Overlap Cycles for Steiner Quadruple Systems

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Abstract: Steiner quadruple systems are set systems in which every triple is contained in a unique quadruple. It is well known that Steiner quadruple systems of order v, or SQS(v), exist if and only if $v \equiv 2, 4 \pmod 6$. Universal cycles, introduced by Chung, Diaconis, and Graham in 1992, are a type of cyclic Gray code. Overlap cycles are generalizations of universal cycles that were introduced in 2010 by Godbole, et al. Using Hanani's SQS constructions, we show that for every $v \equiv 2, 4 \pmod 6$ with $v > 4$ there exists an SQS(v) that admits a 1-overlap cycle.
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1. INTRODUCTION AND DEFINITIONS

A 3-(v, 4, 1)-design is known as a **Steiner quadruple system of order** v, or SQS(v). They may also be defined as a pair $(X, \mathcal{B})$ where X is a set and $\mathcal{B}$ a collection of 4-element subsets of X called **quadruples**, with the property that any three points of X are contained in a unique quadruple [1]. The existence of Steiner quadruple systems was completely determined by H. Hanani in 1960.

Theorem 1.1 [6]. *An SQS(v) exists if and only if $v \equiv 2, 4 \pmod 6$.*

Ordering the blocks of a design and the points within its blocks is an important problem for many applications of design theory. For example, in [3] and [4] the blocks of Steiner triple systems are ordered in a specific manner so as to produce efficient disk erasure correcting codes. Other examples of these orderings include various Gray codes and universal cycles. A **universal cycle** over a set of combinatorial objects represented as strings of length n is an ordering that requires that the last $n-1$ letters of one string to match the first $n-1$ letters of its successor in the listing [2].

To prove Theorem 1.1, Hanani used six different recursive constructions and various base cases. Using these constructions, we create 1-overlap cycles for each SQS(v). An s-**overlap cycle**, or s-**ocycle**, is a generalization of a universal cycle that relaxes the almost complete $n-1$ overlap between successive elements. Instead of requiring that the last $n-1$ letters of one string match the first $n-1$ letters of its successor, an s-ocycle requires just the last s letters of one to match the first s letters of the next. See [5] for some background on ocycles, including the construction of s-ocycles for m-ary words, and see [7] for 1-ocycles and Steiner triple systems. In this paper, we achieve the following result using Hanani's six recursive constructions.

Theorem 1.2. *For every $v \equiv 2, 4 \pmod 6$ with $v > 4$, there exists an SQS(v) that admits a 1-ocycle.*

Note that s-ocycles may be thought of as a special type of Gray code. Define a t-**swap Gray code** on a design $(X, \mathcal{B})$ to be an ordering of the blocks in $\mathcal{B}$ as $B_1, B_2, \ldots, B_n$ in which $|B_i \cap B_{i+1}| \geq t$. Then an s-ocycle produces an s-swap Gray code, but the converse is not true. Thus, while Pike, Vandell, and Walsh [8] have solved the 1-swap problem (and more general variations of it), we attack the 1-ocycle problem here.

We begin with some fundamental lemmas for 1-ocycles on Steiner quadruple systems in Section 2, then Section 3 presents Hanani's original recursive SQS constructions from [6] alongside our corresponding 1-ocycle constructions, and finally conclude with some open problems in Section 4.

2. MAIN TECHNIQUES

We now introduce some lemmas about 1-ocycles that will be used frequently. Throughout this section we assume that $(X, \mathcal{B})$ is an SQS(n).

Lemma 2.1. *Let $\mathcal{B}$ be a set of blocks on the point set X with 1-ocycle $\mathcal{O}$ on $\mathcal{B}$. Let $f : X \to Y$ be a bijection. Then $f(\mathcal{O})$ is a 1-ocycle on $f(\mathcal{B})$ over point set $f(X) = Y$.*

Proof. We simply rename the points within the ocycle consistently as given by the function f. □

Lemma 2.2. *Let $\mathcal{B}$ be a set of blocks on the point set X with ℓ-ocycle $\mathcal{O}$. Then the cycle $\overline{\mathcal{O}}$, which is the cycle $\mathcal{O}$ with points in reverse order, is also an ℓ-ocycle for $\mathcal{B}$.*

Proof. Since blocks are unordered sets, it does not matter what order their elements appear in the ocycle so long as they are consecutive. This is preserved when the cycle $\mathcal{O}$ is reversed. □

Lemma 2.3. *Let $\mathcal{B}$ be a set of blocks over point set X such that all blocks $B \in \mathcal{B}$ contain the two distinct points $a, b \in X$. If $|\mathcal{B}|$ is even, then there is a 1-ocycle for $\mathcal{B}$.*

Proof. Since all blocks contain points a and b and there are an even number of blocks, we create the following cycle.

$$\begin{array}{c} a \dots b \\ b \dots a \\ a \dots b \\ \vdots \\ b \dots a \end{array}$$

□

Lemma 2.4. *Let $X = \{0, 1, \dots, x\}$ and $Y = \{0, 1, \dots, y\}$. Then there exists a set of 1-ocycles on the set of all blocks of the form*

$$\{ij, ij', (i+1)j, (i+1)j'\},$$

where $i \in Y$, $j, j' \in X$, and $j \neq j'$.

Proof. Consider one pair $j \neq j' \in X$. Create the following cycle $\mathcal{O}_{j,j'}$.

$$\begin{array}{cccc} 0j & 0j' & 1j' & 1j \\ 1j & 1j' & 2j' & 2j \\ 2j & 2j' & 3j' & 3j \\ & \vdots & & \\ (x-1)j & (x-1)j' & xj' & xj \\ xj & xj' & 0j' & 0j \end{array}$$

Then the set $C = \{\mathcal{O}_{j,j'} \mid j \neq j' \in X\}$ is the desired set of 1-ocycles. □

Lemma 2.5. *Let X be a set of points with $a, b \in X$, $Y = \{0, 1, \dots, y\}$, and $\mathcal{B}$ be a set of blocks over $Y \times X$ such that:*

1. *each block may be represented by a pair $\{ia, (i+1)b\}$ for some $i \in Y$, and*
2. *given (1), every pair $\{ia, (i+1)b\}$ represents exactly one block in $\mathcal{B}$.*

Then there exists a set of 1-ocycles on the set $\mathcal{B}$.

Proof. For each possible pair $[a, b]$, we will create a set of cycles $C_{a,b}$. We have three cases.

a = b: In this case, the set $C_{a,a} = C_{b,b}$ contains only the following cycle.

$$\begin{array}{ll} 0a & \dots 1a \\ 1a & \dots 2a \\ 2a & \dots 3a \\ & \vdots \\ (y-1)a & \dots ya \\ ya & \dots 0a \end{array}$$

a ≠ b and y even: In this case, the set $C_{a,b}$ contains only the following cycle.

$$\begin{array}{c} 0a \ldots 1b \\ 1b \ldots 2a \\ 2a \ldots 3b \\ \vdots \\ ya \ldots 0b \\ 0b \ldots 1a \\ 1a \ldots 2b \\ 2b \ldots 3a \\ \vdots \\ yb \ldots 0a \end{array}$$

a ≠ b and y odd: In this case, the set $C_{a,b}$ contains the following two cycles.

1.
$$\begin{array}{c} 0a \ldots 1b \\ 1b \ldots 2a \\ 2a \ldots 3b \\ \vdots \\ yb \ldots 0a \end{array}$$

2.
$$\begin{array}{c} 0b \ldots 1a \\ 1a \ldots 2b \\ 2b \ldots 3a \\ \vdots \\ ya \ldots 0b \end{array}$$

Then the set that we are looking for is the set

$$C = \{C_{a,b} \mid a, b \in X\}. \qquad \square$$

Lemma 2.6. *Let* $X = \{0, 1, \ldots, x\}$ *and* $Y = \{[r(0), s(0)], [r(1), s(1)], \ldots, [r(t), s(t)]\}$. *Then there exists a set of 1-ocycles on the set of all blocks of the form*

$$\{ir(a), is(a), (i+1)r(b), (i+1)s(b)\},$$

where $i \in X$ *and* $a, b \in \{0, 1, \ldots, t\}$ *(possibly equal).*

Proof. For each $d = 0, 1, \ldots, t$ we partition the set Y into classes of pairs such that $[r(i), s(i)]$ and $[r(j), s(j)]$ are in the same class if and only if $j - i \equiv 0 \pmod d$. Each class corresponds to a set of cycles as follows. The first cycle C_1 begins as follows.

$$\begin{array}{cccc} 0r(0) & 0s(0) & 1s(d) & 1r(d) \\ 1r(d) & 1s(d) & 2s(2d) & 2r(2d) \\ 2r(2d) & 2s(2d) & 3s(3d) & 3r(3d) \\ & \vdots & & \end{array}$$

Eventually this cycle will reach some block

$$\{ir(-d), is(-d), (i+1)s(0), (i+1)r(0)\}.$$

If $i+1 \not\equiv 0 \pmod{x+1}$, then we continue on with the block

$$\{(i+1)r(0), (i+1)s(0), (i+2)s(d), (i+2)r(d)\},$$

and so forth. Continuing we must eventually arrive at the block

$$\{xr(-d), xs(-d), 0s(0), 0r(0)\},$$

at which point we may close off the cycle. Now either every block of the form

$$\{ir(j), is(j), (i+1)s(j+d), (i+1)r(j+d)\}$$

has been covered by C_1, or we are missing some, either because (1) we partitioned Y into more than one class or because (2) $\gcd\left(x+1, \frac{|Y|}{\gcd(|Y|,d)}\right) > 1$. If we are in case (1), then we repeat the procedure and create a cycle beginning with an arbitrary pair $[r(i), s(i)]$ as shown below.

$$\begin{array}{cccc} 0r(i) & 0s(i) & 1s(i+d) & 1r(i+d) \\ 1r(i+d) & 1s(i+d) & 2s(i+2d) & 2r(i+2d) \\ 2r(i+2d) & 2s(i+2d) & 3s(i+3d) & 3r(i+3d) \\ & \vdots & & \\ xr(i-d) & xs(i-d) & 0s(i) & 0r(i) \end{array}$$

If we are in case (2), then we repeat the first cycle using a different choice from X to begin, as follows.

$$\begin{array}{cccc} jr(0) & js(0) & (j+1)s(d) & (j+1)r(d) \\ (j+1)r(d) & (j+1)s(d) & (j+2)s(2d) & (j+2)r(2d) \\ (j+2)r(2d) & (j+2)s(2d) & (j+3)s(3d) & (j+3)r(3d) \\ & \vdots & & \\ (j-1)r(-d) & (j-1)s(-d) & js(0) & jr(0) \end{array}$$

Note that when $d = 0$ or $d = \frac{t}{2}$, this covers all possible blocks for our choice of d. However, for all other values of d we must create more cycles as follows. Given a cycle already created of the form:

$$\begin{array}{cccc}
jr(i) & js(i) & (j+1)s(i+d) & (j+1)r(i+d) \\
(j+1)r(i+d) & (j+1)s(i+d) & (j+2)s(i+2d) & (j+2)r(i+2d) \\
(j+2)r(i+2d) & (j+2)s(i+2d) & (j+3)s(i+3d) & (j+3)r(i+3d) \\
 & \vdots & & \\
(j-1)r(i-d) & (j-1)s(i-d) & js(i) & jr(i)
\end{array}$$

we create the following second cycle.

$$\begin{array}{cccc}
jr(i) & js(i) & (j-1)s(i+d) & (j-1)r(i+d) \\
(j-1)r(i+d) & (j-1)s(i+d) & (j-2)s(i+2d) & (j-2)r(i+2d) \\
(j-2)r(i+2d) & (j-2)s(i+2d) & (j-3)s(i+3d) & (j-3)r(i+3d) \\
 & \vdots & & \\
(j+1)r(i-d) & (j+1)s(i-d) & js(i) & jr(i)
\end{array}$$

From these cycles we define the set C_d to consist of all of these 1-ocycles. Finally, we obtain the complete set of 1-ocycles by taking

$$C = \{C_d \mid d \in \{0, 1, \ldots, t\}\}. \qquad \square$$

3. OCYCLES FOR HANANI'S CONSTRUCTIONS

This section contains seven subsections — the first six for each of Hanani's recursive SQS constructions, and subsection 3.7 proves Theorem 1.2, our main result. For each construction, we will first present Hanani's result and then present our theorem for a 1-ocycle based on his constructions.

Before we begin, we make a note that Hanani defines two systems of unordered pairs in [6]. These systems, referred to as $P_\alpha(m)$ and $\overline{P}_\xi(m)$, are necessary for his constructions of Steiner quadruple systems. However, our methods of constructing ocycles do not depend on the precise definitions of these sets, so we refer the reader to [6] for a complete definition, which will be omitted here.

3.1 Construction: $n \to 2n$

The first construction produces an SQS($2n$) from an SQS(n).

Construction 3.1. *Let $(X, \mathcal{B})$ be an SQS with $|X| = n$. Let $X = \{0, 1, \ldots, n-1\}$, and define a new point set $\mathcal{Y} = \{0, 1\} \times X$. The blocks on $\mathcal{Y}$ that form an SQS($2n$) are as follows:*

1. *$\{a_1x, a_2y, a_3z, a_4t\}$, where $\{x, y, z, t\} \in \mathcal{B}$ and $a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 2$; and*
2. *$\{0j, 0j', 1j, 1j'\}$, where $j \neq j'$.*

Theorem 3.2. *Let $(X, \mathcal{B})$ be an SQS of order n. Then there exists an SQS of order $2n$ that admits a 1-ocycle.*

Proof. We use Construction 3.1. To create cycles on the quadruples we do the following.

1. Fix $a_1, a_2 \in \{0, 1\}$ and $\{x, y, z, t\} \in \mathcal{B}$. We have two choices for a_3, and this choice completely determines a_4. Thus by fixing a_1, a_2 and $\{x, y, z, t\}$, we have identified two blocks of type (1). Then we can apply Lemma 2.3.
2. For each $d \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, define the following cycle.

$$\begin{array}{llll} 00 & 10 & 1d & 0d \\ 0d & 1d & 1(2d) & 0(2d) \\ 0(2d) & 1(2d) & 1(3d) & 0(3d) \\ \vdots & & & \end{array}$$

We continue this cycle until we arrive back at 00. At this point, if $\gcd(d, n) = 1$, we will have covered all blocks of difference d. However, if $\gcd(d, n) > 1$, then we may need multiple cycles to cover all blocks. In this case we just start anew with the first block missed.

To connect these cycles, we look to the blocks of type (2). Note that when $d = 1$ we will obtain one long cycle, call it $\mathcal{C}$, that has all points $0j$ with $j \in \{0, 1, 2, \ldots n-1\}$ as overlap points. Then we can join all cycles from (2) to $\mathcal{C}$. To attach the cycles of type (1), we utilize cycles from (1) with $a_1 = 0$. We can connect these all to $\mathcal{C}$ using the overlap points $a_1x = 0x$. For the remaining cycles with $a_1 = 1$, it is again possible to connect to $\mathcal{C}$ as follows. If $a_2 = 0$ we can attach at the points $a_2y = 0y$ on $\mathcal{C}$, and if $a_2 = 1$ we can attach at the points $a_2y = 1y$ that exist on the cycles from (1) with $a_1 = 0, a_2 = 1$. □

3.2 Construction: $n \to 3n - 2$

The second construction produces an SQS$(3n-2)$ from an SQS(n).

Construction 3.3. *Let $(X, \mathcal{B})$ be an SQS with $|X| = n$. Let*

$$X = \{0, 1, 2, \ldots, n-2\} \cup \{A\}.$$

Let $\mathcal{B} = \mathcal{B}_A \cup \overline{\mathcal{B}_A}$ where $\mathcal{B}_A$ denotes all blocks containing A and $\overline{\mathcal{B}_A}$ denotes all blocks not containing the point A. We construct new blocks on the set

$$\mathcal{Y} = (\{0, 1, 2\} \times \{0, 1, 2, \ldots, n-2\}) \cup \{A\}.$$

Note that $\mathcal{Y}$ has cardinality $1 + 3(n-1) = 3n-2$. The new blocks are as follows:

1. $\{a_1x, a_2y, a_3z, a_4t\}$, *where* $\{x, y, z, t\} \in \overline{\mathcal{B}_A}$ *and* $a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 3$;
2. $\{A, b_1u, b_2v, b_3w\}$, *where* $\{A, u, v, w\} \in \mathcal{B}_A$ *and* $b_1 + b_2 + b_3 \equiv 0 \pmod 3$;
3. $\{iu, iv, (i+1)w, (i+2)w\}$, *where* $i \in \{0, 1, 2\}$ *and* $\{A, u, v, w\} \in \mathcal{B}_A$;
4. $\{ij, ij', (i+1)j, (i+1)j'\}$, *where* $i \in \{0, 1, 2\}$ *and* $j, j' \in \{0, 1, 2, \ldots, n-2\}$ *and* $j \neq j'$; *and*
5. $\{A, 1j, 2j, 3j\}$, *where* $j \in \{0, 1, 2, \ldots n-2\}$.

Theorem 3.4. *Let $(X, \mathcal{B})$ be an SQS of order n. Then there exists an SQS of order $3n - 2$ that admits a 1-ocycle.*

Proof. We use Construction 3.3. To create cycles on the quadruples we do the following.

1. Fix $a_1, a_2 \in \{0, 1, 2\}$ and a block $\{x, y, z, t\} \in \overline{\mathcal{B}_A}$. Note that these choices determine a set of three blocks in the SQS($3n - 2$) since we may choose any $a_3 \in \{0, 1, 2\}$, but then our choices have completely determined a_4. For each $a_1 \in \{0, 1, 2\}$ and $\{x, y, z, t\} \in \overline{\mathcal{B}_A}$, we create the following 1-ocycle.

$$\begin{array}{cccc|c}
0z & a_1x & a_4t & 0y & \\
0y & 1z & a_4t & a_1x & a_2 = 0 \\
a_1x & 0y & a_4t & 2z & \\
\hline
2z & 1y & a_4t & a_1x & \\
a_1x & a_4t & 1z & 1y & a_2 = 1 \\
1y & a_4t & 0z & a_1x & \\
\hline
a_1x & 2z & a_4t & 2y & \\
2y & 1z & a_4t & a_1x & a_2 = 2 \\
a_1x & a_4t & 2y & 0z &
\end{array}$$

2. Fix $\{A, u, v, w\} \in \mathcal{B}_A$ and choose $b_1 \in \{0, 1, 2\}$. This identifies the following three blocks, where $\{b_3^0, b_3^1, b_3^2\} = \{0, 1, 2\}$.

$$\{A, 0v, b_3^0w, b_1u\}, \quad \{A, 1v, b_3^1w, b_1u\}, \quad \text{and} \quad \{A, 2v, b_3^2w, b_1u\}$$

Note that the order of $\{b_3^0, b_3^1, b_3^2\}$ depends entirely on our choice of b_1. We can string together the groups of three blocks for each choice of b_1 to create the following 1-ocycle.

$$\begin{array}{cccc|c}
0v & A & 0w & 0u & \\
0u & 1v & 2w & A & b_1 = 0 \\
A & 0u & 1w & 2v & \\
\hline
2v & 1u & 0w & A & \\
A & 1v & 1w & 1u & b_1 = 1 \\
1u & 0v & 2w & A & \\
\hline
A & 2v & 2w & 2u & \\
2u & 1v & 0w & A & b_1 = 2 \\
A & 2u & 1w & 0v &
\end{array}$$

3. Fix $\{A, u, v, w\} \in \mathcal{B}_A$ and $i \in \{0, 1, 2\}$. The three blocks identified create the following 1-ocycle.

$$\begin{array}{cccc}
iu & (i+1)w & (i+2)w & iv \\
iv & (i+1)u & (i+2)u & iw \\
iw & (i+1)v & (i+2)v & iu
\end{array}$$

4. This is a direct application of Lemma 2.4.

5. We can create the following short strings.

$$\begin{array}{cccc} 0j & 1j & 2j & A \\ A & 2(j+1) & 1(j+1) & 0(j+1) \end{array}$$

To create an ocycle, we will utilize a few of the cycles from (4). For each even $j \in \{0, 1, 2, \ldots, n-1\}$, set $j' = j+1$. Then we modify the corresponding cycle from (4) to include one of the short strings, as shown below.

$$\begin{array}{cccc} 0j & 0j' & 1j' & 1j \\ 1j & 1j' & 2j' & 2j \\ 2j & 2j' & 0j' & 0j \end{array} \rightarrow \begin{array}{cccc} 0j & 0j' & 1j' & 1j \\ 1j & 1j' & 2j' & 2j \\ 2j & 2j' & 0j & 0j' \\ \hline 0j' & 1j' & 2j' & A \\ A & 2j & 1j & 0j \end{array}$$

Note that this also ensures that A appears as an overlap point, so together with $\mathcal{C}$ from (4), we are ensured that *every* point appears as an overlap point.

To connect these cycles and make one ocycle, we consider the blocks of type (3). Fix $i \in \{0, 1, 2\}$ and $u \in \{0, 1, 2, \ldots, n-2\}$ and let v vary through $\{0, 1, 2, \ldots, n-2\} \setminus \{u\}$. Then each of these cycles from (3) containing $\{A, u, v\}$ has the point iu as an overlap point so we can connect all of the cycles to make a long cycle, call it $\mathcal{C}_{i,u}$. To connect $\mathcal{C}_{0,u}, \mathcal{C}_{1,u}$, and $\mathcal{C}_{2,u}$ we use a cycle from (4) with $j = u$. Now we have created a cycle containing every point $ij \in \{0, 1, 2\} \times \{0, 1, 2, \ldots, n-2\}$ as an overlap point, so every other cycle can connect to this one. □

3.3 Construction: $n \rightarrow 3n - 8$

The third construction produces an SQS($3n-8$) from an SQS(n) when $n \equiv 2 \pmod{12}$.

Construction 3.5. *Let $(X, \mathcal{B})$ be an SQS with $|X| = n$ and $n \equiv 2 \pmod{12}$. Let*

$$X = \{0, 1, 2, \ldots, n-5\} \cup \{Ah : h \in \{0, 1, 2, 3\}\}.$$

We will make the assumption that $\{A0, A1, A2, A3\}$ is a block in $\mathcal{B}$. Define

$$\mathcal{Y} = (\{0, 1, 2\} \times \{0, 1, 2, \ldots, n-5\}) \cup \{Ah : h \in \{0, 1, 2, 3\}\}.$$

Note that $\mathcal{Y}$ has cardinality $3(n-4)+4 = 3n-8$. We construct blocks on $\mathcal{Y}$ as follows:

1. $\{A0, A1, A2, A3\}$*;*
2. $\{ix, iy, iz, it\}$*, where $\{x, y, z, t\} \in \mathcal{B} \setminus \{A0, A1, A2, A3\}$ (if one of x, y, z, t is Ah, omit the i) – we denote this operation by $i \oplus (\mathcal{B} \setminus \{A0, A1, A2, A3\})$;*
3. $\{Aa_1, 0a_2, 1a_3, 2a_4\}$*, where $a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{n-4}$;*
4. $\{(i+2)b_3, i(b_1 + 2k + 1 + i(4k+2) - d), i(b_1 + 2k + 2 + i(4k+2) + d), (i+1)b_2\}$*, where $n - 4 = 12k + 10$, $b_1 + b_2 + b_3 \equiv 0 \pmod{n-4}$, and $d \in \{0, 1, \ldots, 2k\}$; and*
5. $\{ir_\alpha, is_\alpha, (i+1)r'_\alpha, (i+1)s'_\alpha\}$*, where $[r_\alpha, s_\alpha], [r'_\alpha, s'_\alpha] \in P_\alpha(6k+5)$ (possible the same) with $\alpha = 4k+2, 4k+3, \ldots, 12k+8$.*

Theorem 3.6. *Let $(X, \mathcal{B})$ be an SQS(n) with $n \equiv 2 \pmod{12}$ that admits a 1-ocycle. Then there exists an SQS$(3n-8)$ that admits a 1-ocycle.*

Proof. We use Construction 3.5. We create cycles as follows on each set of blocks.

1. We will add this block to a cycle over blocks of type (2).
2. By Lemma 2.1, there exists a 1-ocycle C_i on $i \oplus \mathcal{B}$. However, each cycle uses the block $\{A0, A1, A2, A3\}$ so we can only use one of these cycles, say $\mathcal{C}_0$. For $\mathcal{C}_1$ and $\mathcal{C}_2$, we get two strings by removing the block $\{A0, A1, A2, A3\}$. Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ can then be joined together at the endpoints. For example, if the block appears as $A0, A1, A2, A3$ (in order) in $\mathcal{C}$, then we create the cycle
$$A3 \quad \mathcal{C}_1 \quad A0 \quad \overline{\mathcal{C}_2} \quad A3$$
where $\overline{\mathcal{C}_2}$ is the string $\mathcal{C}_2$ listed in reverse order (which is a valid 1-overlap string by Lemma 2.2).
3. If we fix a_1, a_2, then we have restricted our attention to $n-4$ distinct blocks, each of which contain the two points Aa_1 and $0a_2$. By Lemma 2.3 we can construct a 1-ocycle on this set of blocks.
4. Fix $d \in \{0, 1, \ldots, 2k\}$. The blocks corresponding to this choice of d can be uniquely represented by their pair $\{(i+1)b_2, (i+2)b_3\}$. Thus for each choice of d we can produce a set of 1-ocycles using Lemma 2.5. We do this for all blocks except those corresponding to $d = 0$ and $b_2 = 0$, which are reserved for the final step of the proof.
5. Fix α, and arbitrarily order the set $P_\alpha(6k+5) = \{[r(0), s(0)], [r(1), s(1)], \ldots, [r(t), s(t)]\}$. By Lemma 2.6, there exists a set of 1-ocycles on this set of blocks.

To connect all of these cycles, we first focus on the unused blocks of (4). We create one long cycle by fixing $b_2 = 0$ and creating a cycle C_{b_3} for each $b_3 \in \{0, 1, 2, \ldots, n-5\}$. We connect all of these cycles at their points 00 to create one long cycle C that has every overlap of the type
$$\{0, 1, 2\} \times \{0, 1, 2, \ldots, n-5\}.$$
Using this cycle C, we can connect every other cycle to C to make one long cycle containing all quadruples. □

3.4 Construction: $n \to 3n-4$

The fourth construction produces an SQS$(3n-4)$ from an SQS(n) when $n \equiv 10 \pmod{12}$.

Construction 3.7. *Let $(X, \mathcal{B})$ be an SQS with $|X| = n$ and $n \equiv 10 \pmod{12}$. Let*
$$X = \{0, 1, 2, \ldots, n-3\} \cup \{A0, A1\}.$$
Define
$$\mathcal{Y} = (\{0, 1, 2\} \times \{0, 1, \ldots, n-3\}) \cup \{A0, A1\}.$$

Note that $\mathcal{Y}$ has cardinality $3(n-2)+2=3n-4$. *We construct blocks on $\mathcal{Y}$ as follows:*

1. $\{ix, iy, iz, it\}$, *where* $\{x, y, z, t\} \in \mathcal{B}$, *or* $i \oplus \mathcal{B}$ *(if one of x, y, z, t is Ah, omit the i);*
2. $\{Aa_1, 0a_2, 1a_3, 2a_4\}$, *where* $a_1+a_2+a_3+a_4 \equiv 0 \pmod{n-2}$ *and* $a_1 \in \{0,1\}$ *and* $a_2, a_3, a_4 \in \{0, 1, 2, \ldots, n-2\}$*;*
3. $\{(i+2)b_3, i(b_1+2k+1+i(4k+2)-d), i(b_1+2k+2+i(4k+2)+d), (i+1)b_2\}$, *where* $n = 12k+10$, $b_1+b_2+b_3 \equiv 0 \pmod{n-2}$, *and* $d = 0, 1, \ldots, 2k$*; and*
4. $\{ir_\alpha, is_\alpha, (i+1)r'_\alpha, (i+1)s'_\alpha\}$, *where* $[r_\alpha, s_\alpha], [r'_\alpha, s'_\alpha] \in P_\alpha(6k+4)$ *(possible the same) with* $\alpha = 4k+2, 4k+3, \ldots, 12k+6$.

Theorem 3.8. *Let* $(X, \mathcal{B})$ *be an SQS*(n) *with* $n \equiv 10 \pmod{12}$ *that admits a 1-ocycle. Then there exists an SQS*$(3n-4)$ *that admits a 1-ocycle.*

Proof. We use Construction 3.7. To form cycles, we do the following for each type.

1. There is a 1-ocycle on $i \oplus \mathcal{B}$ by Lemma 2.1.
2. If we fix a_1, a_2, then we have restricted our attention to $n-2$ distinct blocks. Note that $n-2$ is even, and so we may apply Lemma 2.3 to find a 1-ocycle on this set of $n-2$ blocks.
3. Fix $d \in \{0, 1, \ldots, 2k\}$. The blocks corresponding to this choice of d can be uniquely represented by their pair $\{(i+1)b_2, (i+2)b_3\}$. Thus for each choice of d we can produce a set of 1-ocycles using Lemma 2.5. We do this for all blocks except those corresponding to $d=0$ and $b_2=0$, which are reserved for the final step of the proof.
4. Fix α, and arbitrarily order the set $P_\alpha(6k+4) = \{[r(0), s(0)], [r(1), s(1)], \ldots, [r(t), s(t)]\}$. By Lemma 2.6, there exists a set of 1-ocycles on this set of blocks.

To connect all of these cycles, we first focus on (3). We create one long cycle by fixing $b_2 = 0$ and creating a cycle C_{b_3} for each $b_3 \in \{0, 1, 2, \ldots, n-3\}$. We connect all of these cycles at their points 00 to create one long cycle C that has every overlap of the type

$$\{0, 1, 2\} \times \{0, 1, 2, \ldots, n-3\}.$$

Using this cycle C, we can connect every other cycle to C to make one long cycle containing all quadruples. □

3.5 Construction: $n \to 4n-6$

The fifth construction produces an SQS$(4n-6)$ from an SQS(n).

Construction 3.9. *Let* $(X, \mathcal{B})$ *be an SQS*(n) *with*

$$X = \{0, 1, \ldots, n-2\} \cup \{A0, A1\}.$$

Define

$$\mathcal{Y} = (\{0,1\} \times \{0,1\} \times \{0, 1, \ldots, n-3\}) \cup \{A0, A1\}.$$

Note that $\mathcal{Y}$ has cardinality $(2)(2)(n-2)+2 = 4n-6$. *We construct blocks on $\mathcal{Y}$ as follows:*

1. $h \oplus i \oplus \mathcal{B}$*, where* $h \in \{0,1\}$ *and* $i \in \{0,1\}$*, and we ignore the prefix hi from the points* $A0, A1 \in X$*;*
2. $\{A\ell, 00(2c_1), 01(2c_2-\epsilon), 1\epsilon(2c_3+\ell)\}$*, where* $\ell, \epsilon \in \{0,1\}$ *and* $c_1+c_2+c_3 \equiv 0 \pmod{k}$*, where* $n = 2k$*;*
3. $\{A\ell, 00(2c_1+1), 01(2c_2-1-\epsilon), 1\epsilon(2c_3+1-\ell)\}$*;*
4. $\{A\ell, 10(2c_1), 11(2c_2-\epsilon), 0\epsilon(2c_3+1-\ell)\}$*;*
5. $\{A\ell, 10(2c_1+1), 11(2c_2-1-\epsilon), 0\epsilon(2c_3+\ell)\}$*;*
6. $\{h0(2c_1+\epsilon), h1(2c_2-\epsilon), (h+1)0\overline{r}_{c_3}, (h+1)0\overline{s}_{c_3}\}$*, where* $[\overline{r}_{c_3}, \overline{s}_{c_3}] \in \overline{P}_{c_3}(k)$ *and* $c_3 \in \{0,1,\ldots,k-1\}$*;*
7. $\{h0(2c_1-1+\epsilon), h1(2c_2-\epsilon), (h+1)1\overline{r}_{c_3}, (h+1)1\overline{s}_{c_3}\}$*;*
8. $\{h0(2c_1+\epsilon), h1(2c_2-\epsilon), (h+1)1\overline{r}_{k+c_3}, (h+1)1\overline{s}_{k+c_3}\}$*;*
9. $\{h0(2c_1-1+\epsilon), h1(2c_2-\epsilon), (h+1)0\overline{r}_{k+c_3}, (h+1)0\overline{s}_{k+c_3}\}$*; and*
10. $\{h0r_\alpha, h0s_\alpha, h1r'_\alpha, h1s'_\alpha\}$*, where* $[r_\alpha, s_\alpha], [r'_\alpha, s'_\alpha] \in P_\alpha(k)$ *and* $\alpha \in \{0,1,\ldots,n-4\}$.

Theorem 3.10. *Let* $(X, \mathcal{B})$ *be an SQS*(n) *that admits a 1-ocycle. Then there exists an SQS*$(4n-6)$ *that admits a 1-ocycle.*

Proof. We use Construction 3.9. To create cycles on these blocks, we do the following.

1. There is a 1-ocycle on $h \oplus i \oplus \mathcal{C}$ by Lemma 2.1.
2. We will combine these with the triples of type (3). Note in particular that each block is completely determined by our choice of $2c_2^{(2)} - \epsilon^{(2)}$ and $2c_3^{(2)} + \ell^{(2)}$ from $\{0,1,2,\ldots,n-3\}$, where superscript (2) denotes a variable corresponding to a block of type (2), and similarly for superscript (3) and blocks of type (3).
3. Note that each block is completely determined by our choice of $2c_2^{(3)} - 1 - \epsilon^{(3)}$ and $2c_3^{(3)} + 1 - \ell^{(3)}$ from $\{0,1,2,\ldots,n-3\}$. Fix $x \in \{0,1,2,\ldots,n-3\}$, and define the cycles as follow, which alternates between blocks of type (2) (where $x = 2c_3^{(2)} + \ell^{(2)}$) and (3) (where $x = 2c_3^{(3)} + 1 - \ell^{(3)}$).
   We will connect pairs of blocks (one of type (2), one of type (3)) with $2c_2^{(2)} - \epsilon^{(2)} = 2c_2^{(3)} - 1 - \epsilon^{(3)}$. Note that this implies that $\epsilon^{(2)} \neq \epsilon^{(3)}$. These blocks are connected to make short strings as shown below by matching a block of type (2) and (3) in which $2c_2^{(2)} - \epsilon^{(2)} = 2c_2^{(3)} - 1 - \epsilon^{(3)}$.

$$\begin{array}{llll} 1\epsilon^{(2)}x, & A\ell^{(2)}, & 00(2c_1^{(2)}), & 01(2c_2^{(2)} - \epsilon^{(2)}) \\ 01(2c_2^{(3)} - 1 - \epsilon^{(3)}), & 00(2c_1^{(3)} + 1), & A\ell^{(3)}, & 1\epsilon^{(3)}x \end{array}$$

To connect these two-block strings, we define $y = 2c_2^{(2)} - \epsilon^{(2)} = 2c_2^{(3)} - 1 - \epsilon^{(3)}$, and let y range from 0 up to $n-3$ to create a cycle covering all of these strings.

$$\begin{array}{lll} 10x & \cdots & 010 \\ 010 & \cdots & 11x \\ 11x & \cdots & 011 \\ 011 & \cdots & 10x \\ 10x & \cdots & 012 \\ 012 & \cdots & 11x \\ \vdots & & \\ 01(n-4) & \cdots & 11x \\ 11x & \cdots & 01(n-3) \\ 01(n-3) & \cdots & 10x \end{array}$$

For each choice of x we will have one cycle, and each cycle has length $2(n-2)$, covering a total of $2(n-2)^2$ blocks.

4. We will combine these with the triples of type (5). Note that if we fix ℓ and c_1, then we have restricted our attention to an even number of blocks since we can choose $\epsilon \in \{0, 1\}$.
5. Fix $\ell \in \{0, 1\}$ and choose $c_1^{(4)}, c_1^{(5)}$ so that $2c_1^{(4)} = 2c_1^{(5)} + 1$. Then all blocks of types (4) and (5) contain the two points $A\ell$ and $10(2c_1^{(4)})$, and this set of blocks has even cardinality. Thus we may apply Lemma 2.3 to find a 1-ocycle on this set of blocks for each choice of $\ell, c_1^{(4)}$.
6. We will combine these with the triples of type (8).
7. We will combine these with the triples of type (9).
8. Note that the first two terms in both (6) and (8) are the same. If we fix $c_1, c_2, c_3, \epsilon, h$, and $[\overline{r}_{c_3}, \overline{s}_{c_3}]$, then we have restricted our attention to an even number of blocks (one of type (6) and one of type (8)), all containing the terms $h0(2c_1 + \epsilon)$ and $h1(2c_2 - \epsilon)$. Thus there is a 1-ocycle on these two blocks by Lemma 2.3.
9. Note that the first two terms in both (7) and (9) are the same. As in the previous step, we fix all variables to restrict our attention to just two blocks, one of type (7) and one of type (9), and then apply Lemma 2.3.
10. We will apply Lemma 2.6 by first renaming the prefixes $00, 01, 10, 11$ as $0, 1, 2, 3$, respectively. Then fix α and write $P_\alpha(k) = \{[r(0), s(0)], [r(1), s(1)], \ldots, [r(t), s(t)]\}$. Now we may apply Lemma 2.6 to create a set of 1-ocycles covering blocks of type (10).

Now we must connect all of these cycles. Note that the cycles described in (3) contain as overlap points all points of the form $11y$ for $y \in \{0, 1, 2, \ldots, n-3\}$, regardless of our choice of x. Thus we can connect all of these cycles together to make one cycle. This one long cycle contains as overlap points all points of the form hiy for $hi \in \{01, 10, 11\}$ and $y \in \{0, 1, 2, \ldots, n-3\}$. Once we attach the cycles from (8) to this by using the overlap points $h1(2c_2 - \epsilon)$, we also have as overlaps all points of the form $00y$ for $y \in \{0, 1, 2, \ldots, n-3\}$. Thus we can connect everything else to this cycle. □

3.6 Construction: $n \to 12n - 10$

The sixth and final construction produces an SQS($12n - 10$) from an SQS(n), and begins with the constructions of an SQS(14) and an SQS(38). **SQS(14)** (listed as a 1-ocycle):

2043	0BC2	C841	38B5	0BD6	B9CA	259C
3162	28B1	19B4	5AC3	6BC1	A05B	CA68
25D3	1250	4095	36AD	18D6	B79D	80B9
37A2	03B1	5134	DBC3	629D	DA7C	9158
28C3	1460	4265	36C4	DB51	CD40	8249
3560	0781	57A4	48B6	17C5	08A4	9368
07C3	19D0	48D5	60A7	51A6	4B07	87A9
38D0	0AC1	57D0	7196	69B5	749C	9CD8
09A3	19A2	08C5	62C7	5C6D	CDA4	87CB
39B2	2CD1	5BC4	73B6	D59A	4B2D	B17A
2680	17D3	4783	64D7	A26B	D872	A964
0792	38A1	3D49	7586	B34A	27B5	4AD1
2AD0	139C	9573	69C0	A8DB	58A2	1472

SQS(38):

Construction 3.11. *We identify the points from the SQS(14) with the set*

$$X = \{0, 1, 2, \dots, 11\} \cup \{A0, A1\},$$

and let $\mathcal{B}$ be the set of blocks form the SQS(14) on X. Then we define our new point set $\mathcal{Y}$ as

$$\mathcal{Y} = (\{0, 1, 2\} \times \{0, 1, 2, \dots, 11\}) \cup \{A0, A1\}.$$

Note that $\mathcal{Y}$ has cardinality $(3)(12) + 2 = 38$. The blocks on $\mathcal{Y}$ are as follows:

1. $i \oplus \mathcal{B}$, *where we omit the prefix i for $A0$ or $A1$;*
2. $\{Ah, 0b_1, 1b_2, 2(b_3 + 3h)\}$, *where* $b_1 + b_2 + b_3 \equiv 0 \pmod{12}$ *and* $h \in \{0, 1\}$*;*
3. $\{i(b_1 + 4 + i), i(b_1 + 7 + i), (i + 1)b_2, (i + 2)b_3\}$*;*
4. $\{ij, (i + 1)(j + 6\epsilon), (i + 2)(6\epsilon - 2j + 1), (i + 2)(6\epsilon - 2j - 1)\}$, *where* $\epsilon \in \{0, 1\}$*;*
5. $\{ij, (i + 1)(j + 6\epsilon), (i + 2)(6\epsilon - 2j + 2), (i + 2)(6\epsilon - 2j - 2)\}$*;*
6. $\{ij, (i + 1)(j + 6\epsilon - 3), (i + 2)(6\epsilon - 2j + 1), (i + 2)(6\epsilon - 2j + 2)\}$*;*
7. $\{ij, (i + 1)(j + 6\epsilon + 3), (i + 2)(6\epsilon - 2j - 1), (i + 2)(6\epsilon - 2j - 2)\}$*;*
8. $\{ij, i(j + 6), (i + 1)(j + 3\epsilon), (i + 1)(j + 6 + 3\epsilon)\}$*;*
9. $\{i(2g + 3\epsilon), i(2g + 6 + 3\epsilon), i'(2g + 1), i'(2g + 5)\}$, *where* $i' \neq i$ *and* $g \in \{0, 1, 2, 3, 4, 5\}$*;*
10. $\{i(2g + 3\epsilon), i(2g + 6 + 3\epsilon), i'(2g + 2), i'(2g + 4)\}$*;*
11. $\{ij, i(j + 1), (i + 1)(j + 3e), (i + 1)(j + 3e + 1)\}$, *where* $e = 0, 1, 2, 3$*;*
12. $\{ij, i(j + 2), (i + 1)(j + 3e), (i + 1)(j + 3e + 2)\}$*;*
13. $\{ij, i(j + 4), (i + 1)(j + 3e), (i + 1)(j + 3e + 4)\}$*; and*
14. $\{ir_\alpha, is_\alpha, i'r'_\alpha, i's'_\alpha\}$, *where* $[r_\alpha, s_\alpha], [r'_\alpha, s'_\alpha] \in P_\alpha(6)$ *for* $\alpha = 4, 5$.

Theorem 3.12. *There exists an SQS(38) that admits a 1-ocycle.*

Proof. To construct a 1-overlap cycle, we utilize Construction 3.11 above and look at each type of block separately.

1. Since we have a 1-overlap cycle for the SQS(14), by Lemma 2.1 there is a 1-ocycle on $i \oplus$ SQS(14).
2. Fix b_1 and b_2 and we have restricted our attention to two blocks that both contain points $0b_1$ and $1b_2$. By Lemma 2.3, we can create a 1-ocycle on these two blocks.
3. The blocks of this type can be uniquely represented by their pair $\{(i+1)b_2, (i+2)b_3\}$. Thus we can produce a set of 1-ocycles using Lemma 2.5.
4. We will combine blocks of type (4) with blocks of type (6).
5. We will combine blocks of type (5) with blocks of type (7).
6. Note that the first term and the third term in the block of type (4) are the same as those for the blocks of type (6). Thus for each choice of i, j, ϵ we apply Lemma 2.3 to create a short two-block 1-ocycle.
7. Note that the first term and the last term in the block of type (5) are the same as those for the blocks of type (7). Thus for each choice of i, j, ϵ we apply Lemma 2.3 to create a short two-block 1-ocycle.
8. For each choice of $\epsilon \in \{0, 1\}$, we have two blocks that contain the points $ij, i(j+6)$ of type (8). We then apply Lemma 2.3 to create a short two-block 1-ocycle.
9. We will combine these with the blocks of type (10).
10. Note that the first term and the second term in the block of type (9) are the same as those for the blocks of type (10). Thus for each choice of i, g, i', ϵ we apply Lemma 2.3 to create a short two-block 1-ocycle.
11. Fix i, j. This restricts us to four blocks that all contain the points ij and $i(j+1)$. Then we apply Lemma 2.3 to create a short four-block 1-ocycle.
12. Same as (11).
13. Same as (11).
14. Since $i, i' \in \{0, 1, 2\}$ are distinct and their order is irrelevant, we can always consider $i' = i + 1$. Then for each $\alpha \in \{4, 5\}$, we arbitrarily order

$$P_\alpha(6) = \{[r(0), s(0)], [r(1), s(1)], \ldots, [r(t), s(t)]\}.$$

   Then we can apply Lemma 2.6 to find a set of 1-ocycles covering these blocks.

To connect all of these cycles, we look to the blocks of type (2). Fix b_1, and connect all of the short cycles corresponding to this b_1 together at the point $0b_1$. Note that this cycle has as overlap points $0b_1$ and every point $1j$ for $j \in \{0, 1, 2, \ldots, 11\}$. Do this for each choice of b_1, and then each of these cycles contains the point 10 as an overlap, so they can all be connected. From this, we have constructed one cycle that contains every point ij with $i, j \in \{0, 1, 2, \ldots, 11\}$ as an overlap point. All other cycles can be connected to this. □

Construction 3.13. *Let $(X, \mathcal{B})$ be an SQS with $|X| = n$. Let*

$$X = \{B\} \cup \{0, 1, 2, \ldots, n-2\}.$$

Let $\mathcal{B}_B$ the subset of $\mathcal{B}$ with blocks containing B, and $\overline{\mathcal{B}_B}$ to be the complement. Define

$$\mathcal{Y} = (\{0, 1, 2, \dots, n-2\} \times \{0, 1, 2, \dots, 11\}) \cup \{A0, A1\}.$$

Note that $\mathcal{Y}$ has cardinality $(n-1)(12) + 2 = 12n - 10$. We construct blocks on $\mathcal{Y}$ as follows:

1. *$i \oplus \mathcal{B}(14)$, where $i \in \{0, 1, \dots, n-2\}$ and where i is omitted if the point is of the form Ah;*
2. (a) *$\{Ah, ub_1, vb_2, w(b_3 + 3h)\}$, where $\{u, v, w, B\} \in \mathcal{B}_B$ and $b_1 + b_2 + b_3 \equiv 0 \pmod{12}$;*
   (b) *$\{u\alpha_1, v\alpha_2, w\alpha_3, w\alpha_4\}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the second indices (in order) of blocks of types (3)–(7) in the SQS(38);*
   (c) *$\{i\beta_1, i\beta_2, i'\beta_3, i'\beta_4\}$, where $\{i, i', B\}$ defines a unique block in $\mathcal{B}_B$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are the second indices (in order) of blocks of type (8)–(14) in the SQS(38); and*
3. *$\{xa_1, ya_2, za_3, ta_4\}$, where $\{x, y, z, t\} \in \overline{\mathcal{B}_B}$ and $a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{12}$.*

Theorem 3.14. *Let $(X, \mathcal{B})$ be an SQS(n) that admits a 1-ocycle. Then there exists an SQS$(12n - 10)$ that admits a 1-ocycle.*

Proof. We use Construction 3.13. To create cycles on these blocks, we do the following.

1. Since we can construct a 1-ocycle on $\mathcal{B}(14)$, by Lemma 2.1 we can construct a 1-ocycle on $i \oplus \mathcal{B}(14)$ for each choice of i.
2. (a) These blocks are completely determined by choosing $b_1, b_2 \in \{0, 1, \dots, 11\}$ and $h \in \{0, 1\}$. If we fix b_1, b_2, then we can connect the two blocks identified as follows.

$$ub_1 \cdots vb_2 \cdots ub_1.$$

   We do this for each choice of b_1 and b_2 to make many short two-block cycles.
   (b) Blocks corresponding to type (4)–(7) in the SQS(38) have the same structure as their SQS(38)-counterparts, therefore, we make cycles in exactly the same way. For blocks of type (3) in the SQS(38), we can use the same structure, since only the last two points are used as overlaps. In this SQS$(12n - 10)$, this corresponds to the points $w\alpha_3$, $w\alpha_4$, and so we can create cycles in the same way.
   (c) We can make cycles in the same way as in the SQS(38) for blocks of type (8)–(14).
3. Fix $\{x, y, z, t\} \in \overline{\mathcal{B}_B}$. Our new block is completely determined b choosing $a_1, a_2, a_3 \in \{0, 1, \dots, 11\}$. If we fix a_1, a_2 and let a_3 run through the set $\{0, 1, \dots, 11\}$, then we have identified 12 blocks, all of which contain the points xa_1 and ya_2. Using these as our overlap points, we have a 1-ocycle covering these twelve blocks, which in compressed form is $xa_1, ya_2, xa_1, ya_2, \dots, xa_1, ya_2$.

To connect all of these cycles, we look to the blocks of type (2). Fix u, b_1, and let everything else vary. These many short cycles can all be connected at the point ub_1, and the cycle created contains every point as an overlap point, except $A0$, $A1$. We can connect all other cycles to this one. □

3.7 Ocycles for Steiner Quadruple Systems

Theorem 3.15. *For $n \equiv 2, 4 \pmod 6$ with $n > 4$, there exists an SQS(n).*

Proof. Proceed by induction on n. Since it is not possible to create a 1-ocycle for an SQS(2) or SQS(4), we begin our base cases with $n = 8, 10$. If $n = 8$, we have the following SQS, arranged in 1-ocycle form.

2148, 8523, 3684, 4578, 8156, 6287, 7813, 3576, 6471, 1572, 2163, 3274, 4135, 5462.

When $n = 10$, we have the following SQS(10), arranged in 1-ocycle form.

2145	6907	1372	7268	2694	7491
5263	7108	2483	8379	4681	1693
3674	8192	3594	9480	1583	3608
4785	2903	4506	0591	3705	8520
5896	3401	6157	1602	5297	0472

Before Construction 3.13, we illustrated an SQS(14) and an SQS(38).

Let $n \geq 16$ with $n \equiv 4, 8 \pmod{12}$. Then $n = 2v$ for some $v \equiv 2, 4 \pmod 6$. Since $n \geq 16$, $v \geq 8$, and hence we use Construction 3.1.

Let $n \geq 22$ with $n \equiv 4, 10 \pmod{18}$. Then $n = 3v - 2$ for some $v \equiv 2, 4 \pmod 6$. Since $n \geq 22$, $v \geq 8$, and hence we use Construction 3.3.

Let $n \geq 26$ with $n \equiv 2, 10 \pmod{24}$. Then $n = 4v - 6$ for some $v \equiv 2, 4 \pmod 6$. Since $n \geq 26$, $v \geq 8$, and hence we use Construction 3.9.

Let $n \geq 26$ with $n \equiv 26 \pmod{36}$. Then $n = 3v - 4$ for some $v \equiv 10 \pmod{12}$. Since $n \geq 26$, $v \geq 10$, and hence we use Construction 3.7.

Let $n \geq 34$ with $n \equiv 34 \pmod{36}$. Then $n = 3v - 8$ for some $v \equiv 2 \pmod{12}$. Since $n \geq 34$, $v \geq 14$, and hence we use Construction 3.5.

Let $n \geq 86$ with $n \equiv 14, 38 \pmod{72}$. Then $n = 12v - 10$ for some $v \equiv 2, 4 \pmod 6$. Since $n \geq 86$, $v \geq 8$, and hence we use Construction 3.13. □

Putting all of the 1-ocycle constructions together with the base cases, we get the following result.

Theorem 3.16. *For all $n \equiv 2, 4 \pmod 6$ with $n \geq 8$, there exists an SQS(n) that admits a 1-ocycle.*

Proof. These constructions, together with Theorem 3.15 and the corresponding base cases, we have a 1-ocycle for each order. □

4. FUTURE WORK

Theorem 3.16 shows that for every allowable n we can construct an SQS(n) that admits a 1-ocycle. Can we find 2-ocycles that correspond to Hanani's SQS constructions?

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