{diam}(T)$).*

Proof. Define x' to be the vertex having $\mathsf{dist}_T(x, x') = \mathsf{ecc}_T(x)$, and denote the xx'-path by P. Because $\mathsf{ecc}_T(y) \leq \mathsf{ecc}_T(x)$ we know that $y \in P$. Let x'' be a vertex having $\mathsf{dist}_{T-y}(x, x'') = a$, with xx''-path Q, and note that $a < \mathsf{ecc}_T(x)$.

Now observe that $\mathsf{dist}_{T'}(r, x') = \mathsf{dist}_T(x, x')$ (witnessed by the rx' path P') and $\mathsf{dist}_{T'}(r, x'') = \mathsf{dist}_T(x, x'') + 1$ (witnessed by the rx'' path Q'). This means that the only changes from the maximum path partition of T with root x to the maximum path partition of T' with root r are that the longest path P from x in T becomes the longest path P' from r in T', and the longest path Q from x in $T-y$ becomes the longest path Q' from r in $T'-y$. Hence we have $\pi_t(T,x)=t2^{\mathrm{ecc}_T(x)}+2^a+F(x)$, for some $F(x)$, and $\pi_t(T',r)=t2^{\mathrm{ecc}_{T'}(r)}+2^{a+1}+F(x)=\pi_t(T,x)+2^{a+1}-2^a$. □

4. 2-paths

In this section we calculate a $\pi_t(G,r)$ for r a simplicial vertex of a 2-path G.

4.1. The lower bound

We now present some general removal techniques for finding lower bounds that may also be of independent interest. For a vertex v, define its *open neighborhood* $N(v)$ to be the set of vertices adjacent to v, and its *closed neighborhood* $N[v]=N(v)\cup\{v\}$. Also, for a set of vertices A write $N(A)=\cup_{v\in A}N(v)$. Along the lines of the definition of twin vertices, for a non-root vertex y we say that y is a *junior sibling of* x (or, more simply, *junior to* x) if $N(y)\subseteq N[x]$, and that y is a *junior* if it is junior to some vertex x.

Lemma 10 (*Junior Removal Lemma*)**.** *Given the rooted graph (G,r) with configuration C, suppose that y is a junior with $C(y)=0$. Then C is t-fold r-solvable if and only if C restricted to $G-y$ is t-fold r-solvable in $G-y$.*

Proof. Sufficiency is obvious, so we only prove necessity. Suppose that σ is an r-solution from C that uses y. Let y be junior to some vertex x. Construct σ' from σ by replacing every pebbling step (u,y) with (u,x) and every pebbling step (y,v) with (x,v). Then σ' t-fold solves r as well. □

We say that a set of vertices W is a *wart* if it is a component of $G-X$ for some clique cutset X, where by *clique* we mean complete subgraph.

Lemma 11 (*Wart Removal Lemma*)**.** *Given the rooted graph (G,r) with configuration C, suppose that W is a wart of G not containing r and that $C(w)\le 1$ for every $w\in W$. Then C is t-fold r-solvable if and only if C restricted to $G-W$ is t-fold r-solvable in $G-W$.*

Proof. Sufficiency is obvious, so we only prove necessity. We show that no minimum r-solution from C uses W.

Suppose instead that σ is a minimum r-solution that uses W. Let X be a clique cutset that witnesses the wart W, and let u be a vertex of X having a pebbling step into W. Because σ is minimum, there is a vertex $v\in X$ that receives a pebble from W and that is different from u. By replacing those two pebbling steps by the single step from u to v we find an r-solution with fewer steps, a contradiction. □

Let G be a 2-path with simplicial root r, pleasant path P, and configuration C. For a given t we say that C is *t-extremal for r* (simply, *extremal* if $t=1$) if there is a I-saturating matching M from the internal spine vertices $I=\{x_1,\dots,x_{d-1}\}$ to the fan vertices $\{v_{i,j}\}$ such that $C(x_d)=t2^d-1$, $C(r)=0$, $C(M)=0$, and $C(v)=1$ otherwise. Notice that $|C|=p_t(G,r)-1$.

If a configuration C on G is t-fold r-solvable if and only if C_H is t-fold r-solvable on the subgraph $H\subset G$, then we say that *G t-fold r-reduces to H for C*. If C, t and r are clear from the context we just write *reduces*.

Lemma 12 (*Extremal Lemma*)**.** *If C is t-extremal for the simplicial root r of a 2-path G then C is not t-fold r-solvable. Moreover, by using Lemmas 10 and 11 (repeatedly removing juniors and warts) G reduces to its spine, the path P_d, where $d=diam(G)$.*

Proof. We use induction on d. The result is trivial for $d=1$. For $d>1$ we suppose that C is t-fold r-solvable and let σ be a t-fold r-solution. Write $y_i=v_{i,j_i}$ for the neighbor of x_i in M and let ℓ be the smallest index i such that y_i is a junior. This exists because if y_i is not a junior then either $y\in F_{i-1}\cap F_i$ or $y\in F_i\cap F_{i+1}$ (it is a *fan intersection*), and there are more fans than fan intersections. Set $y=y_\ell$ and $x=x_\ell$. Then y is junior to x and so, by Lemma 10, C is t-fold r-solvable in $G-y$.

Furthermore, let j^+ be the maximum j such that $v_{\ell,j}\in F_\ell-F_{\ell+1}$. If $j_\ell+1\le j^+$ then $\{v_{\ell,j_\ell+1}\}$ is a wart in $G-y$, and so Lemma 11 says that we can remove it. Once we do, $\{v_{\ell,j_\ell+2}\}$ becomes a wart, and so on, until all the vertices $v_{\ell,j}$ with $j_\ell<j\le j^+$ have been removed. Then the graph $G_{\ell+1}=\cup_{i>\ell}F_i$ is a 2-path, with the restriction, $C_{\ell+1}$, of C to $G_{\ell+1}$ being $2^\ell t$-extremal for x_ℓ. By induction, $C_{\ell+1}$ is not $2^\ell t$-fold x_ℓ-solvable and $G_{\ell+1}$ can be reduced to the path $P_{d-\ell}$.

Similarly, let j^- be the minimum j such that $v_{\ell,j}\in F_\ell-F_{\ell-1}$. If $j_\ell-1\ge j^-$ then the warts $\{v_{\ell,j}\}$ for $j^-\le j<j_\ell$ can be successively removed, leaving the 2-path $G^\ell=\cup_{i\le\ell}F_i$. Since C is t-fold r-solvable and x_ℓ is a cut-vertex of $G-y$, all the pebbles of $G_{\ell+1}$ used by σ must pass through x_ℓ. But because $C_{\ell+1}$ is not $2^\ell t$-fold x_ℓ-solvable, the most number of pebbles that can reach x_ℓ is $2^\ell t-1$. After placing as many pebbles as possible on x_ℓ from $G_{\ell+1}$, the resulting configuration C^ℓ is a subconfiguration of a configuration $\hat{C}^\ell$ that is t-extremal for r on G^ℓ. By induction, $\hat{C}^\ell$ is not t-fold r-solvable, a contradiction. Also, G^ℓ can be reduced to the path P_ℓ, which reduces G to the path P_d. □

Corollary 13. *If r is a simplicial vertex of a 2-path G then $\pi_t(G,r)\ge p_t(G,r)$.* □

4.2. The upper bound

We first note that a diameter two 2-path G is Class 0. Indeed, the following lemma is a corollary of the Class 0 characterization for diameter two graphs from [9] that shows that $\pi(G) = n$ in this case and the t-pebbling bound of [11] that states $\pi_t(G) \le \pi(G) + 4t - 4$ for all diameter two graphs. Equality comes from Corollary 13. The diameter one case is from [11] also.

Lemma 14 ([11]). *If G is a 2-path on n vertices with diameter $d \le 2$ then $\pi_t(G) = t2^d + n - 2d$.*

Theorem 15. *Let G be a 2-path on n vertices with simplicial root vertex r having eccentricity d, and configuration C. If $|C| \ge p(G, r)$ then C is r-solvable.*

Proof. When $d \le 2$, the result is taken care of by Lemma 14. So we will assume that $d > 2$ and use induction. Suppose that $|C| = p(G, r)$ and let $P = r, x_1, \ldots, x_{d-1}, s$ be a pleasant shortest rs-path between the two simplicial vertices of G. Write $x_0 = r$ and $x_d = s$ and label G by its fan graph labeling, so that $V(F_i) = \{x_{i-1}, x_i, x_{i+1}, v_{i,1}, \ldots, v_{i,k_i}\}$ and Q_i is the path $x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1}$. Let G' be the restriction of G to the n' vertices of $\cup_{i\ge 2} V(F_i)$, with C' denoting the restriction of C to G'. We further use the abbreviations $C_1 = C(F_1)$ and $n_1 = |V(F_1)|$. Notice that $\mathrm{diam}(G') = d - 1$, so that the Theorem holds for G'. Define $\phi = 1$ (0) if F_2 is same-sided (opposite-sided) as F_1.

If $C(x_1) \ge 1$, $C(x_2) \ge 2$, or $C(v_{1,j}) \ge 2$ for some j (either [$\phi = 1$ and $j = k_1$] or not), then we can place a pebble on x_1. If $|C'| - (1, 2, 2, 0) \ge p(G', x_1)$, where the coordinates correspond, in order, to the four cases above (first two cases plus two subcases of the third case), then we can place another pebble on x_1, and then one on r. Otherwise, $|C'| - (1, 2, 2, 0) \le p(G', x_1) - 1$. That is, $|C'| \le [2^{d-1} + n' - 2(d-1)] + (0, 1, 1, -1)$. Thus $|C_1| \ge |C| - |C'| + (1, 2, 2, 0) \ge 2^{d-1} + (n_1 - 2 - \phi) - 2 + (1, 1, 1, 1) = n_1 + (2^{d-1} - 3 - \phi) \ge n_1$, which means by Lemma 14 that we can solve r.

On the other hand, if $C(x_1) = 0$, $C(x_2) \le 1$, and $C(v_{1,j}) \le 1$ for all j, then $C(\{r, v_{1,1}, \ldots, v_{1,k_1-1}\}) \le k_1 - 1$. Here we define θ to be the number of zeros in $\{v_{1,1}, \ldots, v_{1,k_1}, x_2\}$, so that $|C_1| = n_1 - 2 - \theta$, and set θ' to be the number of those zeros other than x_2 (i.e. $\theta - \theta' = 1 - C(x_2)$). Now we have

$$\begin{aligned}
|C'| &\ge |C| - |C_1| + C(x_2) \\
&= (2^d + n - 2d) - (n_1 - 2 - \theta) + C(x_2) \\
&= (2)2^{d-1} + (n' - 2 - \phi) - 2d + 2 + \theta + C(x_2) \\
&= [(2)2^{d'} + n' - 2d'] + [C(x_2) + \theta - 2 - \phi] \\
&= p_2(G', x_1) + [\theta' - 1 - \phi].
\end{aligned}$$

If $\theta' - 1 - \phi \ge 0$ then $|C'| \ge p_2(G', x_1) > q(G', x_1)$, which means, by Corollary 7, that we can place one pebble on x_1 cheaply. Because the remaining configuration (after solving x_1 cheaply) has at least $p_2(G', x_1) - 2^{d'} = p(G', x_1)$ pebbles, induction places a second pebble on x_1. Then we move one to r.

Otherwise, we have $\theta' - 1 - \phi < 0$, which means that $\theta' \le \phi$. If $\theta' = 0$, that is $C(v_{1,j}) = 1$ for all j, then we will show that it is possible to place two pebbles on x_2, from which we solve r by moving pebbles from x_2 along Q_1. Indeed, this is so if $C(x_2) = 1$ and Q_2 has a big vertex, or if Q_2 contains either a vertex with four pebbles or two big vertices, so we assume otherwise. In this case, we have $|C((F_1 \cup F_2) - G'')| \le |V((F_1 \cup F_2) - G'')|$, where G'' is the restriction of G to the n'' vertices of $\cup_{i\ge 3} V(F_i)$. For the restriction C'' of C to G'', this implies that

$$\begin{aligned}
|C''| &= |C| - |C((F_1 \cup F_2) - G'')| \\
&\ge 2^d + n'' - 2d \\
&= [(2)2^{d-2} + n'' - 2(d-2)] + [2^{d-1} - 4] \\
&\ge p_2(G'', x_2),
\end{aligned}$$

since $d \ge 3$. As before, since $p_2(G'', x_2) > q(G'', x_2)$ and $p_2(G'', x_2) - 2^{d''} = p(G'', x_2)$, we can place one pebble on x_2 cheaply, followed by a second pebble on x_2.

We are left now with the final case (since $\theta' \le \phi \le 1$) in which $\theta' = 1$ (exactly one $v_{1,j}$ is empty), which means that $\phi = 1$ (F_1 and F_2 are same-sided, so that $v_{1,k_1} = v_{2,1}$).

If v_{1,k_1} is not empty then $k_1 \ge 2$, and so

$$\begin{aligned}
|C'| &= |C| - (k_1 - 2) \\
&= (2)2^{d-1} + (n - k_1) - 2(d-1) \\
&= p_2(G', x_1) \\
&> q(G', x_1).
\end{aligned}$$

As above, this means, by Corollary 7 and induction, that we can place two pebbles on x_1, and hence one on r.

If instead v_{1,k_1} is empty then set $\hat{G} = G' - x_1$ and $\hat{C} = C(\hat{G})$, so that

$$\begin{aligned}|\hat{C}| &= |C| - (k_1 - 1)\\ &= (2)2^{d-1} + (n - k_1 - 1) - 2(d - 1)\\ &= p_2(\hat{G}, v_{1,k_1}).\end{aligned}$$

Again, this means that we can place two pebbles on v_{1,k_1}, and hence one on r (via Q_1).

This completes the proof. □

Corollary 16. *If r is a simplicial vertex of a 2-path G then $\pi_t(G, r) = p_t(G, r)$.*

Proof. The lower bound was stated in Corollary 13. The upper bound for $t = 1$ follows from Theorem 15. If $t > 1$, then for any configuration C of size $p_t(G, r) = p_2(G, r)+(t-2)2^d > q(G, r)+(t-2)2^d$, we can place $t-1$ pebbles on r, each cheaply, by Corollary 7. The remaining configuration has at least $p_t(G, r) - (t - 1)2^d = p(G, r)$ pebbles, from which we can place the tth pebble on r by Theorem 15. □

5. Pebbling number of semi-2-trees

We define the *skeleton* T of a semi-2-tree G to be the union of the spines of its blocks; it is a geodesic tree spanning all of the simplicial vertices of G. Let $e(T)$ denote the number of edges of T, $b(G)$ denote the number of blocks of G, and for a simplicial vertex or cut-vertex r and positive integer t define $p_t(G, r) = \pi_t(T, r) + (n - 1) + b(G) - 2e(T)$ (suppressing t when $t = 1$). Notice that this matches the corresponding formula for 2-paths because $b = 1$ and T is a path. In addition, we have $p_t(G, r) = p_{t-1}(G, r) + 2^{\mathrm{ecc}_G(r)}$ because of Theorem 6. We also define $q(G, r) = \pi(T, r) + n - e(T) - 1$; note that $q(G, r) = \pi(T^*, r)$, where T^* is a spanning tree of G, rooted at r, that contains its skeleton and all its fan vertices as leaves, each one adjacent to its neighbor in the skeleton closest to r. Notice that T^* is an r-greedy spanning tree of G preserving distances to r.

5.1. Simplicial or cut-vertex roots

We begin with another consequence of the Cheap Lemma, generalizing Corollary 7. The proof is similar and is left to the reader.

Corollary 17. *Let r be a simplicial vertex or cut-vertex with eccentricity d of a semi-2-tree G. If C is a configuration of size at least $q(G, r) + (t - 1)2^d$ then C has t distinct cheap r-solutions.* □

For a tree T with maximum r-path partition $\mathcal{P} = \{P_1, \ldots, P_k\}$, each P_i having length a_i (sorted so that $a_i \geq a_{i+1}$), let C_T be its t-extremal configuration for r.

For a semi-2-tree G, call a vertex of the skeleton T *internal* if it is not a simplicial vertex or cut-vertex, and let M be any I-saturating matching from the internal vertices I to the fan vertices of G. For a simplicial or cut vertex r of G, define the configuration C by $C(T) = C_T$, $C(M) = 0$, and $C(v) = 1$ otherwise — such a configuration we call *t-extremal for r*. Note that $|C| = p_t(G, r) - 1$.

As in the proof of the Extremal Lemma 12, we can use the Removal Lemmas 10 and 11 to prove that G reduces to T for C and obtain the following more general extremal lemma, which we leave to the reader.

Lemma 18. *If C is t-extremal for the simplicial or cut-vertex root r of a semi-2-tree G then C is not t-fold r-solvable. Moreover, by using Lemmas 10 and 11, G can be reduced to its skeleton T.* □

Now we state and prove the solvability theorem in this case.

Theorem 19. *Let G be a semi-2-tree on n vertices with simplicial or cut-vertex r and configuration C. If $|C| \geq p(G, r)$ then C is r-solvable.*

Proof. We use induction on n with base case $\mathrm{ecc}(r) = 1$, which is handled by Theorem 3. So we assume that $\mathrm{ecc}(r) > 1$. We may also assume that $C(r) = 0$. We consider two cases.

1. r is a cut-vertex.

   Let $H_1, \ldots, H_k$ be the components of $G - r$, with G_i induced by $V(H_i) \cup \{r\}$; then each G_i is a semi-2-tree, so that the theorem holds for them by induction. Let C_i and T_i be the restrictions of C and T to G_i, with n_i and b_i counting the number of vertices and blocks of G_i. If some $|C_i| \geq p(G_i, r)$ then C_i solves r, so we assume not. Then

$$\begin{aligned}
|C| &= \sum_i |C_i| \\
&\leq \sum_i [p(G_i, r) - 1] \\
&= \sum_i [\pi(T_i, r) + (n_i - 1) + b_i - 2e(T_i) - 1] \\
&= \pi(T, r) + (n - 1) + b(G) - 2e(T) - k \\
&< p(G, r),
\end{aligned}$$

a contradiction. Hence some C_i solves r.

2. r is a simplicial vertex.

Let H be the block of G containing r, with r' the other simplicial vertex of H. If $|C(H)| \geq p(H, r)$ then we solve r directly on H. Otherwise, we assume that $|C(H)| = p(H, r) - s$ for some $s > 0$. Recall that $p(H, r) = 2^{d_H} + n_H - 2d_H$, where $n_H = |H|$ and $d_H = \mathsf{ecc}_H(r)$. Let G' be the subgraph of G induced by $(G - H) \cup \{r'\}$, having $n' = n - n_H + 1$ vertices, b' blocks, and root eccentricity $\mathsf{ecc}_{G'}(r') = d' = d - d_H$, with $T' = T \cap G'$ and $d = \mathsf{ecc}_G(r)$. Define the configuration C' on G' by $C'(r') = 0$ and $C'(v) = C(v)$ for all other $v \in G'$.

Suppose that $s \leq 2^{d_H}$. Then

$$\begin{aligned}
|C'| &= |C| - |C(H)| \\
&= p(G, r) - p(H, r) + s \\
&= [\pi(T, r) + (n - 1) + b(G) - 2e(T)] - [2^{d_H} + n_H - 2d_H] + s \\
&= [\pi_s(T', r') + (n' - 1) + b' - 2e(T')] + [s - 2^{d_H} + 2^d - s2^{d'}] \\
&= p_s(G', r') + (2^{d'} - 1)(2^{d_H} - s) \\
&\geq p_s(G', r'),
\end{aligned}$$

which means that we can place s pebbles on r', so that now there are $p(H, r)$ pebbles in H, enough to solve r.

Suppose that $s \geq 2^{d_H}$; i.e. $|C(H)| \leq n_H - 2d_H$. Then

$$\begin{aligned}
|C'| &= |C| - |C(H)| \\
&\geq [\pi(T, r) + (n - 1) + b(G) - 2e(T)] - [n_H - 2d_H] \\
&= [\pi_{2^{d_H}}(T', r') + (n' - 1) + b' - 2e(T')] \\
&\geq p_{2^{d_H}}(G', r'),
\end{aligned}$$

which means that we can place 2^{d_H} pebbles on r', enough to solve r on T. $\square$

Corollary 20. *If r is a simplicial vertex or cut-vertex of a semi-2-tree G then $\pi_t(G, r) = p_t(G, r)$.*

Proof. As in the proof of Corollary 16. $\square$

Theorem 21. *If r is a simplicial vertex or cut-vertex of a semi-2-tree G and r^* is a simplicial vertex with $\mathsf{ecc}(r^*) = \mathsf{diam}(G)$ then $\pi_t(G, r) \leq \pi_t(G, r^*)$.*

Proof. Let T be a skeleton of G. Because the only term in $p_t(G, r) = \pi_t(T, r) + (n - 1) + b(G) - 2e(T)$ that depends on r is $\pi_t(T, r)$, it follows that $\pi_t(G, r)$ is maximized precisely where $\pi_t(T, r)$ is maximized, which is well-known [8] to be at r^*. $\square$

5.2. Other roots

We begin with two more removal lemmas of general use.

Lemma 22 (*Edge Removal Lemma*)**.** *Let r be a vertex of a connected graph G and suppose e is an edge between two neighbors of r. Then $\pi(G, r) = \pi(G - e, r)$.*

Proof. Given any configuration on $V(G) = V(G - e)$, every minimal r-solution in one graph is a minimal solution in the other. $\square$

Lemma 23. *Let r be a cut-vertex of a graph G, and denote the connected components of $G - r$ by $H_1, \ldots, H_k$. For each i define the graph G_i induced by $H_i \cup \{r\}$. Then $\pi(G, r) = 1 + \sum_i (\pi(G_i, r) - 1)$.*

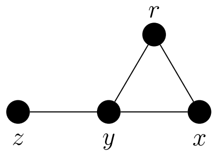

Fig. 2. A non-semi-2-tree G for which $G-x$ is a semi-2-tree. The configuration $C(r,x,y,z)=(0,1,0,3)$ is extremal for r.

Proof. The lower bound follows from the union of the individual maximum-sized r-unsolvable configurations on H_i. The upper bound follows from the pigeonhole principle. □

Lemma 24 (*Neighbor Removal Lemma*)**.** *Let r be a vertex of a connected graph G. Suppose that $A \subseteq N(r)$ such that $N(A) \subseteq N[r]$. Let $\{H_1,\dots,H_k\}$ be the connected components of $(G-r)-A$ and denote by G_i the subgraph of G induced by $V(H_i)\cup\{r\}$. Then $\pi(G,r)=1+|A|+\sum_i(\pi(G_i,r)-1)=|A|+\pi(G-A,r)$.*

Proof. We can remove the edges incident with A by Lemma 22. Then each $v\in A$ is its own component of $G-r$. The result follows from Lemma 23. □

Under the conditions of Lemma 24, if each (G_i,r) is a rooted semi-2-tree, then we say that a configuration C on G is *extremal for r* if $C(x)=1$ for every $x\in A$ and each C_{G_i} is extremal for r on G_i.

A small example of a non-semi-2-tree to which Lemma 24 applies is shown in Fig. 2. This idea is used later in the proof of Corollary 32.

A simple consequence (using the Cheap Lemma and induction) of Lemma 22 is the following.

Corollary 25. *Let G be a semi-2-tree with skeleton T, and suppose that r is a vertex of T that is not a simplicial or cut vertex of G. Let A_r be the set of vertices of the fan centered on r that are in no other fan of G. If A_r is empty and e_r is as defined on Claim 1, then $\pi_t(G,r)=\pi_t(G-e_r,r)$ for all $t\geq 1$.* □

Notice that the previous corollary allows one to calculate the pebbling number for r. In fact, by Claim 1, $G-e_r$ is a semi-2-tree with r a simplicial or cut vertex, then we use Corollary 20 to calculate $\pi(G,r)=\pi(G-e_r,r)$.

Analogously, a consequence (using the Cheap Lemma and induction) of Lemma 24 is the following.

Corollary 26. *Let G be a semi-2-tree with skeleton T, and suppose that r is a vertex of T that is not a simplicial or cut vertex of G. Let A_r be the set of vertices of the fan centered on r that are in no other fan of G. If A_r is non empty then $\pi_t(G,r)=\pi_t(G-A_r,r)+|A_r|$ for all $t\geq 1$.* □

Notice that the previous corollary allows one to calculate the pebbling number for r. In fact, by Claim 1, $G-A_r$ is a semi-2-tree with r a simplicial or cut vertex, then we use Corollary 20 to calculate $\pi(G,r)=\pi(G-A_r,r)$.

Theorem 27. *Let G be a semi-2-tree with skeleton T. Suppose that r is a vertex of T that is not a simplicial or cut vertex of G, and let r^* be a simplicial vertex of G with $\mathrm{ecc}(r^*)=\mathrm{diam}(G)$. Then $\pi_t(G,r)<\pi_t(T,r^*)$.*

Proof. Let A_r be the set of vertices of the fan centered on r that are in no other fan of G. First assume $A_r=\emptyset$ and let e_r be as in Corollary 25. Notice that the skeleton of $G-e_r$ is the same T, while $G-e_r$ has one block less than G, thus (using Corollaries 20 and 25)

$$\begin{aligned}\pi_t(G,r)&=\pi_t(G-e_r,r)\\&=p_t(G-e_r,r)\\&=\pi_t(T,r)+(n(G-e_r)-1)+b(G-e_r)-2e(T)\\&=\pi_t(T,r)+(n(G)-1)+(b(G)-1)-2e(T)\\&\leq\pi_t(T,r^*)+(n(G)-1)+b(G)-2e(T)-1\\&=\pi_t(G,r^*)-1\\&<\pi_t(G,r^*).\end{aligned}$$

Analogously, if $A_r \neq \emptyset$ then (using Corollary 26)

$$\begin{aligned}
\pi_t(G, r) &= \pi_t(G - A_r, r) + |A_r| \\
&= p_t(G - A_r, r) + |A_r| \\
&= \pi_t(T, r) + (n(G - A_r) - 1) + b(G - A_r) - 2e(T) + |A_r| \\
&= \pi_t(T, r) + ((n(G) - |A_r|) - 1) + (b(G) - 1) - 2e(T) + |A_r| \\
&= \pi_t(T, r) + (n(G) - 1) + b(G) - 2e(T) - 1 \\
&\leq \pi_t(G, r^*) - 1 \\
&< \pi_t(G, r^*). \quad \square
\end{aligned}$$

Another consequence (again using the Cheap Lemma and induction) of Lemma 22 is the following. We say that a vertex r is not in any skeleton of G when for every skeleton T of G, r is a fan vertex, i.e. $r \in V(G - T)$.

Corollary 28. *Let r be a vertex of a semi-2-tree G that is not in any skeleton of G. If the root r is in two fans of G, centered on x and y, with edge $e = xy$, then $\pi_t(G, r) = \pi_t(G - e, r)$.* $\square$

Since $G - e$ is a semi-2-tree with cut-vertex root r, the value of $\pi_t(G - e, r)$ is computed by Corollary 20.

Theorem 29. *Let r be a vertex of a semi-2-tree G that is not in any skeleton of G. Suppose that the root r is in two fans of G, centered on x and y, with edge $e = xy$, labeled so that $\mathrm{ecc}(x) \geq \mathrm{ecc}(y)$, and let r^* be a simplicial vertex of G with $\mathrm{ecc}(r^*) = \mathrm{diam}(G)$. Then $\pi_t(G, r) < \pi_t(G, r^*)$.*

Proof. We note that r is a cut vertex of $G - e$, and so $b(G - e) = b(G) + 1$. Also, the skeleton T' of $G - e$ has one more edge than does T; in fact, T' can be seen as the tree obtained from T by subdividing the edge e with the vertex r. Furthermore, $r \notin N(r^*)$, which means that $\mathrm{ecc}_{T'}(r) \leq \mathrm{diam}(T') - 2$. As in Lemma 9, define $a = \mathrm{ecc}_{T-y}(x)$, and set $d = \mathrm{diam}(T)$. Then, by Corollaries 20 and 28, and Lemmas 8 and 9, we have

$$\begin{aligned}
\pi_t(G, r) &= \pi_t(G - e, r) \\
&= p_t(G - e, r) \\
&= \pi_t(T', r) + (n(G - e) - 1) + b(G - e) - 2e(T') \\
&= \pi_t(T', r) + (n(G) - 1) + b(G) + 1 - 2e(T) - 2 \\
&= \pi_t(T, x) + 2^a + (n(G) - 1) + b(G) + 1 - 2e(T) - 2 \\
&\leq \pi_t(T, x) + 2^{d-\mathrm{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1.
\end{aligned}$$

If $\mathrm{ecc}(x) \geq 3$ then we write

$$\begin{aligned}
\pi_t(G, r) &\leq \pi_t(T, x) + 2^{d-\mathrm{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1 \\
&\leq \pi_t(T, r^*) - (2^{d-2} - 2^{d-3}) + (n(G) - 1) + b(G) - 2e(T) - 1 \\
&< \pi_t(G, r^*).
\end{aligned}$$

Otherwise we have $\mathrm{ecc}(x) \leq 2$ and so, with h' defined as in Lemma 8, we find that

$$\begin{aligned}
\pi_t(G, r) &\leq \pi_t(T, x) + 2^{d-\mathrm{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1 \\
&\leq \pi_t(T, r^*) - t(2^d - 2^{\mathrm{ecc}(x)}) + 2^{h'} - 1 + 2^{d-\mathrm{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1 \\
&= \pi_t(G, r^*) - t(2^d - 2^{\mathrm{ecc}(x)}) + 2^{h'} - 1 + 2^{d-\mathrm{ecc}(x)} \\
&\leq \pi_t(G, r^*) - (2^d - 2^{d-2} - 2^{\lfloor d/2 \rfloor} + 1) \\
&< \pi_t(G, r^*),
\end{aligned}$$

since $d \geq 3$. $\square$

We pause to develop some notation that will be used in Corollary 32. Suppose that r is a fan vertex of G, in a unique fan F centered on x. Denote by H_1 and H_2 the two components of $G - \{r, x\}$, and by G_i the subgraph of G induced by $V(H_i) \cup \{r, x\}$. Let V_i be the vertices of $F' \cap H_i$ that are not in any other fan. Define $G'_i = G_i - V_i$. Finally, let the subscripts be labeled either so that V_2 is empty or so that neither V_1 nor V_2 is empty and $\mathrm{ecc}_{G_1}(r) \geq \mathrm{ecc}_{G_2}(r)$. See Fig. 3.

We note that G_i is a semi-2-tree except in the case that the block of G_i containing r, x and their unique common neighbor y is a K_3 (as in Fig. 2), because K_3 is not a 2-path. Observe that this happens if and only if $V_i = \emptyset$ and y is a cut vertex of G, and that in such a case $G_i - x$ is a semi-2-tree. Moreover, by the Neighbor Removal Lemma 24 with $A = \{x\}$, $\pi(G_i, r) = \pi(G_i - x, r) + 1$.

Claim 30. *Let G be a semi-2-tree and suppose that r is a fan vertex of G, in a unique fan F centered on x. Define G_i ($i \in \{1, 2\}$) as above, having n_i vertices. Define $T_i(r)$ to be the skeleton of G_i when G_i is a semi-2-tree and of $G_i - x$ when G_i is not a semi-2-tree. Define $T_i(x)$ to be the skeleton of $G_i - V_i - r$. Then for each $v \in \{r, x\}$ we have $\pi(G_i, v) = \pi(T_i(v), v) + (n_i - 1) + b(G_i) - 2e(T_i(v))$.*

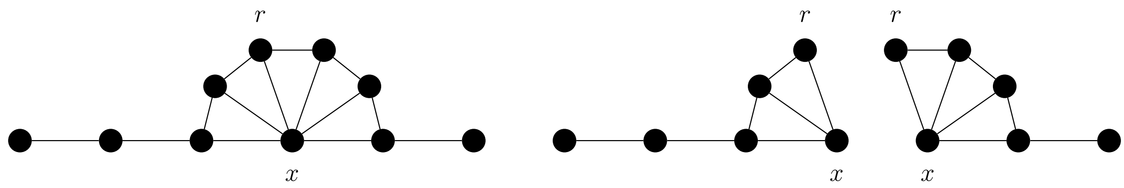

Fig. 3. A semi-2-tree G (left), split into G_1 (center) and G_2 (right).

Proof. For $v = r$ the result is true by Corollary 20 when G_i is a semi-2-tree, because r is simplicial. If G_i is not a semi-2-tree then $|V_i| = 0$ and $\{r, x, y\}$ is a K_3 block, where y is the common neighbor of r and x. In this case $G_i - x$ is a semi-2-tree, and so $\pi(G_i, r) = \pi(G_i - x, r) + 1$ by Lemma 24. This equals $\pi(T_i(r), r) + ((n_i - 1) - 1) + b(G_i - x) - 2e(T_i(r)) + 1 = \pi(T_i(r), r) + (n_i - 1) + b(G_i) - 2e(T_i(r))$.

For $v = x$ we have that x is a simplicial vertex of the semi-2-tree $G_i - V_i - r$, and so by Lemma 24 we obtain that $\pi(G_i, x) = \pi(G_i - V_i - r, x) + |V_i| + 1 = \pi(T_i(x), x) + ((n_i - |V_i| - 1) - 1) + b(G_i - V_i - r) - 2e(T_i(x)) + |V_i| + 1 = \pi(T_i(x), x) + (n_i - 1) + b(G_i) - 2e(T_i(x))$. □

Claim 31. *Under the same hypotheses as in Claim 30 we have $\pi(G_1, r) + \pi(G_2, x) \geq \pi(G_1, x) + \pi(G_2, r)$.*

Proof. Because of the cancellation of common terms, we have

$$\begin{aligned}
&[\pi(G_1, r) + \pi(G_2, x)] - [\pi(G_1, x) + \pi(G_2, r)] \\
&\quad = [\pi(T_1(r), r) + \pi(T_2(x), x)] - [\pi(T_1(x), x) + \pi(T_2(r), r)] \\
&\quad = [\pi(T_1(r), r) - \pi(T_1(x), x)] - [\pi(T_2(r), r) - \pi(T_2(x), x)].
\end{aligned}$$

Because of the cancellation of common branches, this equals

$$\left[2^{\mathrm{ecc}_{G_1}(r)} - 2^{\mathrm{ecc}_{G_1}(x)}\right] - \left[2^{\mathrm{ecc}_{G_2}(r)} - 2^{\mathrm{ecc}_{G_2}(x)}\right]. \tag{1}$$

We note that $\mathrm{ecc}_{G_i}(x) \leq \mathrm{ecc}_{G_i}(r) \leq \mathrm{ecc}_{G_i}(x) + 1$ for each i, with $\mathrm{ecc}_{G_i}(x) = \mathrm{ecc}_{G_i}(r)$ precisely when $V_i = \emptyset$. Thus, the choice of labeling ensures that (1) is non-negative. □

Corollary 32. *Let (G, r) be a rooted semi-2-tree with r not in any skeleton of G. If r is in a unique fan, centered on x, then (using the notation defined above) $\pi(G, r) = \pi(G_1, r) + \pi(G_2, x) - 2$.*

Proof. The lower bound is argued as follows. Let C_1 be an extremal configuration for r on G_1, C_2 be an extremal configuration for x on $G_2 - r$ (which is defined by using $A = V_2$ in the Neighbor Removal Lemma 24), and define the configuration $C = C_1 + C_2$.

Now $|C_1| = \pi(G_1, r) - 1$ and $|C_2| = \pi(G_2 - r, x) - 1 = \pi(G_2, x) - 2$ (by Lemma 24 with $A = \{r\}$), and thus $|C| = |C_1| + |C_2| = \pi(G_1, r) + \pi(G_2, x) - 3$. Furthermore, we claim that C is r-unsolvable. Indeed, C_1 cannot solve r by itself and cannot receive another pebble from C_2 through x, and C_2 (without its pebble already on r) cannot solve r by itself (any step to r can be replaced by a step to x, which would be a contradiction).

For the upper bound, assume that $|C| = \pi(G_1, r) + \pi(G_2, x) - 2$. Let $i \in \{1, 2\}$ and $j = 3 - i$. Define C_i to be the restriction of C to G_i. If $|C_i| \geq \pi(G_i, r)$ then C_i can solve r, so we assume otherwise. Then $|C_j| = |C| - |C_i| + C(x) \geq [\pi(G_1, r) + \pi(G_2, x) - 2] - [\pi(G_i, r) - 1] + C(x) \geq \pi(G_j, x) - 1 + C(x)$. Indeed, this follows trivially for $j = 2$, and from Claim 31 for $j = 1$. If $C(x) \geq 2$ then we can move a pebble to r. If $C(x) = 1$ then, since we may assume that $C(r) = 0$, we have $|C(G_j - r - x)| \geq \pi(G_j, x) - 1 = \pi(G_j - r, x)$ by Lemma 24, and so we can move a second pebble to x and then one to r. Hence we will assume that $C(x) = 0$.

If $|C(V_i)| \geq |V_i| + 2$ then V_i either has a huge vertex or two big vertices in which case it can solve r through x, or it has a big vertex with a path of all ones to r, which also solves r. Hence we assume that each $|C(V_i)| \leq |V_i| + 1$. Thus we have that

$$\begin{aligned}
|C(G_i')| &= |C_i| - |C(V_i)| \\
&\geq \pi(G_i, x) - |V_i| - 2 \\
&= \pi(G_i', x) - 1,
\end{aligned}$$

for each i.

Also, if some $|C(G_i')| \geq \pi(G_i', x)$ then we could place a pebble on x. This implies that $|C(V_j)| \leq |V_j|$ since a big vertex in V_j could place a second pebble on x, and then one on r. Then we would have

$$\begin{aligned}
|C(G_j')| &= |C_j| - |C(V_j)| \\
&\geq \pi(G_j, x) - |V_j| - 1 \\
&= \pi(G_j', x),
\end{aligned}$$

so that we could place a second pebble on x and solve r. Thus we must have $|C(G_i')| = \pi(G_i', x) - 1$ for each i.

Finally we see that

$$\begin{aligned}|V_1| + |V_2| + 2 &\geq |C(F)|\\ &= |C| - |C(G_1')| - |C(G_2')|\\ &= [\pi(G_1, r) - \pi(G_1', x)] + [\pi(G_2, x) - \pi(G_2', x)]\\ &= [\pi(G_1, r) - \pi(G_1, x)] + [\pi(G_1, x) - \pi(G_1', x)] + [|V_2| + 1]\\ &= [2^{\mathrm{ecc}_{G_1}(r)} - 2^{\mathrm{ecc}_{G_1}(x)}] + [|V_1| + 1] + [|V_2| + 1],\end{aligned}$$

which means that $2^{\mathrm{ecc}_{G_1}(x)} \geq 2^{\mathrm{ecc}_{G_1}(r)}$, and hence $2^{\mathrm{ecc}_{G_1}(x)} = 2^{\mathrm{ecc}_{G_1}(r)}$. That is, $V_1 = \emptyset$, which implies by our labeling that $V_2 = \emptyset$. Define x^- and x^+ to be the common neighbors of r and x. Then in the skeleton of G we can replace the path x^-xx^+ by the path x^-rx^+ to obtain a new skeleton containing r, which is a contradiction, completing the proof. □

Notice that the previous corollary allows one to calculate the pebbling number for r. In fact, one can use Corollaries 20 and 26 to calculate $\pi(G_1, r)$ and $\pi(G_2, x)$, respectively.

As with Corollary 7, the following is a simple consequence of Lemma 5.

Corollary 33. *Let (G, r) be a rooted semi-2-tree with r not in any skeleton of G. If C is a configuration of size at least $\pi(G, r) + (t-1)2^{\mathrm{ecc}(r)}$ then C has t distinct cheap r-solutions.* □

Similarly, Corollaries 28, 32 and 33 yield the following result.

Corollary 34. *Let (G, r) be a rooted semi-2-tree with r not in any skeleton of G. Then $\pi_t(G, r) = \pi(G, r) + (t-1)2^{\mathrm{ecc}(r)}$.* □

Theorem 35. *Let (G, r) be a rooted semi-2-tree with r not in any skeleton of G, and let r^* be a simplicial vertex of G with $\mathrm{ecc}(r^*) = \mathrm{diam}(G)$. Then $\pi_t(G, r) \leq \pi_t(G, r^*)$, with equality if and only if $\mathrm{ecc}(r) = \mathrm{diam}(G)$.*

Proof. We prove that $\pi(G, r) \leq \pi(G, r^*)$; then $\pi_t(G, r) = \pi(G, r) + (t-1)2^{\mathrm{ecc}(r)} \leq \pi(G, r^*) + (t-1)2^{\mathrm{ecc}(r^*)} = \pi_t(G, r^*)$ will follow.

First we analyze the case in which $\mathrm{ecc}(r) = \mathrm{ecc}(r^*)$. Define x to be the center of the fan containing r. Then we can suppose that x is in first (longest) path P^* in the maximum path partition of T with root r^*. If s^* is the other endpoint of P^* then x is adjacent to s^*. Hence $\mathrm{ecc}_{G_2}(x) = 1$ and so $\pi(G_2, x) = n(G_2) = |V_2| + 3$. Thus

$$\begin{aligned}\pi(G, r) &= \pi(G_1, r) + \pi(G_2, x) - 2\\ &= \pi(G_1, r) + |V_2| + 3 - 2\\ &= \pi(G_1, r) + |V_2| + 1.\end{aligned}$$

Also,

$$\begin{aligned}\pi(G, r^*) &= \pi(T, r^*) + (n(G) - 1) + b(G) - 2e(T)\\ &= \pi(T_1, r) + (n(G_1) + |V_2| + 1 - 1) + b(G_1) - 2e(T_1)\\ &= \pi(G_1, r) + |V_2| + 1.\end{aligned}$$

Henceforth we will assume that $\mathrm{ecc}(r) < \mathrm{ecc}(r^*)$. In this case we will make use of Lemma 8 and Corollaries 32 and 34. We will also use the facts that $n(G_1) + n(G_2) = n(G) + 2$, $b(G_1) + b(G_2) = b(G) + 1$, and $e(T_1) + e(T_2) = e(T) + \epsilon$, where ϵ, depends on some cases.

Notice that T_2 does not include the edge xr because x is the root, while T_1 does include the edge xr unless $V_1 = \emptyset$ and r and x have a common neighbor in $T \cap V_1$. Hence $\epsilon = 1$ except in this latter case, in which $\epsilon = 0$; that is, $\epsilon = 1 - |N(r) \cap N(x) \cap T \cap V_1|$. Now

$$\begin{aligned}\pi(G, r) &= \pi(G_1, r) + \pi(G_2, x) - 2\\ &= \pi(T_1, r) + (n(G_1) - 1) + b(G_1) - 2e(T_1)\\ &\quad + \pi(T_2, x) + (n(G_2) - 1) + b(G_2) - 2e(T_2) - 2\\ &= \pi(T_1, r) + \pi(T_2, x) + (n(G) - 1) + b(G) - 2e(T) - 2\epsilon.\end{aligned}$$

Analogous to the proof of Lemma 8, let P^* be a path $v_0v_1 \cdots v_d$ with $v_0 = r^*$ and $v_d = s^*$, labeled so that $\mathrm{dist}(x, s^*) \leq \mathrm{dist}(x, r^*) = \mathrm{ecc}(x)$. Denote by P the path from r^* to x, and set $P^* \cap P = v_0 \cdots v_{h'}$. Define $\overline{h} = d - h'$. let $\mathcal{P}^*$ be a maximum path partition of T with root r^*. Define $P_0^* = P^*, P_1^*, \ldots, P_k^*$ to be the sequence of paths of $\mathcal{P}^*$ that are used sequentially in P, and set $d_i^* = \mathrm{length}(P_i^*)$ for each $0 \leq i \leq k$ (so $d_0^* = d$). Next define $P_i' = P \cap P_i^*$, with $h_i' = \mathrm{length}(P_i')$ and $\overline{h}_i = d_i^* - h_i'$ (so $h_0' = h'$ and $\overline{h}_0 = \overline{h}$). Note that $1 \leq h_i' \leq d_i^* \leq d/2$.

Suppose that $k > 0$. Then, since $\mathrm{ecc}(r) < \mathrm{ecc}(r^*)$, we have

$$\begin{aligned}\pi(T_1, r) + \pi(T_2, x) &\le \pi(T_1, r^*) - 2^{d-2} + \pi(T_2, x)\\ &= \pi(T, r^*) + (2^{h'_k+1} - 1) + (2^{\bar{h}_k} - 1) - (2^{d^*_k} - 1) - 2^{d-2}\\ &\le \pi(T, r^*) + 1 - 2^{d-2}\\ &< \pi(T, r^*),\end{aligned}$$

because $d \ge 3$ and $2^{a+1} + 2^b - 2^{a+b} \le 2$ for all $a, b \ge 1$.

Suppose instead that $k = 0$. Define Q to be the longest path in a maximum path partition of T_1 (choosing the partition to contain P, if possible). If $Q = P$ then we have

$$\begin{aligned}\pi(T_1, r) + \pi(T_2, x) &\le \pi(T_1, r^*) - 2^{(h'_0+1)-2} + \pi(T_2, x)\\ &= \pi(T, r^*) - (2^d - 1) + (2^{h'_0+1} - 1) + (2^{\bar{h}_0} - 1) - 2^{h'_0-1}\\ &< \pi(T, r^*) - 2^d - 2^{h'_0+1} + 2^{\bar{h}_0} - 2^{h'_0-1}\\ &< \pi(T, r^*) - 2^d + 2^{\bar{h}_0}\\ &\le \pi(T, r^*) - 2^d + 2^{d-1}\\ &< \pi(T, r^*).\end{aligned}$$

Otherwise, when $Q \ne P$ we define $Q_0 = Q \cap P$ and $Q_1 = Q - Q_0$, having lengths q_0 and q_1, respectively, and set $\hat{h}_0 = h'_0 - q_0$. Here we will use that $q_0 + q_1 < d$, $q_1 > \hat{h}_0$, and $\bar{h}_0 \le d/2 \le q_0 + \hat{h}_0 \le q_0 + q_1$. Hence

$$\begin{aligned}\pi(T_1, r) + \pi(T_2, x) &\le \pi(T_1, r^*) - 2^{q_0+q_1-2} + \pi(T_2, x)\\ &= \pi(T, r^*) - (2^d - 1) - (2^{q_1} - 1) + (2^{q_0+q_1} - 1) + (2^{\hat{h}_0+1} - 1) + (2^{\bar{h}_0} - 1) - 2^{q_0+q_1-2}\\ &< \pi(T, r^*) - 2^d - 2^{q_1} + 2^{q_0+q_1} + 2^{\hat{h}_0+1} + 2^{q_0+q_1} - 2^{q_0+q_1-2}\\ &< \pi(T, r^*) - (2^d - 2^{q_0+q_1+1}) - (2^{q_1} - 2^{\hat{h}_0+1})\\ &\le \pi(T, r^*).\end{aligned}$$

In all cases, then, we see that

$$\begin{aligned}\pi(G, r) &= \pi(T_1, r) + \pi(T_2, x) + (n(G) - 1) + b(G) - 2e(T) - 2\epsilon.\\ &< \pi(T, r^*) + (n(G) - 1) + b(G) - 2e(T)\\ &= \pi(G, r^*),\end{aligned}$$

and the result follows. □

Theorem 36. *If G is a semi-2-tree then $\pi_t(G) = \pi_t(G, r^*)$, where r^* is a simplicial vertex with $\mathrm{ecc}(r^*) = \mathrm{diam}(G)$.*

Proof. Use Theorems 21, 27, 29 and 35. □

Theorem 37. *If G is a semi-2-tree then $\pi_t(G)$ can be computed in linear time.*

Proof. A breadth-first search from any simplicial vertex finds r^*, a simplicial vertex with $\mathrm{ecc}(r^*) = \mathrm{diam}(G)$. Indeed, this is true for trees, and the result extends to semi-2-trees as follows. Let T be the skeleton of G and let **A** be a breadth-first search algorithm on G. Then **A** is also a breadth-first search algorithm on T and so finds a simplicial vertex r with $\mathrm{ecc}_T(r) = \mathrm{diam}(T)$. Because T is a geodesic tree spanning all of the simplicial vertices of G, we have $\mathrm{ecc}_G(r) = \mathrm{ecc}_T(r)$ and $\mathrm{diam}(G) = \mathrm{diam}(T)$, and so $r^* = r$.

At this point, we do not yet know T. However, we realize that T can be constructed during **A** because it is a geodesic tree spanning all of the simplicial vertices of G. Once we have T we can remove its cut-vertices S (those having degree bigger than 2) to reveal b, which equals the number of components of $T - S$.

Then $\pi_t(T, r)$ can be computed in linear time, according to Theorem 3 of [6]. □

6. Remarks

The obvious pressing question is how to extend this work to 2-trees. The *pyramid* is the graph on 6 vertices formed by adjoining a 2-simplicial vertex onto each of the three sides of a triangle. The pyramid is the key structure that forms the basis in the Class 0 characterization of diameter two graphs found in [9] and is what causes the extra 1 in their pebbling numbers — the configuration with 3 pebbles at two of the simplicial vertices cannot reach the third. The pyramid is also the smallest example of a 2-tree that is not a semi-2-tree, and it hints at the complexity that can ensue in a more general 2-tree.

Another natural question in the direction of this research program regards other simple examples of chordal graphs, such as interval graphs. It would seem that tackling k-paths is a necessary investigation toward approaching interval graphs. One interesting thing about the 2-path pebbling number is that both of the standard lower bounds of $n(G)$ and $2^{\mathrm{diam}(G)}$ for general graphs G appear in its formula. This is encouraging in light of the manner in which the size of k can determine which of those two terms is dominant.

It appears that parameters such as pathwidth and treewidth may figure prominently in the determination of pebbling numbers of general graphs. Other authors have made similar remarks, for example in [7]. Thus considering these classes of graphs seems the most productive direction of research.

Our final thought points to the many lemmas developed in this paper that should be of very general use, including the Cheap Lemma 5 and the four Removal lemmas: Junior 10, Wart 11, Edge 22, and Neighbor 24. We anticipate their ability to simplify the analysis of many future problems.

Acknowledgments

The authors wish to thank the referees for excellent feedback that improved the exposition in this article greatly. The third author's research was partially supported by Simons Foundation Grant #246436.

References

[1] L. Alcón, M. Gutierrez, G. Hurlbert, Pebbling in split graphs, SIAM J. Discrete Math. 28 (3) (2014) 1449–1466.
[2] L. Alcón, M. Gutierrez, G. Hurlbert, Pebbling in 2-paths, Electron. Notes Discrete Math. 50 (2015) 145–150.
[3] A. Bekmetjev, C. Cusack, Pebbling algorithms in diameter two graphs, SIAM J. Discrete Math. 23 (2) (2009) 634–646.
[4] A. Blasiak, J. Schmitt, Degree sum conditions in graph pebbling, Australas. J. Combin. 42 (2008) 83–90.
[5] H.L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth, Theoret. Comput. Sci. 209 (1–2) (1998) 1–45.
[6] D. Bunde, E. Chambers, D. Cranston, K. Milans, D. West, Pebbling and optimal pebbling in graphs, J. Graph Theory 57 (2008) 215–238.
[7] M. Chan, A. Godbole, Improved pebbling bounds, Discrete Math. 308 (11) (2008) 2301–2306.
[8] F.R.K. Chung, Pebbling in hypercubes, SIAM J. Discrete Math. 2 (1989) 467–472.
[9] T. Clarke, R. Hochberg, G. Hurlbert, Pebbling in diameter two graphs and products of paths, J. Graph Theory 25 (2) (1997) 119–128.
[10] C.A. Cusack, T. Lewis, D. Simpson, S. Taggart, The complexity of pebbling in diameter two graphs, SIAM J. Discrete Math. 26 (2012) 919–928.
[11] D. Herscovici, B. Hester, G. Hurlbert, t-pebbling and extensions, Graphs Combin. 29 (4) (2013) 955–975.
[12] G. Hurlbert, Graph pebbling, in: D. Jaume, S. Eliahou (Eds.), Modern Methods in Combinatorics, in: 2nd Puntana School of Combinatorics, Centre International de Mathématiques Pures et Appliquées, 2013.
[13] G. Hurlbert, Graph pebbling, in: J. Gross, J. Yellen, P. Zhang (Eds.), Handbook of Graph Theory, second ed., in: Discrete Mathematics and its Applications, CRC Press, Boca Raton, 2014, pp. 1428–1454.
[14] G. Hurlbert, H. Kierstead, Graph pebbling complexity and fractional pebbling, unpublished (2005).
[15] T. Lewis, C.A. Cusack, L. Dion, The complexity of pebbling reachability and solvability in planar and outerplanar graphs, Discrete Appl. Math. 172 (2014) 62–74.
[16] K. Milans, B. Clark, The complexity of graph pebbling, SIAM J. Discrete Math. 20 (2006) 769–798.
[17] L. Pachter, H. Snevily, B. Voxman, On pebbling Graphs, Congr. Numer. 107 (1995) 65–80.