The Pi-Pebbling Function

T. Baillie Arnold*

Department of Mathematics
Bowdoin College, Brunswick, Maine
tarnold2@bowdoin.edu

Abstract

Recent research in graph pebbling has introduced the notion of a cover pebbling number. Along this same idea, we develop a more general pebbling function $\pi_P^t(G)$. This measures the minimum number of pebbles needed to guarantee that any distribution of them on G can be transformed via pebbling moves to a distribution with pebbles on t target vertices. Furthermore, the P part of the function gives the ability to change how many pebbles are needed to pebble from one vertex to another. Bounds on the π-pebbling function are developed, as well as its exact value for several families of graphs.

Introduction

The idea of graph pebbling was first introduced in a paper by Chung [1]. If G is a connected graph and $C(G)$ is a distribution of pebbles onto the vertices of G, a pebbling move consists of removing two pebbles from a vertex and placing a pebble on an adjacent vertex. The pebbling number $\pi(G)$ is the minimum number such that given any configuration C of $\pi(G)$ pebbles, C can be transformed via pebbling moves to a configuration with a pebble on any target vertex. This question has been studied extensively and the exact answer is known for a large set of graphs [3]. Recently, several papers have introduced the concept of a cover pebbling number [2] [4] [7] . This is the minimum number of pebbles such that any distribution on G can be transformed via pebbling moves (i.e. pebbled) to end with a distribution where every vertex has at least one pebble on it.

In this paper, we generalize this concept by asking how many pebbles are needed to guarantee that a distribution of them can be pebbled to any t target vertices. This is called the t-th pebbling number. Furthermore, we extend the notion of a pebbling move

by allowing the number of pebbles that are removed at each vertex to be a function of the vertex.

In this context the first pebbling number is what is classically called the pebbling number, and the n-th pebbling number is what is called the cover pebbling number.

In [2] the case where P is the standard price function, π_P^t is called the weighted pebbling number. The paper defines the π-function in terms of taking the maximum over a specific set of weighted cover pebbling numbers, but does not study it. The problem with using this approach is that previous research has only calculated positive weighted cover pebbling numbers, and the maximum must be taken over some non-negative weighted covering pebbling numbers.

We begin in Section 1 by formalizing the notion of the π-pebbling function. Then in Section 2 we prove a theorem about the cover pebbling number which eliminates degenerate cases from several proofs. In Section 3 we find appropriate bounds on the π-pebbling function for any given graph. These bounds are then used to calculate the exact value for the complete graph, the path graph, and star graph in Section 4. In Section 5, we use the standard price function and study a π-pebbling sequence. Section 6 discusses the weighted cover pebbling number and the cover pebbling theorem. Finally, in Section 7 several open questions involving the π-pebbling function are presented.

1   Preliminaries

We begin by formalizing the notion of a configuration of pebbles on the vertices of a graph G.

Definition 1. *A configuration $C(G)$ of k pebbles on the graph G with vertex set $V(G)$ of size n, is a function $C : V(G) \to \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^n C(v_i) = k$. The set of all configurations on a graph are represented by $\mathcal{C}(G)$, and the set of all configurations of size k by $\mathcal{C}_k(G)$. The value of C at a particular vertex i is written as c_i.*

Definition 2. *A price function on a graph G is a function $P : V(G) \to \mathbb{Z}_{\geq 2}$. The set of all connected graphs with all possible price functions is $\mathcal{G}$*

The value of P at a vertex i is written as either p_i or p^i. The former is normally used when the vertices are labelled in some natural way from the graph G, and the latter when we label the vertices such that $p^i \leq p^{i+1}$. In either case, the exact labelling will be made explicit. We now can define a pebbling move.

Definition 3. *A pebbling move is a function $M : C(G) \to \mathcal{C}(G)$ such that $M(c_h) = c_h - p_h$, $M(c_k) = c_k + 1$ and $M(c_i) = c_i$ for all $i \neq h, k$ where the edge (v_h, v_k) is contained in the edge set $E(G)$. The set $\mathcal{P}(C)$ is the set of all pebbling moves on the configuration C.*

One thing to be careful about this definition is that the resulting M(C) must be in $\mathcal{C}$ and thus can have no negatively valued vertices. If we want to talk about mapping a configuration C to another configuration C' via a pebbling move, we say that C is pebbled to C'. With the notion of a pebbling move, comes the notion of derivability and solvability.

Definition 4. *[6] We say that the configuration C' is derivable from C if there are a series of pebbling moves $M_1, M_2, ...M_n$ such that $M_n \circ ...M_2 \circ M_1(C) = C'$.*

Definition 5. *A configuration C is said to cover a subset of $V(G)$ if c_i is non-zero on the entire subset. A configuration C is t-solvable [1], if given a set of t vertices, there exists a configuration that covers those t vertices and is derivable from C.*

Now we finally can define the notion of a π-pebbling function.

Definition 6. *The π-pebbling function is a map $\pi : \mathcal{G} \times [1, n] \cap \mathbb{Z} \to \mathbb{Z}_{\geq 1}$. The value of $\pi(G, P, t)$, denoted by $\pi_P^t(G)$, is the minimum number k such that any configuration C contained in $\mathcal{C}_k(G)$ with price function p is t-solvable.*

2 Theorem on the Cover Pebbling Number

One difficulty in dealing with the general pebbling function, is that oftentimes the n-th pebbling number, the cover pebbling number, does not fit directly into the formula for the other pebbling numbers. Instead of dealing with this case by case, there is a nice theorem which relates π_P^{n-1} and π_P^n.

Theorem 7. *For any graph G with n vertices, $\pi_P^{n-1}(G) + 1 = \pi_P^n(G)$*

Proof: Let $k = \pi_P^{n-1}(G)$, and let C be a configuration of $k+1$ pebbles on G. We show that there is a way to pebble C to a covering of G. Pick a vertex v_1 that has at least one pebble from C, and mark one of the pebbles on this vertex. Now, since $k = \pi_P^{n-1}(G)$, there is a way to pebble C onto $G - v_1$ without moving the marked pebble. This results in a covering of G. The fact that $\pi_P^n(G) > \pi_P^{n-1}(G)$ is obvious. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

When the cover pebbling number does not fit into the general pattern, it will be noted. This theorem can then be used to get it from the given formula.

3 Bounds on the π-Pebbling Function

We begin by finding the obvious general upper and lower bounds on the π-pebbling function. The first lower bound comes from assuming that the pebbles are spread out over all but t of the vertices.

Theorem 8. *Let d be the diameter of G and the vertices of G be numbered such that $p^i \leq p^{i+1}$, then for $n \neq t$*

$$\sum_{i=t+1}^{n} (p^i - 1) + p^n(t - 1) + 1 \leq \pi_C^t(G)$$

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[1]This is not to be confused with the definition of t-solvability given in [3]

Proof: Assume $(p^i - 1)$ pebbles are placed on their respective vertices for all i such that $t + 1 \leq i \leq n - 1$, and $p^n t - 1$ pebbles are placed on the p^nth vertex. Then there are t unpebbled vertices and $\sum_{i=t+1}^{n-1}(p^i - 1) + p^n t - 1$ pebbles. The only vertex that can be pebbled from is the p^nth, and there are only enough pebbles to pebble to $t - 1$ vertices, so we have $\pi_P^t(G) \geq \sum_{i=t+1}^{n-1}(p^i - 1) + p^n t$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

The second lower bound comes from assuming all the pebbles are placed on the same vertex. This requires two different results, one for $t \leq d$ and one for $t > d$.

Theorem 9. *Let the diameter d of G be greater than or equal to t, and Γ be a path of length d in G. Furthermore, let the path Γ be numbered such that p_i is adjacent to p_{i+1} for all i and under this restriction the product $p_1 p_2 ... p_{n-d}$ is maximal. Then for $t < d$,*

$$\sum_{j=1}^{t}(\prod_{i=1}^{n-j} p_j) \leq \pi_P^t(G)$$

Now if we have t is greater than d and $n \neq t$ then,

$$\sum_{j=1}^{t}(\prod_{i=1}^{n-j} p_j) + (p_1 p_2)(d - t) \leq \pi_P^t(G)$$

Proof: $(t \leq d)$ Since the graph has diameter d, this means that there is a path in G with length d. Consider $\sum_{j=1}^{t}(\prod_{i=2}^{n-j+1} p^j) - 1$ pebbles placed on p_1. Even if the price function on the graph is as low as possible, there are not enough pebbles to pebble to the farthest t vertices on the path.

$(t > d)$ Using the same argument, place $\sum_{j=1}^{d}(\prod_{i=2}^{n-j+1} p^j) + (p^1 p^2)(d - t) - 1$ pebbles on an endpoint of a length d path. At the least it will take $\sum_{j=1}^{d}(\prod_{i=2}^{n-j+1} p^j)$ pebbles to fill up the other d vertices in the path. Then, in order to fill up $t - d$ vertices not on the path, the least number of pebbles needed would be $(p^1 p^2)(d - t)$ since none of these vertices could be directly adjacent to the initial vertex or else there would be a path of length $d + 1$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$

There is also a corresponding higher bound on the π-pebbling function. This comes from putting all the pebbles on one vertex and assuming the price function is as high as possible from this vertex.

Theorem 10. *Let d be the diameter of a graph G, then*

$$\pi_P^t(G) \leq t[(\prod_{i=n-(d-1)}^{n}(p^i) - 1)(n - 1) + 1]$$

Proof: Looking at the case of $t = 1$, assume $[(\prod_{i=n-(d-1)}^{n}(p^i) - 1)(n - 1) + 1]$ pebbles are distributed on the graph G. Then either every vertex has a pebble on it or some vertex has $\prod_{i=n-(d-1)}^{n}(p^i)$ pebbles on it by the pigeonhole principle. This implies the theorem

for $t = 1$. Now notice that since $\pi_P^t(G)$ is the number of pebbles needed to pebble to any vertex, we have the inequality $\pi_P^t(G) \leq t\pi_p^1(G)$. This finishes the proof. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

These three inequalities are nice because in the case of $t = 1$ and $p_i = 2$ for all i they all collapse to the inequalities for the classical pebbling number given in [3].

Corollary 11. *Let G be a graph with diameter d, then $max[n, 2^d] \leq \pi^1(G) \leq (2^d - 1)(n - 1) + 1$.*

This inequality immediately gives the first pebbling number for K_n and P_n. These two graphs in turn show the sharpness for the case of $t = 1$ of all three bounds. It is natural to ask whether or not the general bounds are sharp for all values of t. As will be shown in the next section, the complete graph and path graph make Theorem 8 and Theorem 9 sharp respectively for all values of t and n. Using Theorem 7, it is not hard however to show that the inequality it Theorem 10 is equal only when n is 1.

4   Complete Graphs, Path Graphs, and Star Graphs

The first family of graphs we look at are the complete graphs. These are not too complicated since any set of t vertices are indistinguishable from another set of t vertices. The complete graphs are also nice because they will show that the lower bound given in Theorem 8 is sharp.

Theorem 12. *If the vertices of K_n are numbered such that $p^i \leq p^{i+1}$, then for $n \neq t$*

$$\pi_P^t(K_n) = \sum_{i=t+1}^{n} (p^i - 1) + p^n(t - 1) + 1.$$

Proof: Let C be a configuration of $\sum_{i=t+1}^{n}(p^i - 1) + p^n(t - 1) + 1$ pebbles on K_n. Assume that t target vertices have been selected. We can assume that there is at least one target vertex v_1 such that $c_1 = 0$, and then there are $\sum_{i=t+1}^{n-1}(p^i - 1) + p^n t$ pebbles on $n - 1$ vertices. The pigeonhole principle says that there exists a vertex such that $c_i \geq p_i$. This implies that there is a way to get a pebble to v_1. Now if we continue to fill up empty target vertices in this way, the pigeonhole principle says that this can continue for at least t steps. The only question is whether or it is possible that in the process of coving a target vertex, another target vertex becomes uncovered. It turns out that this can only happen if the target vertex was covered by the original configuration, and not by pebbling. This is because in each pebbling step the chosen target vertex ends up with only 1 pebble, so there is no way to pebble off of it in future pebbling moves. Therefore, using this algorithm until there are no uncovered target vertices will take at most t steps, and thus there are enough pebbles to carry it out. Theorem 8 gives the lower bound and completes the result. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

While the path graph is not symmetric, it turns out that using a convenient numbering of the vertices the pebbling number is rather simple. The path graph also shows that Theorem 9 is sharp, which is natural since it involves putting all the pebbles on the end of a maximal path in G.

Theorem 13. *If the vertices on the path of n elements are labelled such that v_i is adjacent to v_{i+1} for all i and under this restriction $p_1 p_2 ... p_t$ is as large as possible, then for $n \neq t$*

$$\pi_P^t(P_n) = \sum_{j=1}^{t} (\prod_{i=1}^{n-j} p_i)$$

Proof: Let D be a distribution of $\sum_{j=1}^{t}(\prod_{i=1}^{n-j} P_i)$ pebbles on P_n. If all the vertices have pebbles, then we are done, so assume there exists a vertex without any pebbles. Then there are $\sum_{j=1}^{t}(\prod_{i=1}^{n-j} p_i)$ pebbles on $n-1$ vertices. Using the pigeonhole principle, it can be shown that there is a way to pebble to any open vertex. Now assume we have a distribution of size $\sum_{j=1}^{t}(\prod_{i=1}^{n-j} p_i)$ with at least $k \leq t$ vertices without pebbles and we can pebble to any $k-1$ of them. If we pebble to all but the lowest numbered vertex, then there are at the least $\sum_{j=k}^{t}(\prod_{i=1}^{n-j} p_i)$ pebbles left on the other $n-(k-1)$ vertices and the pigeonhole principle again shows that these can be pebbled to the k-th vertex. This upper bound is the same as the lower bound in Theorem 9, and thus the result follows. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Now we are ready to look at the star graph S_n. The star graph consists of a central node, connected to n degree one vertices. This graph is not regular and has to be dealt with in two cases.

Theorem 14. *If the vertices of the star graph are labelled such that p_0 is the price of the center vertex and $p^i \leq p^{i+1}$ for all other i, then for $(t < n)$*

$$\pi_P^t(S_n) = \sum_{i=t+1}^{n-1} (p^i - 1) + tp_0 p_n$$

for $(t = n)$,

$$\pi_P^t(S_n) = np_0 p^n + p^n$$

Proof:$(t < n)$ Assume $\sum_{i=t+1}^{n-1}(p^i-1)+tp_0 p_n-1$ pebbles are placed on the vertex p^n and (p^i-1) are put on there respective vertices for for $t+1 \leq i \leq n-1$. There are then t empty outer vertices. The only vertex that can be pebbled from is p_n, and there are not enough pebbles to pebble to all t empty vertices. Therefore $\sum_{i=t+1}^{n-1}(p^i - 1) + tp_0 p_n \leq \pi_P^t(S_n)$.

Now assume that there is a configuration of $\sum_{i=t+1}^{n-1}(p^i-1)+tp_0 p_n$ pebbles on S_n. We assume that there are no pebbles on the central vertex. Assume there are $k \leq t$ empty outer vertices. Then there are $\sum_{i=t+1}^{n-1}(p^i-1)+tp_0 p_n$ on $n-k$ vertices. The pigeonhole principle says that as long as $k \leq t$ there is a way to pebble $p_0 t$ pebbles on to the central node, and thus a way to cover the k empty outer vertices. The case of $c_0 \neq 0$ or the central node being a target vertex is obvious given the above proof.

$(t = n)$ Assume $np_0 p^n + p^n - 1$ pebbles are placed on the vertex p^n. There are then n empty outer vertices. The only vertex that can be pebbled from is p_n, and there are not enough pebbles to pebble to all t empty vertices. Therefore $np_0 p^n + p^n \leq \pi_P^t(S_n)$. Showing the other inequality is the same as above. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

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5 π-Pebbling Ratio Series

In [2] a covering ratio of a graph G [2] is defined as $\pi^n(G)/\pi^1(G)$. Since we have a whole set of pebbling numbers, the natural generalization of this is a sequence.

Definition 15. *The π-pebbling ratio series of the graph G is the series $(\pi^{i+1}(G)/\pi^i(G))_{i=1}^{n-1}$. We denote it by ρ and the i-th term by ρ_i.*

If we have an entire family of graphs, such as the complete graphs, there is a natural way to define an infinite sequence on the whole family.

Definition 16. *Let $\mathcal{F}$ be a family of graphs such that for every natural number n there is an unique graph in the family with n vertices. Then the pebbling ratio sequence of the family is the sequence $(\rho_t(F_i))_{i=t}^{\infty}$ where F_n is the unique representative with n vertices. We denote this sequence as $\alpha^t(\mathcal{F})$.*

The nice thing about the pebbling ratio sequence of a family of graphs is that it is infinite, and so we can talk about when it converges and what it converges to.

Now using our results on complete graphs, path graphs, and star graphs, we can determine what the ratio sequence of these families converge too for various values of t.

Theorem 17. *Let $\mathcal{P}$ be the family of paths on n vertices. Then $\alpha^t(\mathcal{P})$ is a constant series and the constant is $\frac{2^{t+2}-1}{2(2^t-1)}$.*

Proof: We see from Theorem 13, that $\pi^t(P_n) = 2^n - 2^{n-t}$. Therefore

$$\alpha_n^t(\mathcal{P}) = \frac{2^n - 2^{n-(t+1)}}{2^n - 2^{n-t}}$$

By multiplying the top and bottom by 2 we get

$$\alpha_n^t(\mathcal{P}) = \frac{2^{n+1} - 2^{n+1-(t+1)}}{2^{n+1} - 2^{n+1-t}} = \alpha_{n+1}^t(\mathcal{P})$$

And therefore the series is constant. To get the constant set n=1 and simplify since the series is constant. $\mathbb{Q.E.D.}$

Theorem 18. *Let $\mathcal{K}$ be the family of complete graphs on n vertices. Then $\alpha^t(\mathcal{K})$ converges to 2 for all values of t.*

Proof: We see from Theorem 12, that $\pi^t(K_n) = n - t + 2^n(t-1) + 1$. Therefore we can take the limit

$$\lim_{n \to \infty} \frac{n + 1 - t + 2^{n+1}(t-1) + 1}{n - t + 2^n(t-1) + 1}$$

Taking the limit yields the result of 2. $\mathbb{Q.E.D.}$

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[2]In this section we assume that G has the standard price function $p_i = 2$ for all i and omit the P from the π-pebbling function

Theorem 19. *Let $\mathcal{S}$ be the family of star graphs on n vertices. Then $\alpha^t(\mathcal{S})$ converges to 1 for all values of t.*

Proof: We see from Theorem 14, that $\pi^t(S_n) = 3t + n - 1$. Therefore we can take the limit

$$\lim_{n \to \infty} \frac{3t + n}{3t + n - 1}$$

Dividing by n and taking the limit yields the result. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Now that we have shown that these sequences converge, we define the secondary pebbling ratio sequence.

Definition 20. *If $\alpha^t(\mathcal{F})$ converges for all t for some family of graphs, we can define the secondary (or Beta) pebbling ratio sequence as the sequence $\beta(\mathcal{F}) = (\alpha^t(\mathcal{F}))_{t=1}^{\infty}$.*

The β pebbling sequence can be thought of as the ρ pebbling sequence of F_∞. Now we can calculate $\beta(\mathcal{F})$ for these three families and get $\beta(\mathcal{P})$ converges to 2, $\beta(\mathcal{K})$ converges to 2, and $\beta(\mathcal{S})$ converges to 1. What this means is the given a large enough n and a large enough t, in order to be able to pebble to $t+1$ vertices takes about twice as many pebbles in the complete graph or the path graph but takes about the same number in the star graph.

These two sequences appear to have an extraordinary amount of structure. In order to fully understand them more π-pebbling numbers must be constructed for more families of graphs. One approach to doing this is described in the next section. One thing to note is that some families of graphs, such as the family of all trees or all odd cycles, do not have the property that there is one unique graph on n vertices for every natural number n. In the case of odd cycles when existence is the problem, one could simply define the pebbling ratio sequence in the normal way, but simply skip terms that are not defined. In the case where uniqueness is a problem, such as trees, the fix is more difficult.

6 Weighted Cover Pebbling Number

A natural way to generalize our results on the π-pebbling function would be to ask what is the minimum number of pebbles needed to guarantee that a sequence of pebbling moves can pebble to a configuration with a minimum number of pebbles on each vertex. This idea is worked out in [2].

Definition 21. *A weight function is a map: $W : V(G) \to \mathbb{Z}_{\geq 0}$. Its value at a vertex i is denoted by w_i. A weight function is said to be positive if $w_i \geq 1$ for all i. The size of W is the sum of its values over all vertices*

Definition 22. *The weighted cover pebbling number $\gamma_W(G)$ is the minimum number k such that any $C \in \mathcal{C}_k$ can be pebbled to a configuration C' such that $c_i' \geq w_i$.*

We see that by taking the max over all weighted cover pebbling numbers of size k on a graph G, we would get $\pi^t(G)$. But while the weighted cover pebbling number has been

studied for the case of a positive weight, it has not been calculated for non-positive weight functions. This is because all positive weight functions share a nice property that makes their cover pebble number easy to calculate.

Theorem 23. *[5] (Cover Pebbling Theorem) If $\gamma_W(G) = k$, then there exists a simple configuration of size $k-1$ that is not solvable for the weight function W.*

We have seen from the complete graph and the star graph that this is not the case for non-positive weight functions. A very important result in studying the π-pebbling function would be to find an appropriate generalization of the Cover Pebbling Theorem. This would allow one to calculate the number for more complicated graphs like wheels without too much difficulty. We provide a conjecture on what a possible generalization might be.

Conjecture 24. *Let $\pi^t(G) = k$. Then there exists a non-t-solvable configuration C of $k-1$ pebbles on G such that c_i is either 0, or $p_i - 1$ except for possibly one vertex c_r. For the standard price function, the point r should be an element that has another point s a distance d away, where d is the diameter of the graph, and there should exist a minimal path from r to one such s that has no pebbles on it other than those on r.*

7 Open Questions

There are so many unknown things about the π-pebbling function that we present only a couple problems that the author sees to be the most important.

Open Question 25. *Prove some variation on Corollary 24.*

Open Question 26. *Determine the weighted pebbling number for weights that are not necessarily positive.*

Open Question 27. *Determine when the primary and secondary pebbling ratio series converge, and determine what values they can converge to (natural numbers?).*

Open Question 28. *Using the full π-pebbling function, determine bounds on the pebbling threshold of various families of graphs.*

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