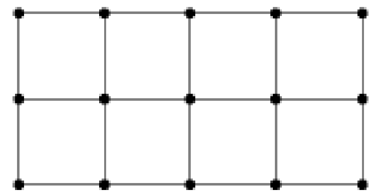


Figure 11.2: Street map for Gridburg

11.2 Covers

Problem 11.2.1 *The Police Chief of Gridburg decides to place* 15 *policemen at the* 15 *street corners of his town, as shown in the map of Figure 11.2. A policemen has the ability to see the activity of people on the streets leading from his street corner, but only as far as one block. The Mayor fires the Chief for spending over budget, and offers the job to whomever can, with the fewest possible policemen, make sure that every street can be seen by some policeman.*

Workout 11.2.2 *Become the new Police Chief.*

Problem 11.2.3 *The Commerce Secretary of Gridburg decides to place* 3 *hot dog vendors on the* 3 *East-West streets of his town (see the map of Figure 11.2). The Hotdogger's Union requires that no two vendors can be on street blocks that share an intersection. The Mayor fires the Secretary for not generating enough commerce in town, and offers the job to whomever can place the most vendors, subject to the union restriction.*

Workout 11.2.4 *Become the new Commerce Secretary.*

Recall from Section 1.2 that a cover $\mathcal{C}$ in a graph $\mathcal{G}$ is a set of vertices such that every edge of $\mathcal{G}$ has at least one of its endpoints in $\mathcal{C}$. For example, the set of all vertices is a cover, and the job of the Police Chief of Gridburg in Problem 11.2.1 is to place policemen at the vertices of a cover in the Figure 11.2 graph. It is not difficult to find large covers in graphs; the challenge is to find small ones. Similarly, it is the job of the Commerce Secretary of Gridburg in Problem 11.2.3 to place vendors on the edges of a matching in the Figure 11.2 graph. It is likewise not difficult to find small matchings (such as the empty matching); the challenge is in finding large ones.

Workout 11.2.5 *Prove that every graph has* $|\mathcal{M}| \leq |\mathcal{C}|$ *for every matching* $\mathcal{M}$ *and cover* $\mathcal{C}$.

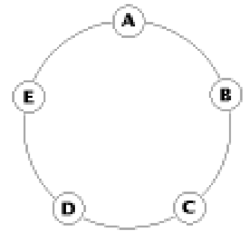
A matching $\mathcal{M}$ is **maximum** if no other matching contains more edges than $\mathcal{M}$. The new Commerce Secretary in Problem 11.2.3 must find a maximum matching in the Figure 11.2 graph. A cover $\mathcal{C}$ is **minimum** if no other cover contains fewer vertices than $\mathcal{C}$. The new Police Chief in Problem 11.2.1 must find a minimum cover in the Figure 11.2 graph. As usual, we use the notations $\mathcal{M}^*$ and $\mathcal{C}^*$ to denote any maximum matching and minimum cover, respectively.

maximum matching

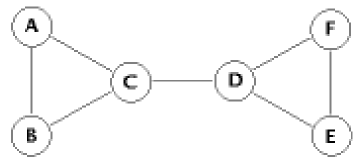
minimum cover

Workout 11.2.6 *For each of the following two graphs, find a maximum matching* $\mathcal{M}^*$ *and a minimum cover* $\mathcal{C}^*$ *possible.*

(a)



(b)



Workout 11.2.7 *Find a sequence of graphs $\{\mathcal{G}_n\}$ for which $|\mathcal{C}^*(\mathcal{G}_n)| - |\mathcal{M}^*(\mathcal{G}_n)| \to \infty$ as $n \to \infty$.*

Workout 11.2.6 illustrates that odd cycles in a graph may cause $|\mathcal{M}^*| < |\mathcal{C}^*|$. Exercises 11.5.2 and 11.5.19 show that a graph is bipartite if and only if it contains no odd cycles. Thus, Workouts 11.2.2 and 11.2.4 illustrate the following theorem.

König-Egerváry Theorem

Theorem 11.2.8 *Every bipartite graph has $|\mathcal{M}^*| = |\mathcal{C}^*|$.*

Proof. In light of Workout 11.2.5, we need only prove that $|\mathcal{C}^*| \leq |\mathcal{M}^*|$. We will do this by associating a network to a given bipartite graph. From the optimal solution to the network we will find both $\mathcal{M}^*$ and $\mathcal{C}^*$, and then show that the inequality holds. Several illustrations along the way should help prove the various steps.

Let $\mathcal{B}$ be a bipartite graph with left vertices $L_1, \ldots L_s$ and right vertices $R_1, \ldots, R_t$. Without loss of generality we assume that $\mathcal{B}$ is connected. We define the network $\mathcal{N} = \mathcal{N}(\mathcal{B})$ to have supply nodes $L_1, \ldots, L_s$ (each of supply 1) and demand nodes $R_1, \ldots, R_t$ (each of demand 1), with two extra "infinity" nodes L_∞ (having demand s) and R_∞ (having supply t). There is a 0-cost arc $L_iR_j \in \mathcal{N}$ for every edge $\{L_i, R_j\} \in \mathcal{B}$, and there are 0-cost arcs L_iL_∞ and $R_\infty R_j$ in $\mathcal{N}$ for every $1 \leq i \leq s$ and $1 \leq j \leq t$. Finally, there is a special "infinity" arc $R_\infty L_\infty \in \mathcal{N}$, having cost -1.

Workout 11.2.9 *Draw the network $\mathcal{N}_0$ that corresponds to the bipartite graph $\mathcal{B}_0$, below.*