An Application of Graph Pebbling to Zero-Sum Sequences in Abelian Groups

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Abstract

A sequence of elements of a finite group G is called a zero-sum sequence if it sums to the identity of G. The study of zero-sum sequences has a long history with many important applications in number theory and group theory. In 1989 Kleitman and Lemke, and independently Chung, proved a strengthening of a number theoretic conjecture of Erdős and Lemke. Kleitman and Lemke then made more general conjectures for finite groups, strengthening the requirements of zero-sum sequences. In this paper we prove their conjecture in the case of abelian groups. Namely, we use graph pebbling to prove that for every sequence $(g_k)_{k=1}^{|G|}$ of $|G|$ elements of a finite abelian group G there is a nonempty subsequence $(g_k)_{k\in K}$ such that $\sum_{k\in K} g_k = 0_G$ and $\sum_{k\in K} 1/|g_k| \leq 1$, where $|g|$ is the order of the element $g \in G$.

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1 Introduction

A sequence of elements of a finite group G is called a zero-sum sequence if it sums to the identity of G. A standard pigeonhole principle argument shows that any sequence of $|G|$ elements of G contains a zero-sum subsequence; in fact having consecutive terms (one can instead stipulate that the zero-sum subsequence has at most N terms — where $N = N(G)$ is the exponent of G, i.e. the maximum order of an element of G — which is best possible).

First considered in 1956 by Erdős [15], the study of zero-sum sequences has a long history with many important applications in number theory and group theory. In 1961 Erdős et al. [16] proved that every sequence of $2|G|-1$ elements of a cyclic group G contains a zero-sum subsequence of length exactly $|G|$. In 1969 van Emde Boas and Kruyswijk [14] proved that any sequence of $N(1 + \log(|G|/N))$ elements of a finite abelian group contains a zero-sum sequence. In 1994 Alford et al. [1] used this result and modified Erdős's arguments to prove that there are infinitely many Carmichael numbers. Much of the recent study has involved finding Davenport's constant $D(G)$, defined to be the smallest D such that every sequence of D elements contains a zero-sum subsequence [28]. Applications of the wealth of results on this problem [5, 18, 19, 21, 22, 30] and its variations [20, 27] to factorization theory and to graph theory can be found in [2, 6].

In 1989 Kleitman and Lemke [25], and independently Chung [7], proved the following strengthening of a number theoretic conjecture of Erdős and

Lemke (see also [8, 13]).

Result 1 *For any positive integer n, every sequence $(a_k)_{k=1}^n$ of n integers contains a nonempty subsequence $(a_k)_{k\in K}$ such that $\sum_{k\in K} a_k \equiv 0 \bmod n$ and $\sum_{k\in K} \gcd(a_k, n) \le n$.*

Kleitman and Lemke then made more general conjectures for finite groups, strengthening the requirements of zero-sum sequences. In this paper we prove their conjecture in the case of abelian groups. Namely, we use graph pebbling (and Result 1) to prove the following theorem (we use $|g|$ to denote the order of the element $g \in G$).

Theorem 2 *For every sequence $(g_k)_{k=1}^{|G|}$ of $|G|$ elements of a finite abelian group G there is a nonempty subsequence $(g_k)_{k\in K}$ such that $\sum_{k\in K} g_k = 0_G$ and $\sum_{k\in K} 1/|g_k| \le 1$.*

Notice that Result 1 is the special case of Theorem 2 in which G is cyclic. Also notice that the condition on the sum of the orders implies that $|K| \le N(G)$, with equality if and only if $|g_k| = N$ for every $k \in K$.

2 Preliminaries

2.1 Graph Pebbling

Let $\Gamma = (V, E)$ be a graph with vertices V and edges (unordered pairs of edges) E. Given a configuration of pebbles on V, a pebbling step consists of removing two pebbles from a vertex u and placing one pebble on an adjacent vertex v ($uv \in E$). The pebbling number $\pi = \pi(\Gamma)$ is the smallest number π such that, from every configuration of π pebbles on V it is possible to place a pebble on any specified target vertex after a sequence of pebbling moves. There is a rapidly growing literature on graph pebbling [10, 12, 23], including variations such as optimal pebbling [17, 26, 29], pebbling thresholds [3, 4, 11] and cover pebbbling [9, 24, 31].

One variation of graph pebbling involves labelling the edges $uv \in E$ by positive integer weights $w(uv)$, so that a pebbling step from u to v removes $w(uv)$ pebbles from u before placing one pebble on v. In this light, standard graph pebbling has weight 2 on every edge. Let $\mathcal{B}^n$ be the graph of the n dimensional boolean algebra — its vertices are all binary n-tuples; its edges are the pairs of n-tuples that differ by a single digit. For every edge between vertices that differ in the i^{th} digit, let w_i be its weight. Finally, let $\boldsymbol{w} = \langle w_i \rangle_{i=1}^n$ and denote the resulting weighted graph by $\mathcal{B}^n(\boldsymbol{w})$. Then Chung's theorem [7] is as follows.

Theorem 3 *The generalized pebbling number of the weighted graph $\mathcal{B}^n(\boldsymbol{w})$ is $\pi(\mathcal{B}^n(\boldsymbol{w})) = \prod_{i=1}^n w_i$.*

2.2 Group Structure

Let $\mathbb{Z}_n$ denote the finite cyclic group on n elements. The standard representation for an abelian group G has the form $\mathbb{Z}_{N_1} \oplus \mathbb{Z}_{N_2} \oplus \ldots \oplus \mathbb{Z}_{N_r}$, where $N_i | N_{i-1}$ for $1 < i \leq r$ (although, purposely, we've written the order of the cycles in reverse to the standard). Thus the exponent of G is $N(G) = N_1$

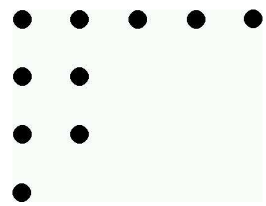

Figure 1: Ferrer's diagram for $(5, 2, 2, 1)$

and the rank of G is $r(G) = r$. One of the useful techniques in this paper is to break down each cycle $\mathbb{Z}_{N_i}$ into products of cycles of distinct prime powers. We write $G = \oplus_{i=1}^{t} \oplus_{j=1}^{m_i} \mathbb{Z}_{p_i^{e_{i,j}}}$ for some primes p_i, multiplicities m_i, and exponents $e_{i,j}$. Thus G can be coordinatized so that elements g have the form $\boldsymbol{g} = \langle g_{i,j} \rangle$, and addition is coordinatewise with the $(i, j)^{\text{th}}$ coordinate computed modulo $p_i^{e_{i,j}}$. Further, instead of writing the primes p_i in increasing order, we write them so that $e_{i,1} \geq \cdots \geq e_{i,m_i}$ for every $1 \leq i \leq t$. Hence the exponent of G can be written $N = N(G) = \prod_{i=1}^{t} p_i^{e_{i,1}}$.

2.3 Notation

As already witnessed, we will adopt the convention that bold fonts will denote vectors. Let $\boldsymbol{e}_i = \langle e_{i,j} \rangle_{j=1}^{m_i}$, $\boldsymbol{e} = \langle \boldsymbol{e}_i \rangle_{i=1}^{t}$ and $m = \sum_{i=1}^{t} m_i$. Then $\boldsymbol{e}_i$ can be thought of as a partition of the exponent of p_i in the prime factorization of $|G|$. Define $\boldsymbol{d}_i$ to be the dual partition that arises from the Ferrer's diagram of $\boldsymbol{e}_i$. For example, Figure 1 shows the Ferrer's diagram for $(5, 2, 2, 1)$ (dots per row) and its dual $(4, 3, 1, 1, 1)$ (dots per column), both partitions of 10.

Next define $\boldsymbol{f}_{i,r} = \langle \mathbf{1}^r, \mathbf{0}^{m_i - r} \rangle$, and let

$$\boldsymbol{F}_{i,r} = \langle \boldsymbol{f}_{1,0}, \cdots, \boldsymbol{f}_{i-1,0}, \boldsymbol{f}_{i,r}, \boldsymbol{f}_{i+1,0}, \cdots, \boldsymbol{f}_{m,0} \rangle = \langle \mathbf{0}^a, \boldsymbol{f}_{i,r}, \mathbf{0}^b \rangle,$$

$$
\begin{array}{rcccl}
\multicolumn{5}{c}{\boldsymbol{e} = \langle 5,4,3,1;2,2;3;4,1,1\rangle\ , \quad \boldsymbol{e}_1 = \langle 5,4,3,1\rangle\ , \quad \boldsymbol{d}_1 = \langle 4,3,3,2,1\rangle} \\
\hline
\boldsymbol{e}(0,0,0,0) & = & & & \langle 5,4,3,1;2,2;3;4,1,1\rangle \\
\boldsymbol{e}(1,0,0,0) & = & \boldsymbol{e}(0,0,0,0) - \boldsymbol{F}_{1,d_{1,u_1}} & = & \langle 4,3,2,0;2,2;3;4,1,1\rangle \\
\boldsymbol{e}(1,1,0,0) & = & \boldsymbol{e}(1,0,0,0) - \boldsymbol{F}_{2,d_{2,u_2}} & = & \langle 4,3,2,0;1,1;3;4,1,1\rangle \\
\boldsymbol{e}(1,1,0,1) & = & \boldsymbol{e}(1,1,0,0) - \boldsymbol{F}_{4,d_{4,u_4}} & = & \langle 4,3,2,0;1,1;3;3,0,0\rangle \\
\boldsymbol{e}(2,1,0,1) & = & \boldsymbol{e}(1,1,0,1) - \boldsymbol{F}_{1,d_{1,u_1}} & = & \langle 3,2,1,0;1,1;3;3,0,0\rangle \\
\boldsymbol{e}(3,1,0,1) & = & \boldsymbol{e}(2,1,0,1) - \boldsymbol{F}_{1,d_{1,u_1}} & = & \langle 2,1,0,0;1,1;3;3,0,0\rangle \\
 & \vdots & & \vdots & \\
\boldsymbol{e}(5,2,3,4) & = & \boldsymbol{e}(5,2,2,4) - \boldsymbol{F}_{3,d_{3,u_3}} & = & \langle 0,0,0,0;0,0;0;0,0,0\rangle
\end{array}
$$

Figure 2: $\boldsymbol{e}(\boldsymbol{u})$ for $\boldsymbol{e} = \langle 5,4,3,1;2,2;3;4,1,1\rangle$ and various $\boldsymbol{u}$

where $a = \sum_{i<r} m_i$ and $b = \sum_{i>r} m_i$. For vectors $\boldsymbol{u} = \langle u_k\rangle_{k=1}^{s}$, $\boldsymbol{v} = \langle v_k\rangle_{k=1}^{s}$ and $\boldsymbol{w} = \langle w_k\rangle_{k=1}^{s}$ denote $\boldsymbol{u}\boldsymbol{v} = \langle u_k^{v_k}\rangle_{k=1}^{s}$ and $\boldsymbol{u}^{\cdot\boldsymbol{v}} = \prod_{k=1}^{s} u_k^{v_k}$. Now let $\boldsymbol{p}_i = \langle p_i\rangle_{j=1}^{m_i}$, $\boldsymbol{p} = \langle \boldsymbol{p}_i\rangle_{i=1}^{t}$ and $\boldsymbol{p}_0 = \langle p_i\rangle_{i=1}^{t}$, and define $\boldsymbol{n} = \langle n_i\rangle_{i=1}^{t} = \langle e_{i,1}\rangle_{i=1}^{t}$ and $n = \sum_{i=1}^{t} n_i$. Note that $\boldsymbol{p}_0^{\cdot\boldsymbol{n}} = N(G)$ and $\boldsymbol{p}^{\cdot\boldsymbol{e}} = |G|$. We also write $\boldsymbol{u} \le \boldsymbol{v}$ when $u_k \le v_k$ for every k, $\boldsymbol{u} \equiv \boldsymbol{v} \bmod \boldsymbol{w}$ when $u_k \equiv v_k \bmod w_k$ for every k, and $\boldsymbol{u}\boldsymbol{v} = \boldsymbol{w}$ (or $\boldsymbol{u} = \boldsymbol{w}/\boldsymbol{v}$) when $u_k v_k = w_k$ for every k.

Finally, let $\boldsymbol{e}(\boldsymbol{0}^m) = \boldsymbol{e}$, and denote the k^{th} characteristic vector $\boldsymbol{\chi}_k$, having all zeros with a single one in the k^{th} entry. For $\boldsymbol{0}^m \le \boldsymbol{u} \le \boldsymbol{n}$ define

$$\boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{e}(\boldsymbol{u} - \boldsymbol{\chi}_i) - \boldsymbol{F}_{i,d_{i,u_i}}.$$

(Note that this definition is valid for every $1 \le i \le t$.) Figure 2 shows an

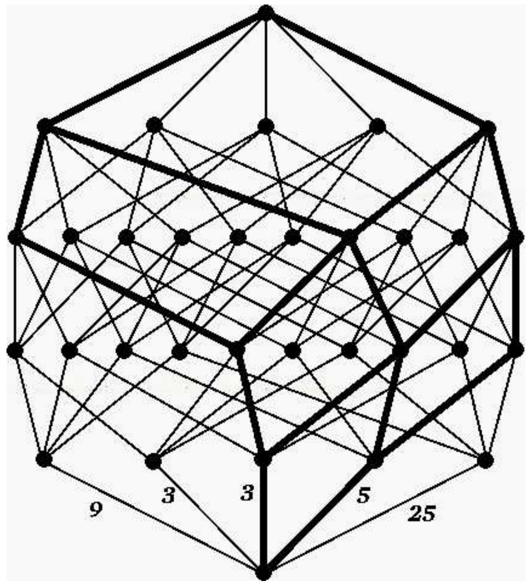

Figure 3: $L(G)$ as a retract of $\mathcal{B}^5(9,3,3,25,5)$ for the group $G = \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_5$

example for these definitions. Note that we always have $\boldsymbol{e}(\boldsymbol{n}) = \mathbf{0}^n$.

2.4 Lattice Graph and Pebbling Number

Define the lattice $L = L(G) = \prod_{i=1}^{t} P_{n_i+1}$ (the cartesian product of paths with $n_i + 1$ vertices). Note that L is isomorphic to the divisor lattice of $N = N(G) = \prod_{i=1}^{t} p_i^{e_{i,1}}$ (having height $n = \sum_{i=1}^{t} e_{i,1}$) and label the vertices of L accordingly. Next consider an edge of L between vertex $p_i^k q$ and vertex $p_i^{k-1} q$, where $p_i \not| q$. Label such an edge by weight $p_i^{d_{i,k}}$.

Because L and its labelling is a retract (see Figure 3 for an example) of the n-dimensional boolean lattice $\mathcal{B}^n(\boldsymbol{w})$, having edge labels $\boldsymbol{w} = \langle p_i^{d_{i,j}} \rangle_{i,j}$, we have that the generalized (pebbling operations obey the edge labels) pebbling number $\pi(L) = \pi(\mathcal{B}^n(\boldsymbol{w}))$. (This is the same argument used in [7].) Notice that

$$\pi(\mathcal{B}(\boldsymbol{w})) = \prod_{i=1}^{t}\prod_{j=1}^{n_i} p_i^{d_{i,j}} = \prod_{i=1}^{t}\prod_{j=1}^{m_i} p_i^{e_{i,j}} = |G| \ .$$

Given a sequence of elements of G, $(g_1, \ldots, g_{|G|})$, define a configuration by placing corresponding pebbles $\{g_1\}, \ldots, \{g_{|G|}\}$ on L, with pebble $\{g_k\}$ on vertex $|g_k| \in V(L)$. Because $\pi(L) = |G|$, the configuration is solvable to the bottom vertex labelled 1. As was noted in [8], L is greedy, meaning that we may assume that every pebbling step moves toward the root 1.

We will now use the solution of the configuration to construct a subsequence $(g_k)_{k\in K}$ that satisfies $\sum_{k\in K} g_k = 0_G$ and $\sum_{k\in K} 1/|g_k| \leq 1$. (We will follow somewhat the structure of the argument presented in [8], with a few necessary tricks thrown in.)

2.5 Well Placed Pebbles

We now make several useful recursive definitions. For a pebble A define

- $\mathrm{Set}(A) = \bigcup_{B\in A} \mathrm{Set}(B)$, where $\mathrm{Set}(\{g_k\}) = \{g_k\}$,
- $\mathbf{Val}(A) = \sum_{B\in A} \mathbf{Val}(B)$, where $\mathbf{Val}(\{g_k\}) = \boldsymbol{g}_k$, and
- $\mathrm{Ord}(A) = \sum_{B\in A} \mathrm{Ord}(B)$, where $\mathrm{Ord}(\{g_k\}) = 1/|g_k|$.

Note that $\mathbf{Val}(A) = \sum_{g\in \mathrm{Set}(A)} \boldsymbol{g}$ and $\mathrm{Ord}(A) = \sum_{g\in\mathrm{Set}(A)} 1/|g|$. We say a pebble A is *well placed* at vertex $\boldsymbol{p}^{\cdot\boldsymbol{u}}$ if

1. $\mathbf{Val}(A) \equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u})}$ and
2. $\mathrm{Ord}(A) \leq 1/\boldsymbol{p}^{\cdot\boldsymbol{u}}$.

Thus each pebble in the initial configuration is well placed.

We will interpret each pebbling step from x to y as follows: first remove a collection of pebbles $A_1, A_2, \ldots, A_s$ of the appropriate size (the edge weight of xy) from x, then for some carefully chosen index set $K_x \subseteq \{1, \ldots, s\}$ place the new pebble $A = \{A_k\}_{k \in K_x}$ on y. We will show that if each A_k is well placed at x then A is well placed at y. Any pebble A that is well placed at vertex $1 = \boldsymbol{p}^{\cdot \mathbf{0}}$ yields the solution $\mathrm{Set}(A)$ to Theorem 2.

3 Proof of Theorem 2

For the purposes of notational readability, we will first give the proof of Theorem 2 in the case of p-groups. Once established, the general case will be straightforward.

3.1 Characteristic p

Here we have $t = 1$ so that $i = 1$ always. For ease of notation we will simply drop the 1; thus $G = \prod_{j=1}^{m} \mathbb{Z}_{p^{e_j}}$ for some prime p, multiplicity m, and exponents e_j ($e_1 \geq \cdots \geq e_m$). For $\boldsymbol{e} = \langle e_j \rangle_{j=1}^{m}$ recall that $\boldsymbol{p}^{\cdot \boldsymbol{e}} = \prod_{j=1}^{m} p^{e_j} = |G|$.

Lemma 4 *Theorem 2 holds for groups of the form* $G = \mathbb{Z}_p^m = \oplus_{j=1}^{m} \mathbb{Z}_p$.

Proof. This result will follow from Theorem 1. View G as the m-dimensional vector space over $\mathbb{F}_p$. Then assign to $\mathbb{F}_p^m$ the natural correspondence with field $\mathbb{F}_{p^m}$, and partition $\mathbb{F}_{p^m} - \{0\}$ into $(p^m - 1)/(p - 1)$ lines of size $p - 1$.

$$
\begin{array}{lcccc}
\multicolumn{5}{c}{\boldsymbol{e} = \langle 5,2,2,1\rangle \ , \quad \boldsymbol{d} = \langle 4,3,1,1,1\rangle} \\
\hline
\boldsymbol{e}(0) & & & = & \langle 5,2,2,1\rangle \\
\boldsymbol{e}(1) & = & \langle 5,2,2,1\rangle - \boldsymbol{f}_4 & = & \langle 4,1,1,0\rangle \\
\boldsymbol{e}(2) & = & \langle 4,1,1,0\rangle - \boldsymbol{f}_3 & = & \langle 3,0,0,0\rangle \\
\boldsymbol{e}(3) & = & \langle 3,0,0,0\rangle - \boldsymbol{f}_1 & = & \langle 2,0,0,0\rangle \\
\boldsymbol{e}(4) & = & \langle 2,0,0,0\rangle - \boldsymbol{f}_1 & = & \langle 1,0,0,0\rangle \\
\boldsymbol{e}(5) & = & \langle 1,0,0,0\rangle - \boldsymbol{f}_1 & = & \langle 0,0,0,0\rangle
\end{array}
$$

Figure 4: $\boldsymbol{e}(u)$ for $\boldsymbol{e} = \langle 5,2,2,1\rangle$ and $u = 0,\ldots,5$

With p^m pebbles, none of which is at $\mathbf{0}$ (otherwise we are done), the pigeonhole principle forces some line to have at least p pebbles. Since a line plus the origin forms the cycle $\mathbb{Z}_p$, Theorem 1 completes the proof. $\square$

Theorem 5 *Theorem 2 holds for groups of the form* $G = \oplus_{j=1}^{m} \mathbb{Z}_{p^{e_j}}$.

Proof. We use Lemma 4 to show that each pebbling step preserves the well placed property. Given a sequence of $|G| = \boldsymbol{p}^{\cdot\boldsymbol{e}} = \prod_{j=1}^{m} p^{e_j}$ elements of G place, as discussed in Section 2.5, corresponding pebbles on the lattice $L = L(G) = P_{e_1+1}$, having edge label p^{d_k} between vertices p^k and p^{k-1}, where $\boldsymbol{d} = \langle d_k\rangle_{k=1}^{e_1}$ is the dual partition to $\boldsymbol{e}$. For $r \geq 0$ recall that $\boldsymbol{f}_r = \langle \mathbf{1}^r, \mathbf{0}^{n-r}\rangle$. Let $\boldsymbol{e}(0) = \boldsymbol{e}$, and for $0 < u \leq e_1$ define $\boldsymbol{e}(u) = \boldsymbol{e}_{u-1} - \boldsymbol{f}_{d_u}$ (see Figure 4 for an example). recall that we always have $\boldsymbol{e}(e_1) = \mathbf{0}^m$ because of the Ferrer's duality.

Given p^{d_u} well placed pebbles $\{A_r\}_{r=1}^{p^{d_u}}$ on vertex p^u, we know that each $\mathbf{Val}(A_r) \equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(u)}$ and each $\mathrm{Ord}(A_r) \leq 1/p^u$. Consider, for each r, $\boldsymbol{B}_r = \mathbf{Val}(A_r)/\boldsymbol{p}^{\boldsymbol{e}(u)}$. By Lemma 4 we can find a nonempty index set R so that for $B = \{\boldsymbol{B}_r\}_{r\in R}$ we have $\mathbf{Val}(B) \equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{f}_{d_u}}$ and $\mathrm{Ord}(B) \leq 1$.

Now let $A = \{A_r\}_{r\in R}$. Then

$$\begin{aligned}
\mathbf{Val}(A) &= \sum_{r\in R} \mathbf{Val}(A_r) \\
&= \sum_{r\in R} \boldsymbol{p}^{\boldsymbol{e}(u)} \boldsymbol{B}_r \\
&= \boldsymbol{p}^{\boldsymbol{e}(u)} \mathbf{Val}(B) \\
&\equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(u)+\boldsymbol{f}_{d_u}} \\
&= \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(u-1)}\ .
\end{aligned}$$

Also, $\mathrm{Ord}(A) = \sum_{r\in R} \mathrm{Ord}(A_r) \leq |R|/p^u = 1/p^{u-1}$. Hence A is well placed on vertex p^{u-1}.

Since the pebbling number guarantees that some pebble A reaches vertex $1 = p^0$, and since the previous argument ensures that A is well placed, we find, for some $K \neq \emptyset$ that

$$\sum_{k\in K} \boldsymbol{g}_k \;=\; \mathbf{Val}(A) \;\equiv\; \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(\mathbf{0})} \;=\; \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}} \;=\; \mathbf{0}_G$$

(i.e. $\sum_{k\in K} g_k = 0_G$) and

$$\sum_{k\in K} 1/|g_k| \;=\; \mathrm{Ord}(A) \;\le\; 1/\boldsymbol{p}^{\cdot\mathbf{0}} \;=\; 1\ .$$

3.2 General Case

As expected, the same proof carries through; only the notation generalizes. Given $p_i^{d_{i,u_i}}$ well placed pebbles $\{A_r\}_{r=1}^{p_i^{d_{i,u_i}}}$ on vertex $\boldsymbol{p}^{\cdot\boldsymbol{u}}$, we know that each $\mathbf{Val}(A_r) \equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u})}$ and each $\mathrm{Ord}(A_r) \le 1/\boldsymbol{p}^{\cdot\boldsymbol{u}}$. Consider, for each r, $\boldsymbol{B}_r = \mathbf{Val}(A_r)/\boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u})}$. By Lemma 4 we can find a nonempty index set R so that for $B = \{\boldsymbol{B}_r\}_{r\in R}$ we have $\mathbf{Val}(B) \equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{F}_{i,d_{i,u_i}}}$ and $\mathrm{Ord}(B) \le 1$.

Now let $A = \{A_r\}_{r\in R}$. Then

$$\begin{aligned}
\mathbf{Val}(A) &= \sum_{r\in R} \mathbf{Val}(A_r) \\
&= \sum_{r\in R} \boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u})}\boldsymbol{B}_r \\
&= \boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u})}\mathbf{Val}(B) \\
&\equiv \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u})+\boldsymbol{F}_{i,d_{i,u_i}}} \\
&= \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(\boldsymbol{u}-\boldsymbol{\chi}_i)}\ .
\end{aligned}$$

Also, $\mathrm{Ord}(A) = \sum_{r\in R} \mathrm{Ord}(A_r) \le |R|/\boldsymbol{p}^{\cdot\boldsymbol{u}} = 1/\boldsymbol{p}^{\cdot(\boldsymbol{u}-\boldsymbol{\chi}_i)}$. Hence A is well placed on vertex $\boldsymbol{p}^{\boldsymbol{u}-\boldsymbol{\chi}_i}$.

Since the pebbling number guarantees that some pebble A reaches vertex $1 = \boldsymbol{p}^{\mathbf{0}}$, and since the previous argument ensures that A is well placed, we find, for some $K \neq \emptyset$ that

$$\sum_{k \in K} \boldsymbol{g}_k \;=\; \mathbf{Val}(A) \;\equiv\; \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}(\mathbf{0})} \;=\; \mathbf{0}^m \bmod \boldsymbol{p}^{\boldsymbol{e}} \;=\; \mathbf{0}_G$$

(i.e. $\sum_{k \in K} g_k = 0_G$) and

$$\sum_{k \in K} 1/|g_k| \;=\; \mathrm{Ord}(A) \;\leq\; 1/\boldsymbol{p}^{\cdot \mathbf{0}} \;=\; 1\ .$$

4 Further Comments

For cyclic groups Theorem 2 is best possible. However, for other groups it is conceivable that shorter sequences of elements may suffice. It had been conjectured that $D(G) = 1 + \sum_{i=1}^{r}(N_i - 1)$ for abelian G [28]. While this was shown true for groups of rank at most 2 and for p-groups, among other special cases, it has been shown false in general [14, 22]. One may ask for the generalized Davenport constant for the minimum length of a sequence required to force a zero-sum subsequence with the extra condition on its orders.

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