Diameter Bounds, Fractional Pebbling, and Pebbling with Arbitrary Target Distributions

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Abstract

We give upper bounds on t-pebbling numbers of graphs of small diameter and investigate the relationships between these results and those relating to fractional pebbling. Furthermore, we present a connection between optimal pebbling and linear optimization and utilize this relationship in analyzing various graphs. We introduce the optimal fractional pebbling number of a graph and provide a combinatorial interpretation. Finally, we explore a generalization of graph pebbling by allowing the target to be any distribution of a given size.

1 Introduction

For a graph $G = (V, E)$, a function $D : V \to \mathbb{N}$ is called a *distribution on the vertices of G*, or a *distribution on G*. We usually imagine that $D(v)$ pebbles are placed on v for each vertex $v \in V$. Let $|D|$ denote the *size* of D, i. e. $|D| = \sum_{v \in V} D(v)$. For two distributions D and D' on G, we say that D *contains* D'

if $D'(v) \leq D(v)$ for all $v \in V$. We also use $D(G)$ to denote the diameter of G, but it should be clear throughout whether D refers to a distribution of pebbles or the diameter of a graph. We write $\delta(v, w)$ for the distance from v to w in G. We use n to represent the number of vertices in a graph. The following definition stipulates how pebbles can be transferred from one vertex to another.

Definition: A *pebbling move* in G takes two pebbles from a vertex $v \in V$, which contains at least two pebbles, and places a pebble on a neighbor of v.

For two distributions D and D', we say that D' is *reachable* from D if there is some sequence of pebbling moves beginning with D and resulting in a distribution which contains D'. We have the following definition.

Definitions: For an integer $t \geq 1$, we say a distribution D on a graph G is *t-fold solvable* if every distribution with t pebbles on a single vertex is reachable from D. If $t = 1$ we say the distribution is *solvable*; otherwise it is *unsolvable*. The *t-pebbling number* of a graph G, denoted $\pi_t(G)$, is the smallest integer k such that every distribution D with $|D| \geq k$ is t-fold solvable. The *pebbling number* of G is $\pi_1(G)$, and we denote it $\pi(G)$.

In a sequence of pebbling moves, a distribution we are attempting to reach is called a *target* and a vertex we are attempting to reach is called a *target vertex* or a *root*.

Suppose that, instead of considering all possible distributions of a given size, we desire the smallest t-fold solvable distribution. In this spirit, we give the definition of the optimal t-pebbling number of a graph.

Definition: For an integer $t \geq 1$, the *optimal t-pebbling number* of a graph G, denoted $\pi_t^*(G)$, is the smallest integer k such that there exists a t-fold solvable distribution D of pebbles on V with $|D| = k$. The *optimal pebbling number* of G is $\pi_1^*(G)$, and we denote it $\pi^*(G)$.

2 Trees and Cycles

In this section we describe prior work on pebbling in trees and cycles. To find the pebbling number of a vertex r in a tree T, Chung [3] defined T_r^* as the directed graph in which all edges in T are directed toward r. She then described *path partitions* and *maximal path partitions* in the tree T_r^*, which we generalize to describe path partitions in the undirected tree T as well.

Definition [3]: A *path partition* of a undirected tree T or of a tree T_r^* in which all edges are directed toward the vertex r is a partition of the edges of the tree into sets in such a way that the edges in each set in the partition form a path in T, or a path directed toward r in T_r^*. The *path-size sequence* of a path partition is the sequence of lengths of the paths in nonincreasing order, $a_1 \geq a_2 \geq \cdots \geq a_k$. A *maximal path partition* in T or in T_r^* is a path partition whose path-size sequence is lexicographically greatest.

Chung found $\pi_t(T, r)$, and Bunde *et al.* [2] gave $\pi_t(T)$. We present these results as Theorem 2.1.

Theorem 2.1 (Chung [3]; Bunde *et al.* [2]) *If r is a vertex in a tree T, then $\pi_t(T, r)$ is given by*

$$\pi_t(T, r) = 2^{a_1}t + 2^{a_2} + \ldots + 2^{a_k} - k + 1,$$

where $a_1, a_2, \ldots, a_k$ is the path-size sequence of a maximal path partition of T_r^*. The t-pebbling number of T is given by

$$\pi(T) = 2^{a_1}t + 2^{a_2} + \ldots + 2^{a_k} - k + 1,$$

where $a_1, a_2, \ldots, a_k$ is the path-size sequence of a maximal path partition of T.

Although it was certainly clear from Chung's work, it appears that no one has formally stated and proved that moving a pebble to r costs at most 2^{a_1} pebbles from the rest of the graph. We do so now.

Proposition 2.2 *Let r be any vertex in the tree T and suppose D is a distribution on T from which t pebbles can be moved to r. Then it is possible to move t pebbles to r at a cost of at most $2^{a_1}t$ pebbles from the rest of*

the graph, where $a_1 = \max\limits_{v \in V(T)} \delta(r, v)$. In particular, t pebbles can be moved to any vertex at a cost of at most $2^{D(G)}t$ pebbles from the rest of the graph.

Proof: Let S be a minimal sequence of pebbling moves that places t pebbles on r. For every $i \in \{0, 1, \ldots, a_1\}$ let $L_i = \{u \in V(G) \mid \delta(r, u) = i\}$, so L_i is the ith level in the tree rooted at r. For $i < a_1$ let n_i denote the number of pebbling moves in S from L_{i+1} to L_i. Now $n_0 = t$, and for larger i we need at most $2n_{i-1}$ moves onto L_i to make n_{i-1} moves onto L_{i-1}; therefore, by induction, we have $n_i \le 2^i t$. Thus, the number of pebbling moves in S is at most $\sum\limits_{i=0}^{a_1-1} 2^i t = (2^{a_1} - 1)t = 2^{a_1}t - t$. Each such pebbling move results in the loss of a pebble. Thus, along with the t pebbles that ends up on r, at most $2^{a_1}t$ pebbles are removed from the rest of the graph. □

The t-pebbling number $\pi_t(C_n)$ was given in [8].

Proposition 2.3 ([8]) *The t-pebbling number of a cycle is given by*

$$\pi_t(C_{2k}) = 2^k \cdot t$$

$$\pi_t(C_{2k+1}) = \tfrac{2^{k+2}-(-1)^k}{3} + 2^k(t-1).$$

3   A Distance-Based Bound

In this section, we prove the following theorem.

Theorem 3.1 *Suppose r is a vertex in a graph G with the property that $\delta(v, r) \le d$ for every vertex v in G. Then*

$$\pi_t(G, r) \le \frac{2^d - 1}{d}(n - 1) + 2^d(t - 1) + 1. \tag{1}$$

Furthermore, if there are at least $\frac{2^d-1}{d}(n-1) + 1$ pebbles on the graph, moving a pebble to r costs at most 2^d pebbles from the rest of the graph.

We use Lemma 3.2, which shows that $\frac{2^d-1}{d}$ is an increasing function of d.

Lemma 3.2 *The function $f : \mathbb{Z}^+ \to \mathbb{Q}^+$ defined by $f(d) = \frac{2^d-1}{d}$ is an increasing function of d.*

Proof: We simply show $f(d+1) > f(d)$ for every positive integer d. Toward that end, we have

$$f(d+1) = \frac{2^{d+1} - 1}{d+1} = \frac{2(2^d - 1) + 1}{d+1} > \frac{2(2^d - 1)}{d+d} = \frac{2(2^d - 1)}{2d} = \frac{2^d - 1}{d} = f(d),$$

as desired. $\qquad\square$

Proof of Theorem 3.1: Let T be a spanning tree of G obtained by doing a breadth-first search from r. Since T is a spanning subgraph of G, we have $\pi_t(G, r) \leq \pi_t(T, r)$. Because T was obtained by a breadth-first search from r, we have $\delta_G(r, v) = \delta_T(r, v)$ for every vertex v in G. Therefore, it suffices to show that (1) holds when $G = T$. Let $a_1, a_2, \ldots, a_k$ be the path-size sequence of a maximal path partition of T_r^*. In particular, we have each $a_i \leq d$. Then we have

$$\sum_{i=1}^{k} a_i = |E(T)| = |V(T)| - 1 = n - 1, \tag{2}$$

since each edge in the tree is in exactly one part in the partition. From Theorem 2.1, we also have

$$\pi_t(T, r) = 2^{a_1} t + 2^{a_2} + \ldots + 2^{a_k} - k + 1,$$

which, we can rewrite as

$$\pi_t(T, r) = \sum_{i=1}^{k} (2^{a_i} - 1) + 2^{a_1}(t - 1) + 1 = \sum_{i=1}^{k} \left[\left(\frac{2^{a_i} - 1}{a_i} \right) a_i \right] + 2^{a_1}(t - 1) + 1.$$

Now since each a_i is at most d, we apply Lemma 3.2 to obtain

$$\pi_t(T, r) \leq \sum_{i=1}^{k} \left[\left(\frac{2^d - 1}{d} \right) a_i \right] + 2^d(t - 1) + 1 = \left(\frac{2^d - 1}{d} \right) \sum_{i=1}^{k} a_i + 2^d(t - 1) + 1.$$

But from (2), we find

$$\pi_t(G, r) \leq \pi_t(T, r) \leq \frac{2^d - 1}{d}(n - 1) + 2^d(t - 1) + 1.$$

Finally, since each move is made along a directed edge in T_r^*, by Proposition 2.2, at most 2^d pebbles from the rest of the graph are consumed. $\qquad\square$

Curtis *et al.* prove Theorem 3.3:

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Theorem 3.3 (Curtis *et al.* [5]) *For any integer $t \geq 1$, if G is a graph with diameter 2, then $\pi_t(G) \leq n + 7t - 6$.*

The proof of Theorem 3.3 can be generalized to prove Theorem 3.4.

Theorem 3.4 (Curtis *et al.* [5]) *If r is a vertex in G such that $\delta(r, v) \leq 2$ for every vertex v in G, then $\pi_t(G, r) \leq n + 7t - 6$.*

Using Theorem 3.1 with $d = 2$ gives the bound $\pi_t(G, r) \leq 1.5n + 4t - 4.5$. Thus, Theorem 3.1 represents an improved bound on that given by Theorem 3.4 when $t > \frac{n+3}{6}$. In Section 4, we further improve this bound when the diameter of the graph is 2.

4   Graphs of Diameter 2

We give a bound on the t-pebbling number of diameter 2 graphs. That is, we prove the following theorem.

Theorem 4.1 *If G is a graph with $D(G) = 2$ then $\pi_t(G) \leq \pi(G) + 4t - 4$.*

Star graphs, denoted K_{1p}, feature prominently in our proof, so we define them now.

Definition: If $p \geq 2$, the *star on $p + 1$ vertices*, denoted K_{1p}, is the graph whose vertex and edge sets are given by $V(K_{1p}) = \{u, v_1, v_2, \ldots, v_p\}$ and $E(K_{1p}) = \{(u, v_i) : 1 \leq i \leq p\}$. We call u the *center* of the star and we call the v_i's *outer vertices*. By abuse of language, we identify the vertex set V with the star K_{1p} if $|V| = p + 1$ and the subgraph induced by V contains K_{1p}.

To prove Theorem 4.1, we let D be a distribution of $\pi(G) + 4t - 4$ pebbles on G for some $t \geq 2$ (there is nothing to show if $t = 1$), and we show that t pebbles can be moved to the vertex r. We assume by induction that $\pi_{t-1}(G) \leq \pi(G) + 4(t - 1) - 4 = \pi(G) + 4t - 8$. Therefore, if we could move a pebble to r at a cost of no more than four pebbles, we could use the remaining $\pi(G) + 4t - 8$ pebbles

to put $t - 1$ additional pebbles on r. We show that if putting a pebble on r always requires using five pebbles, then $n + 4t - 4$ pebbles are sufficient to put t pebbles on r. To do this, we note that if four pebbles are not sufficient to move a pebble onto r, this places certain constraints on D. We formalize this idea as Lemma 4.2.

Lemma 4.2 *Suppose G is a graph with $D(G) = 2$, and D is a distribution of pebbles on G from which any sequence of pebbling moves that puts a pebble on the vertex r requires at least five pebbles. Then D has the following properties:*

1. *The vertex r is not occupied.*

2. *Every vertex in G has at most three pebbles.*

3. *No vertex in G with two or three pebbles is adjacent to r.*

4. *Every vertex that is adjacent both to r and to a vertex with two or three pebbles is unoccupied.*

5. *No vertex adjacent to r is also adjacent to two vertices which each have two or three pebbles.*

That is, for every vertex v_i with two or three pebbles, there is a different unoccupied vertex w_i that is adjacent to both v_i and r.

Proof: If any of the properties did not hold for D, we could put a pebble on r using at most four pebbles. If the vertex v_i has two or more pebbles, it cannot be adjacent to r, but because $D(G) = 2$ there is a vertex w_i that is adjacent to v_i and to r. By property 4, w_i is unoccupied, and by property 5, if $w_i = w_j$ then $v_i = v_j$. □

We want to focus on the vertices which have three pebbles, so we let V_3 denote the set of vertices in G with three pebbles, and we let $n_3 = |V_3|$. We now give an outline of the proof of Theorem 4.1

Outline of proof of Theorem 4.1: If $t = 1$ there is nothing to prove, so we assume $t \geq 2$. We focus on the vertices in V_3 and on vertices with at least two neighbors in V_3. We apply the following steps:

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1. We use pebbles on vertices in V_3 that are adjacent to one another. We let V_3' be the subset of V_3 whose pebbles were not used in this step.

2. We use pebbles on the vertices with $i < 3$ pebbles that are adjacent to at least $3 - i$ vertices in V_3' along with the pebbles on those neighbors in V_3'.

3. We let U be the subset of V_3' whose pebbles were not used in Step 2. We form the vertex set S that consists of V_3, all vertices directly between the vertices of V_3 and r, and all vertices that are adjacent to at least two vertices in U.

4. We put bounds the number of vertices in S and the number of pebbles P_S on those vertices.

5. We count the number of vertices and pebbles on the vertices not in S. We show that if $|D| \geq n + 4t - 4$, then t pebbles are moved to r in Steps 1 and 2.

In Steps 1 and 2, we count the vertices in V_3 whose pebbles are consumed and compare that to the number of pebbles moved to r. We use Lemma 4.3 to perform this analysis in Step 2.

Lemma 4.3 *Suppose that on the star K_{1p} with $p \geq 2$, the center vertex has $i \leq 2$ pebbles, and the outer vertices each have three pebbles. Let Q and R be defined by*

$$Q = p + i \text{ div } 3$$

$$R = p + i \text{ mod } 3,$$

so $p + i = 3Q + R$ and $0 \leq R \leq 2$. Then a fourth pebble can be moved to Q of the outer vertices, after which R outer vertices still have three pebbles. In particular, putting a fourth pebble on Q outer vertices uses $3Q - i$ outer vertices.

Proof: Let x be the center vertex. If $Q = 0$ we have $p = 2$ and $i = 0$. No pebbles can be moved, and after this process, $R = 2$ outer vertices still have three pebbles. If $Q \geq 1$, we first move $2Q - i \geq 0$ pebbles from the outer vertices onto x. There are now $2Q$ pebbles on x, so we can move a pebble from

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x to each of Q different outer vertices which still have three pebbles. This uses $3Q - i$ outer vertices, so the number of unused outer vertices is $p - (3Q - i) = p + i - 3Q = R$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1: As outlined above, we assume we have a distribution D of $n + 4t - 4$ pebbles on G with $t \geq 2$ such that moving any pebble to r requires at least five pebbles. We show that under these conditions, t pebbles can be moved to any vertex r.

Step 1: As long as there is a pair of unmatched, adjacent vertices in V_3, we choose such a pair and match them together. We call these matched pairs *mates*. If $\{v_j, v_k\}$ is a pair of mates, we use this pair to put a pebble on r. To do this, either v_j can donate a fourth pebble to v_k or *vice versa*. The pebbles on the recipient can then be used to put a pebble on r. Note that the choice of donor may be relevant, since the donor still has one pebble remaining even after the pebble has been moved to r, and we may be able to use this pebble later. We postpone the decision as to which vertex should donate a pebble to the other, and we call a pair of mates *unsettled* until we make that choice.

We let V_3' be the subset of unmatched vertices of V_3. We *mark* all vertices in V_3 which may be adjacent to two or more unused vertices in V_3'. Clearly, none of the vertices in V_3' are adjacent, since adjacent vertices in V_3 would be matched with each other if they do not already have a mate. Therefore, only vertices with a mate can be marked. We begin by marking every such vertex.

In Step 4, we want to limit the number of marked vertices, so when we settle a pair of mates, we do it in such a way that at least one vertex of the pair becomes unmarked. To settle the pair $\{v_j, v_k\}$, we pick one of these vertices, say v_j, and we count the number of unused neighbors of v_j that are in V_3'. Suppose there are p such neighbors. Then if $p \equiv 2 \bmod 3$, we let v_j be the donor, so v_j has a pebble left after a pebble is moved from $\{v_j, v_k\}$ to r; otherwise v_k is the donor. This pair is now settled, and we move a pebble to r, leaving one pebble on the donor. If either $p \equiv 2 \bmod 3$ (so v_j has a pebble) or $p \equiv 0 \bmod 3$, we form a star with v_j as its center and all its unused neighbors (if any) in

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V_3' as its outer vertices. If $p \equiv 1 \bmod 3$, we omit one of the unused neighbors of v_j. In any case, the outer vertices of the star can all be used, (so they do not appear in subsequent stars), and once they are used, v_j is adjacent to at most one unused neighbor in V_3', so we remove the mark from v_j. We make the additional moves on these stars in Step 2.

If the number of matched pairs is B, we can put B pebbles on r by using the pebbles from $2B$ vertices in V_3. Furthermore, at most B vertices remain marked.

Step 2: We choose additional vertices v_j which are occupied and are adjacent to at least two unused vertices in V_3', or which are unoccupied and are adjacent to at least three unused vertices in V_3'. We use these vertices to form a star centered at v_j, and we move as many pebbles as possible from this star to r. Any vertex whose pebbles are not used in this process can be part of a subsequent star. We also include the stars formed in Step 1 as part of this process, since their centers have either zero or one pebble. For $i \in \{0, 1, 2\}$ we let s_i be the number of stars whose center has i pebbles. For convenience, we reorder the v_j's so that v_j has two pebbles if $1 \leq j \leq s_2$, one pebble if $s_2 < j \leq s_2 + s_1$, and no pebbles if $s_2 + s_1 < j \leq s_2 + s_1 + s_0$.

If v_j has i pebbles and there are q_j vertices of V_3' in its star, then $c_j = (q_j + i)$ div 3 pebbles can be moved to r using the pebbles on at most $3c_j - i$ vertices in V_3'. We let C be the total number of pebbles can be moved to r from these stars. Then $C = \sum_{j=1}^{s_2+s_1+s_0} c_j$, and the number of vertices in V_3' whose pebbles are used is

$$\sum_{j=1}^{s_2}(3c_j - 2) + \sum_{j=s_2+1}^{s_2+s_1}(3c_j - 1) + \sum_{j=s_2+s_1+1}^{s_2+s_1+s_0} 3c_j = 3C - 2s_2 - s_1. \tag{3}$$

In each case, at most two vertices in any star still have three pebbles on them, and in such a star, the center is no longer occupied. Note also that a total of $B + C$ pebbles are on r as a result of Steps 1 and 2. In the subsequent steps, we show that $B + C \geq t$ if $|D| \geq n + 4t - 4$.

Step 3: Let U be the subset of vertices in V_3' that are still unused at this point, and let $u = |U|$. We note the following facts about U.

Fact 1 *None of the vertices in U are adjacent to each other, and none of them have a mate in Step 1, since U is a subset of V_3'.*

Fact 2 *We used $2B$ vertices in V_3 in Step 1, and from (3), we used $3C - 2s_2 - s_1$ of these vertices in Step 2, so the number of unused vertices in V_3 is $u = n_3 - 2B - 3C + 2s_2 + s_1$.*

Fact 3 *If u_i and u_j are vertices in U, there is some vertex v_{ij} in $V(G)$ that is adjacent to both u_i and u_j, since $D(G) = 2$, and u_i and u_j are not adjacent, by Fact 1.*

Fact 4 *None of the vertices v_{ij} in Fact 3 is adjacent to r, by part 5 of Lemma 4.2.*

Fact 5 *The vertex v_{ij} has no neighbors in U other than u_i and u_j; otherwise v would have been the center of a star with three unused neighbors in Step 2, and some neighbors of v_{ij} would have been used.*

Fact 6 *If v_{ij} initially had three pebbles, then it remains marked after Step 1. Therefore, there are at most B such vertices.*

Fact 7 *If v_{ij} initially had one or two pebbles, it is the center of a star in Step 2, since any occupied vertex with two unused neighbors in V_3 would form a star in that step.*

Fact 8 *There are at least $\binom{u}{2}$ vertices in $V(G)$ with exactly two neighbors in U, since every pair of vertices in U has a common neighbor (Fact 3), and those common neighbors are all different (Fact 5).*

We let V_{3r} denote the set of vertices that are adjacent to r and also to a vertex that initially had three pebbles. We also let U^* be the set of vertices in G that are adjacent to two vertices in U, and we define the vertex set S by

$$S = V_3 \cup V_{3r} \cup U^*.$$

Step 4: We find a lower bound for the number of vertices in S, and an upper bound for the number of pebbles P_S that started on those vertices. There are n_3 vertices in V_3, at least n_3 vertices in V_{3r} (by

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Lemma 4.2), and at least $\binom{u}{2}$ vertices in U^*. By part 3 of Lemma 4.2, $V_3 \cap V_{3r}$ is empty. By Fact 4, $V_{3r} \cap U^*$ is empty as well, and by Fact 6, $|V_3 \cap U^*| \le B$. Therefore, by the principle of inclusion-exclusion,

$$|S| = |V_3| + |V_{3r}| + |U^*| - |V_3 \cap U^*| \ge 2n_3 + \binom{u}{2} - B \ge 2n_3 + (u-1) - B,$$

since $\binom{u}{2} \ge u - 1$ for all $u \ge 0$. Now from Fact 2, $u = n_3 - 2B - 3C + 2s_2 + s_1$, so

$$|S| \ge 3n_3 - 3B - 3C + 2s_2 + s_1 - 1. \tag{4}$$

Counting pebbles, we started with $3n_3$ pebbles on the vertices of V_3, no pebbles on the vertices of V_{3r} (by part 4 of Lemma 4.2), and at most $2s_2 + s_1$ on the vertices of U^* that are not in V_3 (by Fact 7). Therefore,

$$P_S \le 3n_3 + 2s_2 + s_1. \tag{5}$$

Step 5: We examine the vertices that are not in S. For $i \in \{0, 1, 2, 3\}$, we let x_i be the number of vertices in $V(G) - S$ that start with i pebbles. In particular, $x_3 = 0$, since every vertex in V_3 is included in S. Thus, we have

$$n = |S| + x_2 + x_1 + x_0$$

$$|D| = P_S + 2x_2 + x_1.$$

Also note the only vertices adjacent to r that are in S are those that are also adjacent to a vertex with three pebbles. Therefore, a vertex adjacent to r and to a vertex with two pebbles is not in S. Furthermore, r itself is not in S, so $x_2 + 1 \le x_0$. Therefore, using (4) and (5), we have

$$|D| - n = P_S - |S| + x_2 - x_0 \le 3(B + C).$$

Now if $|D| \ge n + 4t - 4$, we have $3(B + C) \ge 4t - 4$, so

$$B + C \ge \left\lceil \frac{4t - 4}{3} \right\rceil = t - 1 + \left\lceil \frac{t - 1}{3} \right\rceil \ge t,$$

since $t \ge 2$. But we have already moved $B + C$ pebbles to r, so we are done. $\qquad \square$

Pachter, Snevily, and Voxman [12] found the pebbling number for graphs with diameter 2. In particular, they showed:

Theorem 4.4 (Pachter *et al.* [12]) *If G is a graph with diameter 2, then $\pi(G) \leq n + 1$.*

Putting Theorems 4.1 and 4.4 together, gives us Corollary 4.5, first conjectured in [5].

Corollary 4.5 *For any integer $t \geq 1$, if G is a graph such that $D(G) = 2$, then $\pi_t(G) \leq n + 4t - 3$.*

Proof: Applying Theorems 4.1 and 4.4, gives $\pi_t(G) \leq \pi(G) + 4t - 4 \leq (n+1) + 4t - 4 = n + 4t - 3$. □

One might ask whether there are any diameter 2 graphs G and any values of t where the inequality in Theorem 4.1 is strict, i. e. $\pi_t(G) < \pi(G) + 4t - 4$. Indeed there are. If $G = W_4$ is the wheel graph with four outside vertices (i. e. K_{14} with the four outside vertices connected in a cycle), it is straightforward to show that $\pi(W_4) = 5$, but $\pi_t(W_4) = 4t$ if $t \geq 2$.

It would be interesting to expand these methods to graphs with larger diameter. Postle, Streib, and Yerger [13] announced Theorem 4.6, strengthening a result of Bukh [1].

Theorem 4.6 (Postle *et al.* [13]) *If G is a graph with diameter 3, then $\pi(G) \leq 1.5n + 2$.*

Let $\pi(n, d)$ denote the maximum pebbling number of an n-vertex graph which has diameter d. Bukh proved Theorem 4.7, and Postle *et al.* postulated Conjecture 4.8.

Theorem 4.7 (Bukh [1]) *We have $\pi(n, d) \in \left(\frac{2^{\lceil \frac{d}{2} \rceil} - 1}{\lceil \frac{d}{2} \rceil} \right) n + \Omega(1)$.*

Conjecture 4.8 (Postle *et al.* [13]) *We have $\pi(n, d) \in \left(\frac{2^{\lceil \frac{d}{2} \rceil} - 1}{\lceil \frac{d}{2} \rceil} \right) n + O(f(d))$ for some function $f(d)$ that is independent of n.*

We show that if Conjecture 4.8 holds, then $f(d)$ is at least exponential in d, for all n. We do this by creating n-vertex graphs of diameter d that have large pebbling numbers for all n and d. Given positive integers n and d, we build the graph $G_{n,d}$ as follows.

If $d = 2k$, choose a vertex v and build $\left\lfloor \frac{n-1}{k} \right\rfloor$ paths of length k beginning at v which are disjoint (except of course at v). If the number of vertices at this point is smaller than n, add one more path of length $n - k \left\lfloor \frac{n-1}{k} \right\rfloor - 1$ which begins at v and is disjoint from the rest of the graph. This is the graph $G_{n,2k}$.

If $d = 2k + 1$, build a clique K_m, where $m = \left\lfloor \frac{n}{k+1} \right\rfloor$. From each vertex in this K_m, build a path of length k which is disjoint from the rest of the graph. If the number of vertices at this point is smaller than n, choose any $v \in V(K_m)$ and add one more path of length $n - m(k + 1)$ which begins at v and is disjoint from the rest of the graph. This is the graph $G_{n,2k+1}$.

It is easy to check that $G_{n,2k}$ and $G_{n,2k+1}$ have n vertices and diameters $2k$ and $2k + 1$, respectively.

Theorem 4.9 $\pi(G_{n,d}) \geq \left(\frac{2^{\lceil \frac{d}{2} \rceil} - 1}{\lceil \frac{d}{2} \rceil} \right) n + \left(2^d - 3 \left(2^{\lceil \frac{d}{2} \rceil} - 1 \right) \right)$ *for all n and d. In particular, if Conjecture 4.8 holds, then $f(d) \in \Omega(2^d)$.*

Proof: If $d = 2k$ for some k, then $G_{n,d}$ is a tree. In a maximal path partition, there is one path of length $2k$ and there are $\left\lfloor \frac{n-1}{k} \right\rfloor - 2$ paths of length k. Thus, from Theorem 2.1, we have

$$
\begin{aligned}
\pi(G_{n,d}) &\geq 2^{2k} + 2^k \left(\left\lfloor \frac{n-1}{k} \right\rfloor - 2 \right) - \left(\left\lfloor \frac{n-1}{k} \right\rfloor - 1 \right) + 1 \\
&= \left(2^k - 1 \right) \left(\left\lfloor \frac{n-1}{k} \right\rfloor \right) + \left(2^{2k} - 2^{k+1} + 2 \right) \\
&\geq \left(2^k - 1 \right) \left(\frac{n}{k} - 1 \right) + \left(2^{2k} - 2^{k+1} + 2 \right) \\
&= \left(\frac{2^{\lceil \frac{d}{2} \rceil} - 1}{\lceil \frac{d}{2} \rceil} \right) n + \left(2^d - 3 \left(2^{\lceil \frac{d}{2} \rceil} - 1 \right) \right).
\end{aligned}
$$

If $d = 2k + 1$ for some k, we build an unsolvable distribution D on $G_{n,d}$. Let the root r be any leaf which is the endpoint of a maximum induced path P. Place $2^{2k+1} - 1$ pebbles on the other endpoint of P. Now, for every vertex (disjoint from P) that is distance k from K_m, place $2^{k+1} - 1$ pebbles. There are $\left\lfloor \frac{n}{k+1} \right\rfloor - 2$ such vertices. It is easy to verify that D cannot send a pebble to r. Thus, since $\pi(G_{n,d}) \geq |D| + 1$,

$$
\pi(G_{n,d}) \geq 2^{2k+1} + \left(2^{k+1} - 1 \right) \left(\left\lfloor \frac{n}{k+1} \right\rfloor - 2 \right)
$$

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$$
\begin{aligned}
&= \left(2^{k+1}-1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right) + \left(2^{2k+1}-2^{k+2}+2\right) \\
&\geq \left(2^{k+1}-1\right)\left(\frac{n}{k+1}-1\right) + \left(2^{2k+1}-2^{k+2}+2\right) \\
&= \left(\frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil}\right)n + \left(2^d - 3\left(2^{\lceil \frac{d}{2} \rceil}-1\right)\right),
\end{aligned}
$$

as desired. □

Conjecture 4.8 would follow from Conjecture 4.10.

Conjecture 4.10 *Let G be any n-vertex graph with diameter d. Then $\pi(G) \leq \pi(G_{n,d})$.*

5 Fractional Pebbling

In this section, we discuss how the t-pebbling number of a graph increases as t increases. We note that for complete graphs, trees, cycles, and indeed for all other graphs G for which $\pi_t(G)$ is known, we have $\pi_{t+1}(G) \leq \pi_t(G) + 2^{D(G)}$ for all t. We raise this observation to the status of a conjecture, and we prove it for large enough t. Conjecture 5.2 is a weaker version of Conjecture 5.1.

Conjecture 5.1 *For every graph G and for every $t \geq 1$, we have $\pi_{t+1}(G) \leq \pi_t(G) + 2^{D(G)}$.*

Conjecture 5.2 *For every graph G and for every $t \geq 1$, we have $\pi_t(G) \leq \pi(G) + 2^{D(G)}(t-1)$.*

Theorem 4.1 proves Conjecture 5.2 for all graphs with diameter 2. Combining Conjecture 5.2 with Theorem 4.6 gives us Conjecture 5.3, and combining it with Conjecture 4.8 gives Conjecture 5.4.

Conjecture 5.3 *If G is a graph with diameter 3, then $\pi_t(G) \leq 1.5n + 8t - 6$.*

Conjecture 5.4 *If G is a graph with diameter d, then $\pi_t(G) \in \left(\frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil}\right)n + 2^d(t-1) + O(f(d))$.*

We show Conjecture 5.1 holds for sufficiently large t after giving one lemma.

Lemma 5.5 *For any graph G and any integer $t \geq 1$, we have $\pi_t(G) \geq 2^{D(G)}t$.*

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Proof: We simply note that placing $2^{D(G)}t - 1$ pebbles on some vertex v would create a situation from which we could not move t pebbles onto another vertex whose distance from v is $D(G)$. □

Theorem 5.6 *For every graph nontrivial graph G with n vertices, and for every $t \geq \left\lceil \frac{n-1}{D(G)} \right\rceil$, we have* $\pi_{t+1}(G) \leq \pi_t(G) + 2^{D(G)}$.

Proof: We let $d = D(G)$. By Lemma 5.5 we have $\pi_t(G) \geq 2^d t$, so

$$\pi_t(G) + 2^d \geq 2^d(t+1) \geq 2^d \left(\frac{n-1}{d} + 1 \right) = \frac{2^d}{d}(n-1) + 2^d \geq \frac{2^d - 1}{d}(n-1) + 1.$$

Therefore, by Theorem 3.1, if we have $\pi_t(G) + 2^d$ pebbles on G, putting the first pebble on any target vertex costs at most 2^d pebbles, so we can use the remaining $\pi_t(G)$ pebbles to put t additional pebbles on the target. □

The fractional pebbling number was defined in [9] as follows:

Definition: The *fractional pebbling number* $\hat{\pi}(G)$ is given by $\hat{\pi}(G) = \liminf_{t \to \infty} \frac{\pi_t(G)}{t}$.

We use Theorem 5.6 to prove that $\hat{\pi}(G) = 2^{D(G)}$ for every graph G, as conjectured in [9].

Theorem 5.7 *For any graph G, we have $\hat{\pi}(G) = 2^{D(G)}$.*

Proof: We let $s = \left\lceil \frac{n-1}{D(G)} \right\rceil$. Applying Theorem 5.6 inductively on t gives $\pi_t(G) \leq \pi_s(G) + (t-s)2^{D(G)}$ for all $t \geq s$. Given $\epsilon > 0$, we let $x = \pi_s(G) - 2^{D(G)}s \geq 0$. Then for any $t \geq \max(\frac{x}{\epsilon}, s)$ we have

$$2^{D(G)}t \leq \pi_t(G) \leq \pi_s(G) + (t-s)2^{D(G)} = 2^{D(G)}t + x.$$

Dividing by t gives

$$2^{D(G)} \leq \frac{\pi_t(G)}{t} \leq 2^{D(G)} + \frac{x}{t} \leq 2^{D(G)} + \epsilon.$$

Thus, $\hat{\pi}(G) = \liminf_{t \to \infty} \frac{\pi_t(G)}{t} = 2^{D(G)}$. □

6   Optimal Fractional Pebbling

In this section, we analyze the concepts of optimal pebbling and fractional pebbling from a combina-

torial standpoint. We see that optimal pebbling can be modeled nicely as an optimization problem.

This in turn leads to a nice combinatorial interpretation of the optimal fractional pebbling number

of a graph. We use this interpretation to obtain a resulting property of vertex-transitive graphs. We

begin by generalizing the definition of a distribution to allow non-integral amounts of pebbles to be

placed on each vertex.

Definition: For a graph G, a function $D : V \to \mathbb{R}^{\geq 0}$ is called a *fractional distribution* on G. As in an

integer-valued distribution, the *size* of D is given by $|D| = \sum_{v \in V} D(v)$.

We give the following definition, which serves to generalize the notion of a pebbling move.

Definition: A *fractional pebbling move* of size $\alpha \in \mathbb{R}^+$ from a vertex v, which has at least 2α pebbles, to

a vertex $u \in N(v)$ removes 2α pebbles from v and places α pebbles on u.

Thus, the pebbles are no longer discrete objects. Instead, they can be viewed as infinitely divisible

"piles." Nevertheless, for a vertex v, a fractional distribution D, and a nonnegative real number α, if

$D(v) = \alpha$, then we say that there are α pebbles on v under D.

Definition: A fractional distribution D on a graph G is called *optimal* if the following two conditions

hold.

1. For every $v \in V$, a pebble can be moved to v after some sequence of fractional pebbling moves,

   starting from D.

2. If D' is a fractional distribution on G with $|D'| < |D|$, then there is some $v \in V$ which cannot be

   reached with a pebble after any sequence of fractional pebbling moves, starting from D'.

   Recall that for a graph G and an integer $t \geq 1$, $\pi_t^*(G)$ is the size of the smallest t-fold solvable

distribution of pebbles on G. Thus, given a t-fold solvable distribution D on G, every $v \in V$ must

have a corresponding sequence of pebbling moves that places t pebbles on v, starting from D. Let $V = \{v_1, v_2, \ldots, v_n\}$. For all i, j, and k, let $p_i(v_j, v_k)$ denote the number of pebbling moves from v_j to v_k in the sequence of moves which places a pebble on v_i. Let us refer to the following integer optimization problem as OPT.

The OPT Integer Optimization Problem: Minimize $\sum_{k=1}^{n} D(v_k)$ subject to the following constraints for each i with $1 \leq i \leq n$, and for all vertices v and w:

$$D(v_i) + \sum_{x \sim v_i} (p_i(x, v_i) - 2p_i(v_i, x)) \geq t$$

$$D(v) + \sum_{x \sim v} (p_i(x, v) - 2p_i(v, x)) \geq 0$$

$$D(v_i) \in \mathbb{N}$$

$$p_i(v, w) \in \mathbb{N}$$

Clearly, every t-fold solvable distribution on G results in a feasible solution to OPT. Indeed, in order to produce such a solution, one would simply record all pebbling moves made from the initial distribution to reach each root. Conversely, Watson [14] shows that every feasible solution to OPT results in a t-fold solvable distribution on G. Thus, the solution to OPT is equal to $\pi_t^*(G)$.

We give the following definition, which is similar to that of $\hat{\pi}(G)$.

Definition: We define the *optimal fractional pebbling number* $\hat{\pi}^*(G)$ by

$$\hat{\pi}^*(G) = \liminf_{t \to \infty} \frac{\pi_t^*(G)}{t}.$$

Suppose that we desire a combinatorial interpretation for $\hat{\pi}^*(G)$. In this spirit, suppose we relax the integer constraints in OPT and set $t = 1$. Let us refer to the following optimization problem as FRAC OPT, and denote its solution by $\overline{\pi}^*(G)$. Moews [11] called $\overline{\pi}^*(G)$ the *continuous optimal pebbling number of G*.

The FRAC OPT Optimization Problem: Minimize $\sum_{k=1}^{n} D(v_k)$ subject to the following constraints for

each i with $1 \leq i \leq n$, and for all vertices v and w:

$$D(v_i) + \sum_{x \sim v_i} (p_i(x, v_i) - 2p_i(v_i, x)) \geq 1$$

$$D(v) + \sum_{x \sim v} (p_i(x, v) - 2p_i(v, x)) \geq 0$$

$$D(v_i) \geq 0$$

$$p_i(v, w) \geq 0$$

We show that $\overline{\pi}^*(G)$ is equal to the optimal fractional pebbling number of G.

Theorem 6.1 *For any graph G, $\overline{\pi}^*(G) = \hat{\pi}^*(G)$.*

Proof: Let G be a graph, with $V = \{v_1, v_2, \ldots, v_n\}$. We first show $\overline{\pi}^*(G) \leq \hat{\pi}^*(G)$. For an integer $t \geq 1$, let D be a t-fold solvable distribution on G with $|D| = \pi_t^*(G)$. Then, for every $v_i \in V$, there are $D(v_i)$ pebbles initially on v_i and there is some sequence of pebbling moves which places t pebbles on v_i. This gives a solution to OPT. In this solution, let $p_i(v_j, v_k)$ be defined as above. Now, let $D'(v_i) = \frac{D(v_i)}{t}$ and let $p_i'(v_j, v_k) = \frac{p_i(v_j, v_k)}{t}$ for all i, j, and k. This gives a feasible solution to FRAC OPT with $|D'| = \frac{\pi_t^*(G)}{t}$. This solution may or may not be optimal. Since this holds for any integer $t \geq 1$, we have $\overline{\pi}^*(G) \leq \hat{\pi}^*(G)$.

We now show that $\overline{\pi}^*(G) \geq \hat{\pi}^*(G)$. Suppose we have a feasible solution to FRAC OPT, with values denoted $\overline{D}(v_i)$ and $\overline{p}_i(v_j, v_k)$ for all i, j, and k. We may assume that every $\overline{D}(v_i)$ and $\overline{p}_i(v_j, v_k)$ is rational, since all of the coefficients are integers. Let t be the least common multiple of the denominators of these values. Let $D'(v_i) = t\overline{D}(v_i)$ and let $p_i'(v_j, v_k) = t\overline{p}_i(v_j, v_k)$ for all i, j, and k. This gives a feasible solution to OPT and thus a t-fold solvable distribution D' on G. Clearly, $\frac{|D'|}{t}$ is the value of the rational solution we were given. However, D' may not be the smallest t-fold solvable distribution on G. Furthermore, we can let $D''(v_i) = ts\overline{D}(v_i)$ and let $p_i''(v_j, v_k) = ts\overline{p}_i(v_j, v_k)$ for all i, j, and k for any positive integer s to obtain a ts-fold solvable distribution on G. Thus, $\overline{\pi}^*(G) \geq \hat{\pi}^*(G)$. □

We can now dispense with the notation $\overline{\pi}^*(G)$. The following corollary provides a combinatorial interpretation for $\hat{\pi}^*(G)$.

Corollary 6.2 *The size of an optimal fractional distribution on a graph G is equal to $\hat{\pi}^*(G)$.*

Proof: From the definition, we see that the size of an optimal fractional distribution on a graph G is equal to the solution to the optimization problem FRAC OPT. The result follows from Theorem 6.1.□

Theorem 6.6 shows that every vertex-transitive graph has an optimal fractional distribution which is uniform. We start with some lemmas, beginning with the following weight argument.

Lemma 6.3 *Let D be a fractional distribution on a graph G. Then there is a sequence of fractional pebbling moves starting from D which places a pebble on $r \in V$ if and only if $\sum_{v \in V} D(v)2^{-\delta(v,r)} \geq 1$.*

Proof: For the "if" direction, we note that a vertex v at distance i from r can send exactly $D(v)2^{-i}$ pebbles to r by making fractional pebbling moves toward r. Thus, if all pebbling moves are directed toward r, then it is possible to send a pebble to r by using fractional pebbling moves.

Conversely, any fractional distribution D' which can be obtained from D satisfies

$$\sum_{v \in V} D(v)2^{-\delta(v,r)} \geq \sum_{v \in V} D'(v)2^{-\delta(v,r)},$$

since any fractional pebbling move toward r keeps this sum constant, whereas all other such moves decrease the sum. In order to place a pebble on r, we must be able to reach a fractional distribution D^* satisfying $D^*(r) = 1$. In particular, we have

$$\sum_{v \in V} D(v)2^{-\delta(v,r)} \geq \sum_{v \in V} D^*(v)2^{-\delta(v,r)} \geq 1,$$

as desired. □

The following lemma is obvious, but useful.

Lemma 6.4 *If $G = (V, E)$ is a vertex-transitive graph, then the function $f : V \to \mathbb{R}^+$ given by $f(r) = \sum_{v \in V} 2^{-\delta(v,r)}$ is constant for all r.* □

Lemma 6.5 *If D and D' are fractional distributions on a vertex-transitive graph G and*

$$\sum_{r \in V} D(r) 2^{-\delta(v,r)} \leq \sum_{r \in V} D'(r) 2^{-\delta(v,r)} \qquad (6)$$

for all $v \in V$, then $|D| \leq |D'|$.

Proof: Let $G = (V, E)$ be a vertex-transitive graph. Summing both sides of (6) over all $v \in V$, we find

$$\sum_{v \in V} \sum_{r \in V} D(r) 2^{-\delta(v,r)} \leq \sum_{v \in V} \sum_{r \in V} D'(r) 2^{-\delta(v,r)}$$

Switching the order of the summation gives us

$$\sum_{r \in V} D(r) \sum_{v \in V} 2^{-\delta(v,r)} \leq \sum_{r \in V} D'(r) \sum_{v \in V} 2^{-\delta(v,r)}.$$

But by Lemma 6.4, $\sum_{v \in V} 2^{-\delta(v,r)}$ is a constant for all $r \in V$, so dividing by this constant gives us $\sum_{r \in V} D(r) \leq \sum_{r \in V} D'(r)$, or $|D| < |D'|$ $\qquad\qquad\square$

We are now ready to show the main result for this section.

Theorem 6.6 *If G is a vertex-transitive graph, an optimal fractional distribution on G is obtained by putting $\frac{1}{m}$ pebbles on each vertex in G, where m is the constant $\sum_{v \in V} 2^{-\delta(v,r)}$ from Lemma 6.4. Therefore, $\hat{\pi}^*(G) = \frac{n}{m}$.*

Proof: Let D be the distribution in question. Note that for all $r \in V$, we have

$$\sum_{v \in V} D(v) 2^{-\delta(v,r)} = \frac{1}{m} \sum_{v \in V} 2^{-\delta(v,r)} = 1,$$

so by Lemma 6.3, starting from D, each root r can receive a pebble by making fractional pebbling moves toward r. Therefore, $\hat{\pi}^*(G) \leq |D|$.

Now let D' be another fractional distribution from which one pebble can be moved to any vertex. By Lemma 6.3, we have

$$\sum_{v \in V} D'(v) 2^{-\delta(v,r)} \geq 1 = \sum_{v \in V} D(v) 2^{-\delta(v,r)}$$

for all $v \in V$, and by Lemma 6.5, this implies $|D'| \geq |D|$. Therefore, D is optimal, so $\hat{\pi}^*(G) = |D|$. $\quad\square$

Corollary 6.7 gives $\hat{\pi}^*(G)$ for several vertex-transitive graphs. Moews [11] also proved part 1.

Corollary 6.7 *Let k and n be positive integers. Then we have the following.*

1. $\hat{\pi}^*(Q^k) = \left(\dfrac{4}{3}\right)^k$ *where Q^k denotes the k-dimensional hypercube.*

2. $\hat{\pi}^*(K_n) = \dfrac{2n}{n+1}$.

3. *If $k \geq 2$, then $\hat{\pi}^*(C_{2k}) = \dfrac{k2^{k+1}}{3(2^k - 1)}$.*

4. $\hat{\pi}^*(C_{2k+1}) = \dfrac{(2k+1)(2^{k-1})}{3(2^{k-1}) - 1}$.

Proof: By Theorem 6.6, in each case it suffices to find the value of m. For the hypercube, if we fix a target r, there are $\binom{k}{i}$ vertices whose distance from r is i. We compute m as follows, using the Binomial Theorem:

$$m = \sum_{v \in V} 2^{-\delta(v,r)} = \sum_{i=1}^{k} \binom{k}{i} \frac{1}{2^i} = \left(\frac{3}{2}\right)^k.$$

Therefore, $\hat{\pi}^*(Q^k) = \frac{n}{m} = \frac{2^k}{\left(\frac{3}{2}\right)^k} = \left(\frac{4}{3}\right)^k$.

For K_n every vertex $v \neq r$ has $\delta(v,r) = 1$, so $m = 1 + \displaystyle\sum_{v \in V; v \neq r} \frac{1}{2} = 1 + \frac{n-1}{2} = \frac{n+1}{2}$, and $\hat{\pi}^*(K_n) = \frac{n}{\frac{n+1}{2}} = \frac{2n}{n+1}$.

For C_n we assume the vertex set is $\{x_0, x_1, \ldots, x_{n-1}\}$ and that $r = x_0$ is the target. If $n = 2k$, we let $A = \{x_i : i < k\}$, and we note that for every x_{k+i} with $0 \leq i \leq k-1$ we have $\delta(x_{k+i}, x_0) = k - i$. Therefore, computing m gives

$$m = \sum_{v \in V} 2^{-\delta(v,x_0)} = \sum_{v \in A} 2^{-\delta(v,x_0)} + \sum_{v \notin A} 2^{-\delta(v,x_0)} = \sum_{i=0}^{k-1} 2^{-i} + \sum_{i=0}^{k-1} 2^{-(k-i)}.$$

Substituting $j = k - 1 - i$ in the last summation gives

$$m = \sum_{i=0}^{k-1} 2^{-i} + \sum_{j=0}^{k-1} 2^{-(j+1)} = \sum_{i=0}^{k-1} 2^{-i} + \frac{1}{2}\sum_{j=0}^{k-1} 2^{-j} = \frac{3}{2}\left(2 - \frac{1}{2^{k-1}}\right) = \frac{3(2^k - 1)}{2^k}.$$

Therefore, $\hat{\pi}^*(C_{2k}) = \frac{n}{m} = \frac{2k(2^k)}{3(2^k-1)} = \frac{k2^{k+1}}{3(2^k-1)}$.

Finally, for C_{2k+1} we let $A = \{x_i : 1 \le i \le k\}$ and $B = \{x_i : k+1 \le i \le 2k\}$. Now $\delta(x_{k+i}, x_0) = k - i + 1$, so we have

$$m = \sum_{v \in V} 2^{-\delta(v,x_0)} = 2^{-\delta(x_0,x_0)} + \sum_{v \in A} 2^{-\delta(v,x_0)} + \sum_{v \in B} 2^{-\delta(v,x_0)} = 1 + \sum_{i=1}^{k} 2^{-i} + \sum_{i=1}^{k} 2^{-(k-i+1)}.$$

Now substituting $j = k - i + 1$ gives

$$m = 1 + \sum_{i=1}^{k} 2^{-i} + \sum_{j=1}^{k} 2^{-j} = 1 + 2 \sum_{i=1}^{k} 2^{-i} = 1 + 2 \left(1 - \frac{1}{2^k}\right) = 3 - \frac{1}{2^{k-1}} = \frac{3(2^{k-1}) - 1}{2^{k-1}}.$$

Therefore, $\hat{\pi}^*(C_{2k+1}) = \frac{n}{m} = \frac{(2k+1)(2^{k-1})}{3(2^{k-1})-1}$. $\square$

7   Arbitrary Target Distributions with t Pebbles

In this section, we consider how many pebbles are required to reach an arbitrary target distribution with t pebbles.

Definition: We define $\pi(G, t)$ as the smallest number of pebbles such that any target distribution D with $|D| = t$ is reachable from every distribution D' with $|D'| \ge \pi(G, t)$.

Clearly if we can reach any distribution with t pebbles starting from D, we can reach any distribution with t pebbles on a single vertex. Therefore, $\pi_t(G) \le \pi(G, t)$ for every positive integer t. Conversely, it seems reasonable to believe that if we have a distribution of pebbles from which we can put t pebbles on any single vertex of G, then any other distribution of t pebbles is likewise reachable. For example, if we can put two pebbles either on the vertex x or the vertex y, then we should be able to put one pebble each on x and y. This suggests the following conjecture.

Conjecture 7.1 *For every graph G and every positive integer t, we have $\pi(G, t) = \pi_t(G)$.*

We prove this conjecture for some common graphs. We start with two lemmas.

Lemma 7.2 *Suppose G is a graph with the property that, for some t, whenever $\pi_{t+1}(G)$ pebbles are on G, one*

pebble can be moved to any vertex at a cost of at most $\pi_{t+1}(G) - \pi(G,t)$ pebbles. Then $\pi_{t+1}(G) = \pi(G, t+1)$.

Proof: Let D be a distribution on G with $t + 1$ pebbles. Given a distribution of $\pi_{t+1}(G)$ pebbles on G, choose one occupied vertex v in D, and spend $\pi_{t+1}(G) - \pi(G,t)$ pebbles to move a pebble to v. The remaining $\pi(G,t)$ pebbles can be used to move t additional pebbles to fill out the rest of D. $\qquad\square$

Lemma 7.3 *Suppose G is a graph with the property that for every t, if $\pi_{t+1}(G)$ pebbles are on G, one pebble can be moved to any vertex at a cost of at most $\pi_{t+1}(G) - \pi_t(G)$ pebbles. Then $\pi_t(G) = \pi(G, t)$ for all t.*

Proof: We use induction on t. When $t = 1$ we have $\pi_1(G) = \pi(G, 1) = \pi(G)$ since the target distributions are the same in either case. For larger t, if $\pi_t(G) = \pi(G,t)$, then $\pi_{t+1}(G) - \pi_t(G) = \pi_{t+1}(G) - \pi(G,t)$, so by Lemma 7.2, $\pi_{t+1}(G) = \pi(G, t+1)$.

Theorems 7.4 and 7.5 gives some classes of graphs for which Conjecture 7.1 holds.

Theorem 7.4 *Let G be any graph such that $\pi(G) = 2^{D(G)}$. Then for any $t \geq 1$, $\pi_t(G) = \pi(G,t) = 2^{D(G)}t$. In particular, Conjecture 7.1 holds for even cycles and hypercubes.*

Proof: By Lemma 5.5, $2^{D(G)}t \leq \pi_t(G)$. Conversely, given $2^{D(G)}t$ pebbles, we can split them into t groups of $2^{D(G)}$ pebbles each. Then each group can be matched to a different pebble in any target distribution with t pebbles. Thus, $2^{D(G)}t \leq \pi_t(G) \leq \pi(G,t) \leq 2^{D(G)}t$, so $\pi_t(G) = \pi(G,t)$. $\qquad\square$

Theorem 7.5 *If G is a complete graph, a tree, or a cycle, then $\pi_t(G) = \pi(G,t)$.*

Proof: If $G = K_n$ we have $\pi_t(G) = n + 2t - 2$, and whenever $\pi_{t+1}(K_n) = n + 2t$ pebbles are on K_n, one pebble may be moved to any vertex at a cost of at most two pebbles. If G is a tree, by Theorem 2.1 and Proposition 2.2, the cost of putting a pebble on any vertex is at most $2^{D(G)} = \pi_{t+1}(G) - \pi_t(G)$.

If G is an even cycle we can apply Theorem 7.4, so suppose $G = C_n$ is an odd cycle with vertices $\{x_1, x_2, \ldots, x_n\}$ in order with $n = 2k + 1$. We may assume without loss of generality that x_n is

the target vertex. By Proposition 2.3, $\pi_t(C_n)$ is given by $\pi_t(C_n) = \frac{2^{k+2}-(-1)^k}{3} + 2^k(t-1)$. Thus, $\pi_{t+1}(C_n) - \pi_t(C_n) = 2^k$, and $\pi_t(G) \geq 2^{k+1}$ when $t \geq 2$. In particular, if we have $\pi_{t+1}(G) \geq 2^{k+1}$ pebbles on C_n, either we have 2^k pebbles on the vertices $\{x_n, x_1, x_2, \ldots, x_k\}$ or we have 2^k pebbles on the vertices $\{x_n, x_{n-1}, x_{n-2}, \ldots, x_{k+1}\}$. Since these vertex sets each induce the subgraph P_{k+1}, we can move a pebble to x_n at a cost of at most $\pi(P_{k+1}) = 2^k$ pebbles. $\qquad\square$

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