Now suppose we are given a LOP in the following general form.

Problem 6.1.6

$$\begin{array}{llrcll}
\text{Max.} & z & = & \sum_{j=1}^{n} c_j x_j & & \\
\\
\text{s.t.} & & \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i & (i \in I_P) \\
& & \sum_{j=1}^{n} a_{ij}x_j & = & b_i & (i \in E_P) \\
\\
\& & & x_j & \geq & 0 & (j \in R_P)
\end{array}$$

Here, $I_P \subseteq \{1,\ldots,m\}$ identifies those (primal) constraints that use an *Inequality*, while $E_P = \{1,\ldots,m\} - I_P$ identifies those that use an *Equality*. For example, in Problem 6.1.1 we have $I = \{1,2,4\}$ and $E = \{3\}$. Likewise, $R_P \subseteq \{1,\ldots,n\}$ identifies those variables that are *Restricted* to be nonnegative, while $F_P = \{1,\ldots,n\} - R_P$ will denote those that are *Free*. We have $R = \{1,3\}$ and $F = \{2\}$ in Problem 6.1.1. Define the analogous sets I_D, E_D, R_D, F_D for dual LOPs, and then notice their values for Problem 6.1.3.

Problem 6.1.7

$$\begin{array}{llrcll}
\text{Min.} & w & = & \sum_{i=1}^{m} b_i y_i & & \\
\\
\text{s.t.} & & \sum_{i=1}^{m} a_{ij}y_i & \geq & c_j & (j \in I_D) \\
& & \sum_{i=1}^{m} a_{ij}y_i & = & c_j & (j \in E_D) \\
\\
\& & & y_i & \geq & 0 & (i \in R_D)
\end{array}$$

Workout 6.1.8 *Suppose that Problem 6.1.7 is the dual of Problem 6.1.6 (in other words the Weak Duality Inequality holds: every primal-feasible z and dual-feasible w satisfy $z \leq w$). Prove that*

- **(a)** $I_D = R_P$,
- **(b)** $E_D = F_P$,
- **(c)** $R_D = I_P$, *and*
- **(d)** $F_D = E_P$.

6.2 General Simplex and Phase 0

Let us return to an analysis of Problem 6.1.1. Below is its initial tableau (we have made note of the free variable x_2 as shown).

Tableau 6.2.1

$$\begin{array}{c}
\begin{array}{ccc} & F & \end{array} \\
\left[\begin{array}{rrr|rrrr|r}
7 & 6 & 2 & 1 & 0 & 0 & 0 & 419 \\
-5 & 3 & 8 & 0 & 1 & 0 & 0 & 528 \\
3 & -1 & 0 & 0 & 0 & 0 & 0 & 272 \\
0 & -9 & -4 & 0 & 0 & 1 & 0 & 168 \\
\hline
-36 & 31 & -37 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

Notice that one cannot write down the dictionary for Tableau 6.2.1 because only $4, 5, 6 \in \beta$ so far — the third constraint, because it is an equality, delivers no basic variable. Fear not, however: just pick one! In fact, since both phases of the Simplex algorithm include mechanisms for keeping basic variables from becoming negative, why not throw in a basic variable that you don't care whether it is negative or not? That is, let's make x_2 basic in the third constraint; i.e. pivot on the -1, obtaining the next tableau.

Tableau 6.2.2

$$
\begin{array}{c}
\begin{array}{cccccccc} & F & & & & & & \end{array} \\
\left[\begin{array}{rrr|rrrr|r}
25 & 0 & 2 & 1 & 0 & 0 & 0 & 2051 \\
4 & 0 & 8 & 0 & 1 & 0 & 0 & 1344 \\
-3 & 1 & 0 & 0 & 0 & 0 & 0 & -272 \\
-27 & 0 & -4 & 0 & 0 & 1 & 0 & -2280 \\
\hline
57 & 0 & -37 & 0 & 0 & 0 & 1 & 8432
\end{array}\right]
\end{array}
$$

Phase 0 In any LOP having equality constraints, the first part of what we will call **Phase 0** will be to *fill the basis*, choosing nonbasic variables (in least subscript order) to be basic in those constraints. As above, preference is given to free variables in these choices. In fact, after the basis is filled, any LOP having free variables may still have some that are nonbasic. It makes the same sense to put those in the basis, replacing basic restricted variables (both in least subscript order).

Workout 6.2.3 *Restate Problem 6.1.1 in standard form by solving for x_2 in constraint 3 and substituting its formula back into the other constraints and objective function. Compare your result with Tableau 6.2.2 and explain.*

Recall from Section 2.2 the notation $i \mapsto j$ that denotes the pivot operation which replaces the basic variable x_j by the parameter x_i. By $i \mapsto \emptyset$ we mean the Phase 0 pivot in which x_i replaces nothing because the current basis is not full.

Workout 6.2.4 *For each of the following instances of LOPs write the pending Phase 0 pivot $i \mapsto j$.*

(a) $n = 6$, $m = 5$, $I = \{1, 2, 4\}$, $R = \{1, 3, 4\}$ *and* $\beta = \{2, 7, 8, 9\}$.

(b) $n = 4$, $m = 9$, $I = \{1, 3, 4, 6, 7, 9\}$, $R = \{3, 4\}$ *and* $\beta = \{1, 2, 5, 6, 7, 8, 9\}$.

(c) $n = 6$, $m = 7$, $I = \{1, 3, 4, 7\}$, $R = \{2, 3, 6\}$ *and* $\beta = \{1, 3, 5, 7, 8, 9, 10\}$.

(d) $n = 3$, $m = 8$, $I = \{2, 5, 6, 7\}$, $R = \{\}$ *and* $\beta = \{1, 2, 3, 4, 5, 6, 7\}$.

After Phase 0 has been completed, we will assume that every free variable is in the basis, and that some basic variable is restricted (see Exercise 6.5.19). We now need to see the effect of free variables on Phases I and II. It is useful to think in terms of Workout 6.2.3, which suggests that a LOP with free variables is equivalent to a smaller LOP, derived from the first but without those free variables. In that smaller LOP all pivots are exchanges of restricted variables, and so then they should be in the original LOP. Let's see how this holds true.

In Phase I we first look for negative basic current values because they signal infeasibility. But now free variables can be ignored because their negativity is allowed. Thus Phase I will never kick a free variable out of a basis. In Phase II, once an incoming variable is found, the ratio calculations place upper bounds on its increasing value that are induced by preventing basic variables from becoming negative. But not caring that free variables go negative means that their ratios are irrelevant. Therefore Phase II will never kick out a free variable either. One can keep floating in the back of one's mind, then, the guideline of the general Simplex algorithm that all free variables go in to the basis and never come out.

It's time to continue with Tableau 6.2.2 from Problem 6.1.1. Phase I works as usual in this case since -2280 is the most negative current basic value among restricted variables. Hence we pivot $1 \mapsto 6$.

Tableau 6.2.5

$$\begin{array}{c} \begin{array}{cccccccc} & F & & & & & & \end{array} \\ \left[\begin{array}{ccc|cccc|c} 0 & 0 & -46 & 27 & 0 & 25 & 0 & -1623 \\ 0 & 0 & 200 & 0 & 27 & 4 & 0 & 27168 \\ 0 & 27 & 12 & 0 & 0 & -3 & 0 & -504 \\ 27 & 0 & 4 & 0 & 0 & -1 & 0 & 2280 \\ \hline 0 & 0 & -1227 & 0 & 0 & 57 & 27 & 97704 \end{array}\right] \end{array}$$

The next pivot is likewise as usual: $3 \mapsto 4$.

Tableau 6.2.6

$$\begin{array}{c} \begin{array}{cccccccc} & F & & & & & & \end{array} \\ \left[\begin{array}{ccc|cccc|c} 0 & 0 & 46 & -27 & 0 & -25 & 0 & 1623 \\ 0 & 0 & 0 & 200 & 46 & 192 & 0 & 34264 \\ 0 & 46 & 0 & 12 & 0 & 6 & 0 & -1580 \\ 46 & 0 & 0 & 4 & 0 & 2 & 0 & 3644 \\ \hline 0 & 0 & 0 & -1227 & 0 & -1039 & 46 & 240215 \end{array}\right] \end{array}$$

Finally we arrive at Phase 0 — Tableau 6.2.6 is feasible because x_2 is free. At this stage, x_4 enters β, and while the x_2 b-ratio isn't considered because it is negative, it wouldn't be in any case because x_2 is free. Now we pivot $4 \mapsto 5$ to reach the optimal tableau, with $\mathbf{x}^* = (12864, -15808, 27168 \mid 34264, 0, 0)^\mathsf{T}/200$ and $z^* = 1958368/200$.

Tableau 6.2.7

$$\begin{array}{c} \begin{array}{cccccccc} & F & & & & & & \end{array} \\ \left[\begin{array}{ccc|cccc|c} 0 & 0 & 200 & 0 & 27 & 4 & 0 & 27168 \\ 0 & 0 & 0 & 200 & 46 & 192 & 0 & 34264 \\ 0 & 200 & 0 & 0 & -12 & -24 & 0 & -15808 \\ 200 & 0 & 0 & 0 & -4 & -8 & 0 & 12864 \\ \hline 0 & 0 & 0 & 0 & 1227 & 604 & 200 & 1958368 \end{array}\right] \end{array}$$

Workout 6.2.8 *Write an outline for all Phases of the general Simplex algorithm.*

Theorem 6.2.9 *Let P be a LOP in general form (Problem 6.1.6). Then* General Fundamental Theorem

(a) *P is either infeasible, unbounded, or it has a maximum;*

(b) *if P has a feasible solution then it has a basic feasible solution; and*

(c) *if P has an optimal solution then it has a basic optimal solution.*

Workout 6.2.10 *Prove Theorem 6.2.9.*