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Optimal pebbling number of graphs with given minimum degree

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ABSTRACT

Consider a distribution of pebbles on a connected graph G. A pebbling move removes two pebbles from a vertex and places one to an adjacent vertex. A vertex is reachable under a pebbling distribution if it has a pebble after the application of a sequence of pebbling moves. The optimal pebbling number $\pi^*(G)$ is the smallest number of pebbles that we can distribute in such a way that each vertex is reachable.

It was known that the optimal pebbling number of any connected graph is at most $\frac{4n}{\delta+1}$, where δ is the minimum degree of the graph. We strengthen this bound by showing that equality cannot be attained and that the bound is sharp. If $\mathrm{diam}(G) \geq 3$ then we further improve the bound to $\pi^*(G) \leq \frac{3.75n}{\delta+1}$. On the other hand, we show that, for arbitrary large diameter and any $\epsilon > 0$, there are infinitely many graphs whose optimal pebbling number is bigger than $\left(\frac{8}{3} - \epsilon\right) \frac{n}{(\delta+1)}$.

1. Introduction

Graph pebbling is a game on graphs initiated by a method of Saks and Lagarias to answer a number-theoretic question of Erdős and Lemke, which was successfully carried out by Chung in 1989 [2]. Each graph in this paper is simple. We denote the vertex set and the edge set of graph G by $V(G)$ and $E(G)$, respectively. We use n and δ for the order and the minimum degree of G, respectively.

A pebbling distribution D on graph G is a function mapping the vertex set to nonnegative integers. We can imagine that each vertex v has $D(v)$ pebbles. A pebbling move removes two pebbles from a vertex (having at least two pebbles) and places one on an adjacent vertex.

A vertex v is *k-reachable* under a distribution D if there is a sequence of pebbling moves that places at least k pebbles on v. We say that a subgraph H is *k-solvable* under distribution D if each vertex of H is k-reachable under D. When the whole graph is k-solvable under a pebbling distribution, then we say that the distribution is k-solvable.

When $S \subseteq V(G)$ then let $D(S)$ denote the total number of pebbles placed on the vertices of S. We say that $D(V(S))$ is the *size* of D. We use the standard $|X|$ notation to denote the size of X when X is a pebbling distribution or a set. A pebbling distribution D on a graph G will be called *k-optimal* if it is k-solvable and its size is the smallest possible. This size is called the *k-optimal pebbling number* and denoted by $\pi^*_k(G)$. When $k = 1$ we omit the k part from all of the previous definitions and notations.

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The optimal pebbling number of several graph families is known. For example exact values were given for paths and cycles [3,8], ladders [1], caterpillars [5] and m-ary trees [4]. The values for graphs with diameter smaller than four are also characterized by some easily checkable domination conditions [7]. However, determining the optimal pebbling number for a given graph is NP-hard [6].

In [1] the optimal pebbling number of graphs with given minimal degree is studied. This paper contains many great results about the topic. The authors proved that $\pi^*(G) \leq \frac{4n}{(\delta+1)}$ and they also found a version utilizing the girth of the graph. A construction for infinite number of graphs with optimal pebbling number $(2.4 - \frac{24}{5\delta+15} - o(\frac{1}{n}))\frac{n}{\delta+1}$ is also given in the article.

In the present paper we continue the study of graphs with fixed minimum degree. We prove that there are infinitely many diameter two graphs whose optimal pebbling number is close to the $\frac{4n}{\delta+1}$ upper bound. More precisely, for any $\epsilon > 0$ there is a graph with optimal pebbling number greater than $\frac{(4-\epsilon)}{\delta+1}$.

One can ask that what happens if we consider larger diameter? In the second part of Section 2 we use the previous graphs as building blocks to construct a family of graphs with arbitrary large diameter, fixed minimum degree, and high optimal pebbling number. For any d and $\epsilon > 0$ we present a graph G with $\mathrm{diam}(G) > d$ and $\pi^*(G) > \left(\frac{8}{3} - \epsilon\right)\frac{n}{(\delta+1)}$.

In the case when the diameter is at least three we also prove a stronger upper bound in Section 3. It is shown that $\pi^*(G) \leq \frac{15n}{4(\delta+1)}$ holds in this case. Unfortunately, we do not know whether this bound is sharp or not, but it is strong enough to conclude that the original upper bound of [1] does not hold with equality.

Finally, the authors of [1] prove, for $k \geq 6$, that the family of connected graphs G with n vertices and minimum degree k has $\pi^*(G)/n \to 0$ as $\mathrm{girth}(G) \to \infty$. They asked whether the same could be true for $k \in \{3, 4, 5\}$. We prove in Section 4 that this is true for $k \in \{4, 5\}$, leaving the case $k = 3$ unresolved.

2. A family of graphs with large optimal pebbling number

We say that a vertex v is *dominated* by a set of vertices S if v is contained in S or there is a vertex in S that is adjacent to v. A vertex set S *dominates* G (or is a *dominating set* of G) if each vertex of G is dominated by S. An edge $\{u, v\}$ *dominates* G if the set of its endpoints dominates G. We use $G\square H$ to denote the *Cartesian product* of graphs G and H, that is $V(G\square H) = V(G) \times V(H)$ and $\{(g, h), (g', h')\} \in E(G\square H)$ if either $g = g'$, $\{h, h'\} \in E(H)$ or $\{g, g'\} \in E(G)$, $h = h'$.

The *distance* between vertices u and w is the number of edges contained in a shortest path between them. We denote this quantity by $d(u, v)$. The *distance-k open neighborhood* of a vertex v, denoted $N^k(v)$, contains all vertices whose distance from v is exactly k. On the other hand, *the distance-k closed neighborhood* of v contains all vertices whose distance from v is at most k. We denote this set by $N^k[v]$. When $k = 1$ we omit the distance-1 part from the name and the upper index 1 from the notation.

We are going to use small graphs as a building blocks in our construction. We call these graphs *special*.

Definition 2.1. Graph H is *special* if the following properties are fulfilled:

1. The diameter of H is two,
2. H does not have a dominating edge, and
3. H has two vertices u and v – called a *special pair* – such that
   (a) $d(u, v) = 2$,
   (b) if we remove u or v from H then the obtained subgraph does not have a dominating edge, and
   (c) $(N(u) \cap N(v)) \cup \{u, v\}$ is a dominating set of H.

Claim 2.2. *Let K_m be the complete graph with $V(K_m) = \{x_1, \ldots, x_m\}$. Then $\overline{K_m\square K_m}$, the complement of $K_m\square K_m$, is special for any $m \geq 4$.*

Proof. Two vertices are adjacent in $\overline{K_m\square K_m}$ if and only if both of their coordinates are different. Its diameter is two since any two vertices share a common neighbor.

If (x_i, x_j) and (x_k, x_ℓ) are two adjacent vertices then $i \neq k$ and $j \neq \ell$, and so neither (x_i, x_ℓ) nor (x_k, x_j) is dominated by $\{(x_i, x_j), (x_k, x_\ell)\}$. Hence no two adjacent vertices dominate $V(\overline{K_m\square K_m})$. Furthermore, if we remove a vertex, at least one of the two non-dominated vertices remains, which is still not dominated.

Define $u = (x_1, x_1)$ and $v = (x_1, x_2)$ to be the special pair. Then (x_m, x_m) is a common neighbor of u and v and it dominates everything except (x_i, x_m) and (x_m, x_i), where $i < m$. Vertices u and v dominate all of these except (x_1, x_m), but this last one is dominated by (x_2, x_3), which is also a common neighbor of u and v. Therefore $(N(u) \cap N(v)) \cup \{u, v\}$ is a dominating set of $\overline{K_m\square K_m}$. $\square$

There are many other special graphs. In this paper we mention just one more example: the *circulant* graph, where the vertices are labeled from 1 to $2m$, $m \geq 5$ and two vertices i and j are adjacent if and only if $i - j \not\equiv m, m \pm 1 \mod 2m$.

Muntz et al. showed that the optimal pebbling number of a diameter two graph lacking a dominating edge is 4 [7]. Therefore the optimal pebbling number of any special graph is 4. Using this we can prove the following theorem.

Theorem 2.3. *For any $\epsilon > 0$ there is a diameter two graph G on n vertices with $\pi^*(G) > \frac{(4-\epsilon)n}{\delta+1}$.*

Proof. For a fixed $\epsilon > 0$, consider a graph $\overline{K_m \Box K_m}$, where $m > \max(\frac{a}{a-1}, 2)$ and $a = \sqrt{\frac{4}{4-\epsilon}}$. Each degree in $\overline{K_m \Box K_m}$ is $(m-1)^2$ and its order is m^2. Since both a and m are greater than 1 we have that $a > \frac{m}{m-1}$, which implies the following:

$$\pi^*(\overline{K_m \Box K_m}) = 4 = (4-\epsilon)a^2 > \frac{(4-\epsilon)m^2}{(m-1)^2} > \frac{(4-\epsilon)n}{\delta+1}. \quad \Box$$

Now we turn our attention to the case in which the diameter is at least three. For any d and ϵ we exhibit a graph whose diameter is $9d-1$ and its optimal pebbling number is greater than $\left(\frac{8}{3}-\epsilon\right)\frac{n}{\delta+1}$. In fact, we connect $3d$ special graphs sequentially, using the special pairs, and prove that the optimal pebbling number of the obtained graph is $8d$. To obtain the desired result we choose the special graphs to be $\overline{K_m \Box K_m}$ again.

Let $H_1, H_2, \ldots, H_\ell$ be (not necessarily the same) special graphs. Denote the vertex set of H_i by B_i and let u_i and v_i be a special pair of H_i. Connect these graphs sequentially by placing edges between v_i and u_{i+1} to obtain a new graph G_ℓ. Consequently $V(G_\ell) = \cup_{i=1}^{\ell} B_i$ and $E(G_\ell) = \cup_{i=1}^{\ell} E(H_i) \bigcup \cup_{i=1}^{\ell-1}\{\{v_i, u_{i+1}\}\}$. We say that each subgraph H_i of G_ℓ is a *block*.

Claim 2.4. *Let $k \in Z^+$, $r \in \{0, 1, 2\}$, and let G_{3k+r} be the graph defined above. Then*

$$\pi^*(G_{3k+r}) \le \begin{cases} 8k & \text{if } r = 0 \\ 8k+4 & \text{if } r = 1 \\ 8k+6 & \text{if } r = 2. \end{cases}$$

Proof. To construct a solvable distribution with $8k$ pebbles in the $r = 0$ case we place 4 pebbles at vertices v_{3j-2} and u_{3j}, where $j = 1, 2, \ldots, k$. If a vertex of an H_i has 4 pebbles, then each vertex of H_i is reachable because H_i is a diameter two graph. Otherwise u_i and v_i are both adjacent to vertices having 4 pebbles. Therefore each element of $(N(u_i) \cap N(v_i)) \cup \{u_i, v_i\}$ is 2-reachable and, since this is a dominating set of H_i, we can move a pebble to any vertex of H_i. When $r = 1$ we use the same construction and place 4 additional pebbles at a vertex of H_{3k+1}, which solves H_{3k+1}. In the last, $r = 2$ case we start again with the $r = 0$ construction and place 3 pebbles at both v_{3k+1} and u_{3k+2}. These two vertices are 4-reachable, and therefore all vertices in the last two blocks are solvable. $\Box$

Claim 2.5. *Let G_ℓ be a graph defined above, then $\pi^*(G_1) = 4$, $\pi^*(G_2) = 6$, and $\pi^*(G_3) = 8$.*

Proof. The upper bounds have been shown. G_1 is a special graph; therefore its optimal pebbling number is 4.

Let D be a pebbling distribution on G_2 having 5 pebbles. We can assume that H_1 has fewer pebbles than H_2; therefore $D(B_1) \le 2$. Since $\pi^*(H_1) = 4$, we cannot reach each vertex of H_1 without using the pebbles placed at B_2. H_1 and H_2 are connected by only one edge; therefore the only way to use the pebbles of B_2 is to move as many pebbles through this edge as possible. Furthermore, if we do nothing with the pebbles of B_1 and we move as many pebbles from B_2 to v_1 as possible, then the obtained distribution on H_1 has to be solvable if D is solvable on G_2. But this obtained distribution can have at most $D(B_1) + D(B_2)/2 \le 3$ pebbles. Hence D is not solvable.

To prove the last statement we argue that a distribution with 7 pebbles is not solvable on G_3. Define $\ell = D(B_1)$. If $\ell \ge 3$ then we can accumulate on $B_2 \cup B_3$ at most $\ell/2 + (7-l) < 6$ pebbles. If $\ell = 0$ then 4 pebbles must be obtained from 7 on $B_2 \cup B_3$, which is not possible. If $\ell = 1$ then, to be able to solve for B_1, the remaining 6 pebbles must placed on u_2, which is not solvable for B_3. Finally, if $D(B_1) = 2 = D(B_3)$ then only 3 pebbles are on B_2, and it is not possible to move two additional pebbles to B_1. $\Box$

Our goal is to prove that $\pi^*(G_\ell) \ge \frac{8}{3}\ell$ holds, but first we need some preparation. A cut argument will be used, which argues that, in an optimal distribution, several edges cannot transfer pebbles. Therefore these edges can be removed from the graph without changing the optimal pebbling number. We are going to remove edges in such a way that the obtained graph has two connected components, each of which is either a smaller instance of G_ℓ or almost a G_ℓ.

Let G_ℓ^- be the subgraph of G_ℓ that we obtain by deleting v_ℓ. Let G_ℓ^+ be the supergraph of G_ℓ that we obtain by adding a leaf to u_1. Let G_0^+ denote the one vertex graph.

Lemma 2.6. *Let D be a solvable distribution D on G_ℓ such that $|D| < 3\ell - 1$. Then there is a decomposition of G_ℓ into parts isomorphic to either G_k and $G_{\ell-k}$, with $1 \le k < \ell$, or G_k^- and $G_{\ell-k}^+$, with $1 \le k \le \ell$, such that no pebbles can be moved through the edges connecting the two parts.*

Before proving Lemma 2.6, we make note of three results that will be used in its proof. We are going to use the *collapsing technique*, which was introduced in [1]. Let G and H be simple graphs. We say that H is a *quotient* of G if there is a surjective mapping $\phi : V(G) \to V(H)$ such that $\{h_1, h_2\} \in E(H)$ if and only if $h_1 = \phi(g_1)$ and $h_2 = \phi(g_2)$ for some $\{g_1, g_2\} \in E(G)$, We say that ϕ *collapses* G to H and that, if D is a pebbling distribution on G, the *collapsed distribution* $\phi(D)$ on H is defined by $\phi(D)(h) = \sum_{g \in V(G)|\phi(g)=h} D(g)$.

Claim 2.7. *If v is k-reachable under D then $\phi(v)$ is k-reachable under $\phi(D)$.*

This claim is a generalization of the Collapsing Lemma [1]. The proof given by Bunde et al. can be modified to prove our claim.

Proof. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be a sequence of pebbling moves whose application to D yields at least k pebbles at v. We are going to show that there is a corresponding sequence of pebbling moves whose application to $\phi(D)$ yields at least k pebbles at $\phi(v)$.

Let $(u \to w)$ be a pebbling move that removes two pebbles from u and places one pebble at w and let $D_{(u\to w)}$ be the distribution obtained after applying $(u \to w)$ to D. If $\phi(u) \neq \phi(w)$, then clearly $\phi(D_{(u\to v)}) = \phi(D)_{(\phi(u)\to\phi(v))}$. So $\phi(u) \to \phi(w)$ does the same job in the quotient graph under the collapsed distribution as $(u \to w)$ does under the original one.

If $\phi(u) = \phi(w)$ then it is obvious that such a move does not have a corresponding move under the collapsed distribution. Fortunately we do not need such a move, because for any vertex x it is true that $\phi(D_{(u\to w)})(x) \le \phi(D)(x)$. So such moves can be omitted when we are constructing a sequence of pebbling moves that mimics σ under $\phi(D)$. Therefore we define the collapsed version of each move by

$$\phi((u \to w)) = \begin{cases} (\phi(u) \to \phi(w)) & \text{if } \phi(u) \neq \phi(w) \\ \emptyset & \text{if } \phi(u) = \phi(w), \end{cases}$$

and we define $\phi(\sigma) = (\phi(\sigma_1), \phi(\sigma_2), \ldots, \phi(\sigma_k))$. Since $\phi(\sigma)$ mimics σ, its application to $\phi(D)$ results in at least k pebbles at $\phi(v)$. □

We are going to collapse G_ℓ to the path containing 3ℓ vertices, denoted by $P_{3\ell}$. Therefore we state some optimal pebbling results about paths.

Theorem 2.8 ([1]). *Any 2-optimal distribution on the n-vertex path contains at least $n + 1$ pebbles.*

Claim 2.9. *If an inner vertex of the path is not 2-reachable, then neither is one of its neighbors.*

Proof. If a vertex v is not 2-reachable then v cuts the path to two and no pebble can be moved through v. Thus if both neighbors of v are 2-reachable, then each of them can supply a pebble to v, which is a contradiction. □

Proof of Lemma 2.6. Let D' be the following distribution on G_ℓ: $D'(u_1) = D(u_1) + 1$, $D'(v_\ell) = D(v_\ell) + 1$, and $D'(x) = D(x)$ for $x \notin \{u_1, v_\ell\}$. Since D is a solvable distribution, u_1 and v_ℓ are 2-reachable under D'.

Let $P_{3\ell}$ be a path on 3ℓ vertices and denote its vertices with $p_1, p_2, \ldots p_{3\ell}$. Let ϕ be a mapping that collapses G_ℓ to $P_{3\ell}$ in such a way that $\phi(u_i) = p_{3i-2}$, $\phi(v_i) = p_{3i}$, and $\phi(x) = p_{3i-1}$ for $x \in B_i \setminus \{u_i, v_i\}$. $P_{3\ell}$ is a quotient of G_ℓ.

Because $\phi(D')$ has less than $3\ell + 1$ pebbles, according to Theorem 2.8 it is not a 2-solvable distribution on $P_{3\ell}$. Hence there is a vertex p_i in $P_{3\ell}$ that is not 2-reachable under $\phi(D')$. Claim 2.7 implies that both p_1 and $p_{3\ell}$ are 2-reachable, and so $1 < i < 3\ell$. Since p_i is not 2-reachable, no pebble can move from p_i to either p_{i-1} or p_{i+1}. Similarly, pebbles cannot move from both p_{i-1} and p_{i+1} to p_i. Thus, for one of the edges incident with p_i, without loss of generality $\{p_i, p_{i+1}\}$, no pebble can move across it. This also means that p_{i+1} is not 2-reachable.

Claim 2.7 yields that the vertices of $\phi^{-1}(p_i) \cup \phi^{-1}(p_{i+1})$ are also not 2-reachable under D. Therefore no pebbles can be passed between $\phi^{-1}(p_i)$ and $\phi^{-1}(p_{i+1})$. Deleting the edges between them makes the graph disconnected and leaves D solvable.

The two connected components are isomorphic to either G_k and G_{l-k} or G_k^- and G_{l-k}^+, where $i \in \{3k, 3k-1, 3(l-k)+1\}$. This is implied by the collapsing function. □

Lemma 2.6 guides us to make an induction argument. However we also need some information about $\pi^*(G_\ell^+)$ and $\pi^*(G_\ell^-)$. Special graph property 3(b) guarantees that these values are at least $\pi^*(G_\ell)$.

Lemma 2.10. *$\pi^*(G_\ell^+) \ge \pi^*(G_\ell)$ and $\pi^*(G_\ell^-) \ge \pi^*(G_\ell)$.*

Proof. Adding a leaf cannot decrease the optimal pebbling number; hence the first inequality holds. G_1^- is a diameter two graph without a dominating edge; therefore $\pi^*(G_1^-) = 4 = \pi^*(G_1)$. To prove the rest of the assertion, we show that there is an optimal distribution D' of G_ℓ^- that is also a solvable distribution on G_ℓ (where we interpret D' to be a distribution on G_ℓ with no pebbles on v_ℓ).

Let D be an optimal distribution on G_ℓ^-. Denote the last block of G_ℓ^-, where the removed vertex was located, by H'_ℓ. Since H_ℓ was special, H'_ℓ does not have a dominating edge and its diameter is at least two; therefore $\pi^*(H'_\ell) \ge 4$.

H'_ℓ can obtain pebbles from the rest of the graph only through the $\{v_{\ell-1}, u_\ell\}$ edge. Let k be the maximum number of pebbles that can reach u_ℓ using this edge. Since D was solvable, $k + D(V(H'_\ell)) \ge 4$. If we relocate all pebbles of H'_ℓ at u_ℓ and put back v_ℓ, the obtained D' distribution on G_ℓ also satisfies $k + D'(u_\ell) \ge 4$; therefore each vertex of B_ℓ is reachable under D'. The other vertices remained reachable since we moved the pebbles closer to them in one pile. □

Now we are ready to prove the claim we promised earlier.

Claim 2.11. *Let G_ℓ be the graph defined above. Then $\pi^*(G_\ell) \geq \frac{8}{3}\ell$.*

Proof. Assume the contrary. Let G_ℓ be a minimal counterexample. Claim 2.5 implies that $\ell \geq 4$. Let D be an optimal distribution on G_ℓ.

Since G_ℓ is a counterexample, $|D| < \frac{8}{3}\ell \leq 3\ell - 1$. Therefore we can apply Lemma 2.6. Accordingly, we can break G_ℓ into G_k and $G_{\ell-k}$ or G_k^- and $G_{\ell-k}^+$ such that no pebbles can be moved between the two parts. This means that D induces solvable distributions on both parts.

In the first case we have

$$\pi^*(G_\ell) = |D(G_k)| + |D(G_{\ell-k})| \geq \pi^*(G_k) + \pi^*(G_{\ell-k}).$$

But since G_ℓ is a minimal counterexample we have

$$\pi^*(G_k) + \pi^*(G_{\ell-k}) \geq \frac{8}{3}k + \frac{8}{3}(\ell - k) = \frac{8}{3}\ell > |D|,$$

which contradicts our assumption.

In the second case, when $k \neq \ell$ we can use Lemma 2.10 to give the following chain of inequalities,

$$\pi^*(G_\ell) = |D(G_k^-)| + |D(G_{\ell-k}^+)| \geq \pi^*(G_k^-) + \pi^*(G_{\ell-k}^+) \geq \pi^*(G_k) + \pi^*(G_{\ell-k}),$$

which leads to the same contradiction. When $\ell = k$ we have

$$\pi^*(G_\ell) = |D(G_\ell^-)| + |D(G_0^+)| \geq \pi^*(G_\ell^-) + \pi^*(G_0^+) \geq \pi^*(G_\ell) + 1,$$

which is also a contradiction.

Therefore there is no counterexample. □

Corollary 2.12. $\pi^*(G_{3k}) = 8k$

Theorem 2.13. *For any $\epsilon > 0$ and any integer d, there is a graph G such that its diameter is greater than d and $\pi^*(G) \geq (\frac{8}{3} - \epsilon)\frac{n}{\delta+1}$.*

Proof. Consider G_{3d} with blocks $\overline{K_m \square K_m}$, where $m > \max(\frac{a}{a-1}, 3)$ and $a = \sqrt{\frac{8/3}{8/3-\epsilon}}$. Its diameter is $3d - 1$, $\delta(G_{3d}) = \delta(\overline{K_m \square K_m}) = (m-1)^2$, and $|V(G_{3d})| = 3dm^2$. The optimal pebbling number of G_{3d} is $8d$. If we repeat the calculation of Theorem 2.3 we receive the desired result:

$$\pi^*(G_{3d}) = 8d = 3d\left(\frac{8}{3} - \epsilon\right)a^2 > 3d\left(\frac{8}{3} - \epsilon\right)\frac{m^2}{(m-1)^2} > \left(\frac{8}{3} - \epsilon\right)\frac{n}{\delta + 1}. \quad \square$$

3. Improved upper bound when diameter is at least three

In this section we give a construction of a pebbling distribution having at most $\frac{15n}{4(\delta+1)}$ pebbles for any graph whose diameter is at least three.

We are going to discuss several graphs on the same labeled vertex set. To make it clear which graph we are considering in a formula we write the name of the graph as a lower index, i.e. $d_G(u, v)$ is the distance between vertices u and v in graph G.

We define distances between subgraphs in the natural way: If H and K are subgraphs of G, then $d_G(H, K) = \min_{u \in V(H), v \in V(K)} d_G(u, v)$.

We can think about a vertex as a subgraph; therefore, we define the *distance-k open neighborhood of a subgraph H*, denoted $N^k(H)$, to be the set of vertices whose distance from H is exactly k. We define the closed neighborhood similarly, using distance at most k, denoted by $N^k[H]$. Note that $N^k(H) = N^k[H] \backslash N^{k-1}[H]$.

The following property will be useful in our investigations: A vertex $v \in V(G)$ is *strongly reachable under D* if each vertex from the closed neighborhood of v is reachable under D. This property, together with traditional reachability, partitions the vertex set into three sets $\mathcal{T}(D)$, $\mathcal{H}(D)$ and $\mathcal{U}(D)$, where $\mathcal{T}(D)$ includes the strongly reachable vertices, $\mathcal{H}(D)$ contains reachable but not strongly reachable vertices, and $\mathcal{U}(D)$ contains the remaining vertices.

Theorem 3.1. *Let G be a connected graph having diameter at least 3 and with minimum degree δ. Then we have*

$$\pi^*(G) \leq \frac{15n}{4(\delta + 1)}.$$

Let D and D' be pebble distributions. We say that D' is an *expansion* of D ($D' \geq D'$) if $\forall v \in V(G)\, D'(v) \geq D(v)$. If $D \neq D'$, then we write that $D' > D$. If D' is an expansion of D, then let $\Delta_{D,D'}$ be the pebbling distribution defined by $\Delta_{D,D'}(v) = D'(v) - D(v)$ $\forall v \in V(G)$.

If we would like to create a solvable distribution, then we can do it incrementally. We start with the trivial distribution with no pebbles and add more and more pebbles to it. This yields a sequence of distributions $0 < D_1 < D_2 < \cdots < D_{k-1} < D_k$, where D_k is solvable. The number of reachable vertices grows during this process. Note that $\mathcal{T}(D_i) \subseteq \mathcal{T}(D_{i+1})$, while $\mathcal{U}(D_i) \supseteq \mathcal{U}(D_{i+1})$. Furthermore we know that $\mathcal{T}(D_k) = V(G)$ and $\mathcal{H}(D_k) = \mathcal{U}(D_k) = \emptyset$.

If, for each i, the difference $|\Delta_{D_i,D_{i+1}}|$ is relatively small and $|\mathcal{T}(D_{i+1}) \setminus \mathcal{T}(D_i)|$ is relatively big, then it will mean that $|D_k|$ is not too large. To make this intuitive idea precise we define the following *strengthening ratio*.

Definition 3.2. Suppose that we have distributions D and D' on graph G, such that $D' > D$. Denote the difference between the sizes of these distribution by $\Delta p_{D,D'} = |D'| - |D| = |\Delta_{D,D'}|$. We use $\Delta\mathcal{T}_{D,D'}$ for set $\mathcal{T}(D') \setminus \mathcal{T}(D)$, having cardinality $\Delta t_{D,D'}$.

We say that the *strengthening ratio of the expansion* $D' > D$ equals

$$\mathcal{E}(D, D') = \frac{\Delta t_{D,D'}}{\Delta p_{D,D'}}.$$

The *strengthening ratio of distribution* $D \neq 0$ is $\mathcal{E}(0, D)$, and the strengthening ratio of $D = 0$ is ∞.

Fact 3.3. *If D is solvable, then $|D| = \frac{n}{\mathcal{E}(0,D)}$.*

This fact shows that if we want to give a solvable distribution whose size is close to the optimum, then its strengthening ratio must also be also close to that of the optimum. Furthermore, a smaller solvable distribution has bigger strengthening ratio. The next lemma shows that, if we break D_k to a sequence of expansions $0 < D_1 < D_2 < \cdots < D_{k-1} < D_k$, the minimum strengthening ratio among all expansion steps is a lower bound for $\mathcal{E}(0, D_k)$. Therefore we are looking for an expansion chain where each expansion step's strengthening ratio is relatively big.

Fact 3.4. *Let a, b, c, d be nonnegative real numbers, then the following holds:*

$$\frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right).$$

Proof. Without loss of generality we can assume that $\frac{b}{d} \geq \frac{a}{c}$. Then we have that $ac + bc \geq ac + ad$, which is equivalent to $\frac{a+b}{c+d} \geq \frac{a}{c}$. □

Lemma 3.5. *Let D_1, D_2 and D_3 be distributions on G. If $D_1 < D_2$ and $D_2 < D_3$, then*

$$\mathcal{E}(D_1, D_3) \geq \min(\mathcal{E}(D_1, D_2), \mathcal{E}(D_2, D_3)).$$

Proof.

Using Fact 3.4 and the definition of strengthening ratio, we obtain

$$\begin{aligned}\mathcal{E}(D_1, D_3) &= \frac{\Delta t_{D_1,D_3}}{\Delta p_{D_1,D_3}} = \frac{\Delta t_{D_1,D_2} + \Delta t_{D_2,D_3}}{\Delta p_{D_1,D_2} + \Delta p_{D_2,D_3}} \\ &\geq \min\left(\frac{\Delta t_{D_1,D_2}}{\Delta p_{D_1,D_2}}, \frac{\Delta t_{D_2,D_3}}{\Delta p_{D_2,D_3}}\right) = \min\left(\mathcal{E}(D_1, D_2), \mathcal{E}(D_2, D_3)\right). \quad \square\end{aligned}$$

In the next lemma we state that we can construct a distribution D with some special properties. This lemma formalizes the following idea: If there are pairs of adjacent vertices, such that the closed neighborhood of each pair is large, then we can make all vertices of these pairs reachable with few pebbles, while many other vertices become reachable. The connection between few and many is established by strengthening ratio.

Lemma 3.6. *Let G be an arbitrary, simple, connected graph. There is a pebbling distribution D on G that satisfies the following conditions.*

(i) The strengthening ratio of D is at least $\frac{4}{15}(\delta + 1)$, and
(ii) if (u, v) is an edge of G and $|N[u] \cup N[v]| \geq \frac{29}{15}(\delta + 1)$ then both of u and v are reachable under D.

Proof. Our proof is a construction for such a D.

We mark the edges whose vertices must be reachable to fulfill condition *(ii)*: an edge (u, v) is marked if and only if $|N[u] \cup N[v]| \geq \frac{29}{15}(\delta + 1)$.

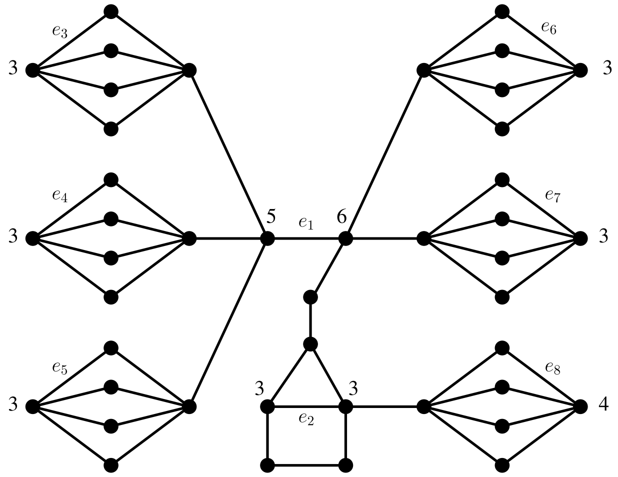

Fig. 1. A distribution constructed by the above algorithm. Edges e_1 and e_2 are selected in step 1 and $e_3, \ldots e_8$ in step 3.

First, if there is no marked edge in G, then the trivial distribution 0 can be D. Otherwise, we have to make reachable each vertex of any marked edge. To achieve this we search for these edges and, if we find a marked edge such that at least one of its vertices is not reachable, then we add some pebbles to D to make it reachable.

We will define sets $H, A, B \subset V(G)$, $P, R \subset V(G) \times V(G)$ and let L_p be a set containing vertices of G for each $p \in P$. These sets, except H, will contain marked edges or their vertices. They will have the following properties at the end of the construction.

- Each element of H will be reachable under D, but not necessarily every reachable vertex is in it.
- Each vertex of B will have a neighbor that has at least two 4-reachable distance-2 neighbors, or has an 8-reachable distance-3 neighbor.
- The elements of P will be edges whose vertices will be 4-reachable.
- The elements of R will be edges whose vertices will be 8-reachable.
- L_p will contain vertices from A whose distance from p is exactly 3.

To define these sets we perform the following construction algorithm.

1. Choose a marked edge (u, v) with $u, v \notin H$. If no such edge exists then move to step 3.
2. Add the elements of $N^2[u] \cup N^2[v]$ to H. Add (u, v) to P. Move to step 1.
3. Search for a marked edge (u, v) with $v \notin H$. If no such edge exists then move to step 6.
4. Add the elements of $N^2[v]$ to H.
5. Count the number of pairs $p \in P$ whose distance from u is 2. If there are more than one then add v to B and move to step 3. Otherwise, add v to A and add v to the set L_p, where p is the unique pair whose distance from u is 2.
6. Do for each $p \in P$: If $|L_p| \geq 5$, then move the elements of L_p from A to B and also move p from P to R.
7. Let D be the following distribution:

$$D(v) = \begin{cases} 4 & \text{if } v \in A; \\ 3 & \text{if either } v \in B \text{ or } v \text{ is an element of a pair } p \in P; \\ 5 & \text{if } v \text{ is the first element of a pair } p \in R; \\ 6 & \text{if } v \text{ is the second element of a pair } p \in R; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A result given by this algorithm is shown in Fig. 1.

First, if we choose a marked edge then both of its vertices are reachable under D. To see this, consider H. Each vertex of H is reachable under D by construction. In each step we expanded H by distance-2 closed neighborhoods of vertices that are 4-reachable. Each vertex of a marked edge is contained in H, A or B. Hence condition *(ii)* is satisfied, so we just need to verify condition *(i)*.

The vertices of the sets A and B, and of the edges contained in P or R, are all 4-reachable. Hence each vertex belonging to their neighborhood is strongly reachable. This implies that

$$\Delta t_{0,D} = |\mathcal{T}(D)| \geq \left| \left(\bigcup_{p \in P \cup R} N[p] \right) \bigcup \left(\bigcup_{v \in A \cup B} N[v] \right) \right| = \sum_{p \in P \cup R} |N[p]| + \sum_{v \in A \cup B} |N[v]|.$$

The second equality holds because the neighborhoods are disjoint by construction. Indeed, the distance between a vertex of $A \cup B$ and a pair p of $P \cup R$ is at least 3. The distance between $p, p' \in P \cup R$ is also at least 3. Both of these are guaranteed by step 2. Also, $d(u, v) \geq 3$ for $u, v \in A \cup B$ because of step 4.

The edges contained in P and R are marked. Therefore

$$\Delta t_{0,D} \geq \sum_{p \in P \cup R} |N[p]| + \sum_{v \in A \cup B} |N[v]| \geq (|P| + |R|) \cdot \frac{29}{15}(\delta + 1) + (|A| + |B|)(\delta + 1),$$

$$\Delta p_{0,D} = |D| = 4|A| + 3|B| + 6|P| + 11|R|, \text{ and}$$

$$\mathcal{E}(0, D) = \frac{\Delta t_{0,D}}{\Delta p_{0,D}} \geq \frac{(|P| + |R|) \cdot \frac{29}{15}(\delta + 1) + (|A| + |B|)(\delta + 1)}{4|A| + 3|B| + 6|P| + 11|R|}.$$

Using $(a + b)/(c + d) \geq \min(a/c, b/d)$, we obtain

$$\mathcal{E}(0, D) \geq \min\left(\frac{(\frac{29}{15}|P| + |A|)(\delta + 1)}{6|P| + 4|A|}, \frac{(\frac{29}{15}|R| + |B|)(\delta + 1)}{11|R| + 3|B|}\right). \tag{1}$$

Then step 6 of the construction implies that $|A| \leq 4|P|$ and $|B| \geq 5|R|$.

Let $|A| = 4x|P|$. In this case $0 \leq x \leq 1$ and we get the following function of x for the first part of (1), which attains its minimum at $x = 1$:

$$\frac{(\frac{29}{15}|P| + |A|)(\delta + 1)}{6|P| + 4|A|} = \frac{(\frac{29}{15} + 4x)(\delta + 1)}{6 + 16x} \geq \frac{89}{330}(\delta + 1).$$

Let $|B| = 5y|R|$ for some $y \geq 1$. Then the second part of (1) attains its minimum at $y = 1$:

$$\frac{(\frac{29}{15}|R| + |B|)(\delta + 1)}{11|R| + 3|B|} = \frac{(\frac{29}{15} + 5y)(\delta + 1)}{11 + 15y} \geq \frac{4}{15}(\delta + 1).$$

This completes the proof of the lemma. □

During the proof of Theorem 3.1 we will show that a non-solvable distribution whose strengthening ratio is above the desired bound always can be expanded to a larger one whose strengthening ratio is still reasonable. To do this, we want to decrease the number of vertices that are not strongly reachable. Usually, we place some pebbles at non-reachable vertices. We know that if a vertex v is not reachable under D and we make it 4-reachable, then all vertices of its closed neighborhood that were not strongly reachable become strongly reachable.

We usually consider a connected component S of the graph induced by $\mathcal{U}(D)$. There are several reasons why we do this. First of all, a chosen S is a small connected part of G where none of the vertices are reachable, hence it is much simpler to work with S instead of the whole graph. A vertex from S has the property that none of its neighbors are strongly reachable. Thus, if we make a vertex from S 4-reachable, then its whole closed neighborhood becomes strongly reachable. Another reason for considering such an S is that, if we add some additional pebbles to S and make sure that all of its vertices become reachable, then these vertices become strongly reachable, too. If we make vertices u and v both 4-reachable with at most 7 pebbles and their closed neighborhoods are disjoint then this is good for us. The disjointness of the neighborhoods happens when $d(v, u) \geq 3$.

While considering such S has its benefits, it can cause challenges when we consider distances. Let u and v be vertices of S. Their distance can be different in G and S. For example if G is the wheel graph on n vertices and we place just one pebble at the center vertex, then S is the outer circle and the distance between two vertices of S can be $\lfloor \frac{n-1}{2} \rfloor$, while their distance in G is not larger than 2.

This difference is important because this shows that we cannot decide the disjointness of closed neighborhoods by distance induced by S. The first idea to handle this is to consider the original distance given by G, but then we have to consider the whole graph, which we would like to avoid. To overcome this problem we make the following compromise: count distances in the graph $N[S]$. Clearly, this distance also can be smaller than the corresponding distance in G, but this happens only for values higher than 3. Hence this $N[S]$ distance determines disjointness of the neighborhoods, which will be enough for our investigation.

The following lemmas will be used in the proof of Theorem 3.1.

Fact 3.7. *Let S and B be induced subgraphs of G such that $V(B) = N_G[V(S)]$. Suppose that $\max_{u,v \in V(S)} d_B(u, v) = 3$ and let a, b, c, d be an induced ad-path with $a, d \in V(S)$. Then either $b, c \in V(S)$ or at least one of b and c is in $V(B) \setminus V(S)$.*

Lemma 3.8. *Let δ be the minimum degree of G. Let S and B be connected, induced subgraphs of G, with $V(B) = N_G[V(S)]$. Suppose that $\max_{u,v \in V(S)} d_B(u, v) = 3$ and let $a, d \in V(S)$ such that $d_B(a, d) = d_S(a, d) = 3$. Then there is an edge (u, v) in S whose closed neighborhood has size at least $\frac{4}{3}(\delta + 1)$.*

Proof. Let a, b, c, d be the vertices of a shortest ad-path that lies in S. If the statement holds for edge (a, b) or (c, d), then we have found the edge that we are looking for. Thus assume the contrary. The Inclusion–Exclusion principle gives us the following result for the vertex pair a, b:

$$|N[a] \cap N[b]| = \underbrace{|N[a]|}_{\geq \delta+1} + \underbrace{|N[b]|}_{\geq \delta+1} - \underbrace{|N[a] \cup N[b]|}_{< \frac{4}{3}\delta + \frac{4}{3}} > \frac{2}{3}\delta + \frac{2}{3}.$$

The analogous result is true for the pair c, d.

The distance between a and d implies that $N[a] \cap N[d] = \emptyset$. Thus $(N[a] \cap N[b]) \cap (N[c] \cap N[d]) = \emptyset$, which implies that

$$\begin{aligned} |N[b] \cup N[c]| \geq & |(N[b] \cap N[a]) \cup (N[c] \cap N[d])| = \\ = & |N[b] \cap N[a]| + |N[c] \cap N[d]| - |(N[a] \cap N[b]) \cap (N[c] \cap N[d])| > \\ > & 2\left(\frac{2}{3}\delta + \frac{2}{3}\right) - 0 = \frac{4}{3}\delta + \frac{4}{3}. \end{aligned}$$

Therefore the edge (b, c) has the required property. □

Lemma 3.9. *Let S and B be connected, induced subgraphs of G, with $V(B) = N_G[V(S)]$. Suppose that there are vertices $u, v \in V(S)$ with $d_B(u, v) = 4$. Then at least one of the following conditions holds.*

1. *There exist $a, b \in V(S)$ with $d_S(a, b) = d_B(a, b) = 4$.*
2. *There exist $c, d \in V(S)$ with $d_B(c, d) = 3$ and some shortest cd-path contains a vertex from $V(B) \setminus V(S)$.*

Proof. Consider a pair of vertices $u, v \in V(S)$ with $d_B(u, v) = 4$. It is clear that $d_S(u, v) \geq 4$. Equality means that the first condition is fulfilled, so assume that $d_S(u, v) > 4$. Let P be a shortest uv-path that lies in S. The length of P is at least five. Label the vertices of P as $u = p_0, p_1, p_2, \ldots, p_k = v$. Let i be the smallest value such p_i does not have a neighbor in $N_B[u]$. The minimality of i implies that $d_B(u, p_i) = 3$.

If $i > 3$, then the shortest up_i-path, which has length three, must contain a vertex from $V(B) \setminus V(S)$, which gives us the second condition.

Otherwise $i = 3$. Let j be the smallest value such that p_j does not have a neighbor in $N_B^2[u]$. The case $j = 4$ gives us $d_B(u, p_j) = 4 = d_S(u, p_j)$, which fulfills the first condition. Otherwise $j > 4$, when $d_B(p_0, p_4) = 3$. This happens if and only if the second condition holds. □

Lemma 3.10. *Let δ be the minimum degree of G, S and B be induced subgraphs of G with $V(B) = N[V(S)]$. If $\max_{u,v \in V(S)} d_B(u, v) \geq 4$, and there exist vertices $a, e \in V(S)$ such that $d_B(a, e) = d_S(a, e) = 4$, then one of the following two conditions holds.*

1. *There exist $u, v \in S$ such that $d_B(u, v) = 2$ and $|N_B[u] \cup N_B[v]| \geq \frac{28}{15}(\delta + 1)$.*
2. *$|N[a] \cup N[e] \cup (N[b] \cap N[d])| \geq \frac{32}{15}(\delta + 1)$, where a, b, c, d, e are the vertices of a path lying in S.*

Proof. Assume that condition 1 does not hold. Because a and d do not have a common neighbor, this yields the following estimate on the size of the common neighborhood of b and d:

$$|N_G[b] \cap N_G[d]| = \underbrace{|N_G[b]|}_{\geq \delta+1} + \underbrace{|N_G[d]|}_{\geq \delta+1} - \underbrace{|N_G[b] \cup N_G[d]|}_{< \frac{28}{15}(\delta+1)} > \frac{2}{15}(\delta + 1).$$

The analogous results hold for the pairs $\{b, e\}$, and $\{a, e\}$, which implies that

$$|N[a] \cup N[e] \cup (N[b] \cap N[d])| = |N[a]| + |N[e]| + |N[b] \cap N[d]| \geq \frac{32}{15}(\delta + 1),$$

which is condition 2. □

The next lemma will be useful to give a lower bound on the number of vertices becoming strongly reachable after the addition of some pebbles to S.

Lemma 3.11. *Let D be a pebbling distribution on G. Let S be a connected component of the subgraph induced by $\mathcal{U}(D)$. Consider D' such that $D \leq D'$. If some $s \in V(S)$ is 2-reachable under $\Delta_{D,D'}$, and each vertex of S is reachable under D', then $N[s] \subseteq \mathcal{T}(D')$; moreover, $N[s] \subseteq \Delta\mathcal{T}_{D,D'}$.*

Proof. We show that each neighbor of s is strongly reachable under D'. Let v be a neighbor of s, and u be a neighbor of v. Since s is 2-reachable under D', v is reachable under D'.

If u is reachable under D or it is a vertex of S, then it is reachable under D'. Otherwise u is in $\mathcal{U}(D) \setminus V(S)$. Thus v separates two connected components in the subgraph induced by $\mathcal{U}(D)$, and so v is reachable under D. Since s is 2-reachable under $\Delta_{D,D'}$, v is also 2-reachable under D' and so u is reachable.

Because s is not reachable under D, its neighbors are not strongly reachable under D. Therefore $N[s] \subseteq \Delta\mathcal{T}_{D,D'}$. □

Proof of Theorem 3.1. Indirectly assume that there is a graph G such that $\pi^*(G) > \frac{15n}{4(\delta+1)}$ and $\mathrm{diam}(G) > 2$. Fact 3.3 means that each solvable distribution has strengthening ratio less than $\frac{4(\delta+1)}{15}$.

Let D_0 be a pebbling distribution that satisfies the properties of Lemma 3.6. Let D be an expansion of D_0 such that the strengthening ratio of D is at least $\frac{4(\delta+1)}{15}$ and, subject to this requirement, $|D|$ is maximal. According to our first assumption, D is not solvable. We will show that either $|D|$ is not maximal or D is not an expansion of D_0. The first case is shown if we give a distribution D' such that $D < D'$ and $\mathcal{E}_{D,D'} \geq \frac{4(\delta+1)}{15}$. We will exhibit $\Delta_{D,D'}$ instead of D'. Clearly D, and $\Delta_{D,D'}$ together determine D'.

For each of the following cases we will assume that the conditions of the previous cases do not hold.

Case A: There exist $u, v \in \mathcal{U}(D)$ with $d(u, v) = 3$ and with some $w \in \mathcal{H}(D)$ on a shortest uv-path.

Without loss of generality, assume that w is a neighbor of v. Then let $\Delta_{D,D'}$ be the following distribution.

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = u, \\ 3 & \text{if } x = v, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then w is reachable under D. It also gets a pebble from u under $\delta_{D,D'}$, so w is 2-reachable (without the three pebbles of v). Thus v is 4-reachable under D'. This means that each vertex of the closed neighborhood of u and v is strongly reachable.

Next, $N[u]$ and $N[v]$ are disjoint subsets of $\Delta\mathcal{T}(D, D')$. Hence

$$\mathcal{E}(D, D') \geq \frac{|N[u] \cup N[v]|}{|\Delta_{D,D'}|} \geq \frac{2(\delta+1)}{7} > \frac{4}{15}(\delta+1),$$

which means that $|D|$ is not maximal, a contradiction.

Case B: $\max_{u,v \in V(S)} d_B(u, v) \geq 4$

The conditions of case A are not satisfied; therefore, by Lemma 3.9, there is a path in S whose length is four in both S and B.

Now apply Lemma 3.10. If there are vertices u and v from $V(S)$ with $d_B(u, v) = 2$ and $|N_B[u] \cup N_B[v]| \geq \frac{28}{15}(\delta+1)$ then let w be a common neighbor of u and v and choose $\Delta_{D,D'}$ as follows.

$$\Delta_{D,D'}(x) = \begin{cases} 2 & \text{if } x \in \{u, v\}, \\ 3 & \text{if } x = w, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Each of u, v, w is 4-reachable, and hence

$$\Delta t_{D,D'} \geq |N[u] \cup N[v]| \geq \frac{28}{15}(\delta+1),$$

Also $|\Delta_{D,D'}| = 7$, and thus $\mathcal{E}(D, D') \geq \frac{4}{15}(\delta+1)$.

If there is no such u, v pair then, by Lemma 3.10, there is a path a, b, c, d, e in S such that $d_B(a, e) = d_S(a, e) = 4$, and $|N(b) \cap N(c)| \geq \frac{2}{15}(\delta+1)$. Consider $\Delta_{D,D'}$ defined as follows.

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x \in \{a, e\} \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The vertices of $N[a] \cup N[e] \cup (N[b] \cap N[d])$ are 2-reachable, and thus they are also strongly reachable. Furthermore,

$$\Delta t \geq |N[a] \cup N[e] \cup (N[b] \cap N[d])| = |N[a]| + |N[e]| + |N[b] \cap N[d]| \geq \frac{32}{15}(\delta+1)$$

and

$$\mathcal{E}(D, D') \geq \frac{32(\delta+1)}{8 \cdot 15} = \frac{4(\delta+1)}{15}.$$

Case C: $\max_{u,v \in V(S)} d_B(u, v) = 3$.

If the conditions of Case A do not hold, then we can use Lemma 3.8 because of Fact 3.7. Let (u, v) be the edge whose neighborhood size is at least $|\frac{4}{3}(\delta+1)|$. We will use this property only in the fourth subcase.

Consider the set $\mathcal{K}$, which is a collection of vertex sets. Set $K \in \mathcal{K}$ if and only if $K \subseteq V(S)$ such that $|K| \geq 2$, $d_B(k, j) \geq 3$ for all $k \neq j \in K$, and K is maximal (we cannot add an element to K). Because $\max_{u,v \in V(S)} d_B(u, v) = 3$ in this case we have that $\mathcal{K}$ is not empty.

The objective in this case is to use Lemma 3.11 for the vertices of K, because this means that the vertices of $\cup_{k \in K} N[k]$ are strongly reachable. Furthermore, $N[k_1]$ and $N[k_2]$ are disjoint if $k_1, k_2 \in K$ and $k_1 \neq k_2$. These imply that $\Delta t \geq \cup_{k \in K} |N(k)| \geq |K|(\delta+1)$. To use this lemma we need to give a proper $\Delta_{D,D'}$ distribution and check that each vertex of S is reachable and each vertex of K is 2-reachable under it.

There are four subcases here.

Subcase 1: $d_B(v, s) \le 2$ for all $s \in V(S)$.
Let K be an arbitrary element of $\mathcal{K}$. Note that $v \notin K$. Define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = v, \\ 1 & \text{if } x \in K, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Each vertex of S is reachable with the pebbles placed at v and the vertices of K are 2-reachable. Also

$$\mathcal{E}(D, D') \ge \frac{|K|(\delta + 1)}{4 + |K|} \ge \frac{1}{3}(\delta + 1).$$

Subcase 2: $\min(d_B(u, s), d_B(v, s)) \le 2$ for all $s \in V(S)$, but $d_B(v, w) = 3$ for some $w \in V(S)$.
Choose K such that $v \in K$. Such a K exists. Define

$$\Delta_{D,D'}(x) = \begin{cases} 3 & \text{if } x \in \{u, v\}, \\ 1 & \text{if } x \in K \setminus \{v\}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then u and v are 4-reachable, and hence all vertices of S are reachable. Furthermore, each vertex of K is 2-reachable, and

$$\mathcal{E}(D, D') \ge \frac{|K|(\delta + 1)}{6 + |K| - 1} \ge \frac{2}{7}(\delta + 1).$$

Subcase 3: $d_B(s, u) = d_B(s, v) = 3$ for some $s \in V(S)$, and $\{s, v\} \notin \mathcal{K}$.
Now $\{s, v\}$ is a subset of some $K \in \mathcal{K}$ with $|K| \ge 3$. Define

$$\Delta_{D,D'}(x) = \begin{cases} 8 & \text{if } x = v, \\ 1 & \text{if } x \in K \setminus \{v\}, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

Each vertex of S is reachable with the pebbles placed at v and the vertices of K are 2-reachable. Also

$$\mathcal{E}(D, D') \ge \frac{|K|(\delta + 1)}{8 + |K| - 1} \ge \frac{3}{10}(\delta + 1).$$

Subcase 4: $d_B(s, u) = d_B(s, v) = 3$ for some $s \in V(S)$, and $\{s, v\} \in \mathcal{K}$.
Set $K = \{s, v\}$ and define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x \in K \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then $K = \{s, v\}$ means that each vertex of S is in $N^2[s] \cup N^2[v]$, and thus each vertex of S is reachable. Also $N[s] \cap (N[u] \cup N[v]) = \emptyset$, and hence

$$\Delta t_{D,D'} \ge |N[s] \cup N[v] \cup N[u]| = |N[s]| + |N[v] \cup N[u]| \ge \frac{7}{3}(\delta + 1)$$

and

$$\mathcal{E}(D, D') \ge \frac{7}{24}(\delta + 1).$$

Case D: $\max_{u,v \in V(S)} d_B(u, v) \le 2$.
In this case, if we put 4 pebbles to an arbitrary vertex s of S, then all vertices of S and $N[s]$ become strongly reachable.

Subcase 1: $|V(S)| \ge \frac{16}{15}(\delta + 1)$.
Let $v \in S$ and define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = v \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{E}(D, D') \ge \frac{16(\delta + 1)}{15 \cdot 4} = \frac{4}{15}(\delta + 1).$$

Subcase 2a: $|N[u] \cup N[v]| \ge \frac{16}{15}(\delta + 1)$ for some $u, v \in V(S)$ with u adjacent to v.

Define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = v \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Each vertex of S is reachable and u and v are 2-reachable under $\Delta_{D,D'}$. Using Lemma 3.11, we get that the neighborhoods of u and v are both strongly reachable. Also

$$\mathcal{E}(D, D') \geq \frac{|N[u] \cup N[v]|}{4} \geq \frac{16(\delta+1)}{15 \cdot 4} = \frac{4}{15}(\delta+1).$$

Subcase 2b: $|N[u] \cup N[v]| \geq \frac{16}{15}(\delta+1)$ for some $u, v \in V(S)$ having a common neighbor $w \in V(S)$.

Define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = w \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The same arguments from the previous case hold here.

Subcase 3a: $|V(S)| \leq \frac{14}{15}(\delta+1)$, and for all $u, v \in V(S)$ there is an $h \in \mathcal{H}(D) \cap N(u) \cap N(v)$.

Choose $v \in S$ and define

$$\Delta_{D,D'}(x) = \begin{cases} 2 & \text{if } x = v \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If s is a vertex of S other than v then choose $h \in \mathcal{H}(D) \cap N(u) \cap N(v)$. Now h is reachable under D and we have two additional pebbles from D', so we can move a pebble to s from v through h. Thus each vertex of S is reachable under D', and so we can apply Lemma 3.11 for vertex v to obtain

$$\mathcal{E}(D, D') \geq \frac{|N[v]|}{2} \geq \frac{\delta+1}{2}.$$

Subcase 3b: $|V(S)| \leq \frac{14}{15}(\delta+1)$, and for some $u, v \in V(S)$ there is no $h \in \mathcal{H}(D) \cap N(u) \cap N(v)$.

The diameter of S (with respect to the distance defined in B) guarantees that either u and v are neighbors or they share a common neighbor $w \in V(B)$. Furthermore, in this subcase we have $w \in V(S)$. Now u has at least $\delta - (\frac{14}{15}(\delta+1) - 1) = \frac{1}{15}(\delta+1)$ neighbors in $\mathcal{H}(D)$, but none of them is a neighbor of v. Hence $|N[u] \cup N[v]| \geq \frac{16}{15}(\delta+1)$. This is subcase 2a or 2b.

Subcase 4: Some $v \in V(S)$ has $d_B(v, s) = 1$ for every $s \in V(S)$.

Define

$$\Delta_{D,D'}(x) = \begin{cases} 2 & \text{if } x = v \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Each vertex of S is reachable under D', and so we apply Lemma 3.11 again to obtain $\mathcal{E}(D, D') \geq \frac{\delta+1}{2}$.

We have handled all cases for which $|V(S)| \leq \frac{14}{15}(\delta+1)$ or $|V(S)| \geq \frac{16}{15}(\delta+1)$, so in the remaining cases we assume that $\frac{14}{15}(\delta+1) < |V(S)| < \frac{16}{15}(\delta+1)$. Before continuing, we introduce one more definition. Let $\mathcal{S}$ be the set of connected components of the graph induced by $\mathcal{U}(D)$. We say that $S \in \mathcal{S}$ is *isolated* in $\mathcal{S}$ if $d_G(S, S') \geq 3$ for every other $S' \in \mathcal{S}$.

Subcase 5: Some $S \in \mathcal{S}$ is not isolated.

Choose $S' \in \mathcal{S}$, $u \in S$, and $v \in S'$ with $d(u, v) = 2$, and define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = u, \\ 3 & \text{if } x = v, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Now u and v are both 4-reachable, and hence all vertices of S and S' are reachable; furthermore, they are strongly reachable. Therefore

$$\mathcal{E}(D, D') \geq \frac{|V(S) \cup V(S')|}{7} \geq \frac{2 \cdot \frac{14}{15}(\delta+1)}{7} = \frac{4(\delta+1)}{15}.$$

Subcase 6: S is isolated in $\mathcal{S}$, and $|N[S]| \geq \frac{16}{15}(\delta+1)$.

Let $s \in S$ and define

$$\Delta_{D,D'}(x) = \begin{cases} 4 & \text{if } x = s \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Each vertex of S becomes strongly reachable. We show that the same is true for any vertex of $N(S)$.

Consider $h \in N(S)$. Then h is not strongly reachable under D but, because S is isolated, all of its non-reachable neighbors under D are contained in S. Thus, under D', h is strongly reachable, which implies that

$$\mathcal{E}(D, D') \geq \frac{|N[V(S)]|}{4} \geq \frac{\frac{16}{15}(\delta+1)}{4} = \frac{4(\delta+1)}{15}.$$

Subcase 7: S is isolated in $\mathcal{S}$, and some $h \in \mathcal{H}(D)$ has $d_G(h, s) \leq 2$ for every $s \in V(S)$.

Define

$$\Delta_{D,D'}(x) = \begin{cases} 3 & \text{if } x = h \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Each vertex of S becomes strongly reachable, and thus

$$\mathcal{E}(D, D') \geq \frac{|V(S)|}{3} > \frac{\frac{14}{15}(\delta+1)}{3} \geq \frac{4(\delta+1)}{15}.$$

Subcase 8: None of the previous cases hold.

In this case we will get a contradiction with $D_0 \leq D$. We summarize what we know about D. The following properties hold for every $S \in \mathcal{S}$:

(a) $\max_{u,v \in V(S)} d_B(u, v) = 2$;
(b) $\frac{14}{15}(\delta+1) < |V(S)| < \frac{16}{15}(\delta+1)$;
(c) S is isolated in $\mathcal{S}$;
(d) $|N[S]| < \frac{16}{15}(\delta+1)$; and
(e) there is no $h \in \mathcal{H}(D)$ such that $d_G(h, s) \leq 2$ for every $s \in S$.

Because the diameter of G is at least 3, some pebbles have been placed, and so $\mathcal{H}(D) \neq \emptyset$. Fix a component $S \in \mathcal{S}$. Because G is connected, $\mathcal{H}(D) \cap N(S)$ is also nonempty. Consider some $h \in \mathcal{H}(D) \cap N(S)$. Property (e) guarantees that there is a vertex $v \in V(S)$ such that $d(v, h) = 3$. Choose some $u \in N(h) \cap V(S)$. Then property (a) and $d(v, h) = 3$ together imply that $u \in N^2(v) \cap S$.

Now $h \in N^3(v)$, and hence $N[h] \cap N[v] = \emptyset$. Also $N[v] \subseteq N[S]$, and so

$$|N[h] \cap N[S]| \leq |N[S] \setminus N[v]| = |N[S]| - |N[v]| < \frac{16}{15}(\delta+1) - (\delta+1) = \frac{1}{15}(\delta+1).$$

Thus we have $|N[h] \setminus N[S]| \geq \frac{14}{15}(\delta+1)$. Moreover, $u \in S$, and so $N(u) \subseteq N[S]$. Therefore

$$|N[u, h]| \geq |N[u]| + |N[h] \setminus N[S]| \geq \frac{29}{15}(\delta+1).$$

Since $u \in S$, it is not reachable under D. But D is an expansion of D_0, and u has to be reachable because $|N[u, h]| \geq \frac{29}{15}(\delta+1)$ and $(u, h) \in E(G)$. This is a contradiction.

We have seen that in each case we have a contradiction, and so our assumption was false. Hence the theorem is true. □

Using this theorem we can prove that the upper bound of Bunde et al. [1] cannot hold with equality.

Claim 3.12. *There is no connected graph G such that $\pi^*(G) = \frac{4n}{\delta+1}$.*

Proof. Theorem 3.1 shows that the optimal pebbling number of graphs whose diameter is at least three is smaller. So we have only to check diameter two graphs and complete graphs, each of whose optimal pebbling number is at most 4. But $\frac{4n}{\delta+1} \geq 4$, with equality only for the complete graph. However $\pi^*(K_n) = 2$. □

Corollary 3.13. *For any connected graph G we have $\pi^*(G) < \frac{4n}{\delta+1}$, and this bound is sharp.*

Muntz et al. [7] characterize diameter three graph graphs whose optimal pebbling number is eight. Their characterization can be reformulated in the following statement.

Claim 3.14. *Let G be a diameter 3 graph. Then $\pi^*(G) = 8$ if and only if there are no vertices x, u, v and w such that $N^2[x] \cup N[u] \cup N[v] \cup N[w] = V(G)$.*

Theorem 3.1 can be used to establish a connection between this unusual domination property and the minimum degree of the graph. Note that this is just a minor improvement of the trivial $\frac{1}{2}n - 1$ upper bound.

Corollary 3.15. *Let G be a diameter 3 graph on n vertices. If there are no vertices x, u, v and w such that $N^2[x] \cup N[u] \cup N[v] \cup N[w] = V(G)$ then the minimum degree of G is at most $\frac{15}{32}n - 1$.*

4. Graphs with high girth and low optimal pebbling number

The authors of [1] proved, for $k \geq 6$, that the family of connected graphs having n vertices, minimum degree k, and girth at least $2t+1$ has $\pi^*(G)/n \to 0$ as $t \to \infty$. They ask (Question 6.3) whether the same is true for $k \in \{3, 4, 5\}$. We answer this affirmatively for $k \in \{4, 5\}$. The case $k = 3$ remains open.

Theorem 4.1. *Suppose G is a connected graph of order n with $\delta(G) \geq k$ and $\mathsf{girth}(G) \geq 2t+1$. If $k \geq 4$ then $\lim_{t\to\infty} \pi^*(G)/n = 0$.*

Proof. For $k, t \in Z^+$ with $k \geq 3$, let $L = 1 + k\left(\frac{(k-1)^t-1}{k-2}\right)$. Suppose that G is a connected graph of order n with $\delta(G) \geq k$ and $\mathsf{girth}(G) \geq 2t+1$, and consider the following experiment, consisting of two steps. In the first step, place 2^t pebbles on a vertex v with probability p, independently for each v. In the second step, place one pebble on every vertex not reachable by pebbles placed in the first step. Clearly the pebbling distribution is solvable.

Let X be the expected number of pebbles used in the experiment. The probability that v is not reachable by the pebbles places in the first step is at most $(1-p)^L$ because, if at least one vertex in the closed ball of radius t around v has 2^t pebbles on it, we can solve v. Thus

$$E(X) \leq 2^t pn + (1-p)^L n,$$

and so $\pi^*(G) \leq (2^t p + (1-p)^L)n$. In particular, for $p = \ln(L/2^t)/L$, using the bound $1 + x \leq e^x$, we get

$$\pi^*(G) \leq (\ln(L/2^t) + 1)\frac{2^t n}{L}.$$

We have $(k-1)^t < L < 3(k-1)^t$, and so $\pi^*(G) < (2 + t\ln(k-1))(\frac{2}{k-1})^t \cdot n$. $\square$

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