THE ANTIPODAL LAYERS PROBLEM

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Abstract. For $n > 2k$ and $[n] = \{1, 2, \ldots, n\}$, let the bipartite graph $\mathcal{M}_{n,k}$ have vertices $\{A \subset [n] \,\big|\, |A| = k \text{ or } n-k\}$ and edges $\{(A, B) \,\big|\, A \subset B\}$. It has been conjectured that $\mathcal{M}_{2k+1,k}$ (the middle two levels of the Boolean Lattice $\mathcal{Q}^{2k+1}$) is Hamiltonian, and we conjecture the same for arbitrary n. Here we show that the conjecture holds for n bigger than roughly k^2, with k large enough. We also define a new product between ranked posets, giving rise to many new representations of $\mathcal{M}_{2k+1,k}$.

1 Introduction

For odd $n = 2k+1$ let $\mathcal{M}_k$ denote the middle two levels of the n-dimensional cube $\mathcal{Q}^n$, i.e., $\mathcal{M}_k$ is the bipartite graph with vertex set $\{S \subset [n] \,\big|\, |S| = k \text{ or } n-k\}$ and edge set $\{(S,T) \,\big|\, S \subset T\}$. What has come to be known as the *Middle Layers Problem* is the following question. Is $\mathcal{M}_k$ Hamiltonian? We discuss this problem in section 2, offering many new representations for $\mathcal{M}_k$, while in section 3 we tackle the following generalization. For $n > 2k$, $\mathcal{M}_{n,k}$ is the bipartite graph with vertex set $\{A \subset [n] \,\big|\, |A| = k \text{ or } n-k\}$ and edge set $\{(A,B) \,\big|\, A \subset B\}$. Notice that $\mathcal{M}_{2k+1,k} \cong \mathcal{M}_k$ (all our isomorphisms will be graph, not poset, isomorphisms). Our conjecture that $\mathcal{M}_{n,k}$ is Hamiltonian is what we call the *Antipodal Layers Problem*, and we will prove the conjecture is true for $n > ck^2 + k$, k large enough.

2 Middle Layers

The *Middle Layers Problem* first appeared in [H] and again as problem 2.5 in [K]. It has been verified [MR] for $k \leq 11$. In [KT] the problem is attacked with the understanding that any Hamiltonian cycle in $\mathcal{M}_k$ is the union of two perfect matchings. There, Kierstead and Trotter generalize the notion of *lexicographic* matchings to *lexical* matchings, in hopes that some pair will work. In [DSW] it is shown that this cannot be the case with lexicographic matchings.

As $\mathcal{M}_k$ is both vertex-transitive and edge-transitive, one would imagine these properties to aid in the construction of a Hamiltonian cycle. In fact, there are exactly $\binom{2k+1}{2t+1}\binom{2(k-t)}{k-t}$ isomorphic copies of $\mathcal{M}_t$ in $\mathcal{M}_k$ (as vertex-induced subgraphs), and $\mathrm{Aut}(\mathcal{M}_k)$ acts transitively on each of these families of subgraphs. (That is, given two isomorphic copies, $\mathcal{M}^1$

and $\mathcal{M}^2$, of $\mathcal{M}_t$ in $\mathcal{M}_k$, there is some $\sigma \in \mathrm{Aut}(\mathcal{M}_k)$ such that $\sigma(\mathcal{M}^1) = \mathcal{M}^2$.) Although symmetry is exploited to some degree in [A], [CDQ], and [DHR], it seems empirically that the greater the degree of symmetry used by an algorithm, the smaller the cycle it constructs. Babai shows in [Ba] that there is always a cycle of length at least $\sqrt{3m}$ in a vertex-transitive graph on m vertices, and recently Savage [S] has found cycles of length roughly $m^{.846}$ in $\mathcal{M}_k$.

Induction has been yet another tool to fall short of the task so far. Figures 1 and 2 hint at how the different subgraphs $\mathcal{M}_t$, $t < k$, can be embedded in $\mathcal{M}_k$. To understand the diagrams, we must define an unusual *bowtie* product, $\bowtie$, between the Hasse diagrams of two ranked posets. Let the *height* of a poset be the largest rank (i.e., one less than the length of the longest chain). If $\mathcal{A}$ and $\mathcal{B}$ are the Hasse diagrams of ranked posets of heights $h+1$ and h, respectively, then we define the product $\mathcal{A} \bowtie \mathcal{B}$ to have the vertex set

$$V(\mathcal{A} \bowtie \mathcal{B}) = \Big\{(a,b) \,\Big|\, R_{\mathcal{A}}(a) = R_{\mathcal{B}}(b) \text{ or } R_{\mathcal{B}}(b)+1\Big\}$$

and edge set

$$\begin{aligned} E(\mathcal{A} \bowtie \mathcal{B}) = &\Big\{\big((a,b_1),(a,b_2)\big) \,\Big|\, (b_1,b_2) \in E(\mathcal{B})\Big\} \\ &\cup \Big\{\big((a_1,b),(a_2,b)\big) \,\Big|\, (a_1,a_2) \in E(\mathcal{A})\Big\}. \end{aligned}$$

(Here, $R_{\mathcal{P}}(x)$ is the rank of x in the poset represented by $\mathcal{P}$). Notice that $\mathcal{A} \bowtie \mathcal{B}$ is well defined whereas $\mathcal{B} \bowtie \mathcal{A}$ is not. Also notice that the bowtie product can be extended to the posets represented by $\mathcal{A}$ and $\mathcal{B}$ in the obvious way (preserving coordinate-wise order), obtaining a ranked poset with $R_{\mathcal{A}\bowtie\mathcal{B}}(a,b) = R_{\mathcal{A}}(a) + R_{\mathcal{B}}(b)$.

Given $t \le n$ and $t \equiv n \bmod 2$ let us now define another graph, $\mathcal{N}_t(n)$, to be the Hasse diagram of the poset whose elements are all subsets of $[n]$ of size s for $(n-t)/2 \le s \le (n+t)/2$, i.e., the middle $t+1$ levels of $\mathcal{Q}^n$. As usual, edges are induced by set inclusion. Of course, $\mathcal{N}_1(2k+1) \cong \mathcal{M}_k$ and $\mathcal{N}_n(n) \cong \mathcal{Q}^n$. The following theorem motivates our discussion of these graphs and of our "bowtie product" as well.

Theorem 1 *For $n = 2k+1$ and $t \leq k$, $\mathcal{M}_k \cong \mathcal{N}_{t+1}(n-t) \bowtie \mathcal{Q}^t$.*

The cases of greatest interest are when $t = 2$ or k. For $t = 2$ we find two copies of $\mathcal{M}_{k-1}$ at the middle two ranks of our diagram and are smacked with thoughts of induction. In general, for $t = 2s$, we find $\binom{2s}{s}$ copies of $\mathcal{M}_{k-s}$ at the middle two ranks of our diagram. For $t = k$ we see that $\mathcal{M}_k \cong \mathcal{Q}^{k+1} \bowtie \mathcal{Q}^k$ and we hope that Hamiltonian cycles in these cubes might somehow be used to construct one in $\mathcal{M}_k$.

Proof of Theorem 1.

Let $S = [n-t]$, $T = \{n-t+1, \ldots, n\}$. We view the vertices of $\mathcal{N} = \mathcal{N}_{t+1}(n-t)$ as the subsets of S of size s, with $k-t \leq s \leq k+1$, and the vertices of $\mathcal{Q} = \mathcal{Q}^t$ as the subsets of T. Define $f : V(\mathcal{M}_k) \to V(\mathcal{N} \bowtie \mathcal{Q})$ as follows. For $A \subset [n]$, $|A| = k$ or $k+1$, let $A_S = A \cap S$, $A_T = A \cap T$, and $A'_T = T \backslash A_T$. Then let $f(A) = (A_S, A'_T)$. To prove that f is a graph isomorphism we first show that it is a vertex isomorphism, then also an edge isomorphism.

In the case of vertices, if $|A| = k$ then $|A_S| = s$ implies that $|A_T| = k-s$ and $|A'_T| = t-k+s$. Also, $R_{\mathcal{N}}(x) = |x| - (k-t)$ since the sets of rank 0 have size $(k-t)$, while $R_{\mathcal{Q}}(x) = |x|$. Thus, $R_{\mathcal{N}}(A_S) = R_{\mathcal{Q}}(A'_T) = t-k+s$, with $k-t \leq s \leq k$. Likewise, if $|A| = k+1$ then $R_{\mathcal{N}}(A_S) - 1 = R_{\mathcal{Q}}(A'_T) = t-k-1+s$, with $k-t+1 \leq s \leq k+1$. Hence f is well-defined, and it is clearly one-to-one and onto.

Now suppose that $(A, B) \in E(\mathcal{M}_k)$ with $|A| = k$ and $|B| = k+1$. Then $B = A \cup \{b\}$ for some $b \in [n] \backslash A$. If $b \in S$ then $B_S = A_S \cup \{b\}$, $B_T = A_T$, and $B'_T = A'_T$, which means that $(B_S, A_S) \in E(\mathcal{N})$ and that $\big((B_S, A_S), (B'_T, A'_T)\big) \in E(\mathcal{N} \bowtie \mathcal{Q})$. Similarly, if $b \in T$ then $B_S = A_S$, $B_T = A_T \cup \{b\}$, and $B'_T = A'_T \backslash \{b\}$, so $(B'_T, A'_T) \in E(\mathcal{Q})$ and

$\big((B_S, A_S), (B'_T, A'_T)\big) \in E(\mathcal{N} \bowtie \mathcal{Q})$. Again f, being well-defined for edges, is clearly one-to-one and onto. $\square$

As an example, in figure 2, the circled vertices in $\mathcal{N}$, $\mathcal{Q}$, and $\mathcal{M}$ are $B_S = \{1, 2, 5\}$, $B'_T = \{7\}$ (so $B_T = \{6\}$), and $B = \{1, 2, 5, 6\}$, respectively.

3 Antipodal Layers

The *Antipodal Layers Problem* was first posed by the author during the problem session of the 4$^{\text{th}}$ SIAM Conference on Discrete Mathematics, June 1988. It is also noted in [Go] and attributed to Roth as a personal communication. The graph $\mathcal{M}_{n,k}$ is vertex- and edge-transitive but cannot be written so nicely as a bowtie product. Nonetheless, we prove the following theorem.

Theorem 2 *Given any $\varepsilon > 0$ there is an integer $k(\varepsilon)$ such that, for $k > k(\varepsilon)$ and*

$$c_1 = \frac{1}{\ln 3} + \varepsilon, \qquad c_2 = \frac{2}{\ln 6} + \varepsilon,$$

$\mathcal{M}_{n,k}$ is Hamiltonian for

$$\text{i)} \quad \binom{n}{k} \ \textit{odd and } n \geq c_1 k^2 + k, \ \textit{and}$$

$$\text{ii)} \quad \binom{n}{k} \ \textit{even and } n \geq c_2 k^2 + k.$$

One might notice that a constant of $c = 1/\ln(3/2) = 2.466\ldots$ can be obtained quite easily by using Fact 2, below, on the graph $\mathcal{M}_{n,k}$ itself. Our motivation here was to lower the constant, obtaining $c_1 = 0.9102\ldots$ and $c_2 = 1.116\ldots$. In fact, one might simplify the statement of theorem 2 somewhat to say for example that, for $k \geq 6$, $\mathcal{M}_{n,k}$ is Hamiltonian for odd $\binom{n}{k}$, $n \geq k^2 + k$, and even $\binom{n}{k}$, $n \geq 1.3k^2 + k$. Of course, an improvement in the exponent of k is preferred.

Proof.

We will use two well-known facts about graphs. Let $\mathcal{G}$ be a graph on m vertices $v_1, \ldots, v_m$ with degrees $d_1, \ldots, d_m$.

Fact 1. (Dirac) If $d_i \geq n/2$ for all i then $\mathcal{G}$ is Hamiltonian (see [BM]).

Fact 2. (Jackson) If $\mathcal{G}$ is 2-connected and $d_i = d \geq n/3$ for all i then $\mathcal{G}$ is Hamiltonian (see [J]).

The *Odd Graph*, $\mathcal{O}_k$, is defined to be the graph with vertices all k-subsets of $[2k+1]$ and edges joining disjoint subsets (see [Bi]). We define the *Generalized Odd Graph* (also known as *Kneser's Graph*), $\mathcal{O}_{n,k}$, to have as vertices all k-subsets of $[n]$, $n > 2k$, with disjoint subsets adjacent. Notice that $\mathcal{O}_{n,k}$ is 2-connected and regular with $d = \binom{n-k}{k}$. In [DHR] it was observed that $\mathcal{M}_k \cong \mathcal{O}_k \times k_2$, where x is the *weak* product $(((x_1, x_2), (y_1, y_2)) \in E(G_1 \times G_2)$ if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2))$. Here we observe that $\mathcal{M}_{n,k} \cong \mathcal{O}_{n,k} \times k_2$ as follows.

For $A, B \subset [n]$, $|A| = |B| = k$, $(A, \overline{B} \in E(\mathcal{M}_{n,k})$ if and only if $A \cap B = \emptyset$. If we denote A by $(A, 0)$ and $\overline{B}$ by $(B, 1)$, then clearly we have $((A, 0), (B, 1)) \in E(\mathcal{O}_{n,k} \times K_2)$ if and only if $A \cap B = \emptyset$ (i.e., $(A, B) \in E(\mathcal{O}_{n,k})$ and $(0, 1) \in E(K_2))$. For convenience we will hereafter let $\mathcal{M} = \mathcal{M}_{n,k}$ and $\mathcal{O} = \mathcal{O}_{n,k}$.

First we define a function

$$g : V(\mathcal{M}) \to V(\mathcal{O})$$

by

$$g(A) = \begin{cases} A & \text{if } |A| = k \\ \overline{A} & \text{if } |A| = n - k, \end{cases}$$

where $\overline{A} = [n] \backslash A$. Then g is two-to-one since $g(A) = g(\overline{A}) = A$ for $|A| = k$. Likewise, g

extended to a function on the edges is two-to-one since, for $|A| = |B| = k$ with $A \cap B = \emptyset$, we have $g\left((\overline{A}, B)\right) = g\left((A, \overline{B})\right) = (A, B)$. When S is a set of edges in $E(\mathcal{O})$ we denote by $g(S)$ its set of preimages in $E(\mathcal{M})$.

Let us take the case that $m = |\mathcal{O}| = \binom{n}{k}$ is odd. Then for $n \geq c_1k^2 + k$ and $k > k(\varepsilon)$, we have

$$\begin{aligned} \frac{d}{m} &= \frac{\binom{n-k}{k}}{\binom{n}{k}} = \frac{(n-k)_k}{(n)_k} > \left(\frac{n-2k}{n-k}\right)^k \\ &= \left(1 - \frac{k}{n-k}\right)^k \geq \left(1 - \frac{1}{c_1k}\right)^k > \frac{1}{3}. \end{aligned}$$

The last inequality holds because $(1 - 1/c_1k)^k$ tends to $e^{-1/c_1} = 3^{-1+\varepsilon'}$ as k tends to infinity. Hence, fact 2 tells us we have a Hamiltonian cycle $\mathcal{C}$ in $\mathcal{O}$. And since m is odd, $g^{-1}(\mathcal{C})$ is a Hamiltonian cycle in $\mathcal{M}$. Indeed, if $\mathcal{C} = (A_1, A_2, \ldots, A_m)$ then $g^{-1}(\mathcal{C}) = (A_1, \overline{A}_2, \ldots, A_m, \overline{A}_1, A_2, \ldots, \overline{A}_m)$.

Figure 3: Combining "half-cycles".

When m is even $g^{-1}(\mathcal{C})$ splits into two "half-cycles," $\mathcal{C}_1$ and $\mathcal{C}_2$, with vertex $A \in \mathcal{C}_1$ if and only if $\overline{A} \in \mathcal{C}_2$. Figure 3 gives an example, with $n = 8$ and $k = 2$, of how we can combine the two into one Hamiltonian cycle $\mathcal{C}_{\mathcal{M}}$ in $\mathcal{M}$. We follow $\mathcal{C}_1$ from the left to $\{1, 2\}$, jump across to $\mathcal{C}_2$ at $\overline{\{5,6\}}$, follow $\mathcal{C}_2$ left and around to $\{7,8\}$, jump back to $\mathcal{C}_1$ at $\overline{\{3,4\}}$, and continue right and around back to $\{1,2\}$.

The crucial ingredient in this construction is the fact that the vertices $\{1,2\},\{3,4\},\{5,6\}$ and $\{7,8\}$ are mutually adjacent in $\mathcal{O}$. The trick will be to guarantee that some Hamiltonian cycle $\mathcal{C}$ has four such vertices in a row.

Define $\mathcal{O}' = \mathcal{O}'_{n,k}$ as follows. Let $A = \{1,2,\ldots,k\}$, $B = \{k+1,k+2,\ldots,2k\}$, $C = \{2k+1,2k+2,\ldots,3k\}$, and $D = \{3k+1,3k+2,\ldots,4k\}$ be four mutually adjacent vertices in $\mathcal{O}$. Let $V(\mathcal{O}') = \{X\} \cup V'$, where $V' = V(\mathcal{O})\backslash\{A,B,C,D\}$. Let $\mathcal{O}'$ have edges (E,F), with $E,F \in V'$ and adjacent in $\mathcal{O}$, and (E,X), with $E \in V'$ and adjacent to at least two of A,B,C, and D in $\mathcal{O}$.

Now if $\mathcal{O}'$ is Hamiltonian then $\mathcal{O}$ has a Hamiltonian cycle of the type we need, as the following argument shows. If $\mathcal{C}' = (\ldots,E,X,F,\ldots)$ is a Hamiltonian cycle in $\mathcal{O}'$ then, without loss of generality, $(E,A),(F,D) \in E(\mathcal{O})$. Hence $\mathcal{C} = (\ldots,E,A,B,C,D,F,\ldots)$ is the desired cycle in $\mathcal{O}$. By replacing $\{1,2\},\overline{\{3,4\}},\{5,6\}$ and $\overline{\{7,8\}}$ in Figure 3 by $A,\overline{B},C,$ and $\overline{D}$ and so on, we see that $\mathcal{M}$ is Hamiltonian. So it remains to prove that $\mathcal{O}'$ is Hamiltonian.

If $E \in V'$ then $d(E) \geq \binom{n-k}{k} - 3$, whereas

$$d(X) = \binom{4}{2}\binom{n-2k}{k} - \binom{4}{1}\binom{n-3k}{k} + \binom{4}{0}\binom{n-4k}{k} \geq 3\binom{n-2k}{k}$$

(since $\binom{n-2k}{k}+\binom{n-4k}{k} \geq 2\binom{n-3k}{k}$). So for $n \geq c_2k^2+k$ and $k > k(\varepsilon)$ we have, for all degrees d in $\mathcal{O}'$,

$$\begin{aligned}\frac{d}{m} &> 3\frac{\binom{n-2k}{k}}{\binom{n}{k}} = 3\frac{(n-2k)_k}{(n)_k} > 3\left(\frac{n-3k}{n-k}\right)^k \\ &= 3\left(1-\frac{2k}{n-k}\right)^k \geq 3\left(1-\frac{2}{c_2k}\right)^k > \frac{1}{2}.\end{aligned}$$

Thus, fact 1 implies that $\mathcal{O}'$ is Hamiltonian, which completes the proof. $\square$

4 Remarks

We are constantly reminded by this problem of a conjecture of Lovász (see [L]) which states that every vertex-transitive graph contains a Hamiltonian path. It has also been conjectured that, other than the Peterson and Coxeter graphs and two related to them, all vertex-transitive graphs are Hamiltonian (see [Bo]). Another conjecture of interest is that all Cayley graphs are Hamiltonian (for the definition of a Cayley graph see chapter 16 of [Bi]). Due to the high degree of symmetry in $\mathcal{M}_{n,k}(\mathrm{Aut}(\mathcal{M}_{n,k}) \cong \mathcal{Z}_2 \times \mathcal{S}_n$ for all $n > 2k)$ we pose the following problem.

Problem 1. Determine for which n and k $\mathcal{M}_{n,k}$ is a Cayley graph.

In [G] this problem was resolved for $\mathcal{O}_{n,k}$. In particular, it was discovered that $\mathcal{O}_{2k+1,k}$ is never a Cayley graph (notice that $\mathcal{O}_{5,2}$ is the Peterson graph). This was to answer a question first raised by Biggs.

As regards the bowtie product we offer

Problem 2. Is it true that for $\mathcal{A}$ and $\mathcal{B}$ Hamiltonian, $\mathcal{A} \bowtie \mathcal{B}$ is also Hamiltonian?

In particular, is it true if regularity or vertex-transitivity also holds? Clearly $\mathcal{A}$ and $\mathcal{B}$ regular or vertex-transitive implies the same of $\mathcal{A} \bowtie \mathcal{B}$. An affirmative answer would settle the *Middle Layers Problem* but not the *Antipodal Layers Problem.*

Naturally, we must also offer

Problem 3. Determine if $\mathcal{N}_t(n)$ is Hamiltonian.

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