On the pebbling numbers of Flower, Blanuša and Watkins snarks[☆]

Matheus Adauto [a], Celina de Figueiredo [a], Glenn Hurlbert [b], Diana Sasaki [c,*]

[a] *Federal University of Rio de Janeiro, Rio de Janeiro, Brazil*
[b] *Virginia Commonwealth University, Richmond, United States of America*
[c] *Rio de Janeiro State University, Rio de Janeiro, Brazil*

ARTICLE INFO

ABSTRACT

Graph pebbling is a game played on graphs with pebbles on their vertices. A pebbling move removes two pebbles from one vertex and places one pebble on an adjacent vertex. The pebbling number $\pi(G)$ is the smallest t so that from any initial configuration of t pebbles it is possible, after a sequence of pebbling moves, to place a pebble on any given target vertex. In this paper, we provide the first results on the pebbling numbers of snarks. Until now, only the Petersen graph had its pebbling number correctly established, although attempts had been made for the Flower and Watkins snarks.

1. Introduction

Graph pebbling is a mathematical game or puzzle that involves moving pebbles along the edges of a connected graph, subject to certain rules. The objective of the game is to place a certain number of pebbles on specific vertices of the graph, typically with the aim of reaching a particular configuration of pebbles or minimizing the number of moves required to achieve a given configuration. Various forms of graph pebbling have applications in number theory, computer science, physics, and combinatorial optimization, and have been studied extensively in mathematics (see [10]).

In this paper, $G = (V, E)$ is always a simple connected graph. The numbers of vertices and edges of G and its diameter are denoted by $n(G)$, $e(G)$, and $D(G)$, respectively. For a vertex w and a positive integer k, denote by $N_k[w]$ the set of all vertices that are at a distance at most k from w. The *girth* is the length of a shortest cycle in the graph.

1.1. Pebbling number

A *configuration* C on a graph G is a function $C : V(G) \to \mathbb{N}$. The value $C(v)$ represents the number of pebbles at vertex v. A vertex with zero, one, at most one, or at least two pebbles on it is called *empty*, *a singleton*, *small*, or *big*, respectively. The *size* $|C|$ of a configuration C is the total number of pebbles on G. A *pebbling move* consists of removing two pebbles from a vertex and placing one pebble on an adjacent vertex. For a *target* vertex r, C is *r-solvable* if one can place a pebble on r after a sequence of pebbling moves, and is *r-unsolvable* otherwise. It was shown in [11,12] that deciding if C is r-solvable on G is NP-complete. The *pebbling number* $\pi(G, r)$ is the minimum number t such that every configuration of size t is r-solvable. The *pebbling number of G* equals $\pi(G) = \max_r \pi(G, r)$.
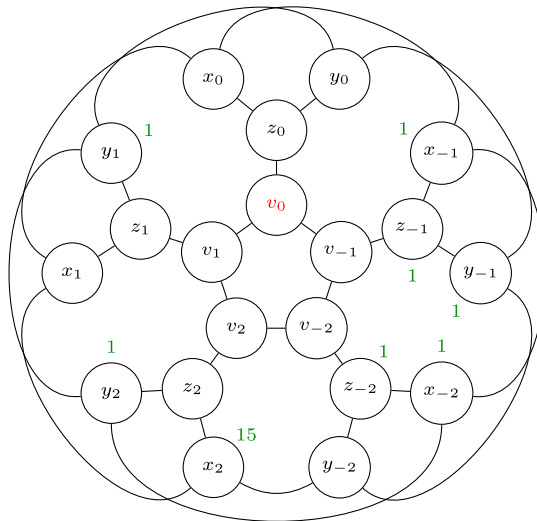

Fig. 1. The graph J_5 and its (green) v_0-unsolvable configuration C of size 22, which equals the configuration C^* with an extra pebble on z_{-1}. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Bounds on the pebbling numbers of several well known snarks.

Snark	$n(G)$	$D(G)$	$\pi(G)$
Petersen	10	2	10
Flower J_3	12	3	$12 \leq \pi(J_3) \leq 13$
Flower J_5	20	4	$23 \leq \pi(J_5) \leq 30$
Flower J_7	28	5	$41 \leq \pi(J_7) \leq 61$
Flower J_m ($m = 2k + 1 \geq 7$)	$4m$	$k + 2$	$2^{k+2} + 9 \leq \pi(J_m) \leq \lfloor 2^{k+2}9/5 + 2k - 18/5 \rfloor + 1$
Blanuša (1 and 2)	18	4	$23 \leq \pi(G) \leq 34$
Loupekine (1 and 2)	22	4	$24 \leq \pi(G) \leq 271$
Double-Star	30	4	$32 \leq \pi(G) \leq 391$
Szekeres	50	7	$128 \leq \pi(G) \leq 5462$
Watkins	50	7	$183 \leq \pi(G) \leq 5462$

The basic lower and upper bounds for every graph are $\max\{n(G), 2^{D(G)}\} \leq \pi(G) \leq (n(G) - D(G))(2^{D(G)} - 1) + 1$ [4,7]. A graph is called *Class 0* if $\pi(G) = n(G)$. Complete graphs, cubes, the Petersen graph, and many other graphs are known to be Class 0, whereas the cycle graphs C_n satisfy: $\pi(C_{2d}) = 2^d$ and $\pi(C_{2d+1}) = \lceil (2^{d+2} - 1)/3 \rceil$. It is not yet known whether or not there exist necessary and sufficient conditions for a graph to be Class 0.

1.2. Snarks

We define the important family of *snark* graphs which are cubic, bridgeless, 4-edge-chromatic graphs. They are important for being related to the Four Color Theorem, which holds if and only if no snark is planar [17]. In [1] we find the origins of the study of the pebbling numbers of chordal graphs. Here we begin the systematic study of the pebbling numbers of snarks.

The Petersen graph is the smallest snark, having 10 vertices, and was discovered in 1898 [14]. Since then, many others have been discovered. There are no snarks of order 12, 14, 16, whereas snarks exist for any even order greater than 16. The Blanuša snarks are the two snarks discovered by Danilo Blanuša in 1946 [3], when only the Petersen snark was known [3]. Both Blanuša 1 and Blanuša 2 snarks have 18 vertices, diameter 4, and girth 5. In this work, they will be denoted by B_1 and B_2, respectively. Myriam Preissmann proved in 1982 that there are exactly two snarks of order 18 [15]. We depict in Fig. 3 the Blanuša 2 snark together with a vertex labeling that makes clear that the graph was obtained from two copies of the Petersen snark. See [3] for a thorough history and Table 1 for a list of several well known snarks.

For odd $m = 2k + 1 \geq 3$, we define the mth *Flower snark* J_m as follows (see Fig. 1 for an example with $m = 5$) [2]. For each $i \in \pm\{0, 1, \ldots, k\}$ we have vertices v_i, x_i, and y_i all adjacent to z_i. Thus the number of vertices of the mth Flower snark is $n(J_m) = 4m$. The vertices $\{v_i\}$ form the cycle C_m, with adjacencies given by consecutive indices modulo m. The vertices $\{x_i\}$ (resp. $\{y_i\}$) form a path given by the cycle without the edge $x_k x_{-k}$ (resp. $y_k y_{-k}$). Finally, we add the edges $x_k y_{-k}$ and $y_k x_{-k}$ so that the two paths now form one cycle C_{2m}. It is easy to see that J_m has m-fold rotational symmetry, with a necessary twist, along with the reflective symmetry that negates subscripts; that is, the automorphisms of J_m yield

three vertex orbits. Thus the only targets necessary to contemplate are, without loss of generality, v_0, z_0, and x_0. Thus the Flower snark J_m ($m = 2k + 1$) has diameter $k + 2$, as well as girth m for $m \in \{3, 5\}$ and 6 if $m \geq 7$.

The Watkins snark is the snark with 50 vertices discovered by John J. Watkins in 1989 [18] depicted in Fig. 4. The Watkins snark has diameter 7 and girth 5.

1.3. Motivation

Studying the pebbling numbers in specific snarks is important because it helps to explore the boundaries of graph properties in complex and challenging contexts. The importance of snarks arises from the fact that famous conjectures would have snarks as minimal counterexamples, for instance Tutte's 5-Flow Conjecture, the 1-Factor Double Cover Conjecture, and the Cycle Double Cover Conjecture. Evaluating pebbling numbers in snarks provides important insights into how pebbling problems behave on graphs with extreme properties, helping to refine and expand understanding of the limits and capabilities of pebbling algorithms. Furthermore, Pebbling concepts can be applied to communication networks and distributed computing. Understanding pebbling in extreme graphs like snarks can improve algorithms and techniques in these areas. Studying pebbling in snarks also tests the limits of current pebbling methods. With infinite families such as powers of paths, k-paths, and interval graphs, one can leverage inductive methods because of hereditary properties. With non-hereditary infinite families such as Kneser and Johnson graphs, there is much symmetry that one can exploit to obtain results. One can hope to be pushed into developing new tools by expanding to well-studied families with less symmetry but still sharing some implicit structure. This work is a preliminary step in that direction, first finding the limits of current pebbling methods on such a family. Hence, this study's importance lies in expanding theoretical knowledge about complex graphs and in the potential application of pebbling concepts in practical contexts and more general algorithms.

1.4. Results

It is known that the Petersen graph is Class 0 [7]. It is the smallest snark and was the only one whose pebbling number was known. We use the Small Neighborhood Lemma presented in Section 2 to prove that the Petersen graph is the only Class 0 snark with at least 23 vertices or girth at least 5.

Theorem 1. *The only* Class 0 *snark of girth at least 5 is the Petersen graph. Moreover, if G is a* Class 0 *snark with girth at most 4, then $n(G) \leq 22$.*

We also prove the following bounds on the pebbling numbers of snarks. Recall that the basic lower and upper bounds for a graph are $\max\{n(G), 2^{D(G)}\} \leq \pi(G) \leq (n(G) - D(G))(2^{D(G)} - 1) + 1$. For the Flower snarks, this means that $12 \leq \pi(J_3) \leq 64$, $20 \leq \pi(J_5) \leq 241$, and $32 \leq \pi(J_7) \leq 691$. Theorem 1 improves the J_5 lower bound to $21 \leq \pi(J_5)$. Theorem 2 provides much tighter bounds, and corrects a claim of [13] that, for $m \geq 5$, $\pi(J_m) = 4m + 1$. In fact, Theorem 2 identifies the correct order of magnitude (2^{k+2}) of $\pi(J_m)$ as m grows, up to some constant between 1 and 1.8.

Theorem 2. *We have $\pi(J_3) \leq 13$, $23 \leq \pi(J_5) \leq 30$, $41 \leq \pi(J_7) \leq 61$, and for all $k \geq 3$ with $m = 2k + 1$, we have $2^{k+2} + 8 \leq \pi(J_m) \leq \lfloor 2^{k+2}9/5 + 2k - 18/5 \rfloor + 1$.*

For the Blanuša snarks the basic bounds give $18 \leq \pi(B_i) \leq 211$. Theorem 1 improves the B_i lower bound to $19 \leq \pi(B_i)$. Theorem 3 provides much tighter bounds for B_i.

Theorem 3. *We have $23 \leq \pi(B_i) \leq 34$.*

For the Watkins snark W the basic bounds give $128 \leq \pi(W) \leq 5462$. Theorem 4 provides a tighter lower bound, and corrects a claim of [16] that $\pi(W) = 166$.

Theorem 4. *The pebbling number of the Watkins graph W satisfies $183 \leq \pi(W)$.*

2. Techniques

2.1. For lower bounds

Given a graph G with vertices u and v such that $N_a[u] \cap N_b[v] = \emptyset$ for some non-negative integers a and b. Define the configuration $C^* = C^*_{u,v}$ by $C^*(v) = 2^{a+b+1} - 1$, $C^*(x) = 0$ for all $x \in (N_a[u] \cup N_b[v]) - \{v\}$, and $C^*(x) = 1$ otherwise. The authors of [6] proved the following lemma (SNL) to provide a lower bound on $\pi(G)$.

Lemma 1 (*Small Neighborhood Lemma [6]*)**.** *Let G be a graph and $u, v \in V(G)$ such that $N_a[u] \cap N_b[v] = \emptyset$ for some non-negative integers a and b. Then C^* is u-unsolvable. Consequently, $\pi(G) \geq \pi(G, u) > |C^*|$. In particular, if $N_a[u] \cap N_b[v] = \emptyset$ and $|N_a[u] \cup N_b[v]| < 2^{a+b+1}$, then G is not Class 0.*

One can see how SNL is, in some sense, a sharpening of the basic exponential lower bound. One immediate consequence we will use here is the following corollary.

Corollary 1. *If G is an n-vertex cubic graph, with diameter at least 4 and girth at least 5, then there is an unsolvable configuration of size $15 + (n - 14)$.*

Another consequence we will use here is the following corollary.

Corollary 2 ([6]). *If G is an n-vertex Class 0 graph with diameter at least 3, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.*

A graph H is a *retract* of a graph G if there is a function $\phi : V(G) \to V(H)$ that preserves edges; that is, if $uv \in E(G)$ then $\phi(u)\phi(v) \in E(H)$.

We will make use of the following lemma in the proof of the lower bound of Theorem 3, since a subgraph of the Blanuša graph has a C_9 retract, and of the lower bound of Theorem 4, since a subgraph of the Watkins graph has a C_{15} retract.

Lemma 2 (*Retract Lemma [5]*). *If H is a retract of G, then $\pi(H) \leq \pi(G)$.*

The idea of the proof of Lemma 2 is straightforward, as any solution along edges in G has a corresponding solution along its retracted edges in H.

Another helpful lemma uses the notion of pebble weights. Given a target vertex r, define the r-*weight* of a pebble on a vertex v to be 2^{-k}, where k is the distance from v to r. Furthermore, the r-*weight* of a configuration C is defined to be the sum of the r-weights of its pebbles.

Lemma 3 ([8]). *Suppose that a configuration C on a graph G is r-solvable for some target vertex r. Then C has r-weight at least 1.*

The lemma is proved by noting that a pebbling step never increases the r-weight of a configuration, and any configuration with a pebble on the target has r-weight at least 1. In fact, the r-weight is maintained if and only if the pebbling step decreases the distance to the target r. Hence, when G is a path and r is one of its leaves, r-weight at least 1 characterizes r-solvable configurations, which is not true for general graphs. However, a generalization of this r-weight concept is introduced as weights for upper bounds in Section 2.2.

2.2. For upper bounds

Here we describe a linear optimization approach introduced in [9]. For an unknown configuration C on a graph G with target vertex r, we consider a connected subgraph H of G that contains r. The intention is to derive a linear inequality in the variables $C(v)$ with $v \in V(H)$ that is satisfied whenever C is r-unsolvable. Given a collection of such inequalities over various choices of H, we can then maximize $|C| = \sum_{v \in V(G)} C(v)$ subject to those constraints, assuming that $C(v) \geq 0$ for all $v \in V(G)$. The optimum value of this linear program is therefore a strict lower bound on $\pi(G, r)$. This value may be tight for some graphs; however, this is really an integer optimization problem, and so typically will yield a result less than the truth. This idea was successfully carried out when H is a tree (generalized to some non-trees in [6]). We introduce the method now.

Let T be a subtree of a graph G rooted at vertex r, with at least two vertices. For a vertex $v \in V(T)$ let v^+ denote the *parent* of v; i.e. the T-neighbor of v that is one step closer to r (we also say that v is a *child* of v^+). We call T an r-*strategy* when we associate with it a non-negative *weight function* w_T with the property that $w_T(r) = 0$ and $w_T(v^+) \geq 2w_T(v)$ for every other vertex v that is not a neighbor of r (and $w_T(v) = 0$ for vertices not in T). Let $\boldsymbol{T}$ be the configuration with $\boldsymbol{T}(r) = 0$, $\boldsymbol{T}(v) = 1$ for all other $v \in V(T)$, and $\boldsymbol{T}(v) = 0$ everywhere else. We now define the T-weight of any configuration C (including $\boldsymbol{T}$) by $w_T(C) = \sum_{v \in V} w_T(v)C(v)$. The following lemma (WFL) is used to provide an upper bound on $\pi(G)$.

Lemma 4 (*Weight Function Lemma [9]*). *Let T be an r-strategy of G with associated weight function w_T. Suppose that C is an r-unsolvable configuration of pebbles on $V(G)$. Then $w_T(C) \leq w_T(\boldsymbol{T})$.*

One way to view T-weights as a generalization of r-weights is as follows. Consider when T is a path $v_0 v_1 \cdots v_n$ and let C be a v_0-unsolvable configuration on T. Then notice that the formula for v_0-weights of pebbles on the vertices of T form a valid weight function w_T if we re-weight v_0 at 0 instead of 1, which changes nothing in practice because $C(v_0) = 0$. In this case, WFL states that $\sum_{i=1}^{n} C(v_i)2^{-i} \leq \sum_{i=1}^{n} 2^{-i} = 1 - 2^{-n}$. Because C is a non-negative integer-valued function, this is equivalent to $\sum_{i=1}^{n} C(v_i)2^{-i} < 1$; i.e. the v_0-weight of C is less than one. Furthermore, the r-weights of a general graph G can be thought of as the T-weights of a breadth-first-search spanning tree T of G, rooted at the target vertex r. Then an r-unsolvable configuration C satisfies the WFL inequality and so, by retracting T onto a path of length equal to the eccentricity of r, we obtain the aforementioned r-weight condition for C.

3. Proofs

3.1. Proof of Theorem 1

Note that every snark has exactly $3n/2$ edges, and that the Petersen graph has diameter equal to 2 and all other snarks have diameter at least 3. Then, by Corollary 2, if $n(G) > 22$ we get $3n/2 < (5n - 11)/3$. Therefore, every snark with $n(G) > 22$ is not Class 0. The remaining non-Petersen snarks with fewer vertices and girth at least 5 (among them, the Flower J_5, the Blanušas, and the Loupekines) all have diameter $4 > 2 + 1$, so for any vertices u and v at distance 4 from each other we have $|N_2[u]| = 10$ and $|N_1[v]| = 4$. Thus $|N_2[u] \cup N_1[v]| = 14 < 16 = 2^{2+1+1}$, and so none of these graphs are Class 0 by SNL. $\square$

3.2. Proof of Theorem 2

First we prove the lower bounds. For these we only need to display a configuration, of size one less than the lower bound, that cannot reach some target.

The diameter of the Flower graph J_m, $m = 2k + 1$, is $k + 2$, and the distance between v_0 and x_k is $k + 2$. The rotational symmetry of the Flower graph J_m gives $2m$ petals, each of which is in a 6-cycle.

For J_5, the diameter is 4 and the girth is 5. The v_0-unsolvable configuration $C^*_{v_0,x_2}$, that is provided by SNL for $a = 2$ and $b = 1$ places 15 pebbles on x_2 and one pebble on each of the 6 vertices not in set $N_2[v_0] \cap N[x_2]$, and has size 21. Notice that we can add a pebble to z_{-1} to obtain the configuration C in Fig. 1. It is not difficult to argue that C is also v_0-unsolvable, since any supposed solution would need to use the pebble at z_{-1}.

For $m \geq 7$ (i.e. $k \geq 3$), we will use $C^* = C^*_{v_0,x_k}$ only. In this case, the girth is equal to 6. So $|N_2[v_0]| = 1 + 3 + 6$, and for any integer $3 \leq i \leq k$, the set of vertices at distance i from v_0 is $\{v_i, v_{-i}, z_{i-1}, z_{-(i-1)}, x_{i-2}, y_{i-2}, x_{-(i-2)}, y_{-(i-2)}\}$. Hence, for any integer $2 \leq i \leq k$, we have $|N_i[v_0]| = 1 + 3 + 6 + (i - 2)8 = 8i - 6$. Similarly, one can enumerate $|N_i[u]| = 1 + 3 + 6 + (i - 2)8 = 8i - 6$ for any vertex u. Let a and b be non-negative integers that are at least 2 and such that $a + b = k + 1$. Vertices v_0 and x_k are at distance $k + 2$, which is the diameter of J_m, from which follows $|N_a[v_0]| + |N_b[x_k]| = 8(a + b) - 12 = 8k - 4$ and $n - |N_a[v_0]| - |N_b[x_k]| = 8$. In fact, one can check that $V(J_m) - (N_a[v_0] \cup N_b[x_k]) = \{y_{a-1}, y_a, x_{-(a-1)}, y_{-(a-1)}, x_{-a}, z_{-a}, x_{-(a+1)}, v_{-(a+1)}\}$. This yields $|C^*| = (2^{a+b+1} - 1) + (n - |N_a[v_0]| - |N_b[x_k]|) = 2^{k+2} + 7$.

Now we prove the upper bounds, using WFL. We shall define strategies with root z_0. We refer to the Appendix for the strategies with roots v_0 and x_0, that give values not larger than the ones given by strategies presented for root z_0.

For J_3, we define three z_0-strategies $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_{-1}$ by

- $\boldsymbol{T}_0(v_0, v_1, v_{-1}, z_1, z_{-1}, x_1, y_1, x_{-1}, y_{-1}) = (8, 4, 4, 2, 2, 1, 1, 1, 1)$,
- $\boldsymbol{T}_1(x_0, x_1, x_{-1}, z_1, z_{-1}, v_{-1}) = (8, 4, 4, 1, 2, 1)$ and
- $\boldsymbol{T}_{-1}(y_0, y_1, y_{-1}, z_1, z_{-1}, v_1) = (8, 4, 4, 2, 1, 1)$,

giving rise to the inequality $|C| \leq \frac{1}{5}(\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}) \leq 64/5$ whenever C is v_0-unsolvable. Hence $\pi(J_3, z_0) \leq 13$. Using the strategies presented in the Appendix for targets v_0 and x_0, we may conclude that $\pi(J_3) \leq 13$.

For J_5, we define three z_0-strategies $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_{-1}$ by

- $\boldsymbol{T}_0(v_0, v_1, v_{-1}, v_2, v_{-2}, z_1, z_{-1}, z_2, z_{-2}, x_2, y_2, x_{-2}, y_{-2}) = (16, 8, 8, 4, 4, 4, 4, 2, 2, 1, 1, 1, 1)$,
- $\boldsymbol{T}_1(x_0, x_1, x_{-1}, x_2, x_{-2}, z_2, z_{-2}, v_2, z_{-1}) = (16, 8, 8, 4, 4, 2, 1, 1, 1)$ and
- $\boldsymbol{T}_{-1}(y_0, y_1, y_{-1}, y_2, y_{-2}, z_2, z_{-2}, v_{-2}, z_1) = (16, 8, 8, 4, 4, 1, 2, 1, 1)$,

giving rise to the inequality $|C| \leq \frac{1}{5}(\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}) \leq 146/5$ whenever C is v_0-unsolvable. Hence $\pi(J_5, z_0) \leq 30$. Using the strategies presented in the Appendix for targets v_0 and x_0, we may conclude that $\pi(J_5) \leq 30$.

For $m \geq 5$ (i.e. $k \geq 2$), please refer to Fig. 2 to see $m = 11$. Using the same pattern we have defined above for the three z_0-strategies for J_5, we define three corresponding z_0-strategies by

- $\boldsymbol{T}_0(v_0, v_1, v_{-1}, v_2, v_{-2}, \ldots, v_k, v_{-k}, z_k, z_{-k}, x_k, y_k, x_{-k}, y_{-k})$
  $= (2^{k+2}, 2^{k+1}, 2^{k+1}, 2^k, 2^k, \ldots, 4, 4, 2, 2, 1, 1, 1, 1)$ and
  $\boldsymbol{T}_0(z_1, z_{-1}, \ldots, z_{k-2}, z_{2-k}, z_{k-1}, z_{1-k}) = (5, 5, \ldots, 5, 5, 4, 4)$;
- $\boldsymbol{T}_1(x_0, x_1, x_{-1}, \ldots, x_k, x_{-k}, z_k, v_k)$
  $= (2^{k+2}, 2^{k+1}, 2^{k+1}, \ldots, 4, 4, 2, 1)$ and $\boldsymbol{T}_1(z_{-k}, z_{1-k}) = (1, 1)$; and
- $\boldsymbol{T}_{-1}(y_0, y_1, y_{-1}, \ldots, y_k, y_{-k}, z_{-k}, v_{-k})$
  $= (2^{k+2}, 2^{k+1}, 2^{k+1}, \ldots, 4, 4, 2, 1)$ and $\boldsymbol{T}_{-1}(z_k, z_{k-1}) = (1, 1)$.

The sum $\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}$ has 3 vertices with coefficient 2^{k+2}, 6 with 2^i (for each $3 \leq i \leq k + 1$), and $2k + 6$ with coefficient 5, giving rise to the inequality

$$5|C| \leq \boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}$$
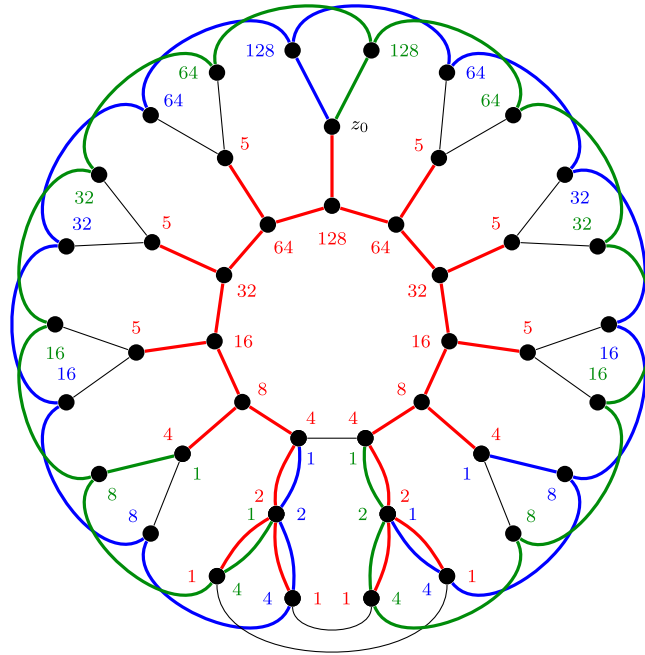$$= 3(2^{k+2}) + 6(2^3 + \cdots + 2^{k+1}) + 5(2k + 6)$$

Fig. 2. The graph J_{11} and its three z_0-strategies T_0 (in red), T_1 (in blue), and T_{-1} (in green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$= 6(2^3 + \cdots + 2^{k+2}) - 3(2^{k+2}) + 5(2k+6)$$
$$= 48(2^k - 1) - 3(2^{k+2}) + 10k + 30$$
$$= 36(2^k) + 10k - 18$$
$$= 9(2^{k+2}) + 10k - 18,$$

whenever C is z_0-unsolvable. Hence $|C| \leq 2^{k+2}9/5 + 2k - 18/5$, and so $\pi(J_m, z_0) \leq \lfloor 2^{k+2}9/5 + 2k - 18/5 \rfloor + 1$. Using the strategies presented in the Appendix for targets v_0 and x_0, we may conclude that $\pi(J_m) \leq \lfloor 2^{k+2}9/5 + 2k - 18/5 \rfloor + 1$. In particular, $\pi(J_7) \leq 61$. $\square$

3.3. Proof of Theorem 3

First we prove the lower bounds. Corollary 1 gives the same lower bound $20 \leq \pi(G)$, for an arbitrary diameter 4, girth 5, cubic graph G with 18 vertices, like B_1 and B_2, by defining an unsolvable configuration C^* of size 19, as follows. Consider a pair of vertices u and v at distance 4 in such a graph G. Define $C^*(v) = 15$, $C^*(x) = 0$ for all $x \in (N_2[u] \cup N_1[v]) - \{v\}$, and $C^*(x) = 1$ otherwise. Hence, $|C^*| = 19$.

Please refer to Fig. 3. Analogous arguments can be done for B_1 and B_2, so we present with no loss of generality the arguments for B_2. We can establish the non trivial lower bound $21 \leq \pi(B_2)$, as an application of the Retract Lemma 2, since a subgraph of B_2 has a C_9 retract and $\pi(C_9) = 21$. We can actually define an x_3-unsolvable configuration of size 22, by organizing B_2 by distance from target x_3. Consider the C_9 induced by $x_3, x_1, z_1, z_2, z_2', x_2', x_5', x_3', z_3$. Adjacent vertices z_2' and x_2' are at distance 4 from target x_3. Place 10 pebbles on z_2' and 10 pebbles on x_2' to get a x_3-unsolvable configuration of size 20. Additionally, place 1 pebble on z_5' and 1 pebble on x_1', to get the desired x_3-unsolvable configuration of size 22, since z_5' and x_1' are among the vertices at distance 3 from target x_3, that are not in a cycle C_9 together with x_3, z_2' and x_2'. Hence $23 \leq \pi(B_2)$.

Now we prove the upper bound $\pi(B_2) \leq 34$, using WFL. Note that we have six different roots, by considering: $z_1 = z_1'$, $z_2 = z_2' = z_5 = z_5'$, $x_1 = x_1'$, $x_2 = x_2' = x_5 = x_5'$, $x_3 = x_3' = x_4 = x_4'$, and $z_3 = z_4$. We shall define strategies with root x_3, since this root gave us the largest upper bound. We refer to the Appendix for the strategies with the other five roots that give values not larger than the ones given by strategies presented for root x_3.

For B_2, we define three x_3-strategies T_1, T_2, and T_3 by

- $T_1(x_1, z_1, x_4, z_2, z_4, z_2', x_4', x_2') = (32, 16, 16, 8, 8, 4, 4, 2)$,
- $T_2(x_5, x_2, z_5, z_5', z_1', z_2', x_2') = (32, 7, 16, 8, 4, 2, 1)$ and
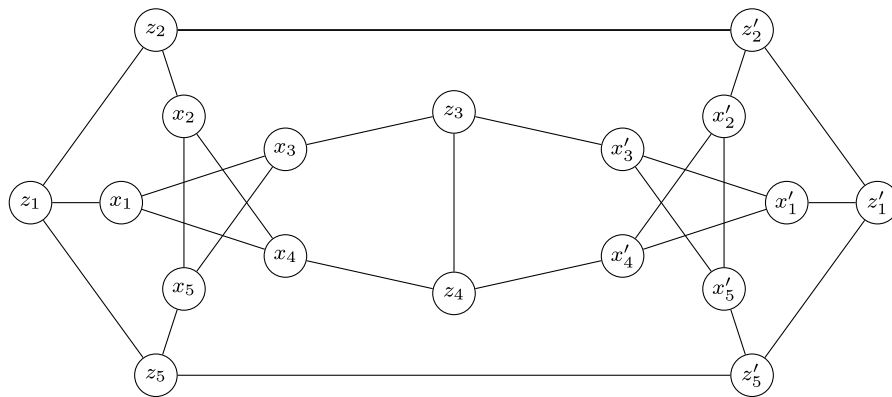- $T_3(z_3, x_3', x_1', x_5', z_1', x_2', x_4') = (32, 16, 8, 8, 4, 4, 3)$,
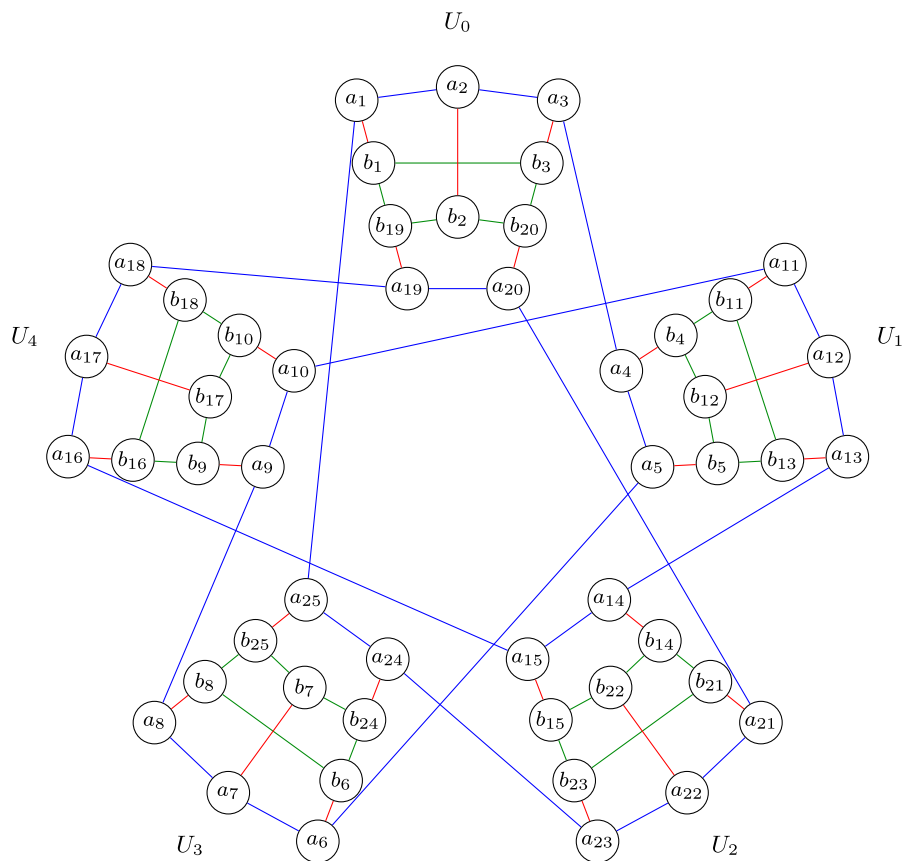
Fig. 3. Blanuša 2 and its labeling.

Fig. 4. The Watkins graph, shown in its traditional drawing. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

giving rise to the inequality $|C| \leq \frac{1}{7}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) \leq 236/7$ whenever C is x_3-unsolvable. Hence $\pi(B_2, x_3) \leq 34$. Using the strategies presented in the Appendix for the other five targets, we may conclude that $\pi(B_2) \leq 34$. □

3.4. Proof of Theorem 4

We first observe an isomorphism between the traditional drawing of W shown in Fig. 4 and the drawing of W in Fig. 5 that partitions its vertices by distance from a_1 (those of distance i are in V_i). Indeed, one can check that, in both drawings, W contains the 25-cycle $a_1, a_2, \ldots, a_{25}, a_1$ (shown with blue edges in Fig. 4 and circular vertex shapes in Fig. 5), five
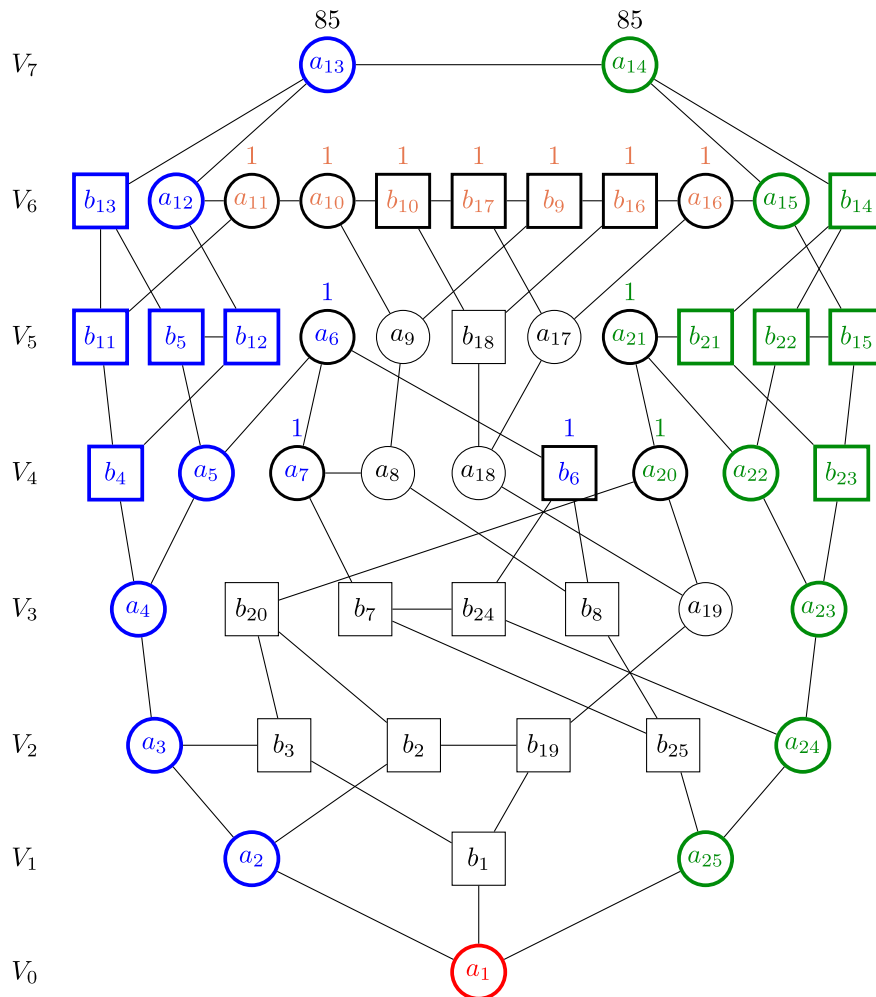
Fig. 5. The Watkins graph, organized by distance from target a_1, along with an a_1-unsolvable configuration C of size 182. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

vertex-disjoint 5-cycles on the vertices $b_1, \ldots, b_{25}$ (shown with green edges in Fig. 4 and square vertex shapes in Fig. 5), and the perfect matching using the 25 pairs $\{a_i, b_i\}$ (shown with red edges in Fig. 4). The point of showing Fig. 4 is to recognize W and its symmetries; the point of showing Fig. 5 is to make lower bound arguments about $\pi(W)$.

Next we prove the lower bound by displaying an a_1-unsolvable configuration of size 182; such a configuration C is shown in Fig. 5, with each vertex of V_7 having 85 pebbles and 12 other vertices (7 orange, 3 blue, and 2 green) having one pebble each. The argument we give will make use of the vertex colors in Fig. 5.

For two vertices u and v, we call a uv-path of minimal length a uv-*geodesic*. Define V_B (resp. V_G) to be the set of vertices in the union of all $a_{13}a_1$-geodesics (resp. $a_{14}a_1$-geodesics); these are shown in Fig. 5 with blue (resp. green) shapes, except for the target vertex a_1 in red. Note that the path P_8 is a retract of the subgraph of W induced by V_B (resp. V_G), and so the graph W_{BG} induced by $V_B \cup V_G$ has the cycle C_{15} as a retract. Hence $\pi(W_{BG}) \geq \pi(C_{15}) = 171$. In particular, by Lemma 2, the configuration C_{BG} — the restriction of the configuration C shown in Fig. 5 to $V_B \cup V_G$ — is a_1-unsolvable in W_{BG}.

Suppose that C_{BG} has a minimal a_1-solution σ in W. If σ uses the edge $e_2 = a_2a_1$ then it does not use either of the edges b_1a_1 or $a_{25}a_1$; hence σ is an a_1-solution in the subgraph $W_2 = W - e_2$. However the distance from a_{14} to a_2 is 7, and so the a_1-weight of C_{BG} equals $85(3)/2^8 = 255/256 < 1$, a contradiction. Similarly, if σ uses the edge $e_1 = b_1a_1$ then we obtain the same a_1-weight calculation, while if σ uses the edge $e_{25} = a_{25}a_1$ then we obtain a smaller a_1-weight calculation; in both cases we have a contradiction. Hence C_{BG} is not a_1-solvable in W. (This already invalidates the claim of [16] that $\pi(W) = 166$.) Therefore, if C is a_1-solvable in W, then any solution must use some of the pebbles in the configuration $S = C - C_{BG}$ of singletons of C.

Now let σ be any minimal a_1-solution and let S_σ denote the set of singletons of S used by σ. We may rearrange the order of pebbling moves so that those moves involving S_σ are performed as early as possible. Now let C_σ denote the configuration that results from halting future pebbling moves of σ once the moves involving S_σ have been performed.

Additionally, denote by C'_σ the configuration $C_\sigma - (S - S_\sigma)$; that is, from C_σ, throw away the singletons of S that were not used by σ. Obviously C'_σ is a_1-solvable. We argue that this is impossible by showing that the a_1-weight of C'_σ is less than 1.

A *uv-slide* is a path $ua_1 \cdots a_k v$ in which u is big and each a_i is nonempty, the use of which allows for a pebble to move from u to v along the slide. A *potential slide* ignores the requirement that u is big; it is "potential" because it becomes a slide if u becomes big. Notice that every singleton vertex is part of a potential slide. Because the a_1-weight of C_{BG} in W equals $1 - 2^{-8}$, C'_σ can only be a_1-solvable if we are able to increase the a_1-weight of C_{BG} by at least 2^{-8} by using the slides of S. In other words, it must be that the a_1-weight of a pebble in C'_σ must be greater than the sum of the a_1-weight of the pebbles of C_{BG} that were used to create it. We show that this cannot happen.

Partition $S = S_6 \cup S_B \cup S_G$, where $S_6 = S \cap V_6$ (the orange pebbles in Fig. 5), $S_B = S \cap V_B$ (the blue pebbles in Fig. 5), and $S_G = S \cap V_G$ (the green pebbles in Fig. 5). Using a slide in S_6 requires two pebbles from $V_5 \cup V_6$ and places one pebble in either $V_5 \cup V_6$. Such a pebble has the same or lesser a_1-weight than the total a_1-weight of the original two pebbles. Using a slide in S_B requires two pebbles from $V_3 \cup V_4$ and places one pebble in $V_3 \cup V_4$; this also cannot increase the a_1-weight. Using a slide in S_G, however, can increase the a_1-weight only if the two pebbles originate from b_{21} and the resulting pebble lands on b_{20}. Still, the resulting pebble must be used in σ to achieve this a_1-weight increase (of $1/2^3 - 2/2^5 = 1/2^4$), although to do so requires at least 2^6 pebbles from V_7, resulting in a a_1-weight loss of $2^7/2^7 - 1/2^3 > 1/2^4$. Hence it is impossible to increase the a_1-weight of C'_σ via pebbling steps, and so C'_σ, and thus C, is a_1-unsolvable. $\square$

4. Final remarks

The previous work of [13] on Flower snarks and of [16] on the Watkins snark are at the foundation of our findings, since after reading their published papers, we have realized we were able to contribute to the subject. We hope the developed techniques and results will motivate more researchers to achieve better bounds on the pebbling numbers of snarks. Table 1 shows the state of art of the pebbling numbers of several well known snarks, using the basic bounds mentioned in the introduction, as well as Lemma 1, and Theorems 1, 2, 3 and 4.

Acknowledgments

Appendix

Missing cases for the proof of Theorem 2

Regarding the upper bounds of Theorem 2, for the two other possible roots v_0 and x_0, we have obtained below strategies that give values not larger than the ones given by strategies presented for root z_0.

For J_3, we define three v_0-strategies $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_{-1}$ by

- $\boldsymbol{T}_0(z_0, x_0, y_0, x_1, y_1, x_{-1}, y_{-1}, z_1, z_{-1}) = (8, 4, 4, 2, 2, 2, 2, 1, 1)$,
- $\boldsymbol{T}_1(v_1, z_1, x_1, y_1, x_0, x_{-1}, y_{-1}) = (8, 4, 2, 2, 1, 1, 1)$ and
- $\boldsymbol{T}_{-1}(v_{-1}, z_{-1}, x_{-1}, y_{-1}, y_0, x_1, y_1) = (8, 4, 2, 2, 1, 1, 1)$,

giving rise to the inequality $|C| \leq \frac{1}{5}(\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}) \leq 64/5$ whenever C is v_0-unsolvable. Hence $\pi(J_3, v_0) \leq 13$.

And we define three x_0-strategies $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_{-1}$ by

- $\boldsymbol{T}_0(z_0, v_0, y_0, v_{-1}, v_1, y_{-1}, y_1) = (8, 4, 4, 2, 2, 1, 1)$,
- $\boldsymbol{T}_1(x_1, z_1, y_{-1}, v_1, z_{-1}, y_0, v_{-1}, v_0) = (8, 4, 4, 2, 1, 1, 1, 1)$ and
- $\boldsymbol{T}_{-1}(x_{-1}, z_{-1}, y_1, v_{-1}, z_1, v_1) = (8, 4, 4, 2, 1, 1)$,

giving rise again to the inequality $|C| \leq \frac{1}{5}(\boldsymbol{T}_0 + \boldsymbol{T}_1 + \boldsymbol{T}_{-1}) \leq 64/5$ whenever C is x_0-unsolvable. Hence $\pi(J_3, x_0) \leq 13$. We may conclude that $\pi(J_3) \leq 13$.

For J_5, we define three v_0-strategies $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_{-1}$ by

- $\boldsymbol{T}_0(z_0, x_0, y_0, x_1, y_1, x_{-1}, y_{-1}, x_2, y_2, x_{-2}, y_{-2}, z_2, z_{-2})$
  $= (16, 8, 8, 4, 4, 4, 4, 2, 2, 1, 1, 1, 1)$,
- $\boldsymbol{T}_1(v_1, v_2, z_1, z_2, x_2, y_2, x_1, y_{-2}, x_{-2}, y_1) = (16, 8, 8, 4, 2, 2, 1, 1, 1, 1)$ and

- $T_{-1}(v_{-1}, v_{-2}, z_{-1}, z_{-2}, x_{-2}, y_{-2}, x_{-1}, y_2, x_2, y_{-1}) =$
  $(16, 8, 8, 4, 2, 2, 1, 1, 1, 1),$

giving rise to the inequality $|C| \leq \frac{1}{5}(T_0 + T_1 + T_{-1}) \leq 146/5$ whenever C is v_0-unsolvable. Hence $\pi(J_5, v_0) \leq 30$.
And we define three x_0-strategies $T_0, T_1,$ and T_{-1} by

- $T_0(z_0, v_0, y_0, v_{-1}, v_1, y_{-1}, y_1, v_2, v_{-2}, y_2, y_{-2}, z_2)$
  $= (16, 8, 8, 4, 4, 4, 4, 2, 2, 1, 1, 1),$
- $T_1(x_1, z_1, x_2, y_{-2}, z_2, y_{-1}, z_{-2}, y_2, v_2, v_1, v_{-2})$
  $= (16, 8, 8, 4, 4, 1, 1, 1, 2, 1, 1)$ and
- $T_{-1}(x_{-1}, z_{-1}, x_{-2}, z_{-2}, y_2, v_{-2}, y_1, v_2, v_{-1}) = (16, 8, 8, 4, 4, 2, 1, 1, 1),$

giving rise again to the inequality $|C| \leq \frac{1}{5}(T_0 + T_1 + T_{-1}) \leq 146/5$ whenever C is x_0-unsolvable. Hence $\pi(J_5, x_0) \leq 30$.
We may conclude that $\pi(J_5) \leq 30$.
　　For J_7, we define three v_0-strategies $T_0, T_1,$ and T_{-1} by

- $T_0(z_0, x_0, y_0, x_1, y_1, x_{-1}, y_{-1}, x_2, z_1, z_{-1}, x_{-2}, y_2, y_{-2}, x_3, x_{-3},$
  $y_3, y_{-3}, z_3, z_{-3}) = (32, 16, 16, 8, 8, 8, 8, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 1, 1),$
- $T_1(v_1, v_2, z_1, v_3, z_2, z_3, x_2, y_2, x_3, y_3, x_{-3}, y_{-3})$
  $= (32, 16, 1, 8, 5, 4, 1, 1, 2, 2, 1, 1)$ and
- $T_{-1}(v_{-1}, v_{-2}, z_{-1}, v_{-3}, z_{-2}, z_{-3}, x_{-2}, y_{-2}, x_{-3}, y_{-3}, y_3, x_3)$
  $= (32, 16, 1, 8, 5, 4, 1, 1, 2, 2, 1, 1),$

giving rise to the inequality $|C| \leq \frac{1}{5}(T_0 + T_1 + T_{-1}) \leq 278/5 < 56$ whenever C is v_0-unsolvable. Hence $\pi(J_7, v_0) \leq 56$.
And we define three x_0-strategies $T_0, T_1,$ and T_{-1} by

- $T_0(z_0, v_0, y_0, v_1, v_{-1}, y_1, y_{-1}, v_2, v_{-2}, y_2, y_{-2}, v_3, v_{-3}, y_3, y_{-3}, z_3)$
  $= (32, 16, 16, 8, 8, 8, 8, 4, 4, 4, 4, 2, 2, 1, 1, 1),$
- $T_1(x_1, z_1, x_2, z_2, x_3, v_2, y_{-3}, z_3, z_{-3}, y_3, v_3, v_{-3})$
  $= (32, 5, 16, 8, 8, 1, 4, 4, 1, 1, 2, 1)$ and
- $T_{-1}(x_{-1}, z_{-1}, x_{-2}, z_{-2}, x_{-3}, y_{-2}, v_{-2}, z_{-3}, y_3, v_{-3}, y_2, v_3)$
  $= (32, 5, 16, 8, 8, 1, 1, 4, 4, 2, 1, 1),$

giving rise to the inequality $|C| \leq \frac{1}{5}(T_0 + T_1 + T_{-1}) \leq 284/5 < 57$ whenever C is x_0-unsolvable. Hence $\pi(J_7, x_0) \leq 57$.
We may conclude that $\pi(J_7) \leq 61$.
　　For J_9, we define three x_0-strategies $T_0, T_1,$ and T_{-1} by

- $T_0(z_0, v_0, y_0, v_1, v_{-1}, y_1, y_{-1}, v_2, v_{-2}, y_2, y_{-2}, v_3, v_{-3}, y_3, y_{-3}, v_4,$
  $v_{-4}, y_4, y_{-4}, z_4)$
  $= (64, 32, 32, 16, 16, 16, 16, 8, 8, 8, 8, 4, 4, 4, 4, 2, 2, 2, 1, 1),$
- $T_1(x_1, z_1, x_2, z_2, x_3, z_3, x_4, v_3, y_{-4}, z_4, z_{-4}, y_4, v_4, v_{-4})$
  $= (64, 5, 32, 16, 16, 5, 8, 1, 2, 4, 1, 1, 2, 1)$ and
- $T_{-1}(x_{-1}, z_{-1}, x_{-2}, z_{-2}, x_{-3}, v_{-2}, z_{-3}, x_{-4}, y_{-3}, v_{-3}, z_{-4}, y_4, v_{-4},$
  $y_3, v_4) = (64, 5, 32, 16, 16, 8, 5, 8, 1, 1, 4, 2, 2, 1, 1).$

　　For $m \geq 7$ (i.e. $k \geq 3$), using the same pattern we have defined above for the three v_0-strategies for J_7, we define three corresponding v_0-strategies giving rise to the inequality

$$5|C| \leq T_0 + T_1 + T_{-1}$$
$$= 3(2^{k+2}) + 4(2^{k+1}) + 6(2^k + \cdots + 2^3) + 5(2k + 8)$$
$$\leq 3(2^{k+2}) + 6(2^3 + \cdots + 2^{k+1}) + 5(2k + 6),$$

whenever C is v_0-unsolvable.
　　For $m \geq 9$ (i.e. $k \geq 4$), using the same pattern we have defined above for the three x_0-strategies for J_9, we define three corresponding x_0-strategies giving rise to the inequality

$$5|C| \leq T_0 + T_1 + T_{-1}$$
$$= 3(2^{k+2}) + 4(2^{k+1}) + 8(2^k) + 6(2^{k-1} + \cdots + 2^3) + 5(2k + 6)$$
$$\leq 3(2^{k+2}) + 6(2^3 + \cdots + 2^{k+1}) + 5(2k + 6),$$

whenever C is x_0-unsolvable.
　　We may conclude that for $m \geq 3$ (i.e. $k \geq 1$), $\pi(J_m) \leq \lfloor 2^{k+2}9/5 + 2k - 18/5 \rfloor + 1$.

Missing cases for the proof of Theorem 3

Regarding the upper bounds of Theorem 3, for the five other possible roots z_1, x_1, z_2, x_2, and z_3, we have obtained below strategies that give values not larger than the ones given by strategies presented for root x_3.

For B_2, we define three x_1-strategies $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_3$ by

- $\boldsymbol{T}_1(x_3, z_3, x_3', x_1', x_5', z_1') = (32, 16, 16, 8, 8, 4)$,
- $\boldsymbol{T}_2(z_1, z_2, z_5, x_2, x_5, z_2', z_5', x_2', x_5', z_1') = (16, 8, 8, 4, 4, 4, 4, 2, 2, 2)$ and
- $\boldsymbol{T}_3(x_4, z_4, x_4', x_1', x_2') = (32, 16, 8, 8, 4)$

giving rise to the inequality $|C| \le \frac{1}{4}(\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}_3) \le 120/4$ whenever C is x_1-unsolvable. Hence $\pi(B_2, x_1) \le 31$.

For B_2, we define three x_2-strategies $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_3$ by

- $\boldsymbol{T}_1(x_4, z_4, x_4', x_1', x_3') = (16, 8, 4, 2, 1)$,
- $\boldsymbol{T}_2(z_2, z_1, z_2', x_1, x_2', z_1', x_1', x_5', x_3') = (16, 8, 8, 4, 4, 4, 2, 2, 1)$ and
- $\boldsymbol{T}_3(x_5, x_3, z_5, z_3, z_5', x_3', x_5') = (16, 8, 4, 4, 2, 2)$

giving rise to the inequality $|C| \le \frac{1}{4}(\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}_3) \le 124/4$ whenever C is x_2-unsolvable. Hence $\pi(B_2, x_2) \le 32$.

For B_2, we define three z_1-strategies $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_3$ by

- $\boldsymbol{T}_1(x_1, x_3, x_4, z_3, z_4, x_3', x_4', x_1') = (16, 8, 8, 4, 4, 2, 2, 1)$,
- $\boldsymbol{T}_2(z_2, x_2, z_2', x_1', z_1', x_4', x_1', x_3', z_4) = (16, 5, 8, 4, 4, 2, 2, 2, 1)$ and
- $\boldsymbol{T}_3(z_5, x_5, z_5', z_1', x_5', x_1', x_3', x_4', z_3) = (16, 5, 8, 4, 4, 2, 2, 1, 1)$

giving rise to the inequality $|C| \le \frac{1}{5}(\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}_3) \le 133/5$ whenever C is z_1-unsolvable. Hence $\pi(B_2, z_1) \le 27$.

For B_2, we define three z_2-strategies $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_3$ by

- $\boldsymbol{T}_1(z_2', z_1', x_2', z_5', x_1', x_4', x_5', x_3') = (16, 8, 8, 4, 4, 4, 4, 2)$,
- $\boldsymbol{T}_2(z_1, z_5, x_1, x_3, z_3, x_3') = (16, 4, 8, 4, 2, 1)$ and
- $\boldsymbol{T}_3(x_2, x_5, x_4, z_4, z_3, x_3') = (16, 4, 8, 4, 2, 1)$

giving rise to the inequality $|C| \le \frac{1}{4}(\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}_3) \le 120/4$ whenever C is z_2-unsolvable. Hence $\pi(B_2, z_2) \le 31$.

For B_2, we define three z_3-strategies $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_3$ by

- $\boldsymbol{T}_1(z_4, x_4, x_4', x_2, x_2', z_2, z_2') = (32, 16, 16, 8, 8, 4, 4)$,
- $\boldsymbol{T}_2(x_3', x_1', x_5', z_1', z_5', z_2') = (32, 16, 16, 8, 8, 4)$ and
- $\boldsymbol{T}_3(x_3, x_1, x_5, z_1, z_5, z_2) = (32, 16, 16, 8, 8, 4)$

giving rise to the inequality $|C| \le \frac{1}{8}(\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}_3) \le 133/5$ whenever C is z_3-unsolvable. Hence $\pi(B_2, z_3) \le 33$.

Data availability

No data was used for the research described in the article.

References

[1] L. Alcón, M. Gutierrez, G. Hurlbert, Pebbling in split graphs, J. Discrete Math. 28 (2014) 1449–1466.
[2] C. Campos, S. Dantas, C. de Mello, The total-chromatic number of some families of snarks, Discrete Math. 311 (12) (2011) 984–988.
[3] A. Cavicchioli, M. Meschiari, B. Ruini, F. Spaggiari, A survey on snarks and new results: Products, reducibility and a computer search, J. Graph Theory 28 (2) (1998) 57–86.
[4] M. Chan, A.P. Godbole, Improved pebbling bounds, Discrete Math. 308 (11) (2008) 2301–2306.
[5] F. Chung, Pebbling in hypercubes, J. Discrete Math. 2 (1989) 467–472.
[6] D.W. Cranston, L. Postle, C. Xue, C. Yerger, Modified linear programming and class 0 bounds for graph pebbling, J. Comb. Optim. 34 (1) (2017) 114–132.
[7] G. Hurlbert, A survey of graph pebbling, Congr. Numer. 139 (1999) 41–64.
[8] G. Hurlbert, General graph pebbling, Discrete Appl. Math. 161 (9) (2013) 1221–1231.
[9] G. Hurlbert, The weight function lemma for graph pebbling, J. Combin. Opt. 34 (2) (2017) 343–361.
[10] G. Hurlbert, F. Kenter, Graph Pebbling: A blend of graph theory, number theory, and optimization, Notices Amer. Math. Soc. 68 (11) (2021) 1900–1913.
[11] G. Hurlbert, H. Kierstead, Graph pebbling complexity and fractional pebbling, 2005, (Unpublished).
[12] K. Milans, B. Clark, The complexity of graph pebbling, J. Discrete Math. 20 (3) (2006) 769–798.
[13] M.C. Monica, A.S. Suganya, Pebbling in flower snark graph, Glob. J. Pure Appl. Math. 13 (6) (2017) 1835–1843.
[14] J. Petersen, Sur le théorème de Tait, Interméd. Math. 15 (1898) 225–227.
[15] M. Preissmann, Snarks of order 18, Discrete Math. 42 (1) (1982) 125–126.
[16] S. Sreedevi, M.S. Anilkumar, Pebbling in Watkins snark graph, Int. J. Res. Adv. Tech. 7 (2) (2019).
[17] P.G. Tait, Remarks on the colourings of maps, Proc. Roy. Soc. Edinburgh 10 (1880) 729.
[18] J.J. Watkins, Snarks, Ann. New York Acad. Sci. 576 (1989) 606–622.