Chapter 8

Geometry Revisited

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8.1 Helly's Theorem

Let $\mathcal{F}$ be a finite family of convex regions in $\mathbb{R}^n$. We say that $\mathcal{F}$ is **k-intersecting** if every k sets of $\mathcal{F}$ intersect ($\forall \mathcal{G} \subseteq \mathcal{F} : |\mathcal{G}| = k \Longrightarrow \cap_{S \in \mathcal{G}} S \neq \emptyset$), and **full-intersecting** if the intersection of all the sets of $\mathcal{F}$ is nonempty. For example, Figure 8.1 shows a 2-intersecting family in $\mathbb{R}^2$ that is not 3-intersecting.

k-intersecting/ full-intersecting family

Game 8.1.1 *Two players alternate drawing distinct convex regions in $\mathbb{R}^2$ under the rule that the resulting family of regions (after the first move) is 2-intersecting but (after the second move) is not full-intersecting. A player unable to draw such a region loses the game.*

For example, Figure 8.1 shows an ellipse, pentagon and line segment as the first three regions drawn in a particular start to Game 8.1.1.

Workout 8.1.2 *Who wins Game 8.1.1? [HINT: Player 1 loses if she starts with a point.]*

Game 8.1.3 *Two players alternate drawing distinct convex regions in $\mathbb{R}^2$ under the rule that the resulting family of regions (after the first move) is 2-intersecting and (after the second move) 3-intersecting but (after the third move) not full-intersecting. A player unable to do so loses the game.*

Workout 8.1.4 *Who wins Game 8.1.3?*

Theorem 8.1.5 *Let $\mathcal{F}$ be a finite family of convex regions in $\mathbb{R}^n$. If $\mathcal{F}$ is $(n+1)$-intersecting then $\mathcal{F}$ is full-intersecting.*

Helly's Theorem

Workout 8.1.6 *Prove Helly's Theorem 8.1.5 for the case $n = 1$. [HINT: Use induction on $|\mathcal{F}|$.]*

Proof. The first part of the proof involves converting the arbitrary convex regions $C_i \in \mathcal{F}$ into polytopes $P_i \subseteq C_i$ such that $\mathcal{G} = \{P_1, \ldots, P_t\}$ is also $(n+1)$-intersecting. The strategy is to prove that if $\mathcal{G}$ is full-intersecting then so is $\mathcal{F}$. Also, it may be easier to prove the result for $\mathcal{G}$ because of the nice structure of its regions.

Workout 8.1.7 *Prove that if $\mathcal{G}$ is full-intersecting, with each $P_i \subseteq C_i$, then so is $\mathcal{F}$.*

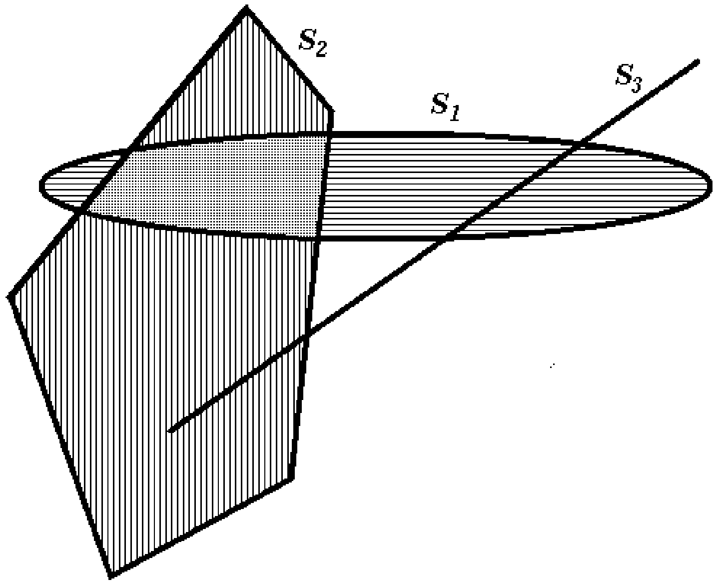

Figure 8.1: A 2-intersecting family of convex regions

Let $H \subseteq \{1, \ldots, t\}$ have size $n+1$. The hypothesis asserts that $\cap_{h \in H} C_h \neq \emptyset$, so choose $\mathbf{p}_H \in \cap_{h \in H} C_h$. Now do the same for every such $(n+1)$-subset of $\{1, \ldots, t\}$. Finally, define $X_h = \{\mathbf{p}_H \mid H \ni h\}$ and let $P_h = \text{vhull}(X_h)$. (Recall that P_h is a polytope by Workout 3.3.2.) For example, Figure 8.2 shows an instance in $\mathbb{R}^2$ with 4 regions.

Workout 8.1.8 *Prove that each $P_i \subseteq C_i$.*

The final task is to show that $\mathcal{G}$ is full-intersecting. For this we recall that every polytope is the intersection of a finite number of half-spaces. This means that for each i there exists a finite system S_i of inequalities whose solution region equals P_i. Now let S be the system of inequalities including every inequality from each system S_i.

Workout 8.1.9 *Let S' be any collection of $n+1$ of the inequalities of S. Prove that S' is solvable.*

Workout 8.1.10 *Use Theorem 7.4.3 to finish the proof.*

$\diamond$

Workout 8.1.11 *Finish the example from Figure 8.2 and find a point $\mathbf{q}$ common to all four regions.*

(c) *Use induction to prove that the intersection of an arbitrary number (finite or infinite) of cones is conic.*

(d) *Prove that all polycones are conic.*

Note that a half-space whose boundary contains the origin is defined by an inequality of the form $\sum_{j=0}^{n} a_j x_j \leq 0$. Thus a polycone can be defined as set of $\mathbf{x}$ satisfying some system $\mathbf{Ax} \leq \mathbf{0}$. In this formulation it is easy to see that polycones are conic, since $\mathbf{A}(r\mathbf{u}+s\mathbf{v}) = r\mathbf{Au}+s\mathbf{Av} \leq \mathbf{0}$ whenever $\mathbf{Au} \leq \mathbf{0}$, $\mathbf{Av} \leq \mathbf{0}$ and $r, s \geq 0$.

A ray $R = \overrightarrow{\mathbf{0v}}$ is an **extreme ray** of the cone C if no line segment, with both its endpoints in $C - R$, intersects R. Of course, for R to be an extreme ray it is necessary that it lie on the boundary of C. In Figure 8.3 one can see that every boundary ray of the generic cone is extreme, while in Figure 8.4 only the five rays (shown) through the extreme points of the pentagons are extreme for the polycone. Naturally, these rays are on the intersections of the bounding planes. It is interesting to note that some polycones have no extreme rays; a cone defined by a single constraint is one such example. We will assume here that our polycones are not so degenerate — such cones contain a line, and hence have no extreme points (see Exercises 8.5.5, 8.5.7, 8.5.18 and 8.5.19) — and leave the more general situation to the reader. In particular, the point $\mathbf{0}$ is extreme. extreme ray

Workout 8.4.2 *Let C be the polycone defined by $\mathbf{Ax} \leq 0$, where*

$$\mathbf{A} = \begin{pmatrix} 3 & -8 \\ -7 & 4 \\ 2 & -9 \end{pmatrix} .$$

Find the extreme rays of C.

Lemma 8.4.3 *Suppose that the LOP P*

$$\begin{array}{ll} \text{Max.} & z = \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{0} \end{array}$$

is unbounded. Then the polycone C defined by $\mathbf{Ax} \leq \mathbf{0}$ has an extreme ray $\overrightarrow{\mathbf{0v}}$ such that $\mathbf{c}^{\mathsf{T}}\mathbf{v} > 0$.

Workout 8.4.4 *Prove Lemma 8.4.3. [HINT: What does the basis look like when Phase II halts?]*

Notice that the polycone in Figure 8.4 can be described as the conic hull of its extreme rays. The same is true in general; the following theorem for cones is the analogue of Workout 3.3.2 and Exercise 3.5.20 for polytopes.

Theorem 8.4.5 *Every polycone C containing no line is the conic hull of some finite set of rays $\overrightarrow{\mathbf{0v}}$.* Minkowski-Weyl Theorem

Proof. We begin by rephrasing the statement as follows, using the result of Exercise 3.5.14. For every system $\mathbf{Ax} \leq \mathbf{0}$ there exist points $\mathbf{v}_1, \ldots, \mathbf{v}_m$ such that $\mathbf{Au} \leq \mathbf{0}$ if and only if $\mathbf{u} = \sum_{i=1}^{m} t_i \mathbf{v}_i$ for some $t_1, \ldots, t_m \geq 0$. We claim that the set of all extreme rays of C have this property.

Workout 8.4.6 *Show that if each $\overrightarrow{\mathbf{0v}_i}$ is an extreme ray of C then any conic combination of the $\mathbf{v}_i$s is in C.*

Now suppose that $\mathbf{u}$ is any point satisfying $\mathbf{Au} \le \mathbf{0}$, and suppose that there is no solution to $\mathbf{u} = \sum_{i=1}^m t_i\mathbf{v}_i$ with each $t_i \ge 0$. By Farkas's Theorem 7.4.5, there must be some $\mathbf{c}$ such that $\mathbf{c}^\mathsf{T}\mathbf{v}_i \le 0$ for all $1 \le i \le m$ and $\mathbf{c}^\mathsf{T}\mathbf{u} > 0$. Now consider the following LOP P.

$$\begin{array}{ll} \text{Max.} & z = \mathbf{c}^\mathsf{T}\mathbf{x} \\ \\ \text{s.t.} & \mathbf{Ax} \le \mathbf{0} \end{array}$$

Workout 8.4.7 *Use Lemma 8.4.3 and the General Fundamental Theorem 6.2.9 to prove that P is optimal at 0.*

Now the fact that $\mathbf{u}$ is feasible and $\mathbf{c}^\mathsf{T}\mathbf{u} > 0$ implies that $z^* > 0$, contradicting the result of Workout 8.4.7. This contradiction proves that $\mathbf{u} \in \text{ncomb}(\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}) = \text{nhull}(\{\mathbf{v}_1, \ldots, \mathbf{v}_m\})$. $\diamond$

The Minkowski-Weyl Theorem can be combined with Workout 3.3.2 and Exercise 3.5.20 to prove that every polyhedron is the sum of a polytope and a polycone — see Exercises 8.5.12, 8.5.13 and 8.5.23.

8.5 Exercises

Practice

8.5.1 *Let* $\mathbf{x}^1 = (-20, 2, 15)^\mathsf{T}$, $\mathbf{x}^2 = (9, -6, 23)^\mathsf{T}$, $\mathbf{x}^3 = (-4, 9, 3)^\mathsf{T}$, $\mathbf{x}^4 = (15, 11, 18)^\mathsf{T}$, $\mathbf{x}^5 = (-8, 16, -9)^\mathsf{T}$, $\mathbf{x}^6 = (1, -5, 8)^\mathsf{T}$, $\mathbf{x}^7 = (10, 3, -1)^\mathsf{T}$, $\mathbf{x}^8 = (-6, -4, -3)^\mathsf{T}$, $\mathbf{x}^9 = (24, -4, -13)^\mathsf{T}$, *and* $\mathbf{x}^{10} = (-7, -12, -17)^\mathsf{T}$, *and define* $X = \{\mathbf{x}^1, \ldots, \mathbf{x}^{10}\}$.

(a) *Prove that* $\mathbf{x}^3$ *is not an extreme point of* $\text{vhull}(X)$.

(b) *Prove that* $\mathbf{x}^4$ *is an extreme point of* $\text{vhull}(X)$.

(c) *Find all extreme points of* $\text{vhull}(X)$.

8.5.2 *Let P be a polytope in $\mathbb{R}^3$ with k extreme points (and that is fully 3-dimensional). At most how many facets does P have?*

8.5.3 *Let P be a polytope in $\mathbb{R}^4$ with k extreme points (and that is full dimensional). At most how many facets does P have?*

8.5.4 *Let P be the polyhedron defined as the region of solutions of the following system.*

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 5x_2 & + & 7x_3 & + & & \le & 782 \\ & + & 8x_2 & - & 4x_3 & + & 9x_4 & \le & 829 \\ 3x_1 & + & & + & 6x_3 & - & 1x_4 & \le & 765 \\ 1x_1 & - & 7x_2 & + & & - & 8x_4 & \le & -94 \\ & + & 4x_2 & - & 3x_3 & - & 5x_4 & \le & -28 \\ -6x_1 & - & 9x_2 & + & 2x_3 & + & & \le & -87 \\ \\ x_1 & , & x_2 & , & x_3 & , & x_4 & \ge & 0 \end{array}$$

(a) *Find all extreme points of P. [HINT: Only 12 of the 210 bases are feasible. Using one of your algorithms from Exercise 2.10.47 would help here.]*