Cops and Robbers Pebbling in Graphs

Nancy E. Clarke[*] Joshua Forkin[†] Glenn Hurlbert[†]

Dedicated to Oleksandr Stanzhytsky, and others like him

who are doing less mathematics than usual at this time.

Abstract

Here we merge the two fields of Cops and Robbers and Graph Pebbling to introduce the new topic of Cops and Robbers Pebbling. Both paradigms can be described by moving tokens (the cops) along the edges of a graph to capture a special token (the robber). In Cops and Robbers, all tokens move freely, whereas, in Graph Pebbling, some of the chasing tokens disappear with movement while the robber is stationary. In Cops and Robbers Pebbling, some of the chasing tokens (cops) disappear with movement, while the robber moves freely. We define the cop pebbling number of a graph to be the minimum number of cops necessary to capture the robber in this context, and present upper and lower bounds and exact values, some involving various domination parameters, for an array of graph classes, including paths, cycles, trees, chordal graphs, high girth graphs, and cop-win graphs, as well as graph products. Furthermore we show that the analogous inequality for Graham's Pebbling Conjecture fails for cop pebbling and posit a conjecture along the lines of Meyniel's Cops and Robbers Conjecture that may hold for cop pebbling. We also offer several new problems.

Keywords: Cops and Robbers, Graph Pebbling, dominating set

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[*]Department of Mathematics and Statistics, Acadia University, Wolfville, NS, Canada. Research support by NSERC grant #2020-06528.

[†]Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, USA

1 Introduction

There are numerous versions of moving tokens in a graph for various purposes. Two popular versions are called *Cops and Robbers* and *Graph Pebbling*. In both cases we have tokens of one type (C) attempting to capture a token of another type (R), and all token movements occur on the edges of a graph. In the former instance, all tokens move freely, whereas in the latter instance, type R tokens are stationary and type C movements come at a cost. In this paper we merge these two subjects to create *Cops and Robbers Pebbling*, wherein type R tokens move freely and type C tokens move at a cost.

We define these three paradigms more specifically in Subsection 1.1 below. The new graph invariant we define to study in this paper is the cop pebbling number of a graph, denoted $\pi^{\mathsf{c}}(G)$; roughly, this equals the minimum number of pebble-cops necessary to capture the robber in the Cops and Robbers Pebbling paradigm. In Subsections 1.2–1.4 we present known results about Cops and Robbers, dominating sets, and optimal pebbling, respectively, that will be used in the sequel. We record in Subsections 2.1–2.3 new theorems on lower bounds, upper bounds, and exact answers for $\pi^{\mathsf{c}}(G)$, respectively, for a range of graph families, including paths, cycles, trees, chordal graphs, high girth graphs, and cop-win graphs, as well as, in some cases, for all graphs. For example, Theorem 14 proves that the cop pebbling number of n-vertex trees is at most $\lceil 2n/3 \rceil$, which is tight for paths (and cycles), and Theorem 15 provides an upper bound involving a domination parameter, as a function of girth. Section 3 contains theorems for Cartesian products of graphs. For example, Theorems 22 and 23 give upper and lower bounds on the cop pebbling numbers of grids and cubes, respectively, while Theorem 24 proves that $\pi^{\mathsf{c}}(G \square K_t) \leq t\pi^{\mathsf{c}}(G)$ for all G. Furthermore, Theorems 26 and 27 show that the analogous inequality from Graham's Pebbling Conjecture fails for cop pebbling. We finish in Section 4 with some natural questions left open from this work, including a version of Meyniel's Cops and Robbers Conjecture that may hold in the Cops and Robbers Pebbling world, namely that $\pi^{\mathsf{c}}(G) \leq 2n/3 + o(n)$ for all G.

1.1 Definitions

We use several standard notations in graph theory, including $V(G)$ for the set of vertices of a graph G (with $n(G) = |V(G)|$), $E(G)$ for its edge set, $\mathsf{rad}(G)$ for its radius, $\mathsf{diam}(G)$ for its diameter, and $\mathsf{gir}(G)$ for its girth, as well as $\mathsf{deg}(v)$ for the degree of a vertex, and $\mathsf{dist}(u, v)$ for the distance between vertices u and v. For a vertex v in a graph G, we use the notations $N_d(v) = \{u \mid \mathsf{dist}(u, v) < d\}$ and $N_d[v] = \{u \mid \mathsf{dist}(u, v) \leq d\}$. If $d = 1$ we drop the subscript; additionally, we write $N[S] = \cup_{v \in S} N[v]$ for a set of vertices S. We often use T to denote a tree, and set P_n, C_n, and K_n to be the path, cycle, and complete graph on n vertices, respectively. (For convenience, we define $C_2 = P_2$.)

For graphs G and H we define the *Cartesian product* $G \Box H$ with vertex set $V(G) \times V(H)$ and edges $(u,v)(w,x)$ if $uw \in E(G)$ and $v = x$ or if $u = w$ and $vx \in E(H)$. The *d-dimensional cube* Q^d is defined by $Q^1 = P_2$ and $Q^d = Q^{d-1} \Box Q_1$ for $d > 1$.

1.1.1 Graph Pebbling

A *configuration* C of pebbles on a graph G is a function from the vertices of G to the non-negative integers. Its *size* equals $|C| = \sum_{v \in G} C(v)$. For adjacent vertices u and v with $C(u) \geq 2$, a *pebbling step* from u to v removes two pebbles from u and adds one pebble to v, while, when $C(u) \geq 1$, a *free step* from u to v removes one pebble from u and adds one pebble to v. In the context of moving pebbles, we use the word *move* to mean *move via pebbling steps.*

The *pebbling number* of a graph G, denoted $\pi(G)$, is the minimum number m so that, from any configuration of size m, one can move a pebble to any specified *target* vertex. The *optimal pebbling number* of a graph G, denoted $\pi^*(G)$, is the minimum number m so that, from some configuration of size m, one can move a pebble to any specified target vertex. We note that in the definitions of the pebbling number and the optimal pebbling number, free moves are not allowed.

1.1.2 Cops and Robbers

In Cops and Robbers, the cops are the pebbles, the robber is the target, and the robber is allowed to move. The Cops and Robbers alternate making moves in *turns.* At each turn, any positive number of cops make one free step, then the robber chooses to make a free step or not. In Graph Pebbling literature, the activity of moving a pebble to a target is called *solving* or *reaching* the target; here we use the analogous Cops and Robbers terminology of *capturing* the robber.

The *cop number* $c(G)$ is defined as the minimum number m so that, from some configuration of m cops, it is possible to capture any robber via free steps. If the cops catch the robber on their t^{th} turn, then we say that the *length* of the game is t; if the robber wins then the length is infinite. When the number of cops used is $c(G)$, the *capture time* of G, denoted $\mathsf{capt}(G)$, is defined to be the length of the game on G when both cops and robbers play optimally. That is, it equals the minimum (over all cop strategies) of the maximum (over all robber strategies) of the length of the game with $c(G)$ cops on G.

1.1.3 Cop Pebbling

The *cop pebbling number* $\pi^{\mathsf{c}}(G)$ is defined as the minimum number m so that, from some configuration of m cops, it is possible to capture any robber via pebbling steps. For example, $c(G) \leq n(G)$, since the placement of one cop on each vertex has already caught any robber. We call an instance of a graph G, configuration

C, and robber vertex v a *game*, and say that the cops win the game if they can capture the robber; else the robber wins. Note that, since we lose a cop with each pebbling step, the cops must catch the robber within at most $|C|-1$ turns to win the game.

We may assume that all graphs are simple. Because games on K_1 are trivial, we will assume that all graph components have at least two vertices. Additionally, because of the following fact, we will restrict our attention in this paper to connected graphs.

Fact. *If G has connected components $G_1, \ldots, G_k$ then $\pi^{\mathsf{c}}(G) = \sum_{i=1}^{k} \pi^{\mathsf{c}}(G_i)$.*

A set $S \subseteq V(G)$ is a *distance-d dominating set* if $\cup_{v \in S} N_d[v] = V(G)$. We denote by $\gamma_d(G)$ the size of the smallest distance-d dominating set.

1.2 Cop Results

Here we list the results on Cops and Robbers that will be used to prove our theorems on cop pebbling. A graph G is *cop-win* if $c(G) = 1$. A vertex u in G is called a *corner* if there is a vertex $v \neq u$ such that $N[u] \subseteq N[v]$. We say that G is *dismantlable* if either G is a single vertex or there is a corner u such that $G - u$ is dismantlable. Note that chordal graphs are dismantlable.

Result 1. *[24] A graph is cop-win if and only if it is dismantlable.*

Result 2. *[1] For $t \geq 1$ let G be a graph with $\mathsf{gir}(G) \geq 8t-3$. Then $c(G) > (\delta(G)-1)^t$.*

Result 3. *[8] For $s \geq 1$ let G be a graph with $\mathsf{gir}(G) \geq 4s+1$. Then $c(G) \geq \frac{1}{se}(\delta(G)-1)^s$.*

The latter two results are useless when $\delta(G) \leq 2$, so we use them only when $\delta(G) \geq 3$. Then, for $s = t = 1$, Result 2 is stronger than Result 3. More generally, for $s = 2t-1$, Result 3 is stronger than Result 2 whenever $(\delta(G)-1)^{t-1} > (2t-1)e$. This holds when $\delta(G) = 3$ and $t \geq 6$, when $\delta(G) = 4$ and $t \geq 4$, when $\delta(G) \geq 5$ and $t \geq 3$, and when $\delta(G) \geq 10$ and $t \geq 2$. Additionally, when $s = 2t$, Result 3 yields the lower bound $\frac{1}{2te}(\delta(G)-1)^{2t}$, compared to the $(\delta(G)-1)^t$ bound from Result 2, and so is stronger whenever $(\delta(G)-1)^t > 2te$. This holds when $\delta(G) = 3$ and $t \geq 5$, when $\delta(G) = 4$ and $t \geq 4$, and when $\delta(G) \geq 5$ and $t \geq 2$.

Result 4. *[5] If G is a chordal graph with radius r, then $\mathsf{capt}(G) \leq r$.*

This bound is tight. For example, P_5 has both radius 2 and $\mathsf{capt}(G) = 2$.

Result 5. *[14] If G is a d-regular Cayley graph then $c = c(G) \leq \lceil \frac{d+1}{2} \rceil$, and $\mathsf{capt}_c(G) \leq |V(G)| \lceil \frac{d+1}{2} \rceil$.*

Result 5 yields the following result as a corollary.

Result 6. *[2] For $d \geq 1$, the d-dimensional cube Q^d satisfies $c(Q^d) = \lceil \frac{d+1}{2} \rceil$.*

Bonato et al. [7] found the correct order of magnitude for the capture time of the cube.

Result 7. *[7] With $c = c(Q^d)$ we have $\mathsf{capt}_c(Q^d) = \Theta(d \lg d)$.*

Result 6 for even $d = 2k$ also follows from the following result because $Q^2 = C_4$.

Result 8. *[23] If $G = \Box_{i=1}^k C_{n_i}$ and each $n_i \geq 4$ then $c(G) = k + 1$. [7] If $n_i = m$ for all i, then $\mathsf{capt}_{2k}(G) \leq k\lfloor \frac{m-1}{2} \rfloor(\lceil \lg k \rceil + 1) + 1 - k$.*

In fact, Result 6 follows as well from the following result because $Q^1 = P_2$ is a tree.

Result 9. *If each T_i is a tree on at least two vertices and $G = \Box_{i=1}^d T_i$ then [21] $c = c(G) = \lceil \frac{d+1}{2} \rceil$ and [7] $\mathsf{capt}_c(G) \leq \lceil \lg d \rceil \sum_{i=1}^d \mathsf{rad}(T_i) - \lfloor \frac{d-1}{2} \rfloor + 1$.*

1.3 Dominating Set Results

In this section we list results on domination that will be used to prove cop pebbling theorems. The definition of a dominating set immediately yields the following result.

Result 10. *If G is a graph with n vertices and maximum degree Δ then $\gamma(G) \geq \frac{n}{\Delta+1}$.*

Result 11. *[6] Almost all cop-win graphs G have $\gamma(G) = 1$.*

Result 12. *[15] If $G = P_k \Box P_m$ with $16 \leq k \leq m$, then $\gamma(G) \leq \left\lfloor \frac{(k+2)(m+2)}{5} \right\rfloor - 4$.*

Result 10 implies that $\gamma(Q^d) \geq 2^d/(d+1)$. The following result shows that the actual value is not much greater, asymptotically.

Result 13. *[16] $\gamma(Q^d) \sim 2^d/d$.*

1.4 Optimal Pebbling Results

Finally, we list the optimal pebbling results we use to establish new cop pebbling theorems.

Result 14. *[9] For every graph G, $\pi^*(G) \leq \lceil 2n/3 \rceil$. Equality holds when G is a path or cycle.*

Fractional pebbling allows for rational values of pebbles. A *fractional pebbling step* from vertex u to one of its neighbors v removes x pebbles from u and adds $x/2$ pebbles to v, where x is an rational number such that $0 < x \leq C(u)$. The *optimal fractional pebbling number* of a graph G, denoted $\hat{\pi}^*(G)$, is the minimum number m so that, from some configuration of size m, one can move, via fractional pebbling moves, a sum of one pebble to any specified target vertex.

Result 15. *[18, 22] For every graph G we have $\pi^*(G) \geq \lceil \hat{\pi}^*(G) \rceil$.*

The authors of [18] prove that $\hat{\pi}^*(G)$ can be calculated by a linear program. Furthermore, they use this result to show that there is a uniform configuration that witnesses the optimal fractional pebbling number of any vertex-transitive graph; that is, the configuration C defined by $C(v) = \hat{\pi}^*(G)/n(G)$ for all v fractionally solves any specified vertex. From this they prove the following.

Result 16. *[18] Let G be a vertex-transitive graph and, for any fixed vertex v, define $m = \sum_{u \in V(G)} 2^{-\mathsf{dist}(u,v)}$. Then $\hat{\pi}^*(G) = n(G)/m$.*

Result 17. *[9] If G is an n-vertex graph with $\mathsf{gir}(G) \geq 2s+1$ and $\delta(G) = k$ then $\pi^*(G) \leq \frac{2^{2s}n}{\sigma_k(s)}$, where $\sigma_k(s) = 1 + k\sum_{i=1}^{s}(k-1)^{i-1}$.*

For a configuration C on a graph G, we say that a vertex v is 2-*reachable* if it is possible to move two pebbles to v via pebbling steps. Then C is 2-*solvable* if every vertex of G is 2-reachable.

Result 18. *[9] If C is a 2-solvable configuration of pebbles on the path P_n then $|C| \geq n+1$.*

For a subset W of vertices in a graph G we define the graph G_W to have vertices $V(G_W) = V(G) - W \cup \{w\}$ (where w is a new vertex) with edges xy whenever $x, y \in V(G) - W$ and $xy \in E(G)$ and xw whenever $x \in V(G) - W$ and $xz \in E(G)$ for some $z \in W$. The process of creating G_W from G is called *collapsing* W. If C is a configuration on G then we define the configuration C_W on G_W by $C_W(w) = \sum_{z \in W} C(z)$ and $C_W(x) = C(x)$ otherwise. Note that $|C| = |C_W|$.

Result 19. *[9] Let W be a subset of vertices in a graph G. If a configuration C on G can reach the configuration D on G then the configuration C_W on G_W can reach the configuration D_W on G_W. In particular, we have $\pi^*(G) \geq \pi^*(G_W)$.*

The next three results involve Cartesian products.

Result 20. *[17] For all $3 \leq k \leq m$ we have $\pi^*(P_k \square P_m) \leq \frac{2}{7}km + 8 \approx 0.2857km$.*

The above result is conjectured to be best possible, while the best known lower bound is below.

Result 21. *[26] For all $k \leq m$ we have $\pi^*(P_k \square P_m) \geq \frac{5092}{28593}km + O(k+m) \approx 0.1781km$.*

Combining Results 15 and 16 yields the lower bound of the following result. The upper bound places 2^k pebbles on each vertex of a distance-k dominating set, with k roughly $d/3$.

Result 22. *[9, 22] For all $d \geq 1$ we have $(4/3)^d \leq \pi^*(Q^d) \leq (4/3)^{d+O(\lg k)}$.*

2 Main Theorems

2.1 Lower Bounds

Theorem 1. *For any graph G, $\pi^{\mathsf{c}}(G) \geq c(G)$, with equality if and only if $G = K_1$.*

Proof. Any configuration of cops that can capture the robber via pebbling steps can also capture the robber via free steps.

If $G = K_1$ then $\pi^{\mathsf{c}}(G) = 1 = c(G)$.

If $\pi^{\mathsf{c}}(G) = c(G)$ then capturing the robber requires no steps. That is, if a pebbling capture requires a pebbling step, then the cop lost during the step is irrelevant in the Cops and Robbers capture, which contradicts the equality. That means that a successful pebbling configuration has no vertex without a pebble; i.e. $\pi^{\mathsf{c}}(G) = n(G)$. However, if $n(G) \geq 3$ then, by placing two cops at a vertex v of degree at least two and one cop on each vertex not adjacent to v, then we can capture any robber in one step; thus $\pi^{\mathsf{c}}(G) < n$, a contradiction. (This is recorded in Corollary 6, below.) If $n(G) = 2$ then $G = K_2$ and $\pi^{\mathsf{c}}(K_2) = 2 > 1 = c(K_2)$, a contradiction. Hence $G = K_1$. □

Theorem 2. *For any graph G, $\pi^{\mathsf{c}}(G) \geq \pi^*(G)$. Equality holds if G is a tree or cycle.*

Proof. Any configuration of k cops, where $k < \pi^*(G)$, will contain a vertex v which is unreachable. The robber can then choose to start and stay on v and thus not be captured.

If G is a cycle then Result 14 and Theorem 18 yield the equality.

If G is a tree then place $\pi^*(G)$ cops according to an optimal pebbling configuration C. The robber beginning at some vertex v defines a subtree T containing v for which every cop in T is on a leaf of T, and any leaf of T with no cop is a leaf of G. Thus the robber can never escape T. Because C can reach every vertex of T, they can capture the robber, regardless of where he moves. Hence $\pi^{\mathsf{c}}(G) \leq \pi^*(G)$, and the equality follows. □

Typically, Theorem 2 gives a sharper lower bound on $\pi^{\mathsf{c}}(G)$ than Theorem 1. However, this may not be true for all graphs.

Theorem 3. *If G is a graph with $\delta = \delta(G) \geq 27$, and with $\mathsf{gir}(G) \geq 4t + 1$ and $n(G) \leq \delta^{2t+1}$ for some $t \geq 3$, then $c(G) > \pi^*(G)$.*

Proof. Given G as above, set $d = \delta - 1$. Then we have $\mathsf{gir}(G) \geq 4t + 1$, and so $c(G) \geq d^t/et$ by Result 3. Also, with $s = 2t$, we have $\mathsf{gir}(G) \geq 2s+1$, and so $\pi^*(G) \leq 2^{2s}n/\sigma_\delta(s)$ by Result 17. Note that $2^{2s}n/\sigma_\delta(s) < 4^s n/d^s \leq (4/d)^{2t}(d+1)^{2t+1}$. Thus the result will be proved by showing that $(4/d)^{2t}(d+1)^{2t+1} \leq d^t/et$.

Since $\delta \geq 27$ we have $d \geq 26$. Then it is easy to calculate that $f(d,t) = [4(1+1/d)]^{2t}(d+1)/etd^t < 1$ when $d = 26$ and $t = 3$, and to observe that $f(d,t)$ decreases in d and t. From this it follows that $(4/d)^{2t}(d+1)^{2t+1} \leq d^t/et$. $\square$

At issue here is that it is not known if there exists a graph that satisfies the hypothesis of Theorem 3. Indeed, Biggs [3] defines a sequence of k-regular graphs $\{G_i\}$ with increasing $n(G_i)$ to have *large girth* if $\mathsf{gir}(G_i) \geq \alpha \log_{k-1}(n(G_i))$ for some constant α. It is known that $\alpha \leq 2$, and the greatest known constant is a construction of [20] that yields $\alpha = 4/3$. However, a graph satisfying the hypothesis of Theorem 3 necessarily has $\alpha = 2$.

Fortunately, there are explicitly constructed girth 5 graphs that have $c(G) > \pi^*(G)$ (in fact, $c(G) \gg \pi^*(G)$), without referring to the conditions of Theorem 3. For examples of larger girth we still need Theorem 3.

Theorem 4. *For all d there exists a graph G such that $\mathsf{gir}(G) = 5$, $\pi^*(G) \leq 4$ and $c(G) \geq d$.*

Proof. Let d be a large integer, and let H be a d-regular graph with $\mathsf{gir}(H) \geq 5$. Create G from H by adding a leaf to each vertex of H, and then add an additional vertex v adjacent to all of these leaves. Beginning with 4 pebbles at v, any vertex of G can be reached in two pebbling steps, and so $\pi^*(G) \leq 4$. A robber, however, can choose to play only on the graph H. Because $\mathsf{gir}(G) = 5$, it follows that if a cop occupies a vertex in G different from the robber's vertex, then this cop guards at most one of the robber's neighbours. So if there are fewer than d cops in G, the robber always has a safe neighbour to which to move and can evade capture indefinitely; i.e. $c(G) \geq d$. $\square$

2.2 Upper Bounds

Theorem 5. *Let G be a graph, S a subset of its vertices, and define $S' = V - N[S]$. Then $\pi^{\mathsf{c}}(G) \leq 2|S| + |S'|$. In particular, $\pi^{\mathsf{c}}(G) \leq 2\gamma(G)$.*

Proof. Place two cops on each vertex of S and one cop on each vertex of S'. In order to not be immediately captured, the robber must start in $N[S] - S$, but then is captured in one step by some pair of cops from S. The second statement follows from choosing S to be a minimum dominating set of G, since $S' = \emptyset$. $\square$

To illustrate the improvement of $2|S| + |S'|$ compared to $2\gamma(G)$, consider the following example.

Example 1. *For positive integers $m \geq 2k \geq 2$, let $Y = \{y_1, \ldots, y_m\}$ and let $Q = \{Q_1, \ldots, Q_k\}$ be a partition of Y with each part size $|Q_i| \geq 2$. Define a bipartite graph G with vertices Y, $Z = \{z_1, \ldots, z_k\}$, and x as follows. For each $1 \leq j \leq k$ set $z_j \sim y_i$ if and only if $y_i \in Q_j$. Also set $x \sim y_i$ for every $1 \leq i \leq m$.*

Then $\gamma(G) = k+1$. Indeed, since the neighborhoods of each z_j are pairwise disjoint, at least k vertices in $Y \cup Z$ are required to dominate Z, one from each $N[z_j]$. Suppose that S is a dominating set of size k. By the above, $|S \cap N[z_j]| = 1$ for all j. But to dominate x, some y_i must be in S. Let $y_i \in N(z_j)$; then y_i does not dominate any other $y_{i'} \in N(z_j)$. Hence $\gamma(G) \geq k+1$. It is easy to see that $Z \cup \{x\}$ is a dominating set, so that $\gamma(G) = k+1$. By choosing $S = \{x\}$ we have $S' = Z$, so that $\pi^{\mathsf{c}}(G) \leq k+2$, much better than $2\gamma(G) = 2k+2$.

An obvious corollary of Theorem 5 (recorded as Corollary 16, below) is that any graph G with a dominating vertex has $\pi^{\mathsf{c}}(G) = 2$, except K_1. A more interesting corollary is the following.

Corollary 6. *Every graph G satisfies $\pi^{\mathsf{c}}(G) \leq n - \Delta(G) + 1$. In particular, if $n(G) \leq 2$ then $\pi^{\mathsf{c}}(G) = n$, if $n(G) \geq 3$ then $\pi^{\mathsf{c}}(G) \leq n-1$, and if $n(G) \geq 6$ then $\pi^{\mathsf{c}}(G) \leq n-2$.*

Proof. Let v be a vertex with $\mathtt{deg}(v) = \Delta(G)$, set $S = \{v\}$, and apply Theorem 5 to obtain the general bound. Next, it is easy to see that $\pi^{\mathsf{c}}(K_n) = n$ for $n \leq 2$. Then, a graph with at least three vertices has a vertex of degree at least two, so that $n - \Delta(G) + 1 \leq n-1$. Finally, if $\Delta(G) \geq 3$ then $n - \Delta(G) + 1 \leq n-2$, while if $\Delta(G) \leq 2$ then G is a path or cycle, for which Theorem 18 yields $\pi^{\mathsf{c}}(G) = \lceil \frac{2n}{3} \rceil$, which is at most $n-2$ when $n \geq 6$. $\square$

All three conditional bounds in Corollary 6 are tight: for example, $\pi^{\mathsf{c}}(P_2) = 2$, $\pi^{\mathsf{c}}(P_5) = 4$, and $\pi^{\mathsf{c}}(P_7) = 5$. Furthermore, its more general bound of $n - \Delta(G) + 1$ is tight for a graph with a dominating vertex (see Corollary 16).

The authors of [12] define a *roman dominating set* of G to be a $\{0,1,2\}$-labeling of $V(G)$ so that every vertex labeled 0 is adjacent to some vertex labeled 2. Note that the construction in Theorem 5 yields a roman dominating set by labeling each vertex by its number of cops. The *roman domination number* $\gamma_R(G)$ is defined to be the minimum sum of labels of a roman dominating set. Hence we obtain the following bound.

Theorem 7. *Every graph G satisfies $\pi^{\mathsf{c}}(G) \leq \gamma_R(G)$.*

Theorem 8. *Let H be an induced subgraph of a graph G. Then, for any s, if $\pi^{\mathsf{c}}(H) \leq n(H) - s$ then $\pi^{\mathsf{c}}(G) \leq n(G) - s$.*

Proof. Suppose that $\pi^{\mathsf{c}}(H) \leq n(H) - s$. Then there is a configuration C_H of $n(H) - s$ cops on H that captures any robber on H. It remains to show that this number of cops can still win when the robber may move off H. Define the configuration C_G of $n(G) - s$ cops on G by placing one cop on each vertex of $G - H$ and $C_H(v)$ cops on each vertex $v \in H$. Then C_H captures any robber on G. $\square$

Corollary 9. *For all $s \geq 2$ there is an $N = N(s)$ such that every graph G with $n = n(G) \geq N$ has $\pi^{\mathsf{c}}(G) \leq n - s$.*

Proof. Suppose that $\pi^{\mathsf{c}}(G) \geq n-s+1$. Then Corollary 6 implies that $\Delta(G) \leq s$. Consider if $\mathsf{diam}(G) \geq 3s$. Then there exists an induced path P of length $3s$ in G. By Theorem 18 we have $\pi^{\mathsf{c}}(P_{3s}) = 2s \leq 2s+1 = n(P) - s$. By Theorem 8, we must have that $\pi^{\mathsf{c}}(G) \leq n-s$, contradicting our assumption that $\pi^{\mathsf{c}}(G) \geq n-s+1$. Thus, we conclude that $\mathsf{diam}(G) < 3s$. Since there are finitely many (at most $\Delta(G)^{\mathsf{diam}(G)}$) such graphs, there must be some N such that $\pi^{\mathsf{c}}(G) \leq n-s$ for all $s \geq N$. □

One might be interested in measuring the gap between the size of a graph and its cop pebbling number. For this we define the *cop deficiency* of a graph G to be $\text{Ï}^{\mathsf{c}}(G) := n(G) - \pi^{\mathsf{c}}(G)$. Then Theorem 8 and Corollary 9 can be restated as follows.

Theorem 10. *Let H be an induced subgraph of a graph G. Then $\text{Ï}^{\mathsf{c}}(G) \geq \text{Ï}^{\mathsf{c}}(H)$.*

Corollary 11. *For all $s \geq 2$ there is an $N = N(s)$ such that every graph G with $n = n(G) \geq N$ has $\text{Ï}^{\mathsf{c}}(G) \geq s$.*

Theorem 12. *If G is a cop-win graph with $\mathsf{capt}(G) = t$, then $\pi^{\mathsf{c}}(G) \leq 2^t$. More generally, if $c(G) = k$ and $\mathsf{capt}_k(G) = t$ then $\pi^{\mathsf{c}}(G) \leq k2^t$.*

Proof. If G is a cop-win graph with $\mathsf{capt}(G) = t$, then there is some vertex v at which the cop begins and the robber can be caught with free steps in at most t moves. If 2^t cops are placed on v, the cops can use the same capture strategy, and there will be sufficiently many cops for up to t pebbling steps. Similarly, by placing 2^t on each of $c(G)$ cops, there will be sufficiently many cops for up to t rounds of pebbling steps. □

For example, let T be a complete k-ary tree of depth t. Then $\mathsf{capt}(T) = t$ by Result 4, and so $\pi^{\mathsf{c}}(T) \leq 2^t$. Theorem 12 is tight for some graphs, as witnessed by any graph G with a dominating vertex (see Corollary 16, below). It is also tight for any complete k-ary tree of depth two, when $k \geq 3$ (see Corollary 20, below).

Corollary 13. *If G is a chordal graph with radius r, then $\pi^{\mathsf{c}}(G) \leq 2^r$.*

Proof. The result follows from Result 4 and Theorem 12. □

Theorem 14. *If T is an n-vertex tree, then $\pi^{\mathsf{c}}(T) \leq \lceil \frac{2n}{3} \rceil$.*

Proof. Consider a maximum length path P in T. Let z be an endpoint of P (necessarily a leaf), let y be the neighbor of z, and let x be the other neighbor of y on P.

Base case: For $n = 3$, the only tree is P_3. Place two cops on the central vertex, and the robber will be caught on the cops' first move.

Inductive Step: Assume that for trees with $n < k$ vertices, $\pi^{\mathsf{c}}(T) \le \lceil \frac{2n}{3} \rceil$. If $d(y) > 2$, form a new tree $T' = T - \{z\} - \{y\} - (\{N[y] - \{x\})\}$. By our inductive hypothesis, we can distribute the cops in such a way that the robber is caught if the robber starts on T'. By placing two cops on y, we can also ensure that the robber is caught on the first move if the robber starts on $T - T'$.

On the other hand, if $d(x) = d(y) = 2$, form a new tree $T' = T - \{x, y, z\}$. By our inductive hypothesis, we can distribute the cops in such a way that the robber is caught if the robber starts on T'. By placing two cops on y, we can also ensure that the robber is caught if the robber starts on $T - T'$.

If $d(y) = 2$ and $d(x) > 2$, and x has a leaf neighbor u, form a new tree $T' = T - \{u, y, z\}$. By our inductive hypothesis, we can distribute the cops in some distribution D' so that the robber is caught if the robber starts on T'. By placing two cops on y, we can also ensure that the robber is caught if the robber starts on vertices y or z. To capture a robber on u, one cop can reach x from D', and another cop can reach x from y. We then can reach x from u.

Finally, suppose $d(y) = 2$ and $d(x) > 2$, and x has no leaf neighbor u. Denote the neighborhood of x which is not on P as $N[x \cap P^c] = N_2[x] \cap T[V(T) \setminus V(P)]$, and let $u \in N[x] \cap T[V(T) \setminus V(P)]$. Since P has maximum length, $N[u] - \{x\}$ consists only of leaves. Let $v \in N[u] - \{x\}$, and let $T' = T - \{v, y, z\}$. By our inductive hypothesis, we can distribute the cops in some distribution D' that the robber is caught if the robber starts on T'. If two cops can reach x in T', we can add 2 more cops to x to catch the robber on the vertices $\{v, y, z\}$. If two cops can reach u in T', then v and x are reachable, so we can add 2 more cops to y to catch the robber on the vertices $\{x, z\}$. Last, if two cops cannot reach x or u, then no sequences of cop moves in T' will use the edge uv (otherwise, we would be able to get two cops on at least one of the two vertices). Thus, we can simultaneously get one cop on x and one cop on u. By adding two cops onto y, the cops can reach the vertices $\{v, y, z\}$. □

Note that, on a tree, cops move greedily toward the robber, so if a cop p can reach a vertex v then the robber cannot ever occupy v, as the robber has no access to v except through p. Hence if G is a tree then $\pi^{\mathsf{c}}(G) = \pi^*(G)$. We note that Theorems 12 and 14 can each be stronger than each other, as the following two examples show. Define the *spider* $S(k, d)$ to be the tree having a unique vertex x of degree greater than 2, all k of whose leaves have distance d from x.

Example 2. *For integers k and d, the spider $S = S(k, d)$ has $c(S) = 1$ and $\mathsf{capt}(S) = d$, with $n = kd + 1$. Thus Theorem 12 yields $\pi^{\mathsf{c}}(S) \le 2^d$, while Theorem 14 yields $\pi^{\mathsf{c}}(S) \le \lceil (2kd + 2)/3 \rceil$. Hence one bound is stronger than the other depending on how k compares, roughly, to $3 \cdot 2^{d-1}/d$.*

Example 3. *For integers $k, t \ge 1$, let T be the complete k-ary tree of depth t. Then $n(T) = \sum_{i=0}^{t} k^i = (k^{t+1} - 1)/(k - 1)$. Thus Theorem 12 is stronger than Theorem 14 for $k \ge 3$ and for $k = 2$ with $t \ge 2$, while*

Theorem 14 is stronger than Theorem 12 when $k = 1$ and $t \geq 5$ (because $\mathsf{capt}(P_t) = \lceil t/2 \rceil$).

Theorem 15. *For any positive integer d, if G is a graph with $\mathsf{gir}(G) \geq 4d-1$, then $\pi^{\mathsf{c}}(G) \leq 2^d \gamma_d(G)$.*

Proof. Let $S = \{v_1, v_2, ...\}$ be a minimum d-distance dominating set of G, and place 2^d cops on each v_i. Suppose the robber starts at vertex v. Since $\mathsf{gir}(G) \geq 4d-1$, we know that $T = N_d[u]$ is a tree for all u. We write $T_i = N_d[v_i]$ and, for each $v \in T$, denote the unique vu-path in T by $P(v)$.

Let J be such that $T \cap T_j \neq \emptyset$ if and only if $j \in J$, and set $Q_j = T \cap T_j$. Note that $\mathsf{gir}(G) \geq 4d-1$ implies that, for each $j \in J$, there is some $v \in T$ such that $Q_j \subseteq P(v)$. Moreover, by the definition of S, we have $\cup_{j \in J} Q_j = T$. In addition, $\mathsf{gir}(G) \geq 4d-1$ implies that, for each $j \in J$, the shortest $v_i v$-path P_i^* is unique.

For each $j \in J$, each cop at v_j adopts the strategy to move at each turn toward v along P_i^* until reaching T, at which time then moving toward the robber along the unique path in T. This strategy ensures the property that, at any point in the game, if some cop is on vertex x while the robber is on vertex z, then the robber can never move to a vertex y for which the unique yz-path in T contains x — which includes x itself. It also implies that the game will last at most d turns. Hence, if we suppose that the robber wins the game, then the game lasted exactly d turns and the robber now sits on some vertex z. However, by the definition of S, some cop reached z within d turns, which implies by the property just mentioned that the robber cannot move to z, a contradiction. Hence the cops win the game, capturing the robber. $\square$

An obvious corollary (recorded as Corollary 16, below) is that any graph G with a dominating vertex has $\pi^{\mathsf{c}}(G) = 2$.

We remark that Theorem 15 applies to trees. P_5 is an example for which this bound is tight. In the case of the spider $S(k, 2)$, this bound is significantly better than Corollary 6 when k is large. The case $d = 1$ yields the same upper bound of $2\gamma(T)$ from Theorem 5, which is better than the bound of Theorem 14 if and only if $\gamma(T) < \lceil (n-1)/3 \rceil$. Since $\gamma(T)$ can be as high as $n/2$, both theorems are relevant. The following example shows that Theorem 15 can be stronger than Theorem 14 for any d.

Example 4. *For $1 \leq i \leq 3$, define the tree T_i to be the complete binary tree of depth $d-1$, rooted at vertex v_i, and define the tree T to be the union of the three T_i with an additional root vertex adjacent to each v_i. Then $\gamma_d(T) = d$, and $n = 3(2^d - 1) + 1$, so that the bound from Theorem 15 is stronger than the bound from Theorem 14.*

Theorem 15 can be stronger than other prior bounds as well, as shown by the following example.

Example 5. *For integers k and d, define the theta graph $\Theta(k, d)$ as the union of k internally disjoint xy-paths, each of length d. Then $\Theta = \Theta(k, 2d)$ has $n = k(d-1) + 2$, $c(\Theta) = 2$, $\mathsf{capt}_2(\Theta) = d$, $\mathsf{gir}(\Theta) = 4d$, $\gamma(\Theta) = k\lceil (2d-3)/3 \rceil$, and $\gamma_d = 2$. Thus Theorem 5 yields an upper bound of roughly $4kd/3$, while Theorems*

12 and 15 both yield the upper bound of 2^{d+1}, which is better or worse than Theorem 5 when k is bigger or less than, roughly, $3 \cdot 2^{d-1}/d$.

The following example illustrates the need for stronger bounds than given by Theorem 15.

Example 6. *Consider the $(3,7)$-cage McGee graph M, defined by $V = \{v_i \mid i \in \mathbb{Z}_{24}\}$, with $v_i \sim v_{i+1}$ for all i, $v_i \sim v_{i+12}$ for all $i \equiv 0 \pmod 3$, and $v_i \sim v_{i+7}$ for all $i \equiv 1 \pmod 3$. We have $\gamma_2(M) \leq 4$ (e.g. $\{v_0, v_6, v_9, v_{15}\}$), and so $\pi^*(M) \leq \pi^{\mathsf{c}}(M) \leq 16$ by Theorem 15. However, this bound is not tight, as $\pi^{\mathsf{c}}(M) \leq 12$: the vertex set $\{v_i \mid i \equiv 0 \pmod 3\}$ induces a matching of size 4 — for each edge, place 2 cops on one of its vertices and 1 cop on the other. Incidentally, this yields $\pi^*(M) \leq 12$; the best known lower bound on π^* comes from Result 16: $\pi^*(M) \geq \lceil \hat{\pi}^*(M) \rceil = \lceil 64/7 \rceil = 10$. Hence we are left with a gap in the bounds for M: $10 \leq \pi^*(M) \leq \pi^{\mathsf{c}}(M) \leq 12$.*

2.3 Exact Results

The following is a corollary of Theorem 5, as well as of Theorem 15.

Corollary 16. *If G is a graph with at least two vertices and a dominating vertex then $\pi^{\mathsf{c}}(G) = 2$.*

The following is a corollary of Result 11 and Theorem 8.

Corollary 17. *Almost all cop-win graphs G have $\pi^{\mathsf{c}}(G) = 2$.*

Theorem 18. *For all $n \geq 1$ we have $\pi^{\mathsf{c}}(P_n) = \pi^{\mathsf{c}}(C_n) = \lceil \frac{2n}{3} \rceil$.*

Proof. We have from Theorem 2 that $\pi^{\mathsf{c}}(P_n) \geq \pi^*(P_n) = \lceil \frac{2n}{3} \rceil$ and $\pi^{\mathsf{c}}(C_n) \geq \pi^*(C_n) = \lceil \frac{2n}{3} \rceil$.

We have from Theorem 14 that $\pi^{\mathsf{c}}(P_n) \leq \lceil \frac{2n}{3} \rceil$. For C_n, partition C_n into $\lfloor \frac{n}{3} \rfloor$ copies of P_3 and, possibly, an extra P_1 or P_2. Place two cops on the center vertex of each P_3, and one cop on each vertex of the remaining one or two vertices. The robber can only choose to start on one of the copies of P_3, where he is next to a pair of cops, and so will be captured on the first move. Thus $\pi^{\mathsf{c}}(C_n) \leq \lceil \frac{2n}{3} \rceil$. □

Theorem 19. *If T is a tree with $\mathsf{rad}(T) = 2$ and $\mathsf{diam}(T) = 4$ then $\pi^{\mathsf{c}}(T) = 4$.*

Proof. The upper bound follows from Result 4 and Theorem 12. The lower bound follows from Theorem 18 since T contains P_5. □

Corollary 20. *If T is a complete k-ary tree of depth 2 with $k \geq 3$, then $\pi^{\mathsf{c}}(T) = 4$.*

3 Cartesian Products

For ladders ($P_2 \square P_m$) we have the following theorem.

Theorem 21. *For all $m \geq 1$ we have $\pi^{\mathsf{c}}(P_2 \square P_m) = m + 1$.*

Proof. Let $G = P_2 \square P_m$ and for each $i \in [m]$ define the vertex subset $E_i = [2] \times \{i\}$. Suppose that C can catch any robber on G. Then C must be able to move two pebbles to any E_i. Indeed, assume that a cop catches the robber on E_i; without loss of generality, on vertex $(0, i)$. If the robber didn't move, it is because each of its neighbors contained a cop. If the robber did move, say from $(1, i)$, then it was because a cop could have moved onto the robber. In each of those cases we see that two cops could be moved onto E_i. If the robber moved instead from, say, $(0, i-1)$ (or symmetrically $(1, i+1)$), then it was because cops could move into both $(0, i-1)$ and $(1, i-1)$ (or $(0, i+1)$ and $(1, i+1)$), and so two cops could be moved onto E_{i-1} (respectively E_{i+1}). Now, by collapsing every E_i we obtain the graph $G' = P_m$, with corresponding collapsed configuration C'. The above argument then shows by Result 19 that C' is 2-solvable. Consequently, Result 18 proves that $|C| = |C'| \geq m + 1$.

For the upper bound, we define $[k] = \{0, 1, \dots, k-1\}$, write m uniquely as $m = 4r + 2s + t$, with $s, t \in [2]$, and let $V(G) = [2] \times [m]$. Next define the sets $S = \{(0,1)\}$ when m is even and $S = \{(0,1), (m-1, s)\}$ when m is odd, and $T = \{(4i + 2j + 1, j) \mid 0 \leq i \leq r, j \in [2], 4i + 2j + 1 \in [m]\}$ — alternately, $T = \{(x, 0) \mid x \equiv 1 \pmod{m}\} \cup \{(x, 1) \mid x \equiv 3 \pmod{m}\}$. Then $S \cup T$ is a dominating set. Now place two cops on each vertex of T and one cop on each vertex of S. It is simple to check that any robber can be captured in one step and that the number of cops in each case equals $m + 1$. $\square$

More general grids have cop pebbling numbers linear in their number of vertices, but there is a gap in the bounds for its coefficient.

Theorem 22. *For all $16 \leq k \leq m$ we have $\frac{5092}{28593}km + O(k+m) \leq \pi^{\mathsf{c}}(P_k \square P_m) \leq 2\left\lfloor \frac{(k+2)(m+2)}{5} \right\rfloor - 8$. The lower bound also holds for all $1 \leq k \leq m$.*

Proof. Result 21 and Theorem 2 produce the lower bound, while Result 12 and Theorem 5 produce the upper bound. $\square$

For cubes, we can use Results 6 and 7 with Theorem 12 to obtain the upper bound $\pi^{\mathsf{c}}(Q^d) \leq \lceil \frac{d+1}{2} \rceil d^{\Theta(d)}$. However, by adding extra cops in the Cops and Robbers game, we can reduce the capture time and therefore also reduce the upper bound for cop pebbling. Still, an exponential gap remains.

Theorem 23. $\left(\frac{4}{3}\right)^d \leq \pi^{\mathsf{c}}(Q^d) \leq \frac{2^{d+1}}{d+1} + o(d)$.

Proof. The lower bound follows from Result 22 and Theorem 2. The upper bound follows from Result 13 and Theorem 5. $\square$

Theorem 24. *For every graph G we have $\pi^{\mathsf{c}}(G\square K_t) \le t\pi^{\mathsf{c}}(G)$.*

Proof. Let C be a configuration of $\pi^{\mathsf{c}}(G)$ cops on G that can capture any robber. Define the configuration C' on $G\square K_t$ by $C'(u,v) = C(u)$ for all $u \in V(G)$ and $v \in V(K_t)$; then $|C'| = t|C|$. Let C'_v be the restriction of C' to the vertices $V_v = \{(u,v) \mid u \in V(G)\}$. Then each C'_v is a copy of C on V_v. Now imagine, for any robber on some vertex (u,v), placing a copy of the robber on each vertex (u',v) and maintaining that property with every robber movement. Then the cops on each V_v will move in unison to catch their copy of the robber in V_v, one of which is the real robber. $\square$

A famous conjecture of Graham [10] postulates that every pair of graphs G and H satisfy $\pi(G\square H) \le \pi(G)\pi(H)$. This relationship was shown by Shiue to hold for optimal pebbling.

Theorem 25. *[29] Every pair of graphs G and H satisfy $\pi^*(G\square H) \le \pi^*(G)\pi^*(H)$.*

One might ask whether or not the analogous relationship holds between $\pi^{\mathsf{c}}(G\square H)$ and $\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$. Theorem 24 shows that this is true for $H = K_2$. However, the inequality is false in general, as the following theorem shows. For any graph G define $G^1 = G$ and $G^d = G\square G^{d-1}$ for $d > 1$.

Theorem 26. *There exist graphs G and H such that $\pi^{\mathsf{c}}(G\square H) > \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$.*

Proof. Suppose that $\pi^{\mathsf{c}}(G\square H) \le \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all G and H. For fixed $k \ge 2$, let $d \ge 25k^2$, $v \in V(C_k^d)$, and $m = \sum_{u\in V(C_k^d)} 2^{-\mathsf{dist}(u,v)}$. Then $\sqrt{d} > \ln d$, so that $d/\ln d > \sqrt{d} \ge 5k > \frac{2}{\ln(3/2)}k$, which implies that $d^{2k} < (3/2)^d$. Also $d \ge \sqrt{k/8} + 2$, so that $\sqrt{k/8} \ge d-2$. Thus

$$\begin{aligned}\left(\frac{2}{3}\right)^d \sum_{u\in V(C_k^d)} 2^{-\mathsf{dist}(u,v)} &\le \left(\frac{2}{3}\right)^d \sum_{i=0}^{kd/2} \binom{i+k-1}{k-1} 2^{-i} \le \left(\frac{2}{3}\right)^d \sum_{i=0}^{kd/2} \binom{i+k-1}{k-1} \le \left(\frac{2}{3}\right)^d \binom{kd/2+k}{k} \\ &\le \left(\frac{2}{3}\right)^d (kd/2+k)^k/k! \le \left(\frac{2}{3}\right)^d (d+2)^k \sqrt{k/2}^k 2^{-k} \le \left(\frac{2}{3}\right)^d (d+2)^k (d-2)^k \\ &\le \left(\frac{2}{3}\right)^d d^{2k} < 1.\end{aligned}$$

Therefore we would have

$$\pi^{\mathsf{c}}(P_k^d) \le \pi^{\mathsf{c}}(P_k)^d \le \left(\frac{2}{3}k\right)^d = \left(\frac{2}{3}\right)^d n(P_k^d) < n(C_k^d)/m = \hat{\pi}^*(C_k^d) \le \hat{\pi}^*(P_k^d) \le \pi^*(P_k^d),$$

by Fact 15 and Theorem 16. This, however, contradicts Theorem 2. $\square$

A more direct example is given by the Cartesian product of wheels. For $n \geq 4$, define the *wheel* W_n by the addition of a dominating vertex x to the cycle C_{n-1}, having vertices $v_0, \ldots, v_{n-2}$.

Theorem 27. *Let $n \geq 4$ and let $G = W_n \Box W_n$. Then $\pi^{\mathsf{c}}(G) \leq 14$ and if $n \geq 67$ then $\pi^{\mathsf{c}}(G) = 14$.*

Proof. First we show the upper bound. Note that because x dominates W_n, the vertex $(x, x) \in G$ has eccentricity 2. By placing 14 cops on (x, x), the robber must choose a vertex outside of $N[(x, x)]$ to start on; by symmetry let it be (v_1, v_1). Then we move 2 cops to each of (v_0, x), (v_1, x), and (v_2, x), and 1 cop to (x, v_1). Because of the two cops on (v_1, x) the robber must move to one of its neighbors without cops; by symmetry let it be (v_2, v_1). Then the cops on (v_2, x) capture the robber. Hence $\pi^{\mathsf{c}}(G) \leq 14$.

Now we prove the lower bound. Define the subgraph H_k of G to be induced by the vertices $\{(v_i, w) \mid i \in [k], w \in V(W_n)\} \cup \{(x, v_j) \mid j \in [k]\}$. Suppose that $n \geq 67$ and 13 pebble-cops are placed on G. Then there is some subgraph H isomorphic to H_5 with no pebble-cops in it. Symmetrically inside of H is a graph H' isomorphic to H_3 (there are three H_3s; we're referring to the middle one of them). Without loss of generality, $H' = H_3$. Now assume that this configuration catches the robber via a set σ of pebbling moves. Suppose some pebble-cop reaches a vertex $u \in H'$ from a vertex other than (x, x), and let S be the set of pebble-cops used to accomplish that. Then we can instead start with S on (x, x) and still reach u. Thus, after making all such modifications, we have that the initial configuration of all 13 pebble-cops on (x, x) catches the robber on G.

But we argue as follows that 13 pebble-cops on (x, x) cannot catch a robber at (v_1, v_1) in G. Compute all subscript arithmetic modulo $n - 1$. We call a robber's move from (v_i, v_j) to (v_{i+1}, v_j) (respectively (v_{i-1}, v_j)) *increasing* (respectively *decreasing*). First, it takes at least 4 pebble-cops to force the robber to move, because doing so requires 2 pebble-cops to be moved to a neighbor of the robber. Then, similarly, it takes at least 4 pebble-cops to prevent the robber from making successive increasing moves, and at least 4 more to prevent successive decreasing moves. Finally it requires at least 2 pebble-cops to block the robber's movement inward, to some vertex (x, v_j). Note that all these sets of pebble-cops are distinct because no pebble-cop can perform two of the above actions simultaneously. But this is a contradiction because only 13 pebble-cops exist. Hence $\pi^{\mathsf{c}}(G) \geq 14$. $\square$

It seems likely that the value 67 can be reduced greatly.

Corollary 28. *For $n \geq 67$ we have $\pi^{\mathsf{c}}(W_n \Box W_n) = \frac{7}{2}\pi^{\mathsf{c}}(W_n)\pi^{\mathsf{c}}(W_n)$.*

4 Open Questions

Question 29. *Can the bounds* $.178 \approx \frac{5092}{28593} \leq \lim_{k,m\to\infty} \pi^{\mathsf{c}}(P_k \Box P_m)/km \leq .4$ *from Theorem 22 be improved?*

Corollary 28 suggests the following two questions.

Question 30. *Is there an infinite family of graphs* $\mathcal{G}$ *for which* $\pi^{\mathsf{c}}(G \Box H) \leq \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ *for all* $G, H \in \mathcal{G}$*?*

Theorem 26 shows that products of paths is not among such a family.

Question 31. *Is there some constant* $a \geq 7/2$ *such that* $\pi^{\mathsf{c}}(G \Box H) \leq a\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ *for all* G *and* H*?*

In addition to chordal graphs and Cartesian products discussed above, it would be interesting to study other graph classes. It was proved in [11] that $c(G) \leq 2$ for outerplanar G, and in [1] that $c(G) \leq 3$ for planar G. Additionally, it was shown in [27] that if G is a planar graph on n vertices then $\mathsf{capt}_3(G) \leq 2n$.

Question 32. *Are there constant upper bounds on* $\pi^{\mathsf{c}}(G)$ *when* G *is planar or outerplanar? If* $k = \pi^{\mathsf{c}}(G)$ *then is* $\mathsf{capt}_k(G)$ *linear?*

Likewise, a result of [25] states that every diameter two graph G on n vertices satisfies $\pi(G) \in \{n, n+1\}$.

Question 33. *Is there a similar, narrow range of values of* $\pi^{\mathsf{c}}(G)$ *over all diameter two graphs* G*?*

Finally, Meyniel [13] conjectured in 1985 that every graph G on n vertices satisfies $c(G) = O(\sqrt{n})$. Some evidence in support of this conjecture is found in [4], where it is proved for $G \in \mathcal{G}_{n,p}$ that when $0 < \epsilon < 1$ and $p > 2(1+\epsilon)\log(n)/n$ we have $c(G) < \frac{10^3}{\epsilon^3} n^{\frac{1}{2}\log(n)}$ almost surely. (In fact, they also show that when $p \gg 1/n$ we have $c(G) > \frac{1}{(pn)^2} n^{\frac{1}{2}\left(\frac{\log\log(pn)-9}{\log\log(pn)}\right)}$ almost surely.) Along these lines, we make the following conjecture.

Conjecture 34. *Every graph* G *on* n *vertices satisfies* $\pi^{\mathsf{c}}(G) \leq 2n/3 + o(n)$*; i.e.,* Ï$^{\mathsf{c}}(G) \geq n/3 - o(n)$.

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