Decide whether each of the following solutions are feasible or infeasible.

(a) $\mathbf{x} = (73, 34, 0)^{\mathsf{T}}/33$;

(b) $\mathbf{x} = (55, 62, 166)^{\mathsf{T}}/109$;

(c) $\mathbf{x} = (1, 1, 1)^{\mathsf{T}}$.

(d) *Draw the set of points that satisfy the constraints of P. [HINT: First reduce the number of constraints by discarding redundant ones (constraints that are implied by a collection of other constraints).] [MORAL: A little 3-dimensional drawing and visualization never hurt anyone!]*

1.5.4 *Consider the following LOP P.*

$$\begin{array}{llrcrcrcrcrcl}
\text{Max.} & z = & 2x_1 & - & 3x_2 & & & + & 4x_4 & - & x_5 & & \\
\text{s.t.} & & x_1 & + & x_2 & & & & & + & 3x_5 & \le & 2 \\
& & 2x_1 & & & & & + & x_4 & - & x_5 & \le & 6 \\
& & & & & & x_3 & + & 2x_4 & + & 3x_5 & \le & 4 \\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \ge & 0
\end{array}$$

Decide whether each of the following solutions are feasible or infeasible.

(a) $\mathbf{x} = (11, 0, 0, 0, 4)^{\mathsf{T}}/3$;

(b) $\mathbf{x} = (2, 0, 0, 2, 0)^{\mathsf{T}}$;

(c) $\mathbf{x} = (0, 2, 0, 2, 0)^{\mathsf{T}}$.

1.5.5 *Consider the following LOP P.*

$$\begin{array}{llrcrcl}
\text{Max.} & z = & 3x_1 & + & 4x_2 & & \\
\text{s.t.} & & 2x_1 & - & 3x_2 & \le & 3 \\
& & 4x_1 & + & x_2 & \le & 6 \\
& & x_1 & + & x_2 & \le & 5 \\
& & x_1 & , & x_2 & \ge & 0
\end{array}$$

(a) *Find a primal feasible solution $\mathbf{x}$ and its corresponding objective value $z = z(\mathbf{x})$.*

(b) *Write the LOP D that is dual to P.*

(c) *Find a dual feasible solution $\mathbf{y}$ and its corresponding objective value $w = w(\mathbf{y})$.*

(d) *What upper and lower bounds do parts a and c produce for z^*?*

1.5.6 *Repeat Exercise 1.5.5 N times, each with a different modestly sized LOP of your own design. [MORAL: Many exercises in the book can be repeated by the reader simply by making up a new LOP.]*

1.5.7 *Consider the following LOP P*

$$\begin{array}{llrcrcrcl}
\text{Max.} & z = & 5x_1 & + & 5x_2 & + & 5x_3 & & \\
\text{s.t.} & & x_1 & + & 2x_2 & + & 3x_3 & \le & 4 \\
& & 4x_1 & + & 3x_2 & + & 2x_3 & \le & 1 \\
& & x_1 & , & x_2 & , & x_3 & \ge & 0
\end{array}$$

(a) *Write the matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}$ which correspond to P.*

(b) *Write the LOP D that is dual to P.*

(c) *Prove, for this particular P and D, that $z \le w$, as in Inequality 1.4.7. Explain each step.*

1.5.8 *Repeat Exercise 1.5.7 N times, each with a different modestly sized LOP of your own design. Try LOPs for which the number of variables is different from the number of problem constraints.*

1.5.9 *Consider the following LOP.*

$$\begin{array}{llcrcrcrcrcr}
\text{Max.} & z & = & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & & \\
\text{s.t.} & & & x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & \le & 6 \\
& & & x_1 & + & 7x_2 & + & 3x_3 & + & 9x_4 & \le & 12 \\
& & & x_1 & + & 6x_2 & + & 5x_3 & + & 9x_4 & \le & 8 \\
& & & x_1 & , & x_2 & , & x_3 & , & x_4 & \ge & 0
\end{array}$$

(a) *Find the dual multipliers that yield the following constraint.*

$$x_1 \quad + \quad 2x_2 \quad + \quad 3x_3 \quad + \quad 4x_4 \quad \le \quad 2$$

(b) *Does it follow from part a that $z^* \le 2$?*

[MORAL: Multipliers of standard form LOP constraints *must* be nonnegative!]

1.5.10 *Consider the following LOP P.*

$$\begin{array}{llcrcrcrcr}
\text{Max.} & z & = & -12x_1 & - & 11x_2 & - & 13x_3 & & \\
\text{s.t.} & & & -x_1 & + & x_2 & & & \le & -2 \\
& & & & - & x_2 & - & x_3 & \le & -3 \\
& & & x_1 & & & + & x_3 & \le & 5 \\
& & & x_1 & , & x_2 & , & x_3 & \ge & 0
\end{array}$$

(a) *Write the LOP D that is dual to P.*

(b) *Consider the point $\mathbf{x} = (2, 0, 3)^\mathsf{T}$.*

[i] *Show that $\mathbf{x}$ is P-feasible.*

[ii] *Find $z(\mathbf{x})$.*

(c) *Consider the point $\mathbf{y} = (12, 13, 0)^\mathsf{T}$.*

[i] *Show that $\mathbf{y}$ is D-feasible.*

[ii] *Find $w(\mathbf{y})$.*

(d) *Use parts b and c to find z^*.*

(e) *Consider the point $\mathbf{y} = (13, 14, 1)^\mathsf{T}$.*

[i] *Show that $\mathbf{y}$ is D-feasible.*

[ii] *Find $w(\mathbf{y})$. [MORAL: a given LOP may have several optimal solutions.]*

1.5.11 *Consider the following LOP P.*

$$\begin{array}{llcrcrcr}
\text{Max.} & z & = & 5x_1 & + & 3x_2 & & \\
\text{s.t.} & & & x_1 & + & 2x_2 & \le & 14 \\
& & & 3x_1 & - & 2x_2 & \le & 18 \\
& & & x_1 & - & 2x_2 & \ge & -10 \\
& & & x_1 & , & x_2 & \ge & 0
\end{array}$$

(a) *Graph the system of constraints of P.*

(b) *Draw the following lines on your graph in part a:*

- **[i]** $z = 12$ *(i.e.* $5x_1 + 3x_2 = 12$*);*
- **[ii]** $z = 24$*;*
- **[iii]** $z = 36$.

(c) *Plot the following points on your graph in part a:*

- **[i]** $\mathbf{x} = (2,3)^\mathsf{T}$*;*
- **[ii]** $\mathbf{x} = (4,6)^\mathsf{T}$*;*
- **[iii]** $\mathbf{x} = (8,3)^\mathsf{T}$.

(d) *Find* $\mathbf{x}^*$ *and* z^*.

(e) *Write the LOP D that is dual to P.*

(f) *Find* $\mathbf{y}^*$. *[HINT: Find dual-feasible* $\mathbf{y}$ *such that* $w(\mathbf{y}) = z^*$*.]*

1.5.12 *Repeat Exercise 1.5.11 N times, each with a different LOP of your own design, having 2 variables and 3 constraints. [NOTE: You may want to reverse-engineer this — that is, derive the constraints from a graph you draw.] You will need to choose your own objective lines to draw (two are enough), and find which points are important to plot.*

Challenges

1.5.13 *Consider the following LOP P.*

$$\begin{array}{llrcrcrcr}
\text{Max.} & z = & & - & 2x_2 & & & & \\
\text{s.t.} & & x_1 & + & 4x_2 & - & x_3 & \leq & 1 \\
& & -2x_1 & - & 3x_2 & + & x_3 & \leq & -2 \\
& & 4x_1 & + & x_2 & - & x_3 & \leq & 1 \\
& & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

(a) *Write the dual D of P.*

(b) *Show that* $\mathbf{y} = (2,3,1)^\mathsf{T}$ *is D-feasible.*

(c) *Use part b to show that P is infeasible.*

(d) *Show that* $\mathbf{y}(t) = (2t, 3t, t)^\mathsf{T}$ *is D-feasible for all $t \geq 0$.*

(e) *Use part d to prove that D is unbounded.*

[MORAL: Simple certificates like $\mathbf{y}$ are very powerful tools in LO.]

1.5.14 *Repeat part a N times, each with a different LOP of your own design.*

(a) *Let P be a LOP in standard max form and let D be its dual LOP. Write D in standard max form as D' and let Q' be its dual. Finally, write Q' in standard max form as Q. Compare P and Q.*

(b) *Prove a statement about the relationship between P and Q in general.*

1.5.15 *Consider the problem:* Max. $\mathbf{c}^\mathsf{T}\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$ *(note that* $\mathbf{x}$ *is free).*

(a) *Let $V = \{\mathbf{v}^i\}_{i=1}^k$ be a basis for the nullspace of $\mathbf{A}$. Write all solutions to $\mathbf{Ax} = \mathbf{b}$ in terms of V. [HINT: Recall Gaussian elimination.]*

(b) *Use part a to prove that if $\mathbf{c}^\mathsf{T}\mathbf{v}^i = 0$ for all $1 \leq i \leq k$ then every feasible point is optimal.*

(c) *Prove that if $\mathbf{Ax} = \mathbf{b}$ is feasible then there is some $\mathbf{v}$ and feasible $\mathbf{x}^0$ such that $\mathbf{x}^0 + t\mathbf{v}$ is feasible for all $t \in \mathbb{R}$.*

(d) *Use part c to prove that if $\mathbf{c}^\mathsf{T}\mathbf{v}^i \neq 0$ for some $1 \leq i \leq k$ then the problem is unbounded.*

(e) *Use parts b and d to prove that every such problem is either infeasible, optimal or unbounded.*

1.5.16 *Write pseudocode for an algorithm that takes as input a LOP with n variables and m constraints and outputs its dual LOP.*

Modeling

1.5.17 *A factory manufactures two products, each requiring the use of three machines. The first machine can be used at most 70 hours; the second machine at most 40 hours; and the third machine at most 90 hours. The first product requires 2 hours on machine 1, 1 hour on machine 2, and 1 hour on machine 3; the second product requires 1 hour on machines 1 and 2 and 3 hours on machine 3. The profit is \$40 per unit for the first product and \$60 per unit for the second product. Write a LOP that will compute how many units of each product should be manufactured in order to maximize profit.*

1.5.18 *An oak table company has an individual who does all its finishing work and it wishes to use him in this capacity at least 36 hours each week. The assembly area can be used at most 48 hours each week. The company has three models of oak tables, T1, T2 and T3. T1 requires 1 hour for assembly, 2 hours for finishing, and 9 board feet of oak. T2 requires 1 hour for assembly, 1 hour for finishing and 9 board feet of oak. T3 requires 2 hours for assembly, 1 hour for finishing and 3 board feet of oak. Write a LOP that will compute how many of each model should be made in order to minimize the board feet of oak used.*

1.5.19 *The system of equations*

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & = & 2 \\ -2x_1 & - & 3x_2 & + & x_3 & = & 1 \\ -3x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 4x_1 & + & x_2 & - & x_3 & = & -1 \end{array}$$

has no solution. A 'best' approximate solution minimizes the error according to some measure. For a given (x_1', x_2', x_3') the error e_1 in the first equation is $e_1 = 2 - x_1 - 4x_2 + x_3$ and similarly for the errors e_2, e_3, e_4 for the other rows.

(a) *Write a LOP that will compute the best L_1 approximation, which minimizes the sum of the absolute values of the errors.*

(b) *Write a LOP that will compute the best L_∞ approximation, which minimizes the maximum absolute value of an error.*

[HINT: Creating a new variable that is an upper bound on both e_i and on $-e_i$ makes it an upper bound on $|e_i|$]

1.5.20 *A housewife asks a butcher to grind up several cuts of beef to form a blend of equal parts of proteins and fats. The butcher, being conscientious, wishes to do this at least cost per pound of meat purchased. The following table gives fat and protein contents and costs in dollars.*

	Chuck	Flank	Porterhouse	RibRoast	Round	Rump	Sirloin
%Protein	19	20	16	17	19	16	17
%Fat	16	18	25	23	11	28	20
cost/lb	.69	.98	1.39	1.29	1.19	1.50	1.65

(a) *Write a LOP that will compute the amounts of meat and how much he should charge.*

(b) *Usually he has extra fat available free per pound. How does this alter the solution?*

(c) *This problem is a slight modification of one from an old textbook. Guess the year in which it was written.*

1.5.21 *A factory buys bags of sand and produces sand castles. Each sand castle requires one bag of sand and the factory has a production capacity of* $3,000$ *sand castles per quarter year. However, sand is available in different amounts and sand castles are required for sale or distribution in different amounts each quarter. Furthermore, storing sand castles is expensive and carrying them over from one quarter to the next is to be minimized. At the beginning of the year,* $3,000$ *sand bags are available and at least this many must be left over at the end of the year. The availability of sand bags and requirements for sand castles per quarter is as follows:*

quarter	*sand bags available for purchase*	*sand castles required for sale*
1	$5,000$	$1,000$
2	$3,000$	$4,000$
3	$1,000$	$3,000$
4	$2,000$	$1,500$

There is storage room available for $10,000$ *sand bags or* $2,000$ *sand castles or any combination in this ratio. (That is, in quarter q, if B_q and C_q respectively represent the number of sand bags and sand castles on hand at the end of the quarter* $B_q + 5C_q \leq 10,000$*. Here we ignore bottlenecks during a quarter.)*

Write down an integer LOP that will compute purchases of sand bags and the number of sand castles made for each quarter, minimizing carryover of sand castles, subject to the availability and requirement constraints.

1.5.22 *Al's refinery can buy two types of gasoline. Boosch Oil has available, at \$60 per barrel, 130,000 barrels of 92 octane gasoline with vapor pressure 4.6 psi and sulfur content 0.58%. Chayni Oil has available, at \$70 per barrel, 140,000 barrels of 85 octane gasoline with vapor pressure 6.5 psi and sulfur content 0.40%. Al needs to blend these two to produce at least 200,000 barrels of a mixture with octane between 87 and 89, with vapor pressure at most 6 psi and sulfur content at most 0.5%.*

Formulate a LOP to determine the proportions of each type he should use to minimize his cost.