The Optimal Pebbling Number of the Caterpillar

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Abstract

Let G be a simple graph. If we place p pebbles on the vertices of G, then a pebbling move is taking two pebbles off one vertex and then placing one on an adjacent vertex. The optimal pebbling number of G, $f'(G)$, is the least positive integer p such that p pebbles are placed suitably on vertices of G and for any target vertex v of G, we can move one pebble to v by a sequence of pebbling moves. In this paper, we find the optimal pebbling number of the caterpillar.

Key word. Optimal pebbling, Caterpillar.

AMS(MOS) subject classification. 05C05

1 Introduction

Throughout this paper, a *configuration* of a graph G means a mapping from $V(G)$ into the set of non-negative integers $N \bigcup \{0\}$. Suppose p pebbles are distributed onto the vertices of G; then we have a so-called distributing configuration(d. c.) δ where we let $\delta(v)$ be the number of pebbles distributed to $v \in V(G)$ and $\delta(H)$ equals $\sum_{v \in V(H)} \delta(v)$ for each induced subgraph H of G. Note that now $\delta(G) = p$.

A pebbling move consists of moving two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a d. c. δ lets us move at least one pebble to each vertex v by applying pebbling moves repeatedly(if necessary), then δ is called a *pebbling* of G. For convenience, let δ be a d. c. of G, we use $\delta_H(v)$ to denote the maximum number of pebbles which can be moved to v by applying pebbling moves on H for each induced subgraph H of G. Therefore, for each $v \in V(G)$ $\delta_G(v) > 0$ if δ is a pebbling of G. The *optimal pebbling number* of G, $f'(G)$, is $\min\{\delta(G) | \delta \text{ is a pebbling of G}\}$, and a d. c. δ is an *optimal pebbling* of G if δ is a pebbling of G such that $\delta(G) = f'(G)$.

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In order to find the optimal pebbling number of the caterpillar, we introduce the notion of *α-pebbling*. Let α be a mapping from $V(G)$ into the set of positive integers. Then a pebbling δ of G is called an α-pebbling if δ lets us move at least $\alpha(v)$ pebbles to the vertex v by applying pebbling moves repeatedly. In what follows, we call α a pebbling type of G and the optimal α-pebbling number of G, $f'_\alpha(G)$, is $\min\{\delta_G|\delta$ is an α-pebbling of $G\}$. Clearly, if $\alpha(v) = 1$ for each $v \in V(G)$, then $f'_\alpha(G) = f'(G)$.

Note here that the *pebbling number* $f(G)$ of a graph G is defined as the minimum number of pebbles p such that any distributing configuration with p pebbles is a pebbling of G. The problem of pebbling graph was first proposed by M. Saks and J. Lagarias[1] as a tool for solving a number theoretic problem by Lemke and Kleitman[6], and some excellent results have been obtained, see [1, 2, 5, 7]. But, the notion of optimal pebbling was introduced later by L. Pachter et. al.[9] and they proved the following result on paths.

Theorem 1.1. [9] *Let P be a path of order 3t+r, i.e., $|V(P)| = 3t + r$. Then $f'(P) = 2t + r$.*

Note that the above theorem is not as easy as it looks. Since then, several results have been obtained.

Theorem 1.2. [10] $f'(C_n) = f'(P_n)$.

Theorem 1.3. [10] *For any graphs G and H, $f'(G \times H) \leq f'(G)f'(H)$.*

Theorem 1.4. [8] $f'(Q_n) = (\frac{4}{3})^{n+O(\log n)}$.

Besides, the optimal pebbling number of the complete m-ary tree is also determined by Fu and Shiue [3].

In this paper, we shall determine the optimal pebbling number of the caterpillar via a special α-pebbling of a path.

2 Main result

A tree T is called a caterpillar if the deletion of all pendent vertices of the tree results in a path P'. For convenience, we shall call a path P with maximum length which contains P' a body of the caterpillar, and all the edges which are incident to pendent vertices are the *legs* of the caterpillar T. Furthermore, the vertex $v \in V(P)$ is a *joint* of T provided that $\deg_T(v) \geq 3$ or v is adjacent to the end vertices, see

Figure 1 for an example.

Now we are ready to prove the first lemma. Mainly, we shall prove that the problem of finding the optimal pebbling number of a caterpillar T is equivalent to the problem of finding the optimal α-pebbling number of a body of T. The following result is of more general form.

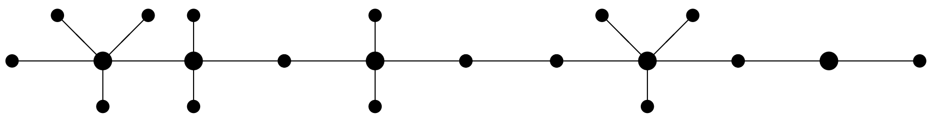


Figure 1. A caterpillar with 5 joints

Lemma 2.1. *Let T be a tree, v be a pendent vertex of T which is adjacent to u, and α be a pebbling type of T satisfying $\alpha(v) = 1$. Then there exits a pebbling type of $T - v$, α', defined by $\alpha'(u) = \max\{2, \alpha(u)\}$ and $\alpha'(w) = \alpha(w)$ for $w \in V(T)\backslash\{u, v\}$ such that $f'_\alpha(T) = f'_{\alpha'}(T - v)$.*

Proof. Since $\alpha'(u) \geq 2$ and $\alpha(v) = 1$, it follows that $f'_\alpha(T) \leq f'_{\alpha'}(T - v)$. Let δ be an optimal α-pebbling of T. It suffices to prove that $f'_{\alpha'}(T - v) \leq \delta_T$. First, if $\alpha'(u) = \alpha(u)$, then $f'_{\alpha'}(T - v) \leq \delta(T - v) + \lfloor \frac{1}{2}\delta(v)\rfloor \leq \delta(T)$, we are done. Otherwise, let $\alpha'(u) = 2$ and $\alpha(u) = 1$. Now, if $\delta_T(u) = 2$, again $f'_{\alpha'}(T - v) \leq \delta(T - v) + \lfloor \frac{1}{2}\delta(v)\rfloor \leq \delta(T)$. Otherwise, $\delta_T(u) = 1$. This implies that $\delta(v) > 0$. Thus $f'_{\alpha'}(T - v) \leq \delta(T - v) + \lfloor \frac{1}{2}\delta(v)\rfloor + 1 \leq \delta(T - v) + (\delta(v) - 1) + 1 = \delta(T)$. We have the proof. ∎

Corollary 2.2. *Let T be a caterpillar of order $n \geq 3$ and P be a body of T. If α is a pebbling type of P defined by $\alpha(v) = 2$ provided that v is a joint of T and $\alpha(v) = 1$ otherwise. Then $f'(T) = f'_\alpha(P)$.*

Proof. It is a direct result of Lemma 2.1 by adding legs to P recursively. ∎

Without mention otherwise, T is a caterpillar, P is a body of T and α is a pebbling type of P which is defined as Corollary 2.2 throughout of this paper. In order to obtain $f'(T)$, we need a good lower bound for $f'(T)$, i. e. , a good lower bound for $f'_\alpha(P)$.

Lemma 2.3. *If δ is an α-pebbling of P, then $\delta(P) \geq |V(P)| - \lfloor \frac{1}{2}|S_1|\rfloor$, where $S_1 = \{v \in V(P)|\delta(v) = 0$ and $\delta_P(v) = 1\}$.*

Proof. Let δ be an α-pebbling of P, and S_0 be the set of vertices v in $V(P)$ such that $\delta(v) = 0$. Then

$$\sum_{v \in S_0} \delta_P(v) \geq |S_1| + 2|S_0 \backslash S_1|. \quad \cdots\cdots \quad (1)$$

Let $P = v_1 v_2 \cdots v_n$ be a path of order $n > 3$. For $i = 1, 2, \cdots n$, we define $L_i = v_1 v_2 \cdots v_i$ and $R_i = v_i v_{i+1} \cdots v_n$. For convenience, we denote $\delta_{L_i}(v_i)$ by $l(v_i)$ and $\delta_{R_i}(v_i)$ by $r(v_i)$ for $1 \leq i \leq n$. It is easy to see that $l(v_i) = \delta(v_i) + \lfloor \frac{l(v_{i-1})}{2} \rfloor$, $r(v_i) = \delta(v_i) + \lfloor \frac{r(v_{i+1})}{2} \rfloor$ for $1 \leq i \leq n$ and $\delta_P(v) = l(v) + r(v)$ for each $v \in S_0$. So we have

$$\sum_{v \in S_0} \delta_P(v) = \sum_{v \in S_0} l(v) + \sum_{v \in S_0} r(v). \quad \cdots\cdots \quad (2)$$

Let s be a positive number. Then we define $\phi_0(s) = s$ and $\phi_i(s) = \lfloor \frac{\phi_{i-1}(s)}{2} \rfloor$ for each positive integer i. Now, consider a subpath of P, $P'' = v_{k+1} v_{k+2} \cdots v_{k+l} v_{k+l+1} \cdots v_{k+l+m}$, which satifies that $\delta(v_{k+i}) > 0$ for $1 \leq i \leq l$ and $\delta(v_{k+l+j}) = 0$ for $1 \leq j \leq m$. Then we have $l(v_{k+l+j}) = \phi_j(l(v_{k+l}))$ for $1 \leq j \leq m$ and $\phi_{m+1}(l(v_{k+l})) = \lfloor \frac{1}{2} l(v_{k+l+m}) \rfloor$. Here, we let $l(v_k) = 0$ if $k = 0$. Then by the fact that for each positive integer s, $\sum_{i=1}^{t} \phi_i(s) \leq s - 1$ for any positive integer t, we have

$$\begin{aligned}
\sum_{j=1}^{m+1} \phi_j(l(v_{k+l})) &\leq l(v_{k+l}) - 1 \\
&= \lfloor \frac{1}{2} l(v_{k+l-1}) \rfloor + \delta(v_{k+l}) - 1 \\
&\leq l(v_{k+l-1}) - 1 + \delta(v_{k+l}) - 1 \\
&= \lfloor \frac{1}{2} l(v_{k+l-2}) \rfloor + \delta(v_{k+l-1}) - 1 + \delta(v_{k+l}) - 1 \\
&\;\;\vdots \\
&\leq \lfloor \frac{1}{2} l(v_k) \rfloor + \sum_{j=1}^{l} (\delta(v_{k+j}) - 1).
\end{aligned}$$

This implies

$$\sum_{v \in V(P'') \bigcap S_0} l(v) = \sum_{j=1}^{m} l(v_{k+l+j}) = \sum_{j=1}^{m} \phi_j(l(v_{k+l})) = \sum_{i=1}^{m+1} \phi_j(l(v_{k+l})) - \left\lfloor \frac{1}{2} l(v_{k+l+m}) \right\rfloor$$

$$\leq \left\lfloor \frac{1}{2} l(v_k) \right\rfloor - \left\lfloor \frac{1}{2} l(v_{k+l+m}) \right\rfloor + \sum_{v \in V(P'') \backslash S_0} (\delta(v) - 1). \quad \cdots\cdots \quad (3)$$

Now, we let $P = P_0 \sim P'_1 \sim P_1 \sim P'_2 \sim \cdots \sim P_{m-1} \sim P'_m \sim P_m$ where $P'_1, P'_2, \cdots, P'_m$ are the maximal subpaths such that for each vertex $v \in V(P'_i)$, $\delta(v) \geq 1$, $1 \leq i \leq m$. Note that $P_0 = \emptyset$ if $v_1 \in V(P'_1)$ and $P_m = \emptyset$ if $v_n \in V(P'_m)$. We also let u_i be the rightmost vertex of P_i for $1 \leq i \leq m$. Obviously, we have $l(u_0) = 0$ and

(i) $\displaystyle\sum_{v \in V(P_0)} l(v) = 0;$

and then by (3), we also have

(ii) for $1 \leq i \leq m-1$,

$$\sum_{v \in V(P_i)} l(v) \leq \left\lfloor \frac{1}{2} l(u_{i-1}) \right\rfloor - \left\lfloor \frac{1}{2} l(u_i) \right\rfloor + \sum_{v \in V(P'_i)} (\delta(v) - 1),$$

and

(iii) if $P_m = \emptyset$,

$$\sum_{v \in V(P_m)} l(v) = \ 0 \ \leq \ \left\lfloor \frac{1}{2} l(u_{m-1}) \right\rfloor + \sum_{v \in V(P'_m)} (\delta(v) - 1),$$

and if $P_m \neq \emptyset$,

$$\sum_{v \in V(P_m)} l(v) \leq \left\lfloor \frac{1}{2} l(u_{m-1}) \right\rfloor - \left\lfloor \frac{1}{2} l(u_m) \right\rfloor + \sum_{v \in V(P'_m)} (\delta(v) - 1)$$
$$\leq \lfloor \frac{1}{2} l(u_{m-1}) \rfloor + \sum_{v \in V(P'_m)} (\delta(v) - 1).$$

Combining (i), (ii) and (iii) and since $S_0 = \bigcup_{i=1}^{m} V(P_i)$, we have

$$\sum_{v \in S_0} l(v) \leq \sum_{i=1}^{m} \sum_{v \in V(P'_i)} (\delta(v) - 1)$$
$$= \sum_{v \in V(P) \setminus S_0} (\delta(v) - 1) = \delta(P) - |V(P)| + |S_0|. \quad \cdots\cdots \quad (4)$$

By the same argument,

$$\sum_{v \in S_0} r(v) \leq \delta(P) - |V(P)| + |S_0|. \quad \cdots\cdots \quad (5)$$

Hence, we have

$$\sum_{v \in S_0} \delta_P(v) \leq 2(\delta(P) - |V(P)| + |S_0|). \quad \cdots\cdots \quad (6)$$

This gives

$$|S_1| + 2|S_0 \backslash S_1| \leq \sum_{v \in S_0} \delta_P(v) \leq 2(\delta(P) - |V(P)| + |S_0|),$$

and the proof follows. ∎

For convenience, we let $S_0 = \{v \in V(P) | \delta(v) = 0\}$ and let $S_1 = \{v \in V(P) | \delta(v) = 0 \text{ and } \delta_P(v) = 1\}$ where δ is a d. c. of P throughout of this paper. Now, the following fact is obvious.

Fact 1. Let δ be a d. c. of P. If $v \in S_1$, then there exists exactly one adjacent vertex u of v such that $2 \leq \delta_P(u) \leq 3$.

The following result is the same as Theorem 1.1. Here we give an independent proof by using our techniques developed in this paper.

Lemma 2.4. *Let $\alpha(v) = 1$ for each $v \in V(P)$. Then $f'_\alpha(P) = |V(P)| - \left\lfloor \frac{|V(P)|}{3} \right\rfloor$.*

Proof: Let $P = v_1 v_2 \cdots v_{3t+r}$. Suppose that δ be an optimal α-pebbling of P. We let $S' = \{v | \delta_P(v) \geq 2\}$ and let $|S'| = x$. Then by Fact 1, there are at most two vertices of S_1 which occur in between two vertices of S'. Hence $|S_1| \leq 2(x-1) + 2 = 2x$. Since $S_1 \subseteq V(P) \backslash S'$, we also have $|S_1| \leq 3t + r - x$. Therefore, $|S_1| \leq 2t + \lfloor \frac{2}{3} r \rfloor (*)$. By Lemma 2.3, $\delta(P) \geq (3t + r) - t - \lfloor \frac{1}{2} \lfloor \frac{2}{3} r \rfloor \rfloor = 2t + r = |V(P)| - \left\lfloor \frac{|V(P)|}{3} \right\rfloor$. Now the proof follows by setting δ be the d. c. of P such that $\delta(v_{3i+1}) = \delta(v_{3i+3}) = 0$, $\delta(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$, $\delta(v_{3t+r}) = r$ for $r > 0$ and $\delta(v_{3t+1}) = 0$ for $r = 2$ (see Figure 2). ∎

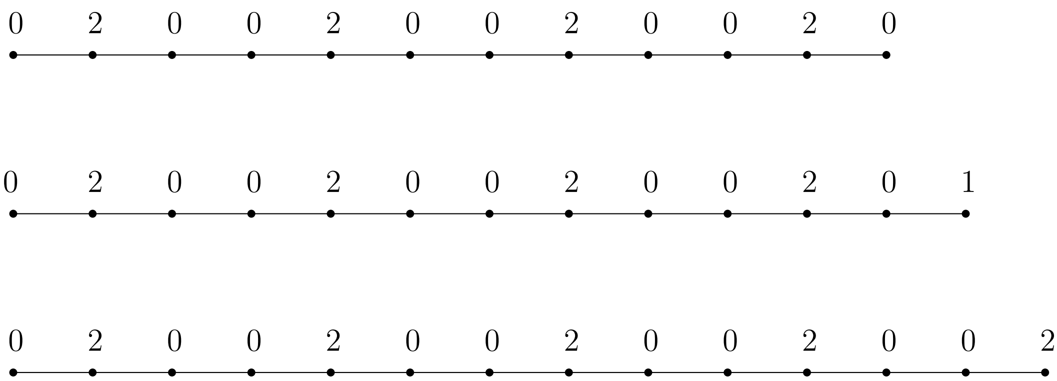


Figure 2.

Lemma 2.5. *Let $P = v_1 v_2 \cdots v_k$, $k \geq 3$ and α be defined as follows: $\alpha(v_1) = \alpha(v_k) = 1$, $\alpha(v_2) = \alpha(v_{k-1}) = 2$ and for $3 \leq i \leq k-2$, $\alpha(v_i) \in \{1, 2\}$ and $\alpha(v_i) \neq \alpha(v_{i+1})$ provided that $\alpha(v_i) = 1$. Then $f'_\alpha(P) = k - 1$.*

Proof. Let δ be an optimal α-pebbling of P. Then by Fact 1 and the definition of α, $v_i \notin S_1$ for $2 \leq i \leq k-1$. This implies that $|S_1| \leq 2$. By Lemma 2.3, we have $f'_\alpha(P) \geq k-1$. Now, by letting δ be the configuration satisfying $\delta(v_1) = \delta(v_k) = 0$, $\delta(v_2) = 2$ and $\delta(v_i) = 1$ for $3 \leq i \leq k-1$, we have $f'_\alpha(P) \leq k-1$. This concludes the proof. ∎

In order to determine $f'_\alpha(P)$, we also need the following notions. A subpath Q of P is said to be 1-*maximal* with respect to α, if Q is a maximal connected subgraph of P such that for each $v \in V(Q)$, $\alpha(v) = 1$ and for each vertex u which is adjacent to v, $\alpha(u) = 1$; and Q is 2-*maximal* with respect to α, if Q is a maximal connected subgraph of P such that for each adjacent pair u and w in $V(Q)$, $\alpha(u) = 1$ implies $\alpha(w) = 2$ or $\alpha(u) = \alpha(w) = 2$. For clearness, we give an example in Figure 2.

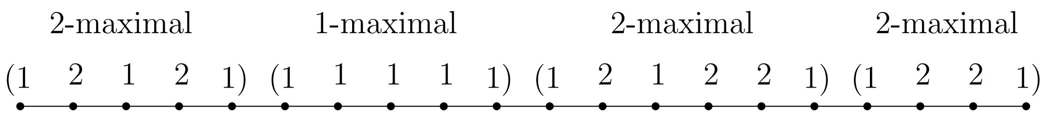


Figure 3. $\alpha(P)$

Now, we are ready for the main theorem.

Theorem 2.6. *Let T be a caterpillar with P a body of T and $|V(P)| = n$. Let $\alpha(v) = 2$ if v is a joint of T and $\alpha(v) = 1$ otherwise. Let $P'_1, P'_2, \cdots, P'_m$ be 2-maximal subpaths of P with respect to α and P_i be a subpath between P'_i and P'_{i+1} for $i = 1, 2, \cdots, m-1$. Then $f'(T) = n - m - \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor$.*

Proof. By corollary 2.2, it suffices to prove $f'_\alpha(P) = n - m - \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor$. First, if $m = 1$, the proof follows by Lemma 2.5. Now, we let $m > 1$. Clearly, for each $i = 1, 2, \ldots, m-1$, P_i is 1-maximal with respect to α. Combining Theorem 2.4 and Lemma 2.5, we have

$$f'_\alpha(P) \leq \sum_{i=1}^{m} (|V(P'_i)| - 1) + \sum_{i=1}^{m-1} \left(|V(P_i)| - \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor \right) = n - m - \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.$$

Now, we will show that the above upper bound is also a lower bound.

For $1 \le i \le m$, let u_i and w_i be the leftmost vertex and the rightmost vertex of P_i' respectively. Note that if $v \in V(P_i')$ is adjacent to u_i or w_i then $\alpha(v) = 2$. If δ is an α-pebbling of P, then by (*) in the proof of Lemma 2.4 and Fact 1,

$$|V(w_i \sim P_i \sim u_{i+1}) \bigcap S_1| \le 2 \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor + 2.$$

Note that it is also true for the cases $u_1 \in S_1$ or $w_m \in S_1$. This implies that

$$|V(P) \bigcap S_1| \le 2 + \sum_{i=1}^{m-1} (2 \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor + 2) = 2m + \sum_{i=1}^{m-1} 2 \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.$$

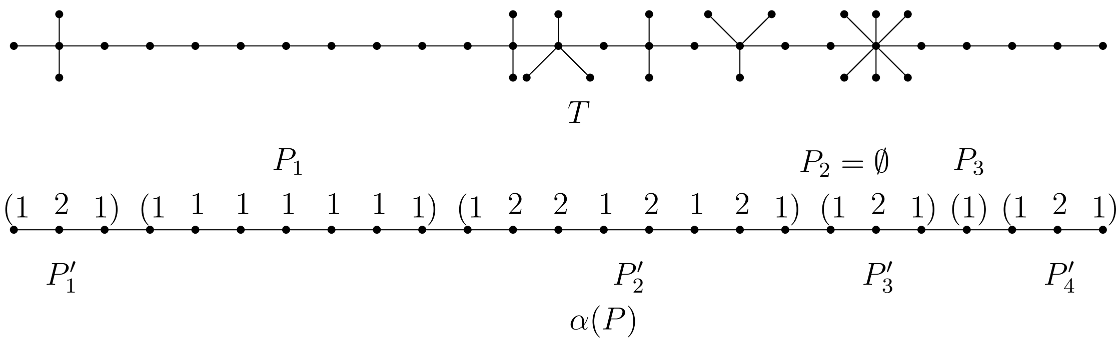
By Lemma 2.3,

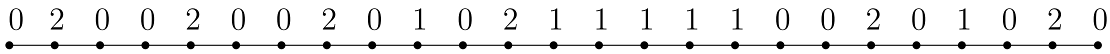
$$\delta(P) \ge |V(P)| - \left\lfloor \frac{1}{2}|S_1| \right\rfloor \ge |V(P)| - \left\lfloor \frac{1}{2}\left|V(P) \bigcap S_1\right| \right\rfloor \ge n - m - \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.$$

This concludes the proof. ∎

Before we finish this paper, we give an example to clarify the idea used in this paper.

Example. Let T be a caterpillar in Figure 4. Here, $n = 25$, $m = 4$, $\sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor =$ 2 and $f'(T) = 25 - 4 - 2 = 19$.





An optimal α-pebbling of P

Figure 4.

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