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The Dimension of Interior Levels of the Boolean Lattice

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Abstract. Let $P(k,r;n)$ denote the containment order generated by the k-element and r-element subsets of an n-element set, and let $d(k,r;n)$ be its dimension. Previous research in this area has focused on the case $k = 1$. $P(1, n-1; n)$ is the standard example of an n-dimensional poset, and Dushnik determined the value of $d(1,r;n)$ exactly, when $r \geqslant 2\sqrt{n}$. Spencer used the Erdös–Szekeres theorem to show that $d(1,2;n) \sim \lg\lg n$, and he used the concept of scrambling families of sets to show that $d(1,r;n) = \Theta(\lg\lg n)$ for fixed r. Füredi, Hajnal, Rödl and Trotter proved that $d(1,2;n) = \lg\lg n + (1/2 + o(1))\lg\lg\lg n$. In this paper, we concentrate on the case $k \geqslant 2$. We show that $P(2, n-2; n)$ is $(n-1)$-irreducible, and we investigate $d(2,r;n)$ when $r \geqslant 2\sqrt{n-1}$, obtaining the exact value for almost all r.

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1. Introduction

For a positive integer n and integers k and r, with $0 \leqslant k < r \leqslant n$, let $\mathcal{K}$ and $\mathcal{R}$ denote (respectively) the families of all k-element and all r-element subsets of $\{1, 2, \ldots, n\}$. Then let $P(k,r;n)$ denote the containment order on $\mathcal{K} \cup \mathcal{R}$. The principal goal of this paper is to investigate the dimension of $P(k,r;n)$, concentrating on the case $k \geqslant 2$.

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In the remainder of this section, we present some background material and give the relevant notation and terminology. In Section 2, we discuss related results for the case $k = 1$. This discussion serves to motivate our results which we state in Section 3. In Section 4, we present some preparatory material on linear extensions, and we give proofs of our principal results in Section 5. In Section 6, we comment on some recent related research and discuss some open problems.

Recall that a family $\mathcal{F} = \{L_1, \ldots, L_t\}$ of linear extensions of an ordered set P is called a *realizer* of P if $\bigcap \mathcal{F} = P$, i.e., $x < y$ in P if and only if $x < y$ in L_i, for each $i = 1, 2, \ldots, t$. Dushnik and Miller [3] defined the *dimension* of P as the least positive integer t for which P has a realizer $\mathcal{F} = \{L_1, \ldots, L_t\}$ of cardinality t. In this paper, we will use the well known alternative formulation of dimension in terms of *critical pairs*. An incomparable pair (x, y) in an ordered set P is a *critical pair* if

(1) $u < y$ in P whenever $u < x$ in P,
(2) $x < w$ in P whenever $y < w$ in P.

A linear extension L *reverses* the critical pair (x, y) if $y < x$ in L. We denote the set of all critical pairs of P by crit(P), and we denote by crit(L, P) the set of critical pairs reversed by L. A family $\mathcal{F}$ of linear extensions of P is a realizer of P if any only if $\bigcup\{\text{crit}(L, P)\colon\ L \in \mathcal{F}\} = \text{crit}(P)$, i.e., every critical pair of P is reversed by some linear extension in $\mathcal{F}$. So the dimension of P is just the least positive integer t for which there exists a family $\mathcal{F} = \{L_1, \ldots, L_t\}$ of linear extensions of P reversing all critical pairs. We refer the reader to [9] for additional background material on dimension theory.

In this paper, we denote by $d(k, r; n)$ the dimension of the containment order $P(k, r; n)$. Note that $d(0, n; n) = 1$; $d(0, r; n) = 2$, if $0 < r < n$; and $d(k, n; n) = 2$, if $0 < k < n$. So in the remainder of this paper, we will always assume $0 < k < r < n$, and now the set of critical pairs of $P(k, r; n)$ is $\{(K, R) \in \mathcal{K} \times \mathcal{R}\colon\ K \not\subset R\}$.

2. The Case $k = 1$

Previous research on the dimension of the containment order $P(k, r; n)$ has focused on the case $k = 1$. In this section, we provide a brief summary of this work. In their seminal 1941 paper introducing the concept of dimension, Dushnik and Miller [3] proved that if $n \geqslant 3$, then $d(1, n-1; n) = n$; and the ordered set $P(1, n-1; n)$ is now known as the *standard example* of an n-dimensional poset. For $n \geqslant 3$, $P(1, n-1; n)$ is n-irreducible, i.e., it has dimension n and the removal of any element leaves a subposet of dimension $n - 1$.

In [2], Dushnik investigated $d(1, r; n)$ when r is relatively large in comparison to n and obtained the following two inequalities.

THEOREM 2.1 (Dushnik). *Let n, j and r be positive integers.*

(1) *If $\lfloor \frac{n}{j} \rfloor + j - 2 \leqslant r < n$, then $d(1, r; n) \geqslant n - j + 1$.*

(2) *If* $j \geqslant 2$, $1 < r < \lfloor \frac{n}{j-1} \rfloor + j - 3$ *and* $n \geqslant (j-1)^2$, *then* $d(1, r; n) \leqslant n - j + 1$.

Dushnik [2] observed that the two inequalities in Theorem 2.1 yield the exact value of $d(1, r; n)$ when r is relatively large. To be more precise, let $n \geqslant 4$ and let j_n be the largest positive integer for which $n \geqslant (j-1)^2$, and

$$\left\lfloor \frac{n}{j} \right\rfloor + j - 2 < \left\lfloor \frac{n}{j-1} \right\rfloor + j - 3,$$

for all j with $2 \leqslant j \leqslant j_n$. Then let

$$r_n = \left\lfloor \frac{n}{j_n} \right\rfloor + j_n - 2.$$

Then $j_n \sim \sqrt{n}$ and $r_n \sim 2\sqrt{n}$. Furthermore, it is easy to see that Theorem 2.1 yields the exact value of $d(1, r; n)$, for all r with $r_n \leqslant r < n$.

In view of the asymptotic values of j_n and r_n, Dushnik made the following observations explicit in [2].

COROLLARY 2.2 (Dushnik).

(1) $d(1, 2m-3; m^2-2) = d(1, 2m-3; m^2-1) = m^2 - m$, *for every* $m \geqslant 3$;
(2) $d(1, 2m-2; m^2+m-1) = d(1, 2m-2; m^2+m) = m^2$, *for every* $m \geqslant 2$.

Corollary 2.2 was extended slightly by Trotter [8].

THEOREM 2.3 (Trotter).

(1) $d(1, 2m-3; m^2) = d(1, 2m-3; m^2+1) = m^2 - m$, *for every* $m \geqslant 3$;
(2) $d(1, 2m-3; m^2+2) = m^2 - m + 1$, *for every* $m \geqslant 3$;
(3) $d(1, 2m-2; m^2+m+1)m^2$, *for every* $m \geqslant 2$;
(4) $d(1, 2m-2; m^2+m+2) = m^2 + 1$, *for every* $m \geqslant 3$.

Trotter also gave *ad-hoc* arguments to complete a table of values for $d(1, r; n)$, for all n and r with $1 < r < n \leqslant 14$.

In [7], Spencer used the Erdös–Szekeres theorem, the Erdös–Ko–Rado theorem and the concept of scrambling families to show

$$\lg \lg n \leqslant d(1, 2; n) \leqslant \lg \lg n + \big(1/2 + o(1)\big) \lg \lg \lg n.$$

More generally, Spencer [7] used probabilistic methods to show $d(1, r; n) = \Theta(\log \log n)$, for fixed r. In [5], Füredi, Hajnal, Rödl, and Trotter establish combinatorial connections with shift graphs and interval orders to show that

$$d(1, 2; n) = \lg \lg n + \big(1/2 + o(1)\big) \lg \lg \lg n.$$

3. The Case $k \geqslant 2$

In this paper, we will concentrate on the problem of determining $d(k, r; n)$ when $k \geqslant 2$. Surprisingly, very little of what is known for the case $k = 1$ carries over, and most of this is extracted from the following elementary proposition.

PROPOSITION 3.1. *Let n, k, and r be integers with $0 < k < r < n$.*

(1) *Let $d(k, k+1, \ldots, r; n)$ be the dimension of the poset $P(k, k+1, \ldots, r; n)$ generated by all t-element subsets of an n-element set with $k \leqslant t \leqslant r$. Then*

$$d_n(k, k+1, \ldots, r) = d(k, r; n).$$

(2) *If $0 < k_1 \leqslant k_2 < r_2 \leqslant r_1 < n$, then $d(k_2, r_2; n) \leqslant d(k_1, r_1; n)$.*

(3) *If $0 \leqslant t \leqslant k < r \leqslant n$, then $d(k-t, r-t; n-t) \leqslant d(k, r; n)$.*

(4) *$d(k, r; n) = d(n-r, n-k; n)$.*

Proof. The proof of part (1) rests on the fact that the $P(k, r; n)$ is a subposet of $P(k, k+1, \ldots, r; n)$, but all the critical pairs of $P(k, k+1, \ldots, r; n)$ belong to $P(k, r; n)$. Part (2) is an immediate corollary to part (1). Part (3) follows from the observation that when $0 < t < k$, $P(k-t, r-t; n-t)$ is isomorphic to the subposet of $P(k, r; n)$ generated by the sets containing $\{1, 2, \ldots, t\}$. Finally, part (4) follows from the fact that $P(n-r, n-k; n)$ is the dual of $P(k, r; n)$. □

It follows from Dushnik's Theorem 2.1 and Proposition 3.1 that $n-2 \leqslant d(2, n-2; n) \leqslant n-1$, for all $n \geqslant 5$. We will obtain the exact value of $d(2, n-2; n)$ with the following result.

THEOREM 3.2. *For $n \geqslant 5$, $d(2, n-2; n) = n-1$.*

The argument used to prove Theorem 3.2 yields the following structural result as an easy corollary.

COROLLARY 3.3. *For $n \geqslant 5$, $P(2, n-2; n)$ is $(n-1)$-irreducible, i.e., $d(2, n-2; n) = n-1$ and the removal of any element from $P(2, n-2; n)$ leaves a subposet of dimension $n-2$.*

Theorem 3.2 may at first glance seem to be a somewhat specialized result. However, this is not entirely the case. We will develop an analogue of Dushnik's Theorem 2.1 for the case $k = 2$, which includes Theorem 3.2 as an exceptional case. The statement of the theorem uses the notation introduced just after the statement of Theorem 2.1.

THEOREM 3.4. *Let n and j be positive integers with $n \geqslant 5$, $2 \leqslant j \leqslant j_{n-1}$ and $\lfloor \frac{n-1}{j-1} \rfloor \geqslant 3$. Then let r be the unique positive integer so that*

$$\left\lfloor \frac{n-1}{j} \right\rfloor + j - 2 \leqslant r - 1 < \left\lfloor \frac{n-1}{j-1} \right\rfloor + j - 3.$$

(1) *If* $n-1 \not\equiv 0 \bmod (j-1)$, *then* $d(1,r-1;n-1) = n-j = d(2,r;n)$.
(2) *If* $n-1 \equiv 0 \bmod (j-1)$, *then* $d(1,r-1;n-1) = n-j \leqslant d(2,r;n) \leqslant n-j+1$.

Theorem 3.4 determines $d(2,r;n)$ exactly, for all r with $r'_{n-1} \leqslant r < n$, except for those values of r for which $r = j-3+(n-1)/(j-1)$, for some j with $2 \leqslant j \leqslant j_{n-1}$. For these values of r, we obtain an estimate of $d(2,r;n)$ accurate to within 1. Note that for $n \geqslant 5$, one of the exceptional values of r is $n-2$ which corresponds to the value $j=2$. In other words, Theorem 3.4 yields only $d(1,n-3;n-1) = n-2 \leqslant d(2,n-2;n) \leqslant n-1$.

4. Linear Orders on Families of Sets

Let $\mathcal{C}$ be a family of sets, let L be a linear extension of the containment order on $\mathcal{C}$ and let $x \in \bigcup \mathcal{C}$. Then let

$$\mathrm{center}(L) = \left\{x \in \bigcup \mathcal{C}\colon\ A > B \text{ in } L \text{ for all } A, B \in \mathcal{C} \text{ with } x \in A - B\right\}.$$

Note that center(L) may be empty; it may also contain more than one point. However, center(L) contains at most one point when L is a linear extension of $P(k,r;n)$ and $0 < k < r < n$.

In what follows, we will require a generalization of the preceding notion. Let $\mathcal{L}$ denote the family of all linear extensions of containment orders. We define a subfamily $\mathcal{P} \subseteq \mathcal{L}$, called the family of *pseudo-lex* linear extensions. The definition is recursive. First, any linear extension $L \in \mathcal{L}$ of a containment order on a family $\mathcal{C}$ of sets is pseudo-lex if $A > B$ in L whenever $A, B \in \mathcal{C}$ and $|A| > |B|$. Also, a linear extension L on $\mathcal{C}$ is pseudo-lex if there exists an element $a \in \bigcup \mathcal{C}$ so that

(1) $\mathcal{C}_1 = \{A \in \mathcal{C}\colon\ a \in A\} \neq \mathcal{C} \neq \mathcal{C}_2 = \{A \in \mathcal{C}\colon\ a \notin A\}$;
(2) $A > B$in L for every $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$.
(3) The restriction of L to $\mathcal{C}_i$ is pseudo-lex, for $i = 1, 2$.

We close this section with some remarks concerning the set of critical pairs reversed by a linear extension. We call L a *maximal* linear extension of P if there is no linear extension L' with $\mathrm{crit}(L,P) \subsetneq \mathrm{crit}(L',P)$. Clearly, if P has dimension t, then there exists a realizer $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ of P with L_i maximal, for all $i = 1, 2, \ldots, t$. We call L a *maximum* linear extension of P if there is no linear extension of P which reverses more critical pairs than L.

In [6], Hurlbert derived a formula for the number of critical pairs reversed by a maximum linear extension of $P(k,r;n)$.

THEOREM 4.1 (Hurlbert). *For all integers n, k and r with $0 < k < r < n$, let $m(k,r;n)$ denote the number of critical pairs reversed by a maximum linear* *extension of $P(k,r;n)$. Then*

$$m(k,r;n)=\sum_{i=0}^{k-1}\sum_{j=0}^{r-1}\binom{i+j}{i}\binom{(n-1)-(i+j)}{(k-1)-i}\binom{(n-1)-(i+j)}{(r-1)-j}.$$

Furthermore, if L is any maximum linear extension of $P(k,r;n)$, then there exists a pseudo-lex linear extension which reverses the same set of critical pairs as L.

In the next section, we will need the special case of Theorem 4.1 for the poset $P(2,n-2;n)$, i.e., a linear extension of $P(2,n-2;n)$ reverses at most n^2-1 critical pairs.

In the arguments which follow, we denote the complement of a set A by A^c. For distinct integers $x,y\in\{1,2,\dots,n\}$, let $L=L(x,y)$ denote a maximal linear extension of $P(2,n-2;n)$ in which

(1) $K>R$ in L, for all $K\in\mathcal{K}$, $R\in\mathcal{R}$ with $1\in K-R$.
(2) $\{1,2\}>\{1,3\}>\cdots>\{1,n\}$ in L.
(3) The lowest two elements of $\mathcal{R}$ in L are $\{1,x\}^c$ and $\{1,y\}^c$ with $\{1,x\}^c>\{1,y\}c$ in L.

Note that center$(L(x,y))=\{1\}$ and that $L(x,y)$ reverses exactly n^2-1 critical pairs. Also note that $L(x,y)$ is pseudo-lex.

The proof of Theorem 3.1 depends on the following two lemmas.

LEMMA 5.1. *Let $n\geqslant 5$ and let L be a maximal linear extension of $P(2,n-2;n)$. If* center$(L)=\varnothing$, *then L reverses at most $6n-9$ critical pairs.*

Proof. Let L be any maximal linear extension of $P=P(k,r;n)$ with center$(L)=\varnothing$. We show that L reverses at most $6n-9$ critical pairs. We may assume that the greatest element of $\mathcal{K}$ in L is $\{1,2\}$ and that the greatest element in $\{\{1,y\}\colon 3\leqslant y\leqslant n\}\cup\{\{2,y\}\colon 3\leqslant y\leqslant n\}$ is $\{1,3\}$. We may further assume that $\{1,2\}>\{1,3\}>\{1,4\}>\cdots>\{1,n\}$ in L. Since center$(L)=\varnothing$, there must be some $\{x,y\}\in\mathcal{K}$ with $2\leqslant x,y\leqslant n$ and $\{1,2\}>\{x,y\}>\{1,n\}$ in L. To see this, observe that if the $n-1$ largest elements of $\mathcal{K}$ in L are $\{1,2\},\{1,3\},\dots,\{1,n\}$, then the maximality of L implies that $1\in$ center(L).

Without loss of generality, $\{x,y\}$ is the largest element of $\mathcal{K}$ in L with $1\notin\{x,y\}$. We may further assume that $\{1,x\}^c>\{1,y\}^c$ in L. Now we claim that the maximality of L requires $\{x,y\}>\{1,4\}$ in L. Suppose to the contrary that there is some k with $r\leqslant k<n$ for which $\{1,k\}<\{x,y\}<\{1,k+1\}$ in L. Observe that crit$(L,P)\subseteq A_1\cup A_2\cup A_3\cup A_4\cup A_5$, where

(1) $A_1=\{(\{1,2\},R)\colon R\in\mathcal{R}$ and $\{1,2\}\not\subseteq R\}$;
(2) $A_2=\{(\{1,j\},R)\colon 3\leqslant j\leqslant k,\ R\in\mathcal{R}$ and $1\in R^c\}$;
(3) $A_3=\{(\{x,y\},\{1,v\}^c)\colon v\in\{x,y\}\}$;
(4) $A_4=\{(K,\{1,v\}^c)\colon K\in\mathcal{K},\ 1\in K,\ v\in\{x,y\}\}$;

(5) $A_5 = \{(K, \{1, y\}^c)\colon\ K \in \mathcal{K},\ K \cap \{1, y\} \neq \varnothing\}$.

It follows that $\mathrm{crit}(L, P) \subsetneq \mathrm{crit}(L', P)$, where $L' = L(x, y)$. The contradiction shows $\{x, y\} > \{1, 4\}$ in L.

Now suppose $\{1, 3\} > \{x, y\}$ in L. Now the only sets from $\mathcal{R}$ which can be under $\{x, y\}$ in L are $R_1 = \{2, 3\}^c$, $R_2 = \{1, x\}^c$ and $R_3 = \{1, y\}^c$. Furthermore, in order for $\{2, 3\}^c$ to be under $\{x, y\}$, we must have $\{2, 3\} \cap \{x, y\} \neq \varnothing$, and in any case, we know that $\{2, 3\}^c > \{1, 4\}$ in L. It is easy to see that L reverses at most $3 + (2n - 3) + n = 3n - 3$ critical pairs having one of R_1, R_2, and R_3 as second coordinate. There are $n - 3$ sets from $\mathcal{R}$ which are between $\{1, 2\}$ and $\{1, 3\}$ in L. Furthermore, there are at least $n - 3$ sets from $\mathcal{R}$ between $\{1, 3\}$ and $\{x, y\}$; in fact, there are $n - 2$ if $\{x, y\} \cap \{2, 3\} = \varnothing$. This implies that L reverses at most $3n - 3 + (n - 3) \cdot 1 + (n - 3) \cdot 2 = 6n - 9$ critical pairs.

Now suppose that $\{x, y\} > \{1, 3\}$ in L. Then $2 < x, y \leqslant n$. Now there are exactly 4 sets from $\mathcal{R}$ which are under $\{x, y\}$ in L. These are $R_1 = \{1, x\}^c$, $R_2 = \{1, y\}^c$, $R_3 = \{2, x\}^c$, and $R_4 = \{2, y\}^c$. At most three of these four sets can be under any additional elements of $\mathcal{K}$. Furthermore, since no three of R_1^c, R_2^c, R_3^c, and R_4^c have a common point, it follows that L reverses at most $(2n - 3) + n + 3 + 2$ critical pairs having one of R_1, R_2, R_3, and R_4 as second coordinate. In addition L reverses $2n - 7$ other critical pairs, all having $\{1, 2\}$ as first coordinate. This implies that L reverses at most $5n - 5$ critical pairs altogether. □

For completeness, we include here a proof of the following lemma which gives the number of critical pairs reversed by a maximum linear extension of $P(2, n - 2; n)$. As remarked previously, the general result is given in Theorem 4.3 and is due to Hurlbert [3].

LEMMA 5.2. *If $n \geqslant 5$, a maximum linear extension of $P(2, n-2; n)$ reverses $n^2 - 1$ critical pairs.*

Proof. We have already demonstrated a linear extension of $P(2, n-2; n)$ reversing $n^2 - 1$ critical pairs. Now let L be a maximal linear extension of $P(2, n-2; n)$, with $n \geqslant 5$. We show L reverses at most $n^2 - 1$ critical pairs. If $\mathrm{center}(L) = \varnothing$, then L reverses at most $6n - 9$ critical pairs. Since $6n - 9 < n^2 - 1$, when $n \geqslant 5$, we may assume that $\mathrm{center}(L) \neq \varnothing$. Without loss of generality, $\mathrm{center}(L) = \{1\}$. Then L reverses all critical pairs in the set $S_1 = \{(K, R) \in \mathcal{K} \times \mathcal{R}\colon\ 1 \in K - R\}$. Note that there are $(n - 1)^2$ critical pairs in S_1.

Without loss of generality the highest two sets from $\mathcal{K}$ in L are $\{1, 2\}$ and $\{1, 3\}$ with $\{1, 2\} > \{1, 3\}$ in L. Now choose $x, y \in \{2, 3, \ldots\}$ so that the lowest two sets from $\mathcal{R}$ in L are $\{1, x\}^c$ and $\{1, y\}^c$ with $\{1, x\}^c > \{1, y\}^c$ in L. Then L reverses all critical pairs with $\{1, 2\}$ as first coordinate; L also reverses all critical pairs with $\{1, y\}^c$ as second coordinate. This yields $2(n - 2) = 2n - 4$ additional critical pairs reversed by L. The only other critical pairs reversed by L are $(\{1, 3\}, \{2, 3\}^c)$ and $(\{x, y\}, \{1, x\}^c)$. This implies that L reverses $(n-1)^2 + (2n-4) + 2 = n^2 - 1$ critical pairs. □

Proof of Theorem 3.2. First, we note that $P(2, n-2; n)$ has

$$\binom{n}{2}(2n-3) = \frac{1}{2}n(n-1)(2n-3)$$

critical pairs. Suppose that $d(2, n-2; n) \leqslant n-2$ and that $\mathcal{F}$ is a realizer with $|\mathcal{F}| = n-2$. Without loss of generality, we may assume that each $L \in \mathcal{F}$ is maximal. Now in view of Theorem 4.5, each linear extension of $\mathcal{F}$ reverses at most $n^2 - 1$ critical pairs.

Now suppose that there is some $L \in \mathcal{F}$ for which center$(L) = \varnothing$. Then by Lemma 5.2, L reverses at most $6n - 9$ critical pairs. Since each critical pair of $P(2, n-2; n)$ is reversed by some linear extension in $\mathcal{F}$, we must have

$$(n-3)(n^2-1) + (6n-9) \geqslant \frac{1}{2}n(2n-1)(2n-3).$$

But this inequality fails when $n \geqslant 5$.

It follows that we may assume that center$(L) \neq \varnothing$, for each $L \in \mathcal{F}$. Now choose distinct elements $x, y \in \{1, 2, \ldots, n\}$ so that $\{x, y\} \cap$ center$(L) = \varnothing$, for every $L \in \mathcal{F}$. It follows that $\{x, y\}^c > \{x, y\}$ in L, for every $L \in \mathcal{F}$. This contradicts the assumption that $\mathcal{F}$ is a realizer. $\square$

There is an alternate approach to proving Theorem 3.2, which is of independent interest. Call a critical pair $(K, R) \in$ crit$(P(2, n-2; n))$ *regular* if $K \cap R \neq \varnothing$.

LEMMA 5.3. *A maximal linear extension of* $P(2, n-2; n)$ *reverses at most* $n^2 - n$ *regular critical pairs. Furthermore, if* L *is any maximal linear extension of* $P(2, n-2; n)$ *reversing exactly* $n^2 - n$ *regular critical pairs, then* center$(L) \neq \varnothing$.

Proof. Let L be a maximal linear extension of $P(2, n-2; n)$. First suppose that center$(L) = \varnothing$. Then the proof of Lemma 5.2 shows that L reverses at most $6n - 9$ critical pairs. Furthermore, L reverses at most $5n - 5$ critical pairs if the largest two sets from $\mathcal{K}$ are disjoint. Now assume that the largest two sets from $\mathcal{K}$ in L are $\{1, 2\}$ and $\{1, 3\}$. Then L reverses $(\{1, 2\}, \{1, 2\}^c)$ and $(\{1, 3\}, \{1, 3\}^c)$, neither of which is regular. We conclude that L reverses at most $6n - 11$ regular pairs. Observe that $6n - 11 < n^2 - n$, when $n \geqslant 5$.

Now let L be any maximal linear extension with center$(L) \neq \varnothing$. Let center$(L) = \{1\}$. Then L reverses the $n - 1$ critical pairs in $\{(K, K^c)\colon K \in \mathcal{K},\ 1 \in K\}$, none of which are regular. Since L reverses at most $n^2 - 1$ critical pairs, it reverses at most $n^2 - n$ regular pairs. $\square$

We note that Theorem 3.2 follows from Lemma 5.3 as there are $n(n-1)(n-2)$ regular critical pairs, so if $\mathcal{F} = \{L_1, L_2, \ldots, L_{n-2}\}$ is a realizer of $P(2, n-2; n)$, and L_i is maximal, for $i = 1, 2, \ldots, t$, then each $L_i \in \mathcal{F}$ has a nonempty center. This yields a contradiction just as before.

Proof of Corollary 3.3. Taking advantage of duality and symmetry, it suffices to show that removing $\{n-1,n\}$ from $P(2,n-2;n)$ leaves a subposet with dimension at most $n-2$. Consider the family $\mathcal{F} = \{L_i\colon 1 \leqslant i \leqslant n\}$ of pseudo-lex linear extensions of $P(2,n-2;n)$ defined as follows. For each $i = 1,2,\ldots,n-2$, $A > B$ in L_i if any of the following conditions holds:

(1) $i \in A - B$;
(2) $i \in A \cap B$ and $n-1 \in A - B$;
(3) $i \in A \cap B$, $n-1 \notin A \cup B$ and $n \in A - B$;
(4) $i \notin A \cup B$ and $n-1 \in A - B$;
(5) $i \notin A \cup B$ and $n-1 \in A \cap B$ and $n \in A - B$;
(6) $i, n-1 \notin A \cup B$ and $n \in A - B$.

It is easy to see that the linear extension in $\mathcal{F}$ reverse every critical pair in $P(2,n-2;n)$ *except* $(\{n-1,n\},\{n-1,n\}^c)$. Furthermore, the restrictions of these linear orders to the subposet formed by removing $\{\{n-1,n\}\}$ from $P(2,n-2;n)$ form a realizer. □

Proof of Theorem 3.4. The conclusion that $d(2,r;n-1) \geqslant d(1,r-1;n-1) = n-j$ follows from Proposition 3.1 and Theorem 2.1. We need only show that the upper bounds claimed for $d(2,r;n)$ are valid.

Let $s = \lfloor \frac{n-1}{j-1} \rfloor$, so that $n-1 = s(j-1)+t$ and $0 \leqslant t < j-1$. We first show that $s \geqslant 3$. Suppose to the contrary that $s = 2$. Then $j \geqslant 2$, $n \geqslant 5$ and $n-1 \geqslant (j-1)^2$ require $n = 5$ and $j = 3$. However, by inspection $j_4 = 2$. We conclude that $s \geqslant 3$.

Next we note that the inequality

$$r-1 < \left\lfloor \frac{n-1}{j-1} \right\rfloor + j - 3$$

can be rewritten as

$$r < s + j - 2. \tag{$*$}$$

Now observe that $n = (j-1)(s-1) + (j-1) + t + 1$. We can then partition $\{1,2,\ldots,n\}$ into 4 pairwise disjoint sets $F \cup M \cup G \cup N$ with $|F| = (j-1)(s-1)$, $|M| = j-1$, $|G| = t$, $|M| = j-1$ and $|N| = 1$. Without loss of generality, $M = \{1,2,\ldots,j-1\}$ and $N = \{n\}$. Relabel the elements in F as $\{f(x,y)\colon 1 \leqslant x \leqslant j-1,\ 1 \leqslant y \leqslant s\}$. If $n-1 \equiv 0 \bmod (j-1)$, then $G = \varnothing$. If $n \not\equiv 0 \bmod (j-1)$, then $|G| = t > 0$ and we relabel the elements of G as $\{g(z)\colon 1 \leqslant z \leqslant t\}$.

We now define a family $\mathcal{F}_1 = \{L(f)\colon f \in F\}$ of linear extensions of $P(2,r;n)$ as follows. Let $f = f(x,y) \in F$. Then $A > B$ in $L(f)$ if any of the following conditions holds:

(1) $f \in A - B$;
(2) $f \in A \cap B$ and $n \in A - B$;

(3) $f \in A \cap B$, $n \notin A \cup B$ and $x \in A - B$;
(4) $f \in A \cap B$, $n, x \notin A \cup B$, $y \leqslant j - 1$, and $y \in A - B$;
(5) $f \notin A \cup B$ and $x \in A - B$;
(6) $f \notin A \cup B$, $x \in A \cap B$ and $n \in A - B$;
(7) $f, n \notin A \cup B$, $x \in A \cap B$, $y \leqslant j - 1$ and $y \in A - B$;
(8) $f, x \notin A \cup B$, $y \leqslant j - 1$ and $y \in A - B$;
(9) $f, x \notin A \cup B$, $y \leqslant j - 1$, $y \in A \cap B$ and $n \in A - B$.

If $G \neq \varnothing$, we define a family $\mathcal{F}_2 = \{L'(g)\colon g \in G\}$ of linear extensions of $P(2, r; n)$ as follows. Let $g = g(z) \in G$. Then $A > B$ in $L'(g)$ if any of the following conditions holds:

(1) $g \in A - B$;
(2) $g \in A \cap B$ and $n \in A - B$;
(3) $g \notin A \cup B$ and $n \in A - B$.

Next, let L_n be any linear extension of $P(2, r; n)$ with $A > B$ in L_n if $n \in A - B$. Let $\mathcal{F} = \mathcal{F}_1 \cup \{L_1\}$, if $G = \varnothing$, and let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, if $G \neq \varnothing$. Then $|\mathcal{F}| = n - j + 1$, if $n - 1 \equiv 0 \bmod (j - 1)$, and $|\mathcal{F}| = n - j$, if $n - 1 \not\equiv 0 \bmod (j - 1)$. We now proceed to show that $\mathcal{F}$ is a realizer of $P(2, r; n)$. We argue by contradiction. Let (K, R) be a critical pair in $P(2, r; n)$ which is not reversed by any linear extension in $\mathcal{F}$. We may assume $K = \{u, v\}$ and $u \notin R$. If $u \in F$, then (K, R) is reversed in $L(u)$. Similarly, if $u \in G$, then (K, R) is reversed in $L'(u)$. We conclude that $u \in M \cup N$.

First we suppose that $u \in N$, i.e., $u = n$. If $n \equiv 0 \bmod (j - 1)$, then (K, R) is reversed in L_n, so we conclude that $n \not\equiv 0 \bmod (j - 1)$. If $v \in F$, then (K, R) is reversed in $L(v)$. Similarly, if $v \in G$, then (K, R) is reversed in $L'(v)$. So we conclude that $v \in M$. Then $f(v, y) \in R$, for every $y = 1, 2, \ldots, s - 1$. Now let $x \in M - \{v\}$. Then (K, R) is reversed in $L(f(x, v))$ unless $R \cap \{f(x, v), x\} \neq \varnothing$. Now let $g \in G$. Then (K, R) is reversed in $L'(g)$ unless $g \in R$. These statements imply that R contains at least $(s - 1) + (j - 2) + |G|$ elements. But, since $G \neq \varnothing$, this requires $r \geqslant s + j - 2$, which contradicts inequality $(*)$.

The contradiction shows that $u \in M$. Now suppose that $v = n$. Then the argument given above implies $v \in R$. Then $f(u, y) \in R$, for all $y = 1, 2, \ldots, s - 1$. Furthermore, R must intersect $\{f(x, u), x\}$, for every $x \in M - \{u\}$. This requires $r \geqslant 1 + (s - 1) + (j - 2)$, which again contradicts $(*)$.

Now suppose that $u \in M$ and $v \neq n$. As before, we observe that $f(u, y) \in R$, for every $y = 1, 2, \ldots, s - 1$, and R intersects $\{f(x, u), x\}$, for every $x \in M - \{u\}$. These two requirements account for all elements of R. In particular, we note that $n \notin R$. Furthermore, we note that R contains exactly one element of $\{f(x, u), x\}$, for every $x \in M - \{u\}$. Also, we note that $v \notin G$. Now suppose $v \in F$. Then $v \in R$, so either $v = f(u, y)$, for some $y \in \{1, 2, \ldots, s - 1\}$, or $f = f(x, u)$, for some $x \in M - \{u\}$. But either of these two possibilities implies $K > R$ in $L(v)$.

It remains to consider the case where $u, v \in M$. Again, $f(u, y) \in R$, for every $y = 1, 2, \ldots, s - 1$, and R contains exactly one element from $\{f(x, u), x\}$, for every

$x \in M - \{u\}$. If $v \in R$, then $f(v, u) \notin R$, so that $K > R$ in $L(v)$. The contradiction shows $v \notin R$. Thus $f(x, v) \in R$, for every $v \in M - \{v\}$. Now the fact that u and v are distinct requires $s = 2$. The contradiction completes the proof. □

6. Recent Results and Open Problems

We conjecture that Theorem 3.4 is tight, i.e., if $n \geqslant 5$, $j \geqslant 2$, $s \geqslant j - 1$, and $n - 1 = s(j - 1)$, then $d(2, s + j - 3; n) = n - j + 1$. If true, this conjecture shows that the formula $d(2, n - 2; n) = n - 1$, for $n \geqslant 5$, is just a special case of a more general result. One possible approach to resolving this conjecture is to develop an analogue of Lemma 5.3 for $2 < r < n - 2$.

The results of this paper suggest that it may be possible to develop a general formula for $d(k, r; n)$ valid for almost all sufficiently large r. As further evidence for this belief, we comment that the second and third authors proved that $d(3, n-3; n) = n - 2$, for $n \geqslant 7$ and that this result has been extended by Füredi [3].

THEOREM 6.1 (Füredi). *If* $k \geqslant 3$ *and* $n > 200k^3$, *then* $d(k, n - k; n) = n - 2$.

In [1], Brightwell, Kierstead, Kostochka and Trotter prove the following results.

THEOREM 6.2. *If* $1 \leqslant k < n$, *then*

$$d(k, k + 1; n) \leqslant (6/\log 3) \log n.$$

THEOREM 6.3. *If* k, s *and* n *are positive integers with* $1 \leqslant k$, $3s \leqslant n$, *and* $k+s \leqslant n$, *then*

$$d(k, k + s; n) \leqslant d(1, 3s; n) + 18s \log n.$$

The general problem of determining $d(k, k + s; n)$ remains an interesting and challenging assignment.

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