On the Existence of de Bruijn Tori with Two by Two Windows

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Abstract

Necessary and sufficient conditions for the existence of de Bruijn Tori (or Perfect Maps) with two by two windows over any alphabet are given. This is the first two-dimensional window size for which the existence question has been completely answered for every alphabet. The techniques used to construct these arrays utilise existing results on Perfect Factors and Perfect Multi-Factors in one and two dimensions and involve new results on Perfect Factors with 'puncturing capabilities'. Finally, the existence question for two-dimensional Perfect Factors is considered and is settled for two by two windows and alphabets of prime-power size.

1 Introduction

A 2-dimensional C-ary *de Bruijn Torus* (or *Perfect Map*) is a 2-dimensional periodic array with symbols drawn from an alphabet of size C having the property that every C-ary array of some fixed size occurs exactly once as a subarray of the array. More precisely, in the notation of [10], an $(R, S; u, v)_C$-dBT is an $R \times S$ periodic array with symbols drawn from a set of size C having the property that every possible $u \times v$ array occurs exactly once as a periodic subarray of the array. The pair (u, v) is often called the *window* of the torus. A variety of notations have been introduced to describe these arrays; we use that of [10].

Perfect Factors are related objects which have proved useful in constructions for de Bruijn Tori: an $(R; u; T)_C$-PF is a set of $T = \frac{C^u}{R}$ C-ary, period R sequences in which every C-ary u-tuple occurs exactly once as a subsequence. The parameter u is often called the *span* of the sequences. Perfect Factors have been extensively studied in [5, 15, 16, 17, 18]. Both de Bruijn Tori and Perfect Factors are generalisations of the classical de Bruijn sequences [1, 4, 7, 8, 9]: a de Bruijn sequence is a Perfect Factor with $T = 1$, i.e. a $(C^u; u; 1)_C$-PF, while de Bruijn Tori are two-dimensional analogues of de Bruijn sequences. We denote a C-ary span u de Bruijn sequence by $(C^u; u)_C$-dBS. It was proved in [4, 7, 9] that there exists a $(C^u; u)_C$-dBS for all integers $C \geq 2$ and $u \geq 1$.

As noted in [2, 10, 20] there are many applications for such objects.

In two dimensions, one can define an $(R, S; u, v; T)_C$-PF as a set of T $R \times S$ periodic arrays, with symbols drawn from a set of size C, having the property that every possible $u \times v$ array occurs exactly once as a periodic subarray in precisely one of the arrays. Of course, an $(R, S; u, v; 1)_C$-PF is simply an $(R, S; u, v)_C$-dBT.

The following necessary conditions on the parameters of an $(R, S; u, v)_C$-dBT are easily established:

Lemma 1 *Suppose there exists an $(R, S; u, v)_C$-dBT. Then*

  i) $RS = C^{uv}$,

  ii) $R > u$ *or* $R = u = 1$,

  iii) $S > v$ *or* $S = v = 1$.

It has been conjectured [20] that these conditions are in fact also sufficient for the existence of de Bruijn Tori. This has been proved in a wide variety of cases and we outline the relevant results next. (In contrast, very little is known about the existence of two-dimensional Perfect Factors.)

Combining ideas from [5] and [6], the conditions of Lemma 1 were shown to be sufficient for the existence of an $(R, S; u, v)_2$-dBT in [19]. This work was extended to alphabets of prime-power size in [20]. Square de Bruijn Tori (i.e. $(R, R; u, u)_C$-dBT) were considered in [10], while some families of arrays with $u = v = 2$ were constructed in [11]. The existence question for de Bruijn Tori over general alphabets was studied in [11, 12, 18, 21], and higher dimensional versions were considered in [13].

The main result of [21] is as follows. Suppose C has prime factorisation

$$C = \prod_{i=1}^{k} p_i^{c_i}.$$

By Lemma 1, $RS = C^{uv}$ and so any prime dividing R or S also divides C. We can therefore write

$$R = \prod_{i=1}^{k} p_i^{r_i}, \quad S = \prod_{i=1}^{k} p_i^{s_i}$$

for some $0 \leq r_i \leq c_i uv$ where $s_i = c_i uv - r_i$, $R > u$, and $S > v$.

Result 2 *[21, Theorem 1.3] Suppose C, R and S have prime factorisations as above and that for some i we have*

$$p_i^{r_i} > u \quad and \quad p_i^{s_i} > v.$$

Then there exists an $(R, S; u, v)_C$-dBT.

The parameter sets satisfying Lemma 1 for which existence of de Bruijn Tori remains unsettled after Result 2 are those where either $p_i^{r_i} \leq u$ or $p_i^{s_i} \leq v$ for each i. The cases where either $p_i^{r_i} \leq u$ for every i or $p_i^{s_i} \leq v$ for every i could be resolved if the existence question for one-dimensional Perfect Factors were positively settled, by using the generalisation of Etzion's construction given in [18]. Recent progress on this problem can be found in [15, 16, 17].

However there would still remain the 'mixed' parameter sets where $p_i^{r_i} \leq u$ for some indices and $p_i^{s_i} \leq v$ for other indices. Our purpose in this paper

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is to develop new construction methods for these mixed parameters in the simplest non-trivial case, where $u = v = 2$. We are then able to settle the existence question for de Bruijn Tori with $u = v = 2$.

In a mixed parameter set with $u = v = 2$, one of the following two cases must occur for each i:

- $p_i = 2$ and $r_i = 0, 1, 4c_i - 1$ or $4c_i$,

- $p_i \geq 3$ and $r_i = 0$ or $4c_i$.

The case where every r_i is either 0 or $4c_i$ is covered by the following result which we prove in Section 3, partially answering a question of [10].

Theorem 3 *Suppose $m > n \geq 2$. Then there exists an $(m^4, n^4; 2, 2)_{mn}$-dBT.*

We consider another subcase in Section 4: where $p_1 = 2$, $r_1 = 1$ and $r_i = 4c_i$ for $i \geq 2$. Our main result in that section is:

Theorem 4 *Suppose $n > 2$ is odd. Then for every $c \geq 1$, there exists a $(2n^4, 2^{4c-1}; 2, 2)_{2^c n}$-dBT.*

The proofs of Theorems 3 and 4 rest on the construction of some special classes of one-dimensional Perfect Factors in Section 2. The sequences of these Factors have what we call 'puncturing capabilities', a generalisation of the notion of the puncturing of de Bruijn sequences. In Section 5, we combine Theorems 3 and 4 with some results from [21] to obtain:

Theorem 5 *The necessary conditions of Lemma 1 are sufficient for the existence of an $(R, S; 2, 2)_C$-dBT.*

Theorem 5 provides the first instance where necessary and sufficient conditions are known for the existence of de Bruijn Tori with a fixed two-dimensional window size over all alphabets.

Having solved the existence problem for de Bruijn Tori with 2×2 windows, we go on to investigate the analogous problem for $(R, S; 2, 2; T)_C$-PFs in Section 6. We obtain a complete answer in the case where C is a prime-power:

Theorem 6 *Let p be a prime and c, r, s and t be integers. The conditions that $p^r, p^s > 2$ and $r + s + t = 4c$ are necessary and sufficient for the existence of a $(p^r, p^s; 2, 2; p^t)_{p^c}$-PF.*

We close with some open questions.

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2   Some Classes of Perfect Factors

2.1   Perfect Multi-Factors

We introduce a class of combinatorial objects which have proved useful in the construction of Perfect Factors. In order to unify the presentation, we use notation different from that found in [15]. Additionally, we say that a v-tuple $\tau = (\tau_0, \tau_1, \ldots, \tau_{v-1})$ *appears in a sequence* $\sigma = \sigma_0, \sigma_1, \ldots$ *at position* z if $(\sigma_z, \sigma_{z+1}, \ldots, \sigma_{z+v-1}) = (\tau_0, \tau_1, \ldots, \tau_{v-1})$.

Definition 7 *[15] Suppose m, n, C and u are positive integers which satisfy $m | C^u$ and $C \geq 2$. An $(R; u; T)_C[n]$ Perfect Multi-Factor, or simply a $(R; u; T)_C[n]$-PMF, is a set of $T = C^u/m$ C-ary sequences of period $R = mn$ with the property that for every C-ary u-tuple τ and integer j with $0 \leq j < n$, τ occurs at a position z with $z \equiv j \pmod{n}$ in one of the sequences.*

Result 8 *[15, Theorem 3.13] Suppose n, C, u are positive integers ($C \geq 2$ and $n \geq u$). Then there exists an $(n; u; C^u)_C[n]$-PMF containing the all-zero sequence.*

Lemma 9 *Suppose n, C, l are positive integers with $n, C \geq 2$. Let $z_0 = 0, z_1, \ldots, z_{l-1}, z_l = n$ be a sequence of integers with $z_{j+1} - z_j \geq 2$ for $0 \leq j \leq l - 1$. Then there exists an $(n; 2; C^2)_C[n]$-PMF with sequences*

$$\sigma(i, j), \quad 0 \leq i, j < C$$

having the property that sequence $\sigma(i, j)$ of the PMF satisfies

$$\sigma(i, j)_{z_0} = \sigma(i, j)_{z_1} = \ldots = \sigma(i, j)_{z_{l-1}} = i$$

(we say that $\sigma(i, j)$ is constant in positions $z_0, z_1, \ldots, z_{l-1}$). Moreover, when $l \geq C$, the PMF can be constructed so that sequence $\sigma(i, j)$ also satisfies

$$\sigma(i, j)_{z_j} = \sigma(i, j)_{z_j+1} = i$$

(we say that $\sigma(i, j)$ is constant in positions z_j and $z_j + 1$).

Proof: Since $R_j = z_{j+1} - z_j \geq 2$ for $1 \leq j \leq l - 1$, there exists for each j an $(R_j; 2; C^2)_C[R_j]$-PMF, A^j, as in Result 8. Our aim is to concatenate

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sequences from each A^j to form sequences of period n which comprise the required PMF.

Let $\sigma^j(a, b)$ denote the unique sequence in A^j which begins with the 2-tuple (a, b) and let $L(\sigma^j(a, b))$ denote the last symbol of that sequence. Let $X_i^j = \{\sigma^j(i, b) : 0 \le b < C\}$. Because of the defining property of A^j, the list

$$L(\sigma^j(i, 0)), L(\sigma^j(i, 1)), \ldots, L(\sigma^j(i, c - 1)),$$

consisting of the final symbols of the sequences in X_i^j, is a permutation of $\{0, 1, \ldots, C - 1\}$. Thus concatenating each sequence in X_i^0 with each sequence of X_i^1 (in any pairing) for each i produces a set of C^2 sequences that constitute a $(z_2; 2; C^2)_C[z_2]$-PMF.

More generally, if we denote the concatenation of a list of l sequences $\sigma(0), \sigma(1), \ldots, \sigma(l - 1)$ by $\sigma(0)\sigma(1) \cdots \sigma(l - 1)$ and if π_j^i is a permutation of $\{0, 1, \ldots, C - 1\}$ for each $0 \le i \le C - 1$ and each $0 \le j \le l - 1$, we have that the C^2 sequences

$$\sigma(i, j) = \sigma^0(i, \pi_0^i(j))\sigma^1(i, \pi_1^i(j)) \cdots \sigma^{l-1}(i, \pi_{l-1}^i(j)), \quad 0 \le i, j < C,$$

constitute an $(n; 2; C^2)_C[n]$-PMF. It is clear that the sequences $\sigma(i, j)$ satisfy $\sigma(i, j)_{z_0} = \sigma(i, j)_{z_1} = \ldots = \sigma(i, j)_{z_{l-1}} = i$, so that the first condition in the statement of the lemma holds.

Now suppose that $l \ge C$. In defining the permutations π_j^i (for $0 \le i \le C - 1$ and $0 \le j \le l - 1$) we additionally specify that

$$\pi_j^i(j) = i, \quad 0 \le i, j < C.$$

Then 2-tuple (i, i) appears in sequence $\sigma(i, j)$ at position z_j and the second condition in the statement of the lemma is satisfied. $\square$

2.2 Puncturing and Joining Sequences

Let K_m denote the complete directed graph (with loops) on m vertices. An m-ary span 2 de Bruijn sequence corresponds to an Eulerian circuit in K_m [1, 4, 8], while an $(R; 2; T)_m$-PF corresponds to a T-set of length R edge-disjoint circuits in K_m. More generally, an m-ary sequence of period k whose 2-tuples within a period are distinct corresponds to a circuit of k distinct edges in K_m. We define a *k-puncture* in K_m to be such a circuit. We say

that a periodic sequence $\sigma = \sigma_0, \sigma_1, \dots$ whose 2-tuples within a period are distinct has a k-puncture at position z if $\sigma_z = \sigma_{z+k}$ (and thus the periodic sequence $\sigma_z, \dots, \sigma_{z+k-1}$ corresponds to a k-puncture in K_m). It is easy to see that if σ has such a puncture, then the terms $\sigma_z, \dots, \sigma_{z+k-1}$ can be removed from the sequence σ leaving a sequence σ' whose 2-tuples are still distinct, the 2-tuples removed being those which occur in the puncture. Naturally, we call the process of removing a k-puncture *puncturing*. As an example, a 1-puncture in σ is just the occurrence of 2 consecutive identical symbols in σ, and corresponds to a loop edge in K_m.

We also need to consider an operation on sequences that is a kind of inverse to puncturing. This process, which we call *joining*, is already well-known [8]. Let C_1 be a circuit of k_1 distinct edges and C_2 a circuit of k_2 distinct edges in K_m. Suppose that C_i has edge set E_i ($i = 1, 2$) and that E_1 and E_2 are disjoint. So C_1 and C_2 correspond to periodic sequences $\sigma(1)$ and $\sigma(2)$ which have disjoint sets of 2-tuples as their subsequences. If C_1 and C_2 have a vertex in common (i.e. $\sigma(1)$ and $\sigma(2)$ have a common term), then we say that C_1 and C_2 (and sequences $\sigma(1)$, $\sigma(2)$) are *adjacent*. Suppose C_1 is comprised of the ordered list of vertices $v_1, v_2, \dots, v_{k_1}$ and C_2 the list of vertices $w_1, w_2, \dots, w_{k_2}$, and that $v_i = w_j$ for some i and j, so that C_1 and C_2 are adjacent. Then it is easy to see that the list

$$v_1, \dots, v_i, w_{j+1}, \dots, w_{k_2}, w_0, \dots, w_j, v_{i+1}, \dots, v_{k_1}$$

is the vertex list of a circuit C of $k_1 + k_2$ distinct edges in K_m whose edge set is $E_1 \cup E_2$. We say that C is obtained by joining C_1 and C_2. The cycle C corresponds to a sequence σ which has period $k_1 + k_2$, contains as subsequences exactly those 2-tuples occurring in $\sigma(1)$ and $\sigma(2)$ and has as a k_i-puncture $\sigma(i)$ ($i = 1, 2$).

Now suppose we have a collection $\mathcal{X}$ of edge disjoint circuits in K_m. We can define an (undirected) adjacency graph $G(\mathcal{X})$ for these circuits using the definition of adjacency for circuits given above. Suppose $G(\mathcal{X})$ is a connected graph. Then repeatedly using the joining process described above on a spanning tree for $G(\mathcal{X})$, we can join all the circuits in $\mathcal{X}$ to form a single circuit that covers exactly the same edges as were covered by the circuits in $\mathcal{X}$.

2.3 Construction of Perfect Factors

In this subsection we will construct some classes of Perfect Factors with various puncturing properties. These are used in the proofs of Theorems 3 and 4. Our main tool will be our construction for PMFs in Lemma 9 and Result 10, below. For any sequence ρ, we denote by $[\rho]^x$ the concatenation of ρ with itself x times.

Result 10 *[15, Construction 5.1 and Theorem 5.2] Suppose there exists an $(n; u; t_1)_C$-PF with sequences $\sigma(0), \ldots, \sigma(t_1 - 1)$ and an $(mn; u; t_2)_D[n]$-PMF with sequences $\tau(0), \ldots, \tau(t_2 - 1)$. Define $\beta(i, j)$ to be the CD-ary sequence $[\sigma(i)]^m + C\tau(j)$ of period mn. Then the set of cycles $\{\beta(i, j) : 0 \le i < t_1, 0 \le j < t_2\}$ is an $(mn; u; t_1 t_2)_{CD}$-PF.*

Lemma 11 *Suppose $m > n \ge 2$. Then there exists an $(m^2; 2; n^2)_{mn}$-PF in which every sequence has a 1-puncture.*

Proof: For $m > n \ge 2$, the finite sequence

$$\sigma = 0, 0, 1, 1, \ldots, n - 1, n - 1$$

corresponds to a path of distinct edges in K_m. Let G denote the graph obtained from K_m by removing these edges of σ from K_m. It is easy to see that G is connected (for every vertex in G is joined to vertex $m - 1$ by a pair of oppositely directed edges). Moreover, every vertex in G has in-degree and out-degree equal, except vertex 0, with in-degree $m - 1$ and out-degree $m - 2$ and vertex $n - 1$, with in-degree $m - 2$ and out-degree $m - 1$. So G has an Eulerian trail τ beginning at vertex $n - 1$ and ending at vertex 0. Concatenating σ and τ results in an Eulerian circuit of K_m. The corresponding de Bruijn sequence α has 1-punctures at positions $0, 2, \ldots, 2n - 2$.

We define $z_j = 2j$, $0 \le j \le n - 1$ and $z_n = m^2$. By Lemma 9, there exists an $(m^2; 2; n^2)_n[m^2]$-PMF in which each sequence is constant in some pair of positions $z_j, z_j + 1$ (where $0 \le j \le n - 1$). We denote the sequences of this PMF by $\tau(0), \ldots, \tau(n^2 - 1)$. We apply Result 10 to α and $\tau(0), \ldots, \tau(n^2 - 1)$ to obtain an $(m^2; 2; n^2)_{mn}$-PF with sequences $\beta(i) = \alpha + m\tau(i)$, $0 \le i < n^2$. Because α is constant in every pair of positions $(z_j, z_j + 1)$ $(0 \le j \le n - 1)$ and each $\tau(i)$ is constant in one of these pairs, we have that each $\beta(i)$ is constant in one of these pairs. Hence each $\beta(i)$ has a 1-puncture and the $(m^2; 2; n^2)_{mn}$-PF has the required properties. $\qquad\square$

Lemma 12 *Suppose $m \geq 2$ and $n \geq 3$. Then there exists an $(n^2; 2; m^2)_{mn}$-PF in which every sequence has a 2-puncture.*

Proof: By considering paths in K_n it is easy to see that, for $n \geq 3$, there exists an n-ary span 2 de Bruijn sequence α that begins $0, 1, 0$, i.e. has a 2-puncture at position 0. We define $z_0 = 0$, $z_1 = 2$ and $z_2 = n^2$. Using Lemma 9, we can construct an $(n^2; 2; m^2)_m[n^2]$-PMF in which each sequence is constant in positions $z_0 = 0$ and $z_1 = 2$. From the definition of a PMF, for every 2-tuple (i, j) $(0 \leq i, j < m)$, there is a unique sequence of the $(n^2; 2; m^2)_m[n^2]$-PMF having (i, j) at position 0. We denote this sequence by $\tau(i, j)$. We apply Result 10 to obtain an $(n^2; 2; m^2)_{mn}$-PF Γ with sequences $\beta(i, j) = \alpha + n\tau(i, j)$, $0 \leq i, j < m$, in which each sequence is constant in positions 0 and 2, i.e. has a 2-puncture at position 0. $\qquad \square$

With notation as in the above proof, let

$$\Gamma_i = \{\beta(i, j) \mid 0 \leq j < m\}, \quad 0 \leq i < m.$$

From the construction of the $\beta(i, j)$, each sequence in Γ_i has symbol ni in position 0. So the adjacency graph for the set of edge-disjoint circuits corresponding to Γ_i is the complete graph on m vertices.

Moreover, for i_1, i_2 and j with $0 \leq i_1, i_2, j < m$, the sequences $\beta(i_1, j)$ and $\beta(i_2, j)$ have symbol $1 + nj$ in position 1, and so any two sets Γ_{i_1} and Γ_{i_2} contain adjacent sequences.

From these observations, it follows that we can select from Γ a connected set of n^2 sequences, provided that $m \geq n$. These sequences can be joined to obtain a single sequence of period n^4, leaving $m^2 - n^2$ sequences of period n^2. We have:

Corollary 13 *Suppose $m \geq n \geq 3$. Then there exists a set of mn-ary sequences, consisting of one sequence of period n^4 and $m^2 - n^2$ sequences of period n^2, with the properties that*

    i) each mn-ary 2-tuple occurs exactly once as a subsequence of a sequence in the set.

    ii) each sequence of period n^2 in the set contains a 2-puncture.

Lemma 14 *Suppose $n \geq 3$ is odd. Then, for every $c \geq 1$, there exists a $(2n^2; 2; 2^{2c-1})_{2^c n}$-PF in which every sequence has a 2-puncture.*

Proof: We initially aim to construct a $(2n^2; 2; 2^{2c-1})_{2^c}[n^2]$-PMF in which every sequence is constant in positions 0 and 2. In the case where $c = 1$, it is easily verified (using the fact that $n^2 \geq 9$ is odd) that the following pair of sequences constitute a $(2n^2; 2; 2)_2[n^2]$-PMF with the required property:

$$\begin{aligned}
\tau(0) &= 0, 1, 0, 1 \ldots, 0, 1, 1, 1 \\
\tau(1) &= 0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0.
\end{aligned}$$

Here sequence $\tau(0)$ consists of $n^2 - 1$ occurrences of $0, 1$ followed by $1, 1$, while sequence $\tau(1)$ consists of $n^2 - 1$ zeros, then $n^2 - 1$ ones and finally 2 zeros. When $c > 1$, we further construct (using Lemma 9) a $(2n^2; 2; 2^{2c-2})_{2^{c-1}}[2n^2]$-PMF with sequences $\sigma(i)$, $0 \leq i < 2^{2c-2}$, in which every sequence is constant in positions 0 and 2. We then combine this PMF and the PMF $\tau(0), \tau(1)$ to obtain a new set of 2^c-ary sequences $\beta(i, j)$, where

$$\beta(i, j) = \sigma(i) + 2^{c-1}\tau(j), \quad 0 \leq i < 2^{2c-2}, \ 0 \leq j < 2.$$

It is easy to see that every sequence in this set is constant in positions 0 and 2. It is only slightly more difficult to show that the set is in fact a $(2n^2; 2; 2^{2c-1})_{2^c}[n^2]$-PMF. This can be shown along the lines of the proof of Theorem 5.2 of [15].

Let α be an n-ary span 2 de Bruijn sequence beginning $0, 1, 0$, i.e. having a 2-puncture in position 0. We apply Result 10 with the $(n^2; 2; 1)_n$-PF α and our $(2n^2; 2; 2^{2c-1})_{2^c}[n^2]$-PMF to obtain a $(2n^2; 2; 2^{2c-1})_{2^c n}$-PF with sequences

$$\gamma(i, j) = [\alpha]^2 + n\beta(i, j), \quad 0 \leq i < 2^{2c-2}, \ 0 \leq j < 2.$$

Because α and each $\beta(i, j)$ are constant in positions 0 and 2, so is each $\gamma(i, j)$. Hence each $\gamma(i, j)$ has a 2-puncture at position 0 and the PF has the required properties. □

3 Construction of an $(m^4, n^4; 2, 2)_{mn}$-dBT

As before, for any sequence ρ we let $[\rho]^x$ denote the concatenation of ρ with itself x times. If ρ has a k-puncture ρ^+ at position 0, then we write $\rho = \rho^+ \rho^-$

and define $\rho^k = [\rho^+]^R[\rho^-]^R$ where R is the period of ρ. We define the shift operator E by $E(\rho) = E(\rho_0, \rho_1, \ldots, \rho_{R-1}) = (\rho_1, \ldots, \rho_{R-1}, \rho_0)$ and define $E^x = E(E^{x-1})$.

Now suppose that ρ and ρ' are C-ary sequences of period R, each containing R distinct 2-tuples, and that ρ has a k-puncture ρ^+ where R and k are coprime. In the next lemma, we consider the 2-tuples occurring in the period R^2 sequences $[\rho']^R$ and $\rho^k = [\rho^+]^R[\rho^-]^R$.

Lemma 15 *With notation as above, let β be a 2-tuple occurring in ρ and β' a 2-tuple occurring in ρ'. Then there exists a unique i with $0 \leq i < R^2$ such that β appears at position i in ρ^k and β' appears at position i in $[\rho']^R$.*

Proof: Suppose that β appears in ρ at position j and β' appears in ρ' at position j'. If $0 \leq j < k$, then β appears in ρ^+ and consequently in positions

$$j + lk, \quad 0 \leq l < R$$

in ρ^k. On the other hand, if $k \leq j < R$, then β appears in ρ^- and so in positions
$$Rk + (j - k) + l(R - k), \quad 0 \leq l < R$$
in ρ^k. In either case, because k is coprime to R, the R positions at which β appears in ρ^k are all distinct modulo R. So exactly one of them is congruent to j' modulo R. But β' appears in every position congruent to j' modulo R in $[\rho']^R$. So β and β' appear exactly once in the same position in the sequences ρ^k and $[\rho']^R$. $\qquad\qquad\square$

It follows from Lemma 15 that if ρ^k and $[\rho']^R$ are placed side-by-side, then every 2-tuple that appears in ρ occurs alongside every 2-tuple that appears in ρ' exactly once. We say that ρ^k and $[\rho']^R$ are *coprime*.

We can now sketch the idea behind our novel construction method for $(m^4, n^4; 2, 2)_{mn}$ de Bruijn Tori. Suppose that k is coprime to m^2 and we can find an $(m^2; 2; n^2)_{mn}$-PF with sequences $\sigma(0), \ldots, \sigma(n^2 - 1)$ in which every sequence has a k-puncture. For each i with $0 \leq i < n^2$, we form the period m^4 sequences $\sigma(i)^0 = [\sigma(i)]^{m^2}$ and $\sigma(i)^k = [\sigma(i)^+]^{m^2}[\sigma(i)^-]^{m^2}$. We arrange these sequences as the columns of an array in such a way that, for every i, j with $0 \leq i, j < n^2$, one of the coprime pairs

$$(\sigma(i)^0, \sigma(j)^k) \quad \text{or} \quad (\sigma(i)^k, \sigma(j)^0)$$

appears as a pair of consecutive columns in the array. Such pairings ensure that all the 2×2 subarrays whose first column comes from $\sigma(i)$ and whose second column comes from $\sigma(j)$ appear in the array. Because the $\sigma(i)$ form a PF and we have pairings for all sequences $\sigma(i), \sigma(j)$, we will obtain a de Bruijn Torus. This idea, and variations on it, lie at the heart of our proof of Theorem 3.

Proof of Theorem 3: Suppose that $m > n \geq 2$. We consider three cases, depending on the parities of m and n.

We begin with the simplest case in which n is even (and m has either parity). Let $\sigma(0), \ldots, \sigma(n^2 - 1)$ be the sequences of the $(m^2; 2; n^2)_{mn}$-PF from Lemma 11, with each sequence having a 1-puncture. Let $\tau = \tau_0, \tau_1, \ldots$ be an $(n^4; 2)_{n^2}$-dBS. We build an array A with columns A_j, $0 \leq j < n^4$, as follows. If j is even, then we take $A_j = \sigma(\tau_j)^0$. If j is odd, then we take $A_j = \sigma(\tau_j)^1$. Notice that that we alternate the use of 0- and 1-punctures, so any pair of consecutive columns of A are coprime. Because n is even, the final column A_{n^4-1} is equal to $\sigma(\tau_{n^4-1})^1$. Thus column A_{n^4-1} and column A_0 are also a coprime pair.

We claim that A is an $(m^4, n^4; 2, 2)_{mn}$-dBT. If B is any 2×2 (mn)-ary matrix whose columns are β_1 and β_2 then, by the definition of an $(m^2; 2; n^2)_{mn}$-PF, there is some i such that β_1 appears in $\sigma(i)$, and there is some j such that β_2 appears in $\sigma(j)$. Also, by the definition of an $(n^4; 2)_{n^2}$-dBS, there is a unique position h with $0 \leq h < n^4$ such that $(\tau_h, \tau_{h+1}) = (i, j)$. Let $h_1 = h \bmod 2$ and $h_2 = h + 1 \bmod 2$. Then $\sigma(i)^{h_1}$ and $\sigma(j)^{h_2}$ occur consecutively in A as columns h and $h + 1$. Because of their coprimality, B appears in these two columns in a unique position.

In the second case we assume that m is even and n is odd. We will construct an $(n^4, m^4; 2, 2)_{mn}$-dBT whose transpose is a torus with the desired parameters. Let $\sigma(0), \ldots, \sigma(m^2 - 1)$ be the sequences of the $(n^2; 2; m^2)_{mn}$-PF from Lemma 12, with each sequence $\sigma(i)$ having a 2-puncture $\sigma(i)^+$. Notice that 2 and n are coprime, so for each i, j with $0 \leq i, j < m^2$, the period n^4 sequences $\sigma(i)^0 = [\sigma(i)]^{n^2}$ and $\sigma(j)^2 = [\sigma(i)^+]^{n^2}[\sigma(i)^-]^{n^2}$ are coprime. Let $\tau = \tau_0, \tau_1, \ldots$ be an $(m^4; 2)_{m^2}$-dBS. We build an array B with columns A_j, $0 \leq j < m^4$, as follows. If j is even, then we take $A_j = \sigma(\tau_j)^0$. If j is odd, then we take $A_j = \sigma(\tau_j)^1$. Once again, any pair of consecutive columns of A are coprime. Also, since m is even, columns A_{m^4-1} and A_0 are a coprime pair. We claim that B is indeed an $(n^4, m^4; 2, 2)_{mn}$-dBT — the proof of this

claim is almost identical to the argument in the first case.

In the third and most complex case we have both m and n odd (so $m > n \geq 3$). We can no longer use alternating punctures to build the columns of our array (as in the first two cases above) because the last and first columns would never be coprime. We use the PF of Corollary 13 to deal with this problem. Let α, of period n^4, and $\sigma(i)$ ($0 \leq i < m^2 - n^2$), each of period n^2, be a set of sequences having properties i) and ii) of Corollary 13. Let $\sigma(i)^+$ denote the 2-puncture in $\sigma(i)$. Notice that 2 and n are coprime, so for each i, j with $0 \leq i, j < m^2 - n^2$, the period n^4 sequences $\sigma(i)^0 = [\sigma(i)]^{n^2}$ and $\sigma(j)^2 = [\sigma(i)^+]^{n^2}[\sigma(i)^-]^{n^2}$ are a coprime pair.

Consider the pairs of sequences in the following list:

$$
\begin{array}{cc}
(\alpha, E^i\alpha), & 0 \leq i < n^4 \\
(\alpha, E^i\sigma(j)^0), & 0 \leq i < n^2, 0 \leq j < m^2 - n^2 \\
(E^i\sigma(j)^0, \alpha), & 0 \leq i < n^2, 0 \leq j < m^2 - n^2 \\
(\sigma(i)^0, \sigma(j)^2) \text{ or } (\sigma(i)^2, \sigma(j)^0), & 0 \leq i, j < m^2 - n^2
\end{array}
$$

The first set of pairs ensure that every 2-tuple of α occurs alongside every 2-tuple from α, while the second and third sets cover the cases where a 2-tuple from α appears with a 2-tuple from a sequence $\sigma(j)$ and vice-versa. In view of the coprimalities established above, the final set covers the cases where a 2-tuple from a sequence $\sigma(i)$ appears with a 2-tuple from another sequence $\sigma(i)$. Thus if we can construct an $n^4 \times m^4$ array C in such a way that the set of pairs of consecutive columns of C are exactly the pairs in the above list, then C will be an $(n^4, m^4; 2, 2)_{mn}$-dBT, as needed.

So it remains to specify how the columns C_j, $0 \leq j < m^4$ of C can be arranged so as to satisfy the above condition. Let $\beta = \beta_0, \beta_1, \ldots, \beta_{n^4}$ be the sequence with $\beta_0 = 0$ and $\beta_i = \sum_{j=0}^{i-1} j \bmod n^4$. Notice that because n is odd, $\beta_{n^4} = 0 \bmod n^4$. We begin by taking $C_j = E^{\beta_j}\alpha$, $0 \leq j \leq n^4$. So C_{n^4} is just the sequence α. As columns C_{n^4+2j}, $1 \leq j < n^2(m^2 - n^2)$, we also take the sequence α, while for columns C_{n^4+2j+1}, $0 \leq j < n^2(m^2 - n^2)$, we take the following ordered list of sequences:

$$
\begin{array}{ll}
& E^1\sigma(0)^0, E^2\sigma(0)^0, \ldots, E^{n^2-1}\sigma(0)^0, \\
E^0\sigma(1), & E^1\sigma(1)^0, E^2\sigma(1)^0, \ldots, E^{n^2-1}\sigma(1)^0, \\
& \quad\quad\quad\quad \vdots \\
E^0\sigma(l), & E^1\sigma(l)^0, E^2\sigma(l)^0, \ldots, E^{n^2-1}\sigma(l)^0, \quad E^0\sigma(0)^0
\end{array}
$$

where $l = m^2 - n^2 - 1$. Finally, let τ be an $((m^2-n^2)^2; 2)_{m^2-n^2}$-dBS beginning and ending with 0. Then for the last $(m^2 - n^2)^2$ columns, we take

$$\sigma(\tau_0)^2, \sigma(\tau_1)^0, \sigma(\tau_2)^2, \sigma(\tau_3)^0, \ldots, \sigma(\tau_{(m^2-n^2)^2-1})^0$$

so that in these columns we alternate the use of 2- and 0-punctures, with the choice of sequence being determined by the terms of τ. Notice that column C_{m^4-1} is equal to $\sigma(0)^0$ (because $(m^2-n^2)^2$ is even), and when matched with column C_0, accounts for the pair $(E^0\sigma(0)^0, \alpha)$ missing from the description of the 'middle third' of C. With this hint, it is straightforward to check that all the required pairings do appear using this specification of columns. This completes the proof in the third case. $\qquad\square$

4   Construction of a $(2n^4, 2^{4c-1}; 2, 2)_{2^c n}$-dBT

Proof of Theorem 4: Let $\gamma(i)$, $0 \leq i < 2^{2c-1}$ be the sequences of a $(2n^2; 2; 2^{2c-1})_{2^c n}$-PF from Lemma 14. Using similar notation as before, we define $\gamma(i)^+$ to be the 2-puncture in $\gamma(i)$. We also write $\gamma(i) = \gamma(i)^+\gamma(i)^-$, and define $\gamma(i)^* = [\gamma(i)^+]^{n^2}[\gamma(i)^-]^{n^2}$ so that $\gamma(i)^*$ has period $2n^4$. Since $\gcd(2n^2, 2n^2 - 2) = 2$, it is easy to see from the construction of $\gamma(i)^*$ that any 2-tuple appearing as a subsequence of $\gamma(i)$ does so in $\gamma(i)^*$ either once in every even position modulo $2n^2$ or once in every odd position modulo $2n^2$. Thus the pair of sequences $\gamma(i)^*$, $E\gamma(i)^*$ have the property that, between them, they contain every 2-tuple appearing in $\gamma(i)$ in every position modulo $2n^2$.

We aim to build an $2n^4 \times 2^{4c-1}$ array A with the property that, for any i, j, there are two pairs of adjacent columns in A which between them account for an occurrence of every 2-tuple in $\gamma(i)$ alongside every 2-tuple in $\gamma(j)$. To do this we use the sequences $[\gamma(i)]^{n^2}$, $\gamma(i)^*$ and $E\gamma(i)^*$ as columns and ensure that for every i, j, A contains as consecutive columns either the pairs

$$[\gamma(i)]^{n^2}, \gamma(j)^* \quad \text{and} \quad [\gamma(i)]^{n^2}, E\gamma(j)^*$$

or the pairs

$$\gamma(i)^*, [\gamma(j)]^{n^2} \quad \text{and} \quad E\gamma(i)^*, [\gamma(j)]^{n^2}.$$

We define the arrangement of columns A_j, $0 \leq j < 2^{4c-1}$, as follows. We take τ to be the concatenation of a $(2^{4c-2}; 2)_{2^{2c-1}}$-dBS with itself (so τ has

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length 2^{4c-1}). If j is even, then we take column A_j to be $[\gamma(\tau_j)]^{n^2}$. If j is odd and $j \leq 2^{4c-2}$ then we take $A_j = \gamma(\tau_j)^*$. If j is odd and $j > 2^{4c-2}$ then we take $A_j = E(\gamma(\tau_j)^*)$. We claim that A is indeed an $(2n^4, 2^{4c-1}; 2, 2)_{2^c n}$-dBT. In view of the above discussion, it suffices to note that using the doubled de Bruijn sequence τ in the way indicated guarantees that all the specified pairs of columns do appear. $\qquad\square$

5 De Bruijn Tori with 2×2 Windows

We begin with some results on 2-dimensional Perfect Multi-Factors from [21]. These will be used in the proof of Theorem 5.

Definition 16 *Let m_1, m_2, n_1, n_2, C, u and v be positive integers with $c \geq 2$ and $m_1 m_2 | C^{uv}$. A 2-dimensional $(m_1 n_1, m_2 n_2; u, v; T)_C[n_1, n_2]$-PMF is a set of $T = C^{uv}/m_1 m_2$ 2-dimensional C-ary arrays of period $(m_1 n_1, m_2 n_2)$ with the property that, for every C-ary $u \times v$ array A and for every i, j with $0 \leq i < n_1$, $0 \leq j \leq n_2$, A occurs at a position (z_1, z_2) with*

$$
\begin{aligned}
z_1 &\equiv i \pmod{n_1} \\
z_2 &\equiv j \pmod{n_2}
\end{aligned}
$$

in one of the arrays.

Result 17 *[21, Lemma 2.2] Suppose there exists an $(n_1, n_2; u, v)_C$-dBT and an $(m_1 n_1, m_2 n_2; u, v; 1)_D[n_1, n_2]$-PMF. Then there exists an $(m_1 n_1, m_2 n_2; u, v)_{CD}$-dBT.*

Result 18 *[21, Theorem 3.2] Suppose $u, v \geq 2$ and m_1, n_1 and n_2 satisfy $m_1 | D^u$, $n_1 \geq u$ and $n_2 \geq v$. Suppose further that in the case where $v = 2$, m_1 is even and n_2 is odd, we have $2m_1 | D^u$. Then there exists an $(m_1 n_1, m_2 n_2; u, v; 1)_D[n_1, n_2]$-PMF.*

Proof of Theorem 5: Suppose C has prime factorisation

$$
C = \prod_{i=1}^{k} p_i{}^{c_i}.
$$

Write

$$R = \prod_{i=1}^{k} p_i^{r_i}, \quad S = \prod_{i=1}^{k} p_i^{4c_i - r_i}$$

with $R, S > 2$. From the discussion in Section 1, to prove Theorem 5 it is sufficient to assume that for each i, one of the following cases occurs:

- $p_i = 2$ and $r_i = 0, 1, 4c_i - 1$ or $4c_i$,

- $p_i \geq 3$ and $r_i = 0$ or $4c_i$.

Consider the case where every r_i is either 0 or $4c_i$. Then writing $m = \prod_{r_i = 4c_i} p_i^{c_i}$ and $n = C/m$, we certainly have $m \neq n$. Because of symmetry, we may assume $m > n$. An application of Theorem 3 then shows that an $(R, S; 2, 2)_C$-dBT exists.

We are left to consider the cases where $p_1 = 2$ and $r_1 = 1$ or $4c_1 - 1$. In what follows, we assume that $r_1 = 1$; tori with $p_1 = 2$ and $r_1 = 4c_1 - 1$ are easily obtained by transposing tori of this type. Note that we cannot have $r_i = 0$ for each $i \geq 2$, since then we would have $R = \prod_{i=1}^{k} p_i^{r_i} = 2$. So $r_i = 4c_i$ for some i. We consider two further cases.

Firstly, suppose that $r_i = 4c_i$, $2 \leq i \leq k$. Then we want to construct a $(2n^4, 2^{4c_1 - 1}; 2, 2)_{2^{c_1}n}$-dBT, where $n = \prod_{i=2}^{k} p_i^{c_i}$ is odd. Such a torus can be obtained from Theorem 4.

Otherwise, some r_i is zero and another is equal to $4c_i$. We write $m = \prod_{r_i = 4c_i} p_i^{c_i}$ and $n = C/2^{c_1}m$ so that $m \neq n$, $m, n \geq 2$ and $C = 2^{c_1}mn$. Using Theorem 3 (and transposition if $n > m$), there exists a $(m^4, n^4; 2, 2)_{mn}$-dBT. From Result 18, we can obtain a $(2m^4, 2^{4c_1 - 1}n^4; 2, 2; 1)_{2^{c_1}}[m^4, n^4]$-PMF. By Result 17, there exists a $(2m^4, 2^{4c_1 - 1}n^4; 2, 2)_{2^{c_1}mn}$-dBT, i.e. a $(2m^4, 2^{4c_1 - 1}n^4; 2, 2)_C$-dBT, as required. $\qquad \square$

6 Two-Dimensional Perfect Factors

Theorem 5 settles the existence question for de Bruijn Tori with 2×2 windows. It is interesting to consider the generalisation of this question to the case of two-dimensional Perfect Factors. As well as being of mathematical interest in their own right, these will be fundamental in building 3-dimensional de Bruijn Tori. Similar conditions to those of Lemma 1 can be derived for the parameters of a Perfect Factor: condition *i)* simply becomes $RST = C^{uv}$.

We also have the following two results, whose proofs use a generalisation of methods dating back to Ma [14] and Etzion [5] (similar methods have also been successfully used in [3, 10, 12, 13, 18, 20]). Let the *weight* of a sequence be the sum of its entries.

Theorem 19 *Suppose that $RST = C^{uv}$ and there exists an $(R; u; T_1)_C$-PF (where $T_1 = C^u/R$). Suppose further that one of the following two cases occurs.*

*   *i) $T | R^{v-1}$ and there exists an $(S; v-1; T)_R[T_1^v]$-PMF in which each sequence has weight $0 \bmod R$, or*

*   *ii) $R^{v-1} | T$ and there exists an $(S; v; T_1^v/S)_{T_1}$-PF, and an $(S; v-1; R^{v-1})_R[S]$-PMF, each sequence of the latter having weight $0 \bmod R$.*

Then there exists an $(R, S; u, v; T)_C$-PF.

Proof: We give a construction for what we claim is an $(R, S; u, v; T)_C$-PF. The construction differs slightly in each of the two cases. Let $\sigma(i)$, $0 \le i < T_1 - 1$ be the sequences of the $(R; u; T_1)_C$-PF. In case i), the condition $T | R^{v-1}$ implies that T_1^v divides S. We let $\tau(0)$ be a $(T_1^v; v)_{T_1}$-dBS concatenated with itself S/T_1^v times. In case ii), the condition $R^{v-1} | T$ implies that S divides T_1^v and we let $\tau(i)$, $0 \le i < T_1^v/S$ be the sequences of the $(S; v; T_1^v/S)_{T_1}$-PF. Finally, in each case, we let $\gamma(i)$ be the sequences of the appropriate PMF and, for each i, define a new sequence $\alpha(i) = \alpha(i)_0, \alpha(i)_1, \dots$ by

$$
\alpha(i)_j = \begin{cases} 0 & \text{if } j = 0 \\ \sum_{k=0}^{j-1} \gamma(i)_k \bmod R & \text{if } j > 0 \end{cases}
$$

Because of the condition on the weight of the PMF sequences, each sequence $\alpha(i)$ has period S.

Next, we define a T-set of $R \times S$ arrays $A(i, j)$. In case i), we have $0 \le i < 1$ and $0 \le j < T$, while in case ii), we have $0 \le i < T_1^v/S$ and $0 \le j < R^{v-1}$. In either case, we take column k of array $A(i, j)$ to be the sequence $E^{\alpha(j)_k}(\sigma(\tau(i)_k))$.

The proof that the set of arrays $A(i, j)$ is a PF with the required parameters is omitted, but as we have already intimated, can be constructed along the lines of the proof of [18, Theorem 6.1], for example. $\qquad \square$

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Proof of Theorem 6: The necessity is obvious. If $p = 2$, the theorem has been proven in [13, Lemma 3.2], so we assume here that $p > 2$. Because of symmetry, we may assume that $r \leq s$, and so $r \leq 2c$. Hence there is a $(p^r; 2; p^{2c-r})_{p^c}$-PF by Theorem 4.6 of [18]. If $t < r$ then we come under case *i)* of Theorem 19. In this case there is a $(p^{s-(r-t)}; 2)_{p^{(2c-r)}}$-dBS. A simple way to obtain the $(S; v - 1; T)_R[T_1^v]$-PMF is to puncture the sequence $0, 1, \ldots, p^r - 1$ at 0. That is, for the sequence $\sigma = [0]^{p^{s-r}}[1, \ldots, p^r - 1]^{p^{s-r}}$, the set of sequences $\{E^{kp^{s-r}}(\sigma) \mid 0 \leq k < p^t\}$ form a $(p^s; 1; p^t)_{p^r}[p^{s-(r-t)}]$-PMF, each sequence of which has weight $0 \bmod p^r$. If $t \geq r$ then we come under case *ii)* of Theorem 19. In this case there is a $(p^s; 2; p^{t-r})_{p^{(2c-r)}}$-PF by Theorem 4.6 of [18]. Finally, the simplest way to find a $(p^s; 1; p^r)_{p^r}[p^s]$-PMF, each sequence of which has weight $0 \bmod p^r$, is to use the sequences $[i]^{p^s}$ for each $0 \leq i < p^r$. In either case, an application of Theorem 19 finishes the proof. □

It is of course possible to combine two-dimensional PFs over different prime-power alphabets to produce PFs over general alphabets: suppose C has prime factorisation $C = \prod_{i=1}^k p_i^{c_i}$ and let $R = \prod_{i=1}^k p_i^{r_i}$, $S = \prod_{i=1}^k p_i^{s_i}$ and $T = \prod_{i=1}^k p_i^{4c_i - r_i - s_i}$, for some r_i, s_i with $0 \leq r_i + s_i \leq 4c_i$ and $R > u$, $S > v$. Then the proof of the following corollary of Theorem 6 is a straightforward generalisation of [18, Lemma 5.1]).

Corollary 20 *Suppose that C, R, S and T are as above and*

$$p_i^{r_i} > 2 \quad and \quad p_i^{s_i} > 2, \quad 1 \leq i \leq k.$$

Then there exists an $(R, S; 2, 2; T)_C$-PF.

This corollary takes us quite some way to answering the existence question for two-dimensional PFs with 2×2 windows. We ask: is it possible to generalise the techniques of Sections 3, 4 and 5 of this paper to settle this question?

We finish with open questions about a much more general problem. Let the *dimension d* be an integer, the *period* $\vec{R} = \langle R_1, \ldots, R_d \rangle$ be a vector of integers, the *window* $\vec{W} = \langle W_1, \ldots, W_d \rangle$ be a vector of integers, the *size T* and *base C* be integers, and the *modulus* $\vec{M} = \langle M_1, \ldots, M_d \rangle$ be a vector of integers satisfying $M_i | R_i$ for each i. Let $R = \prod_i R_i$, $W = \prod_i W_i$, and $M = \prod_i M_i$, and define an $(\vec{R}; \vec{W}; T)_C^d[\vec{M}]$-PMF analogously to Definition 16. Then the following necessary conditions are straightforward to establish.

Lemma 21 *Suppose there exists an $(\vec{R}; \vec{W}; T)_C^d[\vec{M}]$-PMF. Then*

i) $RT = C^W M$, and

ii) for each $1 \le i \le d$, either

      a) $M_i = 1$ and $R_i \ge W_i$ or

      b) $M_i > 1$ and $R_i > W_i$.

It is tempting to conjecture that these conditions are also sufficient for the existence of PMFs in arbitrary dimension d. Conjecture 2.5 of [15] is that the same is true in one dimension, but even this special case is still open. Notice that settling this question for arbitrary d includes as special cases settling Conjecture 5.4 of [10] and the main conjecture of [20] (both on de Bruijn Tori in two dimensions). Also relevant is Conjecture 5.5 of [10]. A few constructions for PMFs in two dimensions can be found in [20] and for higher-dimensional de Bruijn Tori in [13]. Clearly, there is much scope for future work in this area.

Acknowledgement

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