

On the Holroyd-Talbot Conjecture for Sparse Graphs

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Abstract

Given a graph G , let $\mu(G)$ denote the size of the smallest maximal independent set in G . A family of subsets is called a *star* if some element is in every set of the family. A *split* vertex has degree at least 3. Holroyd and Talbot conjectured the following Erdős-Ko-Rado type statement about intersecting families of independent sets in graphs: if $1 \leq r \leq \mu(G)/2$ then there is an intersecting family of independent r -sets of maximum size that is a star. In this paper we prove similar statements for sparse graphs on n vertices: roughly, for graphs of bounded average degree with $r \leq O(n^{1/3})$, for graphs of bounded degree with $r \leq O(n^{1/2})$, and for trees having a bounded number of split vertices with $r \leq O(n^{1/2})$.

1 Introduction

For $0 \leq r \leq n$, let $\binom{n}{r}$ denote the family of all r -element subsets (*r-sets*) of $[n] = \{1, 2, \dots, n\}$. For any family \mathcal{F} of sets, define the shorthand $\cap \mathcal{F} = \cap_{S \in \mathcal{F}} S$.

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If $\cap \mathcal{F} \neq \emptyset$, we say that \mathcal{F} is a *star*; in this case, any $x \in \cap \mathcal{F}$ is called a *center*. The family $\mathcal{F}_x = \{S \in \mathcal{F} \mid v \in S\}$ is called the *full star of \mathcal{F} at x* . Furthermore, we define the notation $\mathcal{F}^r = \{S \in \mathcal{F} \mid |S| = r\}$. The family \mathcal{F} is *intersecting* if every pair of its members intersects.

Erdős, Ko, and Rado [9] proved the following, classical theorem, of central importance in extremal set theory.

Theorem 1. (Erdős-Ko-Rado, 1961) *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting for $r \leq n/2$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. Moreover, if $r < n/2$, equality holds if and only if $\mathcal{F} = \binom{[n]}{r}_x$ for some $x \in [n]$.*

Hilton and Milner [13] proved the following, stronger stability result.

Theorem 2. (Hilton-Milner, 1967) *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting for $r \leq n/2$, and \mathcal{F} is not a star, then $|\mathcal{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.*

For a graph G , let $\mathcal{I}(G)$ denote the family of all independent sets of G . We write $s_r(v) = |\mathcal{I}_v^r(G)|$ when G is understood. Let $\mathcal{F} \subseteq \mathcal{I}^r(G)$ be an intersecting subfamily of maximum size. We say that G is *r -EKR* if some v satisfies $s_r(v) = |\mathcal{F}|$, and *strictly r -EKR* if every such \mathcal{F} equals $\mathcal{I}_v^r(G)$ for some v .

Write $\alpha(G)$ for the independence number of G . Let $\mu(G)$ denote the size of a smallest maximal independent set in G . Equivalently, $\mu(G)$ is the size of the smallest independent dominating set in G . Holroyd and Talbot [15] made the following conjecture.

Conjecture 3. (Holroyd-Talbot, 2005) *For any graph G , if $1 \leq r \leq \mu(G)/2$ then G is r -EKR.*

Of course, this conjecture is true for the empty graph by Theorem 1. While not explicitly stated in graph-theoretic terms, earlier results by Berge [2], Deza and Frankl [8], and Bollobas and Leader [3] support the conjecture. The conjecture has been proven for $\mu(G)$ sufficiently large in terms of r (see [4]), and also

for various graph classes, for example, disjoint unions of complete graphs, paths, and cycles containing at least one isolated vertex (see [14]), chordal graphs containing an isolated vertex (see [16]), and others.

2 Results

Here we prove the following theorem.

Theorem 4. *Let r and d be positive integers. Suppose that G is a graph on $n > \frac{27}{8}dr^2$ vertices, having maximum degree less than d . Then G is r -EKR.*

We can expand the family of graphs beyond bounded degree to bounded average degree at the cost of reducing the range of r from $O(n^{1/2})$ to $O(n^{1/3})$, as follows.

Theorem 5. *Given a positive integer r , let $c \geq e/36$ be a constant. Suppose that G is a graph on $n > 18cr^3$ vertices, having at most cn edges. Then G is r -EKR.*

It is likely that a quadratic bound on n is possible for Theorem 5 as well. Note that the case $c = 1$ in Theorem 5 is especially relevant for trees. In this case, we can retrieve a quadratic lower bound for n for one special class of trees.

A *split vertex* in a graph is a vertex of degree at least three. A *spider* is a tree with exactly one split vertex. For a spider S with split vertex w and leaves v_1, \dots, v_k , we write $S = S(\ell_1, \dots, \ell_k)$, where $\ell_i = \text{dist}(w, v_i)$. The notation is written in *spider order* if:

- if ℓ_i and ℓ_j are both odd and $\ell_i < \ell_j$ then $i < j$;
- if ℓ_i and ℓ_j are both even and $\ell_i < \ell_j$ then $i > j$; and
- if ℓ_i is odd and ℓ_j is even then $i < j$.

Notice that, since every independent set in $S(1, 1, \dots, 1)$ is a subset of its leaves, Conjecture 3 is true for $S(1, 1, \dots, 1)$. In an attempt to prove the Holroyd-Talbot conjecture for spiders by induction, the authors of [17] proved the following result.

Theorem 6. (Hurlbert-Kamat, 2022) *Suppose that $S = S(\ell_1, \dots, \ell_k)$ is a spider written in spider order. Let w be the split vertex of S , for each i let u_i be any vertex on the wv_i -path, and suppose that $r \leq \alpha(S)$. Then*

1. $s_r(w) \leq s_r(v_i)$ for all i ,
2. $s_r(u_i) \leq s_r(v_i)$ for all i , and
3. $s_r(v_j) \leq s_r(v_i)$ for all $i < j$.

Estrugo and Pastine [10] call a tree T r -HK if $s_r(v)$ is maximized at a leaf of T (and HK if r -HK for all $r \leq \alpha(T)$). It is proved in [16] that every tree is r -HK for $r \leq 4$, but Baber [1], Borg [5], and Feghali, Johnson, and Thomas [11] each found counterexamples when $r \geq 5$. However, parts 1 and 3 of Theorem 6 together imply that every spider S is HK. Theorem 5 shows that spiders are r -EKR for $r < (n/18)^{1/3}$. Unfortunately, $\mu/2$ for spiders is roughly $n/6$, so there remains a big gap. Our next theorem shrinks that gap somewhat.

Theorem 7. *Let $S = S(\ell_1, \dots, \ell_k)$ be a spider on n vertices, with split vertex w and leaves v_1, \dots, v_k . Suppose that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$. Then S is r -EKR.*

We note that every spider S has $\alpha(S) \geq 3 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n \leq 16$ and $\alpha(S) \geq (n-1)/3 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n \geq 7$. In other words, the hypothesis of Theorem 7 implies $r \geq \alpha(S)$.

Finally, we prove the following similar result for more general trees.

Theorem 8. *Let T be a tree on n vertices, with exactly s split vertices. Suppose that $s < r/2$ and $r \leq \sqrt{n \ln c} - (\ln c)/2$, where $c = 2 - 2s/r$. Then T is r -EKR.*

3 Technical Lemmas

Proposition 9. *If $0 \leq x \leq 2k/(k+1)^2$ for some $k \geq 1$, then $e^{-x} < 1 - \left(\frac{k}{k+1}\right)x$.*

Proof. Let $0 \leq x \leq 2k/(k+1)^2$ for some $k \geq 1$. Then $|x| < 1$, and so $e^{-x} = \sum_{i \geq 0} (-x)^i / i! < 1 - x + x^2/2$. Also, $(k+1)x < 2$, which implies that $x^2/2 < x/(k+1) = [1 - k/(k+1)]x$. Thus $e^{-x} < 1 - x + x^2/2 < 1 - \left(\frac{k}{k+1}\right)x$. \square

Corollary 10. *If $0 \leq y \leq 2k^2/(k+1)^3$ for some $k \geq 1$ then $1 - y > e^{-(\frac{k+1}{k})y}$.*

Proof. Set $x = \left(\frac{k+1}{k}\right)y$ and apply Proposition 9. \square

Lemma 11. *If $r \geq 2$, $d \geq 2$, and $n \geq \frac{27}{8}dr^2$ then $\prod_{i=1}^{r-1} \left(1 - \frac{r+id}{n}\right) > \frac{r}{n}$.*

Proof. We begin with

$$\prod_{i=1}^{r-1} \left(1 - \frac{r+id}{n}\right) > 1 - \sum_{i=1}^{r-1} \frac{r+id}{n} = 1 - \frac{r(r-1) + d\binom{r}{2}}{n} = 1 - \frac{(d+2)\binom{r}{2}}{n}.$$

Since $d \geq 2$, and by using Corollary 10 with $y = dr^2/n$ and $k = 2$, we have

$$1 - \frac{(d+2)\binom{r}{2}}{n} > 1 - \frac{dr^2}{n} > e^{-3dr^2/2n} > e^{-4/9} > .64.$$

In addition, we calculate

$$\frac{r}{n} \leq \frac{8}{27dr} \leq \frac{2}{27} < .08,$$

which completes the proof. \square

Claim 12. *Let G be a graph with n vertices and maximum degree less than d .*

Then every vertex v satisfies

$$s_r(v) \geq \frac{1}{(r-1)!} (n-d)(n-2d) \cdots (n-(r-1)d).$$

Proof. Let W_0 be the set of vertices of G , and set $w_0 = v$. For each $0 < i < r$, choose $w_i \in W_i$, where $W_{i+1} = W_i - N[w_i]$. Then by induction we have $|W_i| \geq m - id$ for each such i . The resulting set $\{w_0, \dots, w_{r-1}\}$ is independent in G and there are at least $\prod_{0 \leq i < r} (m - id)$ ways to choose such sets, ignoring replication. Accounting for replication, we obtain the result. \square

Lemma 13. *Let H be a graph with at least $m = n(1 - 1/3r)$ vertices and maximum degree less than d . Suppose that $1/3r + rd/n \leq 2k^2/(k+1)^3$ for some $k \geq 1$. Then every vertex v satisfies*

$$s_r(v) \geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}.$$

Proof. We use Claim 12 and Corollary 10 with $y = 1/3r + rd/n$ to obtain

$$\begin{aligned} s_r(v) &\geq \frac{1}{(r-1)!} \prod_{0 \leq i < r} (m - id) \\ &\geq \frac{n^{r-1}}{(r-1)!} \prod_{0 \leq i < r} \left(1 - \frac{1}{3r} - \frac{id}{n}\right) \\ &\geq \frac{n^{r-1}}{(r-1)!} \prod_{0 \leq i < r} \left[1 - \left(\frac{1}{3r} + \frac{rd}{n}\right)\right] \\ &\geq \frac{n^{r-1}}{(r-1)!} \prod_{0 \leq i < r} e^{-\left(\frac{k+1}{k}\right)\left(\frac{1}{3r} + \frac{rd}{n}\right)} \\ &\geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)\left(\frac{k+1}{k}\right)\left(\frac{1}{3r} + \frac{rd}{n}\right)} \\ &\geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}. \end{aligned}$$

\square

4 Proof of Theorem 4

We use the following result of Frankl [12]. For $\mathcal{F} \subseteq \binom{[n]}{r}$, define $\overline{\mathcal{F}}_x = \mathcal{F} - \mathcal{F}_x$.

Theorem 14. (Frankl, 2020) *Suppose that $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting for $r < n/72$. Then there is some x such that $|\overline{\mathcal{F}_x}| \leq \binom{n-3}{r-2}$.*

4.1 Proof of Theorem 4

The result is trivial for $r = 1$ or $d = 1$, so we assume $r \geq 2$ and $d \geq 2$. Let x be as in Theorem 14, and select $E \in \overline{\mathcal{F}_x}$. Via the same counting method as in Claim 12, we have at least

$$\frac{1}{(r-1)!} (n-r-d)(n-r-2d) \cdots (n-r-(r-1)d) \quad (1)$$

r -sets $F \in \mathcal{I}_r(x)$ with $F \cap E = \emptyset$. Since \mathcal{F} is intersecting, these sets are not in \mathcal{F}_x . Therefore, using Theorem 14 and the bound in (1), we have

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_x| + |\overline{\mathcal{F}_x}| \\ &\leq |\mathcal{I}_r(x)| - \frac{(n-r-d) \cdots (n-r-(r-1)d)}{(r-1)!} + \binom{n-3}{r-2}. \end{aligned}$$

This upper bound is at most $|\mathcal{I}_r(x)|$ precisely when

$$\binom{n-3}{r-2} \leq \frac{1}{(r-1)!} \prod_{i=1}^{r-1} (n-r-id),$$

which we rewrite as

$$\prod_{i=1}^{r-1} (n-r-id) \geq (r-1)! \binom{n-3}{r-2} = (r-1) \prod_{i=1}^{r-2} (n-2-i).$$

This inequality will follow from showing that

$$\prod_{i=1}^{r-1} (n-r-id) \geq rn^{r-2},$$

which holds by Lemma 11, and which completes the proof.

5 Proof of Theorem 5

The result is trivial for $r = 1$, so we may assume that $r \geq 2$. Let V_0 be the set of vertices of G . For each $i \geq 0$, choose $v_i \in V(G_i)$ such that $\deg_{G_i}(v_i) \geq 3cr$, where $G_{i+1} = G_i - v_i$. Let t be minimum such that $\Delta(G_t) < 3cr$. The number of edges removed in this process is at least $3tcr$, which must be at most the number of edges of G ; thus $t \leq n/3r$. Hence $V(G_t) = n - t \geq n(1 - 1/3r)$.

Now we set $d = 3cr$, $k = 4r - 7 \geq 1$, and calculate that

$$(k + 3) + \left(\frac{3k + 1}{k^2} \right) \leq k + 7 = 4r,$$

so that $(k + 1)^3 \leq 4k^2r$, which implies that

$$\frac{1}{3r} + \frac{rd}{n} < \frac{1}{3r} + \frac{3cr^2}{18cr^3} = \frac{1}{2r} \leq \frac{2k^2}{(k + 1)^3}.$$

This allows the use of Lemma 13 with $H = G_t$, $m = n(1 - 1/3r)$, and $d = 3cr$.

We obtain that each vertex v of G_t has $s_r(v)$ at least

$$\frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}. \quad (2)$$

Now we use the Hilton-Milner Theorem 2 to show that any intersecting family \mathcal{F} of independent sets that is not a star has size less than (2). First, we note the combinatorial identity $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 = 1 + \binom{n-2}{r-2} + \binom{n-3}{r-2} + \cdots + \binom{n-r-1}{r-2}$. Second, we observe the inequality $r^2/n < e^{-(r-1)2k/(k+1)^2}$. Indeed,

$$\frac{r^2}{n} < \frac{1}{18cr} \leq e^{-1} \leq e^{-(r-1)(8r-14)/(4r-6)^2} = e^{-(r-1)2k/(k+1)^2},$$

because $e \leq 18cr$ and $(4r - 6)^2 > (r - 1)(8r - 14)$ for all $r \geq 2$ and $c \geq e/36$.

Finally, if \mathcal{F} is as above then we have

$$\begin{aligned}
|\mathcal{F}| &< r \binom{n-2}{r-2} \\
&= \frac{r(r-1)}{n-1} \binom{n-1}{r-1} \\
&< \frac{r^2}{n} \cdot \frac{n^{r-1}}{(r-1)!} \\
&< \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}.
\end{aligned}$$

This finishes the proof.

6 Proof of Theorem 7

Lemma 15. *Let $S = S(\ell_1, \dots, \ell_k)$ be a spider on n vertices and let v be a leaf of S . Suppose that $r \leq \alpha(S)$. Then*

$$s_r(v) \geq \binom{n-r-1}{r-1} + \binom{n-k-r-2}{r-2}.$$

Proof. Let $S = S(\ell_1, \dots, \ell_k)$, in spider order. We may assume that $v = v_k$ and then use Theorem 6 for the other leaves. For $S(1, 1, \dots, 1)$ we have $s_r(v) = \binom{n-2}{r-1}$ and $k = n-1$, so that $\binom{n-k-r-2}{r-2} = 0$ and $\binom{n-2}{r-1} \geq \binom{n-r-1}{r-1}$. Thus we may assume that $\ell_k \geq 2$, implying that v and w are not adjacent.

We first count the number of independent r -sets containing v that do not contain the split vertex w . The number of such sets equals

$$|\mathcal{I}_v^r(S-w)| = |\mathcal{I}_v^r(\cup_{i=1}^k P_{\ell_i})|,$$

where P_{ℓ_i} denotes the path on ℓ_i vertices.

Next we add edges to the disjoint union of paths, to reduce the number of independent r -sets that contain v but not w . For each $1 \leq i \leq k$, let u_i be the

neighbor of w on the wv_i -path in S . Now, for each $1 \leq i < k$, add the edge $u_i v_{i+1}$. Finally, remove v and its unique neighbor, resulting in the graph P_m , for $m = n - 3$. This results in the inequality

$$|\mathcal{I}_v^r(\cup_{i=1}^k P_{\ell_i})| \geq |\mathcal{I}^{r-1}(P_m)|.$$

We relabel the vertices of P_m as x_1, \dots, x_m , in order. Observe that $\{x_{a_1}, x_{a_1+a_2}, \dots, x_{a_1+\dots+a_{r-1}}\}$ is independent in P_m if and only if

$$\sum_{i=1}^r a_i = m, \quad a_1 \geq 1, \quad a_i \geq 2 \text{ for } 1 < i < r, \text{ and } a_r = m - a_{r-1} \geq 0. \quad (3)$$

Set $b_1 = a_1 - 1$, $b_i = a_i - 2$ for $1 < i < m$, and $b_r = a_r$. Then system (3) can be rewritten as

$$\sum_{i=1}^r b_i = m - 2r + 3 = n - 2r, \text{ with } b_i \geq 0, \text{ for all } 1 \leq i \leq r. \quad (4)$$

It is well known that the number of integer solutions to system (4) equals

$$\binom{n - 2r + r - 1}{r - 1} = \binom{n - r - 1}{r - 1}.$$

Second, we count the number of independent r -sets containing v that also contain the split vertex w . The number of such sets equals

$$|\mathcal{I}_v^{r-1}(S - N[w])| = |\mathcal{I}_v^{r-1}(\cup_{i=1}^k P_{\ell_i-1})|.$$

As above, we add edges to the disjoint union of paths, to reduce the number of independent r -sets that contain v and w . For each $1 \leq i \leq k$, let u'_i be the neighbor of u_i other than w on the wv_i -path in S . Now, for each $1 \leq i < k$, add the edge $u'_i v_{i+1}$. Finally, remove v and its unique neighbor, resulting in the

graph $P_{m'}$, for $m' = n - 3 - k$. This results in the inequality

$$|\mathcal{I}_v^{r-1}(\cup_{i=1}^k P_{\ell_i-1})| \geq |\mathcal{I}^{r-2}(P_{m'})|.$$

Counting via the same method as above, we obtain

$$|\mathcal{I}^{r-2}(P_{m'})| = \binom{n-k-r-2}{r-2}$$

such sets, which completes the proof. \square

6.1 Proof of Theorem 7

It is easy to check that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$ implies that $r^2 \leq (n-r) \ln 2$. We use this in the calculations below.

Using Lemma 15 with the Hilton-Milner Theorem 2, as in the proof of Theorem 5, the result will follow from proving the inequality

$$\binom{n-1}{r-1} \leq 2 \binom{n-r-1}{r-1}. \quad (5)$$

To accomplish this, we denote $m^{\underline{t}} = m!/(m-t)!$ and calculate the ratio

$$\begin{aligned}
\binom{n-1}{r-1} / \binom{n-r-1}{r-1} &= \frac{(n-1)^{\underline{r-1}}}{(n-r-1)^{\underline{r-1}}} \\
&\leq \frac{(n-r+1)^{r-1}}{(n-2r+1)^{r-1}} \\
&= \left(\frac{n-2r+1}{n-r+1} \right)^{-(r-1)} \\
&= \left(1 - \frac{r}{n-r+1} \right)^{-(r-1)} \\
&\leq e^{r(r-1)/(n-r+1)} \\
&< e^{r^2/(n-r)} \\
&\leq e^{\ln 2} \\
&= 2,
\end{aligned} \tag{6}$$

which finishes the proof.

7 Proof of Theorem 8

Lemma 16. *Let T be a tree on n vertices with exactly $s > 1$ split vertices, and let v be a leaf of T . Suppose that $r \leq \alpha(T)$. Then*

$$s_r(v) \geq \binom{n-r-s}{r-1} + 1.$$

Proof. Let W denote the set of split vertices of T . We need only count the number of independent r -sets containing v that do not contain any split vertex.

The number of such sets equals

$$\begin{aligned}
|\mathcal{I}_v^r(S - W)| &> |\mathcal{I}_v^r(P_{n-s})| \\
&= |\mathcal{I}^{r-1}(P_{n-s-1})| \\
&= \binom{n-r-s}{r-1},
\end{aligned}$$

as in the proof of Lemma 15.

The strict inequality comes from the existence of at least one independent set of $S - W$ that is not independent in P_{n-s} because of the joining of the many paths that create P_{n-s} . For example, let P' and P'' be two paths in $S - W$ that are consecutive in P_{n-s} , with endpoints $u' \in P'$ and $u'' \in P''$ such that u' is adjacent to u'' in P_{n-s} . Let $A \in \mathcal{I}^{r-1}(P_{n-s})$, define a' to be the vertex of A that is closest to u' , a'' to be the vertex of $A - \{a'\}$ that is closest to u'' , and $A' = (A - \{a', a''\}) \cup \{u', u''\}$. Then $A' \in \mathcal{I}^{r-1}(S - W) - \mathcal{I}^{r-1}(P_{n-s})$. \square

7.1 Proof of Theorem 8

Suppose that $s < r/2$ and $r \leq \sqrt{n \ln c} - (\ln c)/2$, where $c = 2 - 2s/r$. It is easy to check that this implies that $n > \frac{1}{2}(r+2)^2 + s$, which we use in the calculations below.

As in the proof of Theorem 7, we use Lemma 16 and the Hilton-Milner Theorem 2, which reduces the proof to certifying the inequality

$$\binom{n-1}{r-1} \leq \binom{n-r-1}{r-1} + \binom{n-r-s}{r-1}. \quad (7)$$

To accomplish this, we note that $n > \frac{1}{2}(r+2)^2 + s$ implies that

$$\frac{r-1}{n-r-s+1} \leq \frac{2(r-1)^2}{r^3}. \quad (8)$$

Next, we derive the following estimates, using Inequality 8 to access Corollary 10 with $y = (r - 1)/(n - r - s + 1)$ and $k = r - 1$.

$$\begin{aligned}
\frac{\binom{n-r-1}{r-1} + \binom{n-r-s}{r-1}}{\binom{n-r-1}{r-1}} &= 1 + \frac{(n-2r-2)^{\underline{s-1}}}{(n-r-1)^{\underline{s-1}}} \\
&> 1 + \left(\frac{n-2r-2-s+2}{n-r-1-s+2} \right)^{s-1} \\
&> 1 + \left(\frac{n-2r-s}{n-r-s+1} \right)^s \\
&= 1 + \left(1 - \frac{r-1}{n-r-s+1} \right)^s \\
&> 1 + e^{-\left(\frac{r}{r-1}\right)\left(\frac{r-1}{n-r-s+1}\right)s} \\
&> 1 + e^{-\left(\frac{r}{r-1}\right)\left(\frac{2(r-1)^2}{r^3}\right)s} \\
&= 1 + e^{-\left(\frac{2(r-1)}{r^2}\right)s} \\
&> 1 + e^{-2s/r} \\
&> 2 - 2s/r.
\end{aligned}$$

The assumption that $s < r/2$ makes the final result greater than 1. Finally, we follow Inequality (6), since $r \leq \sqrt{n \ln c} - (\ln c)/2$ implies that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$, and calculate the ratio

$$\begin{aligned}
\binom{n-1}{r-1} / \binom{n-r-1}{r-1} &< e^{r^2/(n-r)} \\
&\leq e^{\ln(2-2s/r)} \\
&= 2 - 2s/r,
\end{aligned}$$

which finishes the proof.

8 Questions and Remarks

It is clear that improving the orders of magnitude in the upper bound on r in our results will require techniques other than comparison to the Hilton-Milner bounds. To that end, the specificity of spider structure and the knowledge of the location of their biggest stars begs for a proof that they are r -EKR for $r \leq \mu/2$ (or possibly $r \leq \alpha$).

Along these lines, consider the family \mathcal{T} of all trees having no vertex of degree 2. The authors of [17] conjecture that every tree in \mathcal{T} is HK. Naturally, we believe that such trees are EKR as well. As a first step in this direction, for $i \in \{1, 2, 3\}$, let $T_i(h)$ be a complete binary tree of depth h (i.e. having $2^{h+1} - 1$ vertices), with root vertex v_i . Note that v_i is the unique degree-2 vertex in $T_i(h)$. Now define the tree $T(h)$ by $V(T(h)) = \{w\} \cup_{i=1}^3 V(T_i(h))$, with w adjacent to each v_i . Then $T(h) \in \mathcal{T}$.

Problem 17. *Show that $T(h)$ is r -EKR for all $r \leq \mu(T(h))/2$.*

Finally, we observe that the non-uniform case — considering $\mathcal{I}(G)$ instead of $\mathcal{I}^r(G)$ — has yet to be studied specifically for graphs. Of course, this is a special case of Chvátal's conjecture (see [7]) that every subset-closed family \mathcal{F} of sets is EKR — that is, if \mathcal{H} is an intersecting subfamily of \mathcal{F} , then there is some element x such that $|\mathcal{H}| \leq |\mathcal{F}_x|$. Beginning simply, we offer the following problem.

Problem 18. *Show that every path is EKR.*

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