Cup stacking in graphs

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ABSTRACT

Here we introduce a new game on graphs, called cup stacking, following a line of what can be considered as 0-, 1-, or 2-person games such as chip firing, percolation, graph burning, zero forcing, cops and robbers, graph pebbling, and graph pegging, among others. It can be more general, but the most basic scenario begins with a single cup on each vertex of a graph. (This simplification coincides with an earlier game devised by Gordon Hamilton.) For a vertex with k cups on it we can move all its cups to a vertex at distance k from it, provided the second vertex already has at least one cup on it. The object is to stack all cups onto some pre-described target vertex. We say that a graph is stackable if this can be accomplished for all possible target vertices.

In this paper we study cup stacking on many families of graphs, developing a characterization of stackability in graphs and using it to prove the stackability of complete graphs, paths, cycles, grids, the Petersen graph, many Kneser graphs, some trees, cubes of dimension up to 20, "somewhat balanced" complete t-partite graphs, and Hamiltonian diameter two graphs. Additionally we use the Gallai-Edmonds Structure Theorem, the Edmonds Blossom Algorithm, and the Hungarian algorithm to devise a polynomial algorithm to decide if a diameter two graph is stackable.

Our proof that cubes up to dimension 20 are stackable uses Kleitman's Symmetric Chain Decomposition and the new result of Merino, Mütze, and Namrata that all generalized Johnson graphs (excluding the Petersen graph) are Hamiltonian. We conjecture that all cubes and higher-dimensional grids are stackable, and leave the reader with several open problems, questions, and generalizations.

1. Introduction

Cup stacking, also known as sport stacking or speed stacking, has become a popular sport involving the stacking and un-stacking of cups in various formations, in certain order, and in minimum time (see [16,18]). Here we introduce a graph theoretic version of the stacking game.[1]

There are many varieties of what may be called "moving things around in graphs", including cops and robbers [4], graph pebbling [11], zero forcing [1], chip firing [2], graph burning [3], and graph pegging [9], among others. Even network
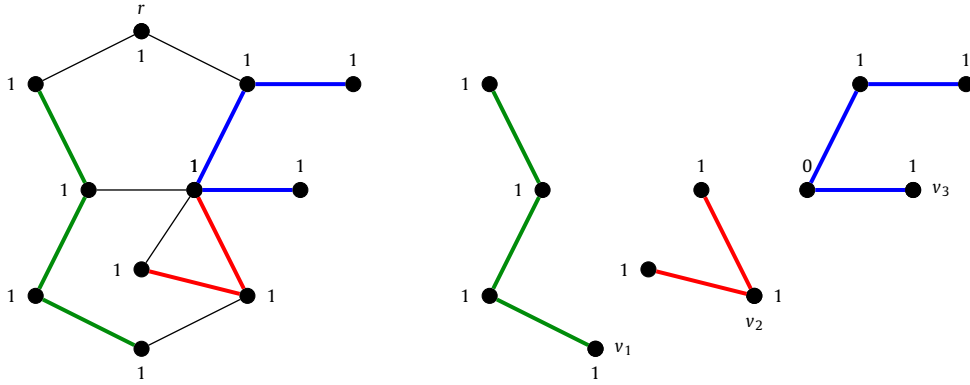

Fig. 1. A graph and configuration (G, C), left, with an r-stacking partition $\{(H_1, D_1), (H_2, D_2), (H_3, D_3)\}$, right.

optimization and network flow can fall under in this umbrella, as can one of the world's oldest board games, Mancala [12,15]. In each scenario, we begin with a configuration of objects that are placed on the vertices of a graph G (modeled as a nonnegative function on the vertices) and a set of target vertices to which we must move the objects. Each scenario has different rules for moving objects along the edges of the graph — movement may cost money, objects may be consumed in transit, edges may have capacities, or conditions on neighboring vertices may exist — and sometimes the target(s) can move, controlled, say, by an adversary. The objective may simply be to reach the target(s), or to do so at minimum cost or in minimum time, for example. Our setup for cup stacking is as follows.

We will work exclusively with connected graphs. Let G be such a graph, with objects called *cups*, which are placed on the vertices of G. A *configuration* C on G is a function $C : V(G) \to \mathbb{N}$ that encodes the number $C(v)$ of cups on each vertex v. A *cup stacking move* from vertex u to vertex v, denoted $u \mapsto v$, can be made if both vertices have at least one cup and $\text{dist}(u, v) = C(u)$, and consists of moving all the cups from u onto v. Because of the distance condition, we know that $u \mapsto v$ can be thought of as carrying the $C(u)$ cups from u to v along a uv-geodesic (minimum length uv-path) $P(u, v)$ of length $C(u)$. The resulting configuration C' satisfies $C'(u) = 0$, $C'(v) = C(u) + C(v)$, and $C'(w) = C(w)$ otherwise. For a target vertex r, we say that G is (C, r)-*stackable* if it is possible to stack every cup on r; it is C-*stackable* if it (C, r)-stackable for every r with $C(r) > 0$.

We define the configuration $\mathbf{1}$ to have one cup on each vertex of G (we will use this notation for any size graph). When $C = \mathbf{1}$, we will drop C from the notation to say r-stackable and stackable, respectively. The focus of this paper is to study the case that $C = \mathbf{1}$. However, it is important to note that, in order to do so, we still need to consider the general case (e.g. see Fig. 1).

A simple observation when $C = \mathbf{1}$, for example, is that the tree $K_{1,m}$ is r-stackable if and only if $m \le 2$ or r is not a leaf, and thus $K_{1,m}$ is stackable if and only if $m \le 2$. More generally we have the following fact.

Fact 1. *If r is a dominating vertex in a graph G on n vertices then G is r-stackable.*

Equally simple is the following.

Fact 2. *If G is a vertex transitive graph and $r \in V(G)$ then G is stackable if and only if G is r-stackable.*

Together, Facts 1 and 2 imply, for example, that all complete graphs are stackable. Interestingly, Veselovac [17] found an infinite family of trees that are not r-stackable for any r.

1.1. Definitions

We mostly follow standard graph theory terminology and notation, viewing a graph $G = (V, E)$ as a set of *vertices* V together with a set of *edges* E, which are unordered pairs of vertices. If necessary to identify to which graph a set of vertices or edges belongs, we may write $V(G)$ and $E(G)$ for the vertices and edges of the graph G. We say that vertices u and v are *adjacent* in G if $uv \in E(G)$ (note that we write uv as a simplification of $\{u, v\}$). We tend to save subscripts to denote the number of vertices of a graph; for example, P_n is the path on n vertices and thus has *length* $n - 1$.

The *open neighborhood* of a vertex v is denoted $N(v)$ and consists of the set of vertices that are adjacent to v, and the *degree* of v is denoted $\deg(v)$, and equals $|N(v)|$. The *closed neighborhood* of a vertex v is denoted $N[v]$ and equals $N(v) \cup \{v\}$. (Each of these can be written with the subscript G — e.g. $N_G(v)$, etc. — if necessary to avoid confusion.) We say that v is a *dominating vertex* in G if $N[v] = V(G)$. More generally, for any $S \subset V(G)$ we write $N(S) = \cup_{v \in S} N(s)$ and $N[S] = N(S) \cup S$; then S is a *dominating set* if $N[S] = V(G)$.

The *distance* between two vertices u and v is denoted $\text{dist}(u, v)$, and counts the length of the shortest path with endpoints u and v. If necessary to distinguish distances in different graphs containing u and v, we will write $\text{dist}_G(u, v)$ to identify distance in the graph G. The *diameter* of a graph G is denoted $\text{diam}(G)$ and counts the largest distance between two vertices of G; i.e. $\text{diam}(G) = \max_{u,v} \text{dist}_G(u, v)$.

The *Cartesian product* of two graphs G and H is denoted $G \square H$ and has vertex set $\{(x, y) \mid x \in V(G), y \in V(H)\}$ and edge set $\{(x, y)(x, y') \mid yy' \in E(H)\} \cup \{(x, y)(x', y) \mid xx' \in V(G)\}$. The *d-dimensional cube* Q^d (or *d-cube*) has vertex set all binary d-tuples $(a_1, \ldots, a_d)$ and edge set all pairs of vertices whose coordinates differ in exactly one position. Thus Q^0 is a single vertex, Q^1 is isomorphic to K_2 (the complete graph on 2 vertices), Q^2 is isomorphic to C_4 (the cycle on 4 vertices), and Q^3 looks like the bordering edges of a 3-dimensional box. As can be readily seen, we can also write $Q^d = Q^1 \square Q^{d-1}$ or, more generally, $Q^d = Q^p \square Q^q$ for any natural numbers for which $p + q = d$. Furthermore, we can identify $V(Q^d)$ with the family of all subsets of $\{1, \ldots, d\}$ by associating the set A with the d-tuple defined by $a_i = 1$ if and only if $i \in A$. In this way, every $uv \in E(Q^d)$ corresponds to a pair of sets U and V for which $|U \triangle V| = 1$. In general, we have $\text{dist}_{Q^d}(u, v) = |U \triangle V|$; in particular we have $\text{dist}(u, \mathbf{0}) = |U|$, where $\mathbf{0}$ denotes the all-zeros vertex.

In this paper we will assume that all variables mentioned are nonnegative integers.

1.2. Results

Let C be any configuration of cups on G. We define the *size* of C to be $|C| = \sum_w C(w)$. A configuration D is called a *sub-configuration* of C if $D(w) \leq C(w)$ for every vertex w. For a target r of G, a subgraph H of G, and a sub-configuration D of C, H is called (D, r)-*feasible* if either $D = \emptyset$ or there is some vertex s of H such that H is (D, s)-stackable and $\text{dist}_G(s, r) = |D|$.

We say that two configurations D_1 and D_2 are *disjoint* (and write $D_1 \cap D_2 = \emptyset$) if, for every vertex w, either $D_1(w) = 0$ or $D_2(w) = 0$. Additionally, we define the configuration $D = D_1 + D_2$ by $D(v) = D_1(v) + D_2(v)$. Thus we say that the set of configurations $\{D_1, \ldots, D_k\}$ *partitions* D if they are pairwise disjoint and $\sum_{i=1}^{k} D_i = D$. Furthermore, we define the configuration $C - r$ to be $(C - r)(r) = 0$ and $(C - r)(v) = C(v)$ otherwise. For a graph G and configuration C on G with $C(r) > 0$, let $\mathcal{P} = \{(H_1, D_1), \ldots, (H_k, D_k)\}$ for some $k \geq 1$, where each H_i is a subgraph of G and each D_i is a sub-configuration of C. The family $\mathcal{P}$ is called an *r-stacking partition* of (G, C) (see Fig. 1) if

1. $V(H_1) \cup \cdots \cup V(H_k) = V(G) - \{r\}$,
2. $\{D_1, \cdots, D_k\}$ partitions $C - r$,
3. each H_i is (D_i, r)-feasible (stacking onto v_i), and
4. each H_i preserves the distances of G along the v_i-stacking moves in part (3).

Our first result is the following characterization lemma, which we will prove in Section 2.1.

Lemma 3 (*Stacking Partition Lemma*). *Let G be a graph, C be any configuration of cups on G, and r be any target vertex. Then G is (C, r)-stackable if and only if (G, C) has an r-stacking partition.*

As a simple corollary we obtain the following fact, which we will prove in Section 2.2. First, a vertex of degree at least three in a graph is called a *split* vertex. A tree with exactly one split vertex is called a *spider*. We denote $S(\ell_1, \ldots, \ell_k)$ to be the spider with $k \geq 3$ paths emanating from the split vertex (also called a *root*), having lengths $\ell_1, \ldots, \ell_k$, for a total of $\ell_1 + \cdots + \ell_k + 1$ vertices. As a slight abuse, we also consider a path to be a spider in the case that the root (now not a split vertex $- k = 2$) is not one of its endpoints; that is, the path $v_1 \cdots v_n$ can be written as $S(i - 1, n - i)$ with root v_i when $1 < i < n$.

Fact 4. *Paths and cycles are stackable, and every spider with split vertex r is r-stackable.*

The Fact 4 case of paths was first proved in [8], using a somewhat complicated induction proof. The Fact 4 case of spiders was first proved in [17]; in fact Veselovac proved more. It is easy to see that the spider $S(1, 1, \ldots)$ is r-stackable if and only if r is the split vertex.

Theorem 5. *[17] For all $k \geq 1$ and all $\ell_i > 1$, $S(\ell_1, \ldots, \ell_k)$ is stackable.*

Left open are the cases for which $k \geq 3$, some $\ell_i = 1$, and some $\ell_j > 1$. In [17] is found a complete solution for the subcase of having a unique j with $\ell_j > 1$; the statement of the theorem that describes for which vertices r the particular graph is r-stackable is not describable succinctly, so we refer the reader to [17].

We also use Lemma 3 to prove the following two theorems, which will be proven in Sections 2.3 and 2.5.

Theorem 6. *There is a polynomial algorithm to determine if a diameter two graph is r-stackable.*

The proof of Theorem 6 uses the Gallai-Edmonds Structure Theorem 9 (Section 2.3) from matching theory.

Theorem 7. *The grid $P_m \square P_k$ is stackable for all m and k.*

Furthermore, Lemma 3 also yields stackability results for complete t-partite graphs and for Johnson graphs (which generalize Kneser graphs, which includes the Petersen graph), which we state in Section 2.3 as well.

Theorem 8. *The d-dimensional cube Q^d is stackable for all $d \leq 20$.*

The proof of Theorem 8 uses the new result (Theorem 18) of Merino, Mütze, and Namrata that generalized Johnson graphs are Hamiltonian and the classic Symmetric Chain Decomposition Theorem for cubes (Theorem 20), both found in Section 2.5.

2. Proofs

2.1. Proof of the stacking partition lemma

Proof of Lemma 3. We begin by assuming that (G, C) has an r-stacking partition $\mathcal{P} = \{(H_1, D_1), \ldots, (H_k, D_k)\}$. We will show that every cup of C on $v \in G - r$ can be placed on r. By property 2 there is a unique i such that $v \in D_i$. By property 3 there is some vertex s of H_i such that all the cups of D_i can be placed on s and $\mathrm{dist}(s, r) = |D_i|$. Hence we can move all those cups from s to r; i.e. G is (C, r)-stackable.

Now we suppose that G is (C, r)-stackable. Consider the sequence of moves that place all the cups on r. Let $s_1 \ldots, s_m$ be the vertices from which cups move onto r and let X_i be the set of cups moved from s_i to r. For each i, let H_i be the union of all geodesics $P(x, y)$ used in moving all the cups in X_i from their original locations to s_i, and let D_i count the number of cups of X_i originally on each vertex of H_i. Next, if $\cup_{i=1}^m H_i = G - r$ then set $k = m$; otherwise, let $(G - r) - \cup_{i=1}^m H_i = \{u_1, \ldots, u_t\}$, $k = m + t$, and $H_{m+i} = \{u_i\}$ for $1 \leq i \leq t$, with $D_{m+i} = 0$ on H_{m+i}. Finally, define $\mathcal{P} = \{(H_1, D_1), \ldots, (H_k, D_k)\}$ — it is clear that $\mathcal{P}$ satisfies each of the four properties of an r-stacking partition. $\square$

2.2. Stackability of paths, cycles, and spiders

In this section we will be assuming that $C = \mathbf{1}$.

Proof of Fact 4. We prove that P_n is stackable by induction. The case $n = 1$ is trivial, so we assume that $n \geq 2$ and that P_k is stackable for all $1 \leq k < n$. Let $r = v_i$ be any target vertex of P_n, with $1 \leq i \leq n$. Partition P_n into two subgraphs $H_1 = \{v_j \in V(P_n) : 1 \leq j \leq i\}$ and $H_2 = \{v_j \in V(P_n) : i \leq j \leq n\}$. Note that $H_1 \cong P_i$ and $H_2 \cong P_{n-i+1}$. By induction $\mathcal{P} = \{(H_1, \mathbf{1}), (H_2, \mathbf{1})\}$ is an r-stacking partition of $(P_n, \mathbf{1})$, and so P_n is r-stackable by Lemma 3.

Now consider the cycle C_n and any target vertex r. By symmetry, we may assume that $r = v_i$, where $i = \lfloor n/2 \rfloor$. Define H_1 and H_2 as above and notice that, for each $j \in \{1, 2\}$, the distance between vertices of H_j are the same in H_j as they are in C_n. Hence $\mathcal{P} = \{(H_1, \mathbf{1}), (H_2, \mathbf{1})\}$ is an r-stacking partition of $(C_n, \mathbf{1})$, and so C_n is r-stackable by Lemma 3.

Finally, let T be an n-vertex spider with root r. Then $T - r$ is a disjoint union of paths $H_{n_1}, \ldots, H_{n_k}$, where $n_1 + \cdots + n_k = n - 1$. Then $\{(H_1, \mathbf{1}), \ldots, (H_k, \mathbf{1})\}$ is an r-stacking partition of $(T, \mathbf{1})$ because paths are stackable, and so T is r-stackable by Lemma 3. $\square$

2.3. An r-stacking algorithm for eccentricity 2 vertices

We begin this section with some definitions leading up to the Gallai-Edmonds Structure Theorem. For a graph $G = (V, E)$ and a subset $U \subseteq V$ define the *neighborhood of U in $V - U$*, denoted $N(U)$, to be the set of vertices in $V - U$ that are adjacent to some vertex of U. Let $\mathcal{M} = \mathcal{M}(G)$ be the set of all maximum matchings of G. A vertex v is called *inessential* if it is not saturated by some $M \in \mathcal{M}$, and let $I = I(G)$ be the set of all inessential vertices of G. Finally set $A = A(G) := N(I)$ and $Z = Z(G) := V - I - A$; thus $\{I, A, Z\}$ partitions V — we call the triple (I, A, Z) the *Gallai-Edmonds partition* of G. For ease of notation, we will also use I, A, and Z to denote the subgraphs of G induced by those vertices. Furthermore, say that a connected graph G is *factor-critical* if the removal of any vertex yields a graph having a perfect matching, and that a matching in G is *near-perfect* if it saturates all but one vertex of G.

Theorem 9 (Gallai-Edmonds Structure Theorem). [5–7] *Let R be a graph and let (I, A, Z) be the Gallai-Edmonds partition of R. Then the following four properties hold.*

1. *Each component of I is factor-critical.*
2. *Every subset $X \subseteq A$ has neighbors in at least $|X| + 1$ components of I.*
3. *Every maximum matching in R consists of*

(a) a near-perfect matching of each component of I,
(b) a perfect matching of Z, and
(c) edges from all vertices in A to distinct components of I.

4. If I has k components, then the number of edges in any maximum matching in R equals $(|V| - k + |A|)/2$.

Moreover, the partition (I, A, Z) can be found in polynomial time.

We begin be proving an r-stacking equivalence for eccentricity two vertices in terms of matchings.

Lemma 10. *Let G be a graph with vertex r having eccentricity 2. Then G is r-stackable if and only if G has a matching that saturates $N_2(r)$.*

Proof. Let G be a graph with vertex r having eccentricity 2.

Suppose that n-vertex G has a matching $M = \{e_1, \ldots, e_k\}$ that saturates $N_2(r)$. Label the vertices of each e_i by x_i and y_i, with $y_i \in N_2(r)$, and define $H_i = e_i$ and $D_i = \mathbf{1}$ for $1 \le i \le k$. Then $|D_i| = 2$ and each H_i is y_i-feasible with $\mathrm{dist}(y_i, r) = 2$. Now label the M-unsaturated vertices of $G - r$ by $z_1, \ldots z_j$, where $j = n - 1 - 2k$, and define $H_{k+i} = z_i$ and $D_{k+i} = \mathbf{1}$ for all $1 \le i \le j$. Since each $z_i \in N(r)$ and $|D_{k+1}| = 1$, H_{k+i} is trivially z_i-feasible with $\mathrm{dist}(z_i, r) = 1$. Hence $\mathcal{P} = \{(H_1, D_1), \ldots, (H_{k+j}, D_{k+j})\}$ is an r-stacking partition, which implies by Lemma 3 that G is r-stackable.

Now suppose that G is r-stackable; by Lemma 3 G has an r-stacking partition $\mathcal{P} = \{(H_1, D_1), \ldots, (H_k, D_k)\}$; let $\{v_1, \ldots, v_k\}$ be the set of vertices for which each H_i is (D_i, v_i)-stackable. Because $\mathrm{dist}(u, r) \le 2$ for all r we have $|D_i| \le 2$ for all $1 \le i \le k$. Consider any vertex $v \in N_2(r)$, and suppose that $v \in H_i$ for some i (which we can denote by $i(v)$ as necessary to emphasize its dependence on v) with $D_i(v) = 1$. If $v = v_i$ then there is a unique vertex u_i adjacent to v such that the cup on u_i is moved to v_i. If $v \ne v_i$ then set $u_i = v$ and note that the cup on u_i is moved to v_i. In either case label the edge $e_i = u_i v_i$. Now the set of edges $M = \{e_i \mid v \in N_2(r), i = i(v)\}$ (ignoring duplicates that occur when both $u_i, v_i \in N_2(r)$) forms a matching that saturates $N_2(r)$. □

Lemma 10 yields the following five corollaries.

Corollary 11. *Let r be a vertex of eccentricity 2 in a tree T. Then T is r-stackable if and only if T is a spider.*

Proof. Let T be a tree with vertex r having eccentricity 2. Note that, because $\mathrm{ecc}(r) = 2$, each $w \in N_2(r)$ is a leaf. Thus each such leaf can be matched simultaneously to its unique neighbor in $N_1(r)$ if and only if each $v \in N_1(r)$ has $\deg(v) \le 2$; that is, if and only if T is a spider. The result then follows from Lemma 10. □

Corollary 12. *Let $G = K_{a_1, \ldots, a_t}$ denote the complete t-partite graph on n vertices with parts $A_1, \ldots, A_t$ having sizes $|A_i| = a_i$, and let r be a vertex in part A_i. Then G is r-stackable if and only if $a_i \le (n+1)/2$. Consequently, G is stackable if and only if $a_i \le (n+1)/2$ for every i.*

Proof. Let G be a complete t-partite graph with partite sets $A_1, \ldots, A_t$ having sizes $|A_i| = a_i$. Notice that G has diameter 2 and so, by utilizing Lemma 10, G is r-stackable if and only if G has a matching that saturates $N_2(r) = A_i - \{r\}$; i.e. there exists a matching M that saturates $N_2(r)$. This happens if and only if $n - a_i = |V(G) - A_i| \ge |M| = |N_2(r)| = a_i - 1$, which is equivalent to $a_i \le (n+1)/2$. □

Corollary 13. *The Petersen graph is stackable.*

Proof. Consider the Petersen graph G_P. By symmetry, we only need to consider one target vertex r. Since G_P has diameter 2, by Lemma 10, G_P is stackable if and only if $N_2(r)$ has a saturating matching. The matching in Fig. 2, saturates $N_2(r)$. □

In fact, we can show that infinitely many Kneser graphs are stackable. We write $[m] = \{1, \ldots, m\}$ and define $\binom{[m]}{k}$ to be the set of all subsets of $[m]$ of size exactly k. For $m \ge 2k + 1 \ge 5$, the *Kneser graph* $K(m, k)$ has vertex set $\binom{[m]}{k}$, with edges between disjoint pairs. For example, the Petersen graph is $K(5, 2)$. We begin by observing a more general corollary to Lemma 10.

Corollary 14. *Let G be a graph with vertex r having eccentricity 2. If G has a Hamiltonian path then G is r-stackable.*

Proof. Consider a Hamiltonian path P of G and let $P - r$ be the disjoint union of paths P_1 and P_2, where P_2 is empty if r is an endpoint of P. For each i let v_i be the endpoint of P_i that is a neighbor of r in P. If $n(P_i)$ is even then let M_i be the perfect matching in P_i, while if $n(P_i)$ is odd then let M_i be the perfect matching in $P_i - v_i$. In both cases M_i saturates $N_2(r) \cap V(P_i)$, and so $M_1 \cup M_2$ saturates $N_2(r)$. Hence G is r-stackable by Lemma 10. □
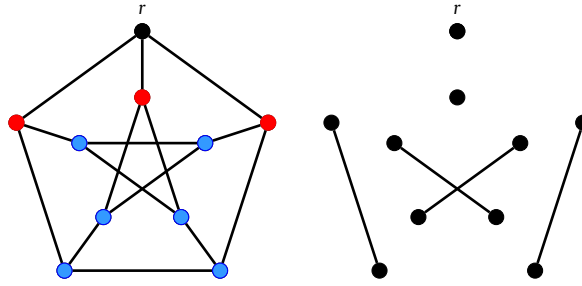
Fig. 2. The Petersen graph, showing $N_1(r)$ and $N_2(r)$, left, and an $N_2(r)$ saturating matching, right.

Notice that the hypothesis of Theorem 14 contains diameter two Hamiltonian graphs. With regard to Kneser graphs, we will use the following fact, which is a special case of Theorem 18 below.

Fact 15. *For all $m \geq 2k + 1 \geq 5$ except the case $(m, k) = (5, 2)$ the Kneser graph $K(m, k)$ is Hamiltonian.*

Corollary 16. *For all $m \geq 3k - 1 \geq 5$ the Kneser graph $K(m, k)$ is stackable.*

Proof. The case $K(5, 2)$ is proven in Theorem 13. In all other cases, we have that $K(m, t)$ is Hamiltonian by Fact 15. Moreover, since $m \geq 3t - 1$, $K(m, t)$ has diameter two. Indeed, if vertices A and B are not adjacent, then $|A \cup B| \leq 2t - 1$, and so there is some $C \subseteq [m] - (A \cup B)$ that is therefore a common neighbor of A and B. Hence Corollary 14 implies that $K(m, t)$ is r-stackable. Because Kneser graphs are vertex transitive, $K(m, t)$ is r-stackable for all r. □

It would be interesting to know whether or not all Kneser graphs are stackable, as Corollary 16 leaves open the cases $5 \leq 2k + 1 \leq m \leq 3k - 2$.

Now we are ready to prove the following theorem.

Theorem 17. *Let G be a graph with vertex r having eccentricity 2. Then it can be decided in polynomial time whether or not G is r-stackable and, in the case that it is, an r-stacking can be found in polynomial time.*

Proof. Let G be a graph with configuration $C = \mathbf{1}$ and target vertex r having eccentricity 2. We use Lemma 10 to determine whether or not G has a matching that saturates $N_2(r)$.

Let (I, A, Z) be a Gallai-Edmonds partition of $G - r$. By part 3b of Theorem 9, Z contains a perfect matching M_Z. For any set S of vertices of G and $i \in \{1, 2\}$, let $S_i = S \cap N_i(r)$. Then G has a matching that saturates $N_2(r)$ if and only if $G - Z$ has a matching that saturates $I_2 \cup A_2$.

Let I^* denote the vertices defined by contracting each component of I to a single vertex. Note by part 2 of Theorem 9 we have $|I^*| > |A|$. Then define the bipartite graph B between I^* and A that arises from the contraction, ignoring the edges within A itself. By part 3c of Theorem 9, B has an A-saturating matching.

Given any A-saturating matching M_A in B, define I_A^* to be the vertices of I^* saturated by M_A. For each edge $ax \in M_A$ we may choose, by definition, some y_x in the component I_x of I represented by $x \in I_A^*$ such that $ay_x \in E(G)$; let M_A' be the matching of all such edges. By part 1 of Theorem 9 we can find a perfect matching M_x of $I_x - y_x$ for every $x \in I_A^*$. For each $x \in I^* - I_A^*$ we can find a near-perfect matching M_x of I_x by part 3a of Theorem 9. Thus $G - Z$ has a matching that saturates $I_2 \cup A_2$ if and only if $G - Z$ has an A-saturating matching M_A that saturates I_2^* (i.e. $I_2^* \subseteq I_A^*$).

Finally, let W be a set of $|I^*| - |A|$ new vertices, define $A^* = A \cup W$, and let B^* denote the balanced complete bipartite graph $I^* \times A^*$ with edge weights $|I^*| + 1$ on every edge of B incident with I_2^*, 1 on every remaining edge of B, and 0 on every edge not in B. Then a matching in B^* has weight at least $|I_2^*|(|I^*| + 1)$ if and only if it saturates I_2^*.

Now let M^* be a perfect matching in B^* of maximum weight and let M_A^* be the set of its edges incident with A. It is easy to see that M_A^* is an A-saturating matching (by part 3c of Theorem 9) that maximizes the number of saturated vertices of I_2^*. Consequently, $G - Z$ has an A-saturating matching M that saturates I_2^* if and only if M_A^* saturates I_2^*.

Define $M_G = M_Z \cup M_A^{*\prime} \cup_{x \in I^*} M_x$. Then by Lemma 10 we see that G is r-stackable if and only if M_G saturates $N_2(r)$. Notice that M_G can be constructed in polynomial time. Indeed, the Gallai-Edmonds partition (I, A, Z) can be found in polynomial time by Theorem 9, Edmonds' Blossom Algorithm finds M_Z and each M_x in polynomial time, and the Hungarian Algorithm finds M_A^*, and hence $M_A^{*\prime}$ in polynomial time. □

Proof of Theorem 6. This follows from Theorem 17 because every vertex of a diameter two graph has eccentricity two. □
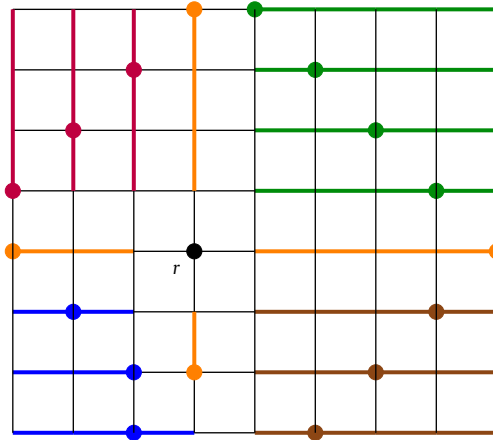
Fig. 3. An r-stacking partition for the grid $P_9 \square P_8$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

2.4. Stackability of grids

Proof of Theorem 7. Let G be the grid $P_m \square P_k$ with arbitrary m, k. By symmetry we may assume that $m \geq k$; a grid with $m = k$ is called *square*. We will first handle the case in which the target vertex r is a corner of G. Then we will reduce all other cases to this case.

Suppose that the target vertex r is on a corner of G − say, r is the bottom left corner − with coordinates $r = (0, 0)$. Consider the case that G is not square (for example, see the upper right green portion of Fig. 3). Partition G into paths $R_0 = P_m - r$, $R^* = P_k - r$ emanating from r (orange in Fig. 3) and paths $\{R_1, \ldots, R_k\}$ parallel to P_m (green in Fig. 3), and let $r_i \in V(R_i)$ have coordinates $r_i = (m - i, i)$ for each $0 \leq i \leq k$, with $r^* = (0, k)$. Clearly, $\text{dist}(r_i, r) = m = |R_i|$ for each $0 \leq i \leq k$ and $\text{dist}(r^*, r) = k = |R^*|$. Moreover, by Fact 4, each R_i is r_i-stackable, and R^* is r^*-stackable, and so G is r-stackable by Lemma 3.

If G is square ($m = k$), we use the same partition as before except that we shorten $R^* = P_{k-1}$ and lengthen $R_k \cong P_m$ (see the lower left blue portion of Fig. 3), with the corresponding changes $r_k = (1, k)$ and $r^* = (0, k - 1)$, with $\text{dist}(r_k, r) = k + 1 = |R_k|$ and $\text{dist}(r^*, r) = k - 1 = |R^*|$. Hence G is r-stackable by Lemma 3.

For the cases in which r is not a corner vertex of G, partition G into two or four smaller grids such that r is a corner of each (i.e. give r the coordinates $(0, 0)$; then the intersection of the grid with the four quadrants gives the partition). Each grid can be partitioned as above whether the quadrants form squares or not, with a slight modification, if necessary. It may be that definitions of the orange paths R^* in neighboring quadrants are not consistent, as is the case in the lower half of Fig. 3. Thus we first partition the interior of each quadrant and subsequently define each R^* to be whatever paths remain. By Lemma 3, the grid $P_m \square P_k$ is stackable for all m and k. □

2.5. Stackability of small cubes

We begin by presenting two theorems that are used in the proof of Theorem 8. Define the *generalized Johnson graph* $J(m, k, s)$ to have vertex set $\binom{[m]}{k}$, with edges between sets sharing exactly s elements. For example, the Petersen graph is isomorphic to both $J(5, 2, 0)$ and $J(5, 3, 1)$, the *Odd graphs* are the family $\{J(2k + 1, k, 0) \mid k \geq 2\}$, the *Kneser graphs* are the family $\{J(m, k, 0) \mid m \geq 2k + 1 \geq 5\}$, and the original *Johnson graphs* are the family $\{J(m, k, k - 1) \mid m \geq k + 1 \geq 2\}$. Settling a conjecture of [10] for Kneser graphs and proving even more, Merino, Mütze, and Namrata [14] proved the following theorem.

Theorem 18. *[14] Except for the Petersen graph, for all values with $s < k \leq (m + s - 1)/2$, the graph $J(m, k, s)$ is Hamiltonian.*

By using Corollary 14 and Theorem 18, we obtain the following result as well.

Corollary 19. *Every diameter two Johnson graph is stackable.*

Next we write $2^{[n]}$ to denote the set of all subsets of $[n]$. Then we define a *chain* to be a sequence $(S_1, \ldots, S_t)$ of subsets of $[n]$ such that $S_i \subseteq S_{i+1}$ for each $1 \leq i < t$; it is *saturated* if $|S_{i+1}| = |S_i| + 1$ for each $1 \leq i < t$. A saturated chain in $2^{[n]}$ is *symmetric* if $|S_1| + |S_t| = n$. A family $\mathcal{C} = \{C_1, \ldots, C_m\}$ of symmetric, saturated chains in $2^{[n]}$ is a *symmetric chain decomposition* if $\mathcal{C}$ partitions $2^{[n]}$.

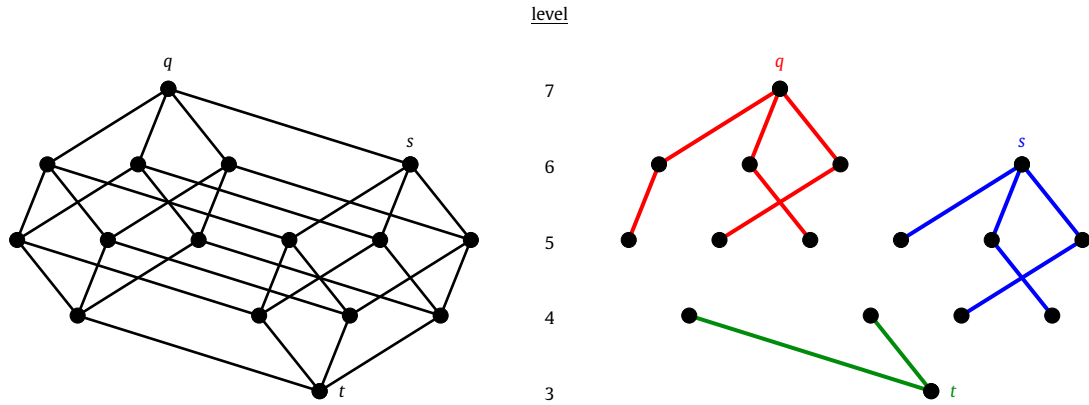**Theorem 20.** *[13] For every n, $2^{[n]}$ has a symmetric chain decomposition.*

level

Fig. 4. A level-3 4-cube T, left, with the **0**-stacking partition into spiders $\{(H_1, D_1), (H_2, D_2), (H_3, D_3)\}$, right.

Theorem 20 quickly gives rise to the following corollary. Define $\binom{[n]}{\geq k}$ (resp. $\binom{[n]}{>k}$) to be the set of all subsets of $[n]$ of size at least (resp. greater than) k. Given a symmetric chain decomposition $\mathcal{C}$, define the function $\phi : \binom{[n]}{>n/2} \to \binom{n}{\geq \lfloor n/2 \rfloor}$ as follows. For $X \in \binom{[n]}{>n/2}$ we find $C \in \mathcal{C}$ such that $X \in C$. Then find $Y \in C$ such that $|Y| = |X| - 1$, and set $\phi(X) = Y$.

Corollary 21. *For every n the function $\phi : \binom{[n]}{>n/2} \to \binom{[n]}{\geq \lfloor n/2 \rfloor}$ is an injection such that $|\phi(X)| = |X| - 1$ for all $X \in \binom{[n]}{>n/2}$.*

Proof. Because of Theorem 20 such a symmetric chain decomposition $\mathcal{C}$ of $2^{[n]}$ exists. Because $\mathcal{C}$ partitions $2^{[n]}$, for every $X \in \binom{[n]}{>n/2}$ the choice of $C \in \mathcal{C}$, such that $X \in C$, exists and is unique. Because C is saturated and $|X| > n/2$, the choice of $Y \in C$, such that $|Y| = |X| - 1 \geq \lfloor n/2 \rfloor$, exists and is unique. Hence ϕ is well-defined and is injective. □

Using the prior-mentioned identification between $2^{[d]}$ and $V(Q^d)$, we apply ϕ to the appropriately associated vertices as well. Recall in this association that each $u = (u_1, \ldots, u_d) \in V(Q^d)$ corresponds to the subset U defined by $i \in U$ if and only if $u_i = 1$. From this we define the *weight* of u by $\mathrm{wt}(u) = |U|$; i.e. $\mathrm{wt}(u) = \mathrm{dist}_{Q^d}(u, \mathbf{0})$. Additionally, the product $Q^d = Q^p \square Q^q$ is a partitioning of Q^d into q-cubes, where each vertex in Q^p (subset of $[p]$) is replaced by a q-cube. Hence we can label each such q-cube A (i.e. not all q-cubes of Q^d, just those arising from this Cartesian-product-generated partition) uniquely and distinctly by its corresponding subset S of $[p]$, and define the *level* of A as $\mathrm{level}(A) = |S|$. Of course this value equals $\min_{a \in A} \mathrm{wt}(a)$. We call the unique vertex of minimum (resp. maximum) weight in A its *bottom* (resp. *top*) vertex.

Lemma 22. *Let $k \leq 3$, $l \leq 2^k$, and $d \geq l + k$, and suppose that if $k = 3$ then $l \neq 4$. Then every level-l k-cube in Q^d is **0**-stackable. In particular, Q^d is stackable for all $d \leq 6$.*

Proof. As noted in Fact 2, Q^d is stackable if and only if it is **0**-stackable. When $k = 0$ the statement is trivially true.

When $k = 1$ we have $Q^k = K_2$, which we know to be stackable. Thus we can stack it immediately onto **0** if $l = 0$ and stack it onto its weight-2 vertex before sending the 2 cups to **0** if $l \in \{1, 2\}$. This implies that Q^2 and Q^3 are stackable since such 1-cubes form a stacking partition of them; that is, for $d \in \{2, 3\}$ we have $Q^d = Q^{d-1} \square Q^1$, with each Q^1 having level at most 2.

When $k = 2$ we have that any level-0 2-cube is **0**-stackable. Also, any level-1 2-cube consists of two 1-cubes that each contain weight-2 vertices, and so two cups can be stacked onto each of those before being sent to **0**. For $2 \leq l \leq 4$ we can stack any level-l 2-cube onto one of its weight-4 vertices before sending the 4 cups to **0**. This implies that Q^d is stackable for $d \leq 6$ since we have $Q^d = Q^{d-2} \square Q^2$, with each Q^2 having level at most 4.

When $k = 3$ we have that any 3-cube of level at most 3 is **0**-stackable because it can be partitioned into two 2-cubes of level at most 4. If $5 \leq l \leq 8$ then any level-l 3-cube can be stacked onto one of its weight-8 vertices before sending all its cups to **0**. This finishes the proof. □

Lemma 23. *For every $l \leq 3$ and $d \geq l + 4$, any level-l 4-cube in Q^d is **0**-stackable. For every $d \geq 8$, the set of all level-4 3-cubes in Q^d is **0**-stackable.*

Proof. Let A be a level-l 4-cube. If $l \leq 2$ then it can be partitioned into two 3-cubes of levels at most 3, so the result follows from Lemma 22. If $l = 3$ then Fig. 4 shows a **0**-stacking partition. If $l = 4$ then we partition it into two 3-cubes of levels 4 and 5. We stack each level-5 3-cube to its top vertex, leaving the set of all level-4 3-cubes in Q^d to stack. These we now consider.
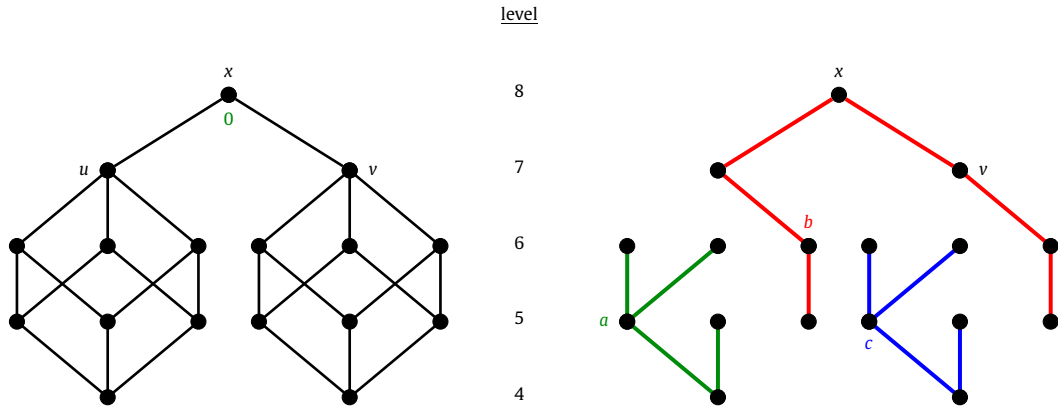
Fig. 5. Coordinating the stacking of two "adjacent" level-4 3-cubes in Q^d for $d \geq 8$.
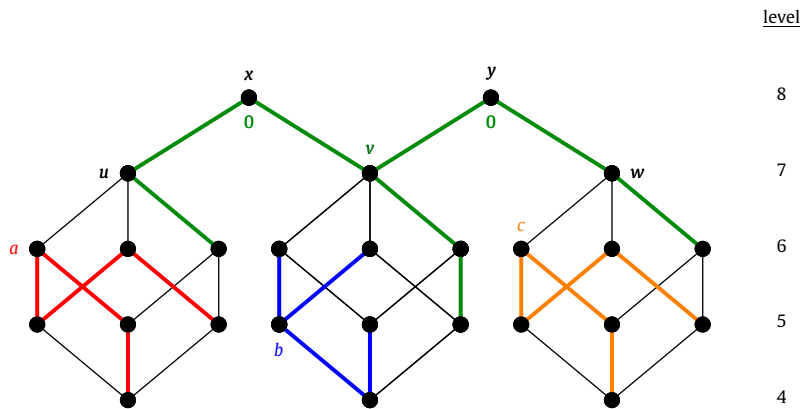

Fig. 6. Coordinating the stacking of three "consecutive" level-4 3-cubes in Q^d for $d \geq 8$.

These cubes can be labeled by their lowest vertex; the set of such labels is $\binom{[d-3]}{4} = V(J(d-3,4,3))$. By Theorem 18 we let H be a Hamilton cycle in $J(d-3,4,3)$, which we partition into paths of length 1, along with a single path of length 2 if $\binom{d-3}{4}$ is odd. For each path P in this partition we show below how the union of cubes associated with the vertices of P is r-stackable.

When P consists of two adjacent vertices U and V we set $X = U \cup V$; recall that $|U| = |V| = 4$ and $|U \cap V| = 3$, and so $|X| = 5$; i.e. X corresponds to a level-5 3-cube. Hence the top vertices u and v of the cubes labeled U and V are each adjacent in Q^d to the top vertex x of the cube labeled X (see the left side of Fig. 5). In this case, the configuration of cups on this structure S has one cup on every vertex except no cup on vertex x, since all the cups in the 3-cube labeled X were already stacked onto its top vertex. The right side of Fig. 5 shows in color the stacking partition of S that places the appropriate number of cups on the appropriate levels of vertices a, b, and c (which are roots of spiders), to then stack them all on r. The point of vertex x is for it to be used to move three cups from v to b.

When P consists of three consecutively adjacent vertices U, V, and W, we set $X = U \cup V$ and $Y = V \cup W$; so that $|X| = |Y| = 4$. Hence the top vertices u and v of the cubes labeled U and V are each adjacent in Q^d to the top vertex x of the cube labeled X, while the top vertices v and w of the cubes labeled V and W are each adjacent in Q^d to the top vertex y of the cube labeled Y (see Fig. 6). In this case, the configuration of cups on this structure S has one cup on every vertex except no cups on vertices x and y, since all the cups in the 3-cubes labeled X and Y were already stacked onto the top vertices of those cubes. Fig. 5 also shows in color the stacking partition of S that places the appropriate number of cups on the appropriate levels of vertices a, b, c, and v (roots of spiders) to then stack them all on r. The point of vertices x and y is for them to be used to move two cups from each of u and w to v.

This completes the proof. ☐

Corollary 24. *For $d \leq 11$, Q^d is stackable.*

Proof. The cases $d \leq 6$ are proven in Lemma 22.

When $d = 7$ we write $Q^7 = Q^3 \square Q^4$, which partitions Q^d into 4-cubes having levels at most 3, which are **0**-stackable by Lemma 23.
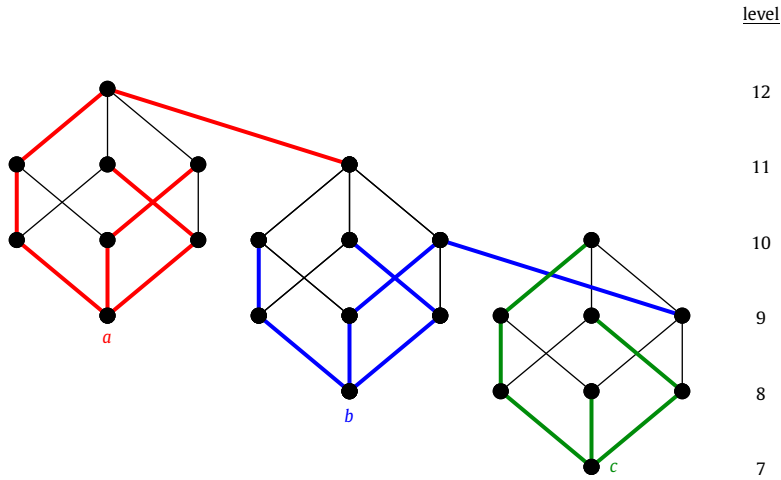
level

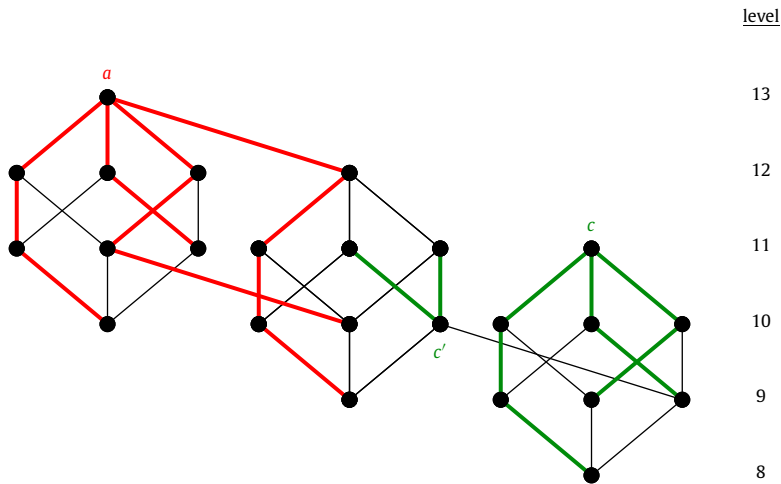**Fig. 7.** Coordinating the **0**-stacking of 3-cubes at levels 7–9 in Q^{12}.

level

Fig. 8. Coordinating the **0**-stacking of 3-cubes at levels 8–10 in Q^{13}.

When $8 \leq d \leq 11$ we write $Q^d = Q^{d-3} \square Q^3$, which partitions Q^d into 3-cubes having levels at most 8, which are **0**-stackable by Lemmas 22 and 23. □

Lemma 25. *Suppose that* $9 \leq l \leq 12$ *and* $l + 3 \leq d \leq 2l$ *and choose some injection* ϕ *from Corollary 21, with* $n = d - 3$. *Let* $A \in \binom{[n]}{l}$ *be the label of a level-l 3-cube in* Q^d, $B = \phi(A)$, *and* $C = \phi(B)$. *Then* $A \cup B \cup C$ *is* **0**-*stackable in* Q^d.

Proof. We note first that the existence of ϕ is given by the hypothesis of Corollary 21: if level$(A) = l$ then level$(B) = l - 1 > (d-3)/2$.

The stacking partitions are given in Figs. 7, 8, 9, and 10.

In Fig. 7, a level-9 3-cube A steals a vertex from the level-8 3-cube B to form a spider rooted at a, with the resulting 9 cups stacked at a before being sent to **0**. Then the "damaged" cube B steals a vertex from the level-7 3-cube C to form a spider rooted at b, with the resulting 8 cups stacked at b before being sent to **0**. Finally, the damaged C stacks its 7 cups at the root of the spider at c before sending them to **0**.

In Fig. 8, A steals 5 vertices from B to form a spider at a, with the resulting 13 cups stacked at a before being sent to **0**. Then C steals the remaining three vertices from B to form spiders at c and c', with the 3 cups stacked at c' being sent to c, joining with the 8 cups stacked at c to send the resulting 11 cups to **0**.

In Figs. 9 and 10, similar schemes are shown, combining stealing, spiders, and supplemental spiders, the details of which are left to the reader. □

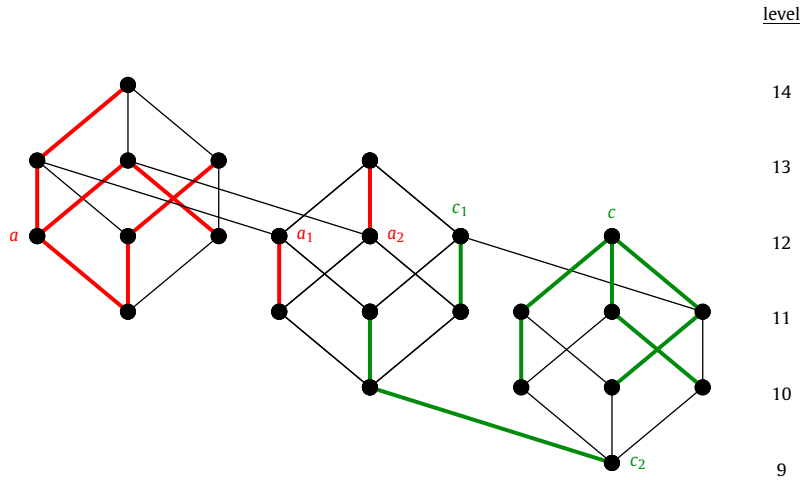**Corollary 26.** *For* $d \leq 15$, Q^d *is stackable.*

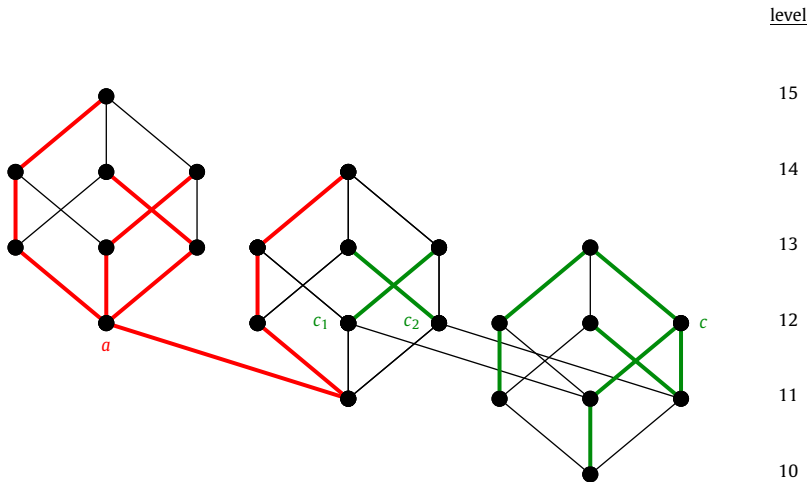**Fig. 9.** Coordinating the **0**-stacking of 3-cubes at levels 9–11 in Q^{14}.

Fig. 10. Coordinating the **0**-stacking of 3-cubes at levels 10–12 in Q^{15}.

Proof. The cases $d \leq 11$ are proven in Corollary 24.

When $12 \leq d \leq 15$ we write $G = Q^d = Q^{d-3} \square Q^3$, which partitions Q^d into 3-cubes having levels at most $d - 3$. We choose ϕ from Corollary 21, with $n = d - 3$, For the highest level cube A we define $B = \phi(A)$ and $C = \phi(B)$ and **0**-stack $A \cup B \cup C$ as in Lemma 25. After removing $A \cup B \cup C$ from G, the remaining graph G', which is still partitioned into 3-cubes, has a highest level cube A', from which we define $B' = \phi(A')$ and $C' = \phi(B')$ and **0**-stack $A' \cup B' \cup C'$ as in Lemma 25. We repeat this process until no cubes at level at least 9 remain; because ϕ comes from a symmetric chain decomposition, each of the 3-cubes in this process are distinct. i.e. the resulting graph is partitioned into 3-cubes having levels at most 8, the union of which is **0**-stackable by Lemmas 22 and 23.

This completes the proof. $\square$

Lemma 27. *Let $k \leq 15$, $2^k - k \leq l \leq 2^k$ and $d \geq l + k$. Then every level-l k-cube in Q^d is **0**-stackable.*

Proof. Corollary 26 shows that Q^k is stackable. Therefore every level-l k-cube in Q^d can be stacked onto one of its weight-2^k vertices, with the resulting 2^k cups then sent to **0**. $\square$

Proof of Theorem 8. The cases $d \leq 15$ are proven in Corollary 26.

For $16 \leq d \leq 20$ we write $Q^d = Q^{d-4} \square Q^4$, which partitions Q^d into 4-cubes. We use Lemma 27 with $k = 4$ to **0**-stack all 4-cubes at levels 12 and above. After removing all such 4-cubes, the remaining graph G is still partitioned into 4-cubes having levels at most 11. Because $Q^4 = Q^1 \square Q^3$, we can more finely partition G into 3-cubes having levels at most 12. Now we choose ϕ from Corollary 21 with $n = d - 3$ (from the representation $Q^d = Q^{d-3} \square Q^3$), and use Lemma 25 and the technique of Corollary 26 to **0**-stack G, finishing the proof. $\square$

3. Conclusion

One can imagine continuing an inductive approach to proving that Q^d is stackable for all d, along the lines of the above. However, one would need to conceive of a way of uniformizing the ad-hoc methods for each level of Lemma 25 in a manner that would readily generalize to higher dimensions. Once achieved, generalizing Lemma 27 to level-l k-cubes with $2^k - l \leq l \leq 2^k$ would be straightforward. For example, now that Theorem 8 has been proven, Lemma 27 is true for $k \leq 20$. We are left believing that there is no mathematical obstruction to the stackability of all cubes and so make the following conjecture.

Conjecture 28. *For all d, Q^d is stackable.*

Moreover, as $Q^d = \square_{i=1}^d P_2$, we believe the same to be true of all higher-dimensional grids, as shown by Theorem 7 for dimension two.

Conjecture 29. *For all d and $k_1, \ldots, k_d$, the grid $\square_{i=1}^d P_{k_i}$ is stackable.*

The following enticing conjecture was posited by Veselovac, who verified it for $h \leq 3$ and also for the root vertex only when $h \leq 5$.

Conjecture 30. *[17] The complete binary tree T_h with depth h is stackable for all $h \geq 1$.*

3.1. Open problems

We mention here a small selection of interesting problems to consider. Readers can likely think of several others.

1. Is there a characterization theorem for the stackability of a graph onto a vertex of eccentricity 3, akin to Lemma 10?
2. Can stackable trees be characterized? For a general tree T, can the vertices r for which T is r-stackable be characterized?
3. Are cartesian products of stackable graphs stackable?
4. Are generalized Johnson graphs of diameter at least 3 stackable?
5. Are there any non-trees that are not r-stackable for any r?[2]
6. What is the computational complexity for determining whether G is r-stackable?

3.2. Variations

Additionally, one can imagine asking a similar range of questions about slightly modified versions of cup stacking. One could:

- generalize the initial configuration — we have seen in Figs. 1, 5, and 6 that this is a necessary generalization from the viewpoint of stacking the parts of an r-stacking partition;
- generalize the stacking target to a set S of vertices — one must stack onto any subset of S, onto exactly S, or onto a prescribed number of cups per vertex of S;
- allow for non-geodesic paths — this would make every super-graph of a stackable graph stackable; for example, graphs with a Hamilton path; or
- allow for traversing trails or walks instead of just paths.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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[2] We thank one of the referees for inspiring this question.

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