On the Pebbling Threshold Spectrum

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Definitions

Suppose p pebbles are distributed onto the vertices of a simple connected graph G having $n(G) = |V(G)|$ vertices. A pebbling step consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be *moved* to a vertex r if we can repeatedly apply pebbling steps so that, in the resulting distribution, r has at least one pebble. We say that a distribution is *solvable* if a pebble can be moved to any given *root* vertex r. Finally we define the *pebbling number*, $pn(G)$, to be the smallest integer m such that every distribution of m pebbles on the vertices of G is solvable.

Origins

The concept of pebbling in graphs arose from an attempt by Lagarias and Saks to give an alternative proof of a theorem of Kleitman and Lemke. An elementary result in number theory which follows from the pigeonhole principle is that for any set $N = \{n_1, \ldots, n_q\}$ of q natural numbers, there is a nonempty index set $I \subset \{1, \ldots, q\}$ such that $q | \sum_{i \in I} n_i$. Erdős and Lemke conjectured in 1987 that the extra condition $\sum_{i \in I} n_i \leq \mathrm{lcm}\{q, n_1, \ldots, n_q\}$ could also be guaranteed. In 1989 Kleitman and Lemke proved the conjecture by replacing the Erdős-Lemke conclusion by the stronger conclusion that $\sum_{i \in I} \gcd(n_i, q) \leq q$.

Theorem 1 *For any set $N = \{n_1, \ldots, n_q\}$ of q natural numbers, there is a nonempty index set $I \subset \{1, \ldots, q\}$ such that $q | \sum_{i \in I} n_i$ and $\sum_{i \in I} \gcd(n_i, q) \leq q$.*

The proof offered by Kleitman and Lemke had many cases and did not seem to be the most natural proof. It was the intention of Lagarias and Saks to introduce graph pebbling as a more intuitive vehicle for proving their result. If the formula for a more general pebbling number of a cartesian product of paths is as was believed then the number theory result would follow easily. It was Chung who finally pinned down such a formula. Denley extended the application of pebbling to prove that if each $n_i | q$ (with $n_i \leq n_{i+1}$) and $\sum_{\text{prime } p|q} 1/p \leq 1$, then there is a nonempty I such that $q = \sum_{i \in I} n_i$ and $n_i | n_j$ for all $i < j$.

Kleitman and Lemke went on to make more general conjectures on groups (we write groups additively).

Conjecture 2 *Let G be a finite group of order q with identity e, and let $|g|$ denote the order of the element g in G. Then for any multisubset $N = \{g_1, \ldots, g_q\}$ of G there is a nonempty I such that $\sum_{i \in I} g_i = e$ and $\sum_{i \in I} 1/|g_i| \leq 1$.*

Their prior theorem is merely the case $G = \mathbf{Z}_q$, and they verified this conjecture for $G = \mathbf{Z}_p^n$, for dihedral G, and also for all $q \leq 15$.

Conjecture 3 *Let H be a subgroup of a group G with $|G|/|H| = q$ and let $N = \{g_1, \ldots, g_q\}$ be any multisubset of G. Then there is a nonempty I such that $\sum_{i \in I} g_i \in H$ and $\sum_{i \in I} 1/|g_i| \leq 1/|\sum_{i=1}^{q} g_i|$.*

The first conjecture is the case $H = \{e\}$ here, and this second conjecture they verified for all $|G| \leq 11$. We have been using pebbling to make progress toward this conjecture in the case that G is simple.

Pebbling Numbers

Here we briefly note some of the basic results of graph pebbling in order to familiarize the reader before leading into the theorems we will present.

If one pebble is placed at each vertex other than the root vertex, r, then no pebble can be moved to r. Also, if w is at distance d from r, and $2^d - 1$ pebbles are placed at w, then no pebble can be moved to r. Thus we have that $pn(g) \geq max\{n(G), 2^{diam(G)}\}$. These bounds are tight, as of course, $pn(K_n) = n$ and $pn(P_n) = 2^{n-1}$, where K_n is the clique on n vertices and P_n is the path with n vertices. The pebbling numbers of cycles is derived by Pachter, Snevily and Voxman. They prove that $pn(C_{2k}) = 2^k$ and $pn(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$ for $k \geq 1$.

The formula for the pebbling number of a tree is worked out by Moews. It involves finding the maximum path partition of the tree. For the n-dimensional cube Q^n, Chung proved that $pn(Q^n) = 2^n$. Chung's result would follow easily from Graham's conjecture, which has generated a great deal of interest. For any two graphs G_1 and G_2, we define the *cartesian product* $G_1 \square G_2$ to be the graph with vertex set $V(G_1 \square G_2) = \{(v_1, v_2) | v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1 \square G_2) = \{\{(v_1, v_2), (w_1, w_2)\} | (v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2)) \text{ or } (v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1))\}$.

Conjecture 4 *(Graham)* $pn(G_1 \square G_2) \leq pn(G_1)pn(G_2)$.

It is worth mentioning that there are some results which verify Graham's conjecture. Among these, the conjecture holds for a tree by a tree (Moews), a cycle by a cycle (Higgins and Herscovici; Pachter, Snevily and Voxman) (with possibly some small exceptions), and a clique by a graph with a special "2-pebbling property" (Chung).

It is fairly easy to argue that if the connectivity $\kappa(G) = 1$ then $pn(G) > n(G)$. Pachter, Snevily and Voxman proved that if $diam(G) = 2$ then $pn(G) = n(G)$ or $n(G) + 1$. Clarke, Hochberg and Hurlbert characterized those graphs having $pn(G) = n(G)$ and proved the corollary that if $diam(G) = 2$, and $\kappa(G) \geq 3$ then $pn(G) = n(G)$. From this it follows that almost all graphs (in the probabilistic sense) are of Class 0, since almost every graph is 3-connected with diameter 2. Czygrinow, Hurlbert, Kierstead, and Trotter proved the following more general result.

Result 5 *If G is a graph with $diam(G) = d$ and $\kappa(G) \geq 2^{2d+3}$ then $pn(G) = n(G)$.*

We also proved that there is a graph with $diam(G) = d$ and $\kappa(G) = 2^{d+1}/d$ for which $pn(G) > n(G)$.

For $m \geq 2t + 1$, the *Kneser graph*, $K(m, t)$, is the graph with vertices $\binom{[m]}{t}$ and edges $\{A, B\}$ whenever $A \cap B = \emptyset$. Using Result 5 Hurlbert proved the following.

Result 6 *For any constant $c > 0$, there is an integer t_0 such that, for $t > t_0$, $s \geq c(t/\log_2 t)^{1/2}$, and $m = 2t + s$, we have that $pn(K(m, t)) = n(K(m, t))$.*

Graph Thresholds

The notion that graphs with very few edges tend to have large pebbling number and graphs with very many edges tend to have small pebbling number can be made precise as follows. Let $\mathcal{G}_{n,p}$ be the random graph model in which each of the $\binom{n}{2}$ possible edges of a random graph having n vertices appears independently with probability p.

Let $\mathcal{P}$ be a property of graphs and consider the probability $Pr(\mathcal{P})$ that the random graph $\mathcal{G}_{n,p}$ has $\mathcal{P}$. For large p it may be that $Pr(\mathcal{P}) \to 1$ as $n \to \infty$, and for small p it may be that $Pr(\mathcal{P}) \to 0$

as $n \to \infty$. More precisely, define the *threshold* of $\mathcal{P}$, $Th(\mathcal{P})$, to be the set of functions t for which $p \gg t$ implies that $Pr(\mathcal{P}) \to 1$ as $n \to \infty$, and $p \ll t$ implies that $Pr(\mathcal{P}) \to 0$ as $n \to \infty$.

It is not clear that such thresholds exist for arbitrary $\mathcal{P}$. However, we observe that "$pn(G) = n(G)$" is a monotone property (adding edges to a Class 0 graph maintains the property), and a theorem of Bollobás and Thomason states that $Th(\mathcal{P})$ exists for every monotone $\mathcal{P}$. It is well known (Erdős and Rényi) that $Th(\text{connected}) = \Theta(\lg n/n)$, and since connectedness is a requirement for pebbling, we see that $Th(\text{Class } 0) \subseteq \Omega(\lg n/n)$. Czygrinow, Hurlbert, Kierstead, and Trotter used Result 5 to prove the following result.

Result 7 *For all $d > 0$, $Th(\text{Class } 0) \subseteq O((n \lg n)^{1/d}/n)$.*

Pebbling Thresholds

Let us fix some notation as follows. The vertex set for any graph on n vertices will be taken to be $\{v_i | i \in [n]\}$, where $[n] = \{0, \ldots, n-1\}$. That way, any distribution $D : V(G_n) \to \mathbf{N}$ is independent of G_n. Let $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$ for G_n, a generic graph on n vertices, and let $D_n : [n] \to \mathbf{N}$ denote a distribution on n vertices.

Let $h : \mathbf{N} \to \mathbf{N}$ and for fixed n consider the probability space X_n of all distributions D_n of size $h = h(n)$, We denote by P_n^+ the probability that D_n is G_n-solvable and let $t : \mathbf{N} \to \mathbf{N}$. We say that t is a *pebbling threshold* for $\mathcal{G}$ if $P_n^+ \to 0$ whenever $h(n) \ll t(n)$ and $P_n^+ \to 1$ whenever $h(n) \gg t(n)$, and we write $th(\mathcal{G})$ for the set of all pebbling thresholds for $\mathcal{G}$. The existence of such thresholds was recently established by Bekmetjev, Brightwell, Czygrinow and Hurlbert.

Result 8 *For every graph sequence $\mathcal{G}$, $th(\mathcal{G})$ is nonempty.*

The first threshold result was found by Clarke. The result is merely an unlabeled version of the so-called "Birthday problem", in which one finds the probability that 2 of t people share the same birthday, assuming n days in a year. He proved that $th(\mathcal{K}) = \Theta(\sqrt{n})$, where K_n is the complete graph on n vertices, and $\mathcal{K} = (K_1, \ldots, K_n, \ldots)$ for K_n. The following results, among others, were proved by Czygrinow, Eaton, Hurlbert and Kayll. Let $\mathcal{P} = (P_1, \ldots, P_n, \ldots)$ and $\mathcal{Q} = (Q^1, \ldots, Q^m, \ldots)$, where P_n is the path on n vertices and Q^m is the cube on $n = 2^m$ vertices.

Result 9 **(a)** *For all $\mathcal{G}$ and all $\epsilon > 0$, $th(\mathcal{G}) = O(n^{1+\epsilon})$,* **(b)** *$th(\mathcal{Q}) = O(n)$, and* **(c)** *$th(\mathcal{P}) = \Omega(n)$.*

Recently Bekmetjev, Brightwell, Czygrinow and Hurlbert proved reasonably tight bounds for the pebbling threshold of paths.

Result 10 *For any constant $c < 1/\sqrt{2}$, we have $th(\mathcal{P}) \subseteq \Omega(n 2^{c\sqrt{\lg n}}) \cap O(n 2^{2\sqrt{\lg n}})$.*

The focus of this presentation is on the spectrum of functions which can be thresholds for some graph sequence. Czygrinow and Hurlbert have proven the following theorem.

Theorem 11 *Let t_1, t_2 be any functions satisfying $\Omega(n^{1/2}) \ni t_1 \ll t_2 \in O(n)$. Then there is some graph sequence $\mathcal{G}$ such that $th(\mathcal{G}) \in \Omega(t_1) \cap O(t_2)$.*

We prove this theorem by letting $G_n = F_l(n)$, where $F_l(n)$ is the path $v_1 \ldots v_l$ with independent vertices $v_{l+1}, \ldots, v_n$ each adjacent to v_l. We show that, for $1/2 < \epsilon < 1$ and $l = (2\epsilon - 1)\lg n$, we have $th(\mathcal{F}_l) = \Theta(n^\epsilon)$, where $\mathcal{F}_l = (F_l(1), \ldots, F_l(n), \ldots)$.