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On pebbling threshold functions for graph sequences

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Abstract

Given a connected graph G, and a distribution of t pebbles to the vertices of G, a pebbling step consists of removing two pebbles from a vertex v and placing one pebble on a neighbor of v. For a particular vertex r, the distribution is r-solvable if it is possible to place a pebble on r after a finite number of pebbling steps. The distribution is solvable if it is r-solvable for every r. The pebbling number of G is the least number t, so that every distribution of t pebbles is solvable. In this paper we are not concerned with such an absolute guarantee but rather an almost sure guarantee. A threshold function for a sequence of graphs $\mathscr{G}=(G_1,G_2,\ldots,G_n,\ldots)$, where G_n has n vertices, is any function $t_0(n)$ such that almost all distributions of t pebbles are solvable when $t\gg t_0$, and such that almost none are solvable when $t\ll t_0$. We give bounds on pebbling threshold functions for the sequences of cliques, stars, wheels, cubes, cycles and paths.

Keywords: Pebbling number; Threshold function

1. Introduction

All graphs considered here are connected and vertex-labeled. Given a graph G_n on n vertices and a distribution $D: V(G_n)\to \mathbf{N}$ of $t=\sum_v D(v)$ pebbles to the vertices of G_n, a pebbling step consists of removing two pebbles from a vertex v and placing one pebble on a neighbor of v. For a particular *root* vertex r, the distribution is *r-solvable* if it is possible to place a pebble on r after a finite number of pebbling steps. The

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distribution is G_n-*solvable* or just *solvable* if it is r-solvable for every r. The *pebbling number* $f(G_n)$ is the least integer $t = t(n)$ such that every distribution of t pebbles is solvable. We always take the vertex set of an n-vertex graph to be $\{v_i \mid i \in [n]\}$ or $[n]$, where $[n] = \{1, \ldots, n\}$. Thus, distributions D are independent of G_n.

Graph pebbling has an interesting history (see [3,5]). Essentially it was invented by Lagarias and Saks to provide an elegant solution to a number-theoretic question of Erdős and Lemke, originally solved by Kleitman and Lemke [10]. The area has grown considerably, with many nice results, open problems, and difficult conjectures (see [7,11]), including the famous conjecture of Graham that the pebbling number of a (cartesian) product of graphs is at most the product of the pebbling numbers of the graphs.

In this paper, we introduce a probabilistic pebbling model, where the pebbling distribution is selected uniformly at random from the set of all distributions with a prescribed number t of pebbles. Since the distribution of t pebbles to n vertices is like the distribution of t unlabeled balls to n labeled urns, our sample space consists of $\binom{n+t-1}{t}$ (equally likely) outcomes. We define and study thresholds so that if t is essentially larger than the threshold, then any distribution is almost surely solvable, and if t is essentially smaller than the threshold, then any distribution is almost surely unsolvable. Of course, the definition mimics the important threshold concept in random graph theory. Unlike the situation in random graphs, however, it does not seem obvious that even "natural" sequences of graphs have pebbling thresholds, one candidate being the sequence of paths. We should emphasize also that, unlike in random graph theory, even the most basic random variables considered here are functions of dependent random variables, and the dependence is not "sparse". This substantially limits the set of tools available for analyzing these random variables.

We now recall some basic asymptotic notation. For two functions f and g, we write $f \ll g$ (equivalently $g \gg f$) when the ratio $f(n)/g(n)$ approaches 0 as n tends to infinity. We use $o(g)$ and $\omega(f)$, respectively, to denote the sets $\{f \mid f \ll g\}$ and $\{g \mid f \ll g\}$, so that $f \in o(g)$ if and only if $g \in \omega(f)$. In addition, we write $f \in O(g)$ (equivalently $g \in \Omega(f)$) when there are positive constants c and k such that $f(n)/g(n) < c$ for all $n > k$, and we write $\Theta(g)$ for $O(g) \cap \Omega(g)$. We also use the shorthand notation $\Theta(f) \leqslant \Theta(g)$ to mean that $f' \in O(g')$ for every $f' \in \Theta(f)$ and $g' \in \Theta(g)$. Finally, we write $f \lesssim g$ for $\limsup f/g \leqslant 1$. To avoid cluttering the paper with floor and ceiling symbols, we adopt the convention that large constants (such as $1/\varepsilon$ when ε is small) are integers.

We are almost ready to define formally our notion of a pebbling threshold function. Let $D_n : [n] \to \mathbf{N}$ denote a distribution of pebbles on n vertices. For a particular function $t = t(n)$, we consider the probability space $\Omega_{n,t}$ of all distributions D_n of *size* t, i.e. with $t = \sum_{i \in [n]} D_n(i)$ pebbles. Given a graph sequence $\mathscr{G} = (G_1, \ldots, G_n, \ldots)$ with $V(G_n) = [n]$, denote by $P_{\mathscr{G}}(n,t)$ the probability that an element of $\Omega_{n,t}$ chosen uniformly at random is G_n-solvable. We call a function g a *threshold* for $\mathscr{G}$, and write $g \in \text{th}(\mathscr{G})$, if the following two statements hold as $n \to \infty$: (i) $P_{\mathscr{G}}(n,t) \to 1$ whenever $t \gg g$, and (ii) $P_{\mathscr{G}}(n,t) \to 0$ whenever $t \ll g$. Notice that $\text{th}(\mathscr{G}) = \Theta(g)$ exactly when

$g \in \mathrm{th}(\mathscr{G})$. Some of the graph sequences under consideration, such as the sequence $\mathscr{Q}$ of cubes below, have the form $\mathscr{G} = (G_1, \ldots, G_m, \ldots)$ with $V(G_m) = [n_m]$, where $\{n_m\}$ is a subsequence of $\mathbf{N}$. Here we put $g \in \mathrm{th}(\mathscr{G})$ if (i) $P_{\mathscr{G}}(n_m, t) \to 1$ whenever $t \gg g$, and (ii) $P_{\mathscr{G}}(n_m, t) \to 0$ whenever $t \ll g$, with these limits now taken as $m \to \infty$.

We shall consider the following sequences of graphs.

- $\mathscr{K} = (K_1, \ldots, K_n, \ldots)$: K_n is the complete graph on n vertices.
- $\mathscr{P} = (P_1, \ldots, P_n, \ldots)$: P_n is the path on n vertices.
- $\mathscr{C} = (C_1, \ldots, C_n, \ldots)$: C_n is the cycle on n vertices.
- $\mathscr{S} = (S_1, \ldots, S_n, \ldots)$: S_n is the star on n vertices.
- $\mathscr{W} = (W_1, \ldots, W_n, \ldots)$: W_n is the wheel on n vertices.
- $\mathscr{Q} = (Q^1, \ldots, Q^m, \ldots)$: Q^m is the m-dimensional cube on $n_m = 2^m$ vertices.

In addition, we will denote generic sequences of graphs by $\mathscr{G} = (G_1, \ldots, G_n, \ldots)$ or $\mathscr{H} = (H_1, \ldots, H_n, \ldots)$. We use $x \sim y$ to indicate that the vertices x and y are adjacent, so, e.g., in P_n, we have $v_i \sim v_j$ if and only if $j = i \pm 1$, and in C_n we add the extra adjacency $v_1 \sim v_n$.

Supposing that thresholds exist for the graph sequences above, we will show that for every $\varepsilon > 0$, we have $\mathrm{th}(\mathscr{G}) \subseteq o(n^{1+\varepsilon})$. For paths, we will show that $\mathrm{th}(\mathscr{P}) \subseteq \Omega(n)$ which shows that $\mathrm{th}(\mathscr{P}) \subseteq \Omega(n) \cap o(n^{1+\varepsilon})$ for any $\varepsilon > 0$. The technique used for paths can be extended to cycles, yielding a similar result. We further prove that $\mathrm{th}(\mathscr{S}) = \Theta(n^{1/2})$, which implies a similar result for wheels. It is interesting to note that, in light of the results for paths and stars, it is conceivable that the set of thresholds for all possible sequences of trees may span the entire range of functions from $n^{1/2}$ to n (or $n^{1+\varepsilon}$, as the case may be).

The rest of the paper is organized as follows. In the next section we make a few observations concerning pebbling thresholds of general graphs. Section 3 contains an overview of some relevant pebbling theorems and their consequences for our probabilistic model. In Section 4, we establish the upper bound, $\mathrm{th}(\mathscr{G}) \subseteq o(n^{1+\varepsilon})$, as well as the results for paths, cycles, stars and wheels mentioned above. The result for paths, Theorem 4.2, invokes Markov's inequality for a certain random variable, so we did not require its second moment. Nevertheless, we did estimate this quantity with the hope of improving Theorem 4.2. This computation appears as an appendix in our final Section 6. In Section 5, we close the main body of the paper with some intriguing open questions and problems.

2. Preliminaries

Here we make some elementary observations about pebbling thresholds of generic sequences of graphs. Our first fact follows easily from the definitions.

Fact 2.1. *If $E(H_n) \subseteq E(G_n)$, then the probability that D_n solves G_n is at least as large as the probability that D_n solves H_n.*

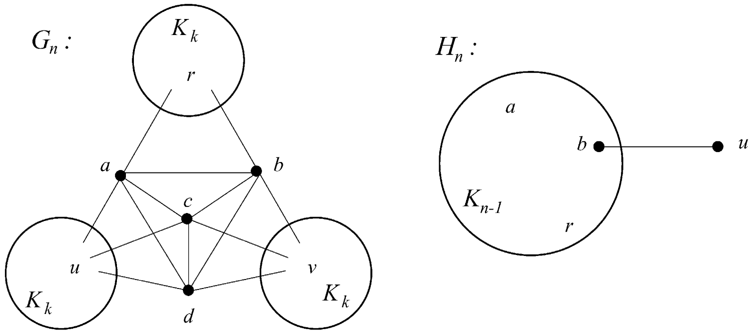

Fig. 1. A counterexample to $f(G_n) \leqslant f(H_n) \Rightarrow P_{\mathscr{G}}(n,t) \geqslant P_{\mathscr{H}}(n,t)$.

Proof. Every H_n-solvable distribution is G_n-solvable. □

A simple consequence of this is

Fact 2.2. *If* $E(H_n) \subseteq E(G_n)$ *for all* n, *and* $\mathrm{th}(\mathscr{G})$ *and* $\mathrm{th}(\mathscr{H})$ *both exist, then* $\mathrm{th}(\mathscr{G}) \leqslant \mathrm{th}(\mathscr{H})$.

A natural generalization leads to the following innocent looking

Question 2.3. *If* $f(G_n) \leqslant f(H_n)$ *for all* n, *and* $\mathrm{th}(\mathscr{G})$ *and* $\mathrm{th}(\mathscr{H})$ *both exist, then is it true that* $\mathrm{th}(\mathscr{G}) \leqslant \mathrm{th}(\mathscr{H})$?

One approach to answering Question 2.3 affirmatively might be to attempt to prove that, if $f(G_n) \leqslant f(H_n)$, then, for all t, $P_{\mathscr{G}}(n,t) \geqslant P_{\mathscr{H}}(n,t)$. However, this statement is false, as the following example shows. In Fig. 1, besides participating in a K_4 with the vertices $b,c,d \in V(G_n)$, the vertex a is joined to every vertex in the two copies of K_k indicated by line segments, and likewise for b, c and d. Our counterexample illustrates one case where the implication fails when the randomly chosen distribution contains $t = n-1$ pebbles (on both G_n and H_n).

It was proved in [12] that every diameter 2 graph has pebbling number either n or $n+1$. In [5], the family $\mathscr{F}$ of 2-connected, diameter 2 graphs having pebbling number $n+1$ was characterized. Since G_n is 2-connected, has diameter 2 and does not belong to $\mathscr{F}$, we have $f(G_n) = n$. Furthermore, if $D = D_n$ has $D(r) = D(a) = D(b) = D(c) = D(d) = 0$, $D(u) = D(v) = 3$, and $D(x) = 1$ otherwise, then D is r-unsolvable in G_n and has $t = n-1$ pebbles. The number of such distributions is in $\Omega(n^3)$.

Turning to H_n, it is easy to see that every graph with a cut vertex has pebbling number at least $n+1$, and since H_n in addition has diameter 2, we have $f(H_n) = n+1$. Since no H_n-unsolvable distribution places more than three pebbles on any single vertex, such distributions $D = D_n$ can be described economically. Indeed, for $i \in \{0,1,2,3\}$, let n_i denote the number of vertices $x \notin \{b,u\}$ with $D(x) = i$. Then the H_n-unsolvable distributions with $n-1$ pebbles have the following "shapes":

n_0	n_1	n_2	n_3	$D(b)$	$D(u)$
2	$n-4$	0	0	0	3
1	$n-4$	0	1	0	0
1	$n-3$	0	0	0	2
0	$n-3$	1	0	0	0
1	$n-3$	0	0	1	1
0	$n-2$	0	0	0	1
0	$n-2$	0	0	1	0.

For example (cf. the first row above), the distribution defined by $D(r)=D(a)=D(b)=0$, $D(u)=3$, and $D(x)=1$ otherwise, is r-unsolvable. On observing that the number of H_n-unsolvable distributions with $n-1$ pebbles is in $\Theta(n^2)$, we arrive at $P_{\mathscr{G}}(n,t) \leqslant P_{\mathscr{H}}(n,t)$.

However, one can imagine that the implication in Question 2.3 may hold with the added hypothesis that $f(G_n)$ is significantly smaller than $f(H_n)$: say $f(H_n)-f(G_n) \to \infty$ as $n \to \infty$, or $\limsup_{n\to\infty} f(G_n)/f(H_n) < 1$.

3. Precursors

Our first threshold result is for cliques. Because of the pigeonhole principle, it is merely an "unordered" reformulation of the so-called "Birthday Problem" (see [13]).

Result 3.1 (Clarke, Hurlbert [4]). *The threshold for the sequence of cliques is* $\mathrm{th}(\mathscr{K}) = \Theta(n^{1/2})$.

Proof. If $D=D_n$ is chosen uniformly at random from among all $\binom{n+t-1}{t}$ distributions on K_n with $t=t(n)$ pebbles, then the favorable distributions for the event $B=\{D$ is not solvable$\}$ are those in which the pebbles occupy distinct vertices. Thus,

$$\Pr(B) = \frac{\binom{n}{t}}{\binom{n+t-1}{t}},$$

which by Stirling's formula (and a little computation), approaches 1 if $t(n) \ll n^{1/2}$ and 0 if $t(n) \gg n^{1/2}$. Since $P_{\mathscr{K}}(n,t) = 1-\Pr(B)$, the result follows. $\square$

Next, we note the following obvious fact.

Fact 3.2. *For given* $\mathscr{G}$ *define the function* $g:\mathbf{N}\to\mathbf{N}$ *by* $g(n)=f(G_n)$ *and suppose that* $\mathrm{th}(\mathscr{G})$ *exists. Then* $\mathrm{th}(\mathscr{G}) \subseteq O(g)$.

This yields the following easy corollaries.

Corollary 3.3. *If* $\mathrm{th}(\mathscr{Q})$ *exists then* $\mathrm{th}(\mathscr{Q}) \subseteq O(n)$.

Proof. Chung proved in [3] that $f(Q^m) = n = 2^m$. □

Corollary 3.4. *Let* G_n *have diameter* 2 *for all* n. *If* $\mathrm{th}(\mathscr{G})$ *exists then* $\mathrm{th}(\mathscr{G}) \subseteq O(n)$.

Proof. It was proved in [12] that $f(G_n) \leqslant n + 1$ (see also [5]). □

In fact, Corollary 3.4 can be generalized by noting that, for an arbitrary graph sequence $\mathscr{G}$, $f(G_n) \leqslant 2^d n$, where d is the diameter of G_n (this follows easily from the pigeonhole principle). Then Fact 3.2 also yields

Corollary 3.5. *Let* $d(n) = \mathrm{diameter}(G_n)$ *and suppose that* $\mathrm{th}(\mathscr{G})$ *exists. Then* $\mathrm{th}(\mathscr{G}) \subseteq O(2^{d(n)}n)$. *In particular, if* $d(n) \leqslant d$ *for all* n, *then* $\mathrm{th}(\mathscr{G}) \subseteq O(n)$.

More recently, it was proved in [6] that $f(G) = n$ for any graph G for which $k \geqslant 2^{2d+3}$, where the connectivity of G is at least k and the diameter of G is at most d.

Corollary 3.6. *Let* $d(n) = \mathrm{diameter}(G_n)$, $k(n) = \mathrm{connectivity}(G_n)$, *and suppose that* $\mathrm{th}(\mathscr{G})$ *exists. If* $k(n) \geqslant 2^{2d(n)+3}$ *for all* n, *then* $\mathrm{th}(\mathscr{G}) \subseteq O(n)$.

4. Thresholds

We begin with an upper bound on $\mathrm{th}(\mathscr{G})$, valid for all $\mathscr{G}$. Then we present our theorems on pebbling thresholds for paths and stars, followed by applications to the thresholds for cycles and wheels.

Theorem 4.1. *For all* $\mathscr{G}$, *if* $\mathrm{th}(\mathscr{G})$ *exists then, for any given* $\varepsilon > 0$, *we have* $\mathrm{th}(\mathscr{G}) \subseteq o(n^{1+\varepsilon})$.

Proof. Let $\varepsilon > 0$ be given, and let $t : \mathbf{N} \to \mathbf{N}$ so that, for each n, $D = D_n$ is chosen uniformly at random from among all distributions on $[n]$ of size $t(n)$. Denote by $\Pr(n)$ the probability that D is solvable. We show that $t(n) \in \Omega(n^{1+\varepsilon})$ implies $\Pr(n) \to 1$ as $n \to \infty$.

One can argue that, for any graph H_l on l vertices, we have $f(H_l) < 2^l$. Indeed, suppose T_l is a spanning tree of H_l, and let P_l be the path on l vertices. Then $f(H_l) \leqslant f(T_l) \leqslant f(P_l) = 2^{l-1}$. The last inequality follows from a result of Moews [11]. We use this observation below.

Let $\delta > 0$, $t \geqslant cn^{1+\varepsilon}$ for some $c > 0$, $l = (1 + \delta)/\varepsilon$, and $k = 2^l$. Fix n and consider the graph $G = G_n$. For each vertex v, choose $G(v)$ to be a connected subgraph of G containing l vertices, including v. Let $|D_{G(v)}|$ denote the number of pebbles on vertices of $G(v)$. We call $G(v)$ an *l-neighborhood* of G; the subgraph $G(v)$ is *k-bounded* in case $|D_{G(v)}| < k$. We claim that the probability that there is a k-bounded l-neighborhood tends to zero. Indeed,

$$
\begin{aligned}
&\Pr[\exists\ k\text{-bounded } l\text{-neighborhood}] \leqslant n \Pr[G(v) \text{ is } k\text{-bounded}]\\
&= n\sum_{i=0}^{k-1} \Pr[|D_{G(v)}| = i] = n\sum_{i=0}^{k-1} \frac{\binom{l+i-1}{i}\binom{n-l+t-i-1}{t-i}}{\binom{n+t-1}{t}}\\
&= \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\binom{n-l+t-1}{t}\prod_{j=0}^{i-1}\left(\frac{t-j}{n-l+t-j-1}\right)\\
&\leqslant \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\binom{n-l+t-1}{t}\left(\frac{t}{n-l+t-1}\right)^{i}\\
&= \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\left(\frac{t}{n-l+t-1}\right)^{i}\binom{n+t-1}{t}\\
&\quad \prod_{j=1}^{l}\left(\frac{n-j}{n+t-j}\right)\\
&\leqslant n\sum_{i=0}^{k-1}\binom{l+i-1}{i}\left(\frac{n}{t}\right)^{l} = n\left(\frac{n}{t}\right)^{l}\binom{l+k-1}{k} \leqslant c^{-l}(n^{1-\varepsilon l})\binom{l+k-1}{k}\\
&= Cn^{-\delta} \to 0.
\end{aligned}
$$

Thus, with probability tending to 1, every l-neighborhood of G contains at least $k = 2^l$ pebbles, from which $\Pr(n) \to 1$ follows. □

Theorem 4.2. *If* $\mathrm{th}(\mathscr{P})$ *exists then, for every* $\varepsilon > 0$, *we have* $\mathrm{th}(\mathscr{P}) \subseteq \Omega(n) \cap o(n^{1+\varepsilon})$.

Proof. The upper bound follows from Theorem 4.1, so it remains only to establish the lower bound. Let $t: \mathbf{N} \to \mathbf{N}$ and, as usual, for each n let $D = D_n$ be chosen uniformly at random from among all distributions of size $t(n)$. Denote by $\Pr(n)$ the probability that D is solvable. We show that $t(n) \ll n$ implies $\Pr(n) \to 0$ as $n \to \infty$.

Let the vertices of P_n be labeled so that $v_1 \sim v_2 \sim \cdots \sim v_n$. For each v_i, let $X_i = D(v_i)$; also, let $X = \sum X_i$ and $Y = \sum X_i/2^{i-1}$. Notice that P_n is v_1-solvable if and only if $Y \geqslant 1$. We proceed to show that the probability of this event tends to 0 when $t \ll n$.

Let $t = n/\omega$ for any $\omega \to \infty$. Then $\mathbf{E}(X_i) = t/n = 1/\omega \to 0$. Consequently, $\mathbf{E}(Y) = \sum \mathbf{E}(X_i)/2^{i-1} = (\sum 1/2^{i-1})/\omega < 2/\omega \to 0$. Now, Markov's inequality (see, e.g. [13] or [14]) shows that

$$\Pr[Y \geqslant 1] \leqslant \mathbf{E}(Y)/1 \ \to \ 0. \qquad \square$$

Theorem 4.3. *If* $\mathrm{th}(\mathscr{C})$ *exists then, for every* $\varepsilon > 0$, *we have* $\mathrm{th}(\mathscr{C}) \subseteq \Omega(n) \cap o(n^{1+\varepsilon})$.

Proof. Label the vertices of C_n as above, with $v_n \sim v_1$. Use the same proof technique as in Theorem 4.2, except replace Y with $Y + Y'$, where $Y' = \sum X_i/2^{n-i+1}$. Then $\mathbf{E}(Y') = \mathbf{E}(Y)$, so

$$\Pr[Y + Y' \geqslant 1] \leqslant \mathbf{E}(Y + Y') = 2\mathbf{E}(Y) \to 0. \qquad \square$$

Theorem 4.4. *The threshold for the sequence of stars is* $\mathrm{th}(\mathscr{S}) = \Theta(n^{1/2})$.

Proof. Let $t : \mathbf{N} \to \mathbf{N}$, and for each n let $D = D_n$ be chosen uniformly at random from among all distributions on $[n]$ of size $t(n)$. Denote by $\Pr(n)$ the probability that D is solvable. It follows from Fact 2.1 and Result 3.1 that $t(n) \ll n^{1/2}$ implies $\Pr(n) \to 0$ as $n \to \infty$. By considering the probability $q = 1 - \Pr(n)$ of the complementary event that D is unsolvable, we show that $t(n) \gg n^{1/2}$ implies $\Pr(n) \to 1$ as $n \to \infty$.

Let $t = t(n) = \omega n^{1/2}$ for any $\omega \to \infty$. Because $f(S_n) = n$ for all n, we may assume that $t < n$. We produce an upper bound for q that tends to 0 by over-counting the number of unsolvable distributions of size t and dividing by the total number $\binom{n+t-1}{t}$ of distributions.

If D is unsolvable, then, for the star's center c, we have $D(c) \leqslant 1$, and there is at most one vertex v with $D(v) > 1$. (Indeed, if both of $D(u)$, $D(v)$ exceed 1, then it is possible to pebble so that the updated $D(c) \geqslant 2$; then one can pebble to any root.) In addition, we have $D(v) < 4$ for all v because each S_n has diameter 2.

The number of such distributions is equal to the number having no v with $D(v) > 1$, plus the number having exactly one v (different from the center) with $1 < D(v) < 4$. (In the latter case, the center contains no pebbles.) This number is exactly

$$\binom{n}{t} + (n-1)\binom{n-2}{t-2} + (n-1)\binom{n-2}{t-3.}.$$

Hence,

$$
\begin{aligned}
q &= \left[1 + \frac{t(t-1)}{n} + \frac{t(t-1)(t-2)}{n(n-t+1)}\right] \frac{(n)_t}{(n+t-1)_t} \\
&< \left[1 + \frac{t^2}{n} + \frac{t^3}{n(n-t)}\right] \left(\frac{n}{n+t}\right)^{t-1} \\
&\lesssim \left[1 + \omega^2 + \frac{\omega^3 n^{1/2}}{n - \omega n^{1/2}}\right] \mathrm{e}^{-t(t-1)/(n+t)} \\
&\lesssim \left[1 + \omega^2 + \frac{\omega^3 n^{1/2}}{n - \omega n^{1/2}}\right] \mathrm{e}^{-\omega^2/2}. \qquad (1)
\end{aligned}
$$

To see that the last expression tends to zero, consider two cases. If $\omega < n^{1/2} - 1$ (so that $n - \omega n^{1/2} > n^{1/2}$), then the right side of (1) is at most $[1 + \omega^2 + \omega^3]\mathrm{e}^{-\omega^2/2}$, which approaches 0 as $\omega \to \infty$. On the other hand, if $\omega \geqslant n^{1/2} - 1$, then, since $n - \omega n^{1/2} = n - t \geqslant 1$, the right side of (1) is at most $[1 + \omega^2 + \omega^3 n^{1/2}]\mathrm{e}^{-\omega^2/2} \leqslant [1 + \omega^2 + \omega^3(\omega + 1)]\mathrm{e}^{-\omega^2/2}$, which also tends to 0 as $\omega \to \infty$. $\square$

Remark. While the preceding case analysis has the advantage of keeping the proof elementary, our decision to employ this strategy hides a cleaner—and we believe more elegant—approach depending on a natural lattice-theoretic property. The *normal* property of the multiset lattice (see, e.g. [8] or [9]) states that if $\mathscr{F}$ is a monotone increasing family of multisets (sub-multisets) of a fixed set, then $\Pr(\mathscr{F}_t)$, the fraction of all multisets of size t belonging to $\mathscr{F}$, is an increasing function of t. Of course, the family of all solvable distributions—where a distribution with t pebbles is viewed as a t-multiset of the vertex set—is monotone increasing, since it is closed under the operation of adding an element (a pebble) to any of its members.

In this context, the normal property guarantees that $\Pr(n)$ in the proof of Theorem 4.4 increases with t. Thus, for this proof, it would have sufficed to consider only small values of t, relative to n, and the reader may easily verify that the assumption $\omega \ll n^{1/2}$ reduces to one of the number of our closing cases.

Fact 2.1—and comparison with stars and cliques—yields the

Corollary 4.5. *The threshold for the sequence of wheels is* $\mathrm{th}(\mathscr{W}) = \Theta(n^{1/2})$.

5. Questions

To close, we add some questions and problems for further research to the list started with Question 2.3. Most fundamental is the

Question 5.1. *Is it true that for any graph sequence* $\mathscr{G}=(G_1,\ldots,G_n,\ldots)$, *its pebbling threshold* $\mathrm{th}(\mathscr{G})$ *exists*?

Based on our results in Sections 3 and 4, it may seem obvious how to construct a sequence which would yield a negative answer to Question 5.1. For example, consider the graph sequence $\mathscr{G}=(K_1,C_2,\ldots,K_{2m-1},C_{2m},\ldots)$ and let $t:\mathbf{N}\to\mathbf{N}$ be such that $n^{1/2}\ll t(n)\ll n/\log\log n$. Then the probability $\Pr(n)$ that a uniformly randomly chosen distribution D_n of size $t(n)$ is solvable has no limit as $n\to\infty$. Indeed, as $m\to\infty$, $\Pr(2m-1)\to 1$ by Result 3.1; while $\Pr(2m)\to 0$ by Theorem 4.3. Nevertheless, provided $\mathrm{th}(\mathscr{C})$ exists, then so does $\mathrm{th}(\mathscr{G})$; it is simply $\mathrm{th}(\mathscr{K})$ for odd n and $\mathrm{th}(\mathscr{C})$ for even n.

Problem 5.2. *Find* $\mathrm{th}(\mathscr{P})$.

Because $\Theta(n)\leqslant\mathrm{th}(\mathscr{P})\leqslant\Theta(n^{1+\varepsilon})$ (provided $\mathrm{th}(\mathscr{P})$ exists), there is room for threshold functions like $n\log n$. However, because the variance of Y (from the proof of Theorem 4.2) is large—see the Appendix—there is also room for no threshold to exist. That is, it may be the case that for some t satisfying $n\ll t\ll n^{1+\varepsilon}$, we have $0<\liminf_{n\to\infty}P_{\mathscr{P}}(n,t)\leqslant\limsup_{n\to\infty}P_{\mathscr{P}}(n,t)<1$.

Problem 5.3. *Find* $\mathrm{th}(\mathscr{Q})$.

Given that the pebbling numbers of the cubes are known exactly (see the proof of Corollary 3.3 or [3]), it seems surprising that our present knowledge leaves open such a wide gap between the lower and upper bounds for $\mathrm{th}(\mathscr{Q})\in\Omega(n^{1/2})\cap O(n)$ (provided this threshold exists).

Finally, we ask

Question 5.4. *Is it true that, for all* $\Omega(n^{1/2})\ni t_1\ll t_2\in O(n)$, *there is a graph sequence* $\mathscr{G}$ *such that* $\mathrm{th}(\mathscr{G})\in\Omega(t_1)\cap O(t_2)$?

In view of Theorems 4.2 and 4.4 (respectively on thresholds for paths and stars, the "extreme cases" of trees), it is conceivable that this is true even within the class of trees.

6. Appendix

Here we compute and estimate the second moment of the random variable Y, defined in the proof of Theorem 4.2. Though it is not used explicitly in this paper, a refinement of our ideas eventually led via the second moment method—see [1]—to an improvement of Theorem 4.2. This is discussed in a forthcoming paper [2].

Let us start with a few combinatorial identities. The first one is standard.

Lemma 6.1.

$$\sum_{l=0}^{t} \binom{l+a}{a} = \binom{t+a+1}{a+1}.$$

Lemma 6.2.

$$\sum_{l=0}^{t} l \binom{l+a}{a} = t\left(\frac{a+1}{a+2}\right)\binom{t+a+1}{a+1}.$$

Proof.

$$\frac{1}{a+1}\sum_{l=0}^{t} l \binom{l+a}{a} = \sum_{l=1}^{t}\binom{l+a}{a+1} = \sum_{l=0}^{t-1}\binom{l+1+a}{a+1}$$

which, by Lemma 6.1, is equal to

$$\binom{t+a+1}{a+2} = \frac{t}{a+2}\binom{t+a+1}{a+1}. \qquad \square$$

Lemma 6.3.

$$\sum_{l=0}^{t} l^2 \binom{l+a}{a} = (a+1)t\left(\frac{t-1}{a+3}+\frac{1}{a+2}\right)\binom{t+a+1}{a+1}.$$

Proof.

$$\begin{aligned}\sum_{l=0}^{t} l^2 \binom{l+a}{a} &= (a+1)\sum_{l=1}^{t} l\binom{l+a}{a+1} = (a+1)\sum_{l=0}^{t-1}(l+1)\binom{l+a+1}{a+1}\\ &= (a+1)\left[\sum_{l=0}^{t-1} l\binom{l+a+1}{a+1} + \sum_{l=0}^{t-1}\binom{l+a+1}{a+1}\right]. \qquad (6.1)\end{aligned}$$

Now, we apply Lemmas 6.1 and 6.2 to conclude that the right-hand side of (6.1) is equal to

$$\begin{aligned}&(a+1)\left[(t-1)\left(\frac{a+2}{a+3}\right)+1\right]\binom{t+a+1}{a+2}\\ &\quad = (a+1)t\left(\frac{t-1}{a+3}+\frac{1}{a+2}\right)\binom{t+a+1}{a+1}. \qquad \square\end{aligned}$$

Let $Y=\sum_{i=1}^{n} Y_i$, where the $Y_i = 2^{1-i}X_i$ are the random variables defined in Section 4. If we consider distributions with t pebbles, then $\mathbf{E}(X_i)=t/n$, whence $\mathbf{E}(Y)=\sum \mathbf{E}(Y_i)=tn^{-1}\sum 2^{1-i} \sim 2t/n$.

Lemma 6.4. $\mathbf{E}(X_i^2)=[2t^2+t(n-1)]/n(n+1)$.

Proof.

$$\mathbf{E}(X_i^2)=\sum_{k=0}^{t}k^2\frac{\binom{t+n-k-2}{t-k}}{\binom{t+n-1}{t}}=\frac{1}{\binom{t+n-1}{n-1}}\sum_{l=0}^{t}(t-l)^2\binom{l+n-2}{n-2}$$

$$=\frac{1}{\binom{t+n-1}{n-1}}\left[t^2\sum_{l=0}^{t}\binom{l+n-2}{n-2}-2t\sum_{l=0}^{t}l\binom{l+n-2}{n-2}\right.$$
$$\left.+\sum_{l=0}^{t}l^2\binom{l+n-2}{n-2}\right]$$

and, applying Lemmas 6.1–6.3 with $a=n-2$, we see that

$$\mathbf{E}(X_i^2)=t^2-2t^2\left(\frac{n-1}{n}\right)+(n-1)t\left(\frac{t-1}{n+1}+\frac{1}{n}\right)=\frac{2t^2+t(n-1)}{n(n+1)}. \qquad \square$$

Lemma 6.5. *For $i\neq j$, we have* $\mathbf{E}(X_iX_j)=(t^2-t)/n(n+1)$.

Proof. Since

$$\mathbf{E}(X_iX_j)=\sum_{k=0}^{t}k\frac{\binom{t+n-k-2}{t-k}}{\binom{t+n-1}{t}}\mathbf{E}(X_j\,|\,X_i=k),$$

and $\mathbf{E}(X_j\,|\,X_i=k)$ is equal to $(t-k)/(n-1)$, it follows that

$$\mathbf{E}(X_iX_j)=\frac{1}{(n-1)\binom{t+n-1}{n-1}}\sum_{k=0}^{t}k(t-k)\binom{t-k+n-2}{n-2}$$

$$=\frac{1}{(n-1)\binom{t+n-1}{n-1}}\sum_{l=0}^{t}(t-l)l\binom{l+n-2}{n-2}$$

$$=\frac{1}{(n-1)\binom{t+n-1}{n-1}}\left[t\sum_{l=0}^{t}l\binom{l+n-2}{n-2}-\sum_{l=0}^{t}l^2\binom{l+n-2}{n-2}\right]$$

which, by Lemmas 6.2 and 6.3, is equal to

$$\left(\frac{1}{n-1}\right)\left[\frac{(n-1)t^2}{n}-(n-1)t\left(\frac{t-1}{n+1}+\frac{1}{n}\right)\right]=\frac{t^2-t}{n(n+1)}. \qquad \square$$

At last we are ready to compute the second moment of Y:

$$\mathbf{E}(Y^2) = \mathbf{E}\left(\left(\sum Y_i\right)^2\right) = \mathbf{E}\left(\sum Y_i^2\right) + \mathbf{E}\left(\sum_{i \neq j} Y_i Y_j\right)$$

$$= \sum \frac{\mathbf{E}(X_i^2)}{4^{i-1}} + \sum_{i \neq j} \frac{\mathbf{E}(X_i X_j)}{2^{i+j-2}}.$$

Note that if $t \geqslant n$ and n is large, we have $\mathbf{E}(X_i X_j) \sim t^2/n^2$ and $2t^2 n^{-2} \leqslant \mathbf{E}(X_i^2) \leqslant 3t^2 n^{-2}$. Thus, for $t \geqslant n$ and some positive constants c_1, c_2, we have

$$c_1 \left(\frac{t}{n}\right)^2 \leqslant \mathbf{E}(Y^2) \leqslant c_2 \left(\frac{t}{n}\right)^2.$$

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