COVER PEBBLING CYCLES AND CERTAIN GRAPH PRODUCTS

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Abstract. A pebbling step on a graph consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. A graph is said to be *cover pebbled* if every vertex has a pebble on it after a series of pebbling steps. The *cover pebbling number* of a graph is the minimum number of pebbles such that the graph can be cover pebbled, no matter how the pebbles are initially placed on the vertices of the graph. In this paper we determine the cover pebbling numbers of cycles, finite products of paths and cycles, and products of a path or a cycle with *good graphs*, amongst which are trees and complete graphs. In the process we provide evidence in support of an affirmative answer to a question posed in a paper by Cundiff, Crull, et al.

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1. Introduction

The game of pebbling was first suggested by Lagarias and Saks as a tool for solving a number-theoretical conjecture of Erdös. Chung successfully used this tool to prove the conjecture and established other results concerning pebbling numbers. In doing so she introduced pebbling to the literature [1].

Begin with a graph G and a certain number of pebbles placed on its vertices. A pebbling step consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. In (regular) pebbling, a target vertex is selected, and the goal is to move a pebble to the target vertex. The minimum number of pebbles such that, regardless of their initial placement and regardless of the target vertex, we can pebble that vertex is called *the pebbling number of G*. In cover pebbling, the goal is to cover all the vertices with pebbles, i.e., to move a pebble to every vertex of the graph simultaneously. The minimum number of pebbles required such that, regardless of their initial placement on G, there is a sequence of pebbling steps at the end of which every vertex has at least one pebble on it is called *the cover pebbling number of G*. In the paper in which the concept of cover pebbling is introduced, the authors find the cover pebbling numbers of several families of graphs, including trees and complete graphs [2]. Hurlbert and Munyan have also announced a proof for the cover pebbling number of the n-dimensional cube.

In this paper we "translate" a distribution on a product of graphs to a distribution on one of the factors by introducing colors. This allows us to find upper bounds for the cover pebbling numbers of $G\Box P_n$ (Corollary 2.5) and $G\Box C_n$ (Corollary 3.5), where G is any graph. As finding lower bounds given a particular graph is generally straightforward, in Corollary 3.6 we establish the cover pebbling number of cycles. It is possible that upper bounds for the cover pebbling numbers of other products can be obtained using this technique.

Let $G = (V, E)$ be any graph. A *distribution* of pebbles to the vertices of G is any initial arrangement of pebbles on some subset S of V. The set S is called the *support* for the distribution; vertices in S are called support vertices. A *simple* distribution is one with a single support vertex. We use $\gamma(G)$ to denote the cover pebbling number of G.

Definition 1.1. *A graph G is* good *if*

$$\gamma(G) = \sum_{w \in V(G)} 2^{dist(w,u)}$$

for some vertex $u \in V(G)$. Any vertex u satisfying this equation is a key *vertex.*

Remark 1.2. A graph is good precisely when its cover pebbling number is equal to the number of pebbles needed to cover pebble the graph from a single (specific) vertex, i.e. from a key vertex. Thus when finding the cover pebbling number of a good graph, we only need to consider simple distributions.

In [2] we see that paths, trees and complete graphs are good, and the authors raise the question of whether every graph is good. We believe this is the case:

Conjecture 1.3. *Every graph is good.*

In support, we show that cycles are good. We also demonstrate that the product of any good graph with a cycle or a path is again good (Corollary 4.4).

Chung's seminal pebbling result relies on products, and she lays out Graham's conjecture, perhaps the best known open question in pebbling. Say graphs G and H have vertex sets $V(G) = \{w_1, \dots, w_g\}$ and $V(H) = \{v_1, \dots, v_h\}$, respectively. The product of G and H, $G \square H$, is the graph with vertex set $V(G) \times V(H)$ (Cartesian product) and with edge set

$$\begin{aligned} E(G\square H) = &\{\big((w_1, v_1), (w_2, v_2)\big) \,|\, w_1 = w_2 \text{ and } (v_1, v_2) \in V(H)\} \\ &\cup \{\big((w_1, v_1), (w_2, v_2)\big) \,|\, v_1 = v_2 \text{ and } (w_1, w_2) \in V(G)\}. \end{aligned}$$

Let $f(G)$ denote the pebbling number of the graph G.

Graham's Conjecture. $f(G\square H) \leq f(G)f(H)$.

There is much evidence in support of Graham's conjecture. (See, for example, [1], [5], and [6].) We believe that the analogous statement for cover pebbling involves equality:

Conjecture 1.4. $\gamma(G\square H) = \gamma(G)\gamma(H)$.

In Theorem 4.2 we show a relationship between Conjectures 1.3 and 1.4 and in Lemma 4.3 we demonstrate that Conjecture 1.4 holds when G is any good graph and H is a path or a cycle. This allows us to easily compute the pebbling numbers of a large family of products, as shown in Theorem 4.5. In particular, we have proven the cover pebbling number of a finite product of cycles and paths which then yields the cover pebbling numbers of hypercubes (P_2^n), web graphs ($C_n \Box P_m$), grids ($P_n \Box P_m$), etc.

2. Cover pebbling $G \Box P_n$

Let G and H be two graphs with vertices $w_1, .., w_g$ and $v_1, \ldots, v_h$ respectively. Recall that their product has vertex set $\{(w_i, v_j) | i = 1, \ldots, g;\ j = 1, \ldots, h\}$. We will associate to each distribution on $G \Box H$ a certain distribution of colored pebbles on H. In some cases, namely when H is a path or a cycle, results about this colored distribution can then be interpreted to obtain upper bounds for the pebbling number of the product.

We will call a distribution t-colored (or a t-distribution) if each pebble in the distribution has been assigned one of t possible colors. A color-respecting pebbling step for a colored distribution consists of taking two pebbles of the same color from some vertex and placing one of these pebbles on an adjacent vertex. When considering colored distributions we allow only color-respecting steps. A distribution is Q-coverable if we can pebble the graph with Q pebbles of any color on each vertex (performing only color-respecting steps). Thus the notions of cover pebbling and coverable distributions correspond to the case where $Q = t = 1$.

To each distribution D on $G \Box H$ we associate a color distribution $\tilde{D}$ on H in the following way: use colors $c_1, c_2, \ldots, c_g$ to assign color c_i to each pebble that D places on vertices (w_i, v_j) (for any j). Collapse $G \Box H$ to a single copy of H, which we call $\tilde{H}$ for clarity, by identifying $G \Box \{v_i\}$ in $G \Box H$ with vertex V_i in $\tilde{H}$. We place all pebbles from $G \Box \{v_i\}$ on V_i.

Lemma 2.1. *Let G and H be graphs and D be a distribution on $G \Box H$. If the associated g-distribution $\tilde{D}$ on $\tilde{H}$ is $\gamma(G)$-coverable, then D is coverable.*

Proof. By hypothesis there is a sequence of color-respecting pebbling steps beginning with $\tilde{D}$ at the end of which there are $\gamma(G)$ pebbles on each vertex of $\tilde{H}$. Because the steps respect color we could have performed them in $G \Box H$: taking two pebbles of color c_i from V_j, discarding one and placing the other one on V_k in $\tilde{H}$ corresponds to taking two pebbles from vertex (w_i, v_j) and placing one of them on (w_i, v_k). So there is a sequence of steps on $G \Box H$, consisting only of moving pebbles from one copy of G to another, at the end of which each copy of G has $\gamma(G)$ pebbles. Now each copy of G may be cover-pebbled using the $\gamma(G)$ pebbles on it, so D is coverable. □

A priori it is possible that there exist coverable distributions on $G \Box H$ that have associated colored distributions on $\tilde{H}$ that are not $\gamma(G)$-coverable. However, in many cases it appears that considering the color distribution on one of the factors is sufficient to find the pebbling number of the product.

Within the usual concept of cover pebbling (t=1), given M pebbles on a vertex v we can always move $\lfloor \frac{M}{2} \rfloor$ pebbles to an adjacent vertex, possibly having to leave one pebble on v in the case when M is odd. The analogous statement holds for colored distributions.

Lemma 2.2. *Suppose a vertex v in the support of a t-colored distribution has $M > t$ pebbles. Given any integer $E \leq M - t$, at least $\lfloor E/2 \rfloor$ pebbles initially on v can be placed on an adjacent vertex using color-respecting steps.*

Proof. Consider the set T of all pebbles on the given vertex v. We will construct a subset S of size at least $M - t$ consisting of pebbles all of which can be placed in same-color pairs. If a color has an odd number of representatives in T remove one pebble of that color. As there are only t colors, at most t pebbles are removed. Let S be the subset of all remaining pebbles, $|S| \geq M - t$. Now by removing pebbles in pairs of the same color we can obtain a smaller set, also containing even numbers of pebbles of each color, of size E if E is even or of size $E - 1$ if E is odd. Half of all pebbles in a given color can be moved to an adjacent vertex while discarding the other half. Thus we can move at least $\lfloor E/2 \rfloor$ pebbles to an adjacent vertex. □

For the rest of this paper we will denote by $|V_s, \ldots, V_t|$ the number of pebbles on the path $V_s, \ldots, V_t$. The path on m vertices will be denoted P_m.

The next proposition is slightly technical. The basic idea is that if we have a path on m vertices and a distribution which places at least Q pebbles on each of $V_2, \ldots, V_m$ and has $Q2^{m-1}$ additional pebbles, then we can use these additional pebbles to get at least Q pebbles on V_1 and thus complete the Q-covering of the path.

Proposition 2.3. *Let P_m be a path with at least 2 vertices, Q and K be integers with $Q > K$ and $Q \geq g$, and $\tilde{D}$ a g-distribution of $Q(m-1) + 2^{m-1}Q$ pebbles such that $|V_i| \geq Q$ for all $i > 1$ and $|V_1| = K$. Then there exists a sequence of color-respecting pebbling steps at the end of which $|V_i| \geq Q$ for all $1 \leq i \leq n$.*

Proof. We will use induction to prove this statement. If $m = 2$, $|V_2| = Q + 2Q - K \geq Q + 2Q - 2K$. By Lemma 2.2 with $E = 2Q - 2K$ we can place $Q - K$ pebbles on V_1 leaving at least Q pebbles on V_2.

Now assume the result holds for all $i < m$. As $|V_1| = K$ and $|V_i| \geq Q$ for $i = 2, \ldots, m-1$, we know $|V_m| \leq Q + 2^{m-1}Q - K$. Let $E = |V_m| - Q \leq 2^{m-1}Q - K$. By Lemma 2.2 we can leave Q pebbles on V_m while moving $\lfloor E/2 \rfloor$ pebbles from V_m to V_{m-1}. Prior to this move,

$$\begin{aligned} |V_1, \ldots, V_{m-1}| =& Q(m-1) + 2^{m-1}Q - |V_m| \\ =& Q(m-1) + 2^{m-1}Q - (E + Q) \end{aligned}$$

After moving $\lfloor E/2 \rfloor$ pebbles to V_{m-1} we have

$$\begin{aligned} |V_1, \ldots, V_{m-1}| =& Q(m-1) + 2^{m-1}Q - (E + Q) + \lfloor E/2 \rfloor \\ =& Q(m-2) + 2^{m-1}Q - \lceil E/2 \rceil \\ \geq& Q(m-2) + 2^{m-1}Q - 2^{m-2}Q \\ =& Q(m-2) + 2^{m-2}Q. \end{aligned}$$

Thus by our induction hypothesis applied to the path $V_1, \ldots, V_{m-1}$ we can place $Q-K$ pebbles on V_1 for a total of Q pebbles on V_1 while keeping at least Q pebbles on each of $V_2, \ldots, V_m$. □

Theorem 2.4. *Let g and Q be positive integers with $g < Q$. If $\tilde{D}$ is any g-distribution of $Q(2^n - 1)$ pebbles on the vertices of $\tilde{P}_n$, then $\tilde{D}$ is Q-coverable.*

Proof. Assume the theorem holds for all $m < n$. Let $\tilde{D}$ be a g-distribution on the vertices of $\tilde{P}_n$. Label the vertices of $\tilde{P}_n$ sequentially and so that $|V_1| \leq |V_n|$ and let $K = |V_1|$.

Case 1: $K \leq Q$

In this case $|V_2, \ldots, V_n| = Q(2^n - 1) - K \geq Q(2^n - 1) - Q = Q(2^n - 2) \geq Q(2^{n-1} - 1)$. By the induction hypothesis we can Q-cover the path $V_2, \ldots, V_n$ using at most $Q(2^{n-1} - 1)$ pebbles. Note that, as we only needed $Q(2^{n-1} - 1)$ pebbles to Q-cover $V_2, \ldots, V_n$, we now have Q pebbles on each of $V_2, \ldots, V_n$ and an additional $Q(2^n - 1) - K - Q(2^{n-1} - 1) = Q2^{n-1} - K$ pebbles lying on the path $V_2, \ldots, V_n$. Thus the path $V_1, \ldots, V_n$ now has a total of $Q(n-1) + Q(2^{n-1})$ pebbles with at least Q on each of $V_2, \ldots, V_n$ and K pebbles on V_1. By Proposition 2.3 we can move $Q-K$ pebbles to V_1 keeping at least Q pebbles at all other vertices, thus completing the Q-covering of $\tilde{P}_n$.

Case 2: $K > Q$

Let s be the largest integer such that for all $i \leq s$ the path $V_1, \ldots, V_i$ contains at least $Q(2^i - 1)$ pebbles. Note that $s \geq 1$ since $K > Q$. By assumption $|V_n| \geq |V_1| > Q$ so V_n is already Q-covered. If $s \geq n-1$ the pebbles on $V_1, \ldots, V_{n-1}$ suffice to Q-cover $V_1, \ldots, V_{n-1}$ by the inductive hypothesis, so the distribution is Q-coverable and we are done. Thus we may assume $s \leq n-2$. Q-cover the path $V_1, \ldots, V_s$ using the pebbles lying on it (as is possible by the induction hypothesis). Note that $|V_1, \ldots, V_s| \leq Q(2^{s+1} - 2)$ otherwise a larger integer s could have been chosen. Now consider the path $V_{s+1}, .., V_n$ which has $n-s$ vertices. It must have $Q(2^n - 1) - |V_1, \ldots, V_s| > Q(2^n - 1) - Q(2^{s+1} - 1) = Q(2^n - 1 - 2^{s+1} + 1) = Q(2^n - 2^{s+1}) \geq Q(2^n - 2^{n-1}) = Q(2^{n-1}) \geq Q(2^{n-s} - 1)$ pebbles, so by hypothesis we can Q-cover $V_{s+1}, \ldots, V_n$.

In either case $\tilde{D}$ is Q-coverable. □

Corollary 2.5. *For any graph G, $\gamma(P_n \Box G) \leq \gamma(G)(2^n - 1)$.*

Proof. Let D be any distribution of $\gamma(G)(2^n - 1)$ pebbles on $(P_n \Box G)$. Letting $\gamma(G) = Q$ and g be the number of vertices in G, by Theorem 2.4 we conclude that the associated g-distribution $\tilde{D}$ on $\tilde{P}_n$ is Q-coverable. The result then follows from Lemma 2.1.

□

By letting G consist of a single vertex we recover a result in [2].

Corollary 2.6. *$\gamma(P_n) = 2^n - 1$ and P_n is good.*

Proof. From Corollary 2.5 we know $\gamma(P_n) \leq 2^n - 1$. To show $\gamma(P_n) \geq 2^n - 1$, label the vertices of P_n sequentially and consider a distribution with v_1 as the only support vertex. Covering v_i from v_1 requires 2^{i-1}, pebbles so covering the whole path requires $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$ pebbles. □

3. Cover Pebbling Number for Cycles

In this section we obtain an upper bound for the cover pebbling number of the product of a cycle with any graph. A special case then gives the cover pebbling number of cycles. Specifically, we show

$$\gamma(C_n \square G) \leq \begin{cases} (2^{(n/2)+1} + 2^{n/2} - 3)\gamma(G), & \text{when } n \text{ is even;} \\ (2^{(n+1)/2} + 2^{(n+1)/2} - 3)\gamma(G), & \text{when } n \text{ is odd.} \end{cases}$$

In particular, taking G to be a single vertex

$$\gamma(C_n) \leq \begin{cases} 2^{(n/2)+1} + 2^{n/2} - 3, & \text{when } n \text{ is even;} \\ 2^{(n+1)/2} + 2^{(n+1)/2} - 3, & \text{when } n \text{ is odd.} \end{cases}$$

Fix some integer $n \geq 3$ and let C_n be a cycle graph with vertices $V = \{v_1, ..., v_n\}$, labeled sequentially. To simplify our discussion we let $r = n/2$ if n is even and $r = (n+1)/2$ if n is odd. Let $P = 2^r + 2^{n-r+1} - 3$; we will show that $\gamma(C_n) = P$. Let G be any graph with g vertices and let $\gamma(G) = Q$.

For the rest of this section we take D be a distribution on $C_n \square G$ and $\tilde{D}$ to be its associated g-distribution on $\tilde{C}_n$. We will refer to a set $V_i, V_{i+1}, \ldots, V_{r+i-1}$ of vertices of $\tilde{C}_n$ as *primary* when V_i is a support vertex; if V_i is not necessarily a support vertex we will refer to the set as *secondary.* Both primary and secondary sets are paths on r vertices. We will call a primary or secondary set *saturated* if it contains at least $Q(2^r - 1)$ pebbles.

Remark 3.1. Note that if, after color-respecting pebbling steps of the pebbles in $\tilde{D}$, there exists a partition of $\tilde{C}_n$ into disjoint paths such that each of these paths has length s_i and contains at least $Q(2^{s_i} - 1)$ pebbles, then $\tilde{D}$ is Q-coverable by Theorem 2.4.

Lemma 3.2. *Suppose D is a non-coverable distribution placing PQ pebbles on $G \square C_n$. Let $\tilde{D}$ be its associated g-distribution on $\tilde{C}_n$. Then there is a sequential numbering of the vertices of $\tilde{C}_n$ such that $V_1, \ldots, V_r$ is saturated. With any such labeling there exists $i \leq r+1$ such that $V_i, \ldots, V_{r+i-1}$ is not saturated.*

Proof. By Lemma 2.1 $\tilde{D}$ is not Q-coverable.

Case 1: n is even

First consider the sets $V_1, \ldots, V_r$ and $V_{r+1}, \ldots, V_n$. They cannot both be saturated otherwise $\tilde{D}$ would be coverable by Remark 3.1. If both sets are unsaturated, then the total number of pebbles on the graph will be at most $Q((2^r-2)+(2^r-2)) < Q(2^{(n-r+1)}-1)+Q(2^r-2) = QP$, leading to a contradiction. Thus one of the sets must be saturated and the other set must be unsaturated. After possibly relabeling the vertices $V_1', \dots, V_n'$ with $V_1' = V_{r+1}$ we have produced a labeling satisfying the conclusion of this lemma.

Case 2: n is odd

First consider the sets $V_1, .., V_r$ and $V_{r+1}, \dots, V_n, V_1$. If both sets are unsaturated then we have at most $Q((2^r - 2) + (2^r - 2)) < QP$ pebbles in $\tilde{D}$, therefore one of them must be saturated. After possibly letting $V_1' = V_r$ we may assume $V_1', \dots, V_r'$ is saturated. If one of $V_r', .., V_n'$ and $V_{r+1}', .., V_n', V_1'$ is unsaturated we would be done, so suppose they are both saturated. By Remark 3.1 we can assume $V_{r+1}', .., V_n'$ contains at most $Q(2^{r-1} - 2)$ pebbles as $V_1, \dots, V_r$ contains at least $Q(2^r - 1)$ pebbles. That means V_1' and V_r' each have at least $Q((2^r - 1) - (2^{r-1} - 2)) = Q(2^{r-1} + 1) \geq 3Q$ pebbles. By Lemma 2.2 with $E = 2Q$ we can remove $2Q$ pebbles from V_1' and place Q of them on V_n'. Now V_1' has at least $Q(2^{r-1} - 1)$ pebbles which is enough to Q-cover $V_1', \dots, V_{r-1}'$ by Theorem 2.4 and V_r' has at least $Q(2^{r-1} + 1)$ pebbles, more than enough to cover $V_r', \dots, V_{n-1}'$. Thus $\tilde{D}$ is Q-coverable, which provides a contradiction. Therefore one of $V_r', \dots, V_n'$ and $V_{r+1}', \dots, V_n', V_1'$ must not be saturated, thus satisfying the conclusion of the lemma. □

Lemma 3.3. *There exists a labeling of the vertices of $\tilde{C}_n$ such that:*

(1) *$V_1, .., V_r$ is primary and saturated,*
(2) *there exists $i \leq r + 1$ such that $V_i, .., V_{i+r-1}$ is unsaturated, and*
(3) *there are no support vertices between V_1 and V_i.*

Proof. By Lemma 3.2 there is a labeling such that the path $V_1, \dots, V_r$ is saturated, so it must contain a support vertex. Let V_k be a support vertex with minimum index k. Then the primary set $V_k, \dots, V_{k+r-1}$ contains at least as many pebbles as $V_1, \dots, V_r$, therefore it is also saturated. Let $V_1' = V_k$. By Lemma 3.2 there is an i satisfying the second condition.

To show that the third property holds we consider the sets

$$
\begin{aligned}
S =& \{V_i' \mid V_i', \dots, V_{i+r-1}' \text{ is a saturated primary set}\} \text{ and} \\
U =& \{V_j' \mid V_j', \dots, V_{j+r-1}' \text{ is an unsaturated set}\}.
\end{aligned}
$$

Let $V_{i^*}' \in S$ and $V_{j^*}' \in U$ be such that $j^* - i^* = \min\{j - i \mid V_j' \in U,\ V_i' \in S,\ j > i\}$. By the construction above at least one such pair i, j exists and satisfies $j - i \leq r$, therefore $j^* - i^* \leq r$. Consider the primary saturated set $V_{i^*}, \dots, V_{i^*+r-1}$ and the unsaturated set $V_{j^*}, \dots, V_{j^*+r-1}$.

Suppose V_s' is a source vertex with $j^* < s < i^*$. If the primary set $V_s', \dots, V_{s+r-1}'$ is saturated, then we should have replaced i^* with s to obtain a lower minimum above. If $V_s', \dots, V_{s+r-1}'$ is unsaturated we should have replaced j^* with s, again giving a lower minimum. Thus no support vertices can lie between V_{i^*}' and V_{j^*}'. Finally, relabel the vertices so that $V_{i^*}' = V_1''$ to obtain the labeling guaranteed by the lemma. (In fact, $i = 2$, but we will not be using this fact.) □

Recall that $P = 2^r + 2^{n-r+1} - 3$, $r = \lceil n/2 \rceil$ and we intend to show that $\gamma(C_n \square G) \leq P\gamma(G)$.

Theorem 3.4. *Given positive integers $g < Q$, if $\tilde{D}$ is any g-distribution of QP pebbles on the vertices of $\tilde{C}_n$, then $\tilde{D}$ is Q-coverable.*

Proof. In search of contradiction suppose $\tilde{D}$ is a g-distribution on $\tilde{C}_n$ that is not Q-coverable. By Lemma 3.3 we can label the vertices of $\tilde{C}_n$ so that $V_1, ..., V_r$ is primary and saturated, $V_i, \ldots, V_{i+r-1}$ is unsaturated, $i \leq r+1$, and there are no pebbles on any vertex V_s for $1 < s < i$. As there are no support vertices between V_1 and V_i, the pebbles in $V_2, \ldots, V_r$ are also pebbles in $V_i, \ldots, V_{i+r-1}$. However, this was an unsaturated set, so $|V_2, \ldots, V_r| \leq |V_i, \ldots, V_{i+r-1}| \leq 2^r - 2$. Thus V_1 must have at least $(2^r - 1) - |V_2, \ldots, V_r|$ pebbles because $V_1, \ldots, V_r$ was chosen to be saturated. Let $a = (2^r - 1) - |V_2, \ldots, V_r|$. Then we can write $|V_1|$ as the sum of two integers, a and b, so that $|V_2, \ldots, V_r| + a = Q(2^r - 1)$ and thus $|V_{r+1}, \ldots, V_n| + b = Q(2^{n-r+1} - 2)$. Use all of the pebbles on $V_2, .., V_r$ and a pebbles from V_1 to Q-cover the path $V_1, \ldots, V_r$. This is possible by Theorem 2.4. Now there are at least Q pebbles on each of $V_1, \ldots, V_r$ and at least $Q + b$ pebbles on V_1. Consider the path $V_{r+1}, \ldots, V_n, V_1$ which contains $n - r + 1$ vertices. On this path there are at least $|V_{r+1}, \ldots, V_n| + b + Q = Q(2^{n-r+1} - 1)$ pebbles and it is therefore Q-coverable by Theorem 2.4. Thus $\tilde{D}$ is Q-coverable, contradicting the assumption.

□

Corollary 3.5. $\gamma(G \square C_n) \leq \gamma(G)(2^r + 2^{n-r+1} - 3)$ *for any graph G.*

Proof. Let D be any distribution of $\gamma(G)P$ pebbles on $(C_n \square G)$. Let $\gamma(G) = Q$ and g be the number of vertices in G. By Theorem 3.4 we conclude that the associated g-distribution $\tilde{D}$ on $\tilde{C}_n$ is Q-coverable. By Lemma 2.1 the distribution D on $(C_n \square G)$ must also be coverable.

□

Corollary 3.6. $\gamma(C_n) = 2^r + 2^{n-r+1} - 3$ *and C_n is good.*

Proof. By Corollary 3.5 with $G = P_1$ we need only show $\gamma(C_n) \geq P$. We number the vertices of C_n sequentially. Consider a distribution with all pebbles placed on v_1. The distance from v_1 to v_i is $i - 1$ when $i \leq r$ and $n - i + 1$ when $i > r$. So we require $\sum_{i=1}^{r} 2^{i-1} + \sum_{i=r+1}^{n} 2^{n-i+1} = (2^r - 1) + (2^{n-r+1} - 2) = P$ pebbles.

□

4. Pebbling numbers for certain products

Recall that a graph G is *good* if there is a distribution with only one support vertex requiring $\gamma(G)$ pebbles to cover pebble G. It was previously known that paths, trees and complete graphs are good [2]. Section 3 establishes that cycles are good. In Theorem 4.2 we will prove that there is a relationship between the cover pebbling version of Graham's conjecture (Conjecture 1.4) and good graphs. First note the following:

Proposition 4.1. *If G and H are good then there is a simple distribution on $G\Box H$ that requires $\gamma(G)\gamma(H)$ pebbles. In particular, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.*

Proof. Let $w_1, \dots, w_g$ and $v_1, \dots, v_h$ be the vertices of G and H respectively, and say w_1 and v_1 are key vertices for G and H. Then $\gamma(G) = \sum_{i=1}^{g} 2^{\mathrm{dist}(w_1,w_i)}$ and $\gamma(H) = \sum_{j=1}^{h} 2^{\mathrm{dist}(v_1,v_j)}$. Consider the distribution on $G\Box H$ consisting of a single support vertex (w_1, v_1). This distribution requires

$$\sum_{i=1,j=1}^{i=g,j=h} 2^{\mathrm{dist}((w_1,v_1),(w_i,v_j))} = \sum_{i=1,j=1}^{i=g,j=h} 2^{\mathrm{dist}(w_1,w_i)+\mathrm{dist}(v_1,v_j)}$$

$$= \sum_{i=1}^{g} 2^{\mathrm{dist}(w_1,w_i)} \sum_{j=1}^{h} 2^{\mathrm{dist}(v_1,v_j)} = \gamma(G)\gamma(H)$$

pebbles. □

Theorem 4.2. *Suppose G and H are good. Then $\gamma(G\Box H) = \gamma(G)\gamma(H)$ if and only if $G\Box H$ is good.*

Proof. If $\gamma(G\Box H) = \gamma(G)\gamma(H)$ then $G\Box H$ is good by Proposition 4.1. If $G\Box H$ is good, then by the same proposition it follows that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

Any simple distribution on $G\Box H$ supported on (w, v) would require

$$\sum_{i=1,j=1}^{h,g} 2^{\mathrm{dist}((w,v),(w_i,v_j))} = \sum_{i=1}^{g} 2^{\mathrm{dist}(w,w_i)} \sum_{j=1}^{h} 2^{\mathrm{dist}(v,v_j)} \leq \gamma(G) \times \gamma(H)$$

pebbles, thus proving the other direction of the inequality and concluding the proof of the theorem. □

Lemma 4.3. *If G is a good graph then $\gamma(P_n\Box G) = \gamma(P_n)\gamma(G)$ and $\gamma(C_n\Box G) = \gamma(C_n)\gamma(G)$.*

Proof. P_n and C_n are good graphs by Corollaries 2.6 and 3.6 respectively. Let H_n indicate P_n or C_n. By Proposition 4.1 we know $\gamma(G\Box H_n) \geq \gamma(G)\gamma(H_n)$. By Corollaries 2.5 and 3.5 we have $\gamma(G\Box H_n) \leq \gamma(G)\gamma(H_n)$. □

Corollary 4.4. *The product of any good graph with P_n or C_n is good.*

Proof. This is a direct result of Lemma 4.3 and Theorem 4.2. □

Now we can easily prove the cover pebbling numbers of some families of graphs as advertised in the introduction.

For quick reference, we collect all known cover pebbling numbers here.

- For any tree T, $\gamma(T) = max_{v\in V(T)}(\sum_{u\in V(T)} 2^{\mathrm{dist}(v,u)})$, as in[2].
- $\gamma(K_n) = 2n - 1$, as in[2].
- $\gamma(P_n) = 2^n - 1$, as in[2].
- $\gamma(C_n) = 2^r + 2^{n-r+1} - 3$, where $r = \lceil n/2 \rceil$, by Corollary 3.6.

As all of the graphs referenced in the above list are good, we also know that the following products are good, with cover pebbling numbers as shown below.

Theorem 4.5. *Let* $H = (\square_i P_{n_i}) \square (\square_j C_{m_j})$.

- $\gamma(H) = \prod_i \gamma(P_{n_i}) \prod_j \gamma(C_{m_j})$
  In particular,
  - $\gamma(\square_i P_{n_i}) = \prod_i (2^{n_i} - 1)$
  - $\gamma(\square_i C_{m_i}) = \prod_i (2^{r_{m_i}} + 2^{m_i - r_{m_i} + 1} - 3)$, *where* $r_{m_i} = \lceil m_i/2 \rceil$
- $\gamma(H \square T) = \gamma(H)\gamma(T)$ *for any tree* T
- $\gamma(H \square K_n) = \gamma(H)\gamma(K_n)$

Proof. In each statement the fact that the product is good follows from Corollary 4.4. The pebbling number then follows from Theorem 4.2. $\square$

We also recover a result announced by Hurlbert:

Corollary 4.6. *The cover pebbling number of the k-hypercube is* 3^k, *i.e.* $\gamma(Q^k) = 3^k$.

Proof. As Q^k is isomorphic to $\square^k P_2$, by Theorem 4.5 (part 1) we have $\gamma(Q^k) = \prod^k \gamma(P_2) = \prod^k (2^2 - 1) = 3^k$. $\square$

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