# Cops and Robbers Pebbling in Graphs

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*Dedicated to Oleksandr Stanzhytsky, and others like him who are doing less mathematics than usual at this time.*

**Abstract**

Here we merge the two fields of Cops and Robbers and Graph Pebbling to introduce the new topic of Cops and Robbers Pebbling. Both paradigms can be described by moving tokens (the cops) along the edges of a graph to capture a special token (the robber). In Cops and Robbers, all tokens move freely, whereas, in Graph Pebbling, some of the chasing tokens disappear with movement while the robber is stationary. In Cops and Robbers Pebbling, some of the chasing tokens (cops) disappear with movement, while the robber moves freely. We define the cop pebbling number of a graph to be the minimum number of cops necessary to capture the robber in this context, and present upper and lower bounds and exact values, some involving various domination parameters, for an array of graph classes, including paths, cycles, trees, chordal graphs, high girth graphs, and cop-win graphs, as well as graph products. Furthermore we show that the analogous inequality for Graham's Pebbling Conjecture fails for cop pebbling and posit a conjecture along the lines of Meyniel's Cops and Robbers Conjecture that may hold for cop pebbling. We also offer several new problems.

**Keywords:** Cops and Robbers, Graph Pebbling, dominating set
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# 1 Introduction

There are numerous versions of moving tokens in a graph for various purposes. Two popular versions are called *Cops and Robbers* and *Graph Pebbling*. In both cases we have tokens of one type (C) attempting to capture a token of another type (R), and all token movements occur on the edges of a graph. In the former instance, all tokens move freely, whereas in the latter instance, type R tokens are stationary and type C movements come at a cost. In this paper we merge these two subjects to create *Cops and Robbers Pebbling*, wherein type R tokens move freely and type C tokens move at a cost.

We define these three paradigms more specifically in Subsection 1.1 below. The new graph invariant we define to study in this paper is the cop pebbling number of a graph, denoted $\pi^{\mathsf{c}}(G)$; roughly, this equals the minimum number of pebble-cops necessary to capture the robber in the Cops and Robbers Pebbling paradigm. In Subsections 1.2–1.4 we present known results about Cops and Robbers, dominating sets, and optimal pebbling, respectively, that will be used in the sequel. We record in Subsections 2.1–2.3 new theorems on lower bounds, upper bounds, and exact answers for $\pi^{\mathsf{c}}(G)$, respectively, for a range of graph families, including paths, cycles, trees, chordal graphs, high girth graphs, and cop-win graphs, as well as, in some cases, for all graphs. The reader already familiar with the prior subjects can skip ahead to Section 2 where we state and prove our results of the cop pebbling numbers of graphs. For example, Theorem 14 proves that the cop pebbling number of $n$-vertex trees is at most $\lceil 2n/3 \rceil$, which is tight for paths (and cycles), and Theorem 15 provides an upper bound involving a domination parameter, as a function of girth. Section 3 contains theorems for Cartesian products of graphs. For example, Theorems 21 and 22 give upper and lower bounds on the cop pebbling numbers of grids and cubes, respectively, while Theorem 23 proves that $\pi^{\mathsf{c}}(G \square K_t) \leq t\pi^{\mathsf{c}}(G)$ for all $G$. Furthermore, Theorems 25 and 26 show that the analogous inequality from Graham's Pebbling Conjecture fails for cop pebbling. We finish in Section 4 with some natural questions left open from this work, including a version of Meyniel's Cops and Robbers Conjecture that may hold in the Cops and Robbers Pebbling world, namely that $\pi^{\mathsf{c}}(G) \leq 2n/3 + o(n)$ for all $G$.

## 1.1 Definitions

In order to describe cop pebbling in Subsection 1.1.3, we need first to share the definitions and notations from each of the other two areas, as well as what we use from basic graph theory.

We use several standard notations in graph theory, including $V(G)$ for the set of vertices of a graph $G$ (with $n(G) = |V(G)|$), $E(G)$ for its edge set, $\mathsf{rad}(G)$ for its radius, $\mathsf{diam}(G)$ for its diameter, and $\mathsf{gir}(G)$ for its girth, as well as $\mathsf{deg}(v)$ for the degree of a vertex, and $\mathsf{dist}(u, v)$ for the distance between vertices $u$ and $v$. For a vertex $v$ in a graph $G$, we use the notations $N_d(v) = \{u \mid \mathsf{dist}(u, v) < d\}$ and $N_d[v] = \{u \mid \mathsf{dist}(u, v) \leq d\}$. If $d = 1$ we drop the subscript; additionally, we write $N[S] = \cup_{v \in S} N[v]$ for a set of vertices $S$. We often use $T$ to denote a tree, and set $P_n$, $C_n$, and $K_n$ to be the path, cycle, and complete graph on $n$ vertices, respectively. (For convenience, we define $C_2 = P_2$.)

For graphs $G$ and $H$ we define the *Cartesian product* $G \square H$ with vertex set $V(G) \times V(H)$ and edges $(u, v)(w, x)$ if $uw \in E(G)$ and $v = x$ or if $u = w$ and $vx \in E(H)$. The *$d$-dimensional cube* $Q^d$ is defined by $Q^1 = P_2$ and $Q^d = Q^{d-1} \square Q_1$ for $d > 1$.

### 1.1.1 Graph Pebbling

A *configuration* $C$ of pebbles on a graph $G$ is a function from the vertices of $G$ to the non-negative integers. Its *size* equals $|C| = \sum_{v \in G} C(v)$. For adjacent vertices $u$ and $v$ with $C(u) \geq 2$, a *pebbling step* from $u$ to $v$

removes two pebbles from $u$ and adds one pebble to $v$, while, when $C(u) \geq 1$, a *free step* from $u$ to $v$ removes one pebble from $u$ and adds one pebble to $v$. In the context of moving pebbles, we use the word *move* to mean *move via pebbling steps.*

The *pebbling number* of a graph $G$, denoted $\pi(G)$, is the minimum number $m$ so that, from any configuration of size $m$, one can move a pebble to any specified *target* vertex. The *optimal pebbling number* of a graph $G$, denoted $\pi^*(G)$, is the minimum number $m$ so that, from some configuration of size $m$, one can move a pebble to any specified target vertex. We note that in the definitions of the pebbling number and the optimal pebbling number, free moves are not allowed.

### 1.1.2 Cops and Robbers

In Cops and Robbers, the cops are the pebbles, the robber is the target, and the robber is allowed to move. The Cops and Robbers alternate making moves in *turns*. At each turn, any positive number of cops make one free step, then the robber chooses to make a free step or not. In Graph Pebbling literature, the activity of moving a pebble to a target is called *solving* or *reaching* the target; here we use the analogous Cops and Robbers terminology of *capturing* the robber.

The *cop number* $c(G)$ is defined as the minimum number $m$ so that, from some configuration of $m$ cops, it is possible to capture any robber via free steps. If the cops catch the robber on their $t^{\text{th}}$ turn, then we say that the *length* of the game is $t$; if the robber wins then the length is infinite. When the number of cops used is $c(G)$, the *capture time* of $G$, denoted $\mathsf{capt}(G)$, is defined to be the length of the game on $G$ when both cops and robbers play optimally. That is, it equals the minimum (over all cop strategies) of the maximum (over all robber strategies) of the length of the game with $c(G)$ cops on $G$.

### 1.1.3 Cop Pebbling

The *cop pebbling number* $\pi^{\mathsf{c}}(G)$ is defined as the minimum number $m$ so that, from some configuration of $m$ cops, it is possible to capture any robber via pebbling steps. For example, $c(G) \leq n(G)$, since the placement of one cop on each vertex has already caught any robber. We call an instance of a graph $G$, configuration $C$, and robber vertex $v$ a *game*, and say that the cops win the game if they can capture the robber; else the robber wins. Note that, since we lose a cop with each pebbling step, the cops must catch the robber within at most $|C| - 1$ turns to win the game.

We may assume that all graphs are simple. Because games on $K_1$ are trivial, we will assume that all graph components have at least two vertices. Additionally, because of the following fact, we will restrict our attention in this paper to connected graphs.

**Fact.** *If $G$ has connected components $G_1, \ldots, G_k$ then $\pi^{\mathsf{c}}(G) = \sum_{i=1}^{k} \pi^{\mathsf{c}}(G_i)$.*

A set $S \subseteq V(G)$ is a *distance-$d$ dominating set* if $\cup_{v \in S} N_d[v] = V(G)$. We denote by $\gamma_d(G)$ the size of the smallest distance-$d$ dominating set.

## 1.2 Cop Results

Here we list the results on Cops and Robbers that will be used to prove our theorems on cop pebbling. A graph $G$ is *cop-win* if $c(G) = 1$. A vertex $u$ in $G$ is called a *corner* if there is a vertex $v \neq u$ such that $N[u] \subseteq N[v]$. We say that $G$ is *dismantlable* if either $G$ is a single vertex or there is a corner $u$ such that $G - u$ is dismantlable. Note that chordal graphs are dismantlable.

**Result 1.** *[23] A graph is cop-win if and only if it is dismantlable.*

**Result 2.** *[5] If $G$ is a chordal graph with radius $r$, then $\mathsf{capt}(G) \le r$.*

This bound is tight. For example, $P_5$ has both radius 2 and $\mathsf{capt}(G) = 2$.

**Result 3.** *[15] If $G$ is a $d$-regular Cayley graph then $c = c(G) \le \lceil \frac{d+1}{2} \rceil$, and $\mathsf{capt}_c(G) \le |V(G)| \lceil \frac{d+1}{2} \rceil$.*

Result 3 yields the following result as a corollary.

**Result 4.** *[2] For $d \ge 1$, the $d$-dimensional cube $Q^d$ satisfies $c(Q^d) = \lceil \frac{d+1}{2} \rceil$.*

Bonato et al. [7] found the correct order of magnitude for the capture time of the cube.

**Result 5.** *[7] With $c = c(Q^d)$ we have $\mathsf{capt}_c(Q^d) = \Theta(d \lg d)$.*

### 1.3 Dominating Set Results

In this section we list results on domination that will be used to prove cop pebbling theorems. The definition of a dominating set immediately yields the following result.

**Result 6.** *If $G$ is a graph with $n$ vertices and maximum degree $\Delta$ then $\gamma(G) \ge \frac{n}{\Delta+1}$.*

**Result 7.** *[6] Almost all cop-win graphs $G$ have $\gamma(G) = 1$.*

**Result 8.** *[16] If $G = P_k \square P_m$ with $16 \le k \le m$, then $\gamma(G) \le \left\lfloor \frac{(k+2)(m+2)}{5} \right\rfloor - 4$.*

Result 6 implies that $\gamma(Q^d) \ge 2^d/(d+1)$. The following result shows that the actual value is not much greater, asymptotically.

**Result 9.** *[19] $\gamma(Q^d) \sim 2^d/d$.*

### 1.4 Optimal Pebbling Results

Finally, we list the optimal pebbling results we use to establish new cop pebbling theorems.

**Result 10.** *[9] For every graph $G$, $\pi^*(G) \le \lceil 2n/3 \rceil$. Equality holds when $G$ is a path or cycle.*

Fractional pebbling allows for rational values of pebbles. A *fractional pebbling step* from vertex $u$ to one of its neighbors $v$ removes $x$ pebbles from $u$ and adds $x/2$ pebbles to $v$, where $x$ is an rational number such that $0 < x \le C(u)$. The *optimal fractional pebbling number* of a graph $G$, denoted $\hat{\pi}^*(G)$, is the minimum number $m$ so that, from <u>some</u> configuration of size $m$, one can move, via fractional pebbling moves, a sum of one pebble to any specified target vertex.

**Result 11.** *[18, 21] For every graph $G$ we have $\pi^*(G) \ge \lceil \hat{\pi}^*(G) \rceil$.*

The authors of [18] prove that $\hat{\pi}^*(G)$ can be calculated by a linear program. Furthermore, they use this result to show that there is a uniform configuration that witnesses the optimal fractional pebbling number of any vertex-transitive graph; that is, the configuration $C$ defined by $C(v) = \hat{\pi}^*(G)/n(G)$ for all $v$ fractionally solves any specified vertex. From this they prove the following.

**Result 12.** *[18] Let $G$ be a vertex-transitive graph and, for any fixed vertex $v$, define $m = \sum_{u \in V(G)} 2^{-\mathsf{dist}(u,v)}$. Then $\hat{\pi}^*(G) = n(G)/m$.*

For a configuration $C$ on a graph $G$, we say that a vertex $v$ is 2-*reachable* if it is possible to move two pebbles to $v$ via pebbling steps. Then $C$ is 2-*solvable* if every vertex of $G$ is 2-reachable.

**Result 13.** *[9] If $C$ is a 2-solvable configuration of pebbles on the path $P_n$ then $|C| \geq n+1$.*

For a subset $W$ of vertices in a graph $G$ we define the graph $G_W$ to have vertices $V(G_W) = V(G) - W \cup \{w\}$ (where $w$ is a new vertex) with edges $xy$ whenever $x, y \in V(G) - W$ and $xy \in E(G)$ and $xw$ whenever $x \in V(G) - W$ and $xz \in E(G)$ for some $z \in W$. The process of creating $G_W$ from $G$ is called *collapsing* $W$. If $C$ is a configuration on $G$ then we define the configuration $C_W$ on $G_W$ by $C_W(w) = \sum_{z \in W} C(z)$ and $C_W(x) = C(x)$ otherwise. Note that $|C| = |C_W|$.

**Result 14.** *[9] Let $W$ be a subset of vertices in a graph $G$. If a configuration $C$ on $G$ can reach the configuration $D$ on $G$ then the configuration $C_W$ on $G_W$ can reach the configuration $D_W$ on $G_W$. In particular, we have $\pi^*(G) \geq \pi^*(G_W)$.*

The next two results involve Cartesian products.

**Result 15.** *[25] For all $k \leq m$ we have $\pi^*(P_k \Box P_m) \geq \frac{5092}{28593}km + O(k+m) \approx 0.1781km$.*

This bound is much smaller than the $\frac{2}{7}km + 8 \approx 0.2857km$ upper bound proved in [17], which is conjectured to be best possible.

Combining Results 11 and 12 yields the lower bound of the following result. The upper bound places $2^k$ pebbles on each vertex of a distance-$k$ dominating set, with $k$ roughly $d/3$.

**Result 16.** *[9, 21] For all $d \geq 1$ we have $(4/3)^d \leq \pi^*(Q^d) \leq (4/3)^{d+O(\lg k)}$.*

# 2 New Theorems on the Cop Pebbling Number

## 2.1 Lower Bounds

The first two theorems give the cop number and optimal pebbling number as lower bounds to the cop pebbling number.

**Theorem 1.** *For any graph $G$, $\pi^{\mathsf{c}}(G) \geq c(G)$, with equality if and only if $G = K_1$.*

*Proof.* Any configuration of cops that can capture the robber via pebbling steps can also capture the robber via free steps.

If $G = K_1$ then $\pi^{\mathsf{c}}(G) = 1 = c(G)$.

If $\pi^{\mathsf{c}}(G) = c(G)$ then capturing the robber requires no steps. That is, if a pebbling capture requires a pebbling step, then the cop lost during the step is irrelevant in the Cops and Robbers capture, which contradicts the equality. That means that a successful pebbling configuration has no vertex without a pebble; i.e. $\pi^{\mathsf{c}}(G) = n(G)$. However, if $n(G) \geq 3$ then, by placing two cops at a vertex $v$ of degree at least two and one cop on each vertex not adjacent to $v$, then we can capture any robber in one step. (We especially highlight this statement, recorded as Corollary 7 in Subsection 2.2, below.) Thus $\pi^{\mathsf{c}}(G) < n$, a contradiction. If $n(G) = 2$ then $G = K_2$ and $\pi^{\mathsf{c}}(K_2) = 2 > 1 = c(K_2)$, a contradiction. Hence $G = K_1$. □

**Theorem 2.** *For any graph $G$, $\pi^{\mathsf{c}}(G) \geq \pi^*(G)$.*

*Proof.* Any configuration of $k$ cops, where $k < \pi^*(G)$, will contain a vertex $v$ which is unreachable. The robber can then choose to start and stay on $v$ and thus not be captured. □

Typically, Theorem 2 gives a sharper lower bound on $\pi^{\mathsf{c}}(G)$ than Theorem 1. However, this may not be true for all graphs.

## 2.2 Upper Bounds

**Theorem 3.** *The cycle $C_n$ satisfies $\pi^{\mathsf{c}}(C_n) \le \lceil \frac{2n}{3} \rceil$.*

*Proof.* For $C_n$, partition $C_n$ into $\lfloor \frac{n}{3} \rfloor$ copies of $P_3$ and, possibly, an extra $P_1$ or $P_2$. Place two cops on the center vertex of each $P_3$, and one cop on each vertex of the remaining one or two vertices. The robber can only choose to start on one of the copies of $P_3$, where he is next to a pair of cops, and so will be captured on the first move. Thus $\pi^{\mathsf{c}}(C_n) \le \lceil \frac{2n}{3} \rceil$. $\square$

**Theorem 4.** *If $G$ is a tree then $\pi^{\mathsf{c}}(G) \le \pi^*(G)$.*

*Proof.* If $G$ is a tree then place $\pi^*(G)$ cops according to an optimal pebbling configuration $C$. The robber beginning at some vertex $v$ uniquely defines the subtree $T$ containing $v$ for which every cop in $T$ is on a leaf of $T$, and any leaf of $T$ with no cop is a leaf of $G$. Thus the robber can never escape $T$.

Now the cops follow the same greedy strategy used by optimal pebbling; i.e. at each state, they make all possible pebbling moves toward $v$. The new tree $T'$ defined by the robber's new vertex $v'$ and resulting cop pebbles has strictly fewer vertices than $T$. Hence this process ends after finitely many steps.

Suppose that we allow, for the moment, that pebbles can be split into fractions, so that a fractional pebbling step moves half the amount of pebbles and removes the other half. Then the process above ends with the capture of the robber at some vertex $v^*$. Because the set of such pebbling steps is greedy with respect to each temporary robber position, they are greedy with respect to vertex $v^*$. That is, they are precisely the steps taken to solve $v^*$ from $C$ if the target $v^*$ was known in advance.

Therefore, because $C$ is an optimal pebbling configuration, each of the fractional steps are actually integral; i.e. $C$ captures the robber with pebbling steps. Hence $\pi^{\mathsf{c}}(G) \le \pi^*(G)$, and the equality follows. $\square$

**Theorem 5.** *Let $G$ be a graph, $S$ a subset of its vertices, and define $S' = V - N[S]$. Then $\pi^{\mathsf{c}}(G) \le 2|S| + |S'|$. In particular, $\pi^{\mathsf{c}}(G) \le 2\gamma(G)$.*

*Proof.* Place two cops on each vertex of $S$ and one cop on each vertex of $S'$. In order to not be immediately captured, the robber must start in $N[S] - S$, but then is captured in one step by some pair of cops from $S$. The second statement follows from choosing $S$ to be a minimum dominating set of $G$, since $S' = \emptyset$. $\square$

The authors of [12] define a *Roman dominating set* of $G$ to be a $\{0, 1, 2\}$-labeling of $V(G)$ so that every vertex labeled 0 is adjacent to some vertex labeled 2. Note that the construction in Theorem 5 yields a Roman dominating set by labeling each vertex by its number of cops. The *Roman domination number* $\gamma_R(G)$ is defined to be the minimum sum of labels of a Roman dominating set. Hence we obtain the following bound.

**Corollary 6.** *Every graph $G$ satisfies $\pi^{\mathsf{c}}(G) \le \gamma_R(G)$.*

To illustrate the improvement of $2|S| + |S'|$ compared to $2\gamma(G)$, consider the following example.

**Example 1.** *For positive integers $m \ge 2k \ge 2$, let $Y = \{y_1, \ldots, y_m\}$ and let $Q = \{Q_1, \ldots, Q_k\}$ be a partition of $Y$ with each part size $|Q_i| \ge 2$. Define a bipartite graph $G$ with vertices $Y$, $Z = \{z_1, \ldots, z_k\}$, and $x$ as follows. For each $1 \le j \le k$ set $z_j \sim y_i$ if and only if $y_i \in Q_j$. Also set $x \sim y_i$ for every $1 \le i \le m$.*

*Then $\gamma(G) = k+1$. Indeed, since the neighborhoods of each $z_j$ are pairwise disjoint, at least $k$ vertices in $Y \cup Z$ are required to dominate $Z$, one from each $N[z_j]$. Suppose that $S$ is a dominating set of size $k$. By the above, $|S \cap N[z_j]| = 1$ for all $j$. But to dominate $x$, some $y_i$ must be in $S$. Let $y_i \in N(z_j)$; then $y_i$ does not dominate any other $y_{i'} \in N(z_j)$. Hence $\gamma(G) \geq k+1$. It is easy to see that $Z \cup \{x\}$ is a dominating set, so that $\gamma(G) = k+1$. By choosing $S = \{x\}$ we have $S' = Z$, so that $\pi^{\mathsf{c}}(G) \leq k+2$, much better than $2\gamma(G) = 2k+2$.*

An obvious corollary of Theorem 5 (recorded as Corollary 16, below) is that any graph $G$ with a dominating vertex has $\pi^{\mathsf{c}}(G) = 2$, except $K_1$. A more interesting corollary is the following.

**Corollary 7.** *Every graph $G$ satisfies $\pi^{\mathsf{c}}(G) \leq n - \Delta(G) + 1$. In particular, if $n(G) \leq 2$ then $\pi^{\mathsf{c}}(G) = n$, if $n(G) \geq 3$ then $\pi^{\mathsf{c}}(G) \leq n-1$, and if $n(G) \geq 6$ then $\pi^{\mathsf{c}}(G) \leq n-2$.*

*Proof.* Let $v$ be a vertex with $\mathsf{deg}(v) = \Delta(G)$, set $S = \{v\}$, and apply Theorem 5 to obtain the general bound. Next, it is easy to see that $\pi^{\mathsf{c}}(K_n) = n$ for $n \leq 2$. Then, a graph with at least three vertices has a vertex of degree at least two, so that $n - \Delta(G) + 1 \leq n-1$. Finally, if $\Delta(G) \geq 3$ then $n - \Delta(G) + 1 \leq n-2$, while if $\Delta(G) \leq 2$ then $G$ is a path or cycle, for which Theorem 18 yields $\pi^{\mathsf{c}}(G) = \lceil \frac{2n}{3} \rceil$, which is at most $n-2$ when $n \geq 6$. □

All three conditional bounds in Corollary 7 are tight: for example, $\pi^{\mathsf{c}}(P_2) = 2$, $\pi^{\mathsf{c}}(P_5) = 4$, and $\pi^{\mathsf{c}}(P_7) = 5$. Furthermore, its more general bound of $n - \Delta(G) + 1$ is tight for a graph with a dominating vertex (see Corollary 16).

**Theorem 8.** *Let $H$ be an induced subgraph of a graph $G$. Then, for any $s$, if $\pi^{\mathsf{c}}(H) \leq n(H) - s$ then $\pi^{\mathsf{c}}(G) \leq n(G) - s$.*

*Proof.* Suppose that $\pi^{\mathsf{c}}(H) \leq n(H) - s$. Then there is a configuration $C_H$ of $n(H) - s$ cops on $H$ that captures any robber on $H$. It remains to show that this number of cops can still win when the robber may move off $H$. Define the configuration $C_G$ of $n(G) - s$ cops on $G$ by placing one cop on each vertex of $G - H$ and $C_H(v)$ cops on each vertex $v \in H$. Then $C_H$ captures any robber on G. □

**Corollary 9.** *For all $s \geq 2$ there is an $N = N(s)$ such that every graph $G$ with $n = n(G) \geq N$ has $\pi^{\mathsf{c}}(G) \leq n - s$.*

*Proof.* Suppose that $\pi^{\mathsf{c}}(G) \geq n - s + 1$. Then Corollary 7 implies that $\Delta(G) \leq s$. Consider if $\mathsf{diam}(G) \geq 3s$. Then there exists an induced path $P$ of length $3s$ in $G$. By Theorem 18 we have $\pi^{\mathsf{c}}(P_{3s}) = 2s \leq 2s+1 = n(P) - s$. By Theorem 8, we must have that $\pi^{\mathsf{c}}(G) \leq n - s$, contradicting our assumption that $\pi^{\mathsf{c}}(G) \geq n - s + 1$. Thus, we conclude that $\mathsf{diam}(G) < 3s$. Since there are finitely many (at most $\Delta(G)^{\mathsf{diam}(G)}$) such graphs, there must be some $N$ such that $\pi^{\mathsf{c}}(G) \leq n - s$ for all $s \geq N$. □

One might be interested in measuring the gap between the size of a graph and its cop pebbling number. For this we define the *cop deficiency* of a graph $G$ to be $\text{Ï}^{\mathsf{c}}(G) := n(G) - \pi^{\mathsf{c}}(G)$. Then Theorem 8 and Corollary 9 can be restated as follows.

**Theorem 10.** *Let $H$ be an induced subgraph of a graph $G$. Then $\text{Ï}^{\mathsf{c}}(G) \geq \text{Ï}^{\mathsf{c}}(H)$.*

**Corollary 11.** *For all $s \geq 2$ there is an $N = N(s)$ such that every graph $G$ with $n = n(G) \geq N$ has $\text{Ï}^{\mathsf{c}}(G) \geq s$.*

**Theorem 12.** *If $G$ is a cop-win graph with $\mathsf{capt}(G) = t$, then $\pi^{\mathsf{c}}(G) \leq 2^t$. More generally, if $c(G) = k$ and $\mathsf{capt}_k(G) = t$ then $\pi^{\mathsf{c}}(G) \leq k2^t$.*

*Proof.* If $G$ is a cop-win graph with $\mathsf{capt}(G) = t$, then there is some vertex $v$ at which the cop begins and the robber can be caught with free steps in at most $t$ moves. If $2^t$ cops are placed on $v$, the cops can use the same capture strategy, and there will be sufficiently many cops for up to $t$ pebbling steps. Similarly, by placing $2^t$ on each of $c(G)$ cops, there will be sufficiently many cops for up to $t$ rounds of pebbling steps. $\square$

For example, let $T$ be a complete $k$-ary tree of depth $t$. Then $\mathsf{capt}(T) = t$ by Result 2, and so $\pi^{\mathsf{c}}(T) \leq 2^t$. Theorem 12 is tight for some graphs, as witnessed by any graph $G$ with a dominating vertex (see Corollary 16, below). It is also tight for any complete $k$-ary tree $T$ of depth two, when $k \geq 3$ (the lower bound follows from Theorem 18 since $T$ contains $P_5$).

**Corollary 13.** *If $G$ is a chordal graph with radius $r$, then $\pi^{\mathsf{c}}(G) \leq 2^r$.*

*Proof.* The result follows from Result 2 and Theorem 12. $\square$

**Theorem 14.** *If $T$ is an $n$-vertex tree, then $\pi^{\mathsf{c}}(T) \leq \lceil \frac{2n}{3} \rceil$.*

*Proof.* This follows from Theorem 4 and Result 10. $\square$

We note that Theorems 12 and 14 can each be stronger than each other, as the following two examples show. Define the *spider* $S(k, d)$ to be the tree having a unique vertex $x$ of degree greater than 2, all $k$ of whose leaves have distance $d$ from $x$.

**Example 2.** *For integers $k$ and $d$, the spider $S = S(k, d)$ has $c(S) = 1$ and $\mathsf{capt}(S) = d$, with $n = kd + 1$. Thus Theorem 12 yields $\pi^{\mathsf{c}}(S) \leq 2^d$, while Theorem 14 yields $\pi^{\mathsf{c}}(S) \leq \lceil (2kd + 2)/3 \rceil$. Hence one bound is stronger than the other depending on how $k$ compares, roughly, to $3 \cdot 2^{d-1}/d$.*

**Example 3.** *For integers $k, t \geq 1$, let $T$ be the complete $k$-ary tree of depth $t$. Then $n(T) = \sum_{i=0}^{t} k^i = (k^{t+1} - 1)/(k - 1)$. Thus Theorem 12 is stronger than Theorem 14 for $k \geq 3$ and for $k = 2$ with $t \geq 2$, while Theorem 14 is stronger than Theorem 12 when $k = 1$ and $t \geq 5$ (because $\mathsf{capt}(P_t) = \lceil t/2 \rceil$).*

**Theorem 15.** *For any positive integer $d$, if $G$ is a graph with $\mathsf{gir}(G) \geq 4d - 1$, then $\pi^{\mathsf{c}}(G) \leq 2^d \gamma_d(G)$.*

*Proof.* Let $S = \{v_1, v_2, ...\}$ be a minimum $d$-distance dominating set of $G$, and place $2^d$ cops on each $v_i$. Suppose the robber starts at vertex $v$. Since $\mathsf{gir}(G) \geq 4d - 1$, we know that $T = N_d[u]$ is a tree for all $u$. We write $T_i = N_d[v_i]$ and, for each $v \in T$, denote the unique $vu$-path in $T$ by $P(v)$.

Let $J$ be such that $T \cap T_j \neq \emptyset$ if and only if $j \in J$, and set $Q_j = T \cap T_j$. Note that $\mathsf{gir}(G) \geq 4d - 1$ implies that, for each $j \in J$, there is some $v \in T$ such that $Q_j \subseteq P(v)$. Moreover, by the definition of $S$, we have $\cup_{j \in J} Q_j = T$. In addition, $\mathsf{gir}(G) \geq 4d - 1$ implies that, for each $j \in J$, the shortest $v_i v$-path $P_i^*$ is unique.

For each $j \in J$, each cop at $v_j$ adopts the strategy to move at each turn toward $v$ along $P_i^*$ until reaching $T$, at which time then moving toward the robber along the unique path in $T$. This strategy ensures the property that, at any point in the game, if some cop is on vertex $x$ while the robber is on vertex $z$, then the robber can never move to a vertex $y$ for which the unique $yz$-path in $T$ contains $x$ — which includes $x$ itself. It also implies that the game will last at most $d$ turns. Hence, if we suppose that the robber wins the game, then the game lasted exactly $d$ turns and the robber now sits on some vertex $z$. However, by the definition of $S$, some cop reached $z$ within $d$ turns, which implies by the property just mentioned that the robber cannot move to $z$, a contradiction. Hence the cops win the game, capturing the robber. $\square$

An obvious corollary (recorded as Corollary 16, below) is that any graph $G$ with a dominating vertex has $\pi^{\mathsf{c}}(G) = 2$.

We remark that Theorem 15 applies to trees. $P_5$ is an example for which this bound is tight. In the case of the spider $S(k, 2)$, this bound is significantly better than Corollary 7 when $k$ is large. The case $d = 1$ yields the same upper bound of $2\gamma(T)$ from Theorem 5, which is better than the bound of Theorem 14 if and only if $\gamma(T) < \lceil (n-1)/3 \rceil$. Since $\gamma(T)$ can be as high as $n/2$, both theorems are relevant. The following example shows that Theorem 15 can be stronger than Theorem 14 for any $d$.

**Example 4.** *For $1 \leq i \leq 3$, define the tree $T_i$ to be the complete binary tree of depth $d-1$, rooted at vertex $v_i$, and define the tree $T$ to be the union of the three $T_i$ with an additional root vertex adjacent to each $v_i$. Then $\gamma_d(T) = d$, and $n = 3(2^d - 1) + 1$, so that the bound from Theorem 15 is stronger than the bound from Theorem 14.*

Theorem 15 can be stronger than other prior bounds as well, as shown by the following example.

**Example 5.** *For integers $k$ and $d$, define the theta graph $\Theta(k, d)$ as the union of $k$ internally disjoint $xy$-paths, each of length $d$. Then $\Theta = \Theta(k, 2d)$ has $n = k(d-1) + 2$, $c(\Theta) = 2$, $\mathsf{capt}_2(\Theta) = d$, $\mathsf{gir}(\Theta) = 4d$, $\gamma(\Theta) = k\lceil (2d-3)/3 \rceil$, and $\gamma_d = 2$. Thus Theorem 5 yields an upper bound of roughly $4kd/3$, while Theorems 12 and 15 both yield the upper bound of $2^{d+1}$, which is better or worse than Theorem 5 when $k$ is bigger or less than, roughly, $3 \cdot 2^{d-1}/d$.*

The following example illustrates the need for stronger bounds than given by Theorem 15.

**Example 6.** *Consider the $(3, 7)$-cage McGee graph $M$, defined by $V = \{v_i \mid i \in \mathbb{Z}_{24}\}$, with $v_i \sim v_{i+1}$ for all $i$, $v_i \sim v_{i+12}$ for all $i \equiv 0 \pmod 3$, and $v_i \sim v_{i+7}$ for all $i \equiv 1 \pmod 3$. We have $\gamma_2(M) \leq 4$ (e.g. $\{v_0, v_6, v_9, v_{15}\}$), and so $\pi^*(M) \leq \pi^{\mathsf{c}}(M) \leq 16$ by Theorem 15. However, this bound is not tight, as $\pi^{\mathsf{c}}(M) \leq 12$: the vertex set $\{v_i \mid i \equiv 0 \pmod 3\}$ induces a matching of size 4 — for each edge, place 2 cops on one of its vertices and 1 cop on the other. Incidentally, this yields $\pi^*(M) \leq 12$; the best known lower bound on $\pi^*$ comes from Result 12: $\pi^*(M) \geq \lceil \hat{\pi}^*(M) \rceil = \lceil 64/7 \rceil = 10$. Hence we are left with a gap in the bounds for $M$: $10 \leq \pi^*(M) \leq \pi^{\mathsf{c}}(M) \leq 12$.*

## 2.3 Exact Results

The following is a corollary of Theorem 5, as well as of Theorem 15.

**Corollary 16.** *If $G$ is a graph with at least two vertices and a dominating vertex then $\pi^{\mathsf{c}}(G) = 2$.*

The following is a corollary of Result 7 and Theorem 8.

**Corollary 17.** *Almost all cop-win graphs $G$ have $\pi^{\mathsf{c}}(G) = 2$.*

**Theorem 18.** *For all $n \geq 1$ we have $\pi^{\mathsf{c}}(P_n) = \pi^{\mathsf{c}}(C_n) = \lceil \frac{2n}{3} \rceil$.*

*Proof.* Result 10 and Theorem 2 provide the lower bound for both graphs. The upper bounds for paths and cycles follow from Theorems 14 and 3, respectively. □

**Theorem 19.** *For any tree $T$ we have $\pi^{\mathsf{c}}(T) = \pi^*(T)$.*

*Proof.* This follows from Theorems 2 and 4. □

A simple corollary to Theorem 19 is that any tree with radius 2 and diameter 4 has cop pebbling number 4. A complete $k$-ary tree of depth 2 with $k \geq 3$ is such an example.

# 3 Cartesian Products

For ladders ($P_2 \Box P_m$) we have the following theorem.

**Theorem 20.** *For all $m \geq 1$ we have $\pi^{\mathsf{c}}(P_2 \Box P_m) = m + 1$.*

*Proof.* Let $G = P_2 \Box P_m$ and for each $i \in [m]$ define the vertex subset $E_i = [2] \times \{i\}$.

For the lower bound, we suppose that $C$ can catch any robber on $G$. Then we claim that $C$ must be able to move two pebbles to any $E_i$. Indeed, assume that a cop catches the robber on $E_i$; without loss of generality, on vertex $(0, i)$. (We shall assume throughout that the robber aims to maximize capture time.) If the robber didn't move, it is because each of its neighbors contained a cop. If the robber did move, say from $(1, i)$, then it was because a cop could have moved onto the robber. In each of those cases we see that two cops could be moved onto $E_i$. If the robber moved instead from, say, $(0, i-1)$ (or symmetrically $(1, i+1)$), then it was because cops could move into both $(0, i-1)$ and $(1, i-1)$ (or $(0, i+1)$ and $(1, i+1)$). At this point, the robber can't move (else wouldn't be captured at $(0, i)$), so there must be at least one cop at each of $(1, i)$ and $(0, i+1)$. The capture at $(0, i)$ implies that at least one of those vertices has two cops; in any case two cops can move to $E_i$. Now, by collapsing every $E_i$ we obtain the graph $G' = P_m$, with corresponding collapsed configuration $C'$. The above argument then shows by Result 14 that $C'$ is 2-solvable. Consequently, Result 13 proves that $|C| = |C'| \geq m + 1$.

For the upper bound, we define $[k] = \{0, 1, \ldots, k-1\}$, write $m$ uniquely as $m = 4r + 2s + t$, with $s, t \in [2]$, and let $V(G) = [2] \times [m]$. Next define the sets $S = \{(0, 1)\}$ when $m$ is even and $S = \{(0, 1), (m-1, s)\}$ when $m$ is odd, and $T = \{(4i + 2j + 1, j) \mid 0 \leq i \leq r, j \in [2], 4i + 2j + 1 \in [m]\}$ — alternately, $T = \{(x, 0) \mid x \equiv 1 \pmod{m}\} \cup \{(x, 1) \mid x \equiv 3 \pmod{m}\}$. Then $S \cup T$ is a dominating set. Now place two cops on each vertex of $T$ and one cop on each vertex of $S$. It is simple to check that any robber can be captured in one step and that the number of cops in each case equals $m + 1$. (In fact, this is a Roman dominating set.) $\square$

More general grids have cop pebbling numbers linear in their number of vertices, but there is a gap in the bounds for its coefficient. We use the following result on the Roman domination number of grids.

**Result 17.** *[13] For all all $k, m \geq 5$ we have $\gamma_R(P_k \Box P_m) \leq \lfloor \frac{2(km+k+m)}{5} \rfloor$.*

**Theorem 21.** *For all $5 \leq k \leq m$ we have $\frac{5092}{28593} km + O(k + m) \leq \gamma_R(P_k \Box P_m) \leq \lfloor \frac{2(km+k+m)}{5} \rfloor$. The lower bound also holds for all $1 \leq k \leq m$.*

*Proof.* Result 15 and Theorem 2 produce the lower bound, while Result 17 and Theorem 5 produce the upper bound. $\square$

Note that this is another example of Roman domination giving a better upper bound than just domination (see Result 8).

For cubes, we can use Results 4 and 5 with Theorem 12 to obtain the upper bound $\pi^{\mathsf{c}}(Q^d) \leq \lceil \frac{d+1}{2} \rceil d^{\Theta(d)}$. However, by adding extra cops in the Cops and Robbers game, we can reduce the capture time and therefore also reduce the upper bound for cop pebbling. Still, an exponential gap remains.

**Theorem 22.** $\left(\frac{4}{3}\right)^d \leq \pi^{\mathsf{c}}(Q^d) \leq \frac{2^{d+1}}{d+1} + o(d)$.

*Proof.* The lower bound follows from Result 16 and Theorem 2. The upper bound follows from Result 9 and Theorem 5. $\square$

**Theorem 23.** *For every graph $G$ we have $\pi^{\mathsf{c}}(G \Box K_t) \leq t\pi^{\mathsf{c}}(G)$.*

*Proof.* Let $C$ be a configuration of $\pi^{\mathsf{c}}(G)$ cops on $G$ that can capture any robber. Define the configuration $C'$ on $G\Box K_t$ by $C'(u,v)=C(u)$ for all $u\in V(G)$ and $v\in V(K_t)$; then $|C'|=t|C|$. Let $C'_v$ be the restriction of $C'$ to the vertices $V_v=\{(u,v)\mid u\in V(G)\}$. Then each $C'_v$ is a copy of $C$ on $V_v$. Now imagine, for any robber on some vertex $(u,v)$, placing a copy of the robber on each vertex $(u',v)$ and maintaining that property with every robber movement. Then the cops on each $V_v$ will move in unison to catch their copy of the robber in $V_v$, one of which is the real robber. □

A famous conjecture of Graham [10] postulates that every pair of graphs $G$ and $H$ satisfy $\pi(G\Box H)\le\pi(G)\pi(H)$. This relationship was shown by Shiue to hold for optimal pebbling.

**Theorem 24.** *[27] Every pair of graphs $G$ and $H$ satisfy $\pi^*(G\Box H)\le\pi^*(G)\pi^*(H)$.*

One might ask whether or not the analogous relationship holds between $\pi^{\mathsf{c}}(G\Box H)$ and $\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$. Theorem 23 shows that this is true for $H=K_2$. However, the inequality is false in general, as the following theorem shows. For any graph $G$ define $G^1=G$ and $G^d=G\Box G^{d-1}$ for $d>1$.

**Theorem 25.** *There exist graphs $G$ and $H$ such that $\pi^{\mathsf{c}}(G\Box H)>\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$.*

*Proof.* Suppose that $\pi^{\mathsf{c}}(G\Box H)\le\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G$ and $H$. For fixed $k\ge 2$, let $d\ge 25k^2$, $v\in V(C_k^d)$, and $m=\sum_{u\in V(C_k^d)}2^{-\mathsf{dist}(u,v)}$. Then $\sqrt{d}>\ln d$, so that $d/\ln d>\sqrt{d}\ge 5k>\frac{2}{\ln(3/2)}k$, which implies that $d^{2k}<(3/2)^d$. Also $d\ge\sqrt{k/8}+2$, so that $\sqrt{k/8}\ge d-2$. Thus

$$\begin{aligned}\left(\frac{2}{3}\right)^d\sum_{u\in V(C_k^d)}2^{-\mathsf{dist}(u,v)} &\le \left(\frac{2}{3}\right)^d\sum_{i=0}^{kd/2}\binom{i+k-1}{k-1}2^{-i} \le \left(\frac{2}{3}\right)^d\sum_{i=0}^{kd/2}\binom{i+k-1}{k-1} \le \left(\frac{2}{3}\right)^d\binom{kd/2+k}{k}\\ &\le \left(\frac{2}{3}\right)^d(kd/2+k)^k/k! \le \left(\frac{2}{3}\right)^d(d+2)^k\sqrt{k/2}^k2^{-k} \le \left(\frac{2}{3}\right)^d(d+2)^k(d-2)^k\\ &\le \left(\frac{2}{3}\right)^d d^{2k} < 1.\end{aligned}$$

Therefore we would have

$$\pi^{\mathsf{c}}(P_k^d)\le\pi^{\mathsf{c}}(P_k)^d\le\left(\frac{2}{3}k\right)^d=\left(\frac{2}{3}\right)^d n(P_k^d)<n(C_k^d)/m=\hat{\pi}^*(C_k^d)\le\hat{\pi}^*(P_k^d)\le\pi^*(P_k^d),$$

by Fact 11 and Theorem 12. This, however, contradicts Theorem 2. □

A more direct example is given by the Cartesian product of wheels. For $n\ge 4$, define the *wheel* $W_n$ by the addition of a dominating vertex $x$ to the cycle $C_{n-1}$, having vertices $v_0,\ldots,v_{n-2}$.

**Theorem 26.** *Let $n\ge 4$ and let $G=W_n\Box W_n$. Then $\pi^{\mathsf{c}}(G)\le 14$ and if $n\ge 67$ then $\pi^{\mathsf{c}}(G)=14$.*

*Proof.* First we show the upper bound. Note that because $x$ dominates $W_n$, the vertex $(x,x)\in G$ has eccentricity 2. By placing 14 cops on $(x,x)$, the robber must choose a vertex outside of $N[(x,x)]$ to start on; by symmetry let it be $(v_1,v_1)$. Then we move 2 cops to each of $(v_0,x)$, $(v_1,x)$, and $(v_2,x)$, and 1 cop to $(x,v_1)$. Because of the two cops on $(v_1,x)$ the robber must move to one of its neighbors without cops; by symmetry let it be $(v_2,v_1)$. Then the cops on $(v_2,x)$ capture the robber. Hence $\pi^{\mathsf{c}}(G)\le 14$.

Now we prove the lower bound. Define the subgraph $H_k$ of $G$ to be induced by the vertices $\{(v_i,w)\mid i\in[k],w\in V(W_n)\}\cup\{(x,v_j)\mid j\in[k]\}$. Suppose that $n\ge 67$ and 13 pebble-cops are placed on $G$. Then there is some subgraph $H$ isomorphic to $H_5$ with no pebble-cops in it. Symmetrically inside of $H$ is a graph $H'$ isomorphic to $H_3$ (there are three $H_3$s; we're referring to the middle one of them). Without loss of generality, $H' = H_3$. Now assume that this configuration catches the robber via a set $\sigma$ of pebbling moves. Suppose some pebble-cop reaches a vertex $u \in H'$ from a vertex other than $(x, x)$, and let $S$ be the set of pebble-cops used to accomplish that. Then we can instead start with $S$ on $(x, x)$ and still reach $u$. Thus, after making all such modifications, we have that the initial configuration of all 13 pebble-cops on $(x, x)$ catches the robber on $G$.

But we argue as follows that 13 pebble-cops on $(x, x)$ cannot catch a robber at $(v_1, v_1)$ in $G$. Compute all subscript arithmetic modulo $n - 1$. We call a robber's move from $(v_i, v_j)$ to $(v_{i+1}, v_j)$ (respectively $(v_{i-1}, v_j)$) *increasing* (respectively *decreasing*). First, it takes at least 4 pebble-cops to force the robber to move, because doing so requires 2 pebble-cops to be moved to a neighbor of the robber. Then, similarly, it takes at least 4 pebble-cops to prevent the robber from making successive increasing moves, and at least 4 more to prevent successive decreasing moves. Finally it requires at least 2 pebble-cops to block the robber's movement inward, to some vertex $(x, v_j)$. Note that all these sets of pebble-cops are distinct because no pebble-cop can perform two of the above actions simultaneously. But this is a contradiction because only 13 pebble-cops exist. Hence $\pi^{\mathsf{c}}(G) \geq 14$. □

It seems likely that the value 67 can be reduced greatly.

**Corollary 27.** *For $n \geq 67$ we have $\pi^{\mathsf{c}}(W_n \Box W_n) = \frac{7}{2}\pi^{\mathsf{c}}(W_n)\pi^{\mathsf{c}}(W_n)$.*

# 4 Open Questions

**Question 28.** *Can the bounds $.178 \approx \frac{5092}{28593} \leq \lim_{k,m\to\infty} \pi^{\mathsf{c}}(P_k \Box P_m)/km \leq .4$ from Theorem 21 be improved?*

Corollary 27 suggests the following two questions.

**Question 29.** *Is there an infinite family of graphs $\mathcal{G}$ for which $\pi^{\mathsf{c}}(G \Box H) \leq \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G, H \in \mathcal{G}$?*

Theorem 25 shows that products of paths is not among such a family.

**Question 30.** *Is there some constant $a \geq 7/2$ such that $\pi^{\mathsf{c}}(G \Box H) \leq a\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G$ and $H$?*

In addition to chordal graphs and Cartesian products discussed above, it would be interesting to study other graph classes. It was proved in [11] that $c(G) \leq 2$ for outerplanar $G$, and in [1] that $c(G) \leq 3$ for planar $G$. Additionally, it was shown in [26] that if $G$ is a planar graph on $n$ vertices then $\mathsf{capt}_3(G) \leq 2n$.

**Question 31.** *Are there constant upper bounds on $\pi^{\mathsf{c}}(G)$ when $G$ is planar or outerplanar? If $k = \pi^{\mathsf{c}}(G)$ then is $\mathsf{capt}_k(G)$ linear?*

Likewise, a result of [24] states that every diameter two graph $G$ on $n$ vertices satisfies $\pi(G) \in \{n, n+1\}$.

**Question 32.** *Is there a similar, narrow range of values of $\pi^{\mathsf{c}}(G)$ over all diameter two graphs $G$?*

Finally, Meyniel [14] conjectured in 1985 that every graph $G$ on $n$ vertices satisfies $c(G) = O(\sqrt{n})$. Some evidence in support of this conjecture is found in [4], where it is proved for $G \in \mathcal{G}_{n,p}$ that when $0 < \epsilon < 1$ and $p > 2(1+\epsilon)\log(n)/n$ we have $c(G) < \frac{10^3}{\epsilon^3} n^{\frac{1}{2}\log(n)}$ almost surely. (In fact, they also show that when $p \gg 1/n$ we have $c(G) > \frac{1}{(pn)^2} n^{\frac{1}{2}\left(\frac{\log\log(pn)-9}{\log\log(pn)}\right)}$ almost surely.) Along these lines, we make the following conjecture.

**Conjecture 33.** *Every graph $G$ on $n$ vertices satisfies $\pi^{\mathsf{c}}(G) \leq 2n/3 + o(n)$; i.e., Ï$^{\mathsf{c}}(G) \geq n/3 - o(n)$.*

## Data Availability Statement

No data were created or analyzed in this study.

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