Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Mathematics 296 (2005) 15–23

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

The cover pebbling number of graphs

Betsy Crull[a], Tammy Cundiff[a], Paul Feltman[a], Glenn H. Hurlbert[b,1], Lara Pudwell[a], Zsuzsanna Szaniszlo[a], Zsolt Tuza[c,d,2]

[a] *Department of Mathematics and Computer Science, Valparaiso University, Valparaiso, Indiana 46383-6493, USA*
[b] *Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287-1804, USA*
[c] *Computer and Automation Institute of the Hungarian Academy of Sciences, Budapest, Hungary*
[d] *Department of Computer Science, University of Veszprém, Hungary*

Received 11 February 2004; received in revised form 7 March 2005; accepted 22 March 2005
Available online 27 April 2005

Abstract

A pebbling move on a graph consists of taking two pebbles off of one vertex and placing one pebble on an adjacent vertex. In the traditional pebbling problem we try to reach a specified vertex of the graph by a sequence of pebbling moves. In this paper we investigate the case when every vertex of the graph must end up with at least one pebble after a series of pebbling moves. The *cover pebbling number* of a graph is the minimum number of pebbles such that however the pebbles are initially placed on the vertices of the graph we can eventually put a pebble on every vertex simultaneously. We find the cover pebbling numbers of trees and some other graphs. We also consider the more general problem where (possibly different) given numbers of pebbles are required for the vertices.

MSC: 05C99; 05C35; 05C05

Keywords: Graph; Pebbling; Coverable

E-mail addresses: hurlbert@asu.edu (G.H. Hurlbert), zsuzsanna.szaniszlo@valpo.edu (Z. Szaniszlo).

[1] Partially supported by National Security Agency grant #MDA9040210095.

[2] Partially supported by OTKA grant T-032969.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.03.009

1. Introduction

The game of pebbling was first suggested by Lagarias and Saks, and introduced to the literature in a paper of Chung [1]. A pebbling move consists of taking two pebbles "off" of one vertex and placing one pebble on an adjacent vertex. Given a graph G, a specified number of pebbles, and a configuration of the pebbles on the vertices of G, the goal is to be able to move at least one pebble to any specified target vertex using a sequence of pebbling moves. The pebbling number $\pi(G)$ is the minimum number of pebbles that are sufficient to reach any target vertex regardless of the original configuration of the pebbles. In the present context it is naturally assumed that *all graphs considered are connected*. Moews [5] found the pebbling number of trees by using a clever path partition of the tree. For a survey of additional results see [3].

In this paper we investigate the following question: How does the pebbling problem change if instead of having a specified target vertex we need to place a pebble simultaneously on every vertex of the graph? In some scenarios this seems to be a more natural question, for example if information needs to be transmitted to several locations of a network, or if army troops need to be deployed simultaneously. We define the *cover pebbling number* $\gamma(G)$ to be the minimum number of pebbles needed to place a pebble on every vertex of the graph using a sequence of pebbling moves, regardless of the initial configuration. We establish the cover pebbling number for several classes of graphs, including complete graphs, paths, fuses (a fuse is a path with leaves attached at one end), and more generally, trees. We also describe the structure of the largest non-coverable configuration on a tree.

More generally, let a weight function w be given that assigns an integer $w(v)$ to each vertex v of G. We say that w is *positive* if $w(v) > 0$ for all v. We define the *weighted cover pebbling number* $\gamma_w(G)$ to be the minimum number k ensuring that, from any initial configuration with k pebbles there is a sequence of pebbling moves after which all the vertices v simultaneously have $w(v)$ pebbles on them. Our main result on trees in Section 4 determines $\gamma_w(T)$ for every tree T and every positive weight function w.

Given a configuration C of pebbles, we will use the following notation. The *size* $|C|$ of the configuration denotes the number of pebbles in C. The *support* $\sigma(C)$ of the configuration is the set of *support vertices*, i.e. those on which there is at least one pebble of C. The number of pebbles on v in C is denoted by $C(v)$ (hence, $v \in \sigma(C)$ if and only if $C(v) > 0$). We call a configuration *simple* if its support consists of a single vertex. We say that a configuration is *cover-solvable*, or simply *coverable* (resp. w-*coverable*), if it is possible to transport at least one pebble (resp. $w(v)$ pebbles) to every vertex v of the graph simultaneously (and *non-coverable* otherwise). As is customary, we denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. If G is of order n, we sometimes denote its vertices by $v_1, v_2, \ldots, v_n$.

2. Preliminary results

We begin with the cover pebbling number of the complete graph K_n on n vertices. Note that the pebbling number for K_n, $\pi(K_n)$, is n (see [3]).

Theorem 1. $\gamma(K_n) = 2n - 1$.

Proof. If $2n-2$ pebbles are placed on vertex v_n, then 2 pebbles will be used to cover each of the $n-1$ other vertices. Thus no pebbles will remain to cover v_n. Hence $\gamma(K_n) \geqslant 2n-1$.

Now suppose that at least $2n - 1$ pebbles are placed on the vertices. We may suppose that some vertex, say v_n, has no pebbles on it, otherwise the graph is already covered. The pigeonhole principle says that some other vertex has at least two pebbles on it; we use those to cover v_n. Since there are now at least $2n-3$ pebbles among the remaining $n-1$ vertices, induction says we can cover them (of course, $\gamma(K_1) = 1$). Hence $\gamma(K_n) \leqslant 2n - 1$. $\square$

A similar inductive proof works also for weighted covering, and yields the following result. Denote the total weight by $|w| = \sum_v w(v)$ and define $\min w = \min_v w(v)$.

Theorem 2. $\gamma_w(K_n) = 2|w| - \min w$ *for every positive weight function* w.

Next we find the cover pebbling number of the path P_n on n vertices $v_1, \ldots, v_n$, with $v_i v_{i+1} \in E$ for $1 \leqslant i < n$. Note that $\pi(P_n) = 2^{n-1}$ (see [3]).

Theorem 3. $\gamma(P_n) = 2^n - 1$.

Proof. If $2^n - 2$ pebbles are placed at vertex v_n, then covering v_1 will use 2^{n-1} pebbles, covering v_2 will use 2^{n-2} pebbles, $\ldots$, and covering v_{n-1} will use 2 pebbles. Then no pebbles will remain to cover v_n. Hence $\gamma(P_n) \geqslant 2^n - 1$.

Now suppose that at least $2^n - 1$ pebbles are placed on the vertices. If there are no pebbles on v_n then we may use at most 2^{n-1} pebbles to cover it, since $\pi(P_n) = 2^{n-1}$. By induction, the remaining $2^{n-1} - 1$ or more pebbles can cover P_{n-1} (of course, $\gamma(P_1) = 1$). If there are pebbles on v_n then move as many of them as possible to v_{n-1}, leaving 1 or 2 on v_n. Either at least $2^{n-1} - 1$ pebbles have been moved to v_{n-1}, or at most $2^{n-1} - 2$ moves have been made and at most two pebbles stay on v_n. In any case, at least $2^{n-1} - 1$ pebbles remain on P_{n-1}. Again, induction shows that $\gamma(P_n) \leqslant 2^n - 1$. $\square$

Note that the upper bound is also mentioned in [2].

Among all graphs on n vertices, the complete graph has the smallest pebbling number (n) and the path has the largest pebbling number (2^{n-1}). In both cases, we have $\gamma(G) = 2\pi(G) - 1$. While this might lead one to guess that such a relation holds for all (connected) graphs, this could not be farther from the truth. As the following theorem shows, the ratio $\gamma(G)/\pi(G)$ is unbounded, even within the class of trees. The subclass of *fuses* is defined as follows. The vertices of $F_l(n)$ ($l \geqslant 2$ and $n \geqslant 3$) are $v_1, \ldots, v_n$, so that the first l vertices form a path from v_1 to v_l, and the remaining vertices are independent and adjacent only to v_l. (The path is sometimes called the *wick*, while the remaining vertices are sometimes called the *sparks*.) For example, $F_2(n)$ is the star S_n on n vertices. The fact that $\gamma(S_n) = 4n - 5$ serves as the base case for the following result.

Theorem 4. $\gamma(F_l(n)) = (n - l + 1)2^l - 1$.

Proof. Following the arguments for the path given above, it is easy to see that so many pebbles are required of a simple configuration sitting on v_1.

Likewise, induction on l shows that so many pebbles suffice to cover the fuse. Indeed, consider the cases whether or not v_1 has pebbles on it and argue as was done for paths, above.

Regarding the base case $l = 2$, we point out that $F_2(n)$ is the star on n vertices, so we can let any leaf play the role of v_1. If all the pebbles are on v_2 then we can cover the star easily. Otherwise, some leaf has at least one pebble on it, and we label that vertex v_1. Now we pebble as many as possible from v_1 to v_2, leaving 1 or 2 on v_1. Induction on the number of leaves finishes the proof. □

We define the *covering ratio* of G to be $\rho(G) = \gamma(G)/\pi(G)$. For a class $\mathscr{F}$ of graphs we define $\rho(\mathscr{F}) = \sup_{G\in\mathscr{F}} \rho(G)$ if it exists, and $\rho(\mathscr{F}) = \infty$, otherwise. Thus, for the families $\mathscr{K}$ of complete graphs and $\mathscr{P}$ of paths, we have $\rho(\mathscr{K}) = \rho(\mathscr{P}) = 2$.

Theorem 5. *Let $\mathscr{T}_n$ be the family of all trees on n vertices. Then $\rho(\mathscr{T}_n) = \infty$.*

Proof. Since $\pi(F_l(n)) = 2^l + n - l - 1$ (see [5]), we see that, for $n = 2^l + l$, $\rho(F_l(n)) > (n - l)2^l/(n - l + 2^l) > (n - \lg n)/2$. □

3. The transition digraph

The main goal of this section is to prove that any sequence of pebbling moves can be replaced by one which is cycle free in a well-defined sense. For this, we introduce the following concept.

Definition. Given a sequence S of pebbling moves on graph G, the *transition digraph* is a *directed multigraph* denoted $T(G, S)$ that has $V(G)$ as its vertex set, and each move $s \in S$ along edge $v_i v_j$ (i.e., where two pebbles are removed from v_i and one placed on v_j) is represented by one directed edge $v_i v_j$.

Theorem 6. *Let S be a sequence of pebbling moves on G, resulting in a configuration C. Then there exists a sequence S^* of pebbling moves, terminating with a configuration C^*, such that*

1. *On each vertex v, the number of pebbles in C^* is at least as large as that in C, and*
2. *$T(G, S^*)$ does not contain any directed cycles.*

Proof. We apply induction on the number of directed cycles in $T(G, S)$. The assertion is trivially true for every S where this number is zero.

Let now S be arbitrary, and consider the shortest prefix S' of S that contains a directed cycle. That is, the last move in S' creates a cycle, say $C' = v_1 v_2 \cdots v_k$, in $T(G, S')$. For $i = 1, 2, \ldots, n$, let us denote by d_i^- and d_i^+ the in-degree and out-degree, respectively, of vertex v_i in $T(G, S')$. In the initial configuration, each v_i has to contain at least $2d_i^+ - d_i^-$ pebbles, otherwise some move of S' could not be performed at v_i.

Let us consider the edge set $F' = E(T(G, S'))\backslash E(C')$. By the choice of S', this F' does not contain any directed cycles. Hence it contains a vertex v_i of in-degree zero. It means $d_i^- = 0$ if $v_i \notin C'$, and $d_i^- = 1$ otherwise. In the former case, v_i initially has at least $2d_i^+$ pebbles and is incident with precisely d_i^+ edges in F'; while in the latter, the number of pebbles at v_i is at least $2d_i^+ - 1$ and that of its incident edges is just $d_i^+ - 1$. In either case, v_i has sufficiently many pebbles so that the pebbling moves for all of its incident edges in F' are feasible before any move belonging to C' has been performed. We now rearrange S' to make all moves of F' involving v_i at the beginning. Analogously, $F' - v_i$ has a vertex v_j of zero in-degree in F'. Hence after the rearrangement of moves at v_i, the moves of edges incident with v_j are feasible completely before C'. Eventually we obtain a rearrangement, say S'' of S' where the moves of C' are performed at the very end, and of course the concatenation of S'' and $S - S'$ terminates in configuration C. Now it is immediately seen that the concatenation S^+ of $S'' - C'$ and $S - S'$ is a feasible sequence of moves that ends up with a configuration C^+ where the vertices $v_1, \ldots, v_k$ have one more pebble than in C, and the other vertices have the same number of pebbles in C and C^+. Since the number of directed cycles in $T(G, S^+)$ is strictly smaller than that in $T(G, S)$, the assertion follows by induction. □

4. Trees

In this section we determine the (weighted) cover pebbling number for an arbitrary tree T. For $v \in V(T)$ define

$$s(v) = \sum_{u \in V(T)} 2^{d(u,v)},$$

where $d(u, v)$ denotes the distance from u to v, and let

$$s(T) = \max_{v \in V(T)} s(v).$$

Analogously, if a positive weight function w is given, we define

$$s_w(v) = \sum_{u \in V(T)} w(u)2^{d(u,v)}$$

and

$$s_w(T) = \max_{v \in V(T)} s_w(v).$$

Clearly, for a *simple* configuration sitting on v, $s_w(v)$ pebbles are necessary and sufficient to cover T. Thus $\gamma_w(T) \geqslant s_w(T)$ for every T and every positive w. We are going to prove that this obvious lower bound is in fact tight.

Theorem 7. *For positive weight functions w we have $\gamma_w(T) = s_w(T)$.*

Proof. The theorem can be reformulated in the following equivalent form:

For every non-coverable configuration C there exists a simple non-coverable configuration C^* such that $|C^*| = |C|$.

The proof of this latter assertion is essentially induction, where we either reduce the tree to another tree with fewer vertices or keep T unchanged but decrease the support $\sigma(C)$ of C without making its size $|C|$ decrease.

We shall use the following terminology concerning a configuration C. We say that a vertex v is a

- D-vertex with demand $D(v) = w(v) - C(v)$ if $w(v) - C(v) > 0$.
- N-vertex (neutral) if $C(v) = w(v)$. Then we define $D(v) = 0$.
- S-vertex with supply $S(v) = C(v) - w(v)$ if $C(v) - w(v) > 0$.

It is immediate by definition that every non-coverable configuration contains at least one D-vertex.

Case 1. *$T = K_1$ or $T = K_2$.*

These are trivial initial cases, handled already in the more general context of Theorem 2.

Case 2. *Some leaf of T is not an S-vertex.*

Let v be such a leaf, and let u be its neighbor in T. We now delete v from T (with all its pebbles), and increase w at u to the value $w'(u) = w(u) + 2D(v)$. Keeping $w'(x) = w(x)$ unchanged for all $x \notin \{u, v\}$, the configuration $C' = C - v$ on the tree $T' = T - v$ with the weight function w' is coverable if and only if so is C on T with w. This follows from Theorem 6, which implies that if T is coverable, then there is a sequence of pebbling moves where no pebble gets moved from v to u. (To make v properly covered, we need to place at least $D(v)$ additional pebbles on it; and this requires taking $2D(v)$ pebbles off of u.)

Case 3. *Every leaf of T is an S-vertex.*

For a given leaf $v = v_1$, define the path $v_1 v_2 \cdots v_m$ so that v_m is the other leaf if T is a path and is the only vertex of the path having degree at least 3 in T otherwise. In the latter case we call v_m the *split* vertex of v_1. If there is a support vertex other than v_1 on this path, we call the one having minimum subscript the *nearest support* vertex of v_1.

Since v_1 is an S-vertex we can move $s_1 = \lfloor S(v_1)/2 \rfloor$ pebbles to v_2. Moreover, if $s_1 > w(v_2) - C(v_2)$ then we can further transmit $s_2 = \lfloor (s_1 + C(v_2) - w(v_2))/2 \rfloor$ pebbles to v_3, and so on. For a vertex v_k on this path we say that v_1 *supplies* v_k if at least one of the pebbles from v_1 can reach v_k in this way. There are three possibilities for v_1: v_1 supplies its split vertex, v_1 supplies its nearest support vertex, or v_1 supplies neither of these. We consider these possibilities in reverse order.

Subcase A. Some leaf supplies neither its split nor its nearest support vertices.

We follow an argument similar to that in Case 2. Let v_1 be such a leaf and let k be the largest subscript so that v_1 supplies v_k (then $k < m$ and v_i is not a support vertex for any $2 \leqslant i \leqslant k$). Let C' and w' be the restrictions of C and w to $T' = T - \{v_1, \ldots, v_k\}$, except that $w'(v_{k+1}) = w(v_{k+1}) + 2D'$, where $D' = w(v_k) - s_{k-1}$ is the resulting demand on v_k after being supplied by v_1. Then C' is non-w'-coverable on T', and since $|T'| < |T|$ there is a simple non-w'-coverable configuration of size $|C'|$ on T'. This yields a non-w-coverable configuration C'' of size $|C|$ on T that sits on two vertices. If T has at least three leaves then some leaf is not an S-vertex and we are done by Case 2. Otherwise T is a path and $\sigma(C'') = \{v_1, v_n\}$. Non-$w$-coverability now means that v_n can supply v_k with strictly fewer than D' pebbles. Finally we test if $k - 1 \geqslant n - k$. If so, then for every j in the range $k \leqslant j \leqslant n$, $d(v_1, v_j) \geqslant d(v_j, v_k)$. Thus, defining $C^*(v_n) = 0$ and $C^*(v_1) = C'(v_1) + C'(v_n) = |C|$, we obtain a simple non-coverable configuration, as required. If $k-1<n-k$ we do the opposite.

Subcase B. Some leaf supplies its nearest support vertex.

Let v_1 be such a vertex and v_k its nearest support vertex (then $v_i \notin \sigma(C)$ for $1<i<k$). We define $C'(v_k)=0$ and $C'(v_1)=C(v_1)+C(v_k)$, keeping C' identical to C on every other vertex. Then $|C'|=|C|$, $|\sigma(C')|<|\sigma(C)|$, and C' is non-coverable whenever C is, because the supply from v_1 yields fewer pebbles on v_k in C' than in C.

Subcase C. Every leaf supplies its split vertex.

By Subcase B we may assume that no leaf supplies its nearest support vertex. There must be some vertex v that is the split vertex for two different leaves (indeed, choose any leaf and let v be any vertex of degree atleast 3 at farthest distance from it—the two leaves past v witness this). Label these leaves v_1 and v_ℓ so that $P=v_1\ldots v_m\ldots v_\ell$ is the unique path between them and $v=v_m$. Recall that v_i is not a support vertex for any $1<i<\ell$ and that both v_1 and v_ℓ supply v_m. Let us denote by s_m their total supply for v_m.

If $s_m>w(v_m)$, then P can supply $T-P$ with $s'=\lfloor\frac{1}{2}(s_m-w(v_m))\rfloor$ pebbles (at most); and otherwise it needs to receive at least $s''=w(v_m)-s_m$ pebbles from $T-P$. In both cases we consider the problem restricted to P, where $w(v_i)$ is kept unchanged for all $i\neq m$, and $w(v_m)$ is modified to s_m+1. This configuration on P is non-coverable. Thus, according to Subcase A, the $C(v_1)+C(v_\ell)$ pebbles can be placed on one vertex (v_1 or v_ℓ), keeping P non-coverable. It follows that the modified configuration, too, either supplies $T-P$ with at most s' pebbles or needs to receive at least s'' pebbles from $T-P$. In either case, the new configuration on T is non-coverable and has at least one D-vertex leaf, thus we are done by Case 2. $\square$

From this proof we see that a non-coverable configuration of maximum size can be assumed to be simple. The next result shows that the single support vertex must be an end of a longest path. (This is the case even for weight functions w where the longest paths are not of maximum weight.)

Theorem 8. *Given a tree T and a positive weight function w, let C be a non-coverable simple configuration of maximum size, with $\sigma(C)=\{v\}$. Then v is a leaf of a longest path in T.*

Proof. Since $\gamma_w(T)=s_w(v)$ for some v, we need to show that the maximum value of $s_w(v)$ is attained only on some endpoints of the longest path(s) of T. We are going to prove something stronger: *every* longest path has at least one endpoint x whose $s_w(x)$ is larger than $s_w(u)$ for *every u* which is not the endpoint of some longest path.

Suppose first that T is just a path $v_1v_2\ldots v_n$. Consider any internal vertex v_k $(1<k<n)$. We compare the partial sums $s^-=\sum_{1\leqslant i<k}w(v_i)2^{d(v_i,v_k)}$ and $s^+=\sum_{k<i\leqslant n}w(v_i)2^{d(v_i,v_k)}$. If $s^-\leqslant s^+$, then $s_w(v_{k-1})>s_w(v_k)$; and if $s^-\geqslant s^+$, then $s_w(v_{k+1})>s_w(v_k)$. Thus, $s_w(k)$ can never be the largest.

Suppose next that T is a tree with precisely three leaves. Applying the previous idea, from any non-leaf vertex we can move to one of its neighbors and find there a larger value of s_w. Hence, let v, v', v'' be the three leaves, and suppose that the longest path P in T is the one connecting v' with v''. We need to show $s_w(v)<\max\{s_w(v'),s_w(v'')\}$. Let u be the unique degree-3 vertex of T. We have $d(u, v) < d(u, v')$ and $d(u, v) < d(u, v'')$ (for otherwise the v–v' path or the v–v'' path were at least as long as the v'–v'' path, contrary to the assumption on v). From this it is easily seen that for every vertex x, at least one of $d(v', x)$ and $d(v'', x)$ is at least $d(v, x) + 1$. Consequently, $s_w(v') + s_w(v'') > 2s_w(v)$, i.e. $s_w(v)$ cannot be the largest.

Finally, let T be a tree with more than three leaves. Let P be one of its longest paths, v^* a leaf that does *not* belong to any longest path of T, and $v \neq v^*$ a leaf not on P (but maybe on some other longest path of T). We apply the transformation on v as described in Case 2 of the proof of Theorem 7. This modification keeps the function s_w unchanged on all vertices of $T - v$, moreover P remains a longest path and v^* does not become the endpoint of any longest path in $T - v$. Thus, by induction on the number of vertices, s_w is larger on some endpoint of P than on v^*. This completes the proof. □

5. Open problems

There are several natural problems and questions to ask.

Problem 9. *Find $\gamma(G)$ for other graphs G, for example cubes, complete r-partite graphs, etc.*

For progress on this question during the year of the refereeing process see [4 and 8].

Question 10. *Is it true for all graphs G that at least one of the largest non-coverable configurations on G is simple*?

For progress on this question during the year of the refereeing process see [6 and 7].

Problem 11. *Find classes of graphs $\mathscr{F}$ whose covering ratio $\rho(\mathscr{F})$ is bounded.*

Question 12. *Can the question, "Is $\rho(G) \leqslant k$?" be answered efficiently*?

These questions extend to positive weight functions in a natural way. Let us note, however, that the situation drastically changes when "positive" is replaced by "nonnegative" for w. This fact is already shown by the complete graph K_n ($n \geqslant 3$) where only one vertex is required to be covered, which corresponds to the weights $1, 0, 0, \ldots, 0$. Here the unique maximal non-coverable configuration has the pebble distribution $0, 1, 1, \ldots, 1$, in striking contrast to the case where $w > 0$ and all pebbles may be concentrated on a suitably chosen single vertex. Such considerations must be tackled in order to pursue the *weighted pebbling number* of a graph G, defined as $\pi_w(G) = \max_w \gamma_w(G)$, where the maximum is taken over all nonnegative weight functions w of size $|w| = w$. The pebbling number $\pi(G)$ is the case $w = 1$.

Problem 13. *Find $\pi_w(T)$ for any tree T and weight w.*

References

[1] F.R.K. Chung, Pebbling in hypercubes, SIAM J. Discrete Math. 2 (1989) 467–472.
[2] D. Hersovici, Graham's conjecture on products of cycles, J. Graph Theory 42 (2003) 141–154.
[3] G. Hurlbert, A survey of graph pebbling, Congr. Numer. 139 (1999) 41–64.
[4] G. Hurlbert, B. Munyan, Cover pebbling hypercubes, Bull. Inst. Combin. Appl., to appear.
[5] D. Moews, Pebbling graphs, J. Combin. Theory (B) 55 (1992) 244–252.
[6] J. Sjostrand, The cover pebbling theorem, Math ArXiv, math.CO/0410129.
[7] A. Vuong, M. Wyckoff, Conditions for weighted cover pebbling of graphs, Math ArXiv, math.CO/0410410.
[8] N. Watson, C. Yerger, Cover pebbling numbers and bounds for certain families of graphs, Math ArXiv, math.CO/0409321.