Graph Pebbling

Definitions:

1. A ***configuration*** C on a connected graph G is a function $C : V(G) \to \mathbb{N}$. The value $C(v)$ signifies the number of pebbles at vertex v. The ***size*** of C is defined by $|C| = \sum_{v \in V(G)} C(v)$.

2. For an edge $\{u, v\} \in E(G)$, if u has at least two pebbles on it, then a ***pebbling step from*** u ***to*** v removes two pebbles from u and places one pebble on v. That is, if C is the original configuration, then the resulting configuration C' has $C'(u) = C(u) - 2$, $C'(v) = C(v) + 1$, and $C'(x) = C(x)$ for all $x \in V(G) - \{u, v\}$.

3. We say that a configuration C on G is r***-solvable*** if it is possible from C to place a pebble on r via pebbling steps; it is r***-unsolvable*** otherwise. C is ***solvable*** if it is r-solvable for every vertex r.

4. For a connected graph G and a particular ***root*** vertex r, the ***rooted pebbling number*** $\pi(G, r)$ is defined to be the minimum number t so that every configuration C on G of size t is r-solvable. The ***pebbling number*** $\pi(G)$ equals $\max_r \pi(G, r)$.

5. A graph G is of ***Class 0*** if $\pi(G) = n(G)$.

6. A sequence of paths $\mathcal{P} = (P[1], \ldots, P[h])$ is a ***maximum*** r***-path partition*** of a rooted tree (T, r) if $\mathcal{P}$ forms a partition of $E(T)$, r is a leaf of $P[1]$ and, for all $1 \le i \le h$, $T_i = \cup_{j=1}^{i} P[j]$ is a tree and $P[i]$ is a maximum length path in $T - T_{i-1}$ among all such paths with one endpoint in T_{i-1}.

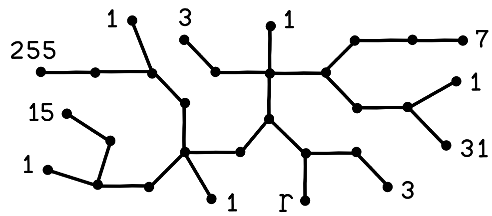
7. For $m \ge 2t + 1$ the ***Kneser*** graph $K(m, t)$ has as vertices all t-subsets of $\{1, 2, \ldots, m\}$ and edges between every pair of disjoint sets.

8. For two graphs G_1 and G_2, define the ***cartesian product*** $G_1 \square G_2$ to be the graph with vertex set $V(G_1 \square G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1 \square G_2) = \{\{(v_1, v_2), (w_1, w_2)\} \mid (v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2)) \text{ or } (v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1))\}$.

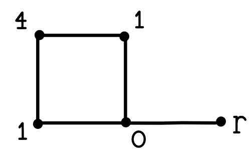
Examples:

1. Two r-unsolvable configurations (of maximum size, right) on the path P_7 are shown below.

2. An maximum sized r-unsolvable configuration on a tree is shown below.



3. An r-solvable configuration on the 4-cycle with pendant edge is shown below.



Facts:

1. Graph pebbling arose as a method to prove a conjecture of Erdős and Lemke in combinatorial number theory. It has since produced the following more general result. If $g_1, \ldots, g_n$ is a sequence of elements of an abelian group $\mathbf{G}$ of size n, then there is a nonempty subsequence $(g_k)_{k\in K}$ such that $\sum_{k\in K} a_k = 0_{\mathbf{G}}$ and $\sum_{k\in K} 1/|g_k| \le 1$, where $|g|$ denotes the order of the element g in $\mathbf{G}$ and $0_{\mathbf{G}}$ is the identity element in $\mathbf{G}$.

2. Every rooted graph (G, r) has $\pi(G, r) \ge \max\{n(G), 2^{\mathsf{ecc}_G(r)}\}$.

3. The path P_n on n vertices has pebbling number $\pi(P_n) = 2^{n-1}$.

4. The cycle C_n on n vertices has pebbling number $\pi(C_{2k}) = 2^k$ and $\pi(C_{2k+1}) = \lceil (2^{k+2} - 1)/3 \rceil$ for all $k \ge 1$.

5. If (T, r) is a rooted tree then $\pi(T, r) = f(T, r) = \sum_{i=1}^{h} 2^{l_i} - h + 1$, where $(l_1, \ldots, l_h)$ is the sequence of lengths $l_i = \mathsf{diam}(P[i])$ in a maximum r-path partition $\mathcal{P}$ of T.

6. Complete graphs, balanced complete bipartite graphs, cubes, the Petersen graph, and split graphs with minimum degree at least 3 are all Class 0.

7. For any constant $c > 0$ there is an integer t_0 such that $K(2t + s, t)$ is Class 0 when $t > t_0$ and $s \ge c(t/\lg t)^{1/2}$.

8. If G is a graph with n vertices and e edges then G is Class 0 when $e \geq \binom{n-1}{2}+2$, while G is Class 0 implies that $\kappa(G) \geq 2$, $e \geq \lfloor 3n/2 \rfloor$, and $\mathsf{gir}(G) \leq 1 + 2\lg n$.

9. If $\mathsf{diam}(G) = 2$ then $\pi(G) \leq n(G) + 1$. If also $\kappa(G) \geq 3$ then G is Class 0.

10. If $\mathsf{diam}(G) = d$ and $\kappa(G) \geq 2^{2d+3}$ then G is Class 0.

11. There exists a Class 0 graph G on at most $n \geq 3$ vertices with $\mathsf{gir}(G) \geq \lfloor \sqrt{(\lg n)/2 + 1/4} - 1/2 \rfloor$.

12. Let G be a random graph on n vertices with edge probability p. If $p \gg (n \lg n)^{1/d}/n$ for fixed d then $\Pr[G$ is Class $0] \to 1$ as $n \to \infty$.

13. Graham's Conjecture states that every pair of connected graphs G and H satisfy $\pi(G \square H) \leq \pi(G)\pi(H)$. This is true when G and H are both complete graphs, both trees, both cycles, both complete bipartite graphs with at least 15 vertices per part, or both connected graphs on n vertices with minimum degree at least $k \geq 2^{12n/k+15}$.

14. Deciding whether a configuration on a graph is solvable is NP-complete, even when restricted to the classes of diameter two graphs or planar graphs, but is in P when restricted to the classes of complete graphs, trees, diameter two planar graphs, or outerplanar graphs.

15. Deciding whether $\pi(G) \leq k$ is Π_2^{P}-complete (i.e. complete for the class of decision problems computable in polynomial time by a co-NP machine equipped with an NP-complete oracle).

16. $\pi(G)$ can be calculated in polynomial time when G is a tree, a diameter two graph, a split graph, or a 2-path.