

# An Application of Graph Pebbling to Zero-Sum Sequences in Abelian Groups

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## Abstract

A sequence of elements of a finite group  $G$  is called a zero-sum sequence if it sums to the identity of  $G$ . The study of zero-sum sequences has a long history with many important applications in number theory and group theory. In 1989 Kleitman and Lemke, and independently Chung, proved a strengthening of a number theoretic conjecture of Erdős and Lemke. Kleitman and Lemke then made more general conjectures for finite groups, strengthening the requirements of zero-sum sequences. In this paper we prove their conjecture (first obtained by Geroldinger) in the case of abelian groups. Namely, we use graph pebbling to prove that for every sequence  $(g_k)_{k=1}^{|G|}$  of  $|G|$  elements of a finite abelian group  $G$  there is a nonempty subsequence  $(g_k)_{k \in K}$  such that  $\sum_{k \in K} g_k = 0_G$  and  $\sum_{k \in K} 1/|g_k| \leq 1$ , where  $|g|$  is the order of the element  $g \in G$ .

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# 1 Introduction

A sequence of elements of a finite group  $G$  is called a zero-sum sequence if it sums to the identity of  $G$ . A standard pigeonhole principle argument shows that any sequence of  $|G|$  elements of  $G$  contains a zero-sum subsequence; in fact having consecutive terms (one can instead stipulate that the zero-sum subsequence has at most  $N$  terms — where  $N = N(G)$  is the exponent of  $G$ , i.e. the maximum order of an element of  $G$  — which is best possible).

First considered in 1956 by Erdős [15], the study of zero-sum sequences has a long history with many important applications in number theory and group theory. In 1961 Erdős et al. [16] proved that every sequence of  $2|G| - 1$  elements of a cyclic group  $G$  contains a zero-sum subsequence of length exactly  $|G|$ . In 1969 van Emde Boas and Kruyswijk [14] proved that any sequence of  $N(1 + \log(|G|/N))$  elements of a finite abelian group contains a zero-sum sequence. In 1994 Alford et al. [1] used this result and modified Erdős's arguments to prove that there are infinitely many Carmichael numbers. Much of the recent study has involved finding Davenport's constant  $D(G)$ , defined to be the smallest  $D$  such that every sequence of  $D$  elements contains a zero-sum subsequence [29]. Applications of the wealth of results on this problem [5, 18, 19, 21, 23, 31] and its variations [20, 28] to factorization theory and to graph theory can be found in [2, 6].

In 1989 Kleitman and Lemke [26], and independently Chung [7], proved the following strengthening of a number theoretic conjecture of Erdős and

Lemke (see also [8, 13]).

**Result 1** *For any positive integer  $n$ , every sequence  $(a_k)_{k=1}^n$  of  $n$  integers contains a nonempty subsequence  $(a_k)_{k \in K}$  such that  $\sum_{k \in K} a_k \equiv 0 \pmod{n}$  and  $\sum_{k \in K} \gcd(a_k, n) \leq n$ .*

Kleitman and Lemke then made more general conjectures for finite groups, strengthening the requirements of zero-sum sequences. In this paper we prove their conjecture (first obtained in [22]) in the case of abelian groups. Namely, we use graph pebbling (and Result 1) to prove the following theorem (we use  $|g|$  to denote the order of the element  $g \in G$ ).

**Theorem 2** *For every sequence  $(g_k)_{k=1}^{|G|}$  of  $|G|$  elements of a finite abelian group  $G$  there is a nonempty subsequence  $(g_k)_{k \in K}$  such that  $\sum_{k \in K} g_k = 0_G$  and  $\sum_{k \in K} 1/|g_k| \leq 1$ .*

Notice that Result 1 is the special case of Theorem 2 in which  $G$  is cyclic. Also notice that the condition on the sum of the orders implies that  $|K| \leq N(G)$ , with equality if and only if  $|g_k| = N$  for every  $k \in K$ .

## 2 Preliminaries

### 2.1 Graph Pebbling

Let  $\Gamma = (V, E)$  be a graph with vertices  $V$  and edges (unordered pairs of edges)  $E$ . Given a configuration of pebbles on  $V$ , a pebbling step consists of removing two pebbles from a vertex  $u$  and placing one pebble on an adjacent

vertex  $v$  ( $uv \in E$ ). The pebbling number  $\pi = \pi(\Gamma)$  is the smallest number  $\pi$  such that, from every configuration of  $\pi$  pebbles on  $V$  it is possible to place a pebble on any specified target vertex after a sequence of pebbling moves. There is a rapidly growing literature on graph pebbling [10, 12, 24], including variations such as optimal pebbling [17, 27, 30], pebbling thresholds [3, 4, 11] and cover pebbling [9, 25, 32].

One variation of graph pebbling involves labelling the edges  $uv \in E$  by positive integer weights  $w(uv)$ , so that a pebbling step from  $u$  to  $v$  removes  $w(uv)$  pebbles from  $u$  before placing one pebble on  $v$ . In this light, standard graph pebbling has weight 2 on every edge. Let  $\mathcal{B}^n$  be the graph of the  $n$  dimensional boolean algebra — its vertices are all binary  $n$ -tuples; its edges are the pairs of  $n$ -tuples that differ by a single digit. For every edge between vertices that differ in the  $i^{\text{th}}$  digit, let  $w_i$  be its weight. Finally, let  $\mathbf{w} = \langle w_i \rangle_{i=1}^n$  and denote the resulting weighted graph by  $\mathcal{B}^n(\mathbf{w})$ . Then Chung's theorem [7] is as follows.

**Theorem 3** *The generalized pebbling number of the weighted graph  $\mathcal{B}^n(\mathbf{w})$  is  $\pi(\mathcal{B}^n(\mathbf{w})) = \prod_{i=1}^n w_i$ .*

## 2.2 Group Structure

Let  $\mathbb{Z}_n$  denote the finite cyclic group on  $n$  elements. The standard representation for an abelian group  $G$  has the form  $\mathbb{Z}_{N_1} \oplus \mathbb{Z}_{N_2} \oplus \dots \oplus \mathbb{Z}_{N_r}$ , where  $N_i | N_{i-1}$  for  $1 < i \leq r$  (although, purposely, we've written the order of the cycles in reverse to the standard). Thus the exponent of  $G$  is  $N(G) = N_1$

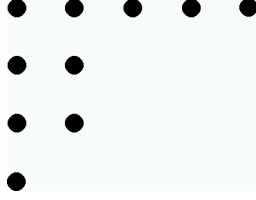


Figure 1: Ferrer's diagram for  $(5, 2, 2, 1)$

and the rank of  $G$  is  $r(G) = r$ . One of the useful techniques in this paper is to break down each cycle  $\mathbb{Z}_{N_i}$  into products of cycles of distinct prime powers. We write  $G = \bigoplus_{i=1}^t \bigoplus_{j=1}^{m_i} \mathbb{Z}_{p_i^{e_{i,j}}}$  for some primes  $p_i$ , multiplicities  $m_i$ , and exponents  $e_{i,j}$ . Thus  $G$  can be coordinatized so that elements  $g$  have the form  $\mathbf{g} = \langle g_{i,j} \rangle$ , and addition is coordinatewise with the  $(i, j)^{\text{th}}$  coordinate computed modulo  $p_i^{e_{i,j}}$ . Further, instead of writing the primes  $p_i$  in increasing order, we write them so that  $e_{i,1} \geq \dots \geq e_{i,m_i}$  for every  $1 \leq i \leq t$ . Hence the exponent of  $G$  can be written  $N = N(G) = \prod_{i=1}^t p_i^{e_{i,1}}$ .

### 2.3 Notation

As already witnessed, we will adopt the convention that bold fonts will denote vectors. Let  $\mathbf{e}_i = \langle e_{i,j} \rangle_{j=1}^{m_i}$ ,  $\mathbf{e} = \langle \mathbf{e}_i \rangle_{i=1}^t$  and  $m = \sum_{i=1}^t m_i$ . Then  $\mathbf{e}_i$  can be thought of as a partition of the exponent of  $p_i$  in the prime factorization of  $|G|$ . Define  $\mathbf{d}_i$  to be the dual partition that arises from the Ferrer's diagram of  $\mathbf{e}_i$ . For example, Figure 1 shows the Ferrer's diagram for  $(5, 2, 2, 1)$  (dots per row) and its dual  $(4, 3, 1, 1, 1)$  (dots per column), both partitions of 10.

Next define  $\mathbf{f}_{i,r} = \langle \mathbf{1}^r, \mathbf{0}^{m_i-r} \rangle$ , and let

$$\mathbf{F}_{i,r} = \langle \mathbf{f}_{1,0}, \dots, \mathbf{f}_{i-1,0}, \mathbf{f}_{i,r}, \mathbf{f}_{i+1,0}, \dots, \mathbf{f}_{m,0} \rangle = \langle \mathbf{0}^a, \mathbf{f}_{i,r}, \mathbf{0}^b \rangle,$$

$$\begin{array}{lcl}
\mathbf{e} = \langle 5, 4, 3, 1; 2, 2; 3; 4, 1, 1 \rangle, & \mathbf{e}_1 = \langle 5, 4, 3, 1 \rangle, & \mathbf{d}_1 = \langle 4, 3, 3, 2, 1 \rangle \\
\hline
\mathbf{e}(0, 0, 0, 0) & = & \langle 5, 4, 3, 1; 2, 2; 3; 4, 1, 1 \rangle \\
\mathbf{e}(1, 0, 0, 0) & = & \mathbf{e}(0, 0, 0, 0) - \mathbf{F}_{1, d_1, u_1} = \langle 4, 3, 2, 0; 2, 2; 3; 4, 1, 1 \rangle \\
\mathbf{e}(1, 1, 0, 0) & = & \mathbf{e}(1, 0, 0, 0) - \mathbf{F}_{2, d_2, u_2} = \langle 4, 3, 2, 0; 1, 1; 3; 4, 1, 1 \rangle \\
\mathbf{e}(1, 1, 0, 1) & = & \mathbf{e}(1, 1, 0, 0) - \mathbf{F}_{4, d_4, u_4} = \langle 4, 3, 2, 0; 1, 1; 3; 3, 0, 0 \rangle \\
\mathbf{e}(2, 1, 0, 1) & = & \mathbf{e}(1, 1, 0, 1) - \mathbf{F}_{1, d_1, u_1} = \langle 3, 2, 1, 0; 1, 1; 3; 3, 0, 0 \rangle \\
\mathbf{e}(3, 1, 0, 1) & = & \mathbf{e}(2, 1, 0, 1) - \mathbf{F}_{1, d_1, u_1} = \langle 2, 1, 0, 0; 1, 1; 3; 3, 0, 0 \rangle \\
& \cdot & \cdot \\
& \cdot & \cdot \\
& \cdot & \cdot \\
\mathbf{e}(5, 2, 3, 4) & = & \mathbf{e}(5, 2, 2, 4) - \mathbf{F}_{3, d_3, u_3} = \langle 0, 0, 0, 0; 0, 0; 0; 0, 0, 0 \rangle
\end{array}$$

Figure 2:  $\mathbf{e}(\mathbf{u})$  for  $\mathbf{e} = \langle 5, 4, 3, 1; 2, 2; 3; 4, 1, 1 \rangle$  and various  $\mathbf{u}$

where  $a = \sum_{i < r} m_i$  and  $b = \sum_{i > r} m_i$ . For vectors  $\mathbf{u} = \langle u_k \rangle_{k=1}^s$ ,  $\mathbf{v} = \langle v_k \rangle_{k=1}^s$  and  $\mathbf{w} = \langle w_k \rangle_{k=1}^s$  denote  $\mathbf{u}\mathbf{v} = \langle u_k^{v_k} \rangle_{k=1}^s$  and  $\mathbf{u}^{\mathbf{v}} = \prod_{k=1}^s u_k^{v_k}$ . Now let  $\mathbf{p}_i = \langle p_i \rangle_{j=1}^{m_i}$ ,  $\mathbf{p} = \langle \mathbf{p}_i \rangle_{i=1}^t$  and  $\mathbf{p}_0 = \langle p_i \rangle_{i=1}^t$ , and define  $\mathbf{n} = \langle n_i \rangle_{i=1}^t = \langle e_{i,1} \rangle_{i=1}^t$  and  $n = \sum_{i=1}^t n_i$ . Note that  $\mathbf{p}_0^{\mathbf{n}} = N(G)$  and  $\mathbf{p}^{\mathbf{e}} = |G|$ . We also write  $\mathbf{u} \leq \mathbf{v}$  when  $u_k \leq v_k$  for every  $k$ ,  $\mathbf{u} \equiv \mathbf{v} \pmod{\mathbf{w}}$  when  $u_k \equiv v_k \pmod{w_k}$  for every  $k$ , and  $\mathbf{u}\mathbf{v} = \mathbf{w}$  (or  $\mathbf{u} = \mathbf{w}/\mathbf{v}$ ) when  $u_k v_k = w_k$  for every  $k$ .

Finally, let  $\mathbf{e}(\mathbf{0}^m) = \mathbf{e}$ , and denote the  $k^{\text{th}}$  characteristic vector  $\chi_k$ , having all zeros with a single one in the  $k^{\text{th}}$  entry. For  $\mathbf{0}^m \leq \mathbf{u} \leq \mathbf{n}$  define

$$\mathbf{e}(\mathbf{u}) = \mathbf{e}(\mathbf{u} - \chi_i) - \mathbf{F}_{i, d_i, u_i}.$$

(Note that this definition is valid for every  $1 \leq i \leq t$ .) Figure 2 shows an

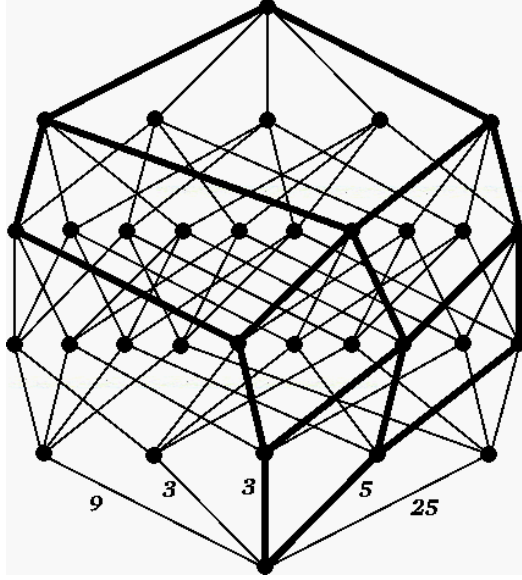


Figure 3:  $L(G)$  as a retract of  $\mathcal{B}^5(9, 3, 3, 25, 5)$  for the group  $G = \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_5$

example for these definitions. Note that we always have  $\mathbf{e}(\mathbf{n}) = \mathbf{0}^n$ .

## 2.4 Lattice Graph and Pebbling Number

Define the lattice  $L = L(G) = \prod_{i=1}^t P_{n_i+1}$  (the cartesian product of paths with  $n_i + 1$  vertices). Note that  $L$  is isomorphic to the divisor lattice of  $N = N(G) = \prod_{i=1}^t p_i^{e_{i,1}}$  (having height  $n = \sum_{i=1}^t e_{i,1}$ ) and label the vertices of  $L$  accordingly. Next consider an edge of  $L$  between vertex  $p_i^k q$  and vertex  $p_i^{k-1} q$ , where  $p_i \nmid q$ . Label such an edge by weight  $p_i^{d_{i,k}}$ .

Because  $L$  and its labelling is a retract (see Figure 3 for an example) of the  $n$ -dimensional boolean lattice  $\mathcal{B}^n(\mathbf{w})$ , having edge labels  $\mathbf{w} = \langle p_i^{d_{i,j}} \rangle_{i,j}$ , we have that the generalized (pebbling operations obey the edge labels) pebbling number  $\pi(L) = \pi(\mathcal{B}^n(\mathbf{w}))$ . (This is the same argument used in [7].) Notice



that

$$\pi(\mathcal{B}(\mathbf{w})) = \prod_{i=1}^t \prod_{j=1}^{n_i} p_i^{d_{i,j}} = \prod_{i=1}^t \prod_{j=1}^{m_i} p_i^{e_{i,j}} = |G| .$$

Given a sequence of elements of  $G$ ,  $(g_1, \dots, g_{|G|})$ , define a configuration by placing corresponding pebbles  $\{g_1\}, \dots, \{g_{|G|}\}$  on  $L$ , with pebble  $\{g_k\}$  on vertex  $|g_k| \in V(L)$ . Because  $\pi(L) = |G|$ , the configuration is solvable to the bottom vertex labelled 1. As was noted in [8],  $L$  is greedy, meaning that we may assume that every pebbling step moves toward the root 1.

We will now use the solution of the configuration to construct a subsequence  $(g_k)_{k \in K}$  that satisfies  $\sum_{k \in K} g_k = 0_G$  and  $\sum_{k \in K} 1/|g_k| \leq 1$ . (We will follow somewhat the structure of the argument presented in [8], with a few necessary tricks thrown in.)

## 2.5 Well Placed Pebbles

We now make several useful recursive definitions. For a pebble  $A$  define

- $\text{Set}(A) = \bigcup_{B \in A} \text{Set}(B)$ , where  $\text{Set}(\{g_k\}) = \{g_k\}$ ,
- $\mathbf{Val}(A) = \sum_{B \in A} \mathbf{Val}(B)$ , where  $\mathbf{Val}(\{g_k\}) = \mathbf{g}_k$ , and
- $\text{Ord}(A) = \sum_{B \in A} \text{Ord}(B)$ , where  $\text{Ord}(\{g_k\}) = 1/|g_k|$ .

Note that  $\mathbf{Val}(A) = \sum_{g \in \text{Set}(A)} \mathbf{g}$  and  $\text{Ord}(A) = \sum_{g \in \text{Set}(A)} 1/|g|$ . We say a pebble  $A$  is *well placed* at vertex  $\mathbf{p}^{\mathbf{u}}$  if

1.  $\mathbf{Val}(A) \equiv \mathbf{0}^m \pmod{\mathbf{p}^{\mathbf{e}(\mathbf{u})}}$  and
2.  $\text{Ord}(A) \leq 1/\mathbf{p}^{\mathbf{u}}$ .

Thus each pebble in the initial configuration is well placed.

We will interpret each pebbling step from  $x$  to  $y$  as follows: first remove a collection of pebbles  $A_1, A_2, \dots, A_s$  of the appropriate size (the edge weight of  $xy$ ) from  $x$ , then for some carefully chosen index set  $K_x \subseteq \{1, \dots, s\}$  place the new pebble  $A = \{A_k\}_{k \in K_x}$  on  $y$ . We will show that if each  $A_k$  is well placed at  $x$  then  $A$  is well placed at  $y$ . Any pebble  $A$  that is well placed at vertex  $1 = \mathbf{p}^{\mathbf{0}}$  yields the solution  $\text{Set}(A)$  to Theorem 2.

### 3 Proof of Theorem 2

For the purposes of notational readability, we will first give the proof of Theorem 2 in the case of  $p$ -groups. Once established, the general case will be straightforward.

#### 3.1 Characteristic $p$

Here we have  $t = 1$  so that  $i = 1$  always. For ease of notation we will simply drop the 1; thus  $G = \prod_{j=1}^m \mathbb{Z}_{p^{e_j}}$  for some prime  $p$ , multiplicity  $m$ , and exponents  $e_j$  ( $e_1 \geq \dots \geq e_m$ ). For  $\mathbf{e} = \langle e_j \rangle_{j=1}^m$  recall that  $\mathbf{p}^{\mathbf{e}} = \prod_{j=1}^m p^{e_j} = |G|$ .

**Lemma 4** *Theorem 2 holds for groups of the form  $G = \mathbb{Z}_p^m = \oplus_{j=1}^m \mathbb{Z}_p$ .*

*Proof.* This result will follow from Theorem 1. View  $G$  as the  $m$ -dimensional vector space over  $\mathbb{F}_p$ . Then assign to  $\mathbb{F}_p^m$  the natural correspondence with field  $\mathbb{F}_{p^m}$ , and partition  $\mathbb{F}_{p^m} - \{0\}$  into  $(p^m - 1)/(p - 1)$  lines of size  $p - 1$ .

$$\begin{array}{l}
\hline \mathbf{e} = \langle 5, 2, 2, 1 \rangle, \quad \mathbf{d} = \langle 4, 3, 1, 1, 1 \rangle \\
\hline
\mathbf{e}(0) = \langle 5, 2, 2, 1 \rangle \\
\mathbf{e}(1) = \langle 5, 2, 2, 1 \rangle - \mathbf{f}_4 = \langle 4, 1, 1, 0 \rangle \\
\mathbf{e}(2) = \langle 4, 1, 1, 0 \rangle - \mathbf{f}_3 = \langle 3, 0, 0, 0 \rangle \\
\mathbf{e}(3) = \langle 3, 0, 0, 0 \rangle - \mathbf{f}_1 = \langle 2, 0, 0, 0 \rangle \\
\mathbf{e}(4) = \langle 2, 0, 0, 0 \rangle - \mathbf{f}_1 = \langle 1, 0, 0, 0 \rangle \\
\mathbf{e}(5) = \langle 1, 0, 0, 0 \rangle - \mathbf{f}_1 = \langle 0, 0, 0, 0 \rangle
\end{array}$$

Figure 4:  $\mathbf{e}(u)$  for  $\mathbf{e} = \langle 5, 2, 2, 1 \rangle$  and  $u = 0, \dots, 5$

With  $p^m$  pebbles, none of which is at  $\mathbf{0}$  (otherwise we are done), the pigeonhole principle forces some line to have at least  $p$  pebbles. Since a line plus the origin forms the cycle  $\mathbb{Z}_p$ , Theorem 1 completes the proof.  $\square$

**Theorem 5** *Theorem 2 holds for groups of the form  $G = \oplus_{j=1}^m \mathbb{Z}_{p^{e_j}}$ .*

*Proof.* We use Lemma 4 to show that each pebbling step preserves the well placed property. Given a sequence of  $|G| = \mathbf{p}^{\mathbf{e}} = \prod_{j=1}^m p^{e_j}$  elements of  $G$  placed, as discussed in Section 2.5, corresponding pebbles on the lattice  $L = L(G) = P_{e_1+1}$ , having edge label  $p^{d_k}$  between vertices  $p^k$  and  $p^{k-1}$ , where  $\mathbf{d} = \langle d_k \rangle_{k=1}^{e_1}$  is the dual partition to  $\mathbf{e}$ . For  $r \geq 0$  recall that  $\mathbf{f}_r = \langle \mathbf{1}^r, \mathbf{0}^{n-r} \rangle$ . Let  $\mathbf{e}(0) = \mathbf{e}$ , and for  $0 < u \leq e_1$  define  $\mathbf{e}(u) = \mathbf{e}_{u-1} - \mathbf{f}_{d_u}$  (see Figure 4 for an example). recall that we always have  $\mathbf{e}(e_1) = \mathbf{0}^m$  because of the Ferrer's duality.

Given  $p^{d_u}$  well placed pebbles  $\{A_r\}_{r=1}^{p^{d_u}}$  on vertex  $p^u$ , we know that each  $\mathbf{Val}(A_r) \equiv \mathbf{0}^m \bmod \mathbf{p}^{e(u)}$  and each  $\text{Ord}(A_r) \leq 1/p^u$ . Consider, for each  $r$ ,  $B_r = \mathbf{Val}(A_r)/\mathbf{p}^{e(u)}$ . By Lemma 4 we can find a nonempty index set  $R$  so that for  $B = \{B_r\}_{r \in R}$  we have  $\mathbf{Val}(B) \equiv \mathbf{0}^m \bmod \mathbf{p}^{f_{d_u}}$  and  $\text{Ord}(B) \leq 1$ .

Now let  $A = \{A_r\}_{r \in R}$ . Then

$$\begin{aligned}
\mathbf{Val}(A) &= \sum_{r \in R} \mathbf{Val}(A_r) \\
&= \sum_{r \in R} \mathbf{p}^{e(u)} B_r \\
&= \mathbf{p}^{e(u)} \mathbf{Val}(B) \\
&\equiv \mathbf{0}^m \bmod \mathbf{p}^{e(u) + f_{d_u}} \\
&= \mathbf{0}^m \bmod \mathbf{p}^{e(u-1)}.
\end{aligned}$$

Also,  $\text{Ord}(A) = \sum_{r \in R} \text{Ord}(A_r) \leq |R|/p^u = 1/p^{u-1}$ . Hence  $A$  is well placed on vertex  $p^{u-1}$ .

Since the pebbling number guarantees that some pebble  $A$  reaches vertex  $1 = p^0$ , and since the previous argument ensures that  $A$  is well placed, we find, for some  $K \neq \emptyset$  that

$$\sum_{k \in K} g_k = \mathbf{Val}(A) \equiv \mathbf{0}^m \bmod \mathbf{p}^{e(0)} = \mathbf{0}^m \bmod \mathbf{p}^e = \mathbf{0}_G$$

(i.e.  $\sum_{k \in K} g_k = 0_G$ ) and

$$\sum_{k \in K} 1/|g_k| = \text{Ord}(A) \leq 1/\mathbf{p}^{\mathbf{0}} = 1.$$

### 3.2 General Case

As expected, the same proof carries through; only the notation generalizes. Given  $p_i^{d_i, u_i}$  well placed pebbles  $\{A_r\}_{r=1}^{p_i^{d_i, u_i}}$  on vertex  $\mathbf{p}^{\mathbf{u}}$ , we know that each  $\mathbf{Val}(A_r) \equiv \mathbf{0}^m \pmod{\mathbf{p}^{e(\mathbf{u})}}$  and each  $\text{Ord}(A_r) \leq 1/\mathbf{p}^{\mathbf{u}}$ . Consider, for each  $r$ ,  $\mathbf{B}_r = \mathbf{Val}(A_r)/\mathbf{p}^{e(\mathbf{u})}$ . By Lemma 4 we can find a nonempty index set  $R$  so that for  $B = \{\mathbf{B}_r\}_{r \in R}$  we have  $\mathbf{Val}(B) \equiv \mathbf{0}^m \pmod{\mathbf{p}^{\mathbf{F}_{i, d_i, u_i}}}$  and  $\text{Ord}(B) \leq 1$ .

Now let  $A = \{A_r\}_{r \in R}$ . Then

$$\begin{aligned} \mathbf{Val}(A) &= \sum_{r \in R} \mathbf{Val}(A_r) \\ &= \sum_{r \in R} \mathbf{p}^{e(\mathbf{u})} \mathbf{B}_r \\ &= \mathbf{p}^{e(\mathbf{u})} \mathbf{Val}(B) \\ &\equiv \mathbf{0}^m \pmod{\mathbf{p}^{e(\mathbf{u}) + \mathbf{F}_{i, d_i, u_i}}} \\ &= \mathbf{0}^m \pmod{\mathbf{p}^{e(\mathbf{u} - \chi_i)}}. \end{aligned}$$

Also,  $\text{Ord}(A) = \sum_{r \in R} \text{Ord}(A_r) \leq |R|/\mathbf{p}^{\mathbf{u}} = 1/\mathbf{p}^{(\mathbf{u} - \chi_i)}$ . Hence  $A$  is well placed on vertex  $\mathbf{p}^{\mathbf{u} - \chi_i}$ .

Since the pebbling number guarantees that some pebble  $A$  reaches vertex  $1 = \mathbf{p}^{\mathbf{0}}$ , and since the previous argument ensures that  $A$  is well placed, we find, for some  $K \neq \emptyset$  that

$$\sum_{k \in K} \mathbf{g}_k = \mathbf{Val}(A) \equiv \mathbf{0}^m \bmod \mathbf{p}^{e(\mathbf{0})} = \mathbf{0}^m \bmod \mathbf{p}^e = \mathbf{0}_G$$

(i.e.  $\sum_{k \in K} g_k = 0_G$ ) and

$$\sum_{k \in K} 1/|g_k| = \text{Ord}(A) \leq 1/\mathbf{p}^{\mathbf{0}} = 1.$$

## 4 Further Comments

For cyclic groups Theorem 2 is best possible. However, for other groups it is conceivable that shorter sequences of elements may suffice. It had been conjectured that  $D(G) = 1 + \sum_{i=1}^r (N_i - 1)$  for abelian  $G$  [29]. While this was shown true for groups of rank at most 2 and for  $p$ -groups, among other special cases, it has been shown false in general [14, 23]. One may ask for the generalized Davenport constant for the minimum length of a sequence required to force a zero-sum subsequence with the extra condition on its orders. It is also worthy of note that Theorem 2 holds for the nonabelian solvable group on 21 elements, as we verified by computer. It would be interesting to consider solvable groups in general, as the methods of this paper rely on the abelian property. However, if one considers subsets rather than subsequences then our method might retain some control over solutions.

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