



# Erdős–Ko–Rado theorems on the weak Bruhat lattice

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## ABSTRACT

Let  $\mathcal{L} = (X, \leq)$  be a lattice. For  $\mathcal{P} \subseteq X$  we say that  $\mathcal{P}$  is  $t$ -intersecting if  $\text{rank}(x \wedge y) \geq t$  for all  $x, y \in \mathcal{P}$ . The seminal theorem of Erdős, Ko and Rado describes the maximum intersecting  $\mathcal{P}$  in the lattice of subsets of a finite set with the additional condition that  $\mathcal{P}$  is contained within a level of the lattice. The Erdős–Ko–Rado theorem has been extensively studied and generalized to other objects and lattices.

In this paper, we focus on intersecting families of permutations as defined with respect to the weak Bruhat lattice. In this setting, we prove analogs of certain extremal results on intersecting set systems. In particular we give a characterization of the maximum intersecting families of permutations in the Bruhat lattice. We also characterize the maximum intersecting families of permutations within the  $r$ th level of the Bruhat lattice of permutations of size  $n$ , provided that  $n$  is large relative to  $r$ .

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## 1. Introduction

Let  $\mathcal{L} = (X, \leq)$  be a lattice. For  $\mathcal{P} \subseteq X$  we say that  $\mathcal{P}$  is  $t$ -intersecting if  $\text{rank}(x \wedge y) \geq t$  for all  $x, y \in \mathcal{P}$ . If  $\mathcal{L}$  is the subset lattice, two subsets  $A, B$  are  $t$ -intersecting exactly when  $|A \cap B| \geq t$  (we will refer to 1-intersecting sets simply as intersecting sets). The problem of finding maximum collections of intersecting sets has a long history, see [6] and the references within. Perhaps the most famous result is the Erdős–Ko–Rado (EKR) Theorem [3]. This theorem gives the size of the largest collection of sets at level  $r$  in the subset lattice such that any two of the sets intersect.

**Theorem 1.1.** *Let  $r$  and  $n$  be integers with  $n \geq 2r$ . If  $\mathcal{A}$  is an intersecting collection of  $r$ -subsets of the set  $\{1, \dots, n\}$  then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

*Moreover, if  $n > 2r$  equality holds if and only if  $\mathcal{A}$  consists of all the  $r$ -subsets that contain a common element.*

Another well-known result is Katona's theorem [9], which gives the size of the largest collection of  $t$ -intersecting sets (with no restriction on the size of the sets) in the subset lattice on  $n$  points. If  $n + t = 2v$ , then this maximum size is achieved by the collection of all subsets with size larger than  $v$ . If  $n + t = 2v - 1$ , an example of a collection of maximum size is the collection that contains all subsets with size at least  $v$ , along with all subsets of size  $v - 1$  that do not contain a fixed

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element. In the special case that  $t = 1$  these sets have size  $2^{n-1}$ . This implies that the collection of all subsets that contain a fixed element is a 1-intersecting set of maximum size. In this case there are also many other (non-isomorphic) collections of maximum size.

One of the reasons that the EKR Theorem is so famous is that versions of it hold for many other objects; for example there are versions for integer sequences, vector spaces over a finite field, perfect matchings, independent sets in graphs, and permutations (these are just a few of the examples; again see [6], and the references within, for more details).

In this paper, we give versions of Katona's Theorem and the Erdős–Ko–Rado Theorem for the weak Bruhat lattice of permutations. We start with some general results for all lattices. Next we define and give properties of the weak Bruhat lattice. Section 4 has information about properties of intersecting sets in the Bruhat lattice and a characterization for the maximum intersecting collections of intersecting permutations in the Bruhat lattice. Section 5 gives a version of the EKR theorem for permutations in level  $r$  of the Bruhat lattice, provided that  $n$  is large relative to  $r$ . In the sequel, collections of sets will be called *systems*, and collections of permutations will be called *families*.

## 2. General lattices

For a subset  $\mathcal{F}$  of elements in a lattice  $\mathcal{L} = (X, \preceq)$ , define the *upset* of  $\mathcal{F}$  to be

$$\text{up}(\mathcal{F}) = \{z \in X \mid x \preceq z, \text{ some } x \in \mathcal{F}\}.$$

Similarly, the *downset* of  $\mathcal{F}$  is defined to be

$$\text{down}(\mathcal{F}) = \{z \in X \mid z \preceq x, \text{ some } x \in \mathcal{F}\}.$$

For any  $\mathcal{F}$ , if  $\text{up}(\mathcal{F}) = \mathcal{F}$ , then we say that  $\mathcal{F}$  is an *upset*. Similarly,  $\mathcal{F}$  is a *downset* if  $\text{down}(\mathcal{F}) = \mathcal{F}$ . Upsets and downsets are also called filters and ideals, respectively.

For a single element  $p \in \mathcal{L}$ , the family  $\text{up}(\{p\})$  (which we abbreviate  $\text{up}(p)$ ) is called a *star* with center  $p$ . If the rank of  $p$  is  $t$ , then  $\text{up}(p)$  is  $t$ -intersecting; we call  $\text{up}(p)$  a *canonical  $t$ -intersecting family*. More generally, for some  $\mathcal{H} \subseteq \mathcal{L}$  we define  $\text{up}_{\mathcal{H}}(\mathcal{F}) = \text{up}(\mathcal{F}) \cap \mathcal{H}$ . In this paper, we only consider the case where  $\mathcal{H}$  is the set of all elements in  $\mathcal{L}$  with rank  $k$ ; this set is denoted by  $\mathcal{L}_k$ . If  $\text{rank}(p) = t$ , then  $\text{up}_{\mathcal{L}_k}(p)$  is called a  *$t$ -star at level  $k$* .

Define  $f_t(\mathcal{L})$  to be the maximum size of a  $t$ -intersecting family from the lattice  $\mathcal{L}$ . We say that  $\mathcal{L}$  has the  *$t$ -EKR property* (or is  *$t$ -EKR*) if  $f_t(\mathcal{L})$  is equal to the size of a  $t$ -star. Equivalently,  $\mathcal{L}$  has the  $t$ -EKR property if  $f_t(\mathcal{L}) = \max_{p \in \mathcal{L}_t} |\text{up}(p)|$ . A lattice that has the 1-EKR property will simply be said to have the EKR property.

Similarly,  $f_t(\mathcal{L}_k)$  is defined to be the maximum size of a  $t$ -intersecting family in level  $k$  of the lattice. The level  $\mathcal{L}_k$  has the  *$t$ -EKR property* (or is  *$t$ -EKR*) if the size of the largest  $t$ -intersecting family of  $\mathcal{L}_k$  can be realized by a  $t$ -star at level  $k$ . This is equivalent to saying that  $f_t(\mathcal{L}_k) = \max_{p \in \mathcal{L}_t} |\text{up}_{\mathcal{L}_k}(p)|$ . Again, we suppress the use of  $t$  when  $t = 1$ .

The Erdős–Ko–Rado Theorem is equivalent to saying that in the subset lattice all levels below  $n/2$  have the 1-EKR property. Wilson [13] proved the subset lattice for  $\{1, \dots, n\}$  has the  $t$ -EKR property at level- $k$  provided that  $n > (t+1)(k-t+1)$  (Frankl [4] had previously proved this with the restriction that  $t > 14$ ). Ahlswede and Khachatrian's complete intersection theorem [1] describes all the largest intersecting sets for all values of  $n, k$  and  $t$ .

We say that a lattice  $\mathcal{L}$  is *uniquely complemented* if, for all  $x \in X$ , there exists a unique  $y \in X$  such that  $\text{rank}(x \wedge y) = 0$  and  $\text{rank}(x \vee y) = \text{rank}(\mathcal{L})$ . A lattice is *distributive* if for all  $x, y, z \in \mathcal{L}$  the following holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

**Theorem 2.1.** *Any uniquely complemented distributive lattice has the EKR property.*

**Proof.** Assume that  $\mathcal{L}$  is a uniquely complemented lattice and denote the least element in  $\mathcal{L}$  by 0. Denoting the complement of an element  $x \in \mathcal{L}$  by  $\bar{x}$ , at most one of the pair  $\{x, \bar{x}\}$  can belong to an intersecting family. Thus an intersecting set can be no larger than  $|\mathcal{L}|/2$ .

Let  $p$  be any element of the lattice with rank 1, and assume that for some  $x \in \mathcal{L}$ , both  $x$  and  $\bar{x}$  are not comparable to  $p$ . Then

$$0 = (p \wedge x) \vee (p \wedge \bar{x}) = p \wedge (x \vee \bar{x}) = p$$

which is a contradiction. Thus for any element  $x \in \mathcal{L}$ , either  $x$  or  $\bar{x}$  is comparable to  $p$ . Since  $p$  is rank 1 this means that either  $x$  or  $\bar{x}$  is above  $p$  in the lattice, and so  $|\text{up}(p)| = |\mathcal{L}|/2$ .  $\square$

Since the subset lattice is uniquely complemented, this theorem implies that the subset lattice has the EKR property. Katona's theorem shows that the subset lattice does not have the  $t$ -EKR property for  $t > 1$ .

It seems to be harder to find general results about maximum intersecting families within a level of a lattice. Many of the standard proofs for the EKR property use regularity conditions on the set. Suda [11] considered the  $t$ -EKR property for levels in semi-lattices with regularity conditions. The Bruhat lattice is different since it does not satisfy these regularity conditions. Specifically, the upsets of different elements at the same level may have different sizes, which makes the Bruhat lattice a particularly interesting lattice to consider. More details are given in the next section.

### 3. Properties of the weak Bruhat lattice

The (right) weak Bruhat ordering on the symmetric group produces a lattice  $\mathcal{B}(n) = (\text{Sym}(n), \leq)$ . For any permutation  $p \in \text{Sym}(n)$  we write  $p = p_1 p_2 \cdots p_n$  to mean  $p(i) = p_i$  (this is the second line in the two line notation). The covering relation  $\prec$  that generates the relation  $\leq$  is defined by  $p_1 p_2 \cdots p_n \prec q_1 q_2 \cdots q_n$  whenever there is some  $i \in \{1, \dots, n-1\}$ , such that  $p_i < p_{i+1}$ ,  $q_i = p_{i+1}$ ,  $q_{i+1} = p_i$ , and  $q_k = p_k$  otherwise. This means that  $p \prec q$  if  $q$  is obtained by reversing two consecutive and increasing elements from  $p$ . See Fig. 1 for  $\mathcal{B}(4)$ .

The transpositions  $(i, i+1)$  for  $i \in \{1, \dots, n-1\}$  are called the *generators*, and we will denote the generator  $(i, i+1)$  by  $g_i$ . We also define  $\mathbf{G}_n = \{g_i \mid 1 \leq i < n\}$  to be the *generating set* for  $\mathcal{B}(n)$ . From another point of view,  $\mathcal{B}(n)$  is the Cayley graph generated by  $\mathbf{G}_n$ , turned into a lattice with ranks given by distance from the identity  $\text{id}$ . The set of all permutations at rank  $\ell$  in  $\mathcal{B}(n)$  is denoted by  $\mathcal{B}_\ell(n)$ .

For a permutation  $p \in \text{Sym}(n)$  we define its set of *inverse descents* by

$$\text{ID}(p) = \{g_i \mid i+1 \text{ precedes } i \text{ in } p\},$$

and for a set of permutations  $\mathcal{P}$ , we define  $\text{ID}(\mathcal{P}) = \{\text{ID}(p) \mid p \in \mathcal{P}\}$ . The set of inverse descents play an important role in the Bruhat lattice, since for any  $p \in \text{Sym}(n)$ , we have  $g_i \in \text{ID}(p)$  if and only if  $g_i \leq p$  in the weak order.

We denote the *inversion set* for a permutation  $p \in \text{Sym}(n)$  by

$$\text{Inv}(p) = \{(p_j, p_i) \mid 1 \leq i < j \leq n, p_j < p_i\}$$

with this definition,  $\text{rank}(p) = |\text{Inv}(p)|$ . Our definition of inverse descents is nonstandard; usually the inverse descents are integers, but we are using the simple transpositions indexed by those integers.

The inversion sets of different permutations are always distinct, while their inverse descent sets are sometimes not. For instance:

$$\begin{aligned} \text{ID}(3214) &= \{g_1, g_2\}, & \text{Inv}(3214) &= \{(1, 2), (1, 3), (2, 3)\}, \\ \text{ID}(3241) &= \{g_1, g_2\}, & \text{Inv}(3241) &= \{(1, 2), (1, 3), (1, 4), (2, 3)\}. \end{aligned} \quad (1)$$

For a set  $A \subseteq \mathbf{G}_n$ , define the *multiplicity of  $A$*  by  $|\{p \in \mathcal{B}(n) \mid \text{ID}(p) = A\}|$  and the *multiplicity of  $A$  at level  $\ell$*  by  $|\{p \in \mathcal{B}_\ell(n) \mid \text{ID}(p) = A\}|$ . For example, consider the set  $\{g_1, g_4\}$ . Its multiplicity in  $\mathcal{B}_2(6)$  is 1 since 213546 is the only permutation with rank 2 and inverse descents set exactly  $\{g_1, g_4\}$ . Its multiplicity in  $\mathcal{B}_3(6)$  is 3 since each of  $(1, 3)$ ,  $(3, 5)$  and  $(4, 6)$  may be reversed to increase the rank without introducing a new inverse descent (the permutations in  $\mathcal{B}_3(6)$  with inverse descents set  $\{g_1, g_4\}$  are 231546, 215346, and 213564).

**Lemma 3.1.** *The multiplicity of any  $k$ -set of generators at level  $\ell$  of  $\mathcal{B}(n)$  is at most  $\binom{\ell+k-1}{k-1} e^{k\pi\sqrt{\frac{2\ell}{3}}}$ .*

**Proof.** Let  $S = \{g_{i_1}, g_{i_2}, \dots, g_{i_k}\}$  be a  $k$ -set of generators and  $B_{\ell,S}$  be the set of permutations at level  $\ell$  of  $\mathcal{B}(n)$  whose inverse descent set is  $S$ . Set  $i_0 = 0$ ,  $i_{k+1} = n$ , and assume  $i_1 < i_2 < \dots < i_k$ . Any permutation in  $B_{\ell,S}$  will preserve the natural order of each of the sets

$$S_0 = \{1, 2, \dots, i_1\}, S_1 = \{i_1 + 1, \dots, i_2\}, \dots, S_k = \{i_k + 1, \dots, n\}$$

and so we can identify every permutation in  $B_{\ell,S}$  with a permutation of the multiset

$$M_S = \{1^{i_1}, 2^{i_2-i_1}, \dots, k^{i_k-i_{k-1}}, (k+1)^{n-i_k}\}.$$

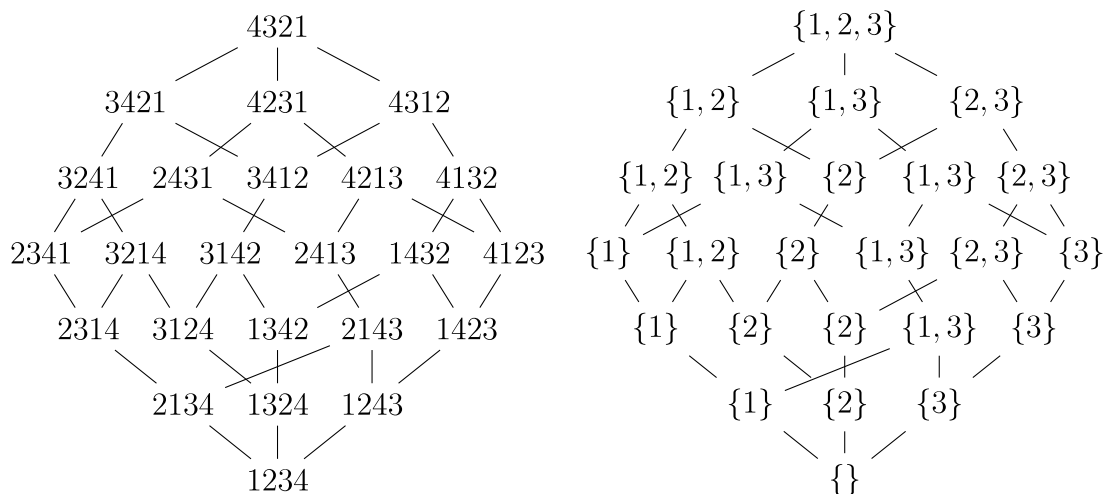
In general, there will be more multiset permutations than elements of  $B_{\ell,S}$ . An inversion in the multiset permutation represents an inversion in the corresponding permutation in  $B_{\ell,S}$ . The  $q$ -multinomial coefficient

$$\begin{bmatrix} n \\ i_1 & i_2 - i_1 & \dots & i_k - i_{k-1} & n - i_k \end{bmatrix} = \begin{bmatrix} n \\ i_1 \end{bmatrix} \begin{bmatrix} n - i_1 \\ i_2 - i_1 \end{bmatrix} \dots \begin{bmatrix} n - i_k \\ n - i_k \end{bmatrix} \quad (2)$$

is the generating function by inversions for permutations of  $M_S$  (see [10]). Therefore,  $|B_{\ell,S}|$  is less than the coefficient of  $q^\ell$  in (2). The coefficient of  $q^\ell$  in (2) is  $\sum_{a_1+a_2+\dots+a_k=\ell} \prod_{j=1}^k A_{a_j}$ , where  $A_{a_j}$  is the coefficient of  $q^{a_j}$  in  $\begin{bmatrix} n - i_{j-1} \\ i_j - i_{j-1} \end{bmatrix}$ . In other words,

$$|B_{\ell,S}| \leq \sum_{a_1+a_2+\dots+a_k=\ell} \prod_{j=1}^k A_{a_j}. \quad (3)$$

Since  $\begin{bmatrix} n - i_{j-1} \\ i_j - i_{j-1} \end{bmatrix}$  is the generating function by size for partitions whose Young diagram fits inside an  $(i_j - i_{j-1}) \times (n - i_j)$  rectangle (see [10]),  $A_{a_j}$  is the number of partitions of  $a_j$  with at most  $(i_j - i_{j-1})$  parts whose first part is at most  $(n - i_j)$ . We



**Fig. 1.** Two representations of the Bruhat lattice  $\mathcal{B}(4)$ . On the left, vertices are labeled by  $p = p_1p_2p_3p_4$ , where  $p(i) = p_i$ . On the right, each vertex  $p$  is labeled by its set of inverse descents  $ID(p)$ .

can crudely bound  $A_{a_j}$  by  $p(\ell)$ , the number of partitions of  $\ell$ , and replace the bound in (3) with

$$|B_{\ell,S}| \leq \sum_{a_1+a_2+\dots+a_k=\ell} (p(\ell))^k. \quad (4)$$

The number of weak compositions of  $\ell$  with  $k$  parts is  $\binom{\ell+k-1}{k-1}$  [10] and  $p(\ell) \leq e^{\pi\sqrt{\frac{2\ell}{3}}}$  [2, Chapter 5], so we can replace (4) by

$$|B_{\ell,S}| \leq \binom{\ell+k-1}{k-1} e^{k\pi\sqrt{\frac{2\ell}{3}}},$$

which is our result.  $\square$

The bound above is often much larger than the actual multiplicity of a set. The next result gives a much stronger bound for the number of permutations in a level with the property that every inversion is a generator. A set of inverse descents is called a *separated set* if it does not contain generators  $g_i$  and  $g_{i+1}$  for any  $i$ .

**Proposition 3.2.** *If a permutation  $p$  in level  $\ell$  of  $\mathcal{B}(n)$  has an inverse descents set of size  $\ell$ , then the inverse descents set must be a separated set.*

**Proof.** Assume that  $A$  is a set of size  $\ell$  that is not a separated set. Then  $i, i+1 \in A$  for some  $i$ . Any permutation that has both  $i$  and  $i+1$  in its inverse descents set must also have  $(i, i+2)$  in its inversion set. Thus the inversion set for any  $p$  with  $ID(p) = A$  must have at least  $\ell+1$  elements, and  $p$  is above rank  $\ell$ . So the multiplicity of  $A$  in  $\mathcal{B}_\ell(n)$  is 0.  $\square$

The next claim follows directly from the definitions of the relation in the Bruhat order, inverse descents sets and inversion sets.

**Claim 3.3.** *Let  $p, q \in \mathcal{B}(n)$ , then the following hold.*

1. If  $p \leq q$ , then  $ID(p) \subseteq ID(q)$ .
2.  $p \leq q$  if and only if  $\text{Inv}(p) \subseteq \text{Inv}(q)$ .
3.  $p \wedge q \neq \text{id}$  if and only if  $ID(p) \cap ID(q) \neq \emptyset$ .

The converse of Statement (1) in the above claim is not true, since it is possible to have incomparable permutations  $p$  and  $q$  with  $ID(p) = ID(q)$ . For example, 2431 and 4213 are incomparable, but have the same set of inverse descents.

**Corollary 3.4.** *Let  $p, q \in \mathcal{B}(n)$ , then  $ID(p \wedge q) = ID(p) \cap ID(q)$ .*

**Proof.** Statement (1) implies that  $ID(p \wedge q) \subseteq ID(p) \cap ID(q)$ . Conversely, if  $g_i \in ID(p) \cap ID(q)$ , then  $(i, i+1) \leq p \wedge q$ , and  $g_i \in ID(p \wedge q)$ . Thus  $ID(p \wedge q) = ID(p) \cap ID(q)$  and the result follows.  $\square$

For a subset  $A \subseteq \mathbf{G}_n$ , let

$$\pi(A) = \bigvee_{g_i \in A} g_i.$$

This permutation is called the *minimal* element for  $A$ . Since  $\mathcal{B}(n)$  is a lattice,  $p$  is well-defined and unique.

**Lemma 3.5.** For any subset  $A \subseteq \mathbf{G}_n$  in the Bruhat ordering  $p = \pi(A)$  is the unique minimum permutation such that  $\text{ID}(p) = A$ . Further, for any  $q$  with  $\text{ID}(q) = A$ , both  $\text{Inv}(p) \subseteq \text{Inv}(q)$  and  $p \leq q$  hold.

**Proof.** Since  $\pi(A) = \bigvee_{g_i \in A} g_i$ , it follows that  $\text{ID}(p) = A$ . Let  $q$  be any other permutation with  $\text{ID}(q) = A$ . Then for any  $g_1 \in A$  it follows that  $g_1 \leq q$ , and so  $p \leq q$  and, by Claim 3.3 (2),  $\text{Inv}(p) \subseteq \text{Inv}(q)$ .  $\square$

Lemma 3.5 implies that

$$\pi(A) = \min\{p \in \mathcal{B}(n) \mid \text{ID}(p) = A\}.$$

Lemma 3.5 also shows that the multiplicity for any inverse descents set is at least 1. However, it is possible to have a set that is not an inversion set for any permutation; for example there is no permutation with the inversion set  $\{(1, 2), (2, 3)\}$  (if 2 is before 1 and 3 is before 2, then 3 is also before 1).

The goal of the next section is to give properties of intersecting families in the Bruhat lattice in order to characterize the extremal intersecting sets.

#### 4. Properties of intersecting sets in $\mathcal{B}(n)$

The Bruhat ordering on permutations is not a uniquely complemented lattice, but the permutations can be paired so that the pairs have no common elements in either their inverse descents or their inversion sets. The permutation  $p = p_1 p_2 \cdots p_n$  is paired with the permutation  $\bar{p} = p_n \cdots p_2 p_1$ . These pairs have the property that

$$\text{Inv}(\bar{p}) = \{(i, j) \mid 1 \leq i < j \leq n\} \setminus \text{Inv}(p),$$

and also that  $\text{ID}(\bar{p}) = \mathbf{G}_n \setminus \text{ID}(p)$ .

With this pairing, a version of Theorem 2.1 holds for the Bruhat lattice. We alter notation slightly to write  $f_t(\mathcal{B}(n))$  as  $f_t(n)$  and  $f_t(\mathcal{B}_k(n))$  as  $f_{t,k}(n)$ .

**Theorem 4.1.** Let  $n$  be an integer with  $n \geq 2$ , then  $f_1(n) = n!/2$ .

**Proof.** Let  $x$  be a permutation at level 1 in  $\mathcal{B}(n)$ ; assume that  $\text{ID}(x) = \{g_i\}$ . The upset of  $x$  is the set of all permutations that have  $i$  in its inverse descents set. For each  $p \in \mathcal{B}(n)$ , exactly one of  $p$  and  $\bar{p}$  will have  $i$  in its inverse descents set.  $\square$

**Corollary 4.2.** The Bruhat lattice has the 1-EKR property.

The next goal is to give a characterization of the maximum intersecting families in  $\mathcal{B}(n)$ . If  $\mathcal{P} \subseteq \mathcal{B}(n)$  is an intersecting family of permutations of maximum size, then  $|\mathcal{P}| = n!/2$  and so, for every permutation  $p$ , either  $p \in \mathcal{P}$  or  $\bar{p} \in \mathcal{P}$ .

An intersecting family  $\mathcal{P}$  is called *maximal* if every permutation that intersects all the permutations in  $\mathcal{P}$  is also in  $\mathcal{P}$ ; equivalently, there are no elements not in  $\mathcal{P}$  that intersect all the elements in  $\mathcal{P}$ .

For a family  $\mathcal{P} \subseteq \mathcal{B}(n)$ , define

$$\min(\mathcal{P}) = \{p \in \mathcal{P} \mid \text{if there exists } q \in \mathcal{P} \text{ with } q \leq p, \text{ then } q = p\}.$$

Observe that  $\mathcal{P}$  is an antichain if and only if  $\min(\mathcal{P}) = \mathcal{P}$ . Further, if  $\mathcal{P}$  is a maximal intersecting family, then  $\text{up}(\min(\mathcal{P})) = \mathcal{P}$ .

**Lemma 4.3.** Let  $\mathcal{P} \subseteq \mathcal{B}(n)$  be a maximal intersecting family. If  $p \in \min(\mathcal{P})$ , then  $p = \pi(\text{ID}(p))$ .

**Proof.** Let  $p \in \min(\mathcal{P})$  and set  $A = \text{ID}(p)$ . Then, for any permutation  $q \in \mathcal{P}$ , we have  $\text{ID}(q) \cap A \neq \emptyset$ . Thus  $\pi(A)$  intersects every  $q \in \mathcal{P}$ . Since  $\mathcal{P}$  is maximal,  $\pi(A) \in \mathcal{P}$ . By definition of  $\pi(A)$  we have  $\pi(A) \leq p$  and, since  $p$  is minimal in  $\mathcal{P}$ , we have  $p = \pi(A)$ .  $\square$

In developing our characterization of maximum intersecting families we will move between permutations and their sets of inverse descents, to do this we will use the following lemma.

**Lemma 4.4.** Let  $\mathcal{P} \subseteq \mathcal{B}(n)$ . Then, the following statements are equivalent.

1.  $\mathcal{P}$  is intersecting.
2.  $\text{up}(\mathcal{P})$  is intersecting.

3.  $\min(\mathcal{P})$  is intersecting.
4.  $\text{ID}(\mathcal{P})$  is an intersecting set system.
5. If  $\mathcal{G}$  is a system of subsets of  $\mathbf{G}_n$ , with  $\mathcal{P} = \{\pi(g) \mid g \in \mathcal{G}\}$ , then  $\mathcal{G}$  is intersecting.

**Proof.** Since  $\mathcal{B}(n)$  is a poset, for all  $p, p', q, q' \in \mathcal{P}$  with  $p \leq p'$  and  $q \leq q'$ , it follows that  $p \wedge q \leq p' \wedge q'$ . This implies that if a family  $\mathcal{P}$  is intersecting, then  $\text{up}(\mathcal{P})$  is also intersecting. This fact shows that Statement (1) implies Statement (2). Statement (2) implies Statement (3) since  $\min(\mathcal{P}) \subseteq \text{up}(\mathcal{P})$ . Statement (3) implies Statement (1) since  $\mathcal{P} \subseteq \text{up}(\min(\mathcal{P}))$ . Thus the first three statements are equivalent.

Finally, Statement (3) of Claim 3.3 implies that Statements are equivalent to (1).  $\square$

For any set system  $\mathcal{A}$  defined on an  $(n-1)$ -set, which we may take to be  $\mathbf{G}_n$ , there is a corresponding subfamily of  $\mathcal{B}(n)$  defined by

$$\mathcal{P}(\mathcal{A}) = \text{up}(\{\pi(A) \mid A \in \mathcal{A}\}).$$

This means that  $\mathcal{P}(\mathcal{A})$  is the family of all permutations  $p$  such that there exists an  $A \in \mathcal{A}$  such that  $A \subseteq \text{ID}(p)$ .

**Corollary 4.5.** If  $\mathcal{A}$  is an intersecting set system on  $\mathbf{G}_n$ , then  $\mathcal{P}(\mathcal{A})$  is an intersecting family in the Bruhat ordering.

**Proof.** This follows from Statement (3) of Claim 3.3 and Part (5) of Lemma 4.4.  $\square$

Further, it is possible to construct a set system from a family of permutations. If  $\mathcal{P}$  is a family of permutations, then  $\text{ID}(\min(\mathcal{P}))$  is a set system on  $\mathbf{G}_n$  called the *generating set* for  $\mathcal{P}$ . The next three lemmas give properties of this set system.

**Lemma 4.6.** Let  $\mathcal{P} \subseteq \mathcal{B}(n)$  be an intersecting family and define  $\mathcal{A} = \text{ID}(\min(\mathcal{P}))$ . Then  $\mathcal{A}$  is an intersecting set system. Further, if  $\mathcal{P}$  is maximal then  $p \in \mathcal{P}$  if and only if  $A \subseteq \text{ID}(p)$  for some  $A \in \mathcal{A}$ .

**Proof.** The first statement is clear from Statement (3) of Claim 3.3.

The implication in the second statement follows from the definition of  $\mathcal{A}$ . In particular, if  $p \in \mathcal{P}$  then  $A \subseteq \text{ID}(p)$  for some  $A \in \mathcal{A}$ . Conversely, for any permutation  $p$ , if  $A \subseteq \text{ID}(p)$  for some  $A \in \mathcal{A}$  then  $\pi(A) \in \mathcal{P}$  and  $\pi(A) \leq p$ . Since  $\mathcal{P}$  is maximal, this implies that  $p \in \mathcal{P}$ .  $\square$

**Corollary 4.7.** Let  $\mathcal{P} \subseteq \mathcal{B}(n)$ , then  $\mathcal{P} \subseteq \mathcal{P}(\text{ID}(\min(\mathcal{P})))$ . If  $\mathcal{P}$  is a maximal intersecting family, then  $\mathcal{P} = \mathcal{P}(\text{ID}(\min(\mathcal{P})))$ .

The next result shows that  $\text{ID}$  is a weak order preserving map on the poset, so it preserves the min operation.

**Corollary 4.8.** Let  $\mathcal{P} \subseteq \mathcal{B}(n)$  be a maximal intersecting family of permutations, then  $\text{ID}(\min(\mathcal{P})) = \min(\text{ID}(\mathcal{P}))$ .

**Proof.** From Part (1) of Claim 3.3, if  $p \leq q$ , then  $\text{ID}(p) \subseteq \text{ID}(q)$ . This implies that if  $\text{ID}(p) \in \min(\text{ID}(\mathcal{P}))$ , then  $p \in \min(\mathcal{P})$ . Thus  $\min(\text{ID}(\mathcal{P})) \subseteq \text{ID}(\min(\mathcal{P}))$ .

Conversely, if  $\text{ID}(p) \in \text{ID}(\min(\mathcal{P}))$ , then  $\text{ID}(p) = \text{ID}(p')$  for some  $p' \in \min(\mathcal{P})$ . Since  $\mathcal{P}$  is maximal, Lemma 4.3 implies that  $p = \pi(\text{ID}(p'))$ .

If there is a set  $\text{ID}(q) \in \text{ID}(\mathcal{P})$  with  $\text{ID}(q) \subset \text{ID}(p')$ , then  $\pi(\text{ID}(q)) \leq \pi(\text{ID}(p'))$ , which contradicts  $p' \in \min(\mathcal{P})$ . This means that  $\text{ID}(p')$  is a minimal set in  $\text{ID}(\mathcal{P})$  and thus  $\text{ID}(p) = \text{ID}(p') \in \min(\text{ID}(\mathcal{P}))$ .  $\square$

Corollary 4.8 implies that  $\text{ID}(\min(\mathcal{P}))$  is an antichain. We call  $\mathcal{A}$  a *maximal intersecting antichain* if  $\mathcal{A}$  is an intersecting antichain with the additional property that, for any set  $B \notin \mathcal{A}$ , there is a set  $A \in \mathcal{A}$  such that either  $A \cap B = \emptyset$ , or  $A \subseteq B$ . Equivalently, an intersecting antichain  $\mathcal{A}$  is a maximal intersecting antichain if, for any  $B$  with  $X \cap B \neq \emptyset$  for all  $X \in \mathcal{A}$ , there exists an  $A \in \mathcal{A}$  with  $A \subseteq B$ .

**Lemma 4.9.** Let  $\mathcal{P} \subseteq \mathcal{B}(n)$  be a maximal intersecting family and define  $\mathcal{A} = \text{ID}(\min(\mathcal{P}))$ . Then  $\mathcal{A}$  is a maximal intersecting antichain.

**Proof.** From Lemma 4.6,  $\mathcal{A}$  must be intersecting. Since  $\mathcal{A} = \text{ID}(\min(\mathcal{P}))$ , Corollary 4.8 implies  $\mathcal{A} = \min(\text{ID}(\mathcal{P}))$  and that  $\mathcal{A}$  is an antichain.

Suppose that  $\mathcal{A}$  is not a maximal intersecting antichain. Then there must be some  $B \notin \mathcal{A}$  such that  $\mathcal{A} \cup \{B\}$  is intersecting and there is no  $A \in \mathcal{A}$  with  $A \subseteq B$ . Let  $p = \pi(B)$ . Since no  $A \in \mathcal{A}$  has the property that  $A \subseteq \text{ID}(p)$ , we have that  $p \notin \mathcal{P}$ . Then  $\mathcal{P} \cup \{p\}$  is an intersecting family, contradicting the maximality of  $\mathcal{P}$ .  $\square$

At this point, to avoid unnecessary technicalities, we associate any set of integers with the set of generators having those integers as subscripts; e.g.  $\{1, 3, 4\} \equiv \{g_1, g_3, g_4\}$ .

There are many maximal intersecting set systems that are not maximal intersecting antichains. For example, consider the following system of  $r$ -subsets of  $M = \{1, \dots, m\}$  (with  $m > 2r$ ) defined by

$$\mathcal{H}_{m,r,k} = \{H \in \binom{M}{r} \mid |H \cap \{1, \dots, 2k-1\}| \geq k\}, \quad (5)$$

where  $1 \leq k \leq r \leq (m-1)/2$ . If  $k = 1$ , this is the canonical intersecting set. But the set system  $\mathcal{H}_{m,r,k}$  is a maximally intersecting antichain if and only if  $k = r$ . If  $r > k$ , consider the set  $\{r, \dots, m\}$ , which intersects each set in  $\mathcal{H}_{m,r,k}$ . Since  $r > k$ , the set  $\{r, \dots, m\}$  is neither contained in, nor contains any of the sets from  $\mathcal{H}_{m,r,k}$ . The following sets are examples of maximally intersecting antichains:

$$\begin{aligned}\mathcal{A} &= \{A \in \binom{M}{r} \mid |A \cap \{1, 2, \dots, 2r-1\}| \geq r\}, \\ \mathcal{B} &= \{12, 13, 14, 15, \dots, 1m, 234 \dots m\}, \text{ and} \\ \mathcal{C} &= \{123, 134, 145, 156, 126, 235, 245, 246, 346, 356\} \text{ on an 6-set.}\end{aligned}$$

The sets below are not maximally intersecting antichains:

$$\begin{aligned}\mathcal{D} &= \{A \in \binom{M}{r} \mid |A \cap \{1, 2, \dots, 2k-1\}| \geq k\}, \quad \text{if } r > k, \text{ and} \\ \mathcal{E} &= \{A \in \binom{M}{r} \mid 1 \in A\}, \quad \text{if } r > 1.\end{aligned}$$

The final theorem of this section is our characterization of the maximum intersecting families in the Bruhat lattice.

**Theorem 4.10.** *Let  $\mathcal{P} \subseteq \mathcal{B}(n)$  be an intersecting family and define  $\mathcal{A} = \text{ID}(\min(\mathcal{P}))$ . Then  $\mathcal{P}$  is maximum if and only if  $\mathcal{A}$  is a maximal intersecting antichain and  $\mathcal{P} = \mathcal{P}(\mathcal{A})$ .*

**Proof.** We have already seen in Lemma 4.9 that if  $\mathcal{P}$  is maximum, then  $\mathcal{A}$  is a maximal intersecting antichain. Further, by Corollary 4.7,  $\mathcal{P} = \mathcal{P}(\mathcal{A})$ .

Assume that  $\mathcal{A}$  is a maximal intersecting antichain and  $\mathcal{P} = \mathcal{P}(\mathcal{A})$ . We will show that, for every permutation  $p$ , either  $p \in \mathcal{P}$  or  $\bar{p} \in \mathcal{P}$ .

For a permutation  $p \in \mathcal{B}(n)$  let  $B = \text{ID}(p)$ . If  $B$  intersects every set in  $\mathcal{A}$  then, by maximality,  $A \subseteq B$  for some  $A \in \mathcal{A}$ . This implies that  $p \in \text{up}(\pi(A)) \subseteq \mathcal{P}$ . If  $B$  does not intersect every set in  $\mathcal{A}$ , then  $B \cap A = \emptyset$  for some  $A \in \mathcal{A}$ . This implies that  $A \subseteq \bar{B}$  so  $\bar{p} \in \text{up}(\pi(A)) \subseteq \mathcal{P}(\mathcal{A}) = \mathcal{P}$ .  $\square$

From Corollary 4.5, the family  $\mathcal{P}(\mathcal{H}_{m,r,k})$  (defined in Eq. (5)) is intersecting. There are many different set systems that are isomorphic to  $\mathcal{H}_{m,r,k}$ , formed by permuting the underlying set. Interestingly, it is possible that isomorphic maximal intersecting antichains give rise to non-isomorphic maximum intersecting families of permutations. For example, when  $(m, r, k) = (6, 2, 2)$ , then  $\mathcal{H}_{6,2,2} = \{(1, 2), (1, 3), (2, 3)\}$ , and  $\mathcal{H} = \{(1, 3), (1, 5), (3, 5)\}$  is isomorphic to  $\mathcal{H}_{6,2,2}$ . Both set systems can be used to construct an intersecting set of permutations in the Bruhat order:

$$\begin{aligned}\mathcal{P}(\mathcal{H}_{6,2,2}) &= \text{up}(\{321456, 214356, 143256\}), \\ \mathcal{P}(\mathcal{H}) &= \text{up}(\{214356, 213465, 124365\}).\end{aligned}$$

Both of these sets have size 360, but they are non-isomorphic families in  $\mathcal{B}(n)$ . This can be seen since the ranks of 321456 and 143256 are 3, whereas the other permutations are of rank 2. One can see that the key distinction here is that the sets in  $\mathcal{H}$  are separated sets.

The final corollary of this section follows from Theorem 4.10.

**Corollary 4.11.** *Let  $\mathcal{H}$  be a set system isomorphic to  $\mathcal{H}_{m,r,k}$ , with  $1 \leq k \leq r < (m-1)/2$ . If  $k = r$  then  $\mathcal{P}(\mathcal{H}) = m!/2$  while, if  $k < r$ , then  $\mathcal{P}(\mathcal{H}) < m!/2$ .*

## 5. Erdős–Ko–Rado theorem for levels in the Bruhat lattice

In this section we prove that an Erdős–Ko–Rado Theorem holds for the  $r^{\text{th}}$  level of  $\mathcal{B}(n)$ , provided that  $n$  is large relative to  $r$ . Recall that  $\mathcal{B}_r(n)$  denotes the set of all permutations of rank  $r$  in the Bruhat lattice  $\mathcal{B}(n)$ , and that  $\mathcal{B}_r(n)$  is EKR if the size of its largest intersecting subfamily is no larger than the largest 1-star at level  $r$ . It is clear that  $\mathcal{B}_2(n)$  is EKR. In this section we give proofs first that  $\mathcal{B}_3(n)$  is EKR for all  $n \geq 3$  and second, more generally, that  $\mathcal{B}_r(n)$  is EKR, provided that  $n$  is large relative to  $r$ .

**Lemma 5.1.** *Let  $\mathcal{H} = \mathcal{B}_3(n)$  and  $p \in \mathcal{B}_1(n)$ . Then  $|\text{up}_{\mathcal{H}}(p)| = \binom{n-1}{2}$ .*

**Proof.** Define the polynomial  $[m] = (1 + x + \dots + x^{m-1})$ ; then the generating function for the sizes of the ranks in  $\mathcal{B}(n)$  is

$$F(x) = [n]! = (1+x)(1+x+x^2) \cdots (1+x+x^2+\dots+x^{n-1}).$$



From [12], the generating function for the sizes of the ranks in  $\text{up}(p)$ , where  $\text{rank}(p) = 1$ , is given by the generating function

$$\frac{F(x)}{1+x} = \frac{[n]!}{1+x} = (1+x+x^2) \cdots (1+x+x^2+\cdots+x^{n-1}).$$

Hence  $|\text{up}_{\mathcal{H}}(p)|$  is the coefficient of  $x^2$  in  $F(x)/(1+x)$ . This coefficient is the same as that in  $(1+x+x^2)^{n-2}$ , which is  $\binom{n-2}{2} + (n-2) = \binom{n-1}{2}$ .  $\square$

We will use the following result of Holroyd, Spencer and Talbot [8] on EKR theorems for separated sets.

**Theorem 5.2 ([8]).** *The largest intersecting system of separated  $r$ -sets from an  $m$ -set has size  $\binom{m-r}{r-1}$  and consists of all the separated sets that contain some fixed point.*  $\square$

**Theorem 5.3.** *The set  $\mathcal{B}_3(n)$  is EKR for all  $n \geq 3$ .*

**Proof.** If  $n = 3$  the result is evident, so we let  $n \geq 4$ . Let  $\mathcal{P} \subseteq \mathcal{B}_3(n)$  be an intersecting family of maximum size. We are done if  $\mathcal{P}$  is a star, so we assume otherwise. Let  $\mathcal{P}_i = \{p \in \mathcal{P} \mid |\text{ID}(p)| = i\}$  where  $i \in \{1, 2, 3\}$ . Clearly,  $\mathcal{P} = \bigcup_{i=1}^3 \mathcal{P}_i$ .

If  $p \in \mathcal{P}_1$ , then, since  $\mathcal{P}$  is intersecting, every permutation in  $\mathcal{P}$  has the single element in  $\text{ID}(p)$  in its inverse descent set. This implies that  $\mathcal{P}$  is a star and we are done. So we may assume that  $\mathcal{P}_1 = \emptyset$ . Now we will consider  $\mathcal{P}_i$  for  $i > 1$ .

Suppose that  $p \in \mathcal{P}_3$ . Then we must have  $\text{ID}(p) = \text{Inv}(p)$ , since  $p$  is a rank 3 permutation with 3 inverse descents. Also, by Proposition 3.2,  $\text{ID}(p)$  must be a separated set. Furthermore,  $\text{Inv}(p)$  has multiplicity 1. Thus the set system  $\{\text{ID}(p) \mid p \in \mathcal{P}_3\}$  is an intersecting system of separated sets that has the same size as  $\mathcal{P}_3$ . By Theorem 5.2 (with  $m = n-1$ ) we have  $|\mathcal{P}_3| \leq \binom{n-4}{2}$ .

Next consider  $\mathcal{P}_2$  and define the set system  $\mathcal{A}_2 = \{\text{ID}(p) \mid p \in \mathcal{P}_2\}$ . Clearly,  $\mathcal{A}_2$  is an intersecting 2-set system. If  $|\mathcal{A}_2| \geq 4$ , then  $\mathcal{A}_2$  must contain four sets isomorphic to

$$\{12, 13, 14, 15\}.$$

Since the inverse descent sets for the permutations in  $\mathcal{P}_2$  and  $\mathcal{P}_3$  are sets of size no more than 3, and these inverse descents sets intersect all the sets in  $\mathcal{A}_2$ , it must be that all the permutations in both  $\mathcal{P}_2$  and  $\mathcal{P}_3$  have a common element in their inverse descent. Thus if  $|\mathcal{A}_2| \geq 4$ , then  $\mathcal{P}$  is a star. So we can assume that  $|\mathcal{A}_2| \leq 3$ . The multiplicity of any set in  $\mathcal{A}_2$  is no more than 4, and so  $|\mathcal{P}_2| \leq 12$ .

Since  $\mathcal{P}$  is not a star, we have

$$|\mathcal{P}| \leq 12 + \binom{n-4}{2} = \frac{1}{2}(n^2 - 9n + 44), \quad (6)$$

which is no more than  $\binom{n-1}{2} = \frac{1}{2}(n^2 - 3n + 2)$ , provided that  $n \geq 7$ .

For  $n \geq 7$ , Eq. (6) provides a bound which is at most the size of a star in the poset. If  $n = 4$  then, from Fig. 1, each set in  $\mathcal{A}_2$  has multiplicity 2. The bound in Eq. (6) becomes  $|\mathcal{P}| \leq 3$ , which is exactly the size of a star in the poset. Similarly, when  $n = 5$  the multiplicities of the sets in  $\mathcal{A}_2$  are no more than 2 and the bound becomes  $|\mathcal{P}| \leq 6$ . This again is the size of a star in the poset. Finally, when  $n = 6$ , the multiplicity is no more than 3 and the bound is  $|\mathcal{P}| \leq (3)(3) + 1 = 10$ . This also is the size of a star in the poset.  $\square$

Next we will use a counting method, similar to the one used in Theorem 5.3, to show that  $\mathcal{B}_r(n)$  is EKR, provided that  $n$  is sufficiently large. First, we will prove a lower bound on the size of a star in  $\mathcal{B}_r(n)$ , and then give two known results that can be used to show that if an intersecting set is sufficiently large then it contains many sets that intersect in exactly one element.

**Lemma 5.4.** *Let  $\mathcal{H} = \mathcal{B}_r(n)$  and  $p \in \mathcal{B}_1(n)$ . Then  $|\text{up}_{\mathcal{H}}(p)| > \binom{n-2}{r-1}$ .*

**Proof.** As we have seen, if  $p \in \mathcal{B}_1(n)$ , then the size of  $\text{up}_{\mathcal{H}}(p)$  is given by the coefficient of  $x^{r-1}$  in

$$\frac{[n]!}{x+1} = (1+x+x^2)(1+x+x^2+x^3) \cdots (1+x+x^2+\cdots+x^{n-1}).$$

There are  $\binom{n-2}{r-1}$  ways to select the term  $x$  from  $r-1$  different factors. So the coefficient of  $x^{r-1}$  is at least  $\binom{n-2}{r-1}$ .  $\square$

The next result is the Hilton–Milner theorem. This result gives a bound on the size of an intersecting set system in which the sets do not all contain a common element.

**Theorem 5.5 ([7]).** *If  $\mathcal{A}$  is an intersecting system of  $k$ -subsets of an  $m$ -set with  $\bigcap_{A \in \mathcal{A}} A = \emptyset$ , then*

$$|\mathcal{A}| \leq \binom{m-1}{k-1} - \binom{m-k-1}{k-1} + 1.$$

We will also use a version of Erdős' matching conjecture due to Frankl.



**Theorem 5.6** ([5]). Let  $\mathcal{A}$  be a system of  $k$ -subsets of an  $m$ -set. Assume that  $\mathcal{A}$  contains no  $r + 1$  pairwise disjoint sets and that  $m \geq (2r + 1)k - r$ . Then

$$|\mathcal{A}| \leq \binom{m}{k} - \binom{m-r}{k}.$$

The previous two results can be used to show that if an intersecting set system is sufficiently large then it contains many sets that intersect in exactly in one common element.

**Lemma 5.7.** Let  $\mathcal{F}$  be an intersecting system of  $\ell$ -sets of an  $(n - 1)$ -set with  $2 \leq \ell$  and  $n \geq (2r + 1)(\ell - 1) - (r - 2)$ . If  $\ell < r$  and

$$|\mathcal{F}| > \binom{n-2}{\ell-1} - \binom{n-2-r}{\ell-1},$$

then some fixed element  $x$  is in every set of  $\mathcal{F}$  and  $\mathcal{F}$  contains at least  $r + 1$  sets that pairwise intersect in exactly the element  $x$ .

**Proof.** First note that, since  $2 \leq \ell < r$ , it follows that

$$|\mathcal{F}| > \binom{(n-1)-1}{\ell-1} - \binom{(n-1)-(\ell+1)}{\ell-1} + 1,$$

and Theorem 5.5 implies that every set in  $\mathcal{F}$  contains some element  $x$ . Every set in  $\mathcal{F}$  contains  $x$ , so let  $\mathcal{F}^x$  be the family of  $(\ell - 1)$ -sets of an  $(n - 2)$ -set formed by removing  $x$  from every set in  $\mathcal{F}$ . Applying Theorem 5.6 to  $\mathcal{F}^x$  yields that, for the stated values of  $n$ , the family  $\mathcal{F}^x$  contains at least  $r + 1$  sets that are pairwise disjoint.  $\square$

The previous lemma will be used to bound the number of permutations  $p$ , in an intersecting family in  $\mathcal{B}_r(n)$ , that have  $|\text{ID}(p)| = \ell < r$ . Theorem 5.2 will be used to bound the number of such  $p$  having  $|\text{ID}(p)| = r$  since, in this case, Proposition 3.2 implies that  $\text{ID}(p)$  is a separated set.

**Theorem 5.8.** The set  $\mathcal{B}_r(n)$  is EKR for  $n$  sufficiently large relative to  $r$ .

**Proof.** We may assume that  $r > 3$ , since we have seen that both  $\mathcal{B}_2(n)$  and  $\mathcal{B}_3(n)$  are EKR. Let  $\mathcal{P} \subseteq \mathcal{B}_r(n)$  be an intersecting family of maximum size. We are done if  $\mathcal{P}$  is a star, so we assume otherwise. As in the proof for  $\mathcal{B}_3(n)$ , let  $\mathcal{P}_i = \{p \in \mathcal{P} \mid |\text{ID}(p)| = i\}$  where  $i \in \{1, 2, \dots, r\}$ . Clearly,  $\mathcal{P} = \bigcup_{i=1}^r \mathcal{P}_i$ . Further set  $\mathcal{A}_i = \{\text{ID}(p) \mid p \in \mathcal{P}_i\}$ .

As in the proof of Theorem 5.3, if  $\mathcal{P}_1 \neq \emptyset$  then  $\mathcal{P}$  is a star, so we may assume that  $\mathcal{P}_1 = \emptyset$ . Now we will consider  $\mathcal{P}_\ell$  for  $\ell \in \{2, \dots, r - 1\}$ .

By Lemma 5.7, if  $|\mathcal{A}_\ell| > \binom{n-2}{\ell-1} - \binom{n-2-r}{\ell-1}$  then  $\mathcal{A}_\ell$  is a star, centered on some  $x$ , which contains at least  $r + 1$  sets that pairwise intersect at exactly  $x$ . Every  $p \in \mathcal{P}$  has  $|\text{ID}(p)| \leq r$  and intersects each of these  $r + 1$  sets. Thus each  $\text{ID}(p)$  contains  $x$ , meaning that  $\mathcal{P}$  is a star. Hence, we may assume that  $|\mathcal{A}_\ell| \leq \binom{n-2}{\ell-1} - \binom{n-2-r}{\ell-1}$  for each  $\ell \in \{2, \dots, r - 1\}$ . Finally, since  $\mathcal{A}_r$  is an intersecting family of separated  $r$ -sets from an  $(n - 1)$ -set, Theorem 5.2 implies that  $|\mathcal{A}_r| \leq \binom{n-1-r}{r-1}$ .

From Lemma 3.1, the multiplicity of any set from  $\mathcal{A}_\ell$  in  $\mathcal{P}_\ell$ , is at most  $\binom{\ell+r-1}{\ell-1} e^{\pi\ell\sqrt{\frac{2r}{3}}}$ . Thus

$$\begin{aligned} |\mathcal{P}| &\leq \binom{n-1-r}{r-1} + \sum_{\ell=2}^{r-1} \binom{\ell+r-1}{\ell-1} e^{\pi\ell\sqrt{\frac{2r}{3}}} \left( \binom{n-2}{\ell-1} - \binom{n-2-r}{\ell-1} \right) \\ &\leq \binom{n-1-r}{r-1} + \sum_{\ell=2}^{r-1} \binom{\ell+r-1}{\ell-1} e^{\pi\ell\sqrt{\frac{2r}{3}}} \left( \sum_{j=3}^{r+2} \binom{n-j}{\ell-2} \right) \end{aligned} \quad (7)$$

$$\leq \binom{n-3}{r-1} + (r-2) \binom{2r-2}{r-2} e^{3r\frac{3}{2}} (r) \binom{n-3}{r-3}. \quad (8)$$

Eq. (7) follows from repeated applications of Pascal's Rule. The bounds used in Eq. (8) are very rough upper bounds. Provided that  $n$  is larger than

$$(r-2) \binom{2r-2}{r-2} e^{3r\frac{3}{2}} r(r-2) + r,$$

the size of  $\mathcal{P}$  is less than  $\binom{n-2}{r-1}$ , which, by Lemma 5.4, is less than the size of the largest star.  $\square$

The previous result uses a very rough counting method to bound the size of an intersecting family of permutations that is not a star. We believe that the lower bound on  $n$  given above is far larger than necessary; in fact we conjecture that all levels below the middle level in the weak Bruhat lattice have the EKR property.

**Conjecture 5.9.** For all values of  $1 \leq r \leq \frac{1}{2} \binom{n}{2}$  the set  $\mathcal{B}_r(n)$  is EKR. Thus, for any  $p \in \mathcal{B}_1(n)$ , we have  $f_{1,r}(n) = |\text{up}(p)|$ , which equals the coefficient of  $x^{r-1}$  in  $F(x)/(1+x)$ .

Fig. 1 reveals that the conjecture holds for  $\mathcal{B}(4)$ . The upper bound on  $r$  is based on the fact that permutations in levels above  $\frac{1}{2} \binom{n}{2}$  have larger sets of inverse descents. For sufficiently large  $r$ , the entire  $r$ -level is intersecting since the size of the inverse descent sets are large. We suspect that for levels above  $\frac{1}{2} \binom{n}{2}$  the sets of all permutations with inverse descent sets of size at least  $(n-1)/2$  are larger than a star.

## 6. $t$ -intersecting families

In this section we conjecture that a version of the EKR theorem holds for some levels of the Bruhat lattice for  $t$ -intersecting permutations. The original version of the EKR theorem was for  $t$ -intersecting sets, and can be stated as follows.

**Theorem 6.1.** Let  $k, t$  and  $n$  be positive integers with  $t < k$ . Assume that  $\mathcal{A}$  is a  $t$ -intersecting system of  $k$ -subsets of  $\{1, \dots, n\}$ . There exists a function  $f(k, t)$  such that, if  $n > f(k, t)$ , then

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, equality holds if and only if  $\mathcal{A}$  is a  $t$ -star at level  $k$ .

Similar to the EKR theorem for  $t$ -intersecting  $k$ -sets, we conjecture that the largest  $t$ -intersecting family of rank  $k$  permutations in the Bruhat lattice is the upset of a permutation at level  $t$ . Unlike the case for sets, the sizes of the upsets of the permutations at level  $t$  are not all equal in the Bruhat lattice. Before stating the conjecture, we need to determine which permutation at level  $t$  we believe has the maximum number of rank  $r$  permutations in its upset.

Consider the permutation  $\rho(t)$  that is formed by starting with the trivial permutation  $(1, 2, \dots, n)$  and reversing exactly  $t$  pairs in  $\{1, \dots, n\}$  as follows. These reversals are done iteratively in  $t$  steps. At each step, the largest element in the permutation that has a smaller element directly before it (this is the largest element with a smaller element on its left), is switched with the element directly to the left of it. For example, for  $n = 6$  we have

$$\begin{array}{llll} \rho(0) = 123456, & \rho(1) = 123465, & \rho(2) = 123645, & \rho(3) = 126345, \\ \rho(4) = 162345, & \rho(5) = 612345, & \rho(6) = 612354, & \rho(7) = 612534, \\ \rho(8) = 615234, & \rho(9) = 651234, & \rho(10) = 651243, & \rho(11) = 651423, \\ \rho(12) = 654123, & \rho(13) = 654132, & \rho(14) = 654312, & \rho(15) = 654321. \end{array}$$

If  $i$  and  $j$  are positive integers with  $i$  chosen as large as possible so that

$$t = (n-1) + (n-2) + \dots + (n-i) + (n-(i+j+1)) \quad (9)$$

then

$$\rho(t) = \langle n, n-1, \dots, n-i+1, 1, 2, \dots, j, n-i, j+1, j+2, \dots, n-i-1 \rangle.$$

The set  $\text{Inv}(\rho(t))$  is the set of the final  $t$  transpositions in the lexicographic ordering (for example, if  $n = 6$ , then  $\text{Inv}(\rho(11)) = \{(2, 4), (3, 4), (1, 5), \dots, (4, 5), (1, 6), \dots, (5, 6)\}$ ) and  $\rho(t)$  is at level  $t$  in the Bruhat lattice. Further, the inverse descent set for  $\rho(t)$  is the set of the final  $i+1$  generators.

By Corollary 3.7 of [12], the generating function for the sizes of the levels in the down set of  $\rho(t)$  is given by

$$[n][n-1] \cdots [n-i+1][n-i-j],$$

where  $i$  and  $j$  are as defined in Eq. (9). By Theorem 3.1 of [12] this implies that generating function of the levels in the upset of  $\rho(t)$  is

$$F_{n,t}(x) := [n-i] \cdots [n-i-j+1][n-i-j-2] \cdots [2][1].$$

For example, with  $n = 4$  and  $t = 4$  (see Fig. 1), we have  $\rho(4) = 4132$ ,  $i = 1$ , and  $j = 1$ . This yields the generating functions  $[4][2] = 1 + 2x + 2x^2 + 2x^3 + x^4$  for its downset and  $F_{4,4}(x) = [3] = 1 + x + x^2$  for its upset.

The inverse descent set of  $\rho(t)$  is very small. This means that there are many different permutations of a given rank with an inverse descent set that contains  $\text{Inv}(\rho(t))$ . We conjecture that  $\rho(t)$  is a rank  $t$  permutation with the maximum number of rank  $r$  permutations in its upset.

**Conjecture 6.2.** Let  $1 \leq t \leq r \leq \binom{n}{2}$ . If  $n$  is sufficiently large relative to  $t$  and  $r$ , then the set  $\mathcal{B}_r(n)$  is  $t$ -EKR. In particular,  $f_{t,r}(n) = |\text{up}(p) \cap \mathcal{B}_r(n)|$ , which equals the coefficient of  $x^{r-t}$  in  $F_{n,t}(x)$ .

Finally we note that the question of  $t$ -intersection for any  $t < n$  can also be considered for the entire poset. In the case of the subsets poset, the largest  $t$ -intersecting sets are given by Katona's theorem [9]. The theorem states that the largest such set is roughly the collection of all set with size greater than  $(n + t)/2$ . At present it is not clear to us what the case will be for the Bruhat poset. Two possible candidates for largest  $t$ -intersecting family are the family of all permutations with at least  $(n + t)/2$  inverse descents and  $\text{up}(\rho(t))$  (this is the family of all permutations that have a common set of size  $t$  in their inverse descent set).

**Question 6.3.** *What is the size and structure of the largest  $t$ -intersecting family of permutations in  $\mathcal{B}(n)$ ?*

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