Domination Cover Pebbling: Graph Families

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Abstract

Given a configuration of pebbles on the vertices of a connected graph G, a *pebbling move* is defined as the removal of two pebbles from some vertex, and the placement of one of these on an adjacent vertex. We introduce the notion of domination cover pebbling, obtained by combining graph cover pebbling ([2]) with the theory of domination in graphs ([3]). The domination cover pebbling number, $\psi(G)$, of a graph G is the minimum number of pebbles that must be placed on

$V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a dominating set of G, regardless of the initial configuration of pebbles. We discuss basic results and determine $\psi(G)$ for paths, cycles and complete binary trees.

1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks and called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hurlbert, who published a survey of pebbling results in [4]. There have been many developments since Hurlbert's survey appeared; some of these are described in this paper.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected. A pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles pebbles on an adjacent vertex. Define the pebbling number, $\pi(G)$ to be the minimum number of pebbles such no matter what their initial configuration, it is possible to move to any root vertex v in G after a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original initial configuration.

In this paper, we will combine two ideas, *cover* pebbling ([2]) and domination ([3]), to introduce a new graph invariant called the domination cover pebbling (DCP) number of a graph, denoted by $\psi(G)$. Recall that a set of vertices D in G is a dominating set if every vertex in G is either in D or adjacent to some element in D. The cover pebbling number, $\lambda(G)$ is defined as the minimum number of pebbles required such that given any initial configuration of at least $\lambda(G)$ pebbles, it is possible to make a series of pebbling moves to place at least one pebble on *every* vertex of G. The domination cover pebbling number of a graph G, proposed by A. Teguia, is the minimum number of pebbles required so that any initial configuration of pebbles can be transformed by a sequence of pebbling moves so that the set of vertices that contain pebbles form a dominating set S of G. The motivation for our definition comes from a hypothetical situation in which one wishes to transport monitors along the edges of a network that could ultimately "watch" each vertex – but half the devices are lost during each move. The pebbles may be placed on any of the vertices of G, and S depends, in general, on the initial configuration – most importantly, however, S need not equal a minimum dominating set. Graphs can be easily constructed to illustrate these facts. Consider the configurations of pebbles on P_4, the path on four vertices, as shown in Figure 1: For the graph on the left, we make pebbling moves so that

Figure 1: An example where two different initial configurations produce two different domination cover solutions.

the first and third vertices (from left to right) form the vertices of the dominating set. However, for the graph on the right, we make pebbling moves so that the second and fourth vertices are selected to be the vertices of the dominating set. In some cases, moreover, it takes more than the minimum dominating set of vertices to form the minimal domination cover solution. For example, in Figure 2 below we consider the case of the binary tree with height two, where the minimum dominating set has two vertices, but the minimal dominating set when creating a domination cover solution has three vertices. This corresponds to several possible starting configurations, e.g., the configuration as pictured; or one with a pebble at the leftmost bottom level vertex and 4 pebbles at the root; or one with 1 and 10 pebbles at the leftmost and rightmost bottom level vertices respectively.

The above two facts constitute the main reason why domination cover pebbling is nontrivial. We refer the reader to [3] for additional exposition on domination in graphs.

The paper is organized as follows. First, we present some basic results about domination cover pebbling. The remainder of the paper will consist of proofs that determine the domination cover pebbling number of P_n, the path on n vertices, C_n, the cycle graph on n vertices, and B_n, the complete

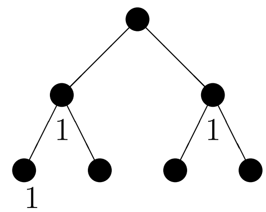

Figure 2: A reachable minimal configuration of pebbles on B_2 that forces a domination cover solution.

binary tree with height n. A companion paper by Watson and Yerger ([5]) examines the relation between $\psi(G)$ and structural properties of the graph.

2 Preliminary Results

We begin by determining the domination cover pebbling number for some families of graphs.

Theorem 1 *For the complete graph K_n on n vertices, $\psi(K_n) = 1$.*

This result is obvious since placing a pebble on any vertex dominates K_n.

Theorem 2 *For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{c_1,c_2,\ldots,c_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively. Then, for $s_1 \geq 3$, $\psi(K_{c_1,c_2,\ldots,c_r}) = s_1$. If $s_1 = 2$, $\psi(K_{c_1,c_2,\ldots,c_r}) = 3$.*

Proof First, the configuration with one pebble on all but one of the vertices in c_1 does not produce a domination cover solution. So $\psi(K_{c_1,c_2,\ldots,c_r}) > s_1 - 1$. Notice next that if there are pebbles on vertices in two different $c_i's$, the configuration contains a domination cover pebbling. Thus, any pair of pebbles on a vertex along with another pebbled vertex can force a domination cover pebbling. So if there are s_1 pebbles, the only configuration that has not been considered is the one with one pebble on every vertex in a vertex class that contains s_i vertices, but this also forces a domination cover pebbling. Hence, $\psi(K_{c_1,c_2,\ldots,c_r}) = s_1$; notice how we used the condition $s_1 \geq 3$. $\square$

For the next theorem we define the wheel graph, denoted W_n, to be the graph with $V(W_n) = h, v_1, v_2, \ldots, v_n$, where h is called the hub of W_n, and $E(W_n) = C_n \cup \{hv_1, hv_2, \ldots, hv_n\}$, where C_n denotes the cycle graph on n vertices.

Theorem 3 *For $n \geq 3$, $\psi(W_n) = n-2$.*

Proof

First, $\psi(W_n) > n-3$ because placing one pebble on each of $n-3$ consecutive outer vertices leaves a vertex of W_n undominated. If there is a pair of pebbles on any vertex, move it to the center, and the domination is complete. Likewise, if there is a pebble in the hub vertex, W_n is dominated. Thus, consider all configurations containing pebbled vertices that each contain only one pebble. If there are $n-2$ vertices containing pebbles, the two non-pebbled outer vertices are forced to be dominated since there are only 3 vertices in all of W_n that contain no pebbles. Therefore, $\psi(W_n) = n-2$. $\square$

3 Domination Cover Pebbling for Paths

Theorem 4

$$\psi(P_n) = 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor,$$

for $n \geq 3$, where $n-2 = \alpha_n + 3k_n \equiv \alpha_n \pmod 3$.

Proof Let $V = V(P_n) = \{v_1, v_2, \ldots, v_n\}$ with $E(P_n) = \{v_1v_2, \ldots v_{n-1}v_n\}$. Consider the configuration where all pebbles are placed on v_1. We need at least 2^{n-2} pebbles to dominate v_n. Likewise, we need at least $2^{n-2} + 2^{n-5} + 2^{n-8} + \cdots + 2^{\alpha_n}$ pebbles to dominate $\{v_n, v_{n-1}, \cdots, v_{\alpha_{n+1}}\}$. If $\alpha_n = 0$ or 1, then we have already dominated P_n. Otherwise, $\alpha_n = 2$ and we need one more pebble on either v_1 or v_2 to dominate P_n. Thus, under this configuration,

$$\begin{aligned} \psi(P_n) &\geq 2^{n-2}\sum_{i=0}^{k_n}\frac{1}{8^i} + \left\lfloor \frac{\alpha_n}{2} \right\rfloor \\ &= 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor, \end{aligned}$$

since $\lfloor \frac{\alpha_n}{2} \rfloor = 0$ for $\alpha_n = 0$ or 1 and $\lfloor \frac{\alpha_n}{2} \rfloor = 1$ for $\alpha_n = 2$.

We now use induction to show that

$$\psi(P_n) \leq 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor.$$

The assertion is clear for $n = 3$. Therefore, we assume it is true for all P_m, where $3 \leq m \leq n-1$. Consider an arbitrary configuration of P_n having $2^{n+1}\left((1-8^{-(k_n+1)})/7\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor$ pebbles. Clearly, we can cover dominate $\{v_{n-2}, v_{n-1}, v_n\}$ in a finite number of moves with 2^{n-2} pebbles or less. Thus, we need to dominate P_{n-3} with the remaining

$$2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor - 2^{n-2} = 2^{(n-3)+1}\left(\frac{1-8^{-(k_{n-3}+1)}}{7}\right) + \left\lfloor \frac{\alpha_{n-3}}{2} \right\rfloor$$

pebbles, since $\forall n$, $k_n = k_{n-3} + 1$ and $\alpha_n = \alpha_{n-3}$. This number of pebbles is enough to dominate P_{n-3} by hypothesis. Thus,

$$\psi(P_n) \leq 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor,$$

completing the proof. □

4 Domination Cover Pebbling for Cycles

We begin by proving that placing all the pebbles on one vertex is the "worst case" configuration that determines the domination cover pebbling number.

Lemma 5 *The value of $\psi(C_n)$ is attained when the original configuration consists of placing all the pebbles on a single vertex.*

Proof The proof relies strongly on the fact that the underlying graph is C_n, and is by contradiction. Assume first that the worst configuration consists of more than one set of consecutively pebbled vertices ("islands"). Consider the cardinality of any such set. This must be at most two, for, were it to be three or more, one could move the pebbles to the inner one or two vertices, thereby causing a larger number of pebbles to be needed to dominate – a contradiction. Thus each such "island" must have at most two vertices. Now consider the effect of moving all the pebbles onto a single such island. Once again one reaches a contradiction since one would now require more pebbles than $\psi(C_n)$ to cover dominate the graph. So we must then have the worst configuration consisting of a single island of two pebbled vertices. Clearly, the worst case is placing $\psi - 1$ pebbles on one vertex, say v_1, and a single pebble on the other vertex, say v_2, since it would now cost more pebbles to reach the v_2 side of the cycle. Had, however, all the pebbles been on v_1, we would need at least two more pebbles to dominate the other vertex closest to v_2, raising a contradiction. The statement follows. $\square$

Since placing all the pebbles on a single vertex is the worst case, we now determine the value of $\psi(C_n)$.

Theorem 6 *Let C_n be a cycle on n vertices. If $n = 2m-1, m \geq 2$,*

$$\psi(C_n) = 2^{m+2}\left(\frac{1-8^{-(k_m+1)}}{7}\right) + \phi_1(m)$$

and if $n = 2m-2, m \geq 3$,

$$\psi(C_n) = 2^{m+1}\left(\frac{1-8^{-(k_m+1)}}{7}\right) + 2^m\left(\frac{1-8^{-(k_{m-1}+1)}}{7}\right) + \phi_2(m),$$

where $\phi_1(m) = 2\lfloor \alpha_m/2 \rfloor - |\alpha_m - 1|$, $\phi_2(m) = \lfloor \alpha_m/2 \rfloor + \lfloor \alpha_{m-1}/2 \rfloor - |\alpha_m - 1||\alpha_{m-1} - 1|$, $m - 2 \equiv \alpha_m \pmod 3$, and $m - 2 = \alpha_m + 3k_m$.

Proof By Lemma 5, we assume all $\psi(C_n)$ pebbles are on $v_1 \in C_n$. If $n = 2m-1$, there are two identical m paths to cover. We can cover these with $2\psi(P_m)$ pebbles. We notice that v_1 may be in both dominating sets; $|\alpha_m - 1| = 1$ if v_1 is double counted. If $n = 2m-2$, there are two paths $P_1, P_2 \in C_n$ with $m - 1 = |P_2| = |P_1| - 1$. Thus we can cover these two paths with $\psi(P_m)+\psi(P_{m-1})$ pebbles. Likewise in this case, we may have double-counted vertex v_1; $|\alpha_m - 1||\alpha_{m-1} - 1| = 1$ in these cases, i.e. $\alpha_m \equiv 0 \pmod 3$; $\alpha_{m-1} \equiv 2 \pmod 3$. Thus we compute the domination cover pebbling number as follows. When $n = 2m-1$

$$\begin{aligned}
\psi(C_n) &= 2\psi(P_m) - |\alpha_m - 1| \\
&= 2^{m+2}((1-8^{-(k_m+1)})/7) + 2\lfloor \alpha_m/2 \rfloor - |\alpha_m - 1|
\end{aligned}$$

and if $n = 2m-2$,

$$\begin{aligned}
\psi(C_n) &= \psi(P_m) + \psi(P_{m-1}) - |\alpha_m - 1||\alpha_{m-1} - 1| \\
&= 2^{m+1}((1-8^{-(k_m+1)})/7) + 2^m((1-8^{-(k_{m-1}+1)})/7) \\
&\quad + \lfloor \alpha_m/2 \rfloor + \lfloor \alpha_{m-1}/2 \rfloor - |\alpha_m - 1||\alpha_{m-1} - 1|,
\end{aligned}$$

as asserted. $\square$

5 Binary Trees

In this section, we will compute the domination cover pebbling number for the family of complete binary trees. Recall that a complete binary tree, denoted by B_n, is a tree of height n, with 2^i vertices at distance i from the root. Each vertex of B_n has two "children", except for the set of 2^n vertices that are distance n away from the root, none of which have children. The root will be denoted by $\rho = \rho_n$

Theorem 7 $\psi(B_0) = 1, \psi(B_1) = 2, \psi(B_2) = 11, \psi(B_3) = 81, \psi(B_4) = 609.$

Proof The fact that $\psi(B_0) = 1$ and $\psi(B_1) = 2$ are obvious. We next show that $\psi(B_2) = 11$, as predicted by the general formula of Theorem 8. In Figure 3, we exhibit a configuration of 10 pebbles on B_2 that does not force a domination cover solution.

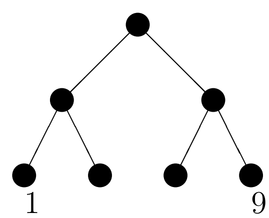

Figure 3: A configuration of 10 pebbles on B_2 that does not force a domination cover solution.

We will now show that $\psi(B_2) \leq 11$. Arbitrarily place 11 pebbles on B_2. Consider the following three subcases based on the number of pebbles on each of the two B_1's connected to the root of B_2.

Case 1: Suppose there are at least two pebbles on each of the two B_1's. It takes at most two pebbles for each of the B_1's to be dominated. Hence, after dominating each of the B_1's there are seven pebbles left. If there is a pebble on either of the two root vertices of the two disjoint copies of B_1, then we have dominated the root of B_2. Otherwise, it is always possible to move a pebble to the root of one of the B_1's, thus dominating the root ρ_2. This process induces a domination cover solution of B_2, completing this case.

Case 2: Suppose that neither B_1 contains two or more pebbles. Then there are at least 9 pebbles on the root of B_2. Pebble the root of each of the B_1's, and this case is complete.

Case 3: Suppose that one copy of B_1 contains two or more pebbles, call it B_1^*, and the other copy does not. Then all of the pebbles on B_1^* except for two can be used, together with any pebbles already on ρ_2, to place two pebbles on ρ_2, which can be used to dominate the other B_1.

We now show that $\psi(B_3) = 81$. First, we have constructed in Figure 4 a configuration of 80 pebbles that does not produce a domination cover pebbling.

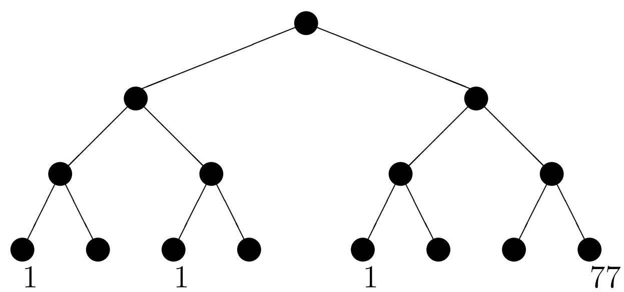

Figure 4: A configuration of 80 pebbles on B_3 that does not force a domination cover solution.

Now suppose that we are given a configuration of 81 pebbles on B_3. We wish to force a domination cover solution on B_3. If there are fewer than 11 pebbles on each of the two disjoint B_2 subtrees in B_3, then we can use 17 of the 61+ pebbles on the root vertex to produce a domination cover solution. If there are at least 11 pebbles on both of the disjoint B_2 subtrees, then we can dominate the root vertex with the 59 remaining pebbles as follows: One subtree must contain 30 of these "extra" pebbles, and thus by the pigeonhole principle one of the 4 paths leading from ρ_3, the root of B_3, to the bottom of the tree must have 8 pebbles on it, enough to send a pebble to the root, since the pebbling number $\pi(P_4)$ of the 4-path (3 edges) is 8.

Next, consider the case when only one of the two disjoint B_2 subgraphs, call it B_2^*, contains at least 11 pebbles. There are at most 70 pebbles somewhere on the graph that must now be succesfully used to dominate the other B_2, call it B_2', and ρ_3. Our strategy will be to move as many pebbles as possible from B_2^* to ρ_3 while still leaving B_2^* dominated. The pebbles placed on ρ_3 in this fashion will then be used to reach the two "middle" vertices in B_2' so as to dominate it. Notice that two single pebbles on the bottom row of

B_2', each with a different parent, does not decrease the number of pebbles required to dominate B_2' when there are no other pebbles on B_2' and all pebbles used in the domination emanate at the root. Also, any preexisting pebbles at ρ_3 [or any pebbles on B_2' other than the above-mentioned two] only make our strategy easier to implement, so assume that 68 of the extraneous pebbles are on B_2^* and the other two on the bottom row of B_2' as specified above.

Call the vertices of B_2^* a (its root); b_1, b_2 (the "middle vertices"); and c_1, c_2, c_3, c_4 (the bottom vertices). We thus need to force 9 pebbles onto ρ_3, failing which we need 8 pebbles on ρ_3 and a pebble on a. Variations of the argument that follows will be used throughout this paper. In order to accomplish our task, we will use (in addition to the 68 extraneous pebbles) the 11 pebbles "reserved" to dominate B_2^*. Now each pebble sent to ρ_3 causes a net reduction of at most 8 pebbles and a pebble sent to a causes at most 4 pebbles to be lost. Since $9 \times 8 = 72 > 68$, it appears that we can't always send 9 pebbles to ρ_3, so let's try to send 8 to the root and one to a. We claim and prove next that *we can send a pebble to ρ_3 as long as B_2^* has a total of 18 or more pebbles on it.* If there are 18 pebbles on B_2^*, one "$a - b - c$" path must contain 5 pebbles, say $a - b_1 - c_1$. The maximum possible number of pebbles on this path is 7, or else we could send a pebble to the ρ_3. If there are exactly 5 pebbles on $a - b_1 - c_1$ (13 left over), one of the remaining 4 vertices must have 4 pebbles on it, one of which can reach a. Another pebble can be put on a using the 5 pebbles on the $a-b-c$ path. We can reach ρ_3. If there are 6 pebbles on the $a - b - c$ path, there is one vertex with 3 pebbles. This must be c_3 or c_4, say c_3, which must, moreover, have *exactly* 3 pebbles (otherwise 2 pebbles can be placed on a.) The next vertex guaranteed to have 3 pebbles on it by the pigeonhole principle can be checked to lead to ρ_3 being pebbleable. Finally, suppose the $a - b - c$ path has 7 pebbles on it. The vertex guaranteed to have (exactly) 3 pebbles on it must be (WLOG) c_3. But there is another such vertex, which can again be checked to lead to ρ_3 being reached.

We had started with 68+11 pebbles on B_2^*. Thus 8 pebbles can be sent to ρ_3 with as few as 15 left on B_2^*. But 15 pebbles do imply that there is an $a - b - c$ path with 4 pebbles, enough to reach a. Thus, $\psi(B_3) = 81$.

We are ready to prove that $\psi(B_4) = 609$. Starting on the left, place one pebble at every alternate vertex on the bottom row with one exception: For the last two vertices, we place no pebbles on the left vertex and 601 pebbles on the right vertex. This construction, similar to those used for $n = 2, 3$ will be a canonical one that we will use in general later. The most efficient pebble domination would be to place a pebble at ρ_4 and one on each vertex at the next to bottom level of the tree. It is elementary to check that this cannot be done. Hence $\psi(B_4) \geq 609$. We now prove that $\psi(B_4) \leq 609$.

If there are fewer than 81 pebbles on each 3-subtree, ρ_4 must have have at least 449 pebbles, but only 64 of these are required to pebble the 8 vertices in the next to bottom row. This, together with one more pebble at ρ_4, completes the pebble domination. If there are 81+ pebbles on each 3-subtree, we use 81 to dominate each subtree, leaving us with 447 pebbles to dominate ρ_4. 224 of these must be on one subtree, so one of the 8 paths leading from ρ_4 to the bottom of this subtree must have 28 pebbles, enough to reach ρ_4. As before the most complicated case is when one subtree, B_3^* has 81+ pebbles and the other doesn't. We employ the same pebbling strategy as for $n = 3$. Put one pebble on each of 4 bottom vertices in B_3' (no two of whom share a parent). Assume that ρ_4 is unpebbled, and that there are just the four aforementioned pebbles on B_4'. We thus have 605 pebbles on B_3^*. Our proof deviates here from the $n = 3$ case, and in general, the case of $n \equiv 1 \pmod 3$ will be seen later to be trickier than the rest. We can clearly assume, given the strategy being used, that the root ρ_3^* of B_3^* is unpebbled. It is a straightforward calculation (similar to the $n = 3$ case) to verify that $\psi(B_3^* \setminus \rho_3^*)$, the domination pebbling number of B_3^* minus its root, equals 77. We will now seek to place 33 pebbles on ρ_4 – adequate to pebble the next to bottom row of B_3' while leaving a pebble at ρ_4 – while never dropping below 77 pebbles on $(B_3^* \setminus \rho_3^*)$. The extra pebble on ρ_4 will serve to dominate ρ_3^*. We have 605 pebbles on $(B_3^* \setminus \rho_3^*)$. We claim that as long as the number of pebbles does not drop to below 77, we can get a pebble to ρ_4 at a loss of ≤ 16 pebbles. Since $(33 \times 16) + 77 = 605$, we will be done if we can show that 93 (=77+16) pebbles on $(B_3^* \setminus \rho_3^*)$ suffice to send a pebble to ρ_4. The pigeonhole principle, the fact that ρ_4 has no pebbles on it, and the observation that no path from ρ_4 to the bottom can have 16+ pebbles, implies that we must have two $a - b - c$ paths with 12+ pebbles each. If these paths are disjoint, we can move two pebbles to ρ_3^* and thus one to ρ_4, so the paths must overlap. There are two possibilities: the paths may be of the form $a - b_1 - c_1$ and $a - b_1 - c_2$, or they may overlap in just the vertex a. In the first case, the worst case scenario is, e.g., when there are no pebbles on a, one on b_1, and 11 on each of the c's. The second case is similar. But these configurations too force a pebble onto ρ_4. Thus $\psi(B_4) = 609$. □

Theorem 8 *For $n \geq 2$,*

$$\begin{aligned}
&\psi(B_n)\\
&= (2^{n-1}-1) + \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} \left(2^{3i+1} + \sum_{j=1}^{n-3i-2} 2^{j-1} 2^{3i+2j+1} \right) + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2} + \gamma_n,\\
&= T_{1,n} + T_{2,n} + T_{3,n} + T_{4,n} \quad \text{say,}
\end{aligned}$$

where $T_{i,n}$ denotes the i^{th} term in the above sum, and $\gamma = \gamma_n = 2^{n-1}$ if $n \equiv 0 \pmod 3$, and $\gamma = 0$ otherwise.

Proof First we will prove that for $n \geq 2$

$$\begin{aligned}
\psi(B_n) \quad > \quad & (2^{n-1}-1) + \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-2} 2^{j-1} 2^{3i+2j+1}]\\
& + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2} + \gamma - 1.
\end{aligned}$$

Consider the following initial configuration of pebbles that generalizes that in Figures 3 and 4. Starting at the left, place one pebble at each of $2^{n-1}-1$ vertices on the bottom row, no two of which share a parent. This leaves the rightmost two vertices on the bottom row unpebbled; we place all the remaining pebbles on one of these vertices, denoted by v. We will endeavor to pebble entire rows in the most efficient way – the rows to be pebbled are specified by working upwards from the bottom of both subtrees – and we make pebbling moves from v, if possible, so that one pebble is placed on every vertex in every third row, starting, in each subtree, with the row that is next to the bottom row. This is clearly the best strategy, since it "costs" the most to reach the bottom row of the left subtree. Note that the γ term enters iff $n \equiv 0 \pmod 3$, since then the three topmost vertices of B_n would be left unpebbled, but we can complete the domination pebbling of the graph by pebbling the root of the right subtree. This requires γ pebbles. If we consider rows as single vertices, this would be analogous to the configuration of pebbles required to get an optimal domination cover pebbling bound for P_n, except that we pebble every third vertex counted from *both* ends, adding a "central γ-correction" if needed.

We start by stating that in order to find a domination cover solution for the subtree that is on the other side of the root vertex v it takes

$$\sum_{k=1}^{\lfloor\frac{n+1}{3}\rfloor} 2^{n-3k+1}2^{2n-3k+2}$$

pebbles as follows: Consider the next to bottom row. There are 2^{n-2} vertices that must have a pebble placed on them. For each vertex, it takes 2^{2n-1} pebbles from vertex v, for a total of 2^{3n-3} pebbles. This is the number of pebbles counted in the $k=1$ term of the sum, since $2^{n-3+1}2^{2n-3+2} = 2^{3n-3}$. We leave the rest of the details – of verifying that the stated expression $\sum_{k=1}^{\lfloor\frac{n+1}{3}\rfloor} 2^{n-3k+1}2^{2n-3k+2}$ does indeed represent the above pebbling process of the left subtree – to the reader. A similar (but somewhat more complicated) computation can be performed to verify the first sum represents the pebbling of the subtree on the same side as v in the manner desired – *except that, due to the last (-1) term, the vertex right above the "pebble source" remains unpebbled.* This configuration leaves the other sibling of v undominated. This proves the claim.

We now proceed to prove the bound by induction. First we note that a simple Maple computation reveals that the values given by the putative formula for $\psi(B_n)$ are, for $n = 2, \ldots 10$, equal to 11, 81, 609, 4777, 38105, 304473, 2434969, 19478809, 155827481; the first three of these have already been proved to be correct in Theorem 7. Note that the asymptotic ratio of the terms appears to be converging to 8 rapidly, reflecting the fact that the dominant term in $\psi(B_n)$ is 2^{3n-3}. Suppose that the value of $\psi(B_{n-1})$ is as stated in the theorem for $n \geq 5$, i.e. $n-1 \geq 4$ (our induction will only work for these cases). Place $\psi(B_n)$ pebbles on B_n. As before, we will consider three cases depending upon whether there are enough pebbles in each of the two disjoint copies of B_{n-1} connected to the root of B_n.

First, suppose that neither copy contains $\psi(B_{n-1})$ pebbles. In this case, there are at least $\psi(B_n) - 2\psi(B_{n-1}) + 1$ pebbles on the root. We claim that this number is at least as large as $4\psi(B_{n-1}) + 1$, a number that would allow us to move $\psi(B_{n-1})$ pebbles onto the root of each subtree while retaining one pebble on the root, thus completing the domination of B_n. It suffices to show that

$$\psi(B_n) \geq 6\psi(B_{n-1}) \tag{1}$$

in order for the above to be true. We have

$$\psi(B_n) \geq 2^{3n-3} \tag{2}$$

by considering only the $k = 1$ term of $T_{3,n}$. Also,

$$6(T_{1,n-1} + T_{4,n-1}) \leq 3 \cdot 2^n; \tag{3}$$

$$6T_{3,n-1} = 6 \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} 2^{3(n-1)-6k+3} \leq \frac{6 \cdot 2^{3n}}{63}; \tag{4}$$

and

$$\begin{aligned} 6T_{2,n-1} &= 6 \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} + 6 \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i} \sum_{j=1}^{n-3i-3} 2^{3j} \\ &\leq \frac{24}{7} 2^n + 6 \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i} \cdot \frac{8(8^{n-3i-3})}{7} \\ &\leq \frac{24}{7} 2^n + \frac{3 \cdot 8^n \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 8^{-2i}}{224} \\ &\leq \frac{24}{7} 2^n + \frac{6}{441} 2^{3n}. \end{aligned} \tag{5}$$

Equations (2) through (5) show that (1) holds if

$$2^{3n-3} \geq 3 \cdot 2^n + \frac{6 \cdot 2^{3n}}{63} + \frac{24}{7} 2^n + \frac{6}{441} 2^{3n}$$

or if

$$\frac{1}{8} - \frac{6}{63} - \frac{6}{441} \geq \frac{45}{7} 2^{-2n},$$

which holds for $n \geq 5$. Of course the fact that $\psi(B_n) \geq 6\psi(B_{n-1})$ holds for $n = 3, 4$ as well.

Next, suppose that both copies contain at least $\psi(B_{n-1})$ pebbles. In this case we can use $\psi(B_{n-1})$ pebbles to construct a domination cover pebbling for each subtree. At least one subtree thus has $2\psi(B_{n-1}) \geq \frac{2^{3n+1}}{64}$ extra pebbles (recalling that $\psi(B_n) \geq 6\psi(B_{n-1})$ and $\psi(B_{n-1}) \geq 2^{3n-6}$), so at least one of the 2^{n-1} $n+1$-paths leading to the root vertex (from the bottom of the subtree) has at least $\frac{2^{2n}}{16}$ pebbles. For $n \geq 4$ this number exceeds 2^n, the (regular) pebbling number of P_{n+1}. We can thus reach the root vertex. [If $n = 3$, the exact values of $\psi(B_3), \psi(B_2)$ show that we must have 30 pebbles on one of the 2-subtrees and we can reach the root as desired since one of the paths to the root must have eight pebbles.]

Finally, suppose that only one copy of B_{n-1}, call it B^*_{n-1}, contains at least $\psi(B_{n-1})$ pebbles. We need to do a more careful analysis, since the strategy to be employed (as in the small cases studied in Theorem 7) is to move all extraneous pebbles in B^*_{n-1} to the root of the tree and then cover pebble dominate the other subtree, say B'_{n-1}, from the root using an "every third row" dominating set. Note that any pebbles in B'_{n-1} can substitute for at least one pebble on the root vertex. We may thus assume the worst case scenario in which *all* the $\psi(B_n)$ pebbles are in B^*_{n-1} – except for the 2^{n-2} non-siblings in the bottom row which can each be assumed to have a single pebble on them, since this does not lessen the pebbling number for the left subtree starting at the root.

We start by showing that we may place a pebble on ρ_n (at a loss of at most 2^n pebbles) if there are at least 2^{3n-6} pebbles on B^*_{n-1} and thus, since

$$\psi(B^*_{n-1}) \geq 2^{3n-6} + 2^{n-2} - 1 + \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} \geq 2^{3n-6} + 2^{n-2},$$

if there are at least $\psi(B^*_{n-1}) - 2^{n-2}$ pebbles on B^*_{n-1} (Of course we will cause problems if we send a pebble to the root when there are fewer than $\psi(B^*_{n-1}) + 2^n$ pebbles on B^*_{n-1}.) To see this, note that one of the 2^{n-1} paths leading to the root from the bottom of B^*_{n-1} must have $2^{3n-6}/2^{n-1} = 2^{2n-5}$ pebbles on it. This number exceeds 2^n if $n \geq 5$, so we can send a pebble to the root. Similarly, a pebble may be sent to the root of B^*_{n-1} if there are at least 2^{3n-6} pebbles on B^*_{n-1}. This costs at most 2^{n-1} pebbles. We have $\psi(B_n) - 2^{n-2}$ pebbles on B^*_{n-1}, and $\psi(B_n) - \psi(B_{n-1}) - 2^{n-2}$ of these are available to pebble B'_{n-1}. Let's compute $D(n) = \psi(B_n) - \psi(B_{n-1}) - 2^{n-2}$; by the above argument, we will, e.g. be able to send $\lfloor D(n)/2^n \rfloor$ pebbles to the root. We get:

$$D(n) \quad = \quad 2^{n-1} + \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-2} 2^{3i+3j}] + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{3n-6k+3} + \gamma_n$$

$$- \left[2^{n-2} + \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-3} 2^{3i+3j}] + \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} 2^{3n-6k} + \gamma_{n-1} \right] - 2^{n-2}$$

$$\begin{aligned}
&\geq \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3i+1} - \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} + \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3n-6i-6} + \frac{7}{8} \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{3n-6k+3} + \gamma_n - \gamma_{n-1} \\
&= \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3i+1} - \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} + \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor + 1} 2^{3n-6i} + \frac{7}{8} \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{3n-6k+3} + \gamma_n - \gamma_{n-1} \\
&\geq \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3i+1} - \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{3n-6k+3} + \gamma_n - \gamma_{n-1}.
\end{aligned}$$

Case 1. If $n \equiv 0 \pmod 3$, we cover dominate B_{n-1}^* using $\psi(B_{n-1})$ pebbles. Note that $\gamma_n - \gamma_{n-1} = 2^{n-1}$ and these 2^{n-1} pebbles are used to pebble the root of B_{n-1}^*, so that ρ_n is dominated in addition to B_{n-1}^*. Also, $\sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3i+1} - \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} = 0$. It follows that $\sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{2n-6k+3}$ pebbles can be placed on the root, and it is easy to check that these suffice to pebble dominate B'_{n-1} by placing pebbles on every third row, starting with the next to bottom row.
Case 2. If $n \equiv 2 \pmod 3$, $\gamma_n - \gamma_{n-1}$ and $\sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3i+1} - \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1}$ both equal zero. We send $\sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{2n-6k+3}$ pebbles to ρ_n. These suffice to dominate the left subtree, and in particular the root of B'_{n-1} is pebbled, so that ρ_n is dominated.
Case 3. $n \equiv 1 \pmod 3$. This is the delicate case. First, $\sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{2n-6k+3}$ pebbles are sent to ρ_n. However,

$$\gamma_n - \gamma_{n-1} + \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{3i+1} - \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^{3i+1} = -2^{n-2} + 2^n = \frac{3}{4} 2^n, \qquad (6)$$

insufficient to place a crucial extra pebble on ρ_n. This is needed since the $\sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{2n-6k+3}$ pebbles on the root are only sufficient to pebble dominate $(B'_{n-1} \setminus \rho'_{n-1})$. The root ρ'_{n-1} of the left subtree could have been dominated by an extra pebble on ρ_n. However such a pebble would cause the root of B_{n-1}^* to be possibly double dominated, which seems sub-optimal. We resolve the problem by mimicking the $n = 4$ case. The induction from $n - 1$ to

$n \equiv 1 \pmod 3$ proceeds as follows: We use $\psi(B^*_{n-1} \setminus \rho^*_{n-1})$ pebbles to dominate $(B^*_{n-1} \setminus \rho^*_{n-1})$. It is easy to check, after all the work done above, that $\psi(B^*_{n-1} \setminus \rho^*_{n-1}) = \psi(B^*_{n-1}) - 2^{n-2}$. We have gained the extra $\frac{1}{4}2^n$ pebbles we need, since we do not pebble ρ^*_{n-1}. The modified value in (6) is 2^n. ρ_n is thus pebbled, and the roots of both subtrees are dominated as a result. This completes the proof. □

6 Open Problems

We are confident that this paper, together with the companion paper [5], will spark interest in the question of domination cover pebbling. Determination of the ψ values for several other families of graphs is an obvious open question. Teresa Haynes has raised the question of determining the domination cover pebbling number when the pebbles *must* reach a *miminum* dominating set. Other open problems are raised in [5].

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