s-Overlap Cycles for Permutations

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Abstract

The goal of this paper is to solve Problem 481 from the list of research problems in the special issue of Discrete Mathematics dedicated to the Banff International Research Station workshop on "Generalizations of de Bruijn Cycles and Gray Codes in 2004. Overlap cycles are generalizations of de Bruijn cycles and Gray codes that were introduced originally in 2010 by Godbole et al. In this paper we prove that s-overlap cycles for k-permutations of $[n]$ exist for all $k < n$.

1   Introduction

Many applications require a set of combinatorial objects to be ordered in a specific manner. One such ordering is an **s-overlap cycle**, or **s-ocycle**. An s-ocycle is an ordering of a set of objects $\mathcal{C}$, each of which has size n and is represented as a string. The ordering requires that object $b = b_1 b_2 \ldots b_n$ follow object $a = a_1 a_2 \ldots a_n$ only if $a_{n-s+1} a_{n-s+2} \ldots a_n = b_1 b_2 \ldots b_s$. Ocycles were introduced by Godbole, Knisley, and Norwood in 2010 [3].

For the reader familiar with universal cycles, we note that an $(n-1)$-ocycle is a universal cycle. Universal cycles, or ucycles, were introduced in 1992 by Chung, Diaconis, and Graham [1], and have been studied extensively ever since, for example over block designs [2]. Universal cycles for permutations have long been an interesting research problem. While ucycles using the standard permutation representation are impossible, it has been shown that when $n + 1$ symbols are used to represent permutations of $[n]$, ucycles are possible [5, 8]. Another alternative is to consider k-permutations, as done by Jackson [7]. Jackson proved the existence of ucycles for k-permutations of $[n]$ with $3 \leq k < n$. As a natural extension to the problem of finding ucycles for permutations, we consider ocycles.

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12345, 34512, 51234, 23415, 41523, 52314, 31425, 42513, 51324, 32415, 41532, 53214, 21435,
43512, 51243, 24315, 31524, 52413, 41325, 32514, 51423, 42315, 31542, 54213, 21345, 34521,
52134, 13425, 42531, 53124, 12435, 43521, 52143, 14325, 32541, 54123, 12354, 35412, 41235,
23514, 51432, 43215, 21534, 53412, 41253, 25314, 31452, 45213, 21354, 35421, 42135, 13524,
52431, 43125, 12534, 53421, 42153, 15324, 32451, 45132, 13245, 24513, 51342, 34215, 21543,
54312, 31245, 24531, 53142, 14235, 23541, 54132, 13254, 25413, 41352, 35214, 21453, 45312,
31254, 25431, 43152, 15243, 24351, 35124, 12453, 45321, 32145, 14523, 52341, 34125, 12543,
54321, 32154, 15432, 43251, 25134, 13452, 45231, 23145, 14532, 53241, 24153, 15342, 34251,
25143, 14352, 35241, 24135, 13542, 54231, 23154, 15423, 42351, 35142, 14253, 25341, 34152,
15234, 23451, 45123.

Figure 1: List of Permutations of [5] in a 3-Ocycle

For example, Figure 1 gives a 3-ocycle for permutations of [5], which is the largest allowable
overlap for permutations of [5]. Omitting repeated elements and writing it as a string, the
corresponding 3-ocycle is:

$$1234512341523 \cdots 34152345.$$

In studying ocycles, it is often useful to consider the **overlap graph** for the set of
objects. This graph has objects represented as vertices. An edge from object $a_1 a_2 \ldots a_n$
to object $b_1 b_2 \ldots b_n$ exists if and only if $a_{n-s+1} a_{n-s+2} \ldots a_n = b_1 b_2 \ldots b_s$. In this transition
graph, we are looking for a Hamilton cycle, which will correspond to an s-ocycle. In [6], the
following question was posed by Anant Godbole, which we are able to answer affirmatively
in Corollary 7.

Question 1. ([6], Problem 481.) *Let $P(k, n, s)$ be the overlap graph for k-permutations of
$[n] = \{1, 2, \ldots, n\}$. Is $P(k, n, s)$ hamiltonian whenever $k < n$?*

2 Results

We begin with a lemma from our previous paper on universal cycles for weak orders.

Lemma 2. ([4], Lemma 4.3) *Let $n, s \in \mathbb{Z}^+$ with $gcd(n, s) = 1$ and $1 \leq s \leq n - 2$, and let
M be a multiset of size n. Then there is an s-ocycle for all permutations of M.*

We can actually improve this and do even better for $s < \frac{n}{2}$.

Lemma 3. *Let $n, s \in \mathbb{Z}^+$ with $1 \leq s < \frac{n}{2}$. Let M be a multiset of length n. Then there is
an s-ocycle for all permutations of M.*

Proof. If $\gcd(s, n) = 1$, then we are done by Lemma 2. If $\gcd(s, n) > 1$, then $\gcd(s, n-1) = 1$ and so we can find an s-ocycle for these permutations using the prefix representation. The **prefix representation** of a word omits the last letter, as it is completely determined by the first $n - 1$ letters. Normally the prefix representation omits the last letter of each word, but it is clear to see that as long as we are consistent over the entire set, we can choose any index to remove. In this case, we will omit the letter at index $\lfloor \frac{n}{2} \rfloor$. Now we find any s-ocycle for this set of shortened words, and create an s-ocycle for the original, unshortened set of permutations by adding the missing letter back in. This missing letter appears only on the edges in the transition graph and not on the vertices, and so this creates a valid s-ocycle for permutations of M. □

The previous lemmas immediately give us the following partial solution when we consider s-ocycles on permutations.

Result 4. *Let $n, s \in \mathbb{Z}^+$ with $n \geq 2$. If either (1) $1 \leq s < \frac{n}{2}$, or (2) $\gcd(s, n) = 1$ with $\frac{n}{2} \leq s < n - 1$, then there exists an s-ocycle on the set of permutations of $[n]$.*

Proof. This is merely a concise restatement of Lemmas 2 and 3 with $M = \{1, 2, \ldots, n\}$. □

This result gives us an alternative modification for dealing with the problem of finding ucycles for permutations. Instead of increasing the alphabet size from n to $n + 1$ as done in [5, 8], an alternative could be to consider the largest possible overlap, with $s = n - 2$ being the best alternative to a ucycle. Note that this result also shows that under the given conditions, all permutations of an $[n]$-set are connected within a larger transition graph. We can extend the ocycle result to k-permutations of $[n]$ to partially answer Question 1.

Result 5. *Let $n, s, k \in \mathbb{Z}^+$ with $1 \leq k < n$. If either (1) $1 \leq s < \frac{k}{2}$, or (2) $\gcd(s, k) = 1$ with $\frac{k}{2} \leq s < k - 1$, then there exists an s-ocycle on the set of k-permutations of $[n]$.*

Proof. First, construct the transition graph with vertices of length s (s-prefixes of k-permutations) and edges representing k-permutations. We allow an edge from vertex u to vertex v if and only if u is an s-prefix and v is an s-suffix for some k-permutation. We will show that this graph is balanced and weakly connected, and thus is eulerian. From an Euler tour, we can find an s-ocycle.

Define the minimum k-permutation $v = 123\ldots k$, and the minimum vertex, v^{s-}, in the transition graph to be the s-prefix of v. Let $w^{s-} = w_1 w_2 \ldots w_s$ be the prefix of an arbitrary k-permutation $w_1 w_2 \ldots w_k$. By Result 4, all permutations of a k-set are connected under our hypthoses, so we may assume that w is ordered as $w_1 < w_2 < \cdots < w_k$. We will show that there is a path in the graph from w^{s-} to v^{s-}. Define $D = \{w_1, w_2, \ldots, w_k\} \setminus \{1, 2, 3, \ldots, k\}$ and $\overline{D} = \{1, 2, 3, \ldots, k\} \setminus \{w_1, w_2, \ldots, w_k\}$.

Note that if $D = \emptyset$, then w is a permutation of v and so by the comments following Result 4, we know that there exists a path. For $D \neq \emptyset$, we choose d letters $a_1, a_2, \ldots, a_d \in \overline{D}$, where $d = \min\{k - s, |\overline{D}|\}$. If $d < k - s$, then we may also select

letters $a_{d+1}, a_{d+2}, \ldots, a_{k-s} \in \{w_{s+1}, w_{s+2}, \ldots, w_k\}$. Now in our graph we follow the edge corresponding to the k-permutation $w_1 w_2 \ldots w_s a_1 a_2 \ldots a_{k-s}$. By following this edge, we have found a k-permutation with more letters in common with v than w did. Since Result 4 implies that all permutations of a k-set are connected in the transition graph, we may arrange this string in increasing order, and by repeating this procedure we will eventually find a k-permutation that is simply a permutation of v, at which point we are done.

Since we have shown that the graph is connected, we need only show that the graph is balanced in order to prove that an Euler tour exists. However it is clear that the graph is balanced, as any prefix of a k-permutation is also a suffix of a k-permutation. □

We now prove our main result, which will provide a complete solution to Question 1.

Result 6. *For all $n, s, k \in \mathbb{Z}^+$ with $1 \le s < k < n$, there is an s-ocycle for k-permutations of $[n]$.*

Proof. We construct the standard transition graph G where vertices of length s correspond to s-prefixes of k-permutations of $[n]$, and edges correspond to k-permutations of $[n]$. If we can show that this graph is eulerian, then we have shown that there exists an s-ocycle for k-permutations of $[n]$. First, we note that since any s-prefix of a k-permutation is also an s-suffix of a k-permutation, the graph is balanced. All that remains is to show that the graph is connected.

Define the minimum vertex $v^{s-} = 12 \ldots s$, and let $w^{s-} = w_1 w_2 \ldots w_s$ be an arbitrary vertex in the graph, which we assume to be an s-prefix of the k-permutation $w = w_1 w_2 \ldots w_k$. We will frequently refer to **rotations** of a vertex. This is defined as following the edges of the cycle corresponding to rotations of a k-permutation that the vertex is an s-prefix for.

We next compare w^{s-} with v^{s-}. Let i be the first index in which $w_i \neq v_i$ and let $g = \gcd(s, k)$. If $g = 1$, we are done by Result 5. Otherwise, we note that rotations of w partition the string into blocks of length g. All addition in indices will be modulo k. We have two cases.

Case 1: If $i \notin \{w_{i+1}, w_{i+2}, \ldots, w_k\}$:

Rotate w so that w_i is in the first block. This means that we are considering some vertex
$$w_{i-t} w_{i-t+1} \ldots w_{i-t+s-1}$$
with $t \le g - 1$, or $i \in [i - t, i - t + g - 1]$. Follow the edge out of this vertex that corresponds to a rotation of w. This takes us to the vertex
$$w_{i-t+k-s} w_{i-t+k-s+1} \ldots w_{i-t+k-1}.$$

Next we follow the backwards edge corresponding to the k-permutation
$$w_{i-t} w_{i-t+1} \ldots w_{i-1}(i) w_{i+1} \ldots w_{i-t+k-1}.$$

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Now we are at the vertex

$$w_{i-t}w_{i-t+1}\ldots w_{i-1}(i)w_{i+1}\ldots w_{i-t+s-1}.$$

Finally we follow rotations of this vertex to end at the vertex

$$12\ldots(i)w_{i+1}w_{i+2}\ldots s.$$

This vertex is closer to the minimum vertex since the first i letters agree. Repeating this procedure, we will eventually arrive at the minimum vertex.

Case 2: If $i \in \{w_{i+1}, w_{i+2}, \ldots, w_k\}$:

Rotate w so that i is in the first block. We are now considering some vertex

$$a_1 a_2 \ldots a_s$$

with $i \in \{a_1, a_2, \ldots, a_g\}$, so assume that $i = a_j$. Note that

$$|[n] \setminus \{w_1, w_2, \ldots, w_k\}| > 1,$$

so choose some x in this set. We follow the edge from $a_1 a_2 \ldots a_s$ corresponding to a rotation of w

$$a_1 a_2 \ldots a_{j-1}(i)a_{j+1}\ldots a_k.$$

This takes us to the vertex

$$a_{k-s+1}a_{k-s+2}\ldots a_k,$$

which does not contain the letter i. From this vertex we follow the backward edge

$$a_1 a_2 \ldots a_{j-1}(x)a_{j+1}\ldots a_k.$$

Note that i does not appear in this edge, so we can go to Case (1).

When we have finished, we will have arrived at the minimum vertex. Thus the graph is weakly connected, and so is eulerian. □

Finally, the following answer to Question 1 is an obvious Corollary to Result 6.

Corollary 7. *The permutation overlap graph $P(k, n, s)$ is hamiltonian whenever $k < n$.*

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