Optimally pebbling hypercubes and powers

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Abstract. We point out that the optimal pebbling number of the n-cube is $(\frac{4}{3})^{n+O(\log n)}$, and explain how to approximate the optimal pebbling number of the nth cartesian power of any graph in a similar way.

Keywords. optimal pebbling, powers of graphs.

Let G be a graph. By a *distribution of pebbles* on G we mean a function $a : V(G) \to \mathbf{Z}_{\geq 0}$; we usually write $a(v)$ as a_v, and call a_v the *number of pebbles* on v. A *pebbling move* on a distribution changes the distribution by removing 2 pebbles from some vertex with at least 2 pebbles and placing 1 additional pebble on some adjacent vertex. Call a distribution a *good* if, for all vertices v, there is some sequence of pebbling moves starting from a and ending with at least one pebble on v. The *pebbling number* $f(G)$ of a graph G was introduced by Chung [1]; it is the smallest n such that, if a distribution a of pebbles on G uses a total of n pebbles, i.e., $\sum_v a_v = n$, then a is good. Chung answered a question of Lagarias and Saks by showing that the pebbling number $f(Q_n)$ of the n-cube equals 2^n, and used her methods to prove a number-theoretic result of Lemke and Kleitman [1, 4] (also, see [2] for a correction.) Pachtor, Snevily and Voxman [5] introduced the dual concept of the *optimal pebbling number*, $of(G)$, of a graph G; this is the smallest n such that there exists some good distribution a of pebbles on G with a total of n pebbles used. Such a distribution is called an *optimal pebbling*. Pachtor et al. also asked what the optimal pebbling number of Q_n is.

To help compute $of(Q_n)$, and later the optimal pebbling number of the cartesian power of a graph, we define continuous analogs of these concepts. Define a *continuous distribution of pebbles* on G to be a function $a : V(G) \to \mathbf{R}_{\geq 0}$, and a *continuous pebbling move* on a distribution a to be a move that changes the distribution by, for some $\delta \geq 0$ and adjacent vertices v and w, decreasing $a_v \geq \delta$ by δ and adding $\delta/2$ to a_w. We define good continuous distributions just as we defined good distributions; a continuous distribution a on G will evidently be good just when

$$\sum_v a_v 2^{-d(v,w)} \geq 1$$

for all vertices w of G, where $d(v,w)$ is the distance between vertices v and w of G. (We set $d(v,w) = \infty$ if v and w are not connected in G, and we set $2^{-\infty} = 0$.)

We can now define the *continuous optimal pebbling number*, $ofc(G)$, and *continuous optimal pebblings* in a way analogous to $of(G)$ and optimal pebblings. For graphs G and H, let the *cartesian product*, $G \times H$, of G and H have $V(G \times H) = V(G) \times V(H)$ and

$$\begin{aligned} E(G \times H) &= \{\{(v,x),(v,y)\} | v \in V(G), \{x,y\} \in E(H)\} \\ &\cup \{\{(v,x),(w,x)\} | \{v,w\} \in E(G), x \in V(H)\}. \end{aligned}$$

Let the nth *cartesian power* of G, G^n, be the graph obtained by taking the cartesian product of n copies of G.

Theorem 1 *For all G and H, $ofc(G \times H) = ofc(G)ofc(H)$.*

Proof.

($\leq$): If a is a continuous optimal pebbling of G, and b of H, and if we define c by $c_{(v,x)} = a_v b_x$, then c is a good continuous distribution on $G \times H$ with a total of $ofc(G)ofc(H)$ pebbles.

($\geq$): Let c be a good continuous distribution on $G \times H$. Then for all v and x,

$$\begin{aligned} 1 &\leq \sum_{w,y} c_{(w,y)} 2^{-d(v,w)-d(x,y)} \\ &= \sum_w (\sum_y c_{(w,y)} 2^{-d(x,y)}) 2^{-d(v,w)} \end{aligned}$$

so for all x, putting $\sum_y c_{(w,y)} 2^{-d(x,y)}$ pebbles on w is a good continuous distribution on G, and therefore, for all x,

$$\begin{aligned} ofc(G) &\leq \sum_w \sum_y c_{(w,y)} 2^{-d(x,y)} \\ &= \sum_y (\sum_w c_{(w,y)}) 2^{-d(x,y)} \end{aligned}$$

which implies that putting $\sum_w c_{(w,y)}/ofc(G)$ pebbles on y is a good continuous distribution on H; therefore, $\sum_y \sum_w c_{(w,y)}/ofc(G) \geq ofc(H)$, so $\sum_{w,y} c_{(w,y)}$ is at least $ofc(G)ofc(H)$, as desired. ∎

Since a good distribution is also a good continuous distribution, $of(G) \geq ofc(G)$ for all G. Let P_2 be the path with two vertices; then the n-cube, Q_n, is P_2^n. It is easy to see that $ofc(P_2) = \frac{4}{3}$ (a continuous optimal pebbling has $\frac{2}{3}$ of a pebble on each vertex) and consequently $of(Q_n) \geq ofc(Q_n) = (\frac{4}{3})^n$. What is interesting is that this is also an approximate upper bound.

Let the *covering radius* of a subset W of $V(G)$ be the smallest d such that all vertices v of G are at distance no more than d from some member of W. In [3] we find the following theorem:

Theorem 2 *For all n and $0 < \rho < n/2$, there exists a subset W of $V(Q_n)$ with covering radius ρ and $|W| = 2^k$, where $k \leq n(1 - H(\rho/n)) + 2\log_2 n$. Here, $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$.*

We can use this to prove our upper bound.

Corollary 3 $of(Q_n) = (\frac{4}{3})^{n+O(\log n)}$.

Proof. Let W be as in Theorem 2. If we put 2^ρ pebbles on each vertex of W, this will be a good distribution on Q_n, and it will use $2^{\rho+k}$ pebbles. If ρ is approximately αn, we can approximate $\rho + k$ by $n(1+\alpha-H(\alpha))$. The minimum of $\alpha - H(\alpha)$ is at $\alpha = \frac{1}{3}$, so let $\rho = \lceil n/3 \rceil$. Since H is increasing on $[0, \frac{1}{2}]$, for $n \geq 2$, $H(\rho/n) \geq H(\frac{1}{3}) = -\frac{2}{3} + \log_2 3$. Then

$$\begin{aligned} \rho + k &\leq \lceil n/3 \rceil + n(1 - H(\rho/n)) + 2\log_2 n \\ &\leq n/3 + n(1 - (-2/3 + \log_2 3)) + 2\log_2 n + 1 \\ &= (2 - \log_2 3)n + O(\log n). \end{aligned}$$

This completes the proof. ∎

In [3], Theorem 2 is proved probabilistically: W is chosen randomly from a set of cardinality 2^k subsets of $V(Q_n)$, and it is shown that there is a positive probability that W has small enough covering radius. This suggests the possibility that we can find an upper bound on $of(G^n)$ in the same manner, and indeed this is the case.

In the remainder of the paper, we will let $0 \cdot \infty = 0$.

Lemma 4 *Let G be a graph, let $ofc(G) = b$, and let a be a continuous optimal pebbling of G. Then for all vertices w of G,*

$$\frac{\sum_y a_y d(w,y) 2^{-d(w,y)}}{\sum_y a_y 2^{-d(w,y)}} \leq \log_2 b.$$

Proof. If we set $0\log_2 0 = 0$, then $x\log_2 x$ is convex for nonnegative x, so for all nonnegative x_y and c_y with $\sum_y c_y = 1$,

$$\sum_y c_y x_y \log_2 x_y \geq (\sum_y c_y x_y)\log_2(\sum_y c_y x_y).$$

Setting $c_y = a_y/b$ and $x_y = 2^{-d(w,y)}$ and rearranging then gives

$$\frac{\sum_y a_y d(w,y) 2^{-d(w,y)}}{\sum_y a_y 2^{-d(w,y)}} \leq \log_2 \frac{b}{\sum_y a_y 2^{-d(w,y)}}.$$

Since a is good, we have $\sum_y a_y 2^{-d(w,y)} \geq 1$, so we have the desired result. ∎

Theorem 5 *For all graphs G, $of(G^n) = ofc(G)^{n+O(\log n)}$.*

Proof. Let $n \geq 2$, let $V(G) = \{x_1, \ldots, x_m\}$, let D be the maximum diameter of any connected component of G, let $ofc(G) = b$, let a be a continuous optimal pebbling of G, and let $\alpha_v = a_v/b$ for all $v \in V(G)$. Let $\Delta_0 = \lceil n \log_2 b \rceil$, and fix $\theta \in \mathbf{R}_{>0}$. For $\Delta = 0, \ldots, \Delta_0$, let $A_\Delta = b^n 2^{-\Delta} n^\theta$. Define a probability distribution on $V(G^n)$ by giving vertex $(v_1, \ldots, v_n)$ probability $\prod_i \alpha_{v_i}$. For each $\Delta = 0, \ldots, \Delta_0$, independently select, with replacement, $\lceil A_\Delta \rceil$ vertices in $V(G^n)$ according to this probability distribution; call the set of selected vertices S_Δ. For each Δ, place $2^{\Delta + m^2 D}$ pebbles on each vertex in S_Δ. This gives us our distribution of pebbles; we use no more than

$$\begin{aligned} \sum_{\Delta=0}^{\Delta_0} \lceil A_\Delta \rceil 2^{\Delta + m^2 D} &\leq ((\Delta_0 + 1) b^n n^\theta + 2^{\Delta_0 + 1} - 1) 2^{m^2 D} \\ &= b^{n + O(\log n)} \end{aligned}$$

pebbles in all.

The resultant distribution will be good if, for each v, there is some Δ such that v is within distance $\Delta + m^2 D$ of one of the vertices in S_Δ, and this will happen with positive probability if, for each vertex v, the probability of such a Δ failing to exist is less than m^{-n}. Fix a typical vertex $v = (v_1, \ldots, v_n)$, and let i_w be the number of indices i with $v_i = w$. For some other vertex $v' = (v'_1, \ldots, v'_n)$, let j_{wy} be the number of indices i with $v_i = w$ and $v'_i = y$. Consider the set T of all vertices v' such that, for some fixed l_{wy}'s, $j_{wy} = l_{wy}$ for all w and y. Each member of this set has distance $\sum_{w,y} l_{wy} d(w,y)$ from v, and is selected with probability $\prod_y \alpha_y^{\sum_w l_{wy}}$. The probability that no vertex in T is in S_Δ is thus

$$p = \left(1 - |T| \prod_y \alpha_y^{\sum_w l_{wy}}\right)^{\lceil A_\Delta \rceil},$$

and

$$|T| = \prod_w \binom{i_w}{l_{wx_1} \ldots l_{wx_m}}.$$

Fix some nonnegative real λ_{wy}'s; let $\lambda_{wy} = 0$ if $\alpha_y = 0$ or $d(w,y) = \infty$, and let $\sum_y \lambda_{wy} = 1$ for all w. For all w and y, let l_{wy} be $\lambda_{wy} i_w$, rounded either to the next larger or next smaller integer in such a way that the condition $\sum_y l_{wy} = i_w$ holds for all w. We wish to find a bound for p in terms of the λ_{wy}'s.

Since l_{wy} and $\lambda_{wy} i_w$ are both in some interval $[r, r+1]$, $r \in \mathbf{Z}_{\geq 0}$, it follows that $l_{wy}! \leq \Gamma(\lambda_{wy} i_w + 1) \max(l_{wy}, 2/\sqrt{\pi})$, and it follows from Stirling's approximation that for $z \geq 0$,

$$\left(\frac{z}{e}\right)^z \leq \Gamma(z+1) \leq \left(\frac{z}{e}\right)^z (\sqrt{2\pi z} + 1);$$

hence,

$$\begin{aligned}
|T| &= \prod_w \frac{i_w!}{l_{wx_1}! \cdots l_{wx_m}!} \\
&\geq \frac{1}{n^{m^2}} \prod_w \frac{\Gamma(i_w + 1)}{\Gamma(\lambda_{wx_1} i_w + 1) \cdots \Gamma(\lambda_{wx_m} i_w + 1)} \\
&\geq \frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^{m^2}} \prod_w \frac{1}{(\lambda_{wx_1}^{\lambda_{wx_1}} \cdots \lambda_{wx_m}^{\lambda_{wx_m}})^{i_w}} \\
&= \frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^{m^2}} \prod_{\lambda_{wy} \neq 0} \frac{1}{\lambda_{wy}^{\lambda_{wy} i_w}}.
\end{aligned}$$

Also, we will have $\sum_w l_{wy} = \sum_w \lambda_{wy} i_w = 0$ if $\alpha_y = 0$; for other y, $l_{wy} \leq \lceil \lambda_{wy} i_w \rceil \leq \lambda_{wy} i_w + 1$, so $\sum_w l_{wy} \leq m + \sum_w \lambda_{wy} i_w$; therefore,

$$\prod_y \alpha_y^{\sum_w l_{wy}} \geq \prod_{\alpha_y \neq 0} \alpha_y^m \prod_{\alpha_y \neq 0} \alpha_y^{\sum_w \lambda_{wy} i_w},$$

and then

$$p \leq \left(1 - \frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^{m^2}} \prod_{\alpha_y \neq 0} \alpha_y^m \prod_{\lambda_{wy} \neq 0} \left(\frac{\alpha_y}{\lambda_{wy}}\right)^{\lambda_{wy} i_w}\right)^{A_\Delta}. \tag{1}$$

To satisfy our distance constraint, we wish to have

$$\sum_{w,y} l_{wy} d(w,y) \leq \Delta + m^2 D. \tag{2}$$

If $d(w,y) = \infty$, $l_{wy} = 0$. Otherwise, $l_{wy} \leq \lambda_{wy} i_w + 1$, and $d(w,y) \leq D$, so

$$\sum_{w,y} l_{wy} d(w,y) \leq m^2 D + \sum_w i_w \sum_y \lambda_{wy} d(w,y),$$

and to satisfy (2) it will do to have

$$\sum_w i_w \sum_y \lambda_{wy} d(w,y) \leq \Delta. \tag{3}$$

Now set

$$\lambda_{wy} = \frac{\alpha_y 2^{-d(w,y)}}{\sum_z \alpha_z 2^{-d(w,z)}}. \tag{4}$$

Since a is a continuous optimal pebbling on G, for all w there must exist some y in the same connected component as w with $a_y \neq 0$. Hence the denominator in (4) is always nonzero. It is clear that $\sum_y \lambda_{wy} = 1$ for all w, and that $\lambda_{wy} = 0$ if $\alpha_y = 0$ or $d(w, y) = \infty$. It follows from Lemma 4 that with our choice of λ_{wy}'s, the left-hand side of (3) is no bigger than $n \log_2 b \leq \Delta_0$, so we can let Δ be the ceiling of the left-hand side of (3). Then

$$\begin{aligned}
\prod_{\lambda_{wy} \neq 0} \Big(\frac{\alpha_y}{\lambda_{wy}}\Big)^{\lambda_{wy} i_w} &= \prod_{\lambda_{wy} \neq 0} \Big(\frac{\sum_z \alpha_z 2^{-d(w,z)}}{2^{-d(w,y)}}\Big)^{\lambda_{wy} i_w} \\
&\geq 2^{\Delta - 1} \prod_w \Big(\sum_z \alpha_z 2^{-d(w,z)}\Big)^{i_w} \\
&\geq 2^{\Delta - 1} b^{-n},
\end{aligned}$$

since a is a continuous optimal pebbling of G. Substituting this into (1), we then find that

$$p \leq \Big(1 - \frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^{m^2}} 2^{\Delta - 1} b^{-n} \prod_{\alpha_y \neq 0} \alpha_y^m\Big)^{A_\Delta},$$

or, using $1 - x \leq e^{-x}$,

$$p \leq \exp\Big(- \frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^{m^2}} 2^{\Delta - 1} b^{-n} A_\Delta \prod_{\alpha_y \neq 0} \alpha_y^m\Big).$$

We want to have $\log p < -n \log m$ for large n. Recalling that $A_\Delta 2^\Delta = b^n n^\theta$, we see that this will be true if $\theta > \frac{3}{2} m^2 + 1$, so we are done. ∎

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