Generalizations of Graham's Pebbling Conjecture

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Abstract

We investigate generalizations of pebbling numbers and of Graham's pebbling conjecture that $\pi(G\square H) \leq \pi(G)\pi(H)$, where $\pi(G)$ is the pebbling number of the graph G. We develop new machinery to attack the conjecture, which is now twenty years old. We show that certain conjectures imply others that initially appear stronger. We also find counterexamples that shows that Sjöstrand's theorem on cover pebbling does not apply if we allow the cost of transferring a pebble from one vertex to an adjacent vertex to depend on the weight of the edge and we describe an alternate pebbling number for which Graham's conjecture is demonstrably false.

Keywords: Pebbling, Graham's conjecture, generalizations

1. Distributions and Pebbling Numbers

We investigate various generalizations of Graham's pebbling conjecture and relationships between those generalizations.

Definition: Chung [1] defined a *distribution* of pebbles on a graph G as a placement of pebbles on the vertices of the graph. Equivalently, a distribution D is a function $D : V(G) \to \mathbb{N}$, where $D(v)$ represents the number of pebbles on the vertex v. Also, for every distribution D and every positive integer t, we define tD as the distribution given by $(tD)(v) = tD(v)$ for every vertex v in G. Following [7], we also define $|D|$ as the total number of pebbles in the distribution D.

Definition: A *pebbling move* consists of removing two pebbles from some vertex, throwing one of the pebbles away, and moving the other pebble to an adjacent vertex.

The following definitions are motivated by Section 4 in [4]:

Definition: Given two distributions D' and D'' on a graph G, we say that D''

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contains D' if $D'(v) \leq D''(v)$ for every vertex $v \in V(G)$.
Definition: Given two distributions D and D' on a graph G, we say that D' is *reachable* from D if it is possible to use a sequence of pebbling moves to go from D to a distribution D'' that contains D'.

We refer to distributions that we are trying to reach as *target distributions*. Some authors have called such distributions *weight functions* [2, 11] and speak of *weighted cover pebbling numbers*. We avoid this terminology; instead, following [3], we use the term *weighted graphs* to refer to graphs whose edges are weighted (see Section 4.1).

We define our most general pebbling number on unweighted graphs as follows.
Definition: Let $\mathcal{S}$ be a set of distributions on a graph G. The *pebbling number of* $\mathcal{S}$ *in* G, denoted $\pi(G, \mathcal{S})$, is the smallest number such that every distribution $D \in \mathcal{S}$ is reachable from every distribution that starts with $\pi(G, \mathcal{S})$ (or more) pebbles on G.

It is customary to require the graph G to be connected and undirected, but we may dispense with this requirement and allow $\pi(G, \mathcal{S}) = \infty$ if some distribution in $\mathcal{S}$ is unreachable from distributions with arbitrarily many pebbles. In particular, Moews [8] considered trees to be directed graphs with all edges directed toward the target vertex.

There are several ways to specialize the above definition.
Definition: Let D be a distribution of pebbles on a graph G. The *pebbling number of* D *in* G, denoted $\pi(G, D)$, is defined by $\pi(G, D) = \pi(G, \{D\})$, i. e. the smallest number such that D is reachable from every distribution that starts with $\pi(G, D)$ pebbles on G.

We define some specific distributions and sets of distributions.
Definition: For any vertex $v \in V(G)$, we define the distribution δ_v as the function

$$\delta_v(x) = \begin{cases} 1, & x = v \\ 0, & x \neq v \end{cases}$$

We also let $\mathcal{S}_t(G) = \{t\delta_v : v \in V(G)\}$; that is, $\mathcal{S}_t(G)$ is the set of distributions with t pebbles on a single vertex.

The definitions of pebbling numbers in the remainder of this section are consistent with the definitions given by Chung [1] and the rest of the literature on pebbling, but we give definitions in terms of the previous definitions.
Definition: Choose $v \in V(G)$. The *pebbling number of* v *in* G, denoted $\pi(G, v)$, is defined by $\pi(G, v) = \pi(G, \delta_v)$. Thus, $\pi(G, v)$ is the smallest number such that the vertex v can be reached from every distribution of $\pi(G, v)$ pebbles on G.
Definition: The *pebbling number of* G is defined as $\pi(G) = \pi(G, \mathcal{S}_1(G))$. Thus, $\pi(G)$ is the smallest number such that any single vertex is reachable from every distribution of $\pi(G)$ pebbles on G.
Definition: For any $v \in V(G)$ and any positive integer t, the t*-pebbling number of* v *in* G, denoted $\pi_t(G, v)$, is defined by $\pi_t(G, v) = \pi(G, t\delta_v)$. Thus, $\pi_t(G, v)$ is the smallest number such that t pebbles can be moved to the vertex v from every distribution of $\pi_t(G, v)$ pebbles on G.

Definition: The *t-pebbling number of* G is defined as $\pi_t(G) = \pi(G, \mathcal{S}_t(G))$. Thus, $\pi_t(G)$ is the smallest number such that t pebbles can be moved to any single vertex from every distribution of $\pi_t(G)$ pebbles on G.

Proposition 1.1 notes some straightforward relationships between these definitions.

Proposition 1.1. *Let G be any graph, and let $\mathcal{S}$ and $\mathcal{S}'$ be two sets of distributions on G. The various pebbling numbers are related as follows.*

1. *We have* $\pi(G, \mathcal{S}) = \max_{D \in \mathcal{S}} \pi(G, D)$.
2. *In particular,* $\pi(G) = \max_{v \in V(G)} \pi(G, v)$*, and* $\pi_t(G) = \max_{v \in V(G)} \pi_t(G, v)$.
3. *Furthermore, if* $\mathcal{S} \subseteq \mathcal{S}'$*, then* $\pi(G, \mathcal{S}) \leq \pi(G, \mathcal{S}')$.

Crull et. al. [2] originally defined the *cover pebbling number* of a graph G, denoted $\gamma(G)$. We define a distribution Γ_G, and use it to define $\gamma(G)$ as follows:
Definitions: The distribution Γ_G is the constant function $\Gamma_G(x) = 1$ for every vertex x in $V(G)$. The *cover pebbling number of* G, denoted $\gamma(G)$, is defined as $\gamma(G) = \pi(G, \Gamma_G)$. Thus, $\gamma(G)$ is the smallest number such that one pebble can be moved to every vertex simultaneously from every distribution of $\gamma(G)$ pebbles on G.

Theorem 1.2, proved by Sjöstrand [10], states that for every target distribution in which all vertices are occupied, we only have to consider starting distributions in which all pebbles are on the same vertex. In particular, we can compute $\gamma(G)$ efficiently, since we can compute how many pebbles we would need to place on each potential starting vertex to be able to reach every target simultaneously, and $\gamma(G)$ is simply the largest of these numbers.

Theorem 1.2 (Sjöstrand [10]). *If D is a distribution of pebbles on the graph G such that $D(v) \geq 1$ for every vertex v in $V(G)$, then $\pi(G, D)$ is the smallest number n with the property that if n pebbles are placed on a* single *vertex, then D is reachable, regardless of which vertex contained the initial pebbles.*

2. Cartesian products

Definition: If $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are two graphs, their Cartesian product is the graph $G \square H$ whose vertex set is the product

$$V_{G \square H} = V_G \times V_H = \{(x, y) : x \in V_G, y \in V_H\}$$

with (x, y) adjacent to (x', y') when $x = x'$ and $yy' \in E(H)$ or when $y = y'$ and $xx' \in E(G)$.

We first define the product of two distributions. This definition appeared with slightly different notation in [4].
Definition: If D_g and D_h are distributions on G and H respectively, then we define $D_g \cdot D_h$ as the distribution on $G \square H$ such that

$$(D_g \cdot D_h)((x, y)) = D_g(x) D_h(y)$$

for every vertex $(x, y) \in V(G\Box H)$. Similarly, if $\mathcal{S}_G$ and $\mathcal{S}_H$ are sets of distributions on G and H respectively, then $\mathcal{S}_G \cdot \mathcal{S}_H$ is the set of distributions on $G\Box H$ given by

$$\mathcal{S}_G \cdot \mathcal{S}_H = \{D_g \cdot D_h : D_g \in \mathcal{S}_G \text{ and } D_h \in \mathcal{S}_H\}.$$

The following conjectures generalize Graham's Conjecture (Conjecture 2.7).

Conjecture 2.1. *For all graphs G and H, and all sets of distributions $\mathcal{S}_G$ and $\mathcal{S}_H$ on G and H respectively, we have $\pi(G\Box H, \mathcal{S}_G \cdot \mathcal{S}_H) \leq \pi(G, \mathcal{S}_G)\pi(H, \mathcal{S}_H)$.*

By choosing specific sets of distributions $\mathcal{S}_G$ and $\mathcal{S}_H$, Conjecture 2.1 generates several more conjectures. Conjecture 2.2 first appeared as Conjecture 4.1 in [4].

Conjecture 2.2. *For all graphs G and H, and all distributions D_g and D_h on G and H respectively, we have $\pi(G\Box H, D_g \cdot D_h) \leq \pi(G, D_g)\pi(H, D_h)$.*

In particular, Sjöstrand [10] proved that Theorem 2.3 follows as a consequence of Theorem 1.2.

Theorem 2.3. *Let D_g be a distribution on the graph G such that $D_g(v) \geq 1$ for every vertex v in $V(G)$, and let D_h be a distribution on H with the same property. Then $\pi(G\Box H, D_g \cdot D_h) = \pi(G, D_g)\pi(H, D_h)$.*

For positive integers s and t, and vertices $x \in V(G)$ and $y \in V(H)$, we let $D_g = s\delta_x$ and $D_h = t\delta_y$ in Conjecture 2.2 to obtain Conjecture 2.4.

Conjecture 2.4. *For all graphs G and H, all positive integers s and t, and all vertices $x \in V(G)$ and $y \in V(H)$, we have $\pi_{st}(G\Box H, (x, y)) \leq \pi_s(G, x)\pi_t(H, y)$.*

Letting $s = t = 1$ in Conjecture 2.4, we can specialize to Conjecture 2.5, which first appeared in [6].

Conjecture 2.5. *For all graphs G and H and all vertices $x \in V(G)$ and $y \in V(H)$, we have $\pi(G\Box H, (x, y)) \leq \pi(G, x)\pi(H, y)$.*

By not specifying a target vertex, we have Conjectures 2.6 and 2.7. Conjecture 2.6 first appeared in [5], and Chung [1] attributed Conjecture 2.7 to Graham.

Conjecture 2.6. *For all graphs G and H and all positive integers s and t, we have $\pi_{st}(G\Box H) \leq \pi_s(G)\pi_t(H)$.*

Conjecture 2.7 (Graham's Conjecture). *For all graphs G and H, we have*

$$\pi(G\Box H) \leq \pi(G)\pi(H).$$

3. Equivalent conjectures

We now establish some equivalences and logical relationships among the Conjectures from Section 2. We first note that Conjectures 2.1 and 2.2 are equivalent. We then use a similar argument to show that Conjectures 2.4 and 2.5 imply Conjectures 2.6 and 2.7, respectively. We then establish equivalences within Conjecture 2.4 for different values of s and t. In particular, we show that we can factor out powers of two. This suggests two more conjectures, one that is equivalent to Conjecture 2.4, and another that is equivalent to Conjecture 2.5.

Proposition 3.1. *For any fixed graphs G and H, the following conjectures are equivalent:*

1. $\pi(G\Box H, \mathcal{S}_G \cdot \mathcal{S}_H) \leq \pi(G, \mathcal{S}_G)\pi(H, \mathcal{S}_H)$ *for all sets of distributions $\mathcal{S}_G$ on G and $\mathcal{S}_H$ on H.*
2. $\pi(G\Box H, D_g \cdot D_h) \leq \pi(G, D_g)\pi(H, D_h)$ *for all individual distributions D_g on G and D_h on H.*

In particular, Conjectures 2.1 and 2.2 are equivalent.

Proof: If statement 1 holds, then applying it with $\mathcal{S}_G = \{D_g\}$ and $\mathcal{S}_H = \{D_h\}$ implies statement 2. Conversely, if statement 2 holds, then Proposition 1.1 yields

$$\pi(G\Box H, \mathcal{S}_G \cdot \mathcal{S}_H) = \max_{D \in \mathcal{S}_G \cdot \mathcal{S}_H} \pi(G\Box H, D).$$

Let $D_g \cdot D_h$ be a distribution for which this maximum is achieved and apply statement 2 to obtain

$$\pi(G\Box H, \mathcal{S}_G \cdot \mathcal{S}_H) = \pi(G\Box H, D_g \cdot D_h) \leq \pi(G, D_g)\pi(H, D_h).$$

Clearly, this product does not exceed

$$\max_{D_g \in \mathcal{S}_G} \pi(G, D_g) \max_{D_h \in \mathcal{S}_H} \pi(H, D_h) = \pi(G, \mathcal{S}_G)\pi(H, \mathcal{S}_H),$$

by Proposition 1.1. □

Proposition 3.2 shows that Conjecture 2.6 and Conjecture 2.7 follow from Conjectures 2.4 and Conjecture 2.5, respectively.

Proposition 3.2. *If G and H are graphs, and s and t are positive integers with the property that $\pi_{st}(G\Box H, (x, y)) \leq \pi_s(G, x)\pi_t(H, y)$ for every pair of vertices $x \in V(G)$ and $y \in V(H)$, then $\pi_{st}(G\Box H) \leq \pi_s(G)\pi_t(H)$. Thus, Conjecture 2.4 implies Conjecture 2.6 and Conjecture 2.5 implies Conjecture 2.7.*

Proof: From Proposition 1.1, we know

$$\pi_{st}(G\Box H) = \max_{(x,y) \in V(G\Box H)} \pi_{st}(G\Box H, (x, y)).$$

If (x, y) is a vertex for which this maximum is achieved, then

$$\pi_{st}(G\Box H) = \pi_{st}(G\Box H, (x, y)) \leq \pi_s(G, x)\pi_t(H, y),$$

and again by Proposition 1.1, we have

$$\pi_s(G,x)\pi_t(H,y) \le \max_{x\in V(G)} \pi_s(G,x) \max_{y\in V(H)} \pi_t(H,y) = \pi_s(G)\pi_t(H).$$

□

The proof of Proposition 3.2 is similar to that for Proposition 3.1; however, Proposition 3.2 is a one-directional implication. Since the sets of distributions used to define $\pi(G)$ and $\pi_t(G)$ are not arbitrary, there is no easy way to reverse the implication in Proposition 3.2 as there was in Proposition 3.1.

We now investigate equivalences within Conjecture 2.4 involving different values of s and t. Theorem 3.5 states that if Conjecture 2.4 holds for all graphs for a given choice of s and t, then it also holds if we double either s or t and keep the other the same. The basic idea of the proof is as follows: given a graph G and a target vertex x_i, we construct a new graph G_i' and choose a target vertex x' in G_i' whose s-pebbling number equals the $2s$-pebbling number of x_i in G. Next, given a target vertex y_j in a graph H, we compute the $2st$-pebbling number of (x_i, y_j) in $G\square H$ in terms of the st-pebbling number of (x', y_j) in $G_i'\square H$. We begin by defining G_i'. Then we use Propositions 3.3 and 3.4 to establish Theorem 3.5.

Definition: Given a graph G and a vertex $x_i \in V(G)$, we let G_i' be the graph obtained by adding a single vertex x' to $V(G)$ and a single edge (x_i, x'). Thus, G is a subgraph of G_i'.

Now given another graph H, we define a function f from distributions on $G_i'\square H$ to distributions on $G\square H$.

Definition: Given a distribution D on $G_i'\square H$, we let $f(D)$ be the distribution on $G\square H$ obtained by replacing every pebble on (x', y) with two pebbles on (x_i, y), i. e.

$$f(D)((x,y)) = \begin{cases} D((x,y)) + 2D((x',y)), & x = x_i \\ D((x,y)), & x \neq x_i \end{cases}$$

Proposition 3.3. *If D_0 and D_n are distributions on $G_i'\square H$ such that D_n is reachable from D_0, then $f(D_n)$ is reachable from $f(D_0)$ on $G\square H$.*

Proof: For each m with $0 \le m < n$, suppose the distribution D_{m+1} can be obtained from D_m by making a single pebbling move. Suppose by induction that $f(D_m)$ is reachable from $f(D_0)$, for some $m \ge 0$, and consider the move from D_m to D_{m+1}. That move replaces two pebbles from some vertex (x_1, y_1) with one pebble on an adjacent vertex (x_2, y_2). If neither x_1 nor x_2 is x', the same move in $G\square H$ is a pebbling move from $f(D_m)$ to $f(D_{m+1})$. If $x_1 = x_2 = x'$, then $D_m((x', y_1)) \ge 2$, so $f(D_m)((x_i, y_1)) \ge 4$. In this case, we use two pebbling moves to replace four pebbles on (x_i, y_1) with two pebbles on (x_i, y_2), and these moves go from $f(D_m)$ to $f(D_{m+1})$.

The only other cases to consider are pebbling moves from (x_i, y) to (x', y), or from (x', y) to (x_i, y) for some vertex y. In a move from (x_i, y) to (x', y), we have $f(D_m) = f(D_{m+1})$, and in a move from (x', y) to (x_i, y), we have

$f(D_m)((x_i,y)) = f(D_{m+1})((x_i,y)) + 3$, since the two pebbles on (x',y) contribute four to $f(D_m)((x_i,y))$ and the single pebble on (x_i,y) only contributes one to $f(D_{m+1})((x_i,y))$. We also have $f(D_m)((x,y)) = f(D_{m+1})((x,y))$ for all other vertices, $x \neq x_i$. Thus, the distribution $f(D_m)$ contains $f(D_{m+1})$. $\square$

Proposition 3.4. *For any vertex x_i in the graph G and any positive integer s, we have $\pi_{2s}(G,x_i) = \pi_s(G_i',x')$.*

Proof: We consider a distribution D on G_i'. We assume x' is unoccupied in the original distribution, since replacing any pebbles on x' with two pebbles on x_i does not help us reach x' with additional pebbles. Thus, we may consider D to be a distribution on the subgraph G of G_i', and we also have $f(D) = D$ for every vertex $x \in V(G)$. Now applying Proposition 3.3 with H equal to the trivial graph, shows that if D is a distribution on $G_i \square H \cong G_i'$ from which s pebbles can be moved to x', then $f(D) = D$ is also a distribution on $G \square H \cong G$ from which $2s$ pebbles can be moved to x_i. The converse also holds: if we can put $2s$ pebbles on x_i, then we can then move s pebbles to x'. Thus, we can move $2s$ pebbles onto x_i if and only if we can move s pebbles onto x', and $\pi_{2s}(G,x_i) = \pi_s(G_i',x')$. $\square$

Theorem 3.5. *If there are values s and t such that*

$$\pi_{st}(G\square H,(x,y)) \leq \pi_s(G,x)\pi_t(H,y)$$

for all graphs G and H and all vertices $(x,y) \in V(G\square H)$, then

$$\pi_{2st}(G\square H,(x,y)) \leq \pi_{2s}(G,x)\pi_t(H,y)$$

and

$$\pi_{2st}(G\square H,(x,y)) \leq \pi_s(G,x)\pi_{2t}(H,y)$$

for all graphs G and H and all vertices $(x,y) \in V(G\square H)$.

Proof: Let D be any distribution on $G\square H$ from which $2st$ pebbles cannot be moved to the target vertex (x_i,y_j). Since $G\square H$ is a subgraph of $G_i'\square H$, we may also regard D as a distribution on $G_i'\square H$. Furthermore, we have $f(D) = D$. By Proposition 3.3, we cannot move st pebbles onto (x',y_j) in $G_i'\square H$ from D. Thus,

$$\pi_{2st}(G\square H,(x_i,y_j)) \leq \pi_{st}(G_i'\square H,(x',y_j)) \leq \pi_s(G_i',x')\pi_t(H,y_j),$$

and by Proposition 3.4, $\pi_s(G_i',x')\pi_t(H,y_j) = \pi_{2s}(G,x_i)\pi_t(H,y_j)$. Similarly, we have

$$\pi_{2st}(G\square H,(x_i,y_j)) \leq \pi_s(G,x_i)\pi_t(H_j',y') = \pi_s(G,x_i)\pi_{2t}(H,y_j),$$

as desired. $\square$

Applying Theorem 3.5 inductively gives Corollary 3.6:

Corollary 3.6. *Suppose there are values s and t such that $\pi_{st}(G\Box H,(x,y)) \leq \pi_s(G,x)\pi_t(H,y)$ for all graphs G and H and all vertices $(x,y) \in V(G\Box H)$. Then for all nonnegative integers a and b,*

$$\pi_{2^{a+b}st}(G\Box H,(x,y)) \leq \pi_{2^a s}(G,x)\pi_{2^b t}(H,y).$$

for all graphs G and H and all vertices $(x,y) \in V(G\Box H)$.

Motivated by this result, we make the following conjectures as additional specializations of Conjecture 2.4.

Conjecture 3.7. *For all graphs G and H, all positive, odd integers s and t, and all vertices $x \in V(G)$ and $y \in V(H)$, we have*

$$\pi_{st}(G\Box H,(x,y)) \leq \pi_s(G,x)\pi_t(H,y).$$

Conjecture 3.8. *For all graphs G and H, all nonnegative integers a and b, and all vertices $x \in V(G)$ and $y \in V(H)$, we have*

$$\pi_{2^{a+b}}(G\Box H,(x,y)) \leq \pi_{2^a}(G,x)\pi_{2^b}(H,y).$$

Theorem 3.9. *Conjecture 2.4 is equivalent to Conjecture 3.7, and Conjecture 2.5 is equivalent to Conjecture 3.8.*

Proof: Conjecture 3.7 follows immediately from Conjecture 2.4, and Conjecture 2.5 follows from Conjecture 3.8 by letting $a = b = 0$. We use Corollary 3.6 to show the converse.

Thus, to show Conjecture 3.7 implies Conjecture 2.4, given s and t, we write $s = 2^a s'$ and $t = 2^b t'$, where s' and t' are odd. Conjecture 3.7 states that s' and t' are values for which $\pi_{s't'}(G\Box H,(x,y)) \leq \pi_{s'}(G,x)\pi_{t'}(H,y)$ for all graphs G and H and all vertices $(x,y) \in V(G\Box H)$. Therefore, if Conjecture 3.7 holds, Corollary 3.6 implies that s and t have the same property. That is, Conjecture 2.4 would hold as well.

To show Conjecture 3.8 follows from Conjecture 2.5, we use the same argument with $s' = t' = 1$. $\Box$

4. Variants on Pebbling

Theorem 3.9 is interesting, but it would be more satisfying to dispense with Conjectures 3.7 and 3.8 and prove that Conjectures 2.4 and 2.5 are equivalent. If we examine the proof of Theorem 3.5, Corollary 3.6, and Theorem 3.9, we find that the powers of 2 in these results arise from the rules of pebbling moves, and in particular, that two pebbles are required from one vertex to put a pebble on an adjacent vertex. In Section 4.1, we define pebbling on weighted graphs. We determine the cost of moving a pebble from one vertex to an adjacent vertex by considering the weight of the edge between them. Using these revised rules, we find that analogues of Conjectures 2.4 and 2.5 are indeed equivalent.

4.1. Pebbling on Weighted Graphs

In a *weighted graph*, we attach positive integral weights to the edges. We use these weights to specify the cost of moving a pebble from one vertex to another.
Definition: A *weighted graph* is a graph G together with a function $w : E(G) \to \mathbb{N}^+$. We say that $w(e)$ is the *weight* of the edge e.
Definition: A *pebbling move along the edge $e = (x, x')$ in a weighted graph* consists of removing $w(e)$ pebbles from x, moving one of the pebbles onto x', and throwing the other pebbles away.
We can now define each of the pebbling numbers $\pi(G, \mathcal{S})$, $\pi(G, D)$, $\pi(G, v)$, $\pi(G)$, $\pi_t(G, v)$, $\pi_t(G)$, $\pi(G, t)$, and $\gamma(G)$ for weighted graphs exactly as we did for unweighted graphs. We note that for this form of pebbling, any connected graph may be regarded as a complete graph, since any missing edge (v, w) may be added with a weight equal to the product of weights on some path from v to w. We may also assume that the weight of each edge is equal to the minimum of all such products; if there is an edge $e = (v, w)$ for which this is not the case, then we may use a path with a smaller product to move a pebble from v to w instead of using e.

We show that the obvious analogue of Sjöstrand's Theorem (Theorem 1.2) is false by answering Question 1 in the negative.

Question 1. *If G is a weighted graph, is $\gamma(G)$ the minimum number of pebbles N such that placing N pebbles on a single vertex allows us to cover G?*

Answer: No. Consider the complete graph K_4 on vertices $\{x_1, x_2, x_3, x_4\}$ in which the weight of the edges (x_1, x_2) and (x_3, x_4) is 2, and the weight of every other edge is 5. We can cover the graph if we start with thirteen pebbles on any single vertex, however, we cannot cover the graph if we start with nine pebbles on x_1 and four pebbles on x_2.

We now define the Cartesian product of two weighted graphs.
Definition: If G and H are two weighted graphs, their Cartesian product is the weighted graph $G \square H$ whose vertex set and edge set are the same as for the corresponding unweighted graph, and whose weight function is given by

$$w((x, y), (x, y')) = w(y, y') \text{ if } (y, y') \in E(H)$$
$$w((x, y), (x', y)) = w(x, x') \text{ if } (x, x') \in E(G).$$

We can now make each of the conjectures in Section 2 for weighted graphs. In each case, the conjecture on weighted graphs is stronger than the corresponding conjecture on unweighted graphs, since we can consider an unweighted graph to be a weighted graph in which the weight of each edge is 2. We limit ourselves to the following conjectures:

Conjecture 4.1. *For all weighted graphs G and H, all positive integers s and t, and all vertices $x \in V(G)$ and $y \in V(H)$, we have $\pi_{st}(G \square H, (x, y)) \leq \pi_s(G, x)\pi_t(H, y)$.*

Conjecture 4.2. *For all weighted graphs G and H and all vertices $x \in V(G)$ and $y \in V(H)$, we have $\pi(G \square H, (x, y)) \leq \pi(G, x)\pi(H, y)$.*

Chung essentially proved Conjecture 4.2 when G and H are powers of K_2, i. e. cubes in which the weights of parallel edges are equal (see [1], Theorem 3). We show that Conjectures 4.1 and 4.2 are equivalent; the proof is similar to the proof of Theorem 3.5. We first modify the required definitions.

Definitions: Given a weighted graph G, a positive integer s, and a vertex $x_i \in V(G)$, we let $G'_{i,s}$ be the graph obtained by adding a vertex x' to $V(G)$ and a single edge (x_i, x') with weight s. Given another graph H, we define the function f from distributions on $G'_{i,s}\Box H$ to distributions on $G\Box H$ by replacing the pebbles on every vertex (x', y) with s pebbles on (x_i, y), i. e.

$$f(D)((x,y)) = \begin{cases} D((x,y)) + sD((x',y)), & x = x_i \\ D((x,y)), & x \neq x_i \end{cases}$$

for every distribution D on $G'_{i,s}\Box H$.

We give the analogues for Propositions 3.3 and 3.4 without proof. The proofs are similar to those of the original propositions. We then prove Theorem 4.5.

Proposition 4.3. *If D_0 and D_n are distributions on $G'_{i,s}\Box H$ such that D_n is reachable from D_0, then $f(D_n)$ is reachable from $f(D_0)$ in $G\Box H$.*

Proposition 4.4. *For any weighted graph G, any positive integers s and t, and any vertex $x_i \in V(G)$, we have $\pi_{st}(G, x_i) = \pi_t(G'_{i,s}, x')$.*

Theorem 4.5. *Conjectures 4.1 and 4.2 are equivalent.*

Proof: Conjectures 4.1 implies Conjecture 4.2 by letting $s = t = 1$.

Conversely, given a weighted graph G, a vertex $x_i \in V(G)$, and an integer s, let D be a distribution on $G\Box H$ from which st pebbles cannot be placed on (x_i, y_j). By Proposition 4.3, we cannot place t pebbles on (x', y_j) starting from D in $G'_{i,s}\Box H$, so

$$\pi_{st}(G\Box H, (x_i, y_j)) \leq \pi_t(G'_{i,s}\Box H, (x', y_j)).$$

Similarly, we form $H'_{j,t}$ by adding a vertex y' and an edge (y_j, y') with a weight of t. By Proposition 4.4, $\pi_t(H, y_j) = \pi(H_{j,t}, y')$. By Proposition 4.3, if D' is a distribution on $G_{i,s}\Box H$ from which t pebbles cannot be placed on (x', y_j), then in $G'_{i,s}\Box H'_{j,t}$ we cannot place one pebble on (x', y') starting from D'. Thus,

$$\pi_t(G'_{i,s}\Box H, (x', y_j)) \leq \pi(G'_{i,s}\Box H'_{j,t}, (x', y')).$$

If Conjecture 4.2 holds for every vertex in every graph, then applying it gives

$$\pi_{st}(G\Box H, (x_i, y_j)) \leq \pi(G'_{i,s}\Box H'_{j,t}, (x', y')) \leq \pi(G'_{i,s}, x')\pi(H'_{j,t}, y'),$$

and by Proposition 4.4, we have $\pi(G'_{i,s}, x')\pi(H'_{j,t}, y') = \pi_s(G, x_i)\pi_t(H, y_j)$, as desired. □

4.2. Target-selectable pebbling numbers

In this section, we define a new pebbling number $\rho(G, \mathcal{S})$ and investigate analogues of the conjectures in Section 2. As with the definition of the usual pebbling number, we do not allow ourselves to choose the starting distribution of $\rho(G, \mathcal{S})$ pebbles, but after those pebbles are placed, we allow ourselves to choose which target distribution from $\mathcal{S}$ we wish to reach. This definition was originally motivated by an attempt to prove a version of Graham's conjecture. We observe that if the vertex v in G is unoccupied, then we can move a pebble onto v if and only if we can move two pebbles onto some neighbor of v in G, or equivalently, in the graph obtained by deleting v from G. Therefore, it seems reasonable to try to prove an analogue of Graham's conjecture with this pebbling number by using a form of induction on the number of vertices in G. However, we give simple counterexamples to show that $\rho(G, \mathcal{S})$ does not satisfy what seems to be the natural analogue to Graham's conjecture.

Definition: Let $\mathcal{S}$ be a set of distributions on a graph G. The *target-selectable pebbling number of* $\mathcal{S}$ *in* G, denoted $\rho(G, \mathcal{S})$, is the smallest number such that some distribution $D \in \mathcal{S}$ is reachable from every distribution starting with $\rho(G, \mathcal{S})$ pebbles on G. We also define $\rho_t(G) = \rho(G, \mathcal{S}_t(G))$ and $\rho(G, v) = \rho(G, \delta_v)$.

We begin by formalizing our previous observation that $\pi(G, v) = \rho(G, v)$ can be computed by determining how many pebbles are required to put two pebbles on a neighbor of v.

Proposition 4.6. *We have* $\pi(G, v) = \rho(G, v) = \rho(G, N_2)$ *where* N_2 *is the set of distributions given by* $N_2 = \{2\delta_w : (v, w) \in E(G)\}$.

We compute some values of $\rho(G, \mathcal{S})$ and relate them to the usual pebbling number.

Observations: If G is a graph with n vertices, then:

1. We have $\rho_1(G) = 1$. Thus, $\rho_1(G \square H) = \rho_1(G)\rho_1(H)$ for every graph G and H, so the analogue of Graham's conjecture for the target-selectable pebbling number holds trivially.
2. We also have $\rho_2(G) = n + 1$. In particular, if H has $m > 1$ vertices, then $\rho_2(G \square H) = mn+1 > \rho_2(G)\rho_1(H) = n+1$. This contradicts the analogues of Conjectures 2.1 and 2.6 for the target-selectable pebbling number.
3. For any distribution D on G, we have $\rho(G, \{D\}) = \pi(G, \{D\})$. Thus, the analogues for Conjectures 2.2, 2.4, and 2.5 are equivalent to the original conjectures.

We also note interesting relationships between this pebbling number for paths and the usual pebbling number for cycles, as given by Proposition 4.8. The pebbling number for cycles was given by Pachter, Snevily and Voxman [9]. We give these numbers in Proposition 4.7.

Proposition 4.7 (Pachter, Snevily, and Voxman [9]). *The pebbling number of the cycles* C_{2k} *and* C_{2k+1} *are*

$$\begin{aligned} f(C_{2k}) &= 2^k \\ f(C_{2k+1}) &= \tfrac{2^{k+2} - (-1)^{k+2}}{3}. \end{aligned}$$

We relate these numbers to the target-selectable pebbling number for certain distributions in paths, which we now define.
Definition: Let the vertices on the path P_n be $\{x_1, \ldots, x_n\}$ in order, and for any positive integer t, let $\mathcal{D}_t$ be the set of distributions given by

$$\mathcal{D}_t = \{t\delta_1, t\delta_n\}.$$

Proposition 4.8. *If $n \geq 2$ and $i \geq 0$, we have $\rho(P_n, \mathcal{D}_{2^i}) = \pi(C_{n+2i-1})$.*

Proof: For $i = 0$, we show $\rho(P_n, \mathcal{D}_1) = \pi(C_{n-1})$. Let the vertices of C_{n-1} be $\{y_1, \ldots, y_{n-1}\}$. Given any distribution D of pebbles on P_n, let D' be the distribution on C_{n-1} given by

$$D'(y_i) = \begin{cases} D(x_1) + D(x_n) & \text{if } i = 1 \\ D(x_i) & \text{if } 2 \leq i \leq n-1 \end{cases}$$

Note that y_1 is reachable from D' if and only if either x_1 or x_n is reachable from D. Thus, $\rho(P_n, \mathcal{D}_1) = \pi(C_{n-1})$.

To prove $\rho(P_n, \mathcal{D}_{2^i}) = \pi(C_{n+2i-1})$, let the vertices of C_{n+2i-1} be

$$\{z, a_{i-1}, \ldots, a_1, y_1, \ldots, y_n, b_1, \ldots, b_{i-1}\}.$$

Let D be a distribution of pebbles on P_n from which 2^i pebbles cannot be moved to either of the vertices x_1 or x_n. Let D' be the distribution on C_{n+2i-1} given by $D'(y_i) = D(x_i)$ and $D'(v) = 0$ for every other vertex in C_{n+2i-1}. Now z is unreachable in C_{n+2i-1}. Thus, $\rho(P_n, \mathcal{D}_{2^i}) \leq \pi(C_{n+2i-1})$.

To show $\pi(C_{n+2i-1}) \leq \rho(P_n, \mathcal{D}_{2^i})$, we construct a distribution containing $\pi(C_{n+2i-1}) - 1$ pebbles in P_n from which 2^i pebbles cannot be moved to either x_1 or x_n. We use the critical distribution on C_{n+2i-1}. Toward that end, if $n = 2k+1$, we have $\pi(C_{n+2i-1}) = 2^{k+i}$. If we put $2^{k+i} - 1$ pebbles on x_{k+1} we cannot move 2^i pebbles to either target.

On the other hand, if $n = 2k$, we analyze separately the cases when $k+i$ is even or odd.
Case 1: $k+i$ is even. Let $m = \frac{k+i}{2}$, so $n + 2i = 2k + 2i = 4m$. In this case, we have $\pi(C_{n+2i-1}) = \pi(C_{4m-1}) = \frac{2^{2m+1}+1}{3}$. If we put $\frac{2^{2m}-1}{3}$ pebbles each on x_k and x_{k+1}, then we have a total of $\frac{2^{2m+1}-2}{3} = \pi(C_{n+2i-1}) - 1$ pebbles on P_n. Since there are an odd number of pebbles on each vertex, one pebble cannot be used in a pebbling move to the other vertex. Therefore, at most $\frac{2^{2m}-4}{3}$ pebbles can be used, so we can transfer at most $\frac{2^{2m-1}-2}{3}$ pebbles from one occupied vertex to the other. Thus, we can put at most $\frac{2^{2m}-1}{3} + \frac{2^{2m-1}-2}{3} = 2^{2m-1} - 1 = 2^{k+i} - 1$ pebbles on either x_k or x_{k+1}. Thus, 2^i pebbles cannot be moved to either x_1 or x_n.
Case 2: $k+i$ is odd. In this case, we let $m = \frac{k+i-1}{2}$, so $n + 2i = 4m + 2$. Now $\pi(C_{n+2i-1}) = \pi(C_{4m+1}) = \frac{2^{2m+2}-1}{3}$. If we put $\frac{2^{2m+1}-2}{3}$ pebbles each on x_k and x_{k+1}, we have $\pi(C_{4m+1}) - 1$ pebbles on P_n, and we can put at most $\frac{2^{2m+1}-2}{3} + \frac{2^{2m}-1}{3} = 2^{2m} - 1 = 2^{k+i-1} - 1$ pebbles on either x_k or x_{k+1}. Once

again, 2^i pebbles cannot be moved to either x_1 or x_n.
Since we can always construct distributions of $\pi(C_{n+2i-1})$ pebbles in P_n from which no distribution of $\mathcal{D}_{2^i}$ is reachable, we have $\pi(C_{n+2i-1}) \leq \rho(P_n, \mathcal{D}_{2^i})$. Therefore, $\pi(C_{n+2i-1}) = \rho(P_n, \mathcal{D}_{2^i})$, as desired. $\square$

This gives rise to another counterexample to the analog of Graham's conjecture. If we let T be the trivial graph with a single vertex v and let $\mathcal{S} = \{2\delta_v\}$, then it is natural to suppose that a definition of multiplying distributions would give $\mathcal{S} \cdot \mathcal{D}_1 = \mathcal{D}_2$. We would then have

$$\rho(T \square P_{4k+2}, \mathcal{S} \cdot \mathcal{D}_1) = \rho(P_{4k+2}, \mathcal{D}_2) = \pi(C_{4k+3}) = \frac{2^{2k+3}+1}{3},$$

and

$$\rho(T, \mathcal{S})\rho(P_{4k+2}, \mathcal{D}_1) = 2\pi(C_{4k+1}) = \frac{2^{2k+3}-2}{3},$$

but an analogue of Graham's conjecture would require

$$\rho(T \square P_{4k+2}, \mathcal{S} \cdot \mathcal{D}_1) \leq \rho(T, \mathcal{S})\rho(P_{4k+2}, \mathcal{D}_1).$$

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