

Cover Pebbling Numbers and Bounds for Certain Families of Graphs

Nathaniel G. Watson
Washington University in St. Louis
ngwatson@wustl.edu

Carl R. Yerger
Harvey Mudd College
cyerger@hmc.edu

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Abstract

Given a configuration of pebbles on the vertices of a graph, a *pebbling move* is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. The cover pebbling number of a graph, $\gamma(G)$, is the smallest number of pebbles such that through a sequence of pebbling moves, a pebble can eventually be placed on every vertex simultaneously, no matter how the pebbles are initially distributed. The cover pebbling number for complete multipartite graphs and wheel graphs is determined. We also prove a sharp bound for $\gamma(G)$ given the diameter and number of vertices of G .

Keywords: graph, pebbling, diameter, coverable

1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research and substantive generalizations. It was first introduced into the literature by Chung[1], and has been developed by many others including Hurlbert, who published a survey of pebbling results in [3]. Given a connected graph G , distribute k pebbles on its vertices in some configuration, C . Specifically, a *configuration* on a graph G is a function from

$V(G)$ to $\mathbb{N} \cup \{0\}$ representing an arrangement of pebbles on G . We call the total number of pebbles k the *size* of the configuration. A *pebbling move* is defined as the simultaneous removal of two pebbles from some vertex and addition of one pebble on an adjacent vertex. A pebble can be moved to a root vertex v if it is possible to place one pebble on v in a sequence of pebbling moves. We define the pebbling number, $\pi(G)$ to be the minimum number of pebbles needed so that for any initial distribution of pebbles, it is possible to move to any root vertex v in G .

The concept of cover solvability was introduced in [2]. We call a configuration on a graph *cover solvable* if, starting with this configuration, it is possible, through a sequence of pebbling moves, to simultaneously place one pebble on every vertex of the graph. The *cover pebbling number* of a graph, $\gamma(G)$, is defined as the smallest number such that every configuration of this size is cover solvable. One application in [2] for $\gamma(G)$ is based on a military application where troops must be distributed simultaneously.

In [2], the cover pebbling number for complete graphs, paths and trees is determined. Also, Hurlbert and Munyan [4] are preparing a proof that determines the cover pebbling number of the d -cube.

This paper will consider various questions related to cover pebbling, including open problem 9 in [2]. Section 2 describes the computation for calculating the cover pebbling number for complete multipartite graphs. In Section 3, we will compute the cover pebbling number of W_n , the wheel graph. We conclude the paper in Section 4 by constructing a tight upper bound for the cover pebbling number of graphs with specified diameter d and number of vertices n .

2 Complete Multipartite Graphs

Definition 2.1. For $s_1 \geq s_2 \geq \dots \geq s_r$, let K_{s_1, s_2, \dots, s_r} be the complete r -partite graph with s_1, s_2, \dots, s_r vertices in vertex classes c_1, c_2, \dots, c_r respectively.

Definition 2.2. For a complete r -partite graph $G = K_{s_1, s_2, \dots, s_r}$, let $\phi(G) = 4s_1 + 2s_2 + \dots + 2s_r - 3$.

Theorem 2.3. $\gamma(K_{s_1, s_2, \dots, s_r}) = \phi(G)$.

Proof. First, we will show that not every configuration of size $\phi(K_{s_1, s_2, \dots, s_r}) - 1$ on K_{s_1, s_2, \dots, s_r} is cover-solvable. Consider the case where all $\phi(K_{s_1, s_2, \dots, s_r}) - 1$ pebbles are on one vertex of c_1 , call it x . There are $k = s_2 + s_3 + \dots + s_r$ vertices that are distance 1 from x and $l = s_1 - 1$ vertices that are distance 2 from x . For the k vertices a distance 1 from x , $2k$ pebbles are required to cover these vertices, and for the l vertices at distance 2 from x , there are $4l$ pebbles

required to cover these vertices. We need one more pebble to remain on x , for a total of $2k + 4l + 1 = \phi(K_{s_1, s_2, \dots, s_r})$ pebbles required, which is one more than we have. Thus, this configuration is not cover-solvable.

Now suppose that there exists some complete r -partite graph K_{s_1, s_2, \dots, s_r} which has a configuration of size $\phi(K_{s_1, s_2, \dots, s_r})$ that is not cover-solvable. Among such graphs, choose one of minimal order (let it be $G' = K_{s'_1, s'_2, \dots, s'_r}$).

First, we will show that G' cannot be a star graph (that is, a $K_{s'_1, 1}$.) To see this, consider:

Definition 2.4. [2] Let T be a tree and let $V(T)$ be the vertex set of T . For $v \in V(T)$, define

$$s(v) = \sum_{u \in V(T)} 2^{d(u, v)},$$

where $d(u, v)$ denotes the distance from u to v , and let

$$s(T) = \max_{v \in V(T)} s(v).$$

In [2] it is shown that for any tree T , $\gamma(T) = s(T)$. Since G' is a tree, we can compute $\gamma(G')$ by evaluating $s(v)$ for all $v \in G'$ to obtain $s(G')$. If $v \in c_1$ then $s(v) = 4s'_1 - 1$, and if $v \in c_2$ then $s(v) = 2s'_1 + 1$. Thus, $s(G') = \gamma(G') = 4s'_1 - 1 = 4s'_1 + 2s'_2 - 3 = \phi(G')$. Hence, for a star, every configuration of size $\phi(G')$ is cover-solvable. Since G' is not a star, further suppose that for any G' , each complete multipartite subgraph G of G' is cover-solvable with $\phi(G)$ pebbles.

Notice that for any complete p -partite graph with $p \geq 2$ other than a star graph, the removal of a vertex from the graph leaves a subgraph that is a complete q -partite graph with $q \geq 2$. Since G' cannot be a star, for any vertex $v \in G'$, $G' - v$ is a complete r^* -partite graph with $r^* \geq 2$. Furthermore, since by our assumption of the minimality of G' , for any complete r -partite graph G smaller than G' , a configuration of size $\phi(G)$ or greater must be cover-solvable, and since clearly $\phi(G' - v) \leq \phi(G') - 2$, any configuration of size $\phi(G') - 2$ or greater on $G' - v$ is cover-solvable.

Let C be a configuration of size $\phi(G')$ on G' . Suppose $C(v) = 1$ or 2 for some $v \in G'$. Then C restricted to $G' - v$ is a configuration of size at least $\phi(G') - 2$ and thus is cover-solvable on $G' - v$. After we carry out the steps of the cover-solution of this subgraph, we will have cover-solved G' , contradicting our hypothesis.

Otherwise, if $C(v) = 0$ or $C(v) \geq 3$ for all $v \in G'$, choose some v' for which $C(v') = 0$ (if no such v' exists, we are done). Then consider the vertices of G' which are in different vertex classes of G' from v' . If at least one of these is

initially occupied, call it v'' . Then since $C(v'') \geq 3$, we can cover v' with pebbles from v'' , while leaving $\phi(G') - 2$ pebbles on $G' - v'$. Thus, the configuration of pebbles on G' after this move, restricted to the subgraph $G' - v'$ is cover solvable, and after we carry out the steps of the cover-solution of this subgraph, we will have cover-solved G' . Otherwise, all the vertices in the vertex classes of G' that are different than the vertex class from v' are empty. Thus, all pebbles are on vertices in the vertex class of v' , and in particular, some vertex w of this class has pebbles on it, so $C(w) \geq 2$. Thus, we can use pebbles on w to cover some vertex w' in another vertex class, as all these vertices are empty. Note that after this move, the configuration of pebbles on $G' - w'$ has size $\phi(G') - 2$, and thus this configuration restricted to the subgraph $G' - w'$ is cover-solvable. Again, after we carry out the steps of the cover-solution of this subgraph, we will have cover-solved G' .

□

3 The Wheel Graph

In this section, we will compute $\gamma(W_n)$, where W_n is the wheel graph. The wheel graph is composed of a cycle consisting of n vertices, v_1, \dots, v_n , which are all connected to a hub vertex, v_0 , for a total of $v = n + 1$ vertices.

Theorem 3.1. *For $n \geq 3$, $\gamma(W_n) = 4n - 5 = 4v - 9$.*

Proof. Consider the configuration of pebbles where all the pebbles are on one vertex of W_n , say x , that is not the hub. In this case, 2 pebbles are required to cover each of the three vertices adjacent to x , and 4 pebbles are required to cover each of the $n - 3$ vertices that are a distance of 2 away from x . The total number of pebbles required to cover-solve these vertices is $4n - 6$. However, we require one more pebble to place on x . Hence, $\gamma(W_n) \geq 4n - 5$.

To complete the proof, we will show that if there is some configuration of pebbles on W_n with at least $4n - 5$ pebbles, then the configuration is cover-solvable. Suppose C is a configuration of pebbles on W_n and consists of at least $4n - 5$ pebbles. We now will describe a sequence of moves that will cover-solve any such configuration. First, if there are outer vertices on W_n that are empty but are adjacent to outer vertices w such that initially $C(w) \geq 3$, then if the adjacent vertices can be covered and w can also remain covered, then these adjacent vertices should be covered. Let k be the number of outer vertices that are covered after this process.

Case 1: Suppose that $k = 0$. In this case, all the pebbles are on the hub vertex. To cover-solve the remaining $v - 1$ vertices, we can cover $\lfloor \frac{4v-10}{2} \rfloor = 2v - 5$

vertices using the excess pebbles already on the hub vertex. Since $v \geq 4$ and $2v - 5 \geq v - 1$, we can cover-solve all of the outer vertices in this manner.

Case 2: Suppose that $k = 1$ or $k = 2$. Each outer vertex covered in the process above requires at most two pebbles to cover it. Since $v \geq 4$, there are at least $4v - 9 - 2k$ pebbles already on the hub vertex. After subtracting 1 pebble for the hub itself, there are $4v - 10 - 2k$ pebbles that can be used such that pebbles can be placed on the remaining $v - k - 1$ uncovered vertices. With these remaining pebbles on the hub, we can cover at least $\lfloor \frac{4v-10-2k}{2} \rfloor = 2v - 5 - k$ vertices. Since $2v - 5 - k \geq v - k - 1$ for $v \geq 4$, there are enough pebbles to cover-solve W_n in this situation.

Case 3: Suppose that $k \geq 3$. Again, each outer vertex in the process above requires at most two pebbles to cover it. If there are any pairs of pebbles remaining on outer vertices such that removing the pairs would not uncover that vertex, those pairs of pebbles should be moved to the hub vertex. After this process, there are at least $\lceil \frac{4v-9-2k}{2} \rceil = 2v - 4 - k$ pebbles on the hub vertex. Notice that this bound is based on the worst case that occurs when no pebbles are initially on the hub vertex. From the hub vertex, it takes exactly 2 pebbles to cover each of the remaining outer vertices and one pebble to cover the hub vertex. So at most $\lfloor \frac{2v-5-k}{2} \rfloor = v - 3 - \lfloor \frac{k}{2} \rfloor$ outer vertices can be pebbled. Since there are at most $v - k - 1$ outer vertices left to be pebbled, and for $k \geq 3$, $v - k - 1 \geq v - 3 - \lfloor \frac{k}{2} \rfloor$, there are enough pebbles to cover-solve W_n in this case, and the proof is complete. \square

4 The Cover Pebbling Number of Graphs of Diameter d

Definition 4.1. A binary weighting on a graph G is a function from $V(G)$ to $\{0, 1\}$. If B is a binary weighting on G , then let the order $|B|$ of B be $\sum_{v \in G} B(v)$.

Definition 4.2. For a graph G and binary weighting B , a configuration C on G will be called permissible (with respect to B) if for all $v \in G$, $B(v) = 0 \implies C(v) = 0$. A permissible configuration on a graph G with a binary weighting B will be called cover-solvable (with respect to B) if we can reach a configuration on which $B(v) = 1 \implies C(v) \geq 1$ for all $v \in G$ by a sequence of pebbling moves.

Lemma 4.3. Let G be a graph of diameter d , B a binary weighting on G , and C a configuration of size at least $(|B| - 1)2^d + 1$ on G which is permissible with

respect to B . Then C is cover-solvable with respect to B .

Proof. Assume the opposite. Then for all pairs $\{G, B\}$ of a graph G together with a binary weighting on G such that there exists a non-cover-solvable configuration of size at least $(|B| - 1)2^d + 1$ (where d is the diameter of G), choose one for which $|B|$ is minimal, and call it $\{G', B'\}$. Let d' be the diameter of G' , let $k = (|B'| - 1)2^{d'} + 1$, and choose some configuration (call it C') on G' which is permissible with respect to B' , has size at least k and is not cover-solvable.

Certainly we cannot have $|B'| = 1$, for then the only permissible configuration of size $|C'| \geq k = 1$ is the function which takes the value $|C'|$ on the lone vertex for which $B' = 1$, and is zero elsewhere. This configuration covers all vertices with non-zero weights, and so is trivially cover-solvable, creating a contradiction.

Now, suppose that $|B'| \geq 2$. If it is true that $C'(v) > 0$ whenever $B'(v) = 1$ we have a contradiction, for C' is then trivially cover-solvable. Otherwise, let v' be some vertex of G' for which $C'(v') = 0$ and $B'(v') = 1$. At most $|B'| - 1$ vertices are initially occupied, and there are at least $(|B'| - 1)2^{d'} + 1$ total pebbles, so by the pigeonhole principle, there are at least $2^{d'} + 1$ pebbles on some vertex (call it v''). Since the diameter of G' is d' , $d(v', v'') \leq d'$. Thus we can move $2^{d'}$ of the pebbles from v'' onto v' , through a series of pebbling moves, losing half of these pebbles for each edge we must move across, but leaving at least one pebble on v' if we move all pebbles via one of the shortest paths.

Now, define a binary weighting B^* on G by

$$B^*(v) = \begin{cases} 0 & : v = v' \\ B'(v) & : v \neq v' \end{cases}$$

and define a configuration C^* on G by

$$C^*(v) = \begin{cases} 0 & : v = v' \\ C'(v'') - 2^{d'} & : v = v'' \\ C'(v) & : \text{otherwise} \end{cases}$$

This is the configuration after we have moved pebbles from v'' onto v' , except that we ignore the pebbles on v' and designate it as a vertex which need not be covered by pebbles. Clearly $|B^*| = |B'| - 1$ and $|C^*| = |C'| - 2^{d'}$ so from $|C'| \geq ((|B'| - 1)2^{d'} + 1)$, we see $|C^*| \geq ((|B^*| - 1)2^{d'} + 1)$. C^* is permissible with respect to B^* , and so by our assumption of the minimality of B' , C^* is cover-solvable with respect to B^* .

If we carry out the moves of the cover-solution of C^* on G starting with the configuration left on G' after our initial movement of pebbles from v'' to v' , (certainly this is possible because this configuration is no smaller than C^* on

any vertex,) we will have covered every vertex of G' for which $B^* = 1$. Also, we must still have $v' \geq 1$, because $C^*(v') = 0$, which does not permit any sequence of moves that decreases the number of pebbles on v' . Thus every vertex for which $B' = 1$ now has $C' \geq 1$, and we have cover-solved C' with respect to B' , which contradicts the assumption that C' was not cover-solvable. \square

Theorem 4.4. *Let G be a graph of order n and diameter d , and let C be a configuration on G of size at least $2^d(n - d + 1) - 1$. Then G is cover-solvable (with respect to the weighting on G which is equal to 1 for each vertex.)*

Proof. First, notice that this bound is sharp because we can exhibit the following class of graphs where $\gamma(G) \geq 2^d(n - d + 1) - 1$. Suppose we have a graph consisting of n vertices and diameter d . Then we will construct a fuse graph (a path connected to a star) whose length is $d - 1$ and has $n - d$ spokes at the end of the fuse. Here is an example for $n = 7$ and $d = 4$.

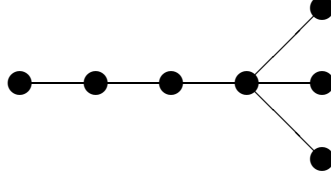


Figure 1: A graph where $n = 7$ and $d = 4$ such that $\gamma(G) = 2^d(n - d + 1) - 1$.

Suppose all the pebbles are on the last vertex of the path, which is at distance d from the spokes (in the figure, the leftmost vertex.) Then each of the $n - d$ spokes requires 2^d pebbles, and the path requires $2^d - 1$ vertices to cover-solve. (Note: In [2], the cover-pebbling number of all trees is found. Thus, we know for these particular trees that $\gamma(G) = 2^d(n - d + 1) - 1$ even before proving this theorem.)

We will prove this theorem by defining an algorithm by induction which will take us to a configuration, the solvability of which we can prove using the lemma. Let $R_0 = \{v \in G : C(v) > 0\}$, let $S_0 = \{v \in G : C(v) = 0\}$, and let $T_0 = \emptyset$. Let $C_0 = C$.

For illustrative purposes, we now describe the first step of the algorithm. If $S_0 = \emptyset$, we are clearly done, for C already covers G . Otherwise, note that since R_0 and S_0 are complementary, there exist vertices $r_0 \in R_0$ and $s_0 \in S_0$ such that $d(r_0, s_0) = 1$. If $C_0(r_0) = 1$ or $C_0(r_0) = 2$, then let $R_1 = R_0 \setminus \{r_0\}$, $S_1 = S_0$ and $T_1 = T_0 \cup \{r_0\} = \{r_0\}$. In this case, let $C_1 = C_0$.

If on the other hand $C_0(r_0) \geq 3$ then we move 2 pebbles from r_0 to s_0 , and instead put s_0 in T_1 and define C_1 according to the following configuration. Explicitly, in this case let $R_1 = R_0$, $S_1 = S_0 \setminus \{s_0\}$, and $T_1 = T_0 \cup \{s_0\} = \{s_0\}$. Define C_1 on G by

$$C_1(v) = \begin{cases} r_0 - 2 & : v = r_0 \\ 1 & : v = s_0 \\ C_0(v) & : \text{otherwise} \end{cases}$$

Define the sequences $R_0, R_1, \dots, R_{d-1}, S_0, S_1, \dots, S_{d-1}, T_0, T_1, \dots, T_{d-1}$, and C_0, C_1, \dots, C_{d-1} , recursively in an analogous manner. Suppose for some $m < d - 1$ we have R_m, S_m, T_m , and C_m , such that the following hold:

1. $|T_m| = m$.
2. R_m, S_m and T_m are disjoint and $R_m \cup S_m \cup T_m = V(G)$.
3. For all $v \in R_m \cup T_m$, $C_m(v) > 0$ and for all $v \in S_m$, $C_m(v) = 0$.
4. C_m is a configuration which can be reached from C by a sequence of pebbling moves.
5. R_m and S_m are both non-empty.
6. $\sum_{v \in R_m} C_m(v) \geq [2^d(n - d + 1) - 1] - [2^{m+1} - 2]$.

Note that all these conditions are trivially true for $m = 0$.

From condition 1, it is evident that the minimum distance between R_m and S_m is at most $m + 1$. Take points $r_m \in R_m$ and $s_m \in S_m$ for which this minimum distance is achieved (and thus $d(r_m, s_m) \leq m + 1$.) If $C_m(r_m) \leq 2^{m+1}$ then let $R_{m+1} = R_m \setminus \{r_m\}$, $S_{m+1} = S_m$ and $T_{m+1} = T_m \cup \{r_m\}$. In this case, let $C_{m+1} = C_m$.

Otherwise, if $C_m(r_m) > 2^{m+1}$ then we can move 2^{m+1} pebbles along a minimal path from r_m to s_m , which is of length at most $m + 1$. We lose half of these pebbles for each edge we must move across, but we will be able to move $2^{(m+1)-d(r_m, s_m)} \geq 1$ onto s_m . Put s_m in T_{m+1} and define C_{m+1} according to the configuration after these moves. Explicitly, in this case let $R_{m+1} = R_m$, $S_{m+1} = S_m \setminus \{s_m\}$ and $T_{m+1} = T_m \cup \{s_m\}$. Define C_{m+1} on G by

$$C_{m+1}(v) = \begin{cases} r_m - 2^{m+1} & : v = r_m \\ 2^{(m+1)-d(r_m, s_m)} & : v = s_m \\ C_m(v) & : \text{otherwise} \end{cases}$$

For $m + 1$, it is clear from our definitions that conditions 1, 2, 3, and 4 still hold. Condition 6 also holds, for in either of the two above cases, the total

number of pebbles left on R_{m+1} is at most 2^{m+1} less than were on R_m . Thus,

$$\begin{aligned} \sum_{v \in R_{m+1}} C_{m+1}(v) &\geq \sum_{v \in R_m} C_m(v) - 2^{m+1} \\ &\geq [2^d(n-d+1) - 1] - [2^{m+1} - 2] - 2^{m+1} \\ &= [2^d(n-d+1) - 1] - [2^{m+2} - 2]. \end{aligned}$$

For condition 5, since always $m+1 < d$ and $n \geq d$, $[2^d(n-d+1) - 1] - [2^{m+1} - 2] > 0$. Thus, the fact that condition 6 is true for $m+1$ necessitates that $R_m \neq \emptyset$. Also, if $S_{m+1} = \emptyset$ then $C_{m+1}(v) > 0$ for all $v \in R_m \cup S_m \cup T_m = V(G)$, and since C_m is attainable from C by a sequence of pebbling moves, we have cover-solved C and we are done. So we may assume $S_{m+1} \neq \emptyset$ and condition 5 holds.

By this recursive definition, we now have R_{d-1} , S_{d-1} , T_{d-1} , and C_{d-1} for which conditions 1-6 hold. Now define a binary weighting B on G by

$$B(v) = \begin{cases} 1 & : v \in R_{d-1} \cup S_{d-1} \\ 0 & : v \in T_{d-1} \end{cases}$$

Also, define C'_{d-1} on G by

$$C'_{d-1}(v) = \begin{cases} C_{d-1}(v) & : v \in R_{d-1} \cup S_{d-1} \\ 0 & : v \in T_{d-1} \end{cases}$$

Clearly C'_{d-1} is permissible with respect to B . From condition 1, $|T_{d-1}| = d-1$ so $|B| = n-d+1$, and from condition 6 we have $|C'_{d-1}| \geq [2^d(n-d+1) - 1] - [2^{(d-1)+1} - 2] = 2^d(n-d) + 1$. Thus, by the lemma, C'_{d-1} is cover-solvable with respect to B .

By condition 4, C_{d-1} is a configuration which can be reached from C by a sequence of pebbling moves. If after we carry out this sequence of moves, we carry out the moves of this cover-solution of C'_{d-1} on G (certainly this is possible because C'_{d-1} is no greater than C_{d-1} on any vertex,) we will have covered every vertex of G for which $B = 1$, that is every vertex in $R_{d-1} \cup S_{d-1}$. Also, every vertex $v \in T_{d-1}$ must remain covered, because for each of these vertices, $C'_{d-1}(v) = 0$, which does not permit any sequence of moves which decreases the number of pebbles on v . Applying, condition 2, we see for every vertex $v \in V(G) = R_{d-1} \cup S_{d-1} \cup T_{d-1}$, our final configuration after this sequence of moves is greater than zero, and so we have cover-solved C . □

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