Chapter 1

Introduction

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1.1 The Diet Problem

Imagine that your entire class is allowed to eat from the following menu. A hamburger, a chicken sandwich, a fish sandwich, and a deluxe cheeseburger (water will be your only beverage). The game is that your meal must satisfy certain percentages of the USRDA of Vitamin A, Vitamin C, Calcium, and Iron, and the winner will be the person who consumes the fewest calories. The table below contains all the necessary information regarding percentages of USRDA and number of calories for each item and for the requirements. Keep in mind that, while you must buy whole number amounts of each item, you are allowed to eat fractional amounts of each item. To add excitement, the winner will receive what Carol Merrill is hiding behind door number 3.

	%A	%C	%Calc	%Iron	Calories
Hamburger	4	4	10	15	250
Chicken	8	15	15	8	400
Fish	2	0	15	10	370
Cheeseburger	15	6	30	20	490
Requirements	10	10	15	15	

For example, if one ate the chicken and the cheeseburger, that would amount to the percentages of 23, 21, 45, and 28, along with 890 calories. Another could eat exactly half of that, but then would fail the Iron requirement, needing 2 more percentage points. Eating an extra one tenth of the cheeseburger would make up that difference, increasing her total calorie intake to 494, a great improvement over 890. Still better, a third person could decide on eating two fifths of each of the hamburger, chicken, and cheeseburger. This satisfies the four requirements with only 456 calories.

Workout 1.1.1 *Find a menu that satisfies the requirements while consuming fewer than 456 calories.*

How low can one go? Is it possible to consume fewer than 400 calories under these conditions? Less than 350? Before we try to answer this, let us first try to set the problem in more precise mathematical terms.

If we set y_1 through y_4 to be the amounts consumed of hamburger through cheeseburger, respectively, then we come upon the following observations. The total percentage of Vitamin A eaten is $4y_1 + 8y_2 + 2y_3 + 15y_4$. By the above discussion, this quantity should be at least 10. We can carry out the same analysis on the other nutrients as well, and in fact, the quantity we would like to minimize is $250y_1 + 400y_2 + 370y_3 + 490y_4$, the number of calories consumed. So we can succinctly display our challenge mathematically as follows.

Problem 1.1.2

$$
\begin{array}{lccccccccccc}
\text{Minimize} & w & = & 250y_1 & + & 400y_2 & + & 370y_3 & + & 490y_4 & & \\
\\
\text{subject to} & & & 4y_1 & + & 8y_2 & + & 2y_3 & + & 15y_4 & \geq & 10 \\
 & & & 4y_1 & + & 15y_2 & & & + & 6y_4 & \geq & 10 \\
 & & & 10y_1 & + & 15y_2 & + & 15y_3 & + & 30y_4 & \geq & 15 \\
 & & & 15y_1 & + & 8y_2 & + & 10y_3 & + & 20y_4 & \geq & 15 \\
\\
\text{and} & & & y_1 & , & y_2 & , & y_3 & , & y_4 & \geq & 0
\end{array}
$$

nonnegativity/ problem constraint

Notice that we haven't forgotten how difficult (actually unsightly) it is to consume a negative amount — these **nonnegativity constraints** are quite common in many similar problems and shouldn't be neglected. The other constraints, called **problem constraints** are all linear inequalities (linear referring to the absence of terms like y_1y_3 and y_4^2 in the sum), the only types of inequalities we will consider here. The final building block of this (and every) linear optimization problem (LOP) is what we call the **objective function**, and we often reserve a separate variable for it, say w.

objective function

As we begin to formalize the LOP in this way, it becomes easier to think about answering the problem of how low we can go with calories. For example, if we multiply the third constraint by 15 we see that $150y_1 + 225y_2 + 225y_3 + 450y_4 \geq 225$. This leads to the following lower bound.

$$
\begin{array}{rcl}
w & = & 250y_1 + 400y_2 + 370y_3 + 490y_4 \\
 & \geq & 150y_1 + 225y_2 + 225y_3 + 450y_4 \\
 & \geq & 225.
\end{array}
$$

Carefully, let's think of why we can reason that $w \geq 150y_1 + 225y_2 + 225y_3 + 450y_4$. It is not simply because $250 \geq 150$, and so on, but because we know that $y_1 \geq 0$, and so on. Because of *both* of these two facts, we have $250y_1 \geq 150y_1$, and so on.

We can improve our lower bound to 350 by multiplying the first, second, and fourth constraints each by 10, and then adding them together. Check closely to see that these next inequalities hold true.

$$
\begin{array}{rcl}
w & = & 250y_1 + 400y_2 + 370y_3 + 490y_4 \\
 & \geq & 230y_1 + 310y_2 + 120y_3 + 410y_4 \\
 & = & 10(4y_1 + 8y_2 + 2y_3 + 15y_4) \\
 & & +10(4y_1 + 15y_2 + 6y_4) \\
 & & +10(15y_1 + 8y_2 + 10y_3 + 20y_4) \\
 & \geq & 10(10) + 10(10) + 10(15). \\
 & = & 350.
\end{array}
$$

With just a minor adjustment (use 11 in place of the second and third 10s) 375 becomes a slightly better lower bound. Can you push the lower bound above 400?

Workout 1.1.3 *Find a lower bound that is greater than 375 calories.*

At this point, it seems we are splashing in the same kind of water we started in. It is nice to know the answer lies somewhere between 375 and 456, but we are resorting to hit-or-miss guessing. It works up to a point, say, right about here. Certainly, if the LOP involved many more variables or constraints, we'd have had to abort much sooner. The aim of this scroll is to consider precisely this kind of analysis and see where it leads us. Soon, it will lead us to an algorithmic solution by the famous Simplex algorithm, to the theory of Duality, and to a myriad of fascinating and powerful applications. While we were considering lower bounds, we were in the process of building the very similar LOP, known as its dual, below.

Problem 1.1.4

$$\begin{array}{llrcrcrcrcr}
\text{Maximize} & z = & 10x_1 & + & 10x_2 & + & 15x_3 & + & 15x_4 & & \\
\text{subject to} & & 4x_1 & + & 4x_2 & + & 10x_3 & + & 15x_4 & \leq & 250 \\
 & & 8x_1 & + & 15x_2 & + & 15x_3 & + & 8x_4 & \leq & 400 \\
 & & 2x_1 & & & + & 15x_3 & + & 10x_4 & \leq & 370 \\
 & & 15x_1 & + & 6x_2 & + & 30x_3 & + & 20x_4 & \leq & 490 \\
\text{and} & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

Notice the similarities, as well as the subtle differences, between Problems 1.1.2 and 1.1.4. We will discuss their dual relationship in detail in Chapter 4. For now be amazed that if z^* is the maximum z under these conditions, and w^* is the minimum w subject to its constraints, then $z^* = w^*$! This impressive result is called the Strong Duality Theorem 4.1.9, and it plays a most central role in this course. It has surprising and powerful implications in fields as dissimilar as Game Theory, Linear Algebra, Combinatorics, and Geometry.

Workout 1.1.5 (Weak Duality Theorem) *Suppose that the set of y_i satisfies the constraints of Problem 1.1.2 and produce the objective value w. Suppose also that the set of x_j satisfies the constraints of its dual Problem 1.1.4 and produce the objective value z. Use all the constraints together to prove that $z \leq w$.*

Now to satisfy your curiosity, let's present the solution. The winner will consume roughly 424.156 calories (exactly $w^* = 766450/1807$) by eating $y_1^* = 535/1807$ of the hamburger, $y_2^* = 810/1807$ of the chicken sandwich, $y_3^* = 0$ of the fish sandwich, and $y_4^* = 630/1807$ of the cheeseburger deluxe. (We use the star superscript to connote optimal values.) One of the charming qualities of Linear Optimization is that we can be easily convinced of the minimality of this particular solution without showing any of the details which led to its discovery. We simply multiply the first constraint by $x_1^* = 27570/1807$, the second by $x_2^* = 24880/1807$, and the fourth by $x_4^* = 16130/1807$, and then add them up (we could say that we also multiply the third constraint by $x_3^* = 0$). How these figures were obtained is for future lectures.

What is significant is that, while the y_is offer a *proposed optimal* solution, the x_js provide a **certificate** of their optimality. The existence of certificates is a hallmark of Linear Optimization. While finding optimal solutions may be time consuming, checking their optimality is trivial. The same cannot be said of Calculus, for example — how do you know your answer is correct without redoing the problem? certificate

One final note: observe that the diet solution included only rational values for the y_i^*s, x_j^*s and $w^* = z^*$. This is no coincidence. While polynomials of degree at least two in one variable with