The first tableau pivot gives rise to Tableau 5.2.1. Let's compare it to what we compute in matrix form, below. We have

$$\mathbf{B} = \begin{pmatrix} -7 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 9 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Pi} = \begin{pmatrix} -1 & -3 & 1 \\ -6 & -8 & 0 \\ 5 & 1 & 0 \\ 4 & 3 & 0 \end{pmatrix} .$$

Thus $d_\beta = 7$,

$$\mathbf{B}' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & 7 & 0 & 0 \\ 4 & 0 & 7 & 0 \\ 9 & 0 & 0 & 7 \end{pmatrix} , \quad \mathbf{B}'\mathbf{\Pi} = \begin{pmatrix} 1 & 3 & -1 \\ -40 & -50 & -2 \\ 31 & -5 & 4 \\ 19 & -6 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{B}'\mathbf{b} = \begin{pmatrix} 23 \\ -52 \\ 517 \\ 577 \end{pmatrix} ,$$

showing an uncanny resemblance to Tableau 5.2.1. The usual rules apply — the second basic variable is replaced by the first parameter: $2 \mapsto 5$. The subsequent tableau and matrices corresponding to $\beta^{(2)} = \{1, 2, 6, 7\}$ follow.

Tableau 5.2.2

$$\left[\begin{array}{rrr|rrrrr|r} 40 & 0 & 10 & -6 & 1 & 0 & 0 & 0 & 124 \\ 0 & 40 & 50 & 2 & -7 & 0 & 0 & 0 & 52 \\ 0 & 0 & -250 & 14 & 31 & 40 & 0 & 0 & 2724 \\ 0 & 0 & -170 & 46 & 19 & 0 & 40 & 0 & 3156 \\ \hline 0 & 0 & 730 & -246 & -59 & 0 & 0 & 40 & 0 \end{array}\right]$$

Also,

$$\mathbf{B} = \begin{pmatrix} -7 & -1 & 0 & 0 \\ -2 & -6 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ 9 & 4 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Pi} = \begin{pmatrix} -3 & 1 & 0 \\ -8 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} .$$

Thus $d_\beta = 40$,

$$\mathbf{B}' = \begin{pmatrix} -6 & 1 & 0 & 0 \\ 2 & -7 & 0 & 0 \\ 14 & 31 & 40 & 0 \\ 46 & 19 & 0 & 40 \end{pmatrix} , \quad \mathbf{B}'\mathbf{\Pi} = \begin{pmatrix} 10 & -6 & 1 \\ 50 & 2 & -7 \\ -250 & 14 & 31 \\ -170 & 46 & 19 \end{pmatrix} \quad \text{and} \quad \mathbf{B}'\mathbf{b} = \begin{pmatrix} 124 \\ 52 \\ 2724 \\ 3156 \end{pmatrix} .$$

Again, the resemblance to Tableau 5.2.2 is unmistakable.

By the looks of $\mathbf{B}'\mathbf{b}$, $\beta^{(2)}$ is feasible, and so the objective row is the place to look next. Equation (5.6) can be rewritten as

$$(\mathbf{c}_\beta^\mathsf{T}\mathbf{B}'\mathbf{\Pi} - d_\beta\mathbf{c}_\pi^\mathsf{T})\mathbf{x}_\pi + d_\beta z = (\mathbf{c}_\beta^\mathsf{T}\mathbf{B}'\mathbf{b}) \tag{5.7}$$

so as to look more like the objective row. The parameter coefficients are thus

$$\mathbf{c}_\beta^\mathsf{T}\mathbf{B}'\mathbf{\Pi} - d_\beta\mathbf{c}_\pi^\mathsf{T} = (730, -246, -59)^\mathsf{T} ,$$

and so the second parameter, x_4, enters the basis.

Now we need to look at the coefficients of x_4 in $\mathbf{B}'\mathbf{\Pi}$, namely $(-6, 2, 14, 46)^\mathsf{T}$, as denominators for their partner terms in $\mathbf{B}'\mathbf{b}$. The ratios

$$\text{“}\mathbf{B}'\mathbf{b}/\mathbf{B}'\mathbf{\Pi}_{x_4}\text{”} = \begin{pmatrix} -124/6 \\ 52/2 \\ 2724/14 \\ 3156/46 \end{pmatrix}$$

show that second term, $52/2$, is the smallest nonnegative. Hence the second basic variable leaves: $4 \mapsto 2$. The subsequent tableau and matrices corresponding to $\beta^{(3)} = \{1, 4, 6, 7\}$ follow.

Tableau 5.2.3

$$\left[\begin{array}{rrr|rrrrr|r} 2 & 6 & 8 & 0 & -1 & 0 & 0 & 0 & 14 \\ 0 & 40 & 50 & 2 & -7 & 0 & 0 & 0 & 52 \\ 0 & -14 & -30 & 0 & 4 & 2 & 0 & 0 & 118 \\ 0 & -46 & -66 & 0 & 9 & 0 & 2 & 0 & 98 \\ \hline 0 & 246 & 344 & 0 & -46 & 0 & 0 & 2 & 644 \end{array}\right]$$

Also,

$$\mathbf{B} = \begin{pmatrix} -7 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 9 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Pi} = \begin{pmatrix} -1 & -3 & 0 \\ -6 & -8 & 1 \\ 5 & 1 & 0 \\ 4 & 3 & 0 \end{pmatrix} .$$

Thus $d_\beta = 2$,

$$\mathbf{B}' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 2 & -7 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 9 & 0 & 2 \end{pmatrix} , \quad \mathbf{B}'\mathbf{\Pi} = \begin{pmatrix} 6 & 8 & -1 \\ 40 & 50 & -7 \\ -14 & -30 & 4 \\ -460 & -66 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{B}'\mathbf{b} = \begin{pmatrix} 14 \\ 52 \\ 118 \\ 98 \end{pmatrix} .$$

As usual, the columns of $\mathbf{B}$ are those of Tableau 5.1.2 that correspond to $\beta^{(3)}$ — technically, column k of $\mathbf{B}$ equals column $\beta_k^{(3)}$ of Tableau 5.1.2. Because $\beta^{(3)}$ is feasible, we next compute the parameter coefficients

$$\mathbf{c}_\beta^\mathsf{T}\mathbf{B}'\mathbf{\Pi} - d_\beta \mathbf{c}_\pi^\mathsf{T} = (246, 344, -46) ,$$

which signal that the third parameter, x_5, enters the basis.

Now we need to look at the coefficients of x_5 in $\mathbf{B}'\mathbf{\Pi}$, namely $(-1, -7, 4, 9)^\mathsf{T}$, as denominators for their partner terms in $\mathbf{B}'\mathbf{b}$. The ratios

$$\text{“}\mathbf{B}'\mathbf{b}/\mathbf{B}'\mathbf{\Pi}_{x_5}\text{”} = \begin{pmatrix} - \\ - \\ 118/4 \\ 98/9 \end{pmatrix}$$

show that fourth term, $98/9$, is the smallest nonnegative. Hence the fourth basic variable leaves: $5 \mapsto 7$. The subsequent tableau and matrices corresponding to $\beta^{(3)} = \{1, 4, 5, 6\}$ follow. (Look for a slight twist!)

Here is an optimal problem.

Problem 5.3.5

$$
\begin{array}{ll}
\text{Min.} & w = 7y_1 + 4y_2 + 2y_3 + y_4 + 8y_5 + 3y_6 + 9y_7 + 5y_8 \\
\\
\text{s.t.} & -y_1 - y_2 + y_4 + y_7 \leq 13 \\
& y_1 - y_3 \geq 11 \\
& y_3 - y_4 - y_5 + y_6 \leq -17 \\
& y_2 - y_6 + y_8 \leq 16 \\
& y_5 - y_7 - y_8 \geq -19 \\
\\
& y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \geq 0
\end{array}
$$

Workout 5.3.6 *Which LOP would you rather solve, Problem 5.3.5 or its dual? Why?*

Workout 5.3.7

- **(a)** *Use the Matrix implementation of the Simplex algorithm to solve Problem 5.3.5. (Maximize $u = -w$ rather than solve its dual.)*
- **(b)** *Find a certificate of optimality.*
- **(c)** *Did you notice anything especially interesting about one of the repeated computations in this problem?*

5.4 Basic Coefficients

As noticed in the examples of Section 5.1, in particular Equations (5.5) and (5.6), it seems as if the basic coefficient for a tableau with basis β equals $d_\beta = |\det(\mathbf{B}_\beta)|$. Of course, while the coefficient d_β does clear all denominators, it is possible that some smaller coefficient does the same. It is our objective now to prove that this is not the case, and also to discuss some of its ramifications.

Theorem 5.4.1 *Let d_T be the basic coefficient for the tableau T having basis β from a standard form LOP. Then $d_T = d_\beta$.*

Proof. We proceed by induction. Clearly the theorem is true at the start. Now suppose that d_T is the basic coefficient for the tableau T having basis β, and that $d_T = d_\beta$. Let T_β the submatrix of T having columns corresponding to β (and not including the objective row). Then $\det(T_\beta) = \rho \det(\mathbf{B}_\beta)$ for some ρ determined by the sequence of row operations that transformed $\mathbf{B}_\beta$ to T_β.

Workout 5.4.2 *Use the induction hypothesis and the structure of T_β to show that $|\rho| = d_T^{m-1}$, where m is the number of rows of T.*

Likewise, let T' be the subsequent tableau derived from T by pivotting from β to β', and let $T'_{\beta'}$ be the analogous submatrix of T'. As above, we have $d_{T'}^m = |\det(T'_{\beta'})| = |\rho' \det(\mathbf{B}_{\beta'})|$ for the analogous ρ', whose value is yet to be determined.

Now suppose that a is the pivot entry that transforms T to T'.

Workout 5.4.3 *Use the form of the row operations of the transformation to prove that $|\rho'| = d_{T'}^{m-1}$. [HINT: Write ρ' in terms of ρ and use Workout 5.4.2.]*

Workout 5.4.4 *Use Workout 5.4.3 to finish the proof of the theorem.*

◇

Workout 5.4.5 *Prove Theorem 5.4.1 for general form LOPs.*

What is the moral of Theorem 5.4.1? One answer lies in Problem 5.3.5. If at any stage we find $d_\beta = 1$ then the corresponding basic solution and objective value are both integer-valued. In particular, if $d_{\beta^*} = 1$ then the optimal LO solution also solves the corresponding ILO. As you may have discovered in Workout 5.3.7, *every* basic determinant has absolute value 1. For such a problem it is guaranteed that its optimal LO solution also solves the corresponding ILO. We say a matrix is **totally unimodular** (TU) if every square submatrix has determinant 0, 1 or −1. Thus we discover that, while general ILOPs are significantly more time consuming than LOPs in general, for the special class of problems having a TU constraint matrix the two problems are identical. Before tossing this class aside as an extreme anomoly, know that the very important subclass of network problems that we will encounter in Chapter 10 are all of this variety. In fact, almost all problems having a TU constraint matrix arise from networks.

totally unimodular matrix

Workout 5.4.6 *Consider Problem 5.3.5.*

- **(a)** *Write its* 5×13 *constraint matrix* $\mathbf{A}$ without *first converting to standard form. That is, add or subtract slack variables as necessary.*
- **(b)** *Write every one of its* $\binom{13}{2}\binom{5}{2} = 780$ 2×2 *submatrices and compute its determinant. Well, okay, write and compute* N *of them.*
- **(c)** *Argue that every* 2×2 *submatrix of* $\mathbf{A}$ *has determinant* 0*,* 1 *or* −1*.*
- **(d)** *Suppose that every* 3×3 *submatrix of* $\mathbf{A}$ *has determinant* 0*,* 1 *or* −1*, and use that supposition to prove that the same holds for every* 4×4 *submatrix. [HINT: Consider the three cases according to the number of nonzero entries of a column of the submatrix.]*
- **(e)** *Use the ideas from part d to prove that* $\mathbf{A}$ *is TU.*

Let us now make the integrality arguments above more formal. We say a nondegenerate polyhedron is **integral** if each of its extreme points is integral. Given a TU matrix $\mathbf{A}$ and integral $\mathbf{b}$, let $Q_\mathbf{b}$ be the polyhedron of solutions to $S_\mathbf{b} = \{\mathbf{Ax} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$. As mentioned above, every basic solution of $S_\mathbf{b}$ is integral, and hence $Q_\mathbf{b}$ is integral. Hence, if $\mathbf{A}$ is TU then $Q_\mathbf{b}$ is integral for every integral $\mathbf{b}$. In 1956 Hoffman and Kruskal proved that the converse is also true. The proof of their result is beyond the scope of this book.

integral polyhedron

Hoffman-Kruskal Theorem

Theorem 5.4.7 *The matrix* $\mathbf{A}$ *is TU if and only if the polyhedron* $Q_\mathbf{b}$ *is integral for every integral vector* $\mathbf{b}$*.* ◇

Workout 5.4.8 *Consider the polytope* $Q_\mathbf{b}$ *for*

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}.$$

- **(a)** *Show that* $\mathbf{A}$ *is not TU.*
- **(b)** *Show that* $Q_\mathbf{b}$ *is integral.*
- **(c)** *Why does this example not contradict Theorem 5.4.7?*