Storming the Castle

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Abstract

Conquering the Kingdom requires traversing bridges guarded by voracious gremlins. How many knights do you need to succeed?

1 The Challenge

The fascinating city of Noccilevbyhein is a collection of islands floating in water on the surface of a torus (donut shape). On one of the islands is a castle.

Unfortunately, the city is ruled by administrators who base their system of educational training on a series of high-stakes exams. For the sake of the children, you realize that the leaders must be removed, and so you begin to plan a coup by parachuting knights onto the islands. With unpredictable winds and unusual gravity, however, you have no control over where they will land.

The islands of Noccilevbyhein are connected by a set of bridges (see the diagram below), under each of which lives a knight-eating gremlin. Because of this, knights must traverse bridges in pairs so that exactly one of them reaches the other side.

Success just requires any knight to reach the castle at A, but because of budget cuts you must guarantee success with the minimum number of paratrooping knights, regardless of where they may land.

You know that 6 knights is not enough because 3 might land on F and 3 might land on L. From F, one knight might be able to move to B, E, G, or J, and from L, one knight might be able to move to D, H, I, or K. In all cases we would be left with at most one knight per island with no more moves possible.

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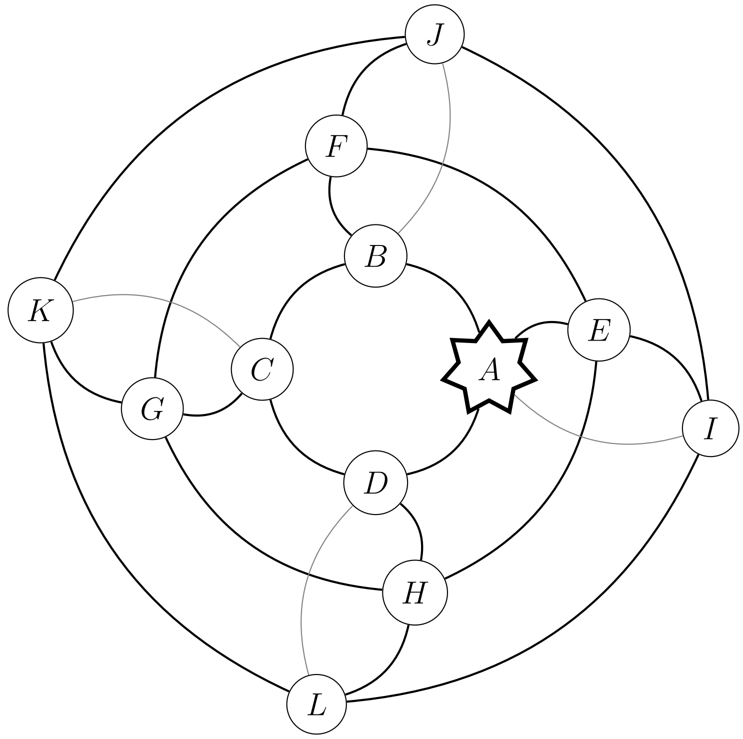


The bridges of Noccilevbyhein.

In fact, 7 is also insufficient because they could all land on K. The most number of knights that could cross to one of its neighboring islands C, G, J, or L is 3, and from any of these none could cross to the castle at A.

On the other hand, if 78 knights parachuted in, then either one of them lands directly on A, or at least 8 of them land together on the same island (at most 7 on each of 11 islands would account for at most 77 knights otherwise). By observation, every island is within 3 crossings from the castle, and so one of those 8 knights will be able to reach A.

Better yet, just 34 knights is sufficient! Indeed, suppose that the knights landed in such a ways as to be unable to reach the castle. Note that if an island is t crossings away from A then 2^t knights on that island will be enough for one of them to be able to reach A. So it must be that such an island has at most $2^t - 1$ knights on it. There are 4 such islands (B, D, E, and I) with $t = 1$; these account for at most $4 \cdot 1 = 4$ knights. There are 5 islands (C, F, H, J, and L) with $t = 2$; these account for at most $5 \cdot 3 = 15$ knights. Finally there are 2 remaining islands (G and K) with $t = 3$; these account for at most $2 \cdot 7 = 14$ knights. In all we have counted at most $4 + 15 + 14 = 33$ knights — hence 34 knights will place enough knights on some island to guarantee storming the castle.

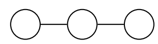
If we write κ for the *castle number* of Noccilevbyhein (i.e. the smallest number of knights which guarantees the ability to storm the castle), then these arguments show that $8 \leq \kappa \leq 34$. What is the actual value of κ?[1]

2 Graphs & Products

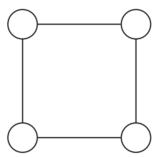
We often call things like the map of Noccilevbyhein a *graph*. The islands are called *vertices* and the bridges are called *edges*. Thus we might use $\mathcal{N}$ for the graph of Noccilevbyhein so that we can write

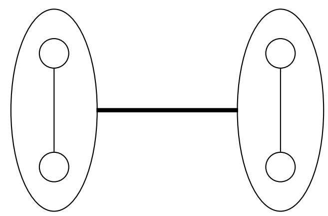
[1]Put down the article if you'd like to try to figure out this puzzle on your own for a while. There is a hint if you wish in Section 3. You can continue to read safely until Section 5, where the solution and its description occurs.

$\kappa(\mathcal{N}, A)$ for the castle number of $\mathcal{N}$ at A. If the castle instead were at B we would write $\kappa(\mathcal{N}, B)$ for the castle number there. For some graphs $\mathcal{G}$, like the one below, these numbers can be different (3 in the middle, but 4 at either end) at different vertices. So we should use $\kappa(\mathcal{G})$ for the worst case; i.e. the maximum of all its different vertex castle numbers. In the case of Noccilevbyhein, however, the symmetry of $\mathcal{N}$ (we can rotate our viewpoint to make any vertex look just like A) implies that all the castle numbers are the same.

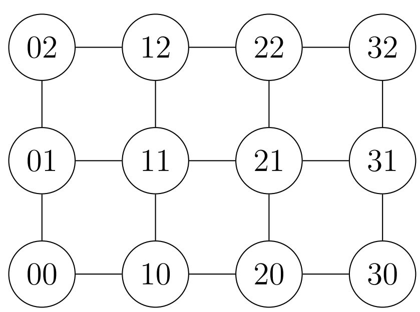


A graph like the above, just a sequence of alternating vertices and edges, is often called a *path*; we write $\mathcal{P}_3$ to signify that it has 3 vertices. Another common graph joins the endpoints of a path to create a *cycle*; the one below (left) is $\mathcal{C}_4$.

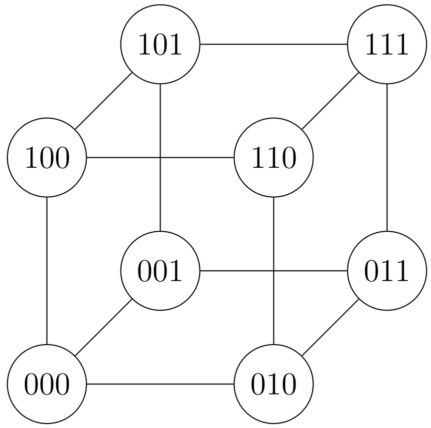




This cycle has an interesting structure. If you group the pair of vertices as shown above (right) then it seems like a big $\mathcal{P}_2$ with each of its vertices containing a little $\mathcal{P}_2$ inside it. The thick edge of the big $\mathcal{P}_2$ represents a set of thin parallel edges in $\mathcal{C}_4$. We call this kind of structure a *Cartesian product*, honoring René Descartes, who gave us the Cartesian plane. An example of the product $\mathcal{P}_3 \times \mathcal{P}_4$ is shown below. The vertex names follow the same pattern we use in high school algebra for Cartesian coordinates.



Thus a product of paths yields a flat grid. One can construct other shapes, like a cylinder from the product of a path and a cycle, or a torus from the product of two cycles. Another nice product is the graph $\mathcal{C}_4 \times \mathcal{P}_2$, below. Because $\mathcal{C}_4 = \mathcal{P}_2 \times \mathcal{P}_2$, we can write $\mathcal{C}_4 \times \mathcal{P}_2 = \mathcal{P}_2 \times \mathcal{P}_2 \times \mathcal{P}_2$, which we can abbreviate as $\mathcal{P}_2^3$; even shorter, people tend to call this graph $\mathcal{Q}^3$ because it is the 3-dimensional *cube*. One can imagine continuing to multiply again and again by $\mathcal{P}_2$ to obtain $\mathcal{Q}^d$ for any dimension d; while it may be difficult to imagine d dimensional geometry, at least we can draw the graph!



One of the interesting patterns that arises here is a relationship between the castle numbers of the individual graphs and their products. For example, $\kappa(\mathcal{P}_2) = 2$ and $\kappa(\mathcal{C}_4) = 4 = \kappa(\mathcal{P}_2)^2$. Also, $\kappa(\mathcal{C}_3 \times \mathcal{P}_2) = 6 = \kappa(\mathcal{C}_3) \cdot \kappa(\mathcal{P}_2)$ and $\kappa(\mathcal{Q}^3) = 8 = \kappa(\mathcal{P}_2)^3$, and it can be a nice challenge to prove these on your own. Based on this sort of evidence, Ron Graham conjectured that every pair of graphs $\mathcal{G}$ and $\mathcal{H}$ satisfy $\kappa(\mathcal{G} \times \mathcal{H}) \leq \kappa(\mathcal{G}) \cdot \kappa(\mathcal{H})$ (there are some examples for which equality does not hold), and the conjecture has been shown to be true for many types of graphs. If the conjecture is true for all graphs, then we would know that 12 knights would suffice for storming the Nochilevbyhein castle, for example. Without knowing that, however, $\mathcal{N}$ remains a bit of a test case.

3 A Hint

It would be easier for an attendee of this conference to guess the correct value of κ without having been told its definition!

As for progress in that direction, one thing that we can do is split the city into districts $U = \{A, B, C, D\}$, $V = \{E, F, G, H\}$, and $W = \{I, J, K, L\}$, with u, v and w being the number of knights landing in their respective districts. Then we can use that $\kappa(\mathcal{C}_4) = 4$ to say that if $u \geq 4$, $v \geq 8$, or $w \geq 8$ then some knight can storm the castle. Otherwise, $u \leq 3$, $v \leq 7$, and $w \leq 7$, which accounts for only 17 knights. Hence $\kappa(\mathcal{N}) \leq 18$. Is that the best we can do?

Also, how many knights can land in Noccilevbyhein in such a way as to not be able to make a single island crossing?

4 Some History

In 1989, Fan Chung proved Graham's conjecture for any number of paths (in particular this means that $\kappa(\mathcal{Q}^d) = 2^d$). Moreover, she showed that it holds more generally when the gremlins can eat more than just one knight. For example, let's say that the gremlins on $\mathcal{P}_3$ eat four knights instead of one, so that it takes 5 knights to traverse an edge for one of them to make it all the way across. Then, as you can imagine, $\kappa_{[5]}(\mathcal{P}_3) = 5^2$ (the subscript shows the gremlin crossing requirement). Similarly, $\kappa_{[7]}(\mathcal{P}_4) = 7^3$. An example of what Chung proved is that $\kappa_{[5,7]}(\mathcal{P}_3 \times \mathcal{P}_4) = \kappa_{[5]}(\mathcal{P}_3) \cdot \kappa_{[7]}(\mathcal{P}_4)$.

And now for something completely different. Suppose that you are given 15 whole numbers, such as $\{-10, 1, 4, 9, 12, 16, 20, 20, 31, 77, 89, 106, 126, 581, 904\}$ (repeats are allowed). No matter what they are, it is always possible to choose from them a subcollection that sums to a multiple of 15: $12 + 20 + 20 + 77 + 126 = 240 = 16 \cdot 15$ is one example. This is called a *zero sum* because its remainder after dividing by 15 is 0. It turns out that, even when the key number 15 is replaced by any positive integer n, this is always possible — every list of n whole numbers contains a zero sum.

However, if we reduce each of the numbers in the zero sum example to its greatest common divisor with 15, we get $3+5+5+1+3=17$. This is considered a *heavy* (as opposed to *light*) sum because $17>15$. In our given list, $-10+1+9=0\cdot 15$ is a light zero sum because $5+1+3\leq 15$. In the late 1980s Paul Erdős and Paul Lemke proposed that every list of n whole numbers contains a light zero sum. This turns out to be true as well and, while it is much more complicated to guarantee the extra light condition, the central reason why it is so for a number like $n=8575$, for instance, happens to be that $\kappa_{[5,7]}(\mathcal{P}_3\times\mathcal{P}_4)=5^2\cdot 7^3=8575$. It's true for $n=216172$ because $\kappa_{[2,11,17]}(\mathcal{P}_3\times\mathcal{P}_2\times\mathcal{P}_4)=\kappa_{[2]}(\mathcal{P}_3)\cdot\kappa_{[11]}(\mathcal{P}_2)\cdot\kappa_{[17]}(\mathcal{P}_4)=2^2\cdot 11^1\cdot 17^3=216172$.

Intriguingly, since antiquity it has been of interest to try to be able to decide reasonably quickly when some number is prime. In our modern, technological era this problem has gained greater practical importance, as many cryptographic and other protocols depend on such things. For a few centuries, a test by Pierre de Fermat was thought to identify primes but, in 1910, Robert Carmichael discovered a counterexample: a composite number that passed Fermat's test. While a handful of counterexamples would not cause too much of a problem, an infinite set of them surely would. In the 1950s Erdős attempted to show that this was the case, and his ideas included questions about zero sums, which have developed into a robust area of research since. In 1994, Alford, Granville, and Pomerance finally proved that infinitely many Carmichael numbers do exist.

Now back to our brave knights!

5 The Solution

The first thing to mention is that it is possible to land 11 knights, one per non-castle island, so that no knight can move at all. This shows that $\kappa(\mathcal{N})\geq 12$. What we'd like to do also is argue that $\kappa(\mathcal{N})\leq 12$. For this, let's explain how to storm the castle from every possible way of parachuting 12 knights into Noccilevbyhein. With the districts defined as before, we start with $u+v+w=12$.

A key observation is that the two districts U and V form the graph $\mathcal{Q}^3$, and so if $u+v\geq 8$ then we can storm the castle. If $u+v=7$ then $w=5$, which means that some knight can cross from W into U or V, so that the resulting 8 knights would be able to storm the castle. So let's assume instead that $u+v\leq 6$, which implies that $w\geq 6$.

Because of the symmetry of $\mathcal{N}$, we can make the same arguments as above to reduce to the case that $u+w\leq 6$ and $v\geq 6$. Put together, we now have $u=0$ and $v=w=6$.

If two knights can cross from V to W then one of the resulting 8 knights on W can storm the castle. Otherwise, no island of V has at least 4 knights and no two islands of V have at least 2 knights each. The only way for this to occur is if the four islands of V contain exactly 1, 1, 1, and 3 knights. But then some knight can cross the appropriate bridge(s) to reach E, and then cross to A.

Happy G4G12!

References

[1] G. Hurlbert, Graph Pebbling. *Handbook of Graph Theory* (2nd ed.), Discrete Mathematics and its Applications, J. Gross, J. Yellen, and P. Zhang, eds., 1428–1454, CRC Press, Boca Raton, 2014.