$$\left[\begin{array}{cccc|cccccc|c} 0 & 0 & 1459 & 0 & 61 & 130 & 0 & 73 & 303 & 0 & -11732 \\ 0 & 0 & -806 & 0 & 0 & -150 & 61 & -103 & -237 & 0 & 13429 \\ 0 & 0 & -164 & 61 & 0 & -6 & 0 & -9 & -29 & 0 & 1174 \\ 0 & 61 & 292 & 0 & 0 & 33 & 0 & 19 & 68 & 0 & -3041 \\ 61 & 0 & 131 & 0 & 0 & 10 & 0 & 15 & 28 & 0 & -1306 \\ \hline 0 & 0 & -83610 & 0 & 0 & -4938 & 0 & -2527 & -13619 & 61 & 560186 \end{array}\right]$$

A certificate that proves Problem 7.1.7 is infeasible is given by the multipliers $\mathbf{y} = (61, 130, 0, 73, 303)^\mathsf{T}$, found as the coefficients of the slack variables in the halting row of Tableau 7.1.8.

Theorem 7.1.9 *Suppose that Problem 7.1.1 halts in Phase I on a row that represents the equation $\sum_{j=1}^{n+m} a'_j x_j = b'$. Then the multipliers $y_i = a'_{n+i}$ $(1 \le i \le m)$ certify that Problem 7.1.1 is infeasible.*

Proof. The proof proceeds in 3 steps.

Lemma 7.1.10 *Every row of a Simplex tableau is a linear combination of the rows of the original tableau.*

Workout 7.1.11 *Prove Lemma 7.1.10.*

Lemma 7.1.12 *Let R be a row of a Simplex tableau have entries a'_j $(1 \le j \le n+m)$, with right hand side b'. Suppose that the multipliers y_i $(1 \le i \le m)$ of the rows of the original tableau produce row R. Then $y_i = a'_{n+i}$ for $1 \le i \le m$.*

Workout 7.1.13 *Prove Lemma 7.1.12.*

Lemma 7.1.14 *The multipliers defined in Lemma 7.1.12 produce a contradiction.*

Workout 7.1.15 *Prove Lemma 7.1.14.*

$\diamond$

Note that $(61, 130, 0, 73, 303)^\mathsf{T}/61$ is not dual-feasible. However, it does have a nice property — it can be used to find a certificate for dual unboundedness. Before explaining how, let's create a small example to study.

Workout 7.1.16

(a) *Draw a diagram of an unbounded LOP D with two variables.*

(b) *Write its corresponding dual P.*

(c) *Obtain a P-infeasible certificate $\mathbf{y}'$.*

(d) *Obtain a D-unbounded certificate $\mathbf{y}(t) = \mathbf{y}^0 + t\overrightarrow{\mathbf{y}}$.*

(e) *Compare $\mathbf{y}'$ and $\overrightarrow{\mathbf{y}}$.*

(f) *Is it possible that $\mathbf{y}' = \overrightarrow{\mathbf{y}}$?*

Workout 7.1.17 *Suppose that $\mathbf{y}'$ is a feasible solution to the dual D of Problem 7.1.7. Use $(61, 130, 0, 73, 303)^\mathsf{T}/61$ to find a certificate for dual unboundedness. Will your method always work in general?*

7.2 Inconsistency

Back at the Varyim Portint Co., one must remember to be prepared — recall from Section 4.4 how swiftly the president can act. Without the offending row in hand, how might one construct an infeasible certificate for the boss on the spot? Is there a method that is independent of the problem?

Because a LOP is infeasible precisely when its system of constraints is unsolvable, we hereafter consider systems of constraints only. Let us now consider the general System 7.2.1 below.

System 7.2.1

$$\begin{array}{ll} \sum_{j=1}^{n} a_{i,j}x_j \leq b_i & (i \in I) \\ \sum_{j=1}^{n} a_{i,j}x_j = b_i & (i \in E) \\ x_j \geq 0 & (j \in R) \end{array}$$

Suppose that System 7.2.1 (S) is unsolvable, and let P be a LOP having S as its system of constraints. Then P is infeasible and the Simplex algorithm will halt in Phase I. What this means, according to Lemma 7.1.10, is that some linear combination of the constraints of S produces an offending row in the halting Simplex tableau of P. What does such a row look like? Phase I means that the right-hand side is negative, and the halting condition means that there is no coefficient on which to pivot. For free variables, this means that their coefficients are zero, and for restricted variables, this means their coefficients are nonnegative. With this in mind we consider the following **alternative** System 7.2.2.

alternative system

System 7.2.2

$$\begin{array}{ll} \sum_{i=1}^{m} b_i y_i = -1 & \\ \sum_{i=1}^{m} a_{i,j}y_i \geq 0 & (j \in R) \\ \sum_{i=1}^{m} a_{i,j}y_i = 0 & (j \in F) \\ y_i \geq 0 & (i \in I) \end{array}$$

Note that part of this system would be the system of constraints for the alternative of P if the objective function for P were identically zero. The added constraint seems to force the dual objective function to be -1. It seems natural, then, that both systems cannot be solvable, a line of reasoning we will explore. The following definition suggests a link between the two systems.

inconsistent system

Definition 7.2.3 *We say that a system is* **inconsistent** *if its alternative system is solvable.*

The above argument almost shows that an unsolvable system is inconsistent. Demanding that the right-hand side equals -1 instead of an arbitrary negative is not much of a stretch: once negative, the multipliers can be scaled to make it any negative number requested. The only thing missing from the discussion is the restriction on some of the y_is. Notice that the presence of equalities means that some of the coefficients expected to appear by Theorem 7.1.9 do not exist. However, those that do correspond to values of $i \in I$, which must be nonnegative in the offending row of the halting tableau.

Workout 7.2.4 *Consider Problem 6.1.4 (P), having constraint system S.*

(a) *Write the alternative system S'.*

(b) *Solve S'.*

(c) *Find the offending row that halts the Simplex algorithm on P.*

(d) *Compare your answers to parts b and c.*

The result of the above discussion is the following theorem, often called the Theorem of the Alternative.

Theorem 7.2.5 *A system is unsolvable if and only if it is inconsistent.*

Theorem of the Alternative

Proof. It is clear that inconsistent implies unsolvable – a solution to the alternative system serves as a certificate. While the converse has been argued above, we offer a more direct inference using duality. Suppose that System 7.2.1 is unsolvable and consider the following LOP, where $s_i = \text{sign}(b_i)$.

Problem 7.2.6

$$\begin{array}{lll} \text{Max.} & z = \sum_{i=1}^{m} -x_{n+i} & \\ \text{s.t.} & \sum_{j=1}^{n} a_{i,j}x_j + s_i x_{n+i} \leq b_i & (i \in I) \\ & \sum_{j=1}^{n} a_{i,j}x_j + s_i x_{n+i} = b_i & (i \in E) \\ \& & x_j \geq 0 & (j \in R) \\ & x_{n+i} \geq 0 & (1 \leq i \leq m) \end{array}$$

Workout 7.2.7 *Use the General Fundamental Theorem 6.2.9 to prove that Problem 7.2.6 is optimal, and that its optimum is negative.*

Workout 7.2.8

(a) *Write the dual of Problem 7.2.6.*

(b) *Use the General Duality Theorem 6.4.1 to draw conclusions about its dual.*

(c) *Use the optimal dual solution $\mathbf{y}^*$ to construct a solution to System 7.2.2.*

◇

7.3 Practice This

Consider the following systems.

System 7.3.1

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & = & 7 \\ -x_1 & + & 3x_2 & + & x_3 & = & -3 \\ x_1 & & & + & x_3 & \leq & 2 \\ 2x_1 & + & 2x_2 & + & 3x_3 & \leq & 5 \\ 4x_1 & - & 2x_2 & - & 3x_3 & \leq & 6 \\ \\ & & x_2 & , & x_3 & \geq & 0 \end{array}$$