NEW RESULTS ON DIMENSION IN THE CUBE

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1. Introduction

Given integers $0 \leq k < r \leq n$, let $[n] = \{1, 2, \ldots, n\}$ and define the partially ordered set $P_n(k, r)$ as follows. Its elements are all subsets of $[n]$ of size k or r, and we set $A < B$ whenever $A \subset B$. Our purpose here is to continue the investigation begun in [H] of the dimension $d_n(k, r)$ of $P_n(k, r)$ ($d_n(1, r)$ was initially investigated in [DM], [O], and [S]).

For those unfamiliar with the terminology of dimension theory, we define a *linear extension* L of a poset P to be a linear order on the same set of elements so that $x < y$ in P implies $x < y$ in L. The intersection $L_1 \cap L_2 \cap \cdots \cap L_t$ of a set $R = \{L_1, L_2, \ldots, L_t\}$ of linear orders is the poset $P(R)$ with $x < y$ in $P(R)$ if and only if $x < y$ in each L_i. If $P(R) = P$ then we say that R *realizes* P, or R is a *realizer* for P. The *dimension* $\dim(P)$ of P is the minimum size of a realizer; i.e., the least t such that there are linear orders $L_1, \ldots, L_t$ with $P = \cap_{i=1}^{t} L_i$.

As for simple observations, the reader might notice that $d_n(0, n) = 1$, $d_n(0, r) =$

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$d_n(r,n) = 2$ for $0 < r < n$, and $d_n(k,r) = d_n(n-r, n-k)$ for $0 \leq k < r \leq n$. It is also easy to show (see [H]) that if $0 < k_1 \leq k_2 < r_2 \leq r_1 < n$ then $d_n(k_1,r_1) \geq d_n(k_2,r_2)$. Dushnik and Miller found in [DM] that $d_n(1,n-1) = n$, which essentially states that the Boolean lattice on n elements has dimension n since it has the same set of critical pairs as $P_n(1,n-1)$.

An ordered pair (x,y) of incomparable elements of a poset P is called *critical* if $a \leq x$ and $y \leq b$ implies $a \leq b$. We say that such a critical pair (x,y) is *reversed* in L if $y < x$ in L, and *reversed* in $R = \{L_1, \ldots, L_t\}$ if $y < x$ in some L_i. It was shown by Kelly and Trotter in [KT] that R realizes P if and only if every critical pair of P is reversed in R. Thus, posets with the same set of critical pairs have the same dimension. (The critical pairs of $P_n(k,r)$ are precisely those pairs (A,B) with $|A| = k$, $|B| = r$, and $A \not\subset B$.)

In the next section we find lower bounds for $d_n(k,n-k)$ when k is far below $n/2$ as well as prove that $d_n(2,n-2) = n-1$. In section 3 we classify which extensions (*pseudolex*) of $P_n(k,r)$ reverse the most critical pairs, and we use them in section 4 to compute the *fractional* dimension (introduction in [B5]) $d_n^f(k,r)$ of $P_n(k,r)$.

2. $d_n(k,n-k)$

Define a set $S = \{\sigma_1, \ldots, \sigma_s\}$ of permutations of [m] to be *t-suitable for m* if, for every t-subset $\{a_1 \ldots, a_t\} \subset [m]$ and every $1 \leq i \leq t$, there is an h such that a_j follows a_i in σ_h for all $j \neq i$. This notion is due to Dushnik [D], who defined $N(m,t)$ to be the size of the smallest t-suitable set for m. (Dushnik actually used "precedes" rather than "follows" in his definition, but "follows" is more suitable (!) for our purposes in section 3.) He showed that $d_m(1,t) = N(m,t+1)$, and then Spencer [S] showed that $N(m,t) \geq \log_2 \log_2 m$. Together, these imply (see [H])

Theorem 2.1. *$d_n(k, n-k) \geq \log_2 \log_2(k+c+1)$ for $n = 2k + c$.*

In [H] we also find

Theorem 2.2. *For each integer k there is an $n_0(k)$ such that for $n \geq n_0(k)$ we have*

$$d_n(k, n-k) \geq n - (k+1)^2/2 + 2.$$

Here we prove

Theorem 2.3.

(i) *$d_n(k, n-k) \geq n - k$ for $k \leq (n+1)/3$.*

(ii) *$d_n(k, n-k) \geq n - k - \sqrt{n}$ for $k \leq n/2 - \sqrt{n}$.*

We use the following result from [D].

Theorem 2.4. *For $2 \leq j \leq \lfloor \sqrt{m} \rfloor$, $m \geq 4$, and*

$$\left\lceil \frac{m + j^2 - j}{j} \right\rceil \leq t < \left\lceil \frac{m + (j-1)^2 - (j-1)}{(j-1)} \right\rceil$$

we have $N(m, t) = m - j + 1$.

Proof of Theorem 2.3. We remember that $d_n(k_1, r_1) \geq d_n(k_2, r_2)$ for $0 < k_1 \leq k_2 < r_2 \leq r_1 < n$, and notice also that $d_n(k, r) \geq d_{n-c}(k', r')$ for $k - c \leq k' < r' \leq r - c$. This second assertion was used to prove Theorem 2.1 and is obtained by considering only those sets of $P_n(k, r)$ which contain some fixed set of size $k - c$. The resulting poset is isomorphic to $P_{n-c}(k - c, r - c)$.

Now let $j = 2$, $m = n - k + 1$, $t = n - 2k + 2$, and $n \geq 3k - 1$. Then

$$\begin{aligned} d_n(k, n-k) &\geq d_{n-k+1}(1, n-2k+2) \\ &= N(n-k+1, n-2k+2) \\ &= n-k. \end{aligned}$$

Similarly, with $k \leq n/2 - \sqrt{n}$ and $j = \sqrt{m}$, we have

$$\begin{aligned} d_n(k, n-k) &\geq N(n-k+1, n-2k+2) \\ &= n-k+1-\sqrt{n-k+1}+1 \\ &\geq (n-k)-\sqrt{n}. \qquad \square \end{aligned}$$

The following theorem was discovered independently by Kostochka and Talysheva [KT]. It was left as an open question in [H].

Theorem 2.4. $d_n(2, n-2) = n-1$.

Proof. Part i) above yields only $d_n(2, n-2) \geq n-2$, so our task is two-fold. We first construct a realizer of size $n-1$ and then show that $n-2$ extensions are insufficient to realizer $P_n(2, n-2)$.

Let $R = \{L_1, \ldots, L_{n-1}\}$ be any set of extensions of $P_n(2, n-2)$ with the properties that, for each i, the greatest 2-subset of L_i is the pair $\{i, n\}$, the $(n-2)$ next-greatest 2-subsets of L_i are the pairs $\{i, j\}$ with $j \neq i$ (in any order), and the $(n-2)$-subsets of $[n]$ are as low as possible, meaning that, for each subset B of size $(n-2)$, the 2-subset immediately below it in L_i is the greatest of all those pairs contained in B. For example, with $n = 5$, we may have $L_3 = \{235 > 135 > 345 > \underline{35} > 125 > 145 > \underline{15} > 245 > \underline{45} > \underline{25} > \cdots\}$.

If R is not a realizer then we have $B > \{x, y\}$ by mistake. That is, there is a set B of size $(n-2)$ and a pair $\{x, y\}$ with $\{x, y\} \not\subset B$ and $\{x, y\} < B$ in each L_i. Without loss of generality, assume that $x \notin B$. But then if $y \neq n$ we have $\{x, y\} > B$ in L_y, and if $y = n$

we have $\{x, n\} > B$ in L_x. Thus, R realizes $P_n(2, n-2)$ and $d_n(2, n-2) \leq n-1$ (and so $d_n(k, r) \leq n-1$ for $2 \leq k < r \leq n-2$).

Now suppose that there is a realizer $R = \{L_1, \ldots, L_{n-2}\}$ of size $(n-2)$. It is proved in [H] that the maximum number of critical pairs reversed in an extension of $P_n(2, n-2)$ is (n^2-1). This is plenty enough since $(n-2)(n^2-1) > \binom{n}{2}[\binom{n}{2} - \binom{n-2}{2}]$, which is the number of critical pairs in $P_n(2, n-2)$, but the numerical considerations are not sufficient in forming realizers. It is proven also that the only extensions which achieve this number have the property that the $(n-1)$ highest 2-subsets have some element x in common. For now, let us call this the *top property*.

If each of our extensions has the top property, then it is safe to assume that the common element regarding L_i is i (we can only do worse if two extensions have the same "common element"). But then we see that $\{1, 2, \ldots, n-2\} > \{n-1, n\}$ in each L_i, and so R does not realize $P_n(2, n-2)$. Thus, at least one extension must fail to have the top property.

Since $(n-2)(n^2-1) - \binom{n}{2}[\binom{n}{2} - \binom{n-2}{2}] = \frac{1}{2}n^2 - \frac{5}{2}n + 2$, we must be sure that such an extension reverses (n^2-1) minus at most that many critical pairs, else we do not realize $P_n(2, n-2)$. We will show that we only reverse at most $(n^2-1) - (n^2-6n+8)$, and with $(n^2-6n+8) > (\frac{1}{2}n^2 - \frac{5}{2}n + 2)$ for $n \geq 5$ ($P_n(2, n-2)$ is only defined for $n \geq 5$) this will complete the proof.

Without loss of generality, (because of symmetry) we may assume an extension L where the top property looks like $\{12 > 13 > 14 > \cdots > 1n > \cdots\}$ when restricting our attention to 2-subsets only. When the $(n-2)$-subsets are pushed down as low as possible, we see that there are $\binom{n-2}{2}$ such elements preceding 12, $\binom{n-3}{1}$ between 12 and 13, $\binom{n-4}{0}$ between 13 and 14, and all remaining $\binom{n-1}{1}$ of them follow $1n$ somewhere. When counting critical pairs we count all (A, B) with $|A| = 2$, $|B| = n-2$, and $A > B$ in L.

Now consider some extension L' not having the top property. We may still assume that $\{12 > 13 > 14 > \cdots > 1n\}$ and that 12 is the highest 2-subset, but we must acknowledge that there may be many 2-subsets interspersed after 12. We make the observation that an extension which begins $\{12 > 13 > 14 > \cdots > xy > \cdots > 1n > \cdots\}$ (in other words, which would have the top property if xy were removed and saved for lower in the extension after $1n$) really can be considered to have the top property. The reason is that there are no $(n-2)$-subsets between xy and $1n$ and so these elements can be written in any order without altering the nature of the extension, in particular, its set of reversed critical pairs. So we may as well write xy after $1n$, thus yielding the top property.

That being said, we see that if xy is the highest 2-subset in L' not containing a 1, then we must have $xy > 14$ in L'. In L there are at least $\binom{n-3}{1}$ $(n-2)$-subsets betweeen $1n$ and xy and these have all been elevated to come before 14 in L'. Hence the conversion from L to L' has lost us at least $(n-3)^2$ reversed critical pairs and gained us at most just 1 if xy is between 13 and 14 (namely, $(xy, 14\cdots n)$ if x or y is either of 2 or 3). This is a loss of at least $(n-3)^2 - 1 = n^2 - 6n + 8$. If xy is between 12 and 13, we lose another $(n-3)$ at least and gain another $(n-3)$ at most, and this completes the proof. $\square$

3. Pseudo-lex extensions.

We begin to generalize the extensions with the top property from the previous section and the lexicographic extensions found in [H] to obtain the following three results.

Theorem 3.1. *If L is an extension of $P_n(k, r)$ reversing the maximum number of critical pairs, then L is a pseudo-lex extension.*

Theorem 3.2. *For fixed k and r there is an integer $n_0(k,r)$ such that for all $n > n_0(k,r)$ we have $d_n(k, n-r) \geq n - (k+1)(r+1)/2 + 2$.*

Theorem 3.3. *For $k > 1$ or $r < n-1$, if $R = \{L_1, \ldots, L_t\}$ realizes $P_n(k,r)$ and each L_i is a lexicographic extension, then $t \geq n-1$.*

Since lexicographic extensiosn are pseudo-lex, it is somewhat surprising that Theorem 3.1 says they make excellent extensions while 3.3 says they make horrible realizers.

We first illustrate the pseudo-lex concept with two examples. Figure 1 shows how to create the lexicographic extension $\mathcal{L}_{\sigma(6)}(2,3)$ of $P_6(2,3)$, and Figure 2 does likewise for the pseudo-lex extension $\mathcal{L}_{\Sigma(6)}(2,3)$ (we have separated the 2-sets from the 3-sets only for readability). Notice that, with the permutation $\sigma(6) = (241653)$, we have listed in Figure 1 both the 2-sets and the 3-sets from top to bottom the way one would find them in a dictionary with the alphabet ordered according to σ. We then combined the two lists into one by pushing the 3-sets as far down as possible into the 2-sets.

In Figure 2, we use a more complicated structure than σ to create $\mathcal{L}_{\Sigma(6)}(2,3)$, thinking of $\Sigma(6)$ as an ordered tree of nested permutations (see Figure 3). Here we can write $\Sigma(6) = (2(4(1536)5(163)3(16)61)4(6(315)5(13)13)1(6(35)35)635)$.

241		241	
246		245	
245		243	
243		246	
	24		24
216		251	
215		256	
213		253	
	21		25
265		231	
263		236	
	26		23
253		261	
	25		26
	23		21
416		463	
415		461	
413		465	
	41		46
465		451	
463		453	
	46		45
453		413	
	45		41
	43		43
165		163	
163		165	
	16		16
153		135	
	15		13
	13		15
653		635	
	65		63
	63		65
	53		53

Figure 1. $\mathcal{L}_{\sigma(6)}(2,3)$

Figure 2. $\mathcal{L}_{\Sigma(6)}(2,3)$

In order to produce a permutation from the tree of $\Sigma(6)$ we start at the root and from any node we proceed either to its leftmost child or to its sibling to its immediate right, finally ending at a leaf. Thus, one such permutation might be (246513). (The reader should be able to find 8 distinct permutations in Σ which begin with $(24\cdots)$, though there could conceivably be 11 with a different Σ. That the tree for Σ has depth 3 corresponds to the fact that we are listing 3-subsets.)

Proof of Theorem 3.1. It should be clear that $\mathcal{L}_{\sigma(n)}(k,v)$ and $\mathcal{L}_{\Sigma(n)}(k,r)$ are structurally isomorphic, as indicated by Figures 1 and 2. The reason that this is so is because in all cases we pick an initial permutation (ordered children of the root) $\sigma = (a_1 a_2 \cdots a_n)$ from which we list all sets containing a_1, then all sets containing a_2, and so on. At any given stage we look at those sets which contain a_j but no previous a_i $(i < j)$ and use the induced isomorphism. From there, the same proof from [H] using the Krushal-Katona theorem (see [K1], [K2]) works here as well. $\square$

Proof of Theorem 3.2. We will use the following two lemmas and postpone their proofs temporarily. Let $c_n(k, n-r)$ be the number of critical pairs of $P_n(k, n-r)$ and $m_n(k, n-r)$ be the number of critical pairs of $P_n(k, n-r)$ reversed in $\mathcal{L}_{\Sigma(n)}(k, n-r)$.

Lemma 3.4. $c_n(k, n-r) = \binom{n}{k}[\binom{n}{r} - \binom{n-k}{r}]$.

Lemma 3.5. $m_n(k, n-r) = \sum_{i=0}^{k-1} \sum_{j=0}^{r-1} \binom{i+j}{i} \binom{(n-1)-(i+j)}{(k-1)-i} \binom{(n-1)-(i+j)}{(r-1)-j}$.

From here we have $m_n(k, n-r) = \binom{n-1}{k-1}\binom{n-1}{r-1} + \binom{n-2}{k-1}\binom{n-2}{r-2} + \binom{n-2}{k-2}\binom{n-2}{r-1} + o(n^{k+r-3})$ and so

$$\begin{aligned}
(k!r!)m_n(k, n-r) &= k[(n-1)\dots(n-k+1)]r[(n-1)\dots(n-r+1)] \\
&\quad + k[(n-2)\dots(n-k)]r(r-1)[(n-2)\dots(n-r+1)] \\
&\quad + k(k-1)[(n-2)\dots(n-k+1)]r[(n-2)\dots(n-r)] + o(n^{k+r-3}) \\
&= (kr)\left(\left[n^{k-1} - \binom{k}{2}n^{k-2}\right]\left[n^{r-1} - \binom{r}{2}n^{r-2}\right] + (k+r-2)_n^{k+r-3}\right) + o(n^{k+r-3}) \\
&= (kr)\left(n^{k+r-2} - \left[\binom{k}{2} + \binom{r}{2} - k - r + 2\right]_n^{k+r-3}\right) + o(n^{k+r-3}) \\
&= (kr)\left(n^{k+r-2} - \frac{1}{2}[k^2 + r^2 - 3k - 3r + 4]n^{k+r-3}\right) + o(n^{k+r-3}).
\end{aligned}$$

Also,

$$\begin{aligned}
(k!r!)c_n(k,n-r) &= [(n)\dots(n-k+1)][(n)\dots(n-r+1)] - [(n)\dots(n-k-r+1)] \\
&= \left[n^k - \binom{k}{2}n^{k-1} + \frac{1}{24}(k)(k-1)(k-2)(3k-1)n^{k-2}\right] \\
&\times \left[n^r - \binom{r}{2}n_n^{r-1}\right. \\
&\quad \left. + \frac{1}{24}(r)(r-1)(r-2)(3r-1)n^{r-2}\right] \\
&\quad - \left[n^{k+r} - \binom{k+r}{2}n^{k+r-1}\right. \\
&\quad \left. + \frac{1}{24}(k+r)(k+r-1)(k+r-2)(3k+3r-1)n^{k+r-2}\right] + o(n^{k+r-2}) \\
&= \left[\binom{k+r}{2} - \binom{k}{2} - \binom{r}{2}\right]n^{k+r-1} \\
&\quad + \frac{1}{24}\left[24\binom{k}{2}\binom{r}{2} + (k)(k-1)(k-2)(3k-1) + (r)(r-1)(r-2)(3r-1)\right. \\
&\quad \left. - (k+r)(k+r-1)(k+r-2)(3k+3r-1)\right]_n^{k+r-2} + o(n^{k+r-2}) \\
&= (kr)\left(n^{k+r-1} - \frac{1}{2}[k^2+r^2+kr-2k-2r+1]_n^{k+r-2}\right) + o(n^{k+r-2}).
\end{aligned}$$

Hence, if $(n-x)m_n(k,n-r) \geq c_n(k,n-r)$ for large n, then

$$\begin{aligned}
(n-x)&\left(n^{k+r-2} - \frac{1}{2}[k^2+r^2-3k-3r+4]n^{k+r-3} + o(n^{k+r-3})\right) \\
&= n^{k+r-1} - \left(x + \frac{1}{2}[k^2+r^2-3k-3r+4]\right)n^{k+r-2} + o(n^{k+r-3}) \\
&\geq n^{k+r-1} - \frac{1}{2}[k^2+r^2+kr-2k-2r+1]^{n^{k+r-2}} + o(n^{k+r-3})
\end{aligned}$$

for large n, and so $x \leq (k+1)(r+1)/2 - 2$. $\square$

The proof of Lemma 3.4 is simple. There are $\binom{n}{k}$ k-sets $\binom{n-k}{r}$ $(n-r)$-sets containing any fixed k-set A, and hence $\binom{n}{r} - \binom{n-k}{r}$ critical pairs involving A. To prove Lemma 3.5 we need a bit more notation (mostly, we follow Bollobás [3]). We will fix $\Sigma(n) = \sigma = (12\cdots n)$ and set $Z = Z_{\sigma(n)}(K, n-r)$, $Z_k = \{\mathcal{A}_1, \dots, \mathcal{A}_{\binom{n}{k}}\}$ the restriction of $\mathcal{L}$ to its k-sets, with

$\mathcal{A}_1 < \mathcal{A}_2 < \ldots$, and likewise $\mathcal{L}(n-r) = \{\mathcal{B}_1, \ldots, \mathcal{B}_{\binom{n}{r}}\}$. (This ordering is often referred to as *colex*: ex. $\{321, 421, 431, 432, 521, 531, 532, 541, 542, 543\}$).

Given $s \le k$ and $s \le t_s < t_{s+1} < \cdots < t_k$, let $b^{(k)}(t_k, \ldots, t_s) = \sum_{j=s}^{k} \binom{t_j}{j}$. It is known that for every integer t there is a unique sequence $s \le t_5 < \cdots < t_k$ such that $t = b^{(k)}(t_k, \ldots, t_s)$. For example, with $k = 4$ we have $106 = \binom{8}{4} + \binom{7}{3} + \binom{2}{2} = b^{(4)}(8, 7, 2)$. Also, let $\mathcal{B}^{(k)}(t_k, \ldots, t_s) < \mathcal{L}^t(k)$ with $t = b^{(k)}(t_k, \ldots, t_s)$, and ask what is $\mathcal{A}_t$? (Here, $\mathcal{L}^t(k) = \{\mathcal{A}_1, \ldots, \mathcal{A}_t\}$, the colexicographically first t elements of $\binom{[n]}{k}$. It is $\{t_s - s + 1, \ldots, t_s - 1, t_s, t_{s+1} + 1, \ldots, t_k + 1\}$. For example, with $k = 4$, we have $\mathcal{A}_{106} = \{t_2 - 1, t_2, t_3 + 1, t_4 + 1\} = \{1, 2, 8, 9\}$. Notice that $105 = b^{(4)}(8, 7)$ so that $\mathcal{A}_{105} = \{5, 6, 7, 9\}$, which immediately precedes $\{1, 2, 8, 9\}$ in $\mathcal{L}(4)$.

From now on we will find it useful to denote $b^{(k)}(t_k, \ldots, t_s)$ by $b^{(k)}(t_k, \ldots, t_s, 0, \ldots, 0)$ so that the number of coordinates is always k. We will call such a representation of t its *k-cascade* form. With $k < n/2$ we will say that a k-cascade is *special* if each of its nonzero coordinates t_j satisfies $t_j \ge n - k - r + j$. For example, $b^{(4)}(8, 6, 5, 0)$ is special with $n = 9$ and $r = 2$, while $b^{(4)}(8, 6, 4, 3)$ is not. The greatest special 4-cascade less than $b^{(4)}(8, 6, 4, 3)$ is $b^{(4)}(8, 6, 0, 0)$.

Given the k-cascade for t, let t' be the greatest integer less than t whose k-cascade is special, and let t^* have the $(n-r)$-cascade obtained from the k-cascade of t' by adjoining $(n - k - r)$ extra zeros. Thus, if $r = 2$, $n = 9$ and $b^{(4)}(8, 6, 4, 3)$ then $t' = b^{(4)}(8, 6, 0, 0)$ and $t^* = b^{(7)}(8, 6, 0, 0, 0, 0, 0)$. Now ask the following question. What is the greatest $(n-r)$-set less than $\mathcal{A}_t$ in $\mathcal{L}$? Well, $\mathcal{A}_t = \{3, 5, 7, 9\}$ so the least 7-set greater than $\mathcal{A}_t$ is $\{1, 2, 3, 4, 5, 7, 9\}$, implying the greatest 7-set less than $\mathcal{A}_t$ is $\{1, 2, 3, 4, 5, 6, 9\}$. Observe that $\mathcal{B}_{t^*} = \{1, 2, 3, 4, 5, 6, 9\}$. We leave it to the reader to prove

Claim 3.6. *Let $\mathcal{L}(k) = \{\mathcal{A}_1 < \cdots < \mathcal{A}_{\binom{n}{k}}\}$ and $\mathcal{L}(n-r) = \{\mathcal{B}_1 < \cdots < \mathcal{B}_{\binom{n}{r}}\}$. For all $1 \le t \le \binom{n}{k}$ the greatest $(n-r)$-set less than $\mathcal{A}_t$ in $\mathcal{L}$ is $\mathcal{B}_{t^*}$.* $\square$

Let $m_t(\mathcal{L})$ be the number of critical pairs of the form $(\mathcal{A}_t, \mathcal{B}_j)$ which are reversed in $\mathcal{L}$.

Corollary 3.7. *For all $1 \le t \le \binom{n}{k}$ we have $m_t(\mathcal{L}) = t^*$.* $\square$

Proof of Lemma 3.5. $m_n(k, n-r) = \sum_{t=1}^{\binom{n}{k}} m_t(\mathcal{L})$. In our example with $t = b^{(4)}(8,6,4,3)$ we found $m_t(\mathcal{L}) = b^{(7)}(8,6,0,0,0,0,0) = \binom{8}{7} + \binom{6}{6}$. So $m_n(k, n-r)$ involves sums of many binomial coefficients. How many times will $\binom{8}{7}$ be counted? Denote this number by $\#\binom{8}{7}$. For every t such that $b^{(4)}(8,0,0,0) < t \le (9,0,0,0)$ we have $t^* = (8, x, y, z, 0, 0, 0)$. Hence $\#\binom{8}{7} = \binom{9}{4} - \binom{8}{4} = \binom{8}{3}$. This contributes $\binom{8}{7}\binom{8}{3}$ to the sum $m_9(4,7)$.

What about $\#\binom{7}{6}$ in $m_9(4,7)$? For every t satisfying $(6)^{(4)}(x,7,0,0) < t \le b^4(x,8,0,0)$ we have $t^* = (x, 7, y, z, 0, 0, 0)$. Hence $\#\binom{7}{6} = \binom{1}{1}[\binom{8}{3} - \binom{7}{3}] = \binom{1}{1}\binom{7}{2}$ since there is only one choice (namely 8) for x. This contributes $\binom{1}{1}\binom{7}{6}\binom{7}{2}$ to $m_9(4,7)$.

In general, we wish to find $\#\binom{a}{b}$, so let $b = (n-r) - i$ for some $0 \le i \le k-1$, and let $a = (n-1-i) - j$ for some $0 \le j \le r-1$ (so that $b \le a < n-i$). Then for each t satisfying

$$b^{(k)}(x_k, \ldots, x_{k-i+1}, a, 0, \ldots, 0) < t \le b^{(k)}(x_k, \ldots, x_{k-i+1}, a+1, 0, \ldots, 0)$$

(each k-cascade having $k-i-1$ zeros), we have

$$t^* = b^{(n-r)}(x_k, \ldots, x_{k-i+1}, a, x_{k-i-1}, \ldots, x_1, 0, \ldots, 0)$$

(with $n-k-r$ zeros). We now have $\binom{n-1-a}{i} = \binom{i+j}{i}$ possible values for the x_j with $k-i+1 \le j \le k$, so that

$$\#\binom{a}{b} = \binom{i+j}{i}\left[\binom{a+1}{k-i} - \binom{a}{k-i}\right] = \binom{i+j}{i}\binom{a}{k-i-1}.$$

This contributes $\binom{i+j}{i}\binom{a}{k-i-1}\binom{a}{b}$ to the sum $m_n(k, n-r)$. Hence,

$$\begin{aligned}
m_n(k, n-r) &= \sum_{i=0}^{k-1}\sum_{j=0}^{r-1}\binom{a}{b}\left[\#\binom{a}{b}\right] \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{r-1}\binom{i+j}{i}\binom{a}{k-i-1}\binom{a}{b} \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{r-1}\binom{i+j}{i}\binom{(n-1)-(i+j)}{(k-1)-i}\binom{(n-1)-(i+j)}{(n-r)-i} \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{r-1}\binom{i+j}{i}\binom{(n-1)-(i+j)}{(k-1)-i)}\binom{(n-1)-(i+j)}{(r-1)-j}. \qquad \square
\end{aligned}$$

Proof of Theorem 3.3. Let $R(S) = \{\mathcal{L}_{\sigma_1}, \ldots, \mathcal{L}_{\sigma_s}\}$ be the set of lexicographic extensions induced from the set $S = \{\sigma_1, \ldots, \sigma_s\}$ of permutations of $[n]$.

Lemma 3.8. *If $R(S)$ realizes $P_n(k, r)$, then S is $(r+1)$-suitable for n.*

Proof. Suppose S is not $(r+1)$-suitable. Then there is a set $\{x_0, x_1, \ldots, x_r\}$ such that, for all i there is a j so that xj precedes x_0 in σ_i. Thus, $\{x_1, \ldots, x_r\} > \{x_0, x_1, \ldots, x_{k-1}\}$ in $\mathcal{L}_{\sigma_i}$ for all i and hence in $P_n(k, r)$, a contradiction. $\square$

Using Theorem 2.4, we discover

Corollary 3.9. *For $r > n/j$, $2 \le j \le \sqrt{n}$, if $R(s)$ realizes $P_n(k, r)$, then $|R| \ge n - j + 1$.* $\square$

In particular, we see that $|R| \ge n-1$ for all $n/2 \le r \le n-1$. In actuality, since $d_n(k, r) = d_n(n-r, n-k)$ ($P_n(n-r, n-k)$ is the dual of $P_n(k, r)$), if $r < n/2$, then $n - k \ge n/2$, and so $|R| \ge n-1$ unless both $k = 1$ and $r = n-1$. $\square$

4. Fractional Dimension.

In [BS] Brightwell and Scheinerman define a *t-fold realizer* of a poset P to be a set of linear extensions $R = \{L_1, \ldots, L_s\}$ with the property that, for every critical pair (x, y) there is a set $|I| \geq t$ with $x > y$ in L_i for all $i \in I$. Then the *fractional dimension* $d^f(P)$ of P is $\inf\limits_{t, R_t} \{|R_t|/t\}$, with each R_t a t-fold realizer of P. We always have $d^f(P) \leq \dim(P)$ since for $t = 1$, $\min_R |R| = \dim(P)$. Also, $d^f(P)$ satisfies the same immediate lower bound as $\dim(P)$. Namely, let $c(P)$ be the number of critical pairs of P and $m(P)$ be the maximum number which can be reversed by a single extension. Then the necessary relation $|R| m(P) \geq tc(P)$ implies $d^f(P) \geq c(P)/m(P)$. Thus, Theorem 3.2 holds for fractional dimension as well (though Theorem 2.3 does not as yet). As for upper bounds, we improve upon the bound $d_n^f(k, r) = d^f(P_n(k, r)) \leq d_n(k, r) \leq n - 1$ with

Theorem 4.1. *For all* $1 \leq k \leq n/2$, $1 \leq r \leq n/2$, $d_n^f(k, n-r) \leq n - r - k + 2$.

Proof. Let R be the set of all $n!$ lexicographic extensions of $P_n(k, n-r)$ ($R = \{\mathcal{L}_\sigma | \sigma \in \mathrm{Sym}(n)\}$).

Claim 4.2. *Given the critical pair* (x, y), *if* σ *is chosen at random then* $Pr[x > y \textit{ in } \mathcal{L}_\sigma] = (k-a)/n - r + k - 2a$, *where* $|x \cap y| = a$.

Proof of Claim 4.2. With regard to distinguishing x and y, none of the elements in $x \cap y$ or $\overline{x \cup y}$ need to be considered. Of the remaining $n - r + k - 2a$ elements in $(x - y) \cup (y - x)$, whichever element occurs first in σ determines whether $x > y$ or $y > x$. Thus $\Pr[x > y \text{ in } L_\sigma] = |x - y|/(n - r + k - 2a)$. $\square$

From this, we learn that the number of times (x, y) is reversed in R is $(n!)(k-a)/(n - r + k - 2a)$, and hence R is a t-fold realizer for

$$t = \min_{0 \leq a \leq k-1} \frac{n!(k-a)}{n - r + k - 2a} = \frac{n!}{n - r - k + 2},$$

and Theorem 4.1 follows. $\square$

Theorem 4.1 highlights two extremes: $d^f_n(1, n-1) = d_n(1, n-1) = n$ since $c/m = n$; $d^f_{2k+1}(k, k+1) = 3$, whereas we know from Theorem 2.1 that $d_{2k+1}(k, k+1) \geq \log_2 \log_2(k+2)$. (Theorem 4.1 only yields $d^f_{2k+1}(k, k+1) \leq 3$, but $P_{2k+1}(k, k+1) \supset P_3(1, 2)$, for all $k \geq 1$, and so $d^f_{2k+1}(k, k+1) \geq d^f_3(1, 2) = 3$.) In addition, the theorem holds even for $k + r = n$, since $P_n(k, n-r)$ is then an antichain.

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