Springer Proceedings in Mathematics & Statistics

Frederick Hoffman · Sarah Holliday ·
Zvi Rosen · Farhad Shahrokhi ·
John Wierman *Editors*

Combinatorics, Graph Theory and Computing

SEICCGTC 2021, Boca Raton, USA,
March 8–12

Springer

Springer Proceedings in Mathematics & Statistics

Volume 448

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including data science, operations research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

Frederick Hoffman · Sarah Holliday · Zvi Rosen ·
Farhad Shahrokhi · John Wierman
Editors

Combinatorics, Graph Theory and Computing

SEICCGTC 2021, Boca Raton, USA, March 8–12

Springer

Editors
Frederick Hoffman
Department of Mathematical Sciences
Florida Atlantic University
Boca Raton, FL, USA

Sarah Holliday
Department of Mathematics
Kennesaw State University
Kennesaw, GA, USA

Zvi Rosen
Department of Mathematics
Florida Atlantic University
Boca Raton, FL, USA

Farhad Shahrokhi
Department of Computer Science
and Engineering
University of North Texas
Denton, TX, USA

John Wierman
Department of Whiting School
of Engineering
Johns Hopkins University
Baltimore, MD, USA

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-031-52968-9 ISBN 978-3-031-52969-6 (eBook)
https://doi.org/10.1007/978-3-031-52969-6

Mathematics Subject Classification: 05-XX, 05-06, 05Cxx, 68Rxx, 94A60

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The Southeastern International Conference on Combinatorics, Graph Theory, and Computing (SEICCGTC) is an international meeting of mathematical scientists, held annually in March, during Spring Break at Florida Atlantic University (FAU) in Boca Raton, Florida. The conference includes a program with plenary lectures by invited speakers, as well as sessions of contributed papers each day. In addition, two or three invitational special sessions are offered each year. A valuable part of the conference is the opportunity afforded for informal conversations about the methods participants employ in their professional work in business, industry, and government and about their current research.

The 52nd meeting was held virtually, March 8–12, 2021, during the pandemic. Six distinguished researchers, at various stages of their careers, accepted invitations to present plenary lectures at the 52nd SEICCGTC: Marni Mishna, Simon Fraser University; Panos Pardalos, University of Florida; Cheryl Praeger, University of Western Australia; Jeroen Schillewaert, University of Auckland; Bryan Shader, University of Wyoming; Peter Winkler, Dartmouth College. There were special sessions on graph labeling, the inverse eigenvalue problem of a graph and zero forcing, and undergraduate research. There were more than 200 presentations in all and 250 participants.

The conference was supported by the department chair and staff, with technical support by Andrew Gultz. Outside support came from the National Security Agency, CRC Press/Taylor Francis, and Springer Nature. Conference coordination and organization was again superbly provided by Dr. Maria Provost. We are again pleased to have our proceedings in Springer PROMS and appreciate their expertise, professionalism, and patience of the people of Springer Nature.

This volume consists of selected papers and is fully refereed.

Boca Raton, USA — Frederick Hoffman
Kennesaw, USA — Sarah Holliday
Boca Raton, USA — Zvi Rosen
Denton, USA — Farhad Shahrokhi
Baltimore, USA — John Wierman

About This Book

We give a brief sketch of the contents. P. J. Slater defined an n-line splitting operation on graphs in 1974. The operation was extended to *es*-splitting on binary matroids by Azanchiler in 2006. Research has continued on applications of the operations on matroids in the years since, and Malavadkar and others have extended the operations to matroids over prime fields, called p-matroids. In their chapter here, he and his co-authors obtain results on the behavior with respect to connectivity, 3-connectivity and Eulerian connectivity of p-matroids under *es*-splitting.

The Cayley table M and the normal multiplication table N of a finite group G are Latin squares. There exists a Latin square orthogonal to both M and N if and only if G admits strong complete mappings. Anthony Evans addresses the natural question of which finite groups admit strong complete mappings. He summarizes work that has been done on this existence problem and establishes new classes of strongly admissible 2-groups. He gives theoretical proofs of the strong admissibility of some groups of order 16 whose strong admissibility had previously been proved only via computer searches.

In 2019, Ullman and Veleman studied functions a from an abelian group G to itself that can be expressed as a difference of two bijections b, c from G to itself. In this chapter, Cruz, Ramos, and Rubio relax the condition that b and c be bijections and instead study functions that can be expressed as the difference of two functions with the *same value multiset*. They construct all possible functions b, c with same value multiset, such that $a = b - c$. As a consequence they obtain a stronger version of Hall's theorem, which gives a description of b and c in terms of a. The chapter concludes with further directions and questions that relate this new approach to applications of Hall's theorem.

For a finite field of order q and v a divisor of $q - 1$, additive translates of a cyclotomic vector yield a $q \times q$ *cyclotomic array* on v symbols. For every positive integer t, for certain q sufficiently large with respect to v, such a cyclotomic array is always a covering array of strength t. For small values of t, this cyclotomic method produces smallest known covering arrays for numerous parameters suitable for practical applications. Yasmeen Akhtar, Charles Colbourn, and Violet Syrotiuk extend these ideas and show that cyclotomy can produce covering arrays of higher index, and locating and detecting arrays with large separation. Computational results also demonstrate that certain cyclotomic arrays for the same order q but different values of v can be juxtaposed to produce mixed-level covering, locating, and detecting arrays.

A classical construction of Bose produces a Steiner triple system of order $3n$ from a symmetric, idempotent latin square of order n whenever n is odd. In an application of access-balancing in storage systems, these Bose triple systems play a central role. A natural question arises: For which orders n does there exist a resolvable Bose triple system? In this chapter, Dylan Lusi and Charles Colbourn determine a resolution of the Bose-averaging triple system of order $3n$ whenever $n = 3p$ and $p \geq 5$ is prime.

A *graph-pair of order* t is a pair of graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. The Johnson graph $J(v, n)$ is the graph whose vertices are the n-element subsets of a v-element set, and two vertices are adjacent if the intersection of the corresponding subsets contains $n - 1$ elements. Atif Abueida and Mike Daven show necessary and sufficient conditions for $J(v, 2)$ to admit a decomposition into graph-pairs of order 4.

Liz Lane-Harvard and Tim Penttila construct new strongly regular graphs with parameters (85, 20, 3, 5) and (4369, 272, 15, 17). These parameters are the same as those arising from the generalized quadrangle $Q(4, 4^h)$ for $h = 1, 2$. The construction relies on the use of quadrics.

A simple, non-edgeless, regular graph is said to be *edge-regular* if the cardinality of the intersection of the neighborhoods of every pair of adjacent vertices is a fixed integer λ. Edge-regular graphs necessarily have a constant number p of non-neighbors for every adjacent vertex pair. Extensive research has been done on extremal and existence problems for edge-regular graphs. Robert McNeilis, Tabitha Parker, and Kenneth Roblee consider the existence problem for the case where $n = 3(\lambda + p) - 4$, with the added condition that the common neighbor set of every pair of adjacent vertices induces $\frac{\lambda}{2}K_2$, a disjoint union of edges. They prove a non-existence result.

The chapter by M. R. Emamy-K and Gustavo A. Melendez Rios reviews a convex geometric connection to threshold logic. They present necessary and sufficient conditions to recognize cut-complexes with 2 or 3 maximal faces from the class of all cubical complexes. This recognition of cut-complexes is closely related to an old proposal on cubical lattices by N. Metropolis and G. C. Rota. They proposed that cubical lattices may also be used for synthesis of Boolean functions parallel to the conventional Boolean algebraic methods. The characterization is applied here to recognize several cut-complexes in the 4-dimensional cube. The cut-complexes of the 4-cube are used to define a new poset that happens to be a distributive lattice.

In their chapter, Lilian Markenzon and Newton Paciornik discuss multicore graphs, a subclass that extends the generalized core-satellite graphs of Estrada and Benzi. They prove that the multicore graphs are the $(P_5, gem, dart)$-free chordal graphs, and they present a characterization of the class which provides a simple linear time recognition algorithm. They also show the interrelation of this class with other subclasses of chordal graphs: the clique-corona and starlike graphs.

Violator spaces were introduced by Gartner et al. in 2008 as a combinatorial framework that embraces linear programming and other geometric optimization problems.

Given a finite set E and an operator on its power set, two subsets of E are *cospanning* if they have the same image under the operator. Yulia Kempner and Vadim Levit investigate cospanning relations on violator spaces. Violator spaces are defined by violator operators. The authors introduce *co-violator spaces*, based on contracting operators. Cospanning characterizations of violator spaces allow the authors to obtain some new properties of violator and coviolator operators. In particular, they show that uniquely generated violator spaces satisfy so-called Krein-Milman properties; i.e., when α, β are a violator operator and a coviolator operator on E, then for all $X \subseteq E, \alpha(\beta(X)) = \alpha(X); \beta(\alpha(X)) = \beta(X)$.

LeRoy Beasley introduced (2, 3)-cordial labelings of directed graphs and conjectured that every orientation of a path of length at least five is (2,3)-cordial and that every tree of max degree five has a cordial orientation. In their chapter, Beasley, Manuel Santana, Jonathan Mousley, and Dave Brown formally define (2, 3)-cordiality from the viewpoint of *quasigroup* cordiality. They show both conjectures to be false, They discuss the (2, 3)-cordiality of orientations of the Petersen graph, and establish an upper bound for the number of edges a graph can have and still be (2, 3)-orientable.

In the next chapter, Mousley, Beasley, Santana, and Brown investigate the existence of (2, 3)-cordial labelings of oriented hypercubes. They determine that there exists a (2, 3)-cordial oriented hypercube for any dimension divisible by 3. They provide examples of (2, 3)-cordial oriented hypercubes of dimension not divisible by 3 and state a conjecture on existence for dimension $3k + 1$. They close by presenting the only 3-dimensional oriented hypercubes (up to isomorphism) that are not (2, 3)-cordial.

The m-ary n-dimensional hypercube is a generalization of the hypercube when $n = 2$. In their chapter, Saranya Anantapantula, Eddie Cheng, and Laszlo Liptak study the structural properties of m-ary n-dimensional hypercubes by considering the structure of the resulting graphs when up to approximately $3n(m-1)$ vertices are deleted from it.

A class of interconnection networks for massively parallel processors are designed by taking copies of a building block network and wiring them together. For Dragonfly networks, the building block network is a complete graph, and the wiring together is done by either a cycle or a complete graph. The resulting graph is known as a replacement graph. One of these constructions leads to a very large number of graphs, some of which are provably not isomorphic. Richard Draper asserts that the construction of a replacement in G by H requires that G be converted to a network. In his chapter, he explains the way the graph of an interconnection network is labeled and presents a table which is analogous to an adjacency matrix of a labeled graph. The graph constructions are presented along with the motivating interconnection network. The author asserts that the graph constructions can be generalized.

Graph pebbling is a network optimization model for satisfying vertex demands with vertex supplies (called pebbles) with partial loss of pebbles in transit. The pebbling number of a demand in a graph is the smallest number for which every placement of that many supply pebbles satisfies the demand. The Target Conjecture of Herscovic-Hester-Hurlbert posits that the largest pebbling number of a demand of fixed size t occurs when the demand is entirely stacked at one vertex. The truth of this conjecture could be useful for attacking many open problems in graph pebbling, including the famous conjecture of Graham in 1989 involving graph products. It has been verified for complete graphs, cubes, and trees. In their chapter, Hurlbert and Seddiq prove the conjecture for 2-paths and Kneser graphs over pairs.

For a graph $G = (V, E)$ embedded in the projective plane, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic projective graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. Call $\{x_{v:} : v \in V(G)\}$ a C_n-face-magic projective labeling on G. Stephen Curran and Stephen Locke consider the $m \times n$ grid graph, denoted by $\mathcal{P}_{m,n}$, embedded in the projective plane in the usual way. It is known that, for $m, n \geq 2$, $\mathcal{P}_{m,n}$ admits a C_4-face-magic projective labeling if and only if m and n have the same parity. When they are both even, a C_n-face-magic projective labeling on $\mathcal{P}_{m,n}$ has C_4-face-magic value $2mn + 2$. The authors show that there are 144 distinct C_4-face-magic projective labelings on the 4×4 projective grid graph $\mathcal{P}_{4,4}$ (up to symmetries of the projective plane).

Mahmud Imrona and his co-authors define rainbow vertex colorings of connected simple finite graphs. These colorings permit a coding of the vertices. If every vertex has a distinct rainbow code, then the coloring is called a *locating rainbow l-coloring*, where l is the number of colors. The smallest positive integer l such that G has a locating rainbow l-coloring is called the *locating rainbow connection number* of G. This chapter provides tight upper and lower bounds for the locating rainbow connection number of the comb product between an arbitrary graph and a complete graph or a tree.

Anna Bachstein, Wayne Goddard, and John Xue define a cycle-compelling coloring of a graph as a proper coloring of the vertices such that every subgraph induced by one vertex of each color contains a cycle. The cycle-compelling number is defined to be the minimum k such that some k-coloring is cycle-compelling. The authors provide some general bounds and algorithmic results on this and related parameters. They also investigate the value in specific graph families including cubic graphs, disjoint unions of cliques, and outerplanar graphs.

A harmonious labeling of a graph G of order n and size m is an injective function $f : V(G) \to Z_m$ that induces an injective function $f' : E(G) \to Z_m$ defined by $f'(uv) = f(u) + f(v)(mod\, m)$. (When G is a tree, f is allowed to repeat one vertex label.) A proper vertex coloring $c : V(G) \to Z_k$ is called a harmonious k-coloring if the induced edge coloring $c' : E(G) \to Z_k$ is also proper. The minimum positive integer k for which G has a harmonious k-coloring is called the harmonious chromatic number of G, χ_h. Alexis Byers and a team of researchers determine the harmonious chromatic number of all trees, cycles, grids, and graphs of diameter at most two, and make connections to existing graph labelings and colorings.

The immersion number of a graph G, denoted im(G), is the largest t such that G has a K_t-immersion. In their chapter, Karen Collins, Megan Heenahan, and Jessica McDonald discuss the determination of the immersion number of the m-Mycielskian of G, denoted $\mu_m(G)$. Given the immersion number of G, they provide a lower bound for im($\mu_m(G)$). To do this they introduce the "distinct neighbor property" of immersions. They include examples of classes of graphs where $\mathrm{im}(\mu_m(G))$ exceeds the lower bound and conclude with a conjecture about $\mathrm{im}(\mu_m(K_t))$.

The star-critical Ramsey number $r_*(G, H)$ was introduced in 2010. It is the least integer k such that every 2-coloring of the edges of $K_r - K_{1,r-k-1}$ contains either a red copy of G or a blue copy of H, where $r = R(G, H)$ is the graph Ramsey number. Research since 2010 has resulted in the discovery of values for the numbers as well as numerous classifications of critical graphs Variants to the number has been introduced to facilitate research. In her chapter, Jonelle Hook discusses recent developments in the research and presents a survey of all known star-critical Ramsey numbers.

In zero forcing, the focus is typically on finding the minimum cardinality of any zero forcing set in the graph; however, the number of cardinalities between 0 and the number of vertices in the graph for which there are both zero forcing sets and sets that fail to be zero forcing sets is not well known. Bonnie Jacob introduces the zero forcing span of a graph, which is the number of distinct cardinalities for which there are sets that are zero forcing and sets that are not. She introduces the span within the context of standard zero forcing and skew zero forcing as well as for standard zero forcing on directed graphs. She characterizes graphs with high span and low span of each type and also investigates graphs with special zero forcing polynomials.

Leyda Almodovar and her colleagues design a collection of tiles that will construct a nanostructure shaped like a target graph G. They employ a method based on the tile method for DNA self-assembly, which involves branched junction molecules whose flexible k-arms are double strands of DNA. They find the minimum number of tile and bond-edge types required to construct complete tripartite graphs and cocktail party graphs in three different scenarios representing distinct levels of laboratory constraints.

A mixed graph D is a graph that can be obtained from an undirected graph by ordering some of its edges. The Hermitian adjacency matrix of a mixed graph is defined to be the matrix $H = [h_{rs}]$, where $h_{rs} = i$ if $v_r v_s$ is an arc in D, $h_{rs} = -i$ if $v_s v_r$ is an arc in D, $h_{rs} = 1$ if $v_r v_s$ is a digon in D, and where $h_{rs} = 0$ otherwise. Omar Alomari and Mohammad Abudayah investigate when the Hermitian adjacency matrix of a bipartite graph is invertible, and they prove for any tree mixed graph T with invertible Hermitian adjacency matrix H that H^{-1} is a $\{0, \pm 1, \pm i\}$-matrix.

The Sierpinski triangle can be modeled using graphs in two different ways, resulting in classes of graphs called Sierpinski triangle graphs and Hanoi graphs. The latter are closely related to the Towers of Hanoi problem, Pascal's triangle and Apollonian networks. Parameters of these graphs have been studied by several researchers. In his chapter, Allan Bickle determines the number of Eulerian circuits of Sierpinski triangle graphs and presents a new and significantly shorter proof of their domination number. He also finds the 2-tone chromatic number and the number of diameter paths for both classes, thereby generalizing the classic Towers of Hanoi problem.

Zack King, Liz Lane-Harvard, and Thomas Milligan introduce new methods of studying properties of iterated line graphs and demonstrate the use of these methods on a class of tree graphs. They describe plans to generalize these methods further.

They derive identities for the number of vertices in the iterated line graphs of arbitrary graphs among other potential extensions of the methodology.

A k-factorization of the complete t-uniform hypergraph $K_v^{(t)}$ is an H-decomposition of $K_v^{(t)}$, where H is a k-regular spanning subhypergraph of $K_v^{(t)}$. It is known that seven of the eight non-isomorphic 3-uniform 2-regular hypergraphs of order $v \leq 9$ factorize $K_v^{(3)}$.

Peter Adams and his co-authors use nauty to generate the 2-regular spanning subhypergraphs of $K_{12}^{(3)}$ and show that they all factorize $K_{12}^{(3)} - I$, where I is a 1-factor.

A 3-uniform linear forest is any hypergraph obtained by starting with a single 3-uniform edge and adding other 3-uniform edges sequentially such that each additional edge intersects with the previous hypergraph at no more than one vertex. There are nine such 3-uniform linear forests with four edges. Ryan Bunge and his co-authors establish necessary and sufficient conditions for a decomposition of a complete 3-uniform hypergraph into isomorphic copies of a linear forest with four edges.

The Motzkin numbers are a well-known sequence with many combinatorial interpretations. One definition is

$$m_n = \frac{1}{n+1}\sum_{i\geq 0}\binom{n+1}{i}\binom{n+1-i}{i+1}, n \geq 0.$$

Toufik Mansour and Jose Ramirez define Motzkin words, which are enumerated by Motzkin numbers, and Motzkin bargraphs, a type of polyominoes. They study the exterior corners of these bargraphs and obtain generating functions and exact combinatorial formulas for them.

A *Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and consists of the same number of North-East steps U and South-East steps D. A *valley* is a subpath of the form DU. A Dyck path is called *restricted d-Dyck* if the difference between any two consecutive valleys is at least d or if it has at most one valley. Rigoberto Florez and his co-authors give some connections between restricted d-Dyck paths and both the non-crossing partitions of $[n]$ and some families of polyominoes. They also provide generating functions for several aspects of the combinatorial objects.

The sequence of balancing numbers $(B_n)_{n\geq 1}$ is given by $B_1 = 1$, $B_2 = 6$, and $B_n = 6B_{n-1} - B_{n-2}$; $n \geq 3$. Jeremiah Bartz and his colleagues consider the exponential Diophantine equations $B_n = 2^n$ and $B_m + B_m = 2^n$. Using Baker's theory of logarithmic forms, Matveev's Theorem, and an additional reduction theorem, they establish bounds on the space of possible solutions. This remaining space is sufficiently small that the problem of classifying solutions is reduced to a computational search which is carried out by a simple computer program.

Aaron Meyerowitz studies the Sprague-Grundy function of certain 2-player combinatorial games with the positive integers as the positions. One such game allows an integer to be reduced by a proper divisor. The game graph has the positive integers as vertices with a directed edge from q to $q - d$ for each proper divisor $d|q$. This is an acyclic directed graph with only finitely many vertices reachable from each given vertex. It has a unique Sprague-Grundy function. It has a strong number-theoretic structure. If the directions of the edges are reversed, we have the game of adding a divisor. The Sprague-Grundy functions of this unbounded game and certain portions of it are considered.

In their chapter, Carlos Agrinsoni, Heeralal Janwa, and Moises Delgado introduce the concept of a degree-gap of a multivariate polynomial. They utilize this concept to present a bound on the number of absolutely irreducible factors of an infinite family of multivariate polynomials, they show how this result can be used to guarantee that a polynomial is absolutely irreducible. They present an algorithm for testing the absolute irreducibility of multivariate polynomials over finite fields. The authors discuss applications of their results in algebraic geometry, coding theory, cryptography, finite geometry, and other research domains.

The covering radius of a q-ary block code C of length n is the smallest integer $R(C)$ such that all vectors in $\mathcal{F}_q^n$ are within Hamming distance R of some codeword of C. An $[n, k, d]R$ code is an $[n, k, d]$ code with covering radius R. Heeralal Janwa established in 1986 an upper bound for $R(C)$. If $n_q(k, d)$ denotes the minimum length of any code of dimension k and distance d over $\mathcal{F}_q$, it was conjectured by Janwa that under certain conditions the Griesmer length in the Janwa bound can be replaced by $n_q(k, d)$. Janwa and Matson proved three of the four cases and conjectured that the final case is true. In their chapter, Janwa and Orozco give a resolution of the conjecture. The resulting bounds help them in determining the exact covering radius for certain codes from Hermitian curves and close bounds for others.

Calderbank and Shor used pairs of binary or quaternary error-control codes for classical channels to control errors in quantum computers. The early results are not completely satisfactory. Researchers searched for "better" code pairs. In their chapter, Heeralal Janwa and Fernando Pinero-Gonzalez use codes from algebraic curves over high degree extensions of $\mathcal{F}_2$ to construct the self-orthogonal binary code or quaternary code pairs. They also present some results on the parameters of the resulting subfield codes over $\mathcal{F}_2$ or $\mathcal{F}_4$ from Hermitian curves, Norm-Trace curves, quasi-Hermitian curves, Castle curves and others. These results represent progress in the quest for practical quantum computing.

The chapter by Eddie Arrieta and Heeralal Janwa involves starting from a set of n codes over a field $\mathcal{F}_q$, for some prime power q, and obtaining an additive or linear code over $\mathcal{F}_{q^n}$. They show under what conditions this code is a self-orthogonal or self-dual code. This enables them to give some new classes of quantum error-correcting codes. These "Go-Up" codes have applications to algebraic coding theory, finite geometries, finite group theory, and strongly regular graphs, as well as codes with few weights.

Jose Velazquez and Heeralal Janwa consider vectorial Boolean functions. They study the constructions of Gold and Kasami-Welch functions of the form $Tr\left(\lambda x^d\right)$; $d = 2^l + 1, 2^{2l} - 2^l + 1$; $\lambda \in \mathcal{F}_{2^n}^*$. These functions' non-linearity property is a measure of their distance to the set of affine functions. The authors generalize a result of Dillon and Dobbertin for conditions under which these functions are bent.

They give algorithms that generate and determine the bentness of the functions. They construct 2-error correcting cyclic codes and enumerate Gold and Kasami-Welch functions. They improve previous algorithms used to determine the minimum distance of these codes.

Contents

The *es*-splitting Operation for Matroids Representable Over Prime Fields $GF(p)$

Prashant Malavadkar, Sachin Gunjal, Uday Jagadale, M. M. Shikare, and B. N. Waphare

Abstract The es-splitting operation for binary matroids is a natural generalization of Slater's n-line splitting operation on graphs. The present paper extends the es-splitting operation to the matroids representable over the field $GF(p)$, called the p-matoids. We characterize circuits, and bases of the resulting matroid in terms of the circuits, and the bases of the original matroid, respectively. It is proved that the es-splitting operation preserves the connectivity of the p-matoids. A Sufficient condition to obtain a 3-connected p-matroid from a 3-connected p-matroid using the es-splitting operation is also provided. In the last section, we obtain a sufficient condition to produce an Eulerian p-matroid from an Eulerian p-matroid using es-splitting operation.

Keywords p-matroid · es-splitting operation · Connected matroid · Eulerian matroid · Bipartite matroid

MSC Classification 05B35 · 05C50 · 05C83

These authors contributed equally to this work.

P. Malavadkar (✉) · S. Gunjal · U. Jagadale
School of Mathematics and Statistics, Dr. Vishwanath Karad MIT World Peace University, Paud road, Pune 411038, MH, India
e-mail: prashant.malavadkar@mitwpu.edu.in

S. Gunjal
e-mail: sachin.gunjal@mitwpu.edu.in

U. Jagadale
e-mail: uday.jagdale@mitwpu.edu.in

M. M. Shikare · B. N. Waphare
School of Computational Sciences & Mathematics, JSPM University, Pune, India
e-mail: mmshikare@gmail.com

B. N. Waphare
e-mail: waphare@yahoo.com

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_1

1 Introduction

The n-line splitting operation on a graph is specified by Slater [20] as follows. Let G be a graph and $e = uv$ be an edge of G with deg $u \geq 2n - 3$ with u adjacent to $v,\ x_1,\ x_2,\ \ldots,\ x_k,\ y_1,\ y_2,\ \ldots,\ y_h$, where k and $h \geq n - 2$. Let H be the graph obtained from G by replacing u with two adjacent vertices u_1 and u_2, with v adj u_1, v adj u_2, u_1 adj x_i $(1 \leq i \leq k)$, and u_2 adj y_j $(1 \leq j \leq h)$, where deg $u_1 \geq n$ and deg $u_2 \geq n$. The transition from G to H is called *n-line splitting operation*. In other words, we say that H is n-line splitting of G. This construction is explicitly illustrated with the help of Fig. 1.

Slater [20] used the n-line splitting operation to characterize n-connected graphs. Azanchiler [1, 2] extended the notion of n-line-splitting operation on graphs to binary matroids and determined the bases of the resulting matroid. The corresponding operation on binary matroids is known as the es-splitting operation. Several results concerning the splitting, element splitting and es-splitting operation on binary matroids have been explored in [1–3, 7, 9, 12, 19, 24, 26]. Malavadkar et al. [6, 13, 14] defined splitting, element splitting and es-splitting operation with two elements on the matroids representable over $GF(p)$; henceforth called p-matroids. We recall the definitions of the generalized splitting and the element splitting operations for p-matroids, defined in [5].

Definition 1 Let M be a p-matroid on a set E, $T \subset E$. Suppose A is a matrix representation of M over the prime field $GF(p)$. Let A_T be the matrix obtained by adjoining an extra row with entries zero everywhere except in the columns corresponding to the members of T where it takes the value 1. The vector matroid of the matrix A_T is denoted by M_T. The transition from M to M_T is called the splitting operation and the matroid M_T is called the splitting matroid. Let A'_T be the matrix obtained from A_T by adjoining an extra column labeled z with entries zero everywhere except in the last row where it takes the value 1. The vector matroid of the matrix A'_T is denoted by M'_T. The transition from M to M'_T is called the element splitting operation and the matroid M'_T is called the element splitting matroid.

Let M be a p-matroid and $T \subset E(M)$. The set of circuits of M is denoted by $\mathcal{C}(M)$ and $[n]$ denotes the set $\{1, 2, \ldots, n\}$. Let $C = \{u_1, u_2, \ldots, u_l, t_1, t_2, \ldots, t_m\}$ be a member of $\mathcal{C}(M)$, where l, m are positive integers, $u_i \in E \setminus T$, $t_j \in T$ for $i \in [l]$

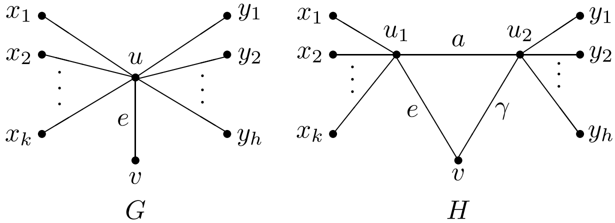


Fig. 1 The ten path with no cordial labeling

and $j \in [m]$. Then there are non zero constants $a_1, a_2, \dots a_l$ and $b_1, b_2, \dots, b_m$ in $GF(p)$ such that $\sum_{i=1}^{i=l} a_i u_i + \sum_{j=1}^{j=m} b_j t_j = 0$. We call a circuit C of M a **PT-circuit** if $\sum_{j=1}^{j=m} b_j \equiv 0 (mod\ p)$ and an **NPT-circuit** if $\sum_{j=1}^{j=m} b_j \not\equiv 0 (mod\ p)$. We use $\mathcal{C}_0$ to denote the collection of PT-circuits and the circuits containing no element of T. A dependent set $D = \{v_1, v_2, \dots, v_j, t_1 = e, t_2, \dots, t_k\}$ of M is called an **ePT-dependent** if there exists non zero constants $\alpha_1, \alpha_2, \dots \alpha_j$ and $\beta_1, \beta_2, \dots, \beta_k$ in $GF(p)$ such that $\sum_{i=1}^{i=j} \alpha_i u_i + \sum_{i=1}^{i=k} \beta_i t_i = 0$, and $\sum_{i=2}^{i=k} \beta_i \equiv 0 (mod\ p)$. We call a minimal ePT-dependent set an **ePT-circuit** of M. Note that an ePT-circuit C is an NPT-circuit of M that contains e, and the $\sum_{x \in (C \setminus e) \cap T} coeff.(x) \equiv 0 \mod p$.

Consider subsets of E of the type $C \cup I$ where $C = \{u_1, u_2, \dots, u_l\}$ is an NPT-circuit of M which is disjoint from an independent set $I = \{v_1, v_2, \dots, v_k\}$ and $T \cap (C \cup I) \neq \phi$. We say $C \cup I$ is a **PT-dependent set** if it contains no member of $\mathcal{C}_0$ and there are non-zero constants $\alpha_1, \alpha_2, \dots, \alpha_l$ and $\beta_1, \beta_2, \dots, \beta_k$ such that $\sum_{i=1}^{l} \alpha_i u_i + \sum_{j=1}^{k} \beta_j v_j = 0$ and $\sum_{x \in T \cap (C \cup I)} coeff.(x) \equiv 0 (mod\ p)$.

Throughout this paper, we consider loopless and coloopless p-matroids. We define the *es*-splitting operation on p-matroids as follow.

Definition 2 Let $M \cong M[A]$ be a p-matroid on ground set E and $T \subset E$ with $e \in T$. Let A'_T represents the element splitting matroid M'_T on $GF(p)$. Construct the matrix A^e_T by adjoining an extra column γ to the matrix A'_T where γ is the subtraction$(e - z)$ of two column vectors corresponding to the elements e and z. Denote $M^e_T \cong M[A^e_T]$. The shift of M to M^e_T is called an *es*-splitting operation. We call the matroid M^e_T the *es*-splitting p-matroid.

For $|T| = 2$, Definition 2 coincides with the definition of the *es*-splitting operation given by Jagadale et al. [6]

Remark 1 Observe that
(1) $rank(A) \leq rank(A^e_T) \leq rank(A) + 1$. If r and r' are the rank functions of M and M^e_T, respectively, then $r(M) \leq r'(M^e_T) \leq r(M) + 1$.
(2) $\Delta = \{z, e, \gamma\}$ is a circuit in *es*- splitting matroid M^e_T.
(3) For an ePT-circuit C of M, the set $(C \setminus e) \cup \gamma$ is dependent in M^e_T.

In Sect. 2, we characterize the circuits, the bases and the rank function of the resulting *es*-splitting matroid M^e_T in terms of the circuits, bases, and the rank function of the original p-matroid M, respectively. In Sect. 3, we prove that the *es*-splitting operation on connected p-matroids yields connected p-matroids. A sufficient condition to produce 3-connected p-matroids from 3-connected p-matroids, after performing the *es*-splitting operation is obtained. In Sect. 4, the effect of the *es*-splitting operation on Eulerian and bipartite matroids is explored. For undefined, standard terminology in graphs and matroids, we refer to Oxley [15].

2 Circuits, Bases and the Rank Function of M_T^e

The next theorem characterizes the circuits of the *es*-splitting *p*-matroid in terms of the circuits of the original *p*-matroid.

Theorem 1 *Let M be a p-matroid on the ground set E and $T \subset E$. Then $\mathcal{C}(M_T^e) = \mathcal{C}(M_T') \cup \mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_8 \cup \Delta$ where*

- $\mathcal{C}_4 = \{C \cup \{e, \gamma\}$:]*C is an NPT-circuit and $e \notin C\}$;*
- $\mathcal{C}_5 = \{(C \setminus e) \cup \{z, \gamma\} : C$ *is a PT-circuit and $e \in C\}$;*
- $\mathcal{C}_6 = \{C \cup \gamma : e \in C, C$ *is an NPT-circuit but not an ePT-circuit* $\}$;
- $\mathcal{C}_7 = \{(C \setminus e) \cup \{z, \gamma\} : e \in C, C$ *is an NPT-circuit but not an ePT-circuit*$\}$;
- $\mathcal{C}_8 =$ *minimal elements of* $\{D \setminus \{e\} \cup \gamma :$ *Dis an ePT dependent set of M, $D \setminus \{e\} \cup \gamma$ contains no member of $\mathcal{C}_i$; $0 \leq i \leq 7$, in M_T^e* $\}$;
- $\Delta = \{e, z, \gamma\}$

Proof The inclusion $\mathcal{C}(M_T') \cup \mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_8 \cup \Delta \subseteq \mathcal{C}(M_T^e)$ follows from the definition 1.1. For the converse part, let $C \in \mathcal{C}(M_T^e)$ and $\gamma \notin C$. Then by Theorem 1.5 of [5], $C \in \mathcal{C}(M_T')$. Suppose that $\gamma \in C$.
Case 1 : $z \in C$. If $e \in C$, then $C = \Delta = \{e, z, \gamma\}$. Now suppose $e \notin C$. Let $C_1 = C \setminus \{z, \gamma\}$. Then C_1 is an independent set of M; otherwise, C_1 is a dependent set of M containing an *NPT*-circuit, say C', then $C' \cup z$ is a circuit of M_T^e contained in C, which is not possible. Observe that the set $C_1 \cup e$ is a dependent set of M. Moreover, it is a circuit of M. If $C_1 \cup e$ is a *PT*-circuit, then $C \in \mathcal{C}_5$. And if $C_1 \cup e$ is an *NPT*-circuit then observe that $\sum_{x \in T \cap (C_1 \cup e)} \text{coeff.}(x) \not\equiv 0 \mod p$. Therefore, $C_1 \cup e$ is not an *ePT*-circuit. Thus $C = (C_1 \cup e) \setminus e \cup \{z, \gamma\}$. Therefore $C \in \mathcal{C}_7$.
Case 2 : $z \notin C$
Case 2.1 : $e \in C \setminus \gamma$. Note that $C_2 = C \setminus \gamma$ is a dependent set of M, and one of the following cases will occur.
Case 2.1.1 : C_2 itself is a circuit of M. Then it must be an *NPT* circuit. If $\sum_{\substack{x \in T \cap C_2 \\ x \neq e}} \text{coeff.}(x) \equiv 0 \mod p$, then $(C_2 \setminus e) \cup \gamma$ is a dependent set in M_T^e contained in C, a contradiction. In other case, if $\sum_{\substack{x \in T \cap C_2 \\ x \neq e}} \text{coeff.}(x) \not\equiv 0 \mod p$, then $C \in \mathcal{C}_6$.
Case 2.1.2 : If C_2 is not a circuit of M, then it contains a circuit say C_3. Note that C_3 must be an *NPT*-circuit.
If $e \notin C_3$ and $C_2 = C_3 \cup e$, then $C = C_3 \cup \{e, \gamma\}$. Hence $C \in \mathcal{C}_4$. If $C_2 = C_3 \cup A, A = C_2 \setminus C_3, |A| \geq 2$, then $C_3 \cup \{e, \gamma\}$ is a circuit of type $\mathcal{C}_4$ contained in C, a contradiction. If $e \in C_3$, then, as proved in Case 2.1.1, $(C_3 \setminus e) \cup \gamma$ is a dependent set in M_T^e or $C_3 \in \mathcal{C}_6$, a contradiction.
Case 2.2 : $e \notin C_2 = C \setminus \gamma$. Then $C_2 \cup e$ is a dependent set of M. Let $D = C_2 \cup e$. In M_T^e, $C = D \setminus \{e\} \cup \gamma$. Therefore the sum $\sum_{\substack{x \in T \cap D \\ x \neq e}} \text{coeff.}(x) \equiv 0 \mod p$, and $D \setminus \{e\} \cup \gamma$ does not contain any member of $\mathcal{C}_i$; $0 \leq i \leq 7$. Therefore $C \in \mathcal{C}_8$.

Theorem 1 is illustrated with the following example.

Example 1 Consider a matroid $M = M[A]$ which is represented over the finite field *GF(3)*.

$$\mathbf{A} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \qquad \mathbf{A}_T^e = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

For $T = \{1, 4, 7\}$ and $e = 7$, the representation of *es*-splitting matroid M_T^e over $GF(3)$ is given by the matrix A_T^e. The collection of circuits of M, M_T and M_T^e is given in the following table.

Circuits of M	**Circuits of** M_T	**Circuits of** M_T^e
$\{1, 2, 6, 7\}$, $\{3, 4, 5, 7\}$, $\{3, 5, 6, 8\}$	$\{1, 2, 6, 7\}$, $\{3, 4, 5, 7\}$, $\{3, 5, 6, 8\}$	$\{1, 2, 6, 7\}$, $\{3, 4, 5, 7\}$, $\{3, 5, 6, 8\}$
$\{1, 2, 3\}$, $\{1, 2, 4, 5, 7\}$, $\{1, 2, 4, 8\}$, $\{1, 2, 5, 6, 8\}$, $\{3, 4, 8\}$, $\{3, 6, 7\}$, $\{4, 5, 6\}$, $\{4, 6, 7, 8\}$, $\{5, 7, 8\}$	–	$\{1, 2, 3, 9\}$, $\{1, 2, 4, 5, 7, 9\}$, $\{1, 2, 4, 8, 9\}$, $\{1, 2, 5, 6, 8, 9\}$, $\{3, 4, 8, 9\}$, $\{3, 6, 7, 9\}$, $\{4, 5, 6, 9\}$, $\{4, 6, 7, 8, 9\}$, $\{5, 7, 8, 9\}$
–	–	$\{1, 2, 3, 7, 10\}$, $\{1, 2, 4, 7, 8, 10\}$, $\{3, 4, 7, 8, 10\}$, $\{4, 5, 6, 7, 10\}$
–	–	$\{1, 2, 6, 9, 10\}$, $\{3, 4, 5, 9, 10\}$
–	–	$\{1, 2, 4, 5, 7, 10\}$, $\{4, 6, 7, 8, 10\}$
–	–	$\{1, 2, 4, 5, 9, 10\}$, $\{4, 6, 8, 9, 10\}$
–	–	$\{1, 2, 3, 4, 5, 10\}$, $\{1, 2, 4, 5, 6, 10\}$, $\{1, 2, 4, 6, 8, 10\}$
–	–	$\{7, 9, 10\}$

We denote by Cl and Cl' the closure operator of M and M_T^e, respectively. The collection of bases of matroid M is denoted by $\mathcal{B}(M)$. The next lemma characterizes the bases of the *es*-splitting matroid in terms of the bases of the original matroid.

Lemma 1 *Let M be a p-matroid, $T \subset E$. Then $\mathcal{B}(M_T^e) = \mathcal{B}(M_T') \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ where*
$\mathcal{B}_1 = \{B \cup \gamma : B \in \mathcal{B}(M)$ *and* $e \in B\}$;
$\mathcal{B}_2 = \{B \cup \gamma : B \in \mathcal{B}(M)$, $e \notin B$, *and* $(B \cup e)$ *does not contain an ePT-circuit.*$\}$;
$\mathcal{B}_3 = \{(C_{NPT} \cup I) \cup \gamma :$ $(C_{NPT} \cup I)$ *is not a PT-dependent set of* M, $rank(C_{NPT} \cup I) = rank(M)$, $e \notin Cl(C_{NPT} \cup I)\}$;
$\mathcal{B}_4 = \{(C_{NPT} \cup I) \cup \gamma : (C_{NPT} \cup I)$ *is not a PT-dependent set of* M, $r(C_{NPT} \cup I) = r(M)$, $e \in Cl(C_{NPT} \cup I)$ *and the circuit containing e is an NPT-circuit but not an ePT-circuit.*$\}$;
$\mathcal{B}_5 = \{I \cup \{z, \gamma\} : I \in \mathcal{I}(M)$, $e \notin Cl(I)$, $|I| = r(M) - 1$ $\}$.

Proof Let $B' \in \mathcal{B}(M_T^e)$
Case 1 : $z \notin B', \gamma \in B'$. Consider $B = B' \setminus \gamma$ in M.
Case 1.1 :Let B be a basis. Suppose $e \in B$. Then $B \in \mathcal{B}_1$.
Let $e \notin B$. Then $e \in Cl(B)$. If there exists an ePT- circuit C in $B \cup e$, then $(C \setminus e) \cup \gamma$ is a dependent set contained in B', a contradiction. Thus, $B \cup e$ does not contain an ePT-circuit and hence $B' \in \mathcal{B}_2$.
Case 1.2 :Next assume, $B$is a dependent set. Then $B = C_{NPT} \cup I$, where C_{NPT} is an NPT circuit disjoint from independent set I in M. If $e \in C_{NPT}$, then we will get a circuit of the type C_6 or C_8. Also, $e \notin I$; otherwise $C_{NPT} \cup \{e, \gamma\}$ is a circuit of the type C_4 contained in B', a contradiction.
If $e \notin Cl(C_{NPT} \cup I)$, then $B' \in \mathcal{B}_3$.

Next assume, $e \in Cl(C_{NPT} \cup I)$. Let C be the circuit of M passing through e, and contained in $(C_{NPT} \cup I) \cup e$. If C is an ePT-circuit, then $(C \setminus e) \cup \gamma$ is a dependent subset of M_T^e contained in B', a contradiction. Therefore $(C_{NPT} \cup I) \cup e$ does not contain an ePT-circuit containing e. Thus $B' \in \mathcal{B}_4$.
Case 2 : $z, \gamma \in B'$. Let $B_1 = B' \setminus \{z, \gamma\}$. If B_1 is a dependent in M, then it must contain an NPT circuit C_{NPT}, which forms a circuit with z in B'. Therefore, B_1 is independent set in M. Further, if $e \in Cl(B_1)$, then we get a circuit which is a member of C_5, C_7, or C_8, which is not possible. Therefore $e \notin Cl(B_1)$. Thus, $B' \in \mathcal{B}_5$.

We now prove the inclusion $\mathcal{B}(M_T') \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \subseteq \mathcal{B}(M_T^e)$. Let $B' \in \mathcal{B}(M_T')$. Then B' is independent in M_T^e, and $|B'| = r(M) + 1$. Therefore $B' \in \mathcal{B}(M_T^e)$. If $B' \in \mathcal{B}_1$, then $B' = B \cup \gamma$, where $B \in \mathcal{B}(M)$. B is also an independent set in M_T^e. Let $\gamma \in Cl'(B)$, and $C \in \mathcal{C}(M_T^e)$ is a circuit passing through e. The circuit C cannot be a member of $\mathcal{C}_4$, $\mathcal{C}_6$, and $\mathcal{C}_8$; for this to happen B must contain a dependent set containing e. Therefore such a circuit cannot exist. Thus, $\gamma \notin Cl'(B)$, and B' is a basis of M_T^e. Suppose $B' \in \mathcal{B}_2$. Then $B' = B \cup \gamma$, where $B \in \mathcal{B}(M)$. Now B is an independent set in M_T^e. Let $\gamma \in Cl'(B)$, and C be a circuit passing through γ. As $e \notin B$, C cannot be a member of $\mathcal{C}_4, \mathcal{C}_6$. For C to be a member of $\mathcal{C}_8$, $(B \cup e)$ must contain an ePT-circuit, which is impossible. Thus $\gamma \notin Cl'(B)$, and hence $B \cup \gamma$ is a basis of M_T^e.

Next assume $B' \in \mathcal{B}_3$. Then $B' = (C_{NPT} \cup I) \cup \gamma$, where$(C_{NPT} \cup I)$ is not a PT-dependent set of M, $r(C_{NPT} \cup I) = r(M)$, and $e \notin Cl(C_{NPT} \cup I)$. Since $(C_{NPT} \cup I)$ is not a PT-dependent set of M, it is an independent set in M_T^e of the size $r(M)$. We claim $\gamma \notin Cl'(C_{NPT} \cup I)$. On the contrary assume $\gamma \in Cl'(C_{NPT} \cup I)$, and C is a circuit containing γ. Such a circuit C must be a member of one of the collection $\mathcal{C}_4$, $\mathcal{C}_6$, and $\mathcal{C}_8$, which implies $e \in Cl(C)$, and hence $e \in Cl(C_{NPT} \cup I)$, a contradiction. Similarly, if $B' \in \mathcal{B}_4$, then $B' = (C_{NPT} \cup I) \cup \gamma$. As $(C_{NPT} \cup I)$ is not a PT-dependent set of M implies it is an an independent set in M_T^e. Let $\gamma \in Cl'(C_{NPT} \cup I)$, and C be a circuit containing γ. Since $e \notin (C_{NPT} \cup I)$, C cannot be a member of $\mathcal{C}_4, \mathcal{C}_6$. Thus the only possibility remains is $C \in \mathcal{C}_8$. But then for this is to be true $((C_{NPT} \cup I \cup e)$ must contain an ePT-circuit, which is not possible. Thus $\gamma \notin Cl'(C_{NPT} \cup I)$, and hence $(C_{NPT} \cup I) \cup \gamma$ is a basis of M_T^e.

$B' \in \mathcal{B}_5$. Then $B' = I \cup \{z, \gamma\}$, where $I \in \mathcal{I}(M)$, and $|I| = r(M) - 1$. I is an independent set in M_T^e. Moreover, $I \cup z$ is independent in M_T', and hence in M_T^e. Now if we assume $\gamma \in Cl'(I \cup z)$, and C be a circuit containing γ. First assume that

$z \in C$. Then C must be a member of one of the collection $\mathcal{C}_5$, or $\mathcal{C}_7$. But this implies $e \in Cl(I)$, a contradiction. If $z \notin C$, then C belongs to on of the collection $\mathcal{C}_4, \mathcal{C}_6$, and $\mathcal{C}_8$. But again this implies $e \in Cl(I)$, which is not possible. Thus, $\gamma \notin Cl'(I \cup z)$. Therefore $(I \cup z) \cup \gamma$ is a basis of M_T^e.

In Theorem 2, the rank function of the *es*-splitting *p*-matroid M_T^e is described in terms of the rank function of the original matroid M.

Theorem 2 *Let r and r' denote the rank functions of the p-matroids M and M_T^e, respectively. Let $X \subseteq E(M)$. Then the following holds*

$$r'(X) = \begin{cases} r(X) + 1, & \textit{if } X \textit{ contains an NPT} - \textit{circuit;} \\ r(X), & \textit{otherwise} \end{cases} \tag{1}$$

$$r'(X \cup z) = r(X) + 1 \tag{2}$$

$$r'(X \ \cup \ \gamma) = \begin{cases} r(X), \ \textit{if } \ e \in Cl(X) \setminus X, X \ \textit{contains no NPT circuit,} \\ \qquad \textit{and } X \cup e \ \textit{contains an ePT-dependent set}; \\ r(X) + 2, \textit{if } e \notin Cl(X) \ \textit{and } X \ \textit{contains an NPT circuit;} \\ r(X) + 1, \textit{otherwise} \end{cases} \tag{3}$$

$$r'(X \ \cup \ \{z, \gamma\}) = \begin{cases} r(X) + 1, \textit{if } e \in Cl(X); \\ r(X) + 2, \textit{if } e \notin Cl(X) \end{cases} \tag{4}$$

Proof The proofs of (1), (2), are given in [5]. To prove (3), consider $e \in Cl(X) \setminus X$. Since X contains no *NPT*, circuit $r'(X) = r(X)$. As $X \cup e$ contains an *ePT*-dependent set of M, say D. The set $(D \setminus e) \cup \gamma$ is dependent in M_T^e. Thus $\gamma \in Cl'(D \setminus e) \subseteq Cl'(X)$. Therefore $r'(X \cup \gamma) = r'(X) = r(X)$.
Now if $e \notin Cl(X)$ and X contains an *NPT*-circuit, then by equation (1), $r'(X) = r(X) + 1$. Further we prove that $\gamma \notin Cl'(X)$. Contrarily, if $\gamma \in Cl'(X)$ then there exists a circuit $C \subset X \cup \gamma$ containing γ in M_T^e. This circuit is one from the type $\mathcal{C}_4, \mathcal{C}_6$ or $\mathcal{C}_8$. But for this to happen e must be in the closure of X, which is not true. Therefore $\gamma \notin Cl'(X)$. Thus $r'(X \cup \gamma) = r'(X) + 1 = r(X) + 2$.
To prove (4), by equation (2) $r'(X \cup \{z\}) = r(X) + 1$. Let $e \in Cl(X)$, and C be the circuit passing through e. Then following are two possible cases:

$e \in Cl(X) \setminus X$: If C is an *ePT* circuit, then $(C \setminus e) \cup \gamma$ is dependent in M_T^e. Therefore $\gamma \in Cl'(X) \subseteq Cl'(X \cup z)$. Thus $r'(X \cup \{z, \gamma\}) = r'(X \cup z) = r(X) + 1$. And if C is not an *ePT*-circuit, then $(C \setminus e) \cup \{z, \gamma\}$ is a circuit. Therefore $r'(X \cup \{z, \gamma\}) = r'(X \cup z) = r(X) + 1$.

$e \in X$: If C is an *ePT* circuit, then as argued in the earlier case, we conclude $r'(X \cup \{z, \gamma\}) = r'(X \cup z) = r(X) + 1$. If C is not an *ePT*-circuit, then $C \cup \gamma$ is a circuit of the type $\mathcal{C}_6$. Therefore $\gamma \in Cl'(X) \subseteq Cl'(X \cup z)$. Thus $r'(X \cup \{z, \gamma\}) = r'(X \cup z) = r(X) + 1$.

If $e \notin Cl(X)$, then γ cannot be in $Cl'(X \cup z)$. Therefore $r'(X \cup \{z, \gamma\}) = r'(X \cup z) + 1 = r(X) + 2$.

3 Connectivity of M_T^e

Recall that a matroid M is connected if and only if for every pair of distinct elements of $E(M)$ there is a circuit containing both. The concept of n-connected matroids was introduced by Tutte [23]. If k is a positive integer, the matroid M is k-separated if there is a subset $X \subset E(M)$ such that $|X| \geq k$, $|E \setminus X| \geq k$ and $r(X) + r(E \setminus X) - r(M) = k - 1$. Connectivity $\lambda(M)$ of M is the least positive integer j such that M is j-separated. If there is no such integer we say $\lambda(M) = \infty$. Note that $\lambda(U_{2,4}) = \infty$. The following result from [15] provides a necessary condition for a matroid to be n-connected.

Lemma 2 *If M is an n-connected matroid and $|E(M)| \geq 2(n-1)$, then all circuits and all cocircuits of M have at least n elements.*

Remark 2 If M is an n-connected p-matroid and $T \subset E(M)$, then M_T^e is at most 3-connected p-matroid. Because; by Theorem 1, *es*-splitting matroid M_T^e contains a 3-circuit $\Delta = \{z, e, \gamma\}$.

Lemma 3 *Let M be a connected p-matroid and $T \subset E(M)$. Then the corresponding es-splitting matroid M_T^e is connected.*

Proof As M is a connected p-matroid, by Theorem 4.3 of [5] the element splitting matroid M_T' is also connected. So it is enough to prove that for every element $x \in E \cup z$, there exists a circuit passing through x and γ.
Case 1 : Suppose $x = z$. Then $\Delta = \{e, z, \gamma\}$ is the required circuit.
Case 2 : Suppose $x \in E(M)$. If $x = e$, then the circuit $\Delta = \{e, z, \gamma\}$ contains x and γ. Now assume $x \neq e$, since M is a connected matroid, there exists a circuit, say C, passing through x and e. Such a circuit C is a PT-circuit or NPT-circuit. If C is a PT-circuit, then $(C \setminus e) \cup \{z, \gamma\}$ is the desired circuit passing through x and γ. Next, assume C is an NPT-circuit. Then the desired circuit passing through x and γ will be a member of one of the collections $\mathcal{C}_6$, $\mathcal{C}_7$ or $\mathcal{C}_8$.

The following result for binary matroids, due to Azanchiler [1], follows immediately from Lemma 3.

Corollary 1 *Let M be a connected binary matroid, $T \subset E(M)$, and $e \in T$. Then M_T^e is a connected binary matroid.*

The n-connected minors of the es-splitting binary matroids is studied in [11]. The following theorem provides a sufficient condition for the *es*-splitting matroid M_T^e to be 3-connected.

Theorem 3 *Let M be a 3-connected p- matroid, $T \subset E(M)$ and $e \in T$ Suppose that M has an NPT-circuit not containing e. Then M_T^e is a 3-connected p- matroid.*

Proof Proof can be given using similar arguments in the proof of Theorem 2.2 of [3], by replacing OX-circuit by an NPT-circuit.

4 Applications

A matroid M is said to be Eulerian if its ground set E can be expressed as a union of disjoint circuits (see Welsh [25]). The next proposition provides a sufficient condition for an Eulerian p-matroid to yield Eulerian *es*-splitting p-matroid M_T^e.

Proposition 1 *Let M be an Eulerian p-matroid, $T \subset E$, and $\{C_1, C_2, \dots, C_n\}$ be a circuit decomposition of $E(M)$ such that $e \in C_1$ but C_1 is not an ePT-circuit; C_2 is an NPT- circuit, and $C_3, C_4, \dots, C_n$ are PT-circuits. Then M_T^e is Eulerian.*

Proof Observe that $C_1 \cup \gamma$, $C_2 \cup z$, and $C_3, C_4, \dots, C_n$ forms a circuit decomposition of M_T^e, hence the proof.

Proposition 2 gives a sufficient condition to produce Eulerian *es*-splitting matroid M_T^e.

Proposition 2 *Let M be a p-matroid, $T \subset E$, and $\{C_1, C_2, \dots, C_n\}$ be a collection of pairwise disjoint circuits of M, except $C_1 \cap C_2 = \{e\}$ such that $E(M) = C_1 \cup C_2 \cup \dots \cup C_n$; C_1 is a PT-circuit, C_2 is an NPT- circuit that is not an ePT-circuit, and $C_3, C_4, \dots, C_n$ are PT-circuits. Then M_T^e is Eulerian.*

Proof Note that, as $C_1, C_3, C_4, \dots, C_n$ are PT-circuits of M, therefore they are also circuits of M_T^e. Further, $(C_2 \setminus e) \cup \{z, \gamma\}$ is a circuit of type $\mathcal{C}_7$. Thus $\{C_1, (C_2 \setminus e) \cup \{z, \gamma\}, C_3, \dots, C_n\}$ forms a circuit decomposition of M_T^e. Therefore, M_T^e is Eulerian matroid.

Remark 3 A matroid M is said to be bipartite if every circuit of M is of even cardinality. Note that the *es*-splitting matroid M_T^e contains a circuit $\{z, e, \gamma\}$; hence M_T^e is not a bipartite p-matroid.

Declarations

- Funding: The authors declare that they did not receive any funding, or grants for the work presented in this paper.
- Conflict of interest/Competing interests: The Authors declare that there is no conflict of interest.
- Consent for publication: We the authors of this paper give our consent for the publication of this paper in the journal Springer Proceedings in Mathematics and Statistics (PROMS).
- Availability of data and materials: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
- Authors' contributions: Each author has made equal contribution for conceptualizing, writing and revising the paper.

References

1. Habib, A.: Extension of line-splitting operation from graphs to binary matroid. Lobachevskii J. Math. **24**, 3–12 (2006)
2. Habib, A.: A characterization of the bases of line-splitting matroids. Lobachevskii J. Math. **26**, 5–15 (2007)
3. Dhotre, S.B., Malavadkar, P.P., Shikare, M.M.: On 3-connected es-splitting binary matroids. Asian-Eur. J. Math. **9**(1), 1650017–26 (2016)
4. Fleischner, H.: Eulerian Graphs and Related Topics, Part 1, vol. 1. North Holland, Amsterdam (1990)
5. Gunjal, S.S., Jagadale, U.V., Malavadkar, P.P.: Generalized splitting and element splitting operations on p-matroids, Preprint. https://doi.org/10.48550/arXiv.2303.01898
6. Jagadale, U.V., Malavadkar, P.P., Gunjal, S.S., Shikare, M.M.: On the characterization of Eulerian *es*-splitting p-matroids, preprint. https://doi.org/10.48550/arXiv.2205.03180
7. Mills, A.D.: On the cocircuits of a splitting matroid. Ars Combin. **89**, 243–253 (2008)
8. Malavadkar, P.P., Shikare, M.M., Dhotre, S.B.: A characterization of n-connected splitting matroids. Asian-Eur. J. Math. **7**(4), 1–7 (2014)
9. Malavadkar, P.P., Shikare, M.M., Dhotre, S.B.: A characterization of cocircuits of an es-splitting matroid. J. Comb. Math. Comb. Comput. **105**, 247–258 (2018)
10. Malavadkar, P.P., Dhotre, S.B., Shikare, M.M.: Forbidden-minors for the class of cographic matroids which yield the graphic element splitting matroids. Southeast Asian Bull. Math. **43**(1), 105–119 (2019)
11. Malavadkar, P.P., Dhotre, S.B., Shikare, M.M.: On n-connected minors of the es-splitting binary matroids. Electron. J. Graph Theory Appl. (EJGTA) **9**(2), 265–275 (2021)
12. Malavadkar, P.P., Dhotre, S.B., Shikare, M.M.: The closure operator, flats and hyperplanes of es-splitting matroid. J. Indian Math. Soc.(JIMS) **88**(3–4), 334–345 (2021)
13. Malavadkar, P.P., Jagadale, U.V., Gunjal S.S.: On the circuits of splitting matroids representable over $GF(p)$, preprint. https://doi.org/10.48550/arXiv.2112.00437
14. Malavadkar P.P., Gunjal S.S., Jagadale U.V.: A note on connectivity preserving splitting operation for matroids representable over $GF(p)$. https://doi.org/10.48550/arXiv.2003.03617
15. Oxley, J.G.: Matroid Theory. Oxford University Press, Oxford (1992)
16. Raghunathan, T.T., Shikare, M.M., Waphare, B.N.: Splitting in a binary matroid. Disc. Math. **184**, 267–271 (1998)
17. Shikare, M.M.: Splitting lemma for binary matroids. Southeast Asian Bull. Math. **32**, 151–159 (2008)
18. Shikare, M.M., Azadi, G.: Determination of bases of a splitting matroid. European J. Combin. **24**, 45–52 (2003)
19. Shikare, M.M., Dhotre, S.B., Malavadkar, P.P.: A forbidden-minor characterization for the class of regular matroids which yield the cographic es-splitting matroids. Lobachevskii J. Math. **34**, 173–180 (2013)
20. Slater, P.J.: A Classification of 4-connected graphs. J. Combin. Theory **17**, 282–298 (1974)
21. Tutte, W.T.: Lectures on matroids. J. Res. Nat. Bur. Standards **B69**, 1–47 (1965)
22. Tutte, W.T.: A theory of 3-connected graphs. Indag. Math. **23**, 441–455 (1961)
23. Tutte, W.T.: Connectivity in matroids. Canad. J. Math. **18**, 1301–1324 (1966)
24. Wagner, D.K.: Bipartite and Eulerian minors. European J. Combin. **74**, 1–10 (2018)
25. Welsh, D.J.A.: Euler and bipartite matroids. J. Combin. Theory **6**, 375–377 (1969)
26. Yaokun, W.: Even poset and a parity result for binary linear code. Linear Algebra Appl **418**, 591–594 (2006)

The Existence Problem for Strong Complete Mappings of Finite Groups

Anthony B. Evans

Abstract The Cayley table M and the normal multiplication table N of a finite group G are Latin squares. There exists a Latin square orthogonal to both M and N if and only if G admits strong complete mappings. A natural question to ask is, which finite groups admit strong complete mappings? We will summarize work done on the existence problem for strong complete mappings of finite groups. We will also establish new classes of strongly admissible 2-groups. We will also give theoretical proofs of the strong admissibility of some groups of order 16, whose strong admissibility has only been proved via computer searches.

Keywords Strong complete mappings · Latin squares

1 Introduction

A *Latin square* of order n is an $n \times n$ matrix with entries from a symbol set of order n in which each symbol appears exactly once in each row and exactly once in each column. Two Latin squares on the same symbol set are *orthogonal* if, when superimposed, each ordered pair of symbols appears exactly once: these squares are *orthogonal mates* of each other. A *transversal* of a Latin square is a set of cells, one from each row, one from each column, in which each symbol occurs exactly once. It is well-known that a Latin square has an orthogonal mate if and only if its cells can be partitioned into transversals. As an example, the Latin squares L_1 and L_2 in Fig. 1 are orthogonal mates. The entries of L_1 in cells corresponding to cells of L_2 with entry 1 are shown in bold: these entries are distinct and, hence, these cells in L_1 form a transversal of L_1. Similarly the cells of L_1 corresponding to cells of L_2 with entry a (resp. b or c) form a transversal of L_1. These four transversals partition the cells of L_1. For more information on Latin squares and orthogonality see [5].

A. B. Evans (✉)
Wright State University, Dayton, OH, USA
e-mail: anthony.evans@wright.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_2

Fig. 1 Two orthogonal Latin squares of order 4

$$L_1 = \begin{pmatrix} \mathbf{1} & a & b & c \\ a & 1 & c & \mathbf{b} \\ b & \mathbf{c} & 1 & a \\ c & b & \mathbf{a} & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & b & c & a \\ a & c & b & 1 \\ b & 1 & a & c \\ c & a & 1 & b \end{pmatrix}.$$

The *Cayley table* of a group $G = \{g_1, \dots, g_n\}$ is the Latin square with ijth entry $g_i g_j$, and the *normal multiplication table* of G is the Latin square with ijth entry $g_i^{-1} g_j$. It is easily seen that the square L_1 in Fig. 1 is the Cayley table for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. A *complete mapping* of a group G is a bijection $\theta\colon G \to G$ for which the mapping $x \mapsto x\theta(x)$ is a bijection. If G admits complete mappings we say that G is *admissible*. A *strong complete mapping* of a group G is a complete mapping θ of G for which the mapping $x \mapsto x^{-1}\theta(x)$ is a bijection. If G admits strong complete mappings we say that G is *strongly admissible*. Complete mappings and strong complete mappings play an important role in determining the existence of an orthogonal mate for the Cayley table of a group, and the existence of Latin squares orthogonal to both the Cayley table and the normal multiplication table of a group.

Theorem 1 *Let M be the Cayley table and N the normal multiplication table of a finite group G.*

1. *M has an orthogonal mate if and only if M has a transversal, if and only if G is admissible.*
2. *There exists a Latin square orthogonal to both M and N if and only if there exists a common transversal of M and N, if and only if G is strongly admissible.*

Proof For convenience, suppose the rows and columns of M and N to be indexed by the elements of G. Let $T = \{(x, \theta(x)) \mid x \in G\}$ be a set of cells in M and N. As the entries in the cells of T in M are $\{x\theta(x) \mid x \in G\}$ we see that T is a transversal of M if and only if θ is a complete mapping of G, and as the entries in the cells of T in N are $\{x^{-1}\theta(x) \mid x \in G\}$ we see that T is a common transversal of M and N if and only if θ is a strong complete mapping of G.

Set $T_g = \{(x, \theta(x)g) \mid x \in G\}$, $g \in G$. If T is a transversal of M (resp. N), then $\{T_g \mid g \in G\}$ is a set of transversals that partitions the cells of M (resp. N). As a Latin square has an orthogonal mate if and only if its cells can be partitioned into transversals, the result follows. □

Note that, in [11, 12], the definition of the normal multiplication of a group is not the same as in this paper: for $G = \{g_1, \dots, g_n\}$ it is instead defined to be the Latin square with ijth entry $g_i g_j^{-1}$. With this definition we can still easily prove that there exists a Latin square orthogonal to both M and N if and only if G is strongly admissible, but the proof is not as transparent: see Theorem 4 in [11].

We are naturally led to ask, which finite groups are admissible and which finite groups are strongly admissible? We should note that, for infinite groups, all infinite groups are admissible [2], and many classes of infinite groups are strongly admissible [14]. Admissible finite groups have been characterized.

Theorem 2 *A finite group is admissible if and only if its Sylow* 2*-subgroup is trivial or non-cyclic.*

In 1955, Hall and Paige [15] proved that any finite group with a non-trivial, cyclic Sylow 2-subgroup is not admissible: they conjectured the converse, which they proved for several classes of groups. The Hall–Paige conjecture was proved in 2009 by Wilcox [20], Evans [8], and Bray: Bray's contribution to this proof is included in [4]. Work done on the Hall-Paige conjecture in the intervening years is described in Part II of [12].

By contrast, finite groups that are strongly admissible have not been characterized. In this paper we will summarize work done on this problem and establish some new classes of strongly admissible 2-groups. Earlier surveys of work on this problem are [6, 10].

In Sect. 2 we will describe known non-existence results, strongly admissible abelian groups, strongly admissible groups of orders relatively prime to 6, and strongly admissible groups of small order; in Sect. 3 we will describe several known classes of strongly admissible groups; and in Sect. 4 we will present some new results for 2-groups.

2 Initial Results

In this section we will set the stage by presenting some known strong admissibility results. In particular we will present non-existence results, the characterization of strongly admissible finite abelian groups, the strong admissibility of groups of orders relatively prime to 6, and the strong admissibility of groups of small order.

Theorem 3 *Let G be a finite group. Then*

1. *G is not strongly admissible if its Sylow* 2*-subgroup is non-trivial and cyclic.*
2. *G is not strongly admissible if its Sylow* 3*-subgroup is non-trivial, cyclic, and a homomorphic image of G.*

Proof Equation (1) is an immediate consequence of Theorem 2 and (2) is proved in [6]. □

From strongly admissible groups we can construct other strongly admissible groups using direct product constructions.

Lemma 1 *If $G_1, \dots, G_n$ are strongly admissible, then $G_1 \times \cdots \times G_n$ is strongly admissible.*

Proof If θ_i is a strong complete mapping of G_i for $i = 1, \dots, n$, then $\theta_1 \times \cdots \times \theta_n$, defined by $\theta_1 \times \cdots \times \theta_n(g_1, \dots, g_n) = (\theta_1(g_1), \dots, \theta_n(g_n))$, is a strong complete mapping of $G_1 \times \cdots \times G_n$. □

The strong admissibility of an abelian group is completely determined by the structure of its Sylow 2-subgroup and its Sylow 3-subgroup.

Theorem 4 (Evans [9]) *A finite abelian group is strongly admissible if and only if*

1. *its Sylow* 2*-subgroup is trivial or non-cyclic and*
2. *its Sylow* 3*-subgroup is trivial or non-cyclic.*

Proof Necessity follows from Theorem 3. By Lemma 1, to prove sufficiency we need to prove that non-cyclic, abelian 2- groups and non-cyclic, abelian 3- groups are strongly admissible. Non-cyclic, abelian 2- groups were shown to be strongly admissible in 1990 independently by Horton [17] and Evans [6], and non-cyclic, abelian 3- groups were shown to be strongly admissible in 2012 by Evans [9]. □

Groups with trivial Sylow 2-subgroups and trivial Sylow 3-subgroups are easily shown to be strongly admissible.

Theorem 5 *If* $2, 3 \nmid |G|$*, then* G *is strongly admissible.*

Proof $\theta : x \mapsto x^2$ is a strong complete mapping of G. □

Theorems 4 and 5 might lead one to suppose that the strong admissibility of a finite group is completely determined by the structures of its Sylow 2-subgroup and its Sylow 3-subgroup. But, it is not so simple, as the following examples show.

Example 1 D_{12}, the dihedral group of order 12 is strongly admissible. A construction of a strong complete mapping of D_{12} was given in [18]: this was the first construction of a strong complete mapping of a group with a non-trivial, cyclic Sylow 3-subgroup.

Example 2 D_8 and Q_8, the non-abelian groups of order 8, are not strongly admissible: a theoretical proof of this fact is given in Theorem 10 in [11].

All strongly admissible groups of order at most 31 have been determined.

Theorem 6 *If* $|G| \leq 31$*, then* G *is strongly admissible unless*

1. G *has a non-trivial, cyclic Sylow* 2*-subgroup,*
2. G *has a non-trivial, cyclic Sylow* 3*-subgroup that is a homomorphic image of* G*, or*
3. $G = D_8$ *or* Q_8.

Proof This is the result of a computer search, using Magma [3], presented in [11].

□

From Theorem 6 we see that the converse of Theorem 3 holds for all groups of order at most 31 except for the two non-abelian groups of order 8. As a working hypothesis we conjecture the following.

Conjecture 1 If $G \neq D_8, Q_8$ is a finite group with a trivial or non-cyclic Sylow 2-subgroup, and a Sylow 3-group that is trivial, non-cyclic, or cyclic but not isomorphic to a homomorphic image of G, then G is strongly admissible.

3 Some Known Classes of Strongly Admissible Groups

The approaches that were most successful in establishing the admissibility of groups involved quotient group constructions and constructing complete mappings of groups from complete mappings of subgroups. If $H \trianglelefteq G$, H and G/H admissible, then G is admissible. There are two principal constructions of complete mappings of groups from complete mappings of subgroups: these involve HP-systems and W-systems. These approaches and the results are described in Part II in [12].

There is no known general quotient group construction for strongly admissible groups, and no analogue for HP-systems and W-systems for general groups. Some of the approaches used to construct strong complete mappings employ quotient group constructions for classes of groups, and finding ways to construct strong complete mappings of groups from strong complete mappings of subgroups for classes of groups.

Here are two quotient constructions for strongly admissible groups. The first is for abelian groups.

Theorem 7 *If G is abelian and $H \trianglelefteq G$, H and G/H both strongly admissible, then G is strongly admissible.*

Proof See Lemma 2.8 in [17] and Theorem 3 in [6]. □

In Theorem 4, strong complete mappings of $\mathbb{Z}_3 \times \mathbb{Z}_{3^m}$ were constructed and the quotient group construction in Theorem 7 was then used to prove all non-cyclic, abelian 3-groups to be strongly admissible. The following is a quotient group construction for groups in general which does require that a technical condition be satisfied.

Theorem 8 *If H is a normal subgroup of G, G/H is strongly admissible, D is a system of distinct coset representatives for H in G, and there exists a strong complete mapping ϕ of H for which the mapping*

$$h \mapsto h^{-1}d^{-1}\phi(h)d$$

is a bijection for all $d \in D$, then G is strongly admissible.

Proof See Theorem 8 in [11]. □

The technical condition in Theorem 8 clearly holds if $H = Z(G)$ and H is strongly admissible (Corollary 2 in [11]). In [1] it was observed that we need only $H \leq Z(G)$.

Corollary 1 *If $H \leq Z(G)$ and H and G/H are strongly admissible, then G is strongly admissible.*

Proof See Lemma 2.1 in [1]. □

Theorem 8 can be used to construct infinite classes of strongly admissible dihedral and quaternion groups.

Corollary 2 *Suppose* $\gcd(m, 6) = 1$. *If* D_n*, the dihedral group of order* n*, is strongly admissible, then* D_{mn} *is strongly admissible, and if* Q_n*, the quaternion group of order* n*, is strongly admissible, then* Q_{mn} *is strongly admissible.*

Proof See Lemma 1 in [11]. □

From constructions of strong complete mappings for small dihedral and quaternion groups we obtain several infinite classes of strongly admissible dihedral and quaternion groups.

Theorem 9 (Evans [11]) *Let* $\gcd(m, 6) = 1$*, let* D_n *denote the dihedral group of order* n*, and let* Q_n *denote the quaternion group of order* n*. Then*

1. D_{4m}, D_{12m}, D_{16m}, *and* D_{24m} *are strongly admissible, and*
2. Q_{16m}, *and* Q_{24m} *are strongly admissible.*

Proof See Theorems 11 and 12 in [11]. □

Recently Akhtar and Gagola [1] proved most non-cyclic 3-groups to be strongly admissible. To do so, they first needed a specialized quotient group construction.

Lemma 2 (Akhtar and Gagola [1]) *If* G *is a* 3*-group,* $N \lhd G$, $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$*, and* G/N *is strongly admissible, then* G *is strongly admissible*

Proof See Proposition 2.4 in [1]. □

Theorem 10 (Akhtar and Gagola [1]) *All finite, non-cyclic* 3*-groups are strongly admissible with the possible exception of the groups*

$$\langle a, b \mid a^{3^{r-1}} = b^3 = 1, bab^{-1} = a^{1+3^{r-2}} \rangle, r \geq 4.$$

Proof See Theorem 3.1 in [1]. □

We see that the smallest 3-group for which strong admissibility has not been determined is of order 81.

Interestingly, infinite classes of strongly admissible groups can be constructed from any finite group, even if not admissible.

Theorem 11 (Evans and Gagola [13]) *If* G *is a finite group, not necessarily admissible or strongly admissible, then* $G \times \cdots \times G \neq G$ *is strongly admissible.*

Proof See Theorems 4 and 5 in [13]. □

The following is an immediate corollary.

Corollary 3 *Finite, characteristically simple groups, that are not simple, are strongly admissible.*

Proof A finite characteristically simple group is either simple or a direct product of isomorphic copies of a simple group. □

4 Some New Results for 2-Groups

In this section we will prove two quotient group results, one specifically for 2-groups and one for even-order groups in general. As examples, we will establish the strong admissibility of some groups of orders 16 and 32. While all non-cyclic, non-abelian groups of order 16 were shown to be strongly admissible in Theorem 6, this proof was the result of computer searches: proofs in this section will be theoretical.

The dihedral group of order $4k$ is

$$D_{4k} = \langle a, b \mid a^{2k} = b^2 = 1, ab = ba^{-1} \rangle.$$

The strong admissibility of dihedral groups has been characterized.

Lemma 3 *D_{4k} is strongly admissible if and only if there exist partitions $\{A, \bar{A}\}$ and $\{B, \bar{B}\}$ of $\mathbb{Z}_{2k}$, $|A| = |\bar{A}| = |B| = |\bar{B}| = k$, and mappings $\alpha\colon A \to \mathbb{Z}_{2k}$, $\beta\colon \bar{A} \to \mathbb{Z}_{2k}$, $\gamma\colon B \to \mathbb{Z}_{2k}$, and $\delta\colon \bar{B} \to \mathbb{Z}_{2k}$ for which*

1. *$\{\alpha(i) + i \mid i \in A\} = \{\alpha(i) - i \mid i \in A\}$,*
2. *$\{\beta(i) + i \mid i \in \bar{A}\} = \{\beta(i) - i \mid i \in \bar{A}\}$,*
3. *$\{\alpha(i) \mid i \in A\}$ and $\{\gamma(i) \mid i \in B\}$ partition $\mathbb{Z}_{2k}$,*
4. *$\{\beta(i) \mid i \in \bar{A}\}$ and $\{\delta(i) \mid i \in \bar{B}\}$ partition $\mathbb{Z}_{2k}$,*
5. *$\{\alpha(i) \pm i \mid i \in A\}$ and $\{\delta(i) - i \mid i \in \bar{B}\}$ partition $\mathbb{Z}_{2k}$, and*
6. *$\{\beta(i) \pm i \mid i \in \bar{A}\}$ and $\{\gamma(i) + i \mid i \in B\}$ partition $\mathbb{Z}_{2k}$.*

Proof See Theorem 9 in [11]. □

From Lemma 3 we can derive the strong admissibility of D_{32}.

Theorem 12 *D_{32} is strongly admissible.*

Proof In Lemma 3, set $k = 8$,

$$A = \{0, 1, 2, 4, 5, 7, 8, 13\},$$

$$\bar{A} = \{3, 6, 9, 10, 11, 12, 14.15\},$$

$$B = \{1, 2, 3, 6, 7, 10, 12, 15\},$$

$$\bar{B} = \{0, 4, 5, 8, 9, 11, 13, 14\},$$

and the mappings α, β, γ, and δ are defined in the following tables.

g	0	1	2	4	5	7	8	13
$\alpha(g)$	0	12	9	8	15	3	13	10
$\alpha(g)+g$	0	13	11	12	4	10	5	7
$\alpha(g)-g$	0	11	7	4	10	12	5	13

and

g	3	6	9	10	11	12	14	15
$\beta(g)$	4	9	8	10	0	7	2	6
$\beta(g)+g$	7	15	1	4	11	3	0	5
$\beta(g)-g$	1	3	15	0	5	11	4	7

g	1	2	3	6	7	10	12	15
$\gamma(g)$	5	7	11	4	1	2	6	14
$\gamma(g)+g$	6	9	14	10	8	12	2	13

and

g	0	4	5	8	9	11	13	14
$\delta(g)$	14	3	11	1	12	13	5	15
$\delta(g)-g$	14	15	6	9	3	2	8	1

It is easily seen that α, β, γ, and δ satisfy the conditions of Lemma 3. □

As a corollary, Theorem 12 yields an infinite class of strongly admissible dihedral groups.

Corollary 4 *If* $\gcd(m, 6) = 1$, *then* D_{32m} *is strongly admissible.*

Proof This follows from Theorem 12 and Corollary 2. □

One other approach to tackle strong admissibility is to consider constructions of complete mappings and ask the question, when are the complete mappings constructed strong complete mappings? Let us consider the construction of complete mappings in Theorem 11 in [7]: see also Theorem 6.22 in [12].

Let H be an admissible, even-order, normal subgroup of a group G, $|G/H| = 2$. Let $a, b \in G \setminus H$ be 2-elements satisfying $xa^ix^{-1} \neq b^j$ for any i, j odd and for any $x \in H$. Let Γ be the graph with vertices the elements of H, and edges $\{\{x, a^\epsilon xb^\delta\} \mid x \in H, \epsilon, \delta = \pm 1\}$. Γ is a bipartite graph. Let X and Y be bipartite classes for Γ. Let ϕ be a complete mapping of H and set

$$A = \{x \mid x \in H, \phi(x) \in X\}, \text{ and } B = \{x \mid x \in H, \phi(x) \in Y\}.$$

Deine $\theta\colon G \to G$ by

$$\theta(x) = \begin{cases} \phi(x) & \text{if } x \in A; \\ \phi(x)b & \text{if } x \in B; \\ a^{-1}\phi(xa^{-1})b & \text{if } x \in Aa; \\ a^{-1}\phi(xa^{-1}) & \text{if } x \in Ba. \end{cases} \tag{1}$$

In Theorem 11 in [7], it was proved that θ is a complete mapping. If ϕ is a strong complete mapping, when is θ a strong complete mapping?

Theorem 13 *In Eq. (1), if ϕ is a strong complete mapping of H, then θ is a strong complete mapping of G if and only if*

1. $\{x^{-1}\theta(x) \mid x \in A\}$ *and* $\{a^{-1}x^{-1}a^{-1}\theta(x) \mid x \in B\}$ *partition* H, *and*
2. $\{x^{-1}\theta(x) \mid x \in B\}$ *and* $\{a^{-1}x^{-1}a^{-1}\theta(x) \mid x \in A\}$ *partition* H.

Proof Routine. □

As a corollary, we obtain the following.

Corollary 5 *If H is strongly admissible, then $H \times \mathbb{Z}_2$ is strongly admissible.*

Proof Pick a to be an involution in $Z(G)$ and b to be any other involution not in H. □

In [15], one of the principal tools used to prove non-cyclic 2-groups to be admissible was extending complete mappings of intersections of maximal subgroups to complete mappings of the group. In the following theorem, we will present a quotient group construction along these lines.

Theorem 14 *Let G be a* 2*-group, and let* M_1 *and* M_2 *be distinct maximal subgroups. If* $K = M_1 \cap M_2$ *is strongly admissible and there exist elements in* $M_1 \setminus K$ *and* $M_2 \setminus K$ *that centralize* K*, then G is strongly admissible.*

Proof The *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G, and if G is a 2-group, then $G/\Phi(G)$ is an elementary abelian 2-group. It follows that G/K is the elementary abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let ϕ be a strong complete mapping of K and let $b \in M_1$ centralize K and $c \in M_2$ centralize K, and define a mapping $\theta\colon G \to G$ by

$$\theta(g) = \begin{cases} \phi(k) & \text{if } g = k, k \in K; \\ \phi(k)c & \text{if } g = bk, k \in K; \\ \phi(k)bc & \text{if } g = ck, k \in K; \\ \phi(k)b & \text{if } g = bck, k \in K. \end{cases}$$

It is easy to verify that θ is a strong complete mapping of G. □

The following is an immediate corollary.

Corollary 6 *Let G be a* 2*-group, and let* M_1 *and* M_2 *be distinct maximal subgroups. If* $K = M_1 \cap M_2$ *is strongly admissible and* M_1 *and* M_2 *are abelian, then G is strongly admissible.*

In what follows, we will give examples of groups of orders 16 and 32 that are strongly admissible along with proofs that these groups are strongly admissible. Groups of orders 16 and 32 are described in [19] and groups of orders 16, 32, and 64 are described in [16].

There are fourteen groups of order 16, five of these are abelian and so are covered by Theorem 4. The nine non-abelian groups of order 16 were shown to be strongly admissible, via computer searches, in Theorem 6; we can now give theoretical proofs for six of these groups.

Example 3 Let G be group $16\Gamma_2 a_1$ in [16], group 16/6 in [19],

$$G = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, ab = ba^{-1} \rangle \cong D_8 \times \mathbb{Z}_2.$$

As D_8 is not strongly admissible by Example 2, Corollary 5 does not apply. Two maximal subgroups of G, M_1 and M_2, and their intersection K are

$$M_1 = \langle a, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$M_2 = \langle b, a^2, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$K = \langle a^2, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

It follows from Corollary 6 that G is strongly admissible.

Example 4 Let G be group $16\Gamma_2 a_2$ in [16], group 16/7 in [19],

$$G = \langle a, b, c \mid a^4 = c^2 = 1, b^2 = a^2, ac = ca, bc = cb, ab = ba^{-1}\rangle \cong Q_8 \times \mathbb{Z}_2.$$

As Q_8 is not strongly admissible by Example 2, Corollary 5 does not apply. Two maximal subgroups of G, M_1 and M_2, and their intersection K are

$$M_1 = \langle a, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$M_2 = \langle b, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$K = \langle a^2, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

It follows from Corollary 6 that G is strongly admissible.

Example 5 Let G be group $16\Gamma_2 c_1$ in [16], group 16/9 in [19],

$$G = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, bc = cb, ac = ca^{-1}b\rangle.$$

Two maximal subgroups of G, M_1 and M_2, and their intersection K are

$$M_1 = \langle a, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$M_2 = \langle b, a^2, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$K = \langle a^2, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

It follows from Corollary 6 that G is strongly admissible.

Example 6 Let G be group $16\Gamma_2 c_2$ in [16], group 16/10 in [19],

$$G = \langle a, b \mid a^4 = b^4 = 1, ab = ba^{-1}\rangle.$$

Two maximal subgroups of G, M_1 and M_2, and their intersection K are

$$M_1 = \langle a, b^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$M_2 = \langle a^2, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$K = \langle a^2, b^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

It follows from Corollary 6 that G is strongly admissible.

Example 7 Let G be group $16\Gamma_3 a_1$ in [16], group 16/12 in [19], $G = D_{16}$ was shown to be strongly admissible in Theorem 11 in [11].

Example 8 Let G be group $16\Gamma_3 a_3$ in [16], group 16/14 in [19], $G = Q_{16}$ was shown to be strongly admissible in Theorem 12 in [11].

There are fifty one groups of order 32, seven of these are abelian and so are covered by Theorem 4. Of the forty four non-abelian groups of order 32 we will prove four of these groups to be strongly admissible.

Example 9 Let G be group $32\Gamma_2 a_1$ in [16], group 32/8 in [19]. $G = (D_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2$. By Example 3, $D_8 \times \mathbb{Z}_2$ is strongly admissible. Hence, by Corollary 5, G is strongly admissible.

Example 10 Let G be group $32\Gamma_2 a_2$ in [16], group 32/9 in [19]. $G = (Q_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2$. By Example 4, $Q_8 \times \mathbb{Z}_2$ is strongly admissible. Hence, by Corollary 5, G is strongly admissible.

Example 11 Let G be group $32\Gamma_3 a_1$ in [16], group 32/23 in [19]. $G = D_{16} \times \mathbb{Z}_2$. By Example 7, D_{16} is strongly admissible. Hence, by Corollary 5, G is strongly admissible.

Example 12 Let G be group $32\Gamma_3 a_3$ in [16], group 32/25 in [19]. $G = Q_{16} \times \mathbb{Z}_2$. By Example 8, Q_{16} is strongly admissible. Hence, by Corollary 5, G is strongly admissible.

References

1. Ahktar, R., Gagola III, S.M.: Strong complete mappings for 3-groups. Discrete Math. **345**, paper 112643 (2022)
2. Bateman, P.T.: A remark on infinite groups. Amer. Math. Mon. **57**, 623–624 (1950)
3. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symbolic Comput. **24**, 235–265 (1997)
4. Bray, J.N., Cai, Q., Cameron, P.J., Spiga, P., Zhang, H.: J. Algebra **545**, 27–42 (2020)
5. Dénes, J., Keedwell, A.D.: Latin Squares and Their Applications, 2nd edn. North-Holland, Amsterdam (2015)
6. Evans, A.B.: On strong complete mappings. Congr. Numer. **70**, 241–248 (1990)

7. Evans, A.B.: The existence of complete mappings of finite groups. Congr. Numer. **90**, 65–75 (1992)
8. Evans, A.B.: The admissibility of sporadic simple groups. J. Algebra **321**, 105–116 (2009)
9. Evans, A.B.: The existence of strong complete mappings. Electronic J. Combin. **19**, # P34 (2012)
10. Evans, A.B.: The existence of strong complete mappings of finite groups: a survey. Discrete Math. **313**, 1191–1196 (2013)
11. Evans, A.B.: The strong admissibility of finite groups: an update. JCMCC **98**, 391–403 (2016)
12. Evans, A.B.: Orthogonal Latin squares based on groups. Developments in Mathematics 57, Springer, Cham (2018)
13. Evans, A.B., Gagola, S.M., III.: The strong admissibility of finite Moufang loops. Discrete Math. **342**, 2717–2723 (2019)
14. Evans, A.B., Martin, G.N., Minden, K., Ollis, M.A.: Infinite latin squares: neighbor balance and orthogonality. Electronic J. Combin. **27**, # P3.52 (2020)
15. Hall, M., Paige, L.J.: Complete mappings of finite groups. Pacific J. Math. **5**, 541–549 (1955)
16. Hall, M., Senior, J.K.: The Groups of Order 2^n $(n \leq 6)$. Macmillan, New York (1964)
17. Horton, J.D.: Orthogonal starters in finite abelian groups. Discrete Math. **79**, 265–278 (1990)
18. Shieh, Y.-P., Hsiang, J., Hsu, D.F.: On the existence problems of complete mappings. preprint
19. Thomas, A.D., Wood, G.V.: Group tables. Devon Print Group, Exeter, Devon (1980)
20. Wilcox, S.: Reduction of the Hall-Paige conjecture to sporadic simple groups. J. Algebra **321**, 1407–1428 (2009)

Differences of Functions with the Same Value Multiset

Dylan Cruz, Andrés Ramos, and Ivelisse Rubio

Abstract In a recent article, Ullman and Velleman studied functions a from an abelian group G to itself that can be expressed as a difference of two bijections b, c from G to itself. In this work we relax the condition that b and c are bijections and instead study functions that can be expressed as the difference of two functions with the *same value multiset*. We construct all possible functions b, c with same value multiset, such that $a = b - c$. As a consequence, we obtain a stronger version of Hall's theorem, which gives a description of b and c in terms of a. We conclude by presenting further directions and questions that relate this new approach to applications of Hall's theorem.

Keywords Same value multiset · Abelian group · Hall's theorem

1 Introduction

Consider a function $a : G \longrightarrow G$ from a finite abelian group G to itself. The problem of expressing a as the difference of two bijections $b, c : G \longrightarrow G,\ a(i) = b(i) - c(i)$, has surprising connections to many problems such as juggling sequences [5], bus scheduling [8], and even to the study of transversals of the Cayley table of the group G [7]. In a recent paper [7], Ullman and Velleman studied this problem over any abelian group. For the case when the group is finite, they noticed that expressing a function as a difference of bijections is solved by Hall's Theorem [6] which can be stated as follows.

Theorem 1 *Let $G = \{g_1, \ldots, g_n\}$ be a finite abelian group of order n and $a : G \longrightarrow G$ be a function. Then $a = b - c$ with b, c bijections on G if and only if $\sum_{i=1}^{n} a(g_i) = 0$.*

D. Cruz · A. Ramos · I. Rubio (✉)
Department of Computer Science, University of Puerto Rico, 17 Ave. Univ. STE 1701, San Juan, PR 00925-2537, USA
e-mail: ivelisse.rubio@upr.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_3

We study the following generalization when G is a finite abelian group. Given a function $a : G \longrightarrow G$ and a multiset M of elements in G, determine conditions that guarantee that a can be expressed as $a = b - c$ with functions $b, c : G \longrightarrow G$ such that $\{\{b(g) \mid g \in G\}\} = \{\{c(g) \mid g \in G\}\} = M$. That is, instead of having G as the value set of b and c, we consider a *value multiset* M, a set that allows repetitions, to be the image of b and c. When $M = G$ we recover the original problem. Note that we use double brackets for multisets and, as usual, single brackets for regular sets.

Looking at the functions a, b, c as sequences $\mathbf{a} = a_1, \ldots, a_n$, $\mathbf{b} = b_1, \ldots, b_n$ and $\mathbf{c} = c_1, \ldots, c_n$ where $a_i = a(g_i)$, $b_i = b(g_i)$ and $c_i = c(g_i)$ for $G = \{g_1, \ldots, g_n\}$, allows for more generality. For example, we can consider sequences of elements in G of arbitrary length k. We write $\mathbf{a} = \mathbf{b} - \mathbf{c}$ instead of $a_i = b_i - c_i$ for $1 \leq i \leq n$. In this context, functions b and c having the same value multiset M are equivalent to sequences $\mathbf{b}$ and $\mathbf{c}$ being reorderings of the elements of M. We also say that the sequences $\mathbf{b}$ and $\mathbf{c}$ have the same value multiset M.

Our main results are as follows. Theorem 3 gives a construction for all possible sequences $\mathbf{b}, \mathbf{c}$ of arbitrary length, same value multiset, such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. From this we obtain all possible multisets M for which it is possible to write $\mathbf{a} = \mathbf{b} - \mathbf{c}$. This theorem generalizes Hall's theorem to sequences $\mathbf{b}, \mathbf{c}$ that do not necessarily represent bijections and provides constructions for these sequences. Moreover, it is used to get a stronger version of Hall's theorem by giving descriptions of $\mathbf{b}$ and $\mathbf{c}$ in terms of $\mathbf{a}$ (Theorem 4).

To get these results we first characterize the sequences $\mathbf{a} = a_1, \ldots, a_k$ that can be expressed as the difference of two sequences with the same value multiset (Proposition 1). The sequence $\mathbf{a}$ is expressed as the concatenation of what we call *zero-sum irreducible components*, and $\mathbf{b}, \mathbf{c}$ are constructed in terms of these components. We conclude our paper with further questions about the structure of the irreducible components which, if resolved, provide concrete constructions for orthomorphisms, complete mappings and transversals of certain Latin squares. We also present connections to parking functions.

We begin by relating the problem of expressing $\mathbf{a} = \mathbf{b} - \mathbf{c}$, $\mathbf{b}, \mathbf{c}$ with the same value multiset, to a family of directed graphs that we call $(M, G, \mathbf{a})$-graphs.

1.1 An Interpretation Using Directed Graphs

The problem of expressing a sequence $\mathbf{a} = a_1, \ldots, a_k$ as the difference of two sequences $\mathbf{b}, \mathbf{c}$ that are rearrangements of the elements in some multiset M is equivalent to constructing certain directed graphs. This provides a very nice visualization that will be used in paper to provide intuition for the algebraic constructions. Ullman and Velleman introduced these directed graphs for the case where $\mathbf{b}, \mathbf{c}$ represent bijections and named them $(G, \mathbf{a})$-graphs. We generalize these directed graphs for our value multiset setting. Define the cardinality of a multiset M as the number of elements in M, counting multiplicity.

Fig. 1 $(M, G, \mathbf{a})$-graph

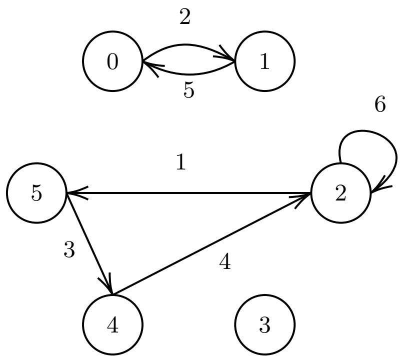


Fig. 2 $\mathbf{a} = \mathbf{b} - \mathbf{c}$

$$\begin{array}{rcl} \mathbf{b} & = & 5,\ 1,\ 4,\ 2,\ 0,\ 2 \\ \mathbf{c} & = & 2,\ 0,\ 5,\ 4,\ 1,\ 2 \\ \hline \mathbf{a} & = & 3,\ 1,\ 5,\ 4,\ 5,\ 0 \end{array}$$

Definition 1 Let $\mathbf{a} = a_1, \ldots, a_k$ be a sequence of elements in a finite abelian group G of order n and M be a multiset of elements in G of cardinality k. An $(M, G, \mathbf{a})$-graph is a directed graph with vertex set G where the in-degree and out-degree of each vertex is the number of repetitions of that element in M. Each edge i is labeled with the numbers from 1 to k, and the **length of** i is the head of i minus its tail and equals a_i.

Every directed graph with labeled edges and vertex set G, where each vertex has the same in and out degrees, is an $(M, G, \mathbf{a})$-graph.

Example 1 Let $G = \mathbb{Z}_6$, and consider the directed graph in Fig. 1. Each connected vertex $g_i \in G$ is an element in the multiset M which has multiplicity equal to its in and out degree. Construct $\mathbf{a}$ by setting a_i to be the length of the edge i. In this example, $M = \{\{0, 1, 2, 2, 4, 5\}\}$ and $\mathbf{a} = 3, 1, 5, 4, 5, 0$.

From an $(M, G, \mathbf{a})$-graph one can construct sequences $\mathbf{b}, \mathbf{c}$ with value multiset M such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Construct b_i to be the vertex g_i that the edge i enters and let c_i be the vertex g_i' that the edge i leaves from. By definition, edge i has length $b_i - c_i = a_i$. Figure 2 shows sequences $\mathbf{b}, \mathbf{c}$ for the $(M, G, \mathbf{a})$-graph in Fig. 1. This equivalence is summarized in the next theorem.

Theorem 2 *Let* $\mathbf{a} = a_1, \ldots, a_k$ *be a sequence of elements in* G *and* M *a multiset with cardinality* k*. Then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *for some sequences* $\mathbf{b}$ *and* $\mathbf{c}$ *with value multiset* M *if and only if there exists an* $(M, G, \mathbf{a})$*-graph.*

Proof Suppose that $\mathbf{a} = \mathbf{b} - \mathbf{c}$ for sequences $\mathbf{b}, \mathbf{c}$ with value multiset M. We first show that we can construct an $(M, G, \mathbf{a})$-graph from $\mathbf{a} = \mathbf{b} - \mathbf{c}$. For each i in the range $1 \le i \le k$, construct an edge that leaves vertex c_i and enters vertex b_i and label it with the number i. Since $a_i = b_i - c_i$, we have that a_i this is the length of edge i. To show that this is an $(M, G, \mathbf{a})$-graph we must check that the in-degree and out-degree of each vertex v is the same and equal to the number of repetitions of v in M. Suppose that $b_i \in M$ with t repetitions in $\mathbf{b}$. This means that there are t edges entering vertex b_i. Since $\mathbf{b}, \mathbf{c}$ have the same value multiset M, there are exactly t repetitions of $c_j = b_i$ in $\mathbf{c}$. This means that there are t edges that leave from $c_j = b_i$. Therefore, vertex b_i has in-degree and out-degree equal to t. Therefore we have constructed an $(M, G, \mathbf{a})$-graph.

Now from an $(M, G, \mathbf{a})$-graph we need to recover sequences $\mathbf{b}, \mathbf{c}$ with value multiset M such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. For each $1 \leq i \leq k$, denote the vertices edge i leaves and enters as b_i and c_i, respectively. Each vertex having the same in and out degrees, means that the sequences $\mathbf{b}$ and $\mathbf{c}$ have the same value multiset. By definition, the length of edge i is given by $a_i = b_i - c_i$. Therefore $\mathbf{a} = \mathbf{b} - \mathbf{c}$. This completes the proof.

This theorem allows us to view what originally was an algebraic problem as a graph theory problem. The (M, G, a)-graphs are used for intuition and for understanding.

2 Difference of Sequences with the Same Value Multiset

In this section we characterize the sequences $\mathbf{a} = a_1, \ldots, a_k$ of elements of a finite abelian group G that can be expressed as the difference of two sequences with the same value multiset. We construct all possible sequences $\mathbf{b}, \mathbf{c}$ with the same value multiset and $\mathbf{a} = \mathbf{b} - \mathbf{c}$ in terms of rearrangements of $\mathbf{a}$ and their corresponding sequences of partial sums. From now on, G represents a finite abelian group of order n.

We start by showing that a condition similar to the one in Hall's Theorem 1 is necessary and sufficient for $\mathbf{a}$ to be written as $\mathbf{a} = \mathbf{b} - \mathbf{c}$ for sequences $\mathbf{b}$ and $\mathbf{c}$ with some value multiset M. To prove this result we first introduce the definition of a sequence of partial sums, which is one of the fundamental building blocks of the results in this paper.

Definition 2 Suppose that $\mathbf{a} = a_1, \ldots, a_k$ is a sequence of elements in G. Define the **sequence of partial sums** $\mathbf{S}(\mathbf{a}) = S(a_1), S(a_2), \ldots, S(a_k)$, where $S(a_j) = \sum_{i=1}^{j} a_i$. Also, for $l \in G$, $l + \mathbf{S}(\mathbf{a})$ denotes the sequence $l + S(a_1), \ldots, l + S(a_k)$, where $+$ is the group operation.

Example 2 Let $\mathbf{a} = 1, 4, 3, 3, 3$ be a sequence of elements in $G = \mathbb{Z}_7$. Then $\mathbf{S}(\mathbf{a}) = 1, 5, 1, 4, 0$ and $3 + \mathbf{S}(\mathbf{a}) = 4, 1, 4, 0, 3$.

Another concept needed is the cyclic right shift of a sequence.

Definition 3 Let $\mathbf{b}, \mathbf{c}$ be two sequences of elements in G of length k. We say that $\mathbf{c}$ is the **cyclic right shift** of $\mathbf{b}$, $\mathbf{c} = r(\mathbf{b})$, if $c_1 = b_k$ and $c_{i+1} = b_i$ for $1 \leq i \leq k - 1$.

Hall's Theorem 1 gives a necessary and sufficient condition for a sequence $\mathbf{a}$ of n elements in G to be written as a difference of two bijections. The next proposition is an analogue to Hall's theorem for sequences of *arbitrary length* k and value multisets.

Proposition 1 *Let* $\mathbf{a} = a_1, \ldots, a_k$ *be a sequence of elements in* G. *Then, there exist sequences* $\mathbf{b}, \mathbf{c}$ *of elements in* G *with the same value multiset such that* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *if and only if* $\sum_{i=1}^{k} a_i = 0$.

Proof Suppose that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Note that $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i - \sum_{i=1}^{k} c_i$. Since $\mathbf{b}, \mathbf{c}$ are reorderings of each other, $\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} c_i$ and $\sum_{i=1}^{k} a_i = 0$.

To prove the other direction let $\mathbf{a} = a_1, \ldots, a_k$ be such that $\sum_{i=1}^{k} a_i = 0$. We construct $\mathbf{b}, \mathbf{c}$ that are reorderings of each other and such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Let $\mathbf{b} = \mathbf{S}(\mathbf{a})$ and $\mathbf{c} = r(\mathbf{b})$ be the cyclic right shift of $\mathbf{b}$. Note that by definition, $c_1 = b_k$ and $c_{j+1} = b_j$ for $1 \leq j \leq k-1$. Therefore $b_1 - c_1 = b_1 - b_k = a_1 - \sum_{i=1}^{k} a_i = a_1 - 0 = a_1$, and $b_j - c_j = b_j - b_{j-1} = \sum_{i=1}^{j} a_i - \sum_{i=1}^{j-1} a_i = a_j$, for $2 \leq j \leq k$. Therefore $\mathbf{a} = \mathbf{b} - \mathbf{c}$ and this completes the proof.

Since $\sum_{i=1}^{k} a_i = 0$ is an important condition in this problem, we give it a name.

Definition 4 Let $\mathbf{a} = a_1, \ldots, a_k$ be a sequence of elements in G. We say $\mathbf{a}$ is a **zero-sum sequence** if $\sum_{i=1}^{k} a_i = 0$.

Zero-sum sequences in finite abelian groups are the object of study in a subfield of additive group theory and combinatorial number theory called zero-sum problems. This study was initiated by Erdös, Ginzburg and Ziv [3], where they proved that, for a cyclic group of order n, $2n - 1$ is the smallest integer such that every sequence of elements has a zero-sum subsequence. Since then zero-sum problems have popped up in various branches of combinatorics, number theory, geometry, graph theory, and cryptographic applications. However, to our knowledge we have not seen results on zero-sum sequences in the context of this paper. For a survey on the algebraic aspects of zero-sum problems on finite abelian groups we refer the reader to [4] and for cryptographic applications to [1, 2].

Given a zero-sum sequence $\mathbf{a}$, we now want to construct all possible sequences $\mathbf{b}$,$\mathbf{c}$ with the same value multiset and whose difference is equal to $\mathbf{a}$. First, let's take a closer look at the implications of the proof of Proposition 1.

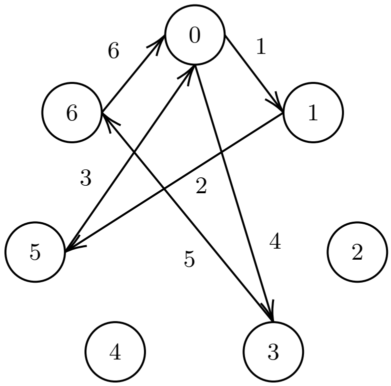
Remark 1 From the proof of Proposition 1 one can see that a zero sum sequence $\mathbf{a}$, can always be written as the difference $\mathbf{a} = \mathbf{b} - \mathbf{c} = \mathbf{S}(\mathbf{a}) - r(\mathbf{S}(\mathbf{a}))$. (We prove a stronger version of this result in Proposition 2.)

Taking $\mathbf{c} = r(\mathbf{b}) = r(\mathbf{S}(\mathbf{a}))$ seems like an arbitrary choice for the construction, but from the point of view of $(M, G, \mathbf{a})$-graphs this choice makes a lot of sense. If we create the $(M, G, \mathbf{a})$-graph associated to our construction, $\mathbf{a} = \mathbf{S}(\mathbf{a}) - r(\mathbf{S}(\mathbf{a}))$, edge 1 will go from vertex c_1 to vertex b_1. Edge 2, following the construction, goes from vertex $c_2 = b_1$ to vertex b_2. Therefore edge 2 starts in the vertex where edge 1 left off. We continue this process where edge $i + 1$ starts in the vertex that edge i ends, until we get to the last edge k. Note that $b_k = c_1$, therefore the last edge will reach the

Fig. 3 $\mathbf{a} = \mathbf{b} - \mathbf{c}$

$$\begin{array}{rcl} \mathbf{b} & = & 1,\ 5,\ 0,\ 3,\ 6,\ 0 \\ \mathbf{c} & = & 0,\ 1,\ 5,\ 0,\ 3,\ 6 \\ \hline \mathbf{a} & = & 1,\ 4,\ 2,\ 3,\ 3,\ 1 \end{array}$$

Fig. 4 $(M, \mathbb{Z}_7, \mathbf{a})$-graph



vertex where the first edge started. This means that our construction defines a circuit inside the $(M, G, \mathbf{a})$-graph. The next example shows the construction in algebraic terms and its corresponding $(M, G, \mathbf{a})$-graph (Fig. 3).

Example 3 Consider the zero-sum sequence $\mathbf{a} = 1, 4, 2, 3, 3, 1$ of elements in $\mathbb{Z}_7$. Let $\mathbf{b} = \mathbf{S}(\mathbf{a}) = 1, 5, 0, 3, 6, 0$ and let $\mathbf{c} = r(\mathbf{b}) = r(\mathbf{S}(\mathbf{a})) = 0, 1, 5, 0, 3, 6$. Figure 4 illustrates the $(M, \mathbb{Z}_7, \mathbf{a})$-graph of $\mathbf{a} = \mathbf{S}(\mathbf{a}) - r(\mathbf{S}(\mathbf{a}))$.

One can decompose a zero-sum sequence in "irreducible components". Suppose that $\mathbf{a}$ is a zero sum sequence. If $j < k$ is the smallest such that $S(a_j) = 0$, we can express $\mathbf{a}$ as $\mathbf{a} = \mathbf{a}_1 * \left(a_{j+1}, \ldots, a_k\right)$ where $\mathbf{a}_1 = a_1, \ldots, a_j$ is a zero-sum sequence and $*$ is a concatenation of sequences. Any zero sum sequence can be expressed as the concatenation of "irreducible" zero-sum sequences.

Definition 5 Let $\mathbf{a} = a_1, a_2, \ldots, a_k$ be a zero-sum sequence of elements in G. We say that $\mathbf{a}$ is **zero-sum irreducible** if $S(a_j) \neq 0$ for $j < k$. The sequence is called **zero-sum reducible** if it is not irreducible.

Example 4 Consider the sequence $\mathbf{a} = 1, 3, 4, 3, 5, 4, 4, 0$ of elements of $\mathbb{Z}_8$. Note that $\mathbf{S}(\mathbf{a}) = 1, 4, 0, 3, 0, 4, 0, 0$. Sequence $\mathbf{a}$ is zero-sum reducible as $S(a_3) = S(a_5) = S(a_7) = S(a_8) = 0$. The sequence can be written as the concatenation zero-sum irreducible sequences: $\mathbf{a} = (1, 3, 4) * (3, 5) * (4, 4) * 0$.

Definition 6 Let $\mathbf{a}$ be a zero-sum sequence of elements in G. We call the concatenation $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$, where $\mathbf{a}_j$ is a zero-sum irreducible sequence, the **irreducible decomposition of a** and the $\mathbf{a}_i$'s its **irreducible components**.

Consider the zero-sum sequence $\mathbf{a}$ and its irreducible decomposition $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$. If $m = 1$, then $\mathbf{a}$ is zero-sum irreducible. On the other extreme, if $m = k$, then $\mathbf{a} = 0, 0, \ldots, 0$.

Starting from the irreducible decomposition of a zero-sum sequence $\mathbf{a}$ and using the partial sums, we give a construction of sequences $\mathbf{b}, \mathbf{c}$ such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. As we mentioned in Remark 1, in the proof of Propoosition 1, we constructed $\mathbf{b} = \mathbf{S}(\mathbf{a})$ and $\mathbf{c} = r(\mathbf{b})$ given a sequence $\mathbf{a}$. If we choose $\mathbf{b} = l + \mathbf{S}(\mathbf{a})$ for some $l \in G$ and $\mathbf{c} = r(\mathbf{b})$ we also get $\mathbf{a} = \mathbf{b} - \mathbf{c}$. We can do the same construction for each irreducible component of $\mathbf{a}$ as the next proposition shows.

Proposition 2 *Let* $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$ *be the irreducible decomposition of a zero-sum sequence* $\mathbf{a}$ *of elements of* G. *For any* $l_j \in G$, *if* $\mathbf{b} = \mathbf{b}_1 * \cdots * \mathbf{b}_m = (l_1 + S(\mathbf{a}_1)) * \cdots * (l_m + S(\mathbf{a}_m))$ *and* $\mathbf{c} = r(\mathbf{b}_1) * \cdots * r(\mathbf{b}_m)$, *then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$.

Proof Let $\mathbf{b} = (l_1 + S(\mathbf{a}_1)) * \cdots * (l_m + S(\mathbf{a}_m))$ for $l_j \in G$, where k_j the length of $\mathbf{b}_j$, and $\mathbf{c}_j = r(\mathbf{b}_j)$. Since $\mathbf{a}_j$ is zero-sum, for $1 \le j \le m$,

$$\mathbf{b}_j - \mathbf{c}_j = l_j + \left(a_{j_1}, a_{j_1} + a_{j_2}, \ldots, \sum_{i=1}^{k_j} a_{j_i}\right) - l_j - \left(\sum_{j=1}^{k_j} a_{j_i}, a_{j_1}, \ldots, \sum_{i=1}^{k_{j-1}} a_{j_i}\right)$$

$$= \left(a_{j_1}, a_{j_2}, \ldots, a_{j_{k_j}}\right) = \mathbf{a}_j.$$

Example 5 Let $\mathbf{a} = 1, 2, 3, 2, 4, 0$ be a zero sum sequence of elements in $\mathbb{Z}_6$ with irreducible decomposition $\mathbf{a}_1 * \mathbf{a}_2 * \mathbf{a}_3 = (1, 2, 3) * (2, 4) * 0$. As seen in Proposition 2, we can choose any l_1, l_2, l_3 for the construction. For example, choose $l_1 = 4$, $l_2 = 3$, $l_3 = 4$ and construct $\mathbf{b} = \mathbf{b}_1 * \mathbf{b}_2 * \mathbf{b}_3 = (4 + \mathbf{S}(\mathbf{a}_1)) * (3 + \mathbf{S}(\mathbf{a}_2)) * (4 + \mathbf{S}(\mathbf{a}_3)) = 5, 1, 4, 5, 3, 4$. Then, $\mathbf{c} = \mathbf{c}_1 * \mathbf{c}_2 * \mathbf{c}_3 = r(\mathbf{b}_1) * r(\mathbf{b}_2) * r(\mathbf{b}_3) = 4, 5, 1, 3, 5, 4$.

This can be seen clearly in the following table, where each box represents a triple $\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j$, and we get $\mathbf{a} = \mathbf{b} - \mathbf{c}$.

$$\begin{array}{r|ccc|cc|c} \mathbf{b}: & 5 & 1 & 4 & 5 & 3 & 4 \\ \mathbf{c}: & 4 & 5 & 1 & 3 & 5 & 4 \\ \hline \mathbf{a}: & 1 & 2 & 3 & 2 & 4 & 0 \end{array}$$

The above proposition and example give a construction of $\mathbf{b}$ and $\mathbf{c}$ in terms of the zero-sum sequence $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$. The components of sequences $\mathbf{b}, \mathbf{c}$ will correspond to the irreducible components of $\mathbf{a}$, this is $\mathbf{a}_i = \mathbf{b}_i - \mathbf{c}_i$. It has the peculiarity that $\mathbf{c}_j = r(\mathbf{b}_j)$ for each irreducible component of $\mathbf{a}$. There are many other possibilities for $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$ not given by the construction in Proposition 2. However, if $\mathbf{a} = \mathbf{b} - \mathbf{c}$, the conditions $\mathbf{b} = l_j + \mathbf{S}(\mathbf{a}_j)$ and $\mathbf{c}_j = r(\mathbf{b}_j)$ are dependent on each other as we show in the next two lemmas.

Lemma 1 *Let* $\mathbf{a} = a_1, \ldots, a_k$ *be a sequence of elements in* G. *Then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$, *for sequences* $\mathbf{b}, \mathbf{c}$, *if and only if* $b_j = l_j + S(a_j)$ *and* $c_j = l_j + S(a_{j-1})$ *for some* $l_j \in G$.

Proof Suppose that $\mathbf{a} = \mathbf{b} - \mathbf{c}$ and let $l_j = b_j - S(a_j) = b_j - \sum_{i=1}^{j} a_i$. Then, $b_j = l_j + \sum_{i=1}^{j} a_i$, and $c_j = b_j - a_j = l_j + \sum_{i=1}^{j-1} a_i = l_j + S(a_{j-1})$. To prove the other direction note that $b_j - c_j = l_j + \sum_{i=1}^{j} a_i - (l_j + \sum_{i=1}^{j-1} a_i) = a_j$.

Note that for this lemma we didn't require $\mathbf{b}, \mathbf{c}$ to have the same value multiset.

Lemma 2 *Let $\mathbf{a} = a_1, \ldots, a_k$ be a zero-sum sequence of elements in G and $\mathbf{b}, \mathbf{c}$ be sequences such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Then $\mathbf{c}$ is the cyclic right shift of $\mathbf{b}$, $\mathbf{c} = r(\mathbf{b})$, if and only if $\mathbf{b} = l + \mathbf{S}(\mathbf{a})$ for some fixed $l \in G$.*

Proof By Lemma 1, $\mathbf{a} = \mathbf{b} - \mathbf{c}$ is equivalent to $b_j = l_j + S(a_j)$ and $c_j = l_j + S(a_{j-1})$ for some $l_j \in G$, $1 \le j \le k$. To prove the forward direction suppose that $\mathbf{c} = r(\mathbf{b})$. This means that $b_j = c_{j+1}$ for $1 \le j \le k-1$ and $b_k = c_1$. Then, $l_{j+1} + S(a_j) = c_{j+1} = b_j = l_j + S(a_j)$, and $l_{j+1} = l_j$ for $1 \le j \le k-1$. Therefore $l_1 = l_2 = \cdots = l_k = l$ for some $l \in G$, and we can write $\mathbf{b} = l + \mathbf{S}(\mathbf{a})$.

To show the other direction, let $\mathbf{b} = l + \mathbf{S}(\mathbf{a})$. Then $c_j = l_j + S(a_{j-1}) = b_j - a_j = l + S(a_j) - a_j = l + S(a_{j-1}) = b_{j-1}$, for $2 \le j \le k$. Also, $c_1 = l_1 = b_1 - a_1 = l = l - S(a_k) = b_k$. Hence $\mathbf{c} = r(\mathbf{b})$, and this completes the proof.

We apply these two lemmas to the irreducible components of a zero-sum sequence.

Proposition 3 *Let $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$ be the irreducible decomposition of a zero-sum sequence $\mathbf{a}$ of elements in G, and $\mathbf{b}, \mathbf{c}$ with the same value multiset, where $\mathbf{a}_j = \mathbf{b}_j - \mathbf{c}_j$. Then, for each $1 \le j \le m$, $\mathbf{c}_j = r(\mathbf{b}_j)$, if and only if $\mathbf{b}_j = l_j + \mathbf{S}(\mathbf{a}_j)$ for some $l_j \in G$.*

Proof Let $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$ be the zero-sum irreducible decomposition of $\mathbf{a}$ and $\mathbf{b} = \mathbf{b}_1 * \cdots * \mathbf{b}_m$ be such that $\mathbf{a}_j = \mathbf{b}_j - \mathbf{c}_j$ for $\mathbf{c}_j = r(\mathbf{b}_j)$. By Lemma 2, $\mathbf{b}_j = l_j + S(\mathbf{a}_j)$ for some $l_j \in G$.

To prove the other direction let $\mathbf{b}_j = l_j + \mathbf{S}(\mathbf{a}_j)$ be a sequence of length k_j, for some $l_j \in G$. Then $\mathbf{c}_j = l_j + \mathbf{S}(\mathbf{a}_j) - \mathbf{a}_j = l_j + \left[a_1, a_1 + a_2, \ldots, \sum_{i=1}^{k_j} a_i\right] - \left[a_1, a_2, \ldots, a_{k_j}\right] = l_j + \left[0, a_1, \ldots, \sum_{i=1}^{k_{j-1}} a_i\right] = b_{k_j}, b_1, \ldots, b_{k_{j-1}}$. This means that $\mathbf{c}_j = r(\mathbf{b}_j)$.

As it was mentioned before, there are many other possibilities for $\mathbf{b}$ and $\mathbf{c}$ for which $\mathbf{a} = \mathbf{b} - \mathbf{c}$, where $\mathbf{c}_i \ne r(\mathbf{b}_i)$. Figure 5 shows sequences $\mathbf{a} = \mathbf{b} - \mathbf{c}$ where $\mathbf{c}_i \ne r(\mathbf{b}_i)$ and Fig. 6 shows the sequences $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$ where $\sigma(\mathbf{c})_i = r(\sigma(\mathbf{b})_i)$. However, we will see that even those cases reduce to our construction when we consider permutations of the sequences. We denote $\sigma(\mathbf{a}) = a_{\sigma(1)}, \ldots, a_{\sigma(k)}$ and we write its irreducible decomposition as $\sigma(\mathbf{a}) = \sigma(\mathbf{a})_1 * \sigma(\mathbf{a})_2 * \cdots * \sigma(\mathbf{a})_m$. One of the main results of this paper (Proposition 4) shows that one can always find a σ such that $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$ and the components of $\sigma(\mathbf{c})$ are right cyclic shifts of the components of $\sigma(\mathbf{b})$, that is, $\sigma(\mathbf{c})_i = r(\sigma(\mathbf{b})_i)$. This means that every possible $\mathbf{b}, \mathbf{c}$ such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$ can be recovered from our construction when we consider permutations. The proof of Proposition 4 gives an algorithm to obtain the permutation σ.

Fig. 5 $\mathbf{c}_i \neq r(\mathbf{b}_i)$

$$\begin{array}{l}\mathbf{b} : 1\ 6\ 7\ 2\ 0\ 2\ 4\ \ 2\\ \mathbf{c} : 0\ 2\ 2\ 7\ 4\ 2\ 1\ \ 6\\ \hline \mathbf{a} : 1\ 4\ 5\ 3\ 4\ 0\ 3\ \ 4.\end{array}$$

Fig. 6 $\mathbf{c}_i = r(\mathbf{b}_i)$

$$\begin{array}{l|l|l|l}\mathbf{b} : 1\ 4\ 0 & 2\ 7 & 2\ 6 & 2\\ \mathbf{c} : 0\ 1\ 4 & 7\ 2 & 6\ 2 & 2\\ \hline \mathbf{a} : 1\ 3\ 4 & 3\ 5 & 4\ 4 & 0.\end{array}$$

Proposition 4 *Let* $\mathbf{a} = a_1, \ldots, a_k$ *be a zero-sum sequence of elements in a group* G. *If* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *where* $\mathbf{b}, \mathbf{c}$ *have the same value multiset, then there exists a permutation* σ *of the elements of* $\mathbf{a}$ *such that* $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$, *where* $\sigma(\mathbf{a})$ *has irreducible decomposition* $\sigma(\mathbf{a}) = \sigma(\mathbf{a})_1 * \cdots * \sigma(\mathbf{a})_m$, $\sigma(\mathbf{b}) = \sigma(\mathbf{b})_1 * \cdots * \sigma(\mathbf{b})_m$, $\sigma(\mathbf{c}) = \sigma(\mathbf{c})_1 * \cdots * \sigma(\mathbf{c})_m$, *where* $\sigma(\mathbf{b})_i = l_i + \mathbf{S}(\sigma(\mathbf{a})_i)$ *and* $\sigma(\mathbf{c})_i = r(\sigma(\mathbf{b})_i)$ *for some* $l_1, \ldots, l_m \in G$.

Before we prove the result, we illustrate how to obtain the permutation σ in Fig. 6, from $\mathbf{a} = \mathbf{b} - \mathbf{c}$ in Fig. 5.

Let $\mathbf{a} = 1, 4, 5, 3, 4, 0, 3, 4$, $\mathbf{b} = 1, 6, 7, 2, 0, 2, 4, 2$ and $\mathbf{c} = 0, 2, 2, 7, 4, 2, 1, 6$ be sequences with elements in $\mathbb{Z}_8$. Note that $\mathbf{c}$ is a reordering of $\mathbf{b}$. We start by writing $\mathbf{a} = \mathbf{b} - \mathbf{c}$ as

$$\begin{array}{l}\mathbf{b} : 1\ 6\ 7\ 2\ 0\ 2\ 4\ \ 2\\ \mathbf{c} : 0\ 2\ 2\ 7\ 4\ 2\ 1\ \ 6\\ \hline \mathbf{a} : 1\ 4\ 5\ 3\ 4\ 0\ 3\ \ 4.\end{array}$$

Recall that, by Proposition 3, $\sigma(\mathbf{b})_j = l_j + S(\sigma(\mathbf{a})_j)$ if and only if $\sigma(\mathbf{c})_j$ is the cyclic right shift of $\sigma(\mathbf{b})_j$. We want to construct the zero-sum irreducible components $\sigma(\mathbf{a})_j$ by switching columns to obtain that $\sigma(\mathbf{c})_j$ is the cyclic right shift of $\sigma(\mathbf{b})_j$. First, move all columns with $a_i = 0$ to the end of the sequences. In this example, swap columns 8 and 6 to obtain sequences $\mathbf{a}^{(1)}, \mathbf{b}^{(1)}, \mathbf{c}^{(1)}$:

$$\begin{array}{l}\mathbf{b}^{(1)} : 1\ 6\ 7\ 2\ 0\ 2\ 4\ \ 2\\ \mathbf{c}^{(1)} : 0\ 2\ 2\ 7\ 4\ 6\ 1\ \ 2\\ \hline \mathbf{a}^{(1)} : 1\ 4\ 5\ 3\ 4\ 4\ 3\ \ 0.\end{array}$$

Next, find a column i where $c_i^{(1)} = b_1^{(1)}$ and swap column i and column 2 to obtain sequences $\mathbf{a}^{(2)}, \mathbf{b}^{(2)}, \mathbf{c}^{(2)}$:

$$\begin{array}{l}\mathbf{b}^{(2)} : 1\ 4\ 7\ 2\ 0\ 2\ 6\ \ 2\\ \mathbf{c}^{(2)} : 0\ 1\ 2\ 7\ 4\ 6\ 2\ \ 2\\ \hline \mathbf{a}^{(2)} : 1\ 3\ 5\ 3\ 4\ 4\ 4\ \ 0.\end{array}$$

Now, find a column i such that $c_i^{(2)} = b_2^{(2)}$ and swap columns i and column 3. Continue until $b_i^{(e)} = c_1^{(e)}$ as in (1). In this example, $\mathbf{b}^{(3)}{}_3 = \mathbf{c}^{(3)}{}_1$. This gives the first zero-sum irreducible component of $\mathbf{a}^{(3)}$, $\mathbf{a}^{(3)}{}_1$, where $\mathbf{c}^{(3)}{}_1$ is the cyclic right shift of $\mathbf{b}^{(3)}{}_1$.

$$\begin{array}{lccc|ccccc}
\mathbf{b}^{(3)}: & 1 & 4 & 0 & 2 & 7 & 2 & 6 & 2 \\
\mathbf{c}^{(3)}: & 0 & 1 & 4 & 7 & 2 & 6 & 2 & 2 \\ \hline
\mathbf{a}^{(3)}: & 1 & 3 & 4 & 3 & 5 & 4 & 4 & 0 \\
 & & \mathbf{a}^{(3)}{}_1 & & & & & &
\end{array} \tag{1}$$

Now repeat the process with the next entries of the sequences to construct the next zero-sum irreducible component of the permutation of $\mathbf{a}$ or the last non-zero entry of $\mathbf{a}$ has been reached.

$$\begin{array}{lccc|cc|cc|c}
\mathbf{b}^{(3)}: & 1 & 4 & 0 & 2 & 7 & 2 & 6 & 2 \\
\mathbf{c}^{(3)}: & 0 & 1 & 4 & 7 & 2 & 6 & 2 & 2 \\ \hline
\mathbf{a}^{(3)}: & 1 & 3 & 4 & 3 & 5 & 4 & 4 & 0 \\
 & & \mathbf{a}^{(3)}{}_1 & & & \mathbf{a}^{(3)}{}_2 & & \mathbf{a}^{(3)}{}_3 & \mathbf{a}^{(3)}{}_4
\end{array} \tag{2}$$

Note that each $\mathbf{c}^{(3)}{}_j$ is the cyclic right shift of $\mathbf{b}^{(3)}{}_j$ and the sequences $\mathbf{a}^{(3)}{}_j$ are zero-sum irreducible. We now give the formal proof of Proposition 4.

Proof (*of Proposition* 4) Let $\mathbf{a} = \mathbf{b} - \mathbf{c}$, where $\mathbf{b}, \mathbf{c}$ have the same value multiset. We will construct a permutation σ such that $\sigma(\mathbf{a}), \sigma(\mathbf{b}), \sigma(\mathbf{c})$ satisfy the following conditions: $\sigma(\mathbf{a})$ has irreducible decomposition $\sigma(\mathbf{a}) = \sigma(\mathbf{a})_1 * \cdots * \sigma(\mathbf{a})_m$ and $\sigma(\mathbf{b}), \sigma(\mathbf{c})$ have decompositions $\sigma(\mathbf{b}) = \sigma(\mathbf{b})_1 * \cdots * \sigma(\mathbf{b})_m,\ \sigma(\mathbf{c}) = \sigma(\mathbf{c})_1 * \cdots * \sigma(\mathbf{c})_m$ with $\sigma(\mathbf{b})_i = l_i + \mathbf{S}(\sigma(\mathbf{a})_i)$ and $\sigma(\mathbf{c})_i = r(\sigma(\mathbf{b})_i)$.

First, we move the entries in position i for which $a_i = 0$ to the end of sequences $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Let h be the last nonzero entry of $\mathbf{a}$. Swap entries a_i, b_i, c_i with entries a_h, b_h, c_h. Continue this process until all the zeroes of $\mathbf{a}$ are moved to the end and call the new sequences $\mathbf{a}^{(1)}, \mathbf{b}^{(1)}, \mathbf{c}^{(1)}$.

Define the set $I_1 = \{1, \dots, h\}$ to store the indices of the entries that haven't been properly rearranged. Since $c_i = b_i$ for $i > h$, we have that $b_1^{(1)}, \dots, b_h^{(1)}$ and $c_1^{(1)}, \dots, c_h^{(1)}$ are reorderings of each other. Therefore, there exists an element $i \in I_1$ such that $b_1^{(1)} = c_i'$. Swap $b_2^{(1)}, c_2^{(1)}, a_2^{(1)}$ with the elements $b_i^{(1)}, c_i^{(1)}, a_i^{(1)}$. Denote the new sequences as $\mathbf{a}^{(2)}, \mathbf{b}^{(2)}, \mathbf{c}^{(2)}$ and let $I_2 = \{3, \dots, h\}$. Again, since $b_3^{(2)}, \dots, b_h^{(2)}$ and $c_3^{(2)}, \dots, c_h^{(2)}$ are reorderings of each other, there is an index $i \in I_2$ such that $b_2^{(2)} = c_i^{(2)}$. Swap $c_3^{(2)}, b_3^{(2)}$ and $a_3^{(2)}$ with entries $c_i^{(3)}, b_i^{(3)}$ and $a_i^{(3)}$, respectively to get sequences $\mathbf{a}^{(3)}, \mathbf{b}^{(3)}, \mathbf{c}^{(3)}$ and let $I_3 = \{4, \dots, h\}$. Continue until $b_s^{(s)} = c_1^{(s)}$. Let $\mathbf{a}^{(s)}{}_1 = a_1^{(s)}, \dots, a_s^{(s)}$, $\mathbf{b}^{(s)}{}_1 = b_1^{(s)}, \dots, b_s^{(s)}$ and $\mathbf{c}^{(s)}{}_1 = c_1^{(s)}, \dots, c_s^{(s)}$.

We rewrite $\mathbf{a}^{(s)} = \mathbf{a}^{(s)}{}_1 * a_{s+1}^{(s)}, \dots, a_h^{(s)} * \cdots * a_k^{(s)}$, $\mathbf{b}^{(s)} = \mathbf{b}^{(s)}{}_1 * b_{s+1}^{(s)}, \dots, b_h^{(s)} * \cdots * b_k^{(s)}$ and $\mathbf{c}^{(s)} = \mathbf{c}^{(s)}{}_1 * c_{s+1}^{(s)}, \dots, c_h^{(s)} * \cdots * c_k^{(s)}$. Observe that $b_s^{(s)} = c_1^{(s)}$ and $c_{i+1}^{(s)} = b_i^{(s)}$ for $1 \le i \le s - 1$. This implies that $\mathbf{c}^{(s)}{}_1$ is the cyclic right shift of $\mathbf{b}^{(s)}{}_1$. Since $\mathbf{b}^{(s)}{}_1$ and $\mathbf{c}^{(s)}{}_1$ are reorderings of each other, this means that $\sum_{i=1}^{s} a_i^{(s)} = \sum_{i=1}^{s} (b_i^{(s)} - c_i^{(s)}) = 0$. Hence $\mathbf{a}^{(s)}{}_1$ is zero-sum, and, since s is the first element such that $b_s^{(s)} = c_1^{(s)}$, then $\mathbf{a}^{(s)}{}_1$ is irreducible.

Let $I_s = \{s+1, \ldots, h\}$ and continue this construction for $a_i^{(s)}, s < i \leq h$ to obtain $\sigma(\mathbf{a}), \sigma(\mathbf{b}), \sigma(\mathbf{c})$ with the desired properties.

Proposition 4 combined with Proposition 3 gives a description of all possible $\mathbf{b}$, $\mathbf{c}$ which sequences can be used to express $\mathbf{a} = \mathbf{b} - \mathbf{c}$ in terms of $\mathbf{a}$.

Theorem 3 *Let* $\mathbf{a} = a_1, \ldots, a_k$ *be a sequence of elements in* G. *Then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *for sequences* $\mathbf{b}, \mathbf{c}$ *with the same value multiset if and only if* $\mathbf{a}$ *is zero-sum,* $\mathbf{b} = \sigma^{-1}([l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)] * [l_2 + \mathbf{S}(\sigma((\mathbf{a})_2)] * \cdots * [l_m + \mathbf{S}(\sigma(\mathbf{a})_m)])$ *and* $\mathbf{c} = \sigma^{-1}[r(l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)) * \cdots * r(l_m + \mathbf{S}(\sigma(\mathbf{a})_m))]$, *where* $\sigma(\mathbf{a})_1 * \cdots * \sigma(\mathbf{a})_m$ *is the irreducible decomposition of* $\sigma(\mathbf{a})$ *for some permutation* σ *of the indices* $\{1, \ldots, k\}$, $l_1, \ldots, l_m \in G$ *and* $1 \leq m \leq k$.

Proof Suppose that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Proposition 4 implies that there exists a permutation σ such that $\sigma(\mathbf{a})$ has irreducible decomposition $\sigma(\mathbf{a}) = \sigma(\mathbf{a})_1 * \cdots * \sigma(\mathbf{a})_m$, $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$, where $\sigma(\mathbf{b}) = \sigma(\mathbf{b})_1 * \cdots * \sigma(\mathbf{b})_m = [l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)] * \cdots * [l_m + \mathbf{S}(\sigma(\mathbf{a})_m)]$ for $l_1, \ldots, l_m \in G$, and $\sigma(\mathbf{c})_j = r(\sigma(\mathbf{b})_j)$. This implies that $\mathbf{b} = \sigma^{-1}([l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)] * \cdots * [l_m + \mathbf{S}(\sigma(\mathbf{a})_m)])$.

To prove the other direction suppose that $\mathbf{a}$ is a zero-sum sequence. Then any permutation $\sigma(\mathbf{a})$ of $\mathbf{a}$ is also zero-sum. Let $\sigma(\mathbf{a}) = \sigma(\mathbf{a})_1 * \cdots * \sigma(\mathbf{a})_m$ be the irreducible decomposition of $\sigma(\mathbf{a})$, and let

$$\mathbf{b} = \sigma^{-1}([l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)] * \cdots * [l_m + \mathbf{S}(\sigma(\mathbf{a})_m)])$$

for $l_1, \ldots, l_m \in G$. Then, $\sigma(\mathbf{b}) = [l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)] * \cdots * [l_m + \mathbf{S}(\sigma(\mathbf{a})_m)]$. Let $\sigma(\mathbf{c})_j = r(\sigma(\mathbf{b})_j)$ and $\sigma(\mathbf{c}) = \sigma(\mathbf{c})_1 * \cdots * \sigma(\mathbf{c})_m$. Since $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$ and we conclude that $\mathbf{a} = \mathbf{b} - \mathbf{c}$.

Given a zero sum sequence $\mathbf{a}$ of elements in G, this theorem gives a complete classification of all possible sequences $\mathbf{b}$, $\mathbf{c}$, with the same value multiset and whose difference is $\mathbf{a}$, in terms of reorderings of $\mathbf{a}$ and its sequence of partial sums. There are several consequences of this theorem. First, it answers the problem that motivated this research: given a multiset M and a function $a : G \longrightarrow G$ determine when a can be written as $a = b - c$ where $b, c : G \longrightarrow G$ have value multiset M. This is stated in the following corollary.

Corollary 1 *Let* G *be a group,* $a : G \longrightarrow G$ *a function,* $\mathbf{a} = a_1, \ldots, a_n$ *the sequence where* $a_i = a(i)$, *and* M *a multiset of elements in* G. *Then* $a = b - c$ *for functions* $b, c : G \longrightarrow G$ *such that* $\{\{b(i)\}\} = \{\{c(i)\}\} = M$ *if and only if* $\mathbf{a}$ *is a zero-sum sequence and*

$$M = \{\{l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)\}\} \cup \cdots \cup \{\{l_m + \mathbf{S}(\sigma(\mathbf{a})_m)\}\}. \tag{3}$$

for some permutation σ *of the indices* $\{1, \ldots, n\}$, $l_1, \ldots, l_m \in G$, $1 \leq m \leq n$, *where* $\sigma(\mathbf{a}) = \sigma(\mathbf{a})_1 * \cdots * \sigma(\mathbf{a})_m$ *is the irreducible decomposition of* $\sigma(\mathbf{a})$.

Remark 2 Corollary 1 can also be stated in terms of $(M, G, \mathbf{a})$-graphs. Given a multiset M and a sequence $\mathbf{a}$ of elements in G, we can construct an $(M, G, \mathbf{a})$-graph if and only if M is as in (3).

The second consequence of Theorem 3 is a stronger version of Hall's theorem in which $\mathbf{b}, \mathbf{c}$ are described in terms of $\mathbf{a}$.

Theorem 4 *Let* $\mathbf{a} = a_1, \ldots, a_n$ *be a sequence of elements in* G*. Then,* $\mathbf{a} = \mathbf{b} - \mathbf{c}$*, where* $\mathbf{b}, \mathbf{c}$ *have value multiset* $M = G$ *if and only if* $\mathbf{a}$ *is zero-sum,* $\mathbf{b} = \sigma^{-1}([l_1 + S(\sigma(\mathbf{a})_1)] * \cdots * [l_m + S(\sigma(\mathbf{a})_m)])$ *and* $\mathbf{c} = \sigma^{-1}[r(l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)) * \cdots * r(l_m + \mathbf{S}(\sigma(\mathbf{a})_m))]$ *for some permutation of the indices* σ *and* $l_1, \ldots, l_m \in G$.

Proof Suppose $\mathbf{a} = \mathbf{b} - \mathbf{c}$, where $\mathbf{b}, \mathbf{c}$ have value multiset $M = G$. By Theorem 3, $\mathbf{a}$ is a zero-sum sequence, $\mathbf{b} = \sigma^{-1}([l_1 + S(\sigma(\mathbf{a})_1)] * \cdots * [l_m + S(\sigma(\mathbf{a})_m)])$ and $\mathbf{c} = \sigma^{-1}[r(l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)) * \cdots * r(l_m + \mathbf{S}(\sigma(\mathbf{a})_m))]$.

To show the other direction, suppose $\mathbf{a}$ is a zero-sum sequence. By Hall's Theorem, there exist $\mathbf{b}, \mathbf{c}$ with value multiset $M = G$ such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Note that, by Theorem 3, $\mathbf{b} = \sigma^{-1}([l_1 + S(\sigma(\mathbf{a})_1)] * \cdots * [l_m + S(\sigma(\mathbf{a})_m)])$ and $\mathbf{c} = \sigma^{-1}[r(l_1 + \mathbf{S}(\sigma(\mathbf{a})_1)) * \cdots * r(l_m + \mathbf{S}(\sigma(\mathbf{a})_m))]$ for some σ and elements $l_1, \ldots, l_m \in G$, and therefore $\mathbf{b}, \mathbf{c}$ have the desired form.

For the above result to be effective, one needs to be able to find the right values for σ and the l_i's. However, finding the σ and the corresponding l's for a given M is difficult. Understanding how to obtain these values in terms of $\mathbf{a}$ is useful for the many applications of Hall's Theorem.

3 Further Directions, Applications and Open Questions

In various applications a sequence $\mathbf{a}$ and a value multiset M is specified beforehand and one needs to construct sequences $\mathbf{b}, \mathbf{c}$ with value multiset M, such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$. One example of such application is discussed in [7] where the authors use the multiset $M = G$ and the functions $\mathbf{a} = \mathbf{b} - \mathbf{c}$ to represent **transversals in Latin squares** obtained from the Cayley table of the group G. The authors also mention applications to **complete mappings** and **orthomorphisms of** G when $\mathbf{a}, \mathbf{b}, \mathbf{c}$ represent bijections of G. For a bijection given by $\mathbf{a}$, concrete constructions of $\mathbf{b}, \mathbf{c}$ with $M = G$ would produce complete mappings, othomorphisms and transversals. The following example illustrates of how we can apply Theorem 3 to these cases.

Example 6 Let $\mathbf{a} = 0, 1, 2, 3, 4, 5, 6$ be a sequence of elements in $\mathbb{Z}_7$. Consider the reordering $\sigma(\mathbf{a}) = 0, 1, 6, 5, 2, 3, 4$ which has irreducible decomposition $\sigma(\mathbf{a}) = (0) * (1, 6) * (5, 2) * (3, 4)$. Applying Theorem 3, we can construct many different $\mathbf{b}, \mathbf{c}$ depending on the elements l_1, l_2, l_3, l_4 that we choose. Suppose that $l_1 = 0,\ l_2 = 4,\ l_3, = 3,\ l_4 = 6$. Then, $\sigma(\mathbf{b}) = (0) * (5, 4) * (1, 3) * (2, 6)$, $\sigma(\mathbf{c}) = (0) * (4, 5) * (3, 1) * (6, 2)$, and $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$. That is, $0, 1, 6, 5, 2, 3, 4 =$

$0, 5, 4, 1, 3, 2, 6 - 0, 4, 5, 3, 1, 6, 2$. By applying the inverse permutation, we obtain $0, 1, 2, 3, 4, 5, 6 = \mathbf{a} = \mathbf{b} - \mathbf{c} = 0, 5, 3, 2, 6, 1, 4 - 0, 4, 1, 6, 2, 3, 5$.

As it is mentioned in [7], $\mathbf{b}$ is an orthomorphism of $\mathbb{Z}_7$ and $\mathbf{c}$ is a complete mapping of $\mathbb{Z}_7$. Sequences $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ also describe a transversal in the Latin square that is the difference table of $\mathbb{Z}_7$, where $\mathbf{b}$ gives the column positions, $\mathbf{c}$ the row positions and $\mathbf{a}$ the respective entries of the transversal.

As we just saw, finding the correct σ and l_i's gives concrete constructions of sequences $\mathbf{b}$ and $\mathbf{c}$ and these can be used to construct orthomorphisms, complete mappings and transversals. A natural question to ask is, for a zero-sum sequence $\mathbf{a}$, which are the permutations σ such that there exists $l_1, \ldots, l_m$ that allows $\{\{\mathbf{b}\}\} = \{\{\mathbf{c}\}\} = M$? When M is general multiset, this would give connections to questions related to **bus scheduling** where there are multiple bus bays. Are there other applications for general multisets M?

There are many approaches to the first question. One is to study it by using $(M, G, \mathbf{a})$-graphs. Another possibility is a more combinatorial approach. Since permutations of a zero-sum sequence $\mathbf{a}$ are concatenations of irreducible components, the question translates to studying which irreducible components can be concatenated such that there exists l_i's that allow $\mathbf{b}$ and $\mathbf{c}$ to have the desired multiset M. This can be done by studying the set of irreducible zero-sum sequences.

We can prove that the number of irreducible zero-sum sequences of length k with entries in a group of order n is $(n-1)^{k-1}$. For the case when the sequences are of length $k = n - 2$, we get that the number of sequences is $(n-1)^{n-3} = (n'+1)^{n'-1}$ where $n' = n - 2$. This means that the set of irreducible zero-sum sequences of length n' is in bijection with the well known set of **parking functions** of length n'. This suggests another problem: to find a natural bijection between these two sets.

As we can see there are many questions and different approaches which are related to differences of functions with the same value multiset that remain to be studied. This paper provides a stronger version of Hall's theorem, with a description of $\mathbf{b}$ and $\mathbf{c}$ in terms of $\mathbf{a}$, that gives new insight to the problem of difference of functions.

Acknowledgements The authors thank Gary Mullen for calling our attention to the paper that motivated this work [7] and for suggesting possible generalizations of some results on that paper. This work was supported in part by the Puerto Rico Louis Stokes Alliance for Minority Participation (PR-LSAMP) Program at the University of Puerto Rico, NSF grant #1400868.

References

1. Aumasson, J.-P., Meier, W.: Zero-sum distinguishers for reduced keccak-f and for the core functions of luffa and hamsi. In: rump session of Cryptographic Hardware and Embedded Systems-CHES, 2009, vol. 67 (2009)
2. Boura, C., Canteaut, A.: Zero-sum distinguishers for iterated permutations and application to keccak-f and hamsi-256. In: Selected Areas in Cryptography: 17th International Workshop, SAC 2010, Waterloo, Ontario, Canada, August 12–13, 2010, Revised Selected Papers 17, pp. 1–17. Springer (2011)

3. Erdös, P., Ginzburg, A., Ziv, A.: Theorem in the additive number theory. Bull. Res. Council Israel F **10**, 41–43 (1961)
4. Gao, W., Geroldinger, A.: Zero-sum problems in finite abelian groups: a survey. Expositiones Mathematicae **24**(4), 337–369 (2006)
5. Graham, R.: Juggling mathematics and magic. In: Notices of the International Congress of Chinese Mathematicians, vol. 1, pp. 7–9. International Press of Boston (2013)
6. Hall, M.: A combinatorial problem on abelian groups. Proc. Amer. Math. Soc. **3**(4), 584–587 (1952)
7. Ullman, D.H., Velleman, D.J.: Differences of bijections. Amer. Math. Mon. **126**(3), 199–216 (2019)
8. Bin, Yu., Ren, S., Enze, W., Zhou, Y., Wang, Y.: Optimization of urban bus operation frequency under common route condition with rail transit. Front. Eng. Manag. **4**(4), 451–462 (2017)

Mixed-Level Covering, Locating, and Detecting Arrays via Cyclotomy

Yasmeen Akhtar, Charles J. Colbourn, and Violet R. Syrotiuk

Abstract For a finite field of order q, and v a divisor of $q-1$, additive translates of a cyclotomic vector yield a $q \times q$ *cyclotomic array* on v symbols. For every positive integer t, for certain q sufficiently large with respect to v, such a cyclotomic array is always a covering array of strength t. Asymptotically such arrays have far too many rows to be competitive with certain other covering array constructions. Nevertheless, for small values of t, this cyclotomic method produces smallest known covering arrays for numerous parameters suitable for practical application. This paper extends these ideas and shows that cyclotomy can produce covering arrays of higher index, and locating and detecting arrays with large separation. Computational results also demonstrate that certain cyclotomic arrays for the same order q but different values of v can be juxtaposed to produce mixed-level covering, locating, and detecting arrays.

Keywords Covering array · Detecting array · Locating array · Cyclotomy · Cyclotomic vector

1 Introduction

Testing is an important but expensive part of the development of any component-based system. In general, the systems are composed of many factors, each of which has many possible levels (input values). A system's performance can be affected not

Y. Akhtar (✉)
Department of Mathematics, Birla Institute of Technology and Science - Pilani K K Birla Goa Campus, Sancoale 403726, India
e-mail: yasmeena@goa.bits-pilani.ac.in

C. J. Colbourn · V. R. Syrotiuk
School of Computing and Augmented Intelligence, Arizona State University, Tempe, AZ 85281, USA
e-mail: colbourn@asu.edu

V. R. Syrotiuk
e-mail: syrotiuk@asu.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*, Springer Proceedings in Mathematics & Statistics 448, https://doi.org/10.1007/978-3-031-52969-6_4

just by individual factor-level choices, but also by the choices of level combinations, i.e., interactions among the factors. Failures or faults caused by interactions among level combinations pose a major challenge in testing. Combinatorial testing aims to verify that no level combination causes a fault, and hence that the system works correctly despite possible interactions. However, the number of level combinations grows exponentially in the number of factors; to treat all interactions, the number of required tests grows exponentially as well. The key insight underlying t-way combinatorial testing is that most failures are triggered by level combinations of relatively few factors [14]. If a particular level combination of factors causes a failure, the outcomes from the tests containing it report failure. When responses are deterministic, Tatsumi [27] observed that interactions need not each be tested the same number of times; instead, each potential interaction needs to be tested at least once. This dramatically reduces the required number of tests while still being very effective in detecting the presence of errors [1, 15].

Covering arrays are extensively used in combinatorial testing of software [14]. A covering array (CA) of strength t (of index one) is an array with the property that for every choice of t columns, all possible ordered t-tuples of levels appear within its rows. When employed in testing, columns of a CA represent factors, and entries within a column specify levels for that factor. Rows represent tests to be performed. A covering array reveals the presence of a fault caused by a level combination by ensuring that the combination arises in at least one test. However, test results for a CA may not suffice to determine the number or location of the faulty level combinations. To address this, Colbourn and McClary [6] introduced locating and detecting arrays to identify level combinations that cause failures. Locating arrays (LAs) and detecting arrays (DAs) determine the location of faulty level combinations by observing them in a unique set of tests, given a prespecified upper bound on the number of faulty level combinations.

Existing constructions for LAs and DAs employ CAs of higher strength [6] and recursive constructions [4, 24, 25]. For ‘small’ numbers of factors, algorithmic methods have been investigated [11–13, 17, 21, 23]. These typically yield arrays with fewer rows than known direct and recursive constructions. However, sophisticated algorithms can be computationally expensive when the number of factors is large. Hence it is important to devise direct and recursive constructions that are competitive. In [3], a construction for covering arrays is given via developing and extending the cyclotomic array. The construction ensures that for a prime power $q \equiv 1 \pmod{s}$, the $q \times q$ cyclotomic array is a covering array of strength t in which each column has s levels if $q > t^2 s^{2t}$. In the binary case, it is a covering array if $q > t^2 2^{2t-2}$.

This paper extends the cyclotomy construction from [3] to yield mixed-level covering, locating, and detecting arrays. In Sect. 2, we provide precise definitions and some background on the construction of LAs and DAs. In Sect. 3, for a prime power q, we use cyclotomic arrays to construct CAs, LAs, and DAs with q runs and numbers of levels dividing $q - 1$. We report on certain arrays and their properties in Sect. 4.

2 Preliminaries

We first provide precise definitions for designs (arrays) used in combinatorial testing. Let $n, k, v_1, \dots, v_k$ be positive integers. A *mixed* or *mixed-level* design of n runs and k factors with numbers of levels $v_1, \dots, v_k$, denoted by $(n, v_1 \cdots v_k)$-*design*, is an $n \times k$ array A in which each row represents a run, each column represents a factor and the jth column takes values from an *admissible* set of v_j symbols, typically $\mathbb{Z}_{v_j}$. When a design has k_j factors with level v_j, we use the exponential notation $(n, v_1^{k_1} \cdots v_m^{k_m})$-*design*. For a $(n, v_1 \cdots v_k)$-design A, let $\mathbf{c}_1, \dots, \mathbf{c}_k$ be its k columns. For a positive integer $t \le k$, a t-*tuple* in A (or t-*way interaction*, or *level combination* of *strength* t), is a selection of t factors and an admissible level for each, denoted by $T = (\mathbf{c}_{j_1} = a_{j_1}, \dots, \mathbf{c}_{j_t} = a_{j_t})$ where $1 \le j_1 < \cdots < j_t \le k$ and $a_{j_i} \in \mathbb{Z}_{v_{j_i}}$. Let $\mathcal{I}_t$ be the set of all t-tuples in A and $\mathcal{T}_{\mathbf{c}_{j_1} \dots \mathbf{c}_{j_t}}$ be the set of all t-tuples for factors $\mathbf{c}_{j_1}, \dots, \mathbf{c}_{j_t}$. Then $|\mathcal{T}_{\mathbf{c}_{j_1}, \dots, \mathbf{c}_{j_t}}| = v_{j_1} \times \cdots \times v_{j_t}$. Let $\rho_A(T)$ be the set of rows in A containing a t-tuple $T \in \mathcal{I}_t$. For a set $\mathcal{T} \subseteq \mathcal{I}_t$, define $\rho_A(\mathcal{T}) = \cup_{T \in \mathcal{T}} \rho_A(T)$. Then $\sum_{T \in \mathcal{T}_{\mathbf{c}_{j_1} \cdots \mathbf{c}_{j_t}}} |\rho_A(T)| = n$.

Covering Arrays (CAs): A *covering array* of strength t, $CA(n, k, t, v_1 \cdots v_k)$, is an $(n, v_1 \cdots v_k)$-design in which $|\rho(T)| \ge 1$ for every $T \in \mathcal{I}_t$ [8]. The *index* λ of a covering array is $\min_{T \in \mathcal{I}_t} |\rho(T)|$; the notation $CA_\lambda(n, k, t, v_1 \cdots v_k)$ is used to indicate that the CA has index (at least) λ. When $|\rho(T)| = n/(v_{j_1} \times \cdots \times v_{j_t})$ for every $T \in \mathcal{T}_{\mathbf{c}_{j_1} \dots \mathbf{c}_{j_t}}$, and every $\mathcal{T}_{\mathbf{c}_{j_1} \dots \mathbf{c}_{j_t}} \subseteq \mathcal{I}_t$, the array is an *orthogonal array* of strength t [10].

The *covering array number* $CAN(k, t, v_1 \cdots v_k)$ is the minimum number of rows n for which a $CA(n, k, t, v_1 \cdots v_k)$ exists. An array with the fewest rows is *optimal*. For fixed strength t, $\max(v_i : 1 \le i \le k) \ge 2$ fixed, and $\min(v_i : 1 \le i \le k) \ge 2$, the covering array number grows as $\Theta(\log k)$, where k is the number of columns [19, 20]. Covering arrays have been extensively studied; see [2, 9, 16, 28] for older survey material.

Locating Arrays (LAs): For a positive integer $d < v_1 \le \cdots \le v_k$, a covering array of strength t is a (d, t)-*locating array* denoted by (d, t)-$LA(n, k, v_1 \cdots v_k)$, if $\rho(\mathcal{T}_1) = \rho(\mathcal{T}_2) \Leftrightarrow \mathcal{T}_1 = \mathcal{T}_2$ whenever $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{I}_t$ and $|\mathcal{T}_1| = |\mathcal{T}_2| = d$ [6]. A locating array has *separation* δ if whenever $\mathcal{T}_1 \ne \mathcal{T}_2$ are two distinct subsets of $\mathcal{I}_t$ and $|\mathcal{T}_1| = |\mathcal{T}_2| = d$, we have that $|(\rho(\mathcal{T}_1) \cup \rho(\mathcal{T}_2)) \setminus (\rho(\mathcal{T}_1) \cap \rho(\mathcal{T}_2))| \ge \delta$ [21]. To record separation greater than 1, the term (d, t, δ)-locating array is used; this guarantees that any two sets of d t-tuples are distinguished in at least δ rows. A locating array must have separation $\delta \ge 1$.

The minimum number of rows n for which a (d, t)-$LA(n, k, v_1 \cdots v_k)$ exists is the *locating array number*, (d, t)-$LAN(k, v_1 \cdots v_k)$. In [6], Colbourn and McClary show that as for CAs, logarithmic growth for the number of rows also applies to LAs [6]. Exploiting the structure of the faulty interactions, Martínez et al. [18] establish feasibility conditions for a locating array to exist. Tang et al. [25] determine a lower bound on the $(1, t)$-locating array number and use transversal designs and optimal LAs with fewer levels to construct optimal LAs whose size attains the lower bound when $v \geq k \geq 2$. A direct construction for optimal LAs with $k \leq 5$ factors is also provided. In [4], Colbourn and Fan develop recursive constructions for LAs when $(d, t) = (1, 2)$. Their technique employs cut-and-paste and column replacement for producing LAs with a large number of factors using small locating and covering arrays. A study of $(1, 1)$-locating arrays shows that when $v > 2$, $(1, 1)$-locating arrays employ substantially fewer rows than do CAs of strength two [5]. Seidel et al. [21] discuss two randomized algorithms for constructing $(1, 2)$-locating arrays based on the Stein–Lovász–Johnson paradigm, and the Lovász Local Lemma. In [17], Lanus et al. develop a partitioned search with column resampling (PSCR) algorithm inspired by Moser–Tardos resampling to construct (d, t)-locating arrays. Konishi et al. [11, 12] formulate the problem of finding a locating array of specified size as a Constraint Satisfaction Problem (CSP) and propose using a CSP solver. In [13], they propose a simulated annealing algorithm to find locating arrays.

Detecting Arrays (DAs): A covering array of strength t is a (d, t)-*detecting array*, (d, t)-$DA(n, k, v_1 \cdots v_k)$, if $\rho(T) \subseteq \rho(\mathcal{T}) \Leftrightarrow T \in \mathcal{T}$, whenever T is a t-tuple and $\mathcal{T}$ is a subset of $\mathcal{I}_t$ of cardinality d [6]. A detecting array has *separation* δ if $|\rho(\mathcal{T}) \setminus \rho(T)| \geq \delta$ for every $\mathcal{T} \subset \mathcal{I}_t$ with $|\rho(\mathcal{T})| = d$ and every t-tuple $T \notin \mathcal{T}$. A detecting array always has separation $\delta \geq 1$. To indicate separation $\delta > 1$, we write (d, t, δ)-detecting array.

The minimum number of rows n for which a (d, t)-$DA(n, k, v_1 \cdots v_k)$ exists is the *detecting array number*, (d, t)-$DAN(k, v_1 \cdots v_k)$. Any $(1, t)$-detecting array is a CA of strength t with index $\lambda \geq 2$ and hence $(1, t)$-$DAN(k, v^t) \geq 2v^t$. Detecting arrays can be constructed from covering arrays of higher strength [6]. Lower bounds on the optimal sizes of DAs and cases for which the bounds can be met are explored in [23, 26, 29]. In [22], Shi et al. present a combinatorial characterization of optimal DAs and prove that an optimum (d, t)-$DA((d + 1)v^t, k, v)$ is equivalent to a super-simple orthogonal array. A construction for DAs using error-correcting codes and separating hash families is presented in [7].

3 Cyclotomy Construction

We now present a construction for mixed-level covering, locating, and detecting arrays using cyclotomic arrays based on finite fields. For fixed t and v, the basic method produces arrays having $\Theta(k)$ rows; in contrast $\Theta(\log k)$ rows suffice asymptotically. One might reasonably ask why there is interest in constructions that appear to fare so poorly asymptotically. At the outset, then, we emphasize that we are concerned with the construction of specific 'small' arrays that have competitive sizes.

Let q be a prime power and ω be a fixed primitive element of the finite field $\mathbb{F}_q$. For v a divisor of $q - 1$, partition the nonzero elements of the field into *cyclotomic classes* $\{C_\beta = \{\omega^{\alpha v + \beta} : 0 \leq \alpha < (q - 1)/v\} : 0 \leq \beta < v\}$. Then place 0 in C_0. As in [3], for such a partition, the *cyclotomic vector* $\mathbf{x}_{q,v} = (x_0, x_1, \ldots, x_{q-1}) \in \mathbb{Z}_v^q$ is indexed by the field elements, and has value β when the index belongs to class C_β.

Fig. 1 The cyclotomic array $D_{13,2}$

$$D_{13,2} = \begin{pmatrix} 0&0&1&0&0&1&1&1&1&0&0&1&0 \\ 0&0&0&1&0&0&1&1&1&1&0&0&1 \\ 1&0&0&0&1&0&0&1&1&1&1&0&0 \\ 0&1&0&0&0&1&0&0&1&1&1&1&0 \\ 0&0&1&0&0&0&1&0&0&1&1&1&1 \\ 1&0&0&1&0&0&0&1&0&0&1&1&1 \\ 1&1&0&0&1&0&0&0&1&0&0&1&1 \\ 1&1&1&0&0&1&0&0&0&1&0&0&1 \\ 1&1&1&1&0&0&1&0&0&0&1&0&0 \\ 0&1&1&1&1&0&0&1&0&0&0&1&0 \\ 0&0&1&1&1&1&0&0&1&0&0&0&1 \\ 1&0&0&1&1&1&1&0&0&1&0&0&0 \\ 0&1&0&0&1&1&1&1&0&0&1&0&0 \end{pmatrix}$$

The additive translates of a cyclotomic vector yield a $q \times q$ *cyclotomic array* $D_{q,v}$ on v symbols, i.e., the rows and columns of array $D_{q,v} = (d_{ij})$ are indexed by elements in $\mathbb{F}_q$ where $d_{ij} = x_{j-i}$ (computing the subscript in $\mathbb{F}_q$).

For every positive integer t, for q sufficiently large with respect to v, such a cyclotomic array is known to form a CA of strength t [3]. Cyclotomic arrays can also produce some of the smallest known locating and detecting arrays for different parameter values. For example, consider a primitive element $\omega = 2$ in $\mathbb{F}_{13}$. Figure 1 shows the cyclotomic array $D_{13,2}$ corresponding to the cyclotomic vector $\mathbf{x}_{13,2} = (x_0, x_1, \ldots, x_{12}) = (0, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0)$. We can verify by computation that $D_{13,2}$ is a locating array, $(1, 2)$-$LA(13, 13, 2)$ with separation one. In contrast, both a strength three covering array and the algorithmic construction in [21], use sixteen rows to generate the LA. Therefore, the cyclotomic array $D_{13,2}$ improves the known upper bound on the locating array number: $(1, 2)$-$LAN(13, 2) \leq 13$. Similarly, for $q = 19$, the cyclotomic array $D_{19,2}$ associated with the cyclotomic vector $\mathbf{x}_{19,2} = (0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 1)$ produces a $(1, 2)$-$DA(19, 19, 2)$ with separation one.

We extend the cyclotomic strategy in three directions. First, we investigate the subgroup-based column selection for subarrays of a cyclotomic array to produce covering, locating, or detecting arrays. Secondly, we explore a level reduction technique for constructing locating and detecting arrays with fewer levels than in the cyclotomic array. Finally, we juxtapose certain cyclotomic arrays having the same order q but different values of v to produce mixed-level covering, locating, and detecting arrays having more factors.

3.1 Column Selection via Subgroup

When a cyclotomic array lacks the desired covering, locating, or detecting property, that property could nonetheless hold for a suitable chosen subarray of the cyclotomic array. But how can one choose a suitable subarray? The number of subarrays grows exponentially; exploring all subarrays is not practical when q is large. A simple way of selecting columns is to employ a subgroup of the multiplicative group $\mathbb{F}_q^*$ on the nonzero elements of the field. Let k_i be a divisor of $q - 1$ and $H_q^{k_i}$ be the subgroup of $\mathbb{F}_q^*$, generated by $\omega^{(q-1)/k_i}$. Construct a subarray $\mathcal{C}_{q,v}^{k_i}$ of the cyclotomic array $D_{q,v}$ with q rows and k_i columns, in which columns are indexed by elements from $H_q^{k_i}$ or any coset of $H_q^{k_i}$. We can further include the column with index 0 to obtain an array with $k_i + 1$ columns. Column selection based on subgroups can produce LAs even though the full cyclotomic array fails to admit a positive separation.

Example 1 For $q = 43, \omega = 3$, and $v = 3$, the cyclotomic array $D_{43,3}$ forms a CA of strength two with index three. However, $D_{43,3}$ is not a $(1, 2)$-locating array because some 2-tuples are covered in the same set of rows. Therefore, we consider subgroups of $\mathbb{F}_{43}^*$. Let $H_{43}^{14} = \{1, 2, 4, 8, 11, 16, 21, 22, 27, 32, 35, 39, 41, 42\}$ be the subgroup generated by ω^3. The columns with indices from H_{43}^{14} in $D_{43,3}$ form a $(1, 2)$-$LA(43, 14, 3)$ with separation one and index three, shown in Fig. 2. In contrast, a covering array of strength three produces a $(1, 2)$-locating array with the same parameters but with forty-five rows.

$$
\begin{pmatrix}
0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0 \\
0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1 \\
0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1 \\
0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2 \\
0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1 \\
0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2 \\
0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0 \\
0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1 \\
0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2 \\
0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2\,0\,2\,1 \\
0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0\,1\,1\,2 \\
0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0\,0\,1 \\
0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0\,0 \\
0\,0\,1\,0\,1\,1\,2\,0\,2\,1\,0\,1\,2\,2\,2\,0\,2\,2\,1\,1\,0\,0\,1\,1\,2\,2\,0\,2\,2\,2\,1\,0\,1\,2\,0\,2\,1\,1\,0\,1\,0\,0\,0
\end{pmatrix}^T
$$

Fig. 2 A locating array $(1, 2)$-$LA(43, 14, 3)$ from the cyclotomic array $D_{43,3}$ via subgroup selection

3.2 Level Reduction

A covering array with v levels can be used to produce a new covering array with $u < v$ levels by replacing the $v - u$ symbols in $\mathbb{Z}_v \setminus \mathbb{Z}_u$ by symbols from $\mathbb{Z}_u$. A level reduction in this fashion preserves the coverage of t-tuples over $\mathbb{Z}_u$. However, in general, level reduction does not preserve separation in locating or detecting arrays.

There are many ways to identify symbols to reduce the number of levels. We employ a simple technique to reduce the number of levels from v to $\lceil v/2 \rceil$: For a positive integer $s = (v + 1)/2$ where v is an odd divisor of $q - 1$, construct a $q \times q$ array on s symbols by replacing symbol $v - j$ by j for $1 \leq j \leq s - 1$. When applied to a cyclotomic array with v levels, this level reduction might provide a covering, locating, or detecting array with $\lceil v/2 \rceil$ levels.

Example 2 The cyclotomic array $D_{43,3}$ is not a $(1, 2)$-locating array. We can identify two symbols (replacing symbol 2 by 1) to construct a 43×43 binary array A. Computation shows that A is a $(1, 2)$-locating array. Indeed, using the idea of Sect. 3.3, horizontally juxtaposing A with the cyclotomic array $D_{43,2}$ (forming $[A \mid D_{43,2}]$) produces a $(1, 2)$-locating array with 43 rows, 86 columns, and separation one.

3.3 Mixed-Level Arrays via Column Juxtaposition

Given a set $\{v_1, \ldots, v_\ell\}$ of divisors of $q - 1$, we employ the associated cyclotomic arrays $D_{q,v_1}, \ldots, D_{q,v_\ell}$, choosing columns from each to construct mixed-level covering, locating, and detecting arrays in two different ways.

1. *When $v_1, \ldots, v_\ell$ are pairwise coprime:* Construct a mixed-level array by juxtaposing the columns from cyclotomic arrays D_{q,v_j} or a subarray $\mathcal{C}^{k_i}_{q,v_j}$.
2. *When v_i and v_j are not coprime:* Choose a set S of column indices for the cyclotomic arrays. Choose the columns in D_{q,v_j} indexed by S, and some or all columns from D_{q,v_i} not indexed by S. Juxtapose the chosen columns from D_{q,v_i} and D_{q,v_j} to obtain an array with q rows and at most q columns. To produce more than q columns with v_i levels, one can first apply the level reduction from Sect. 3.2 on a cyclotomic array $D_{q,2v_i-1}$ with $2v_i - 1$ levels. Then the resulting array with v_i levels and the columns of D_{q,v_j} can be juxtaposed to produce a mixed-level array with more than q columns. Tables 1, 2, 3, 4 and 5 show that certain cyclotomic arrays for the same order q but different values of v can be juxtaposed to produce mixed-level locating, detecting, and covering arrays. When $d = 1$ and $t = 1$, $[D_{q,v_i} \mid D_{q,v_j}]$ can also produce locating arrays with $2q$ columns.

Example 3 Let $q = 151$, $v_1 = 2$, $v_2 = 3$, and $v_3 = 5$. A mixed-level covering array $CA_4(151, 453, 2^{151}3^{151}5^{151})$ is obtained by juxtaposing the cyclotomic arrays $D_{151,2}$, $D_{151,3}$, and $D_{151,5}$.

Table 1 Mixed-level CAs and LAs using column selection via subgroup (Sect. 3.1)

Design	Column index via subgroup
$CA_1(13, 15, 2, 2^{13}3^2)$	$D_{13,2}$, $\{0, 1\}$
$(1, 2, 1)$-$LA(13, 5, 2^4 3^1)$	$\langle 2^3\rangle$, $\{0\}$
$(1, 2, 1)$-$LA(19, 10, 2^9 3^1)$	$\langle 2^2\rangle$, $\{0\}$
$(1, 2, 1)$-$LA(19, 7, 2^4 3^3)$	$\langle 2^6\rangle \cup \{0\}$, $\langle 2^6\rangle$
$(1, 2, 1)$-$LA(29, 29, 2^{28}4^1)$	$\langle 2\rangle$, $\{0\}$
$(1, 2, 1)$-$LA(31, 37, 2^{31}3^6)$	$D_{31,2}$, $\langle 2^6\rangle \cup \{0\}$
$(1, 2, 1)$-$LA(37, 21, 2^{19}4^2)$	$2\langle 2^2\rangle \cup \{0\}$, $\langle 2^{18}\rangle$
$CA_2(41, 41, 2, 2^{40}5^1)$	$\langle 6\rangle$, $\{1\}$
$(1, 2, 1)$-$LA(41, 41, 2^{39}3^2)$	$\langle 6\rangle \setminus \{1\}$, $\{0, 1\}$
$(1, 2, 1)$-$LA(41, 41, 2^{39}4^1 5^1)$	$\langle 6\rangle \setminus \{1\}$, $\{0\}$, $\{1\}$
$(1, 2, 1)$-$LA(43, 57, 2^{43}3^{14})$	$D_{43,2}$, $\langle 3^3\rangle$
$CA_1(43, 57, 2, 2^{42}3^{14}6^1)$	$\langle 3\rangle$, $\langle 3^3\rangle$, $\{0\}$
$CA_1(43, 44, 2, 2^{42}6^1 7^1)$	$\langle 3\rangle$, $\{0\}$, $\{0\}$
$(1, 2, 1)$-$LA(53, 53, 2^{48}4^5)$	$\langle 2\rangle \setminus \langle 2^{13}\rangle$, $\langle 2^{13}\rangle \cup \{0\}$
$(1, 2, 1)$-$LA(61, 61, 2^{56}4^5)$	$\langle 2\rangle \setminus \langle 2^{15}\rangle$, $\langle 2^{15}\rangle \cup \{0\}$
$(1, 2, 1)$-$LA(61, 66, 2^{58}3^5 4^3)$	$D_{61,2} \setminus \langle 2^{20}\rangle$, $\langle 2^{15}\rangle \cup \{0\}$, $\langle 2^{20}\rangle$
$(1, 2, 1)$-$LA(61, 66, 2^{60}3^4 4^1 5^1)$	$D_{61,2} \setminus \{13\}$, $\langle 2^{20}\rangle \cup \{0\}$, $\{13\}$, $\{0\}$
$(1, 2, 1)$-$LA(67, 89, 2^{66}3^{22}6)$	$\langle 2\rangle$, $\langle 2^3\rangle$, $\{0\}$
$(1, 2, 1)$-$LA(71, 73, 2^{71}5^1 7^1)$	$D_{71,2}$, $\{0\}$, $\{0\}$
$(1, 2, 1)$-$LA(73, 80, 3^{73}4^7)$	$D_{73,3}$, $\langle 5^{12}\rangle \cup \{0\}$
$CA_1(73, 109, 2^{72}3^{36}6)$	$\langle 5\rangle$, $\langle 5^2\rangle$, $\{0\}$
$(1, 2, 1)$-$LA(73, 97, 2^{72}3^{24}6^1)$	$\langle 5\rangle$, $\langle 5^3\rangle$, $\{0\}$
$CA_1(73, 97, 2^{72}3^{24}8)$	$\langle 5\rangle$, $\langle 5^3\rangle$, $\{0\}$

Example 4 Consider $\mathbb{F}_{53}$ with $\omega = 2$ a primitive element. The cyclotomic array $D_{53,2}$ is a $(1, 2)$-locating array and $D_{53,4}$ is covering array of strength two. A 53×5 subarray, A of $D_{53,4}$, whose column indices are $\{0, 1, 23, 30, 52\}$ corresponding to the subgroup $H^4_{53,4}$ generated by 2^{13}, is a locating array $(1, 2)$-$LA(53, 5, 4)$. Because $v_1 = 2$ and $v_2 = 4$ are not coprime, for $t = 2$, we consider a subarray of $D_{53,2}$, say B, whose column indices are elements from $\mathbb{F}_{53} \setminus \{0, 1, 23, 30, 52\}$. Then the 53×53 array $[B \mid A]$ is a mixed-level $(1, 2)$-$LA(53, 53, 2^{48}4^5)$.

4 Results

For prime order $q \leq 5000$ and $v \leq 25$, we apply the construction techniques from Sect. 3 to produce (mixed) covering, locating, and detecting arrays based on cyclotomic arrays. Table 1 reports some of the covering and locating arrays produced

Table 2 Mixed-level CAs of Strength 2 by Juxtaposition (Sect. 3.3)

Levels	Prime orders q_λ	Levels	Prime orders q_λ
Mixed-level covering array of strength two $CA(q, 2q, 2, v_1^q v_2^q)$			
2, 3	19_1 31_2 37_2 43_3 61_5	2, 5	131_2 151_4 181_3 191_4 211_6
2, 7	197_1 211_2 239_2 281_2 337_2	2, 9	379_2 397_1 433_2 487_2 541_2
3, 4	61_2 73_2 97_2 109_4 157_7	3, 5	151_4 181_3 211_6 241_6 271_6
3, 7	211_2 337_2 379_5 421_5 463_4	3, 8	193_1 241_1 313_2 409_2 433_4
3, 10	541_2 571_2 601_2 631_2 661_2	4, 5	181_3 241_6 281_8 401_{12} 421_{12}
4, 7	197_1 281_2 337_2 421_5 449_6	4, 9	397_1 433_2 541_2 613_2 757_5
5, 6	151_1 181_1 211_3 241_4 271_2	5, 7	211_2 281_2 421_5 631_8 701_6
5, 8	241_1 281_2 401_3 521_4 601_4	5, 9	541_2 631_4 811_3 991_6 1171_8
6, 7	211_1 337_2 379_4 421_5 463_4	7, 8	281_1 449_2 617_4 673_4 953_{10}
7, 9	379_2 631_4 757_5 883_6 1009_6	7, 10	631_2 701_4 1051_4 1471_{11} 2311_{14}
8, 9	433_1 937_4 1009_6 1153_6 1297_8	9, 10	541_1 631_1 811_2 991_2 1171_4
Mixed-level covering array of strength two $CA(q, 3q, 2, v_1^q v_2^q v_3^q)$			
2, 3, 5	151_4 181_3 211_6 241_6 271_6	2, 3, 7	211_2 337_2 379_5 421_5 463_4
3, 4, 5	181_3 241_6 421_{12} 541_{14} 601_{14}	3, 4, 7	337_2 421_5 673_8 757_{11} 1009_{14}
4, 5, 7	281_2 421_5 701_6 2381_{42} 2521_{36}	4, 5, 9	541_2 1621_8 1801_{16} 2161_{18} 2341_{20}
5, 6, 7	211_1 421_5 631_8 1051_8 1471_{24}		
Mixed-level covering array of strength two $CA(q, 4q, 2, v_1^q v_2^q v_3^q v_4^q)$			
2, 3, 5, 7	211_2 421_5 631_8 1051_8 1471_{24}	3, 4, 5, 7	421_5 2521_{36} 3361_{38} 4201_{68} 4621_{83}
3, 5, 7, 8	2521_{30} 3361_{36} 4201_{55}	4, 5, 7, 9	2521_{18}

via subgroup-based column selection when the cyclotomic arrays themselves do not form the desired array. In subsequent tables, we provide solutions by juxtaposition for covering arrays of strength two (Table 2) and three (Table 3), locating arrays (Table 4) and detecting arrays (Table 5). In each we only report selected results with $v \leq 10$, listing the first few primes for which a solution exists, in order to limit the length. In these tables, the subscript gives the index of the covering array, or the separation of the locating or detecting array. Table 6 reports parameters for mixed-level locating and detecting arrays produced using the level reduction technique in Sect. 3.2 in which the cyclotomic array with v_2 levels is used to produce an array with s levels.

5 Conclusion

The cyclotomic arrays are known to form CAs. Indeed, for fixed strength t and number of levels v, the cyclotomic array $D_{q,v}$ is always a covering array of strength t when $q \equiv 1 \pmod v$ is a large enough prime power. When q is very large, the covering array sizes that result are not competitive with other direct and recursive

Table 3 Mixed-level CA of Strength 3 by Juxtaposition (Sect. 3.3)

Levels	Prime orders q_λ
v	Covering array of strength three $CA(q, q, 3, v)$
2	19_1 23_1 29_2 31_2 37_2 41_2 43_3 47_4 53_4
3	223_2 229_2 271_4 277_2 307_2 313_4 331_4 337_4 349_4
4	461_1 557_1 613_2 653_2 677_2 701_2 733_2 757_4 773_2
5	1511_2 1901_2 2141_2 2371_4 2441_3 2551_2 2671_4 2791_4 2851_4
v_1, v_2	Mixed-level covering array of strength three $CA(q, 2q, 3, v_1^q v_2^q)$
2, 3	223_2 229_2 271_4 277_2 307_2 313_4 331_4 337_4 349_4
2, 5	1511_2 1901_2 2141_2 2371_4 2441_3 2551_2 2671_4 2791_4 2851_4
3, 4	613_2 733_2 757_3 829_2 853_4 877_2 997_4 1021_4 1069_6
3, 5	2371_4 2551_2 2671_4 2791_4 2851_4 2971_4 3001_4 3121_6 3181_4
4, 5	1901_2 2141_2 2441_3 2861_4 3001_4 3121_6 3181_4 3221_4 3301_4
5, 6	2971_2 3271_2 3631_2 3691_2 3931_2 4051_2 4111_2 4231_2 4591_2
v_1, v_2, v_3	Mixed-level covering array of strength three $CA(q, 3q, 3, v_1^q v_2^q v_3^q)$
2, 3, 5	2371_4 2551_2 2671_4 2791_4 2851_4 2971_4 3001_4 3121_6 3181_4
3, 4, 5	3001_4 3121_6 3181_4 3301_4 3361_4 4021_4 4201_8 4261_8 4441_8

Table 4 Mixed-level LAs by Juxtaposition (Sect. 3.3)

Levels	Prime orders q_δ	Levels	Prime orders q_δ
Locating array $(1, 1)$-$LA(q, 2q, v_1^q v_2^q)$			
2, 4	13_2 17_3 29_7 37_9 41_{10}	2, 6	13_1 19_2 31_6 37_6 43_8
3, 6	13_1 19_2 31_5 37_6 43_7	4, 6	13_1 37_6 61_{12} 73_{17} 97_{23}
Locating array $(1, 2)$-$LA(q, q, v^q)$			
2	11_1 13_1 17_2 19_2 23_3	3	37_1 61_4 67_4 73_4 79_5
4	89_2 97_2 101_2 109_2 113_2	5	151_2 181_2 211_2 241_4 251_4
6	193_2 199_2 211_2 241_2 271_2	7	379_2 421_3 449_3 463_2 547_4
8	521_2 569_2 577_2 593_3 601_2	9	631_2 739_4 757_4 811_2 829_4
10	691_2 701_2 751_2 761_2 821_2		
Mixed-level locating array $(1, 2)$-$LA(q, 2q, v_1^q v_2^q)$			
2, 3	37_1 61_4 67_4 73_4 79_4	2, 5	151_6 181_7 211_{11} 241_{12} 251_{11}
2, 7	379_{13} 421_{14} 449_{16} 463_{18} 547_{20}	2, 9	631_{19} 739_{23} 757_{24} 811_{25} 829_{28}
3, 4	97_2 109_2 157_5 181_7 193_8	3, 5	151_3 181_4 211_6 241_7 271_9
3, 7	379_9 421_7 463_{11} 547_{15} 631_{17}	3, 8	577_{10} 601_{12} 673_{15} 769_{17}
4, 5	181_2 241_6 281_8 401_{12} 421_{13}	4, 7	421_6 449_6 617_{12} 673_{12} 701_{12}
5, 6	211_2 241_3 271_1 331_4 421_6	5, 7	421_4 631_{10} 701_{10}
5, 8	521_5 601_7 641_6 881_{12}	6, 7	379_2 421_4 463_4 547_6 631_7

Table 5 Mixed-level DAs by Juxtaposition (Sect. 3.3)

Levels	Prime orders q_δ	Levels	Prime orders q_δ
Detecting array $(1,2)$-$DA(q,q,v^q)$			
2	19_1 23_1 29_2 31_2 37_2	3	61_1 67_1 73_1 79_2 97_3
4	137_1 149_2 157_2 173_2 181_3	5	211_1 241_1 251_2 271_1 281_2
6	241_1 307_1 313_1 331_1 337_1	7	379_1 421_1 449_1 547_2 617_2
8	601_1 617_1 641_2 673_1 769_3	9	739_1 757_1 811_1 829_2 937_2
10	1021_2 1031_2 1051_1 1061_1 1091_2		
Mixed-level detecting array $(1,2)$-$DA(q,2q,v_1^q v_2^q)$			
2, 3	97_1 103_1 109_1 127_1 139_1	2, 5	421_1 431_1 461_2 491_1 521_2
2, 7	911_1 1009_1 1093_1 1163_2 1289_1	3, 4	157_1 181_1 193_1 229_3 241_2
3, 5	241_1 331_2 421_2 541_4 571_5	3, 7	673_1 757_2 883_1 967_2 1009_1
3, 8	937_1 1009_1 1129_1 1201_1 1297_3	3, 10	1471_1 1741_1 1801_1 1831_2 1861_1
4, 5	241_1 281_2 401_4 421_3 461_4	4, 7	701_1 757_2 953_3 1009_3 1093_4
4, 9	937_1 1009_1 1117_1 1153_1 1549_2	5, 6	421_1 541_2 571_3 601_2 631_5
5, 7	631_1 911_3 1051_1 1471_{10} 2311_{20}	5, 8	1201_2 1321_5 1361_4 1481_7 1601_4
5, 9	991_1 1171_2 1531_4 1621_1 1801_5	6, 7	421_1 547_1 631_2 673_2 757_3
7, 8	953_3 1009_3 1289_5 2017_{11} 2129_9	7, 9	757_1 1009_2 2017_7 2143_8 2269_8
7, 10	1471_4 2311_6 2381_3 2521_6 2591_7	8, 9	1009_1 1153_1 1297_2 1657_3 1801_6
9, 10	1801_4 2161_6 2251_4 2341_8 2521_7		
q_δ: Mixed-level detecting array $(1,2)$-$DA(q,3q,v_1^q v_2^q v_3^q)$			
2, 3, 5	421_1 541_2 571_2 601_1 631_1	2, 3, 7	1009_1 1093_1 1303_1 1429_2 1471_2
3, 4, 5	241_1 421_2 541_4 601_3 661_5	3, 4, 7	757_2 1009_1 1093_3 1429_7 1597_7
4, 5, 7	2381_{20} 2521_{17} 2801_{20} 3221_{26} 3361_{17}	4, 5, 9	1621_1 1801_5 2161_6 2341_7 2521_6
4, 7, 9	1009_1 2017_5 2269_5 2521_6 3529_{14}	5, 6, 7	631_1 1051_1 1471_{10} 2311_{20} 2521_{18}
5, 7, 8	2521_{14} 2801_{15} 3361_{19} 4201_{30} 4481_{33}	5, 7, 9	2521_8
5, 8, 9	1801_5 2161_7 2521_8	7, 8, 9	2017_7 2521_9 3529_{17}
q_δ: Mixed-level detecting array $(1,2)$-$DA(q,4q,v_1^q v_2^q v_3^q v_4^q)$			
2, 3, 5, 7	1471_2 2311_8 2521_6 2731_{10} 3361_9	2, 5, 7, 9	2521_3
3, 4, 5, 7	2521_{14} 3361_{16} 4201_{30} 4621_{37}	3, 5, 7, 8	2521_8 3361_{13} 4201_{22}
Detecting array $(1,3)$-$DA(q,q,v^q)$			
2	67_1 71_1 79_1 83_1 89_2	3	487_1 523_1 541_1 571_1 577_1
4	1301_1 1381_1 1429_1 1453_1 1489_1		

constructions. Nevertheless, covering arrays from cyclotomy can arise when q is relatively small; then the resulting covering arrays often have the smallest known size. Moreover, they have a very compact representation and are easy to construct when they do exist.

Table 6 Mixed-level locating and detecting arrays via level reduction from v_2 to s (Sect. 3.2)

$(1, 2, \delta)$-$LA(q, 2q, v_1^q s^q)$										$(1, 2, \delta)$-$DA(q, 2q, v_1^q s^q)$									
q	v_1	v_2	s	δ	q	v_1	v_2	s	δ	q	v_1	v_2	s	δ	q	v_1	v_2	s	δ
31	2	3	2	1	37	2	3	2	2	79	2	3	2	1	97	2	3	2	1
43	2	3	2	2	61	2	3	2	4	103	2	3	2	1	109	2	3	2	1
67	2	3	2	5	71	2	5	3	1	127	2	3	2	1	139	2	3	2	3
73	2	3	2	6	79	2	3	2	7	151	2	3	2	2	157	2	3	2	3
97	2	3	2	9	97	4	3	2	2	163	2	3	2	3	181	2	3	2	3
101	2	5	3	4	103	2	3	2	8	193	2	3	2	3	199	2	3	2	3
109	2	3	2	9	109	4	3	2	2	211	2	3	2	3	211	3	5	3	1
127	2	3	2	12	131	2	5	3	5	223	2	3	2	4	229	2	3	2	3
139	2	3	2	14	151	2	3	2	15	241	2	3	2	3	271	2	3	2	5
151	2	5	3	6	151	3	5	3	5	271	3	5	3	1	277	2	3	2	5
151	5	3	2	3	157	2	3	2	16	281	4	5	3	1	283	2	3	2	3
157	4	3	2	5	163	2	3	2	18	307	2	3	2	6	311	2	5	3	1
181	2	3	2	18	181	2	5	3	8	313	2	3	2	6	331	2	3	2	7
181	3	5	3	5	181	4	3	2	8	331	2	5	3	1	331	3	5	3	2
181	4	5	3	3	181	5	3	2	4	337	2	3	2	7	337	4	3	2	1
191	2	5	3	9	193	2	3	2	22	349	2	3	2	7	349	4	3	2	1
193	4	3	2	8	199	2	3	2	22	367	2	3	2	7	373	2	3	2	7
211	2	3	2	23	211	2	5	3	8	379	2	3	2	8	397	2	3	2	9
211	2	7	4	5	211	3	5	3	7	401	2	5	3	1	401	4	5	3	2
211	3	7	4	4	211	5	3	2	6	409	2	3	2	8	421	2	3	2	9
211	6	5	3	2	223	2	3	2	24	421	2	5	3	2	421	3	5	3	3
229	2	3	2	28	229	4	3	2	10	421	4	3	2	1	421	4	5	3	3
239	2	7	4	7	241	2	3	2	26	431	2	5	3	1	433	2	3	2	9
241	2	5	3	12	241	3	5	3	8	433	4	3	2	1	439	2	3	2	9
241	4	3	2	13	241	4	5	3	7	457	2	3	2	9	457	4	3	2	1
241	5	3	2	7	241	6	5	3	3	461	2	5	3	2	461	4	5	3	4

We have shown here that these ideas extend to covering arrays of higher index, and to detecting and locating arrays of specified separation. We also showed that by subarray selection, level reduction, and juxtaposition, a variety of mixed-level covering, detecting, and locating arrays can be produced.

Acknowledgements This work is supported in part by the U.S. National Science Foundation NeTS grant #1813729.

References

1. Cohen, D.M., Dalal, S.R., Parelius, J., Patton, G.C.: The combinatorial design approach to automatic test generation. IEEE Software **13**(5), 83–88 (1996)
2. Colbourn, C.J.: Combinatorial aspects of covering arrays. Le Matematiche (Catania) **58**, 121–167 (2004)
3. Colbourn, C.J.: Covering arrays from cyclotomy. Designs Codes Cryptogr. **55**, 201–219 (2010)
4. Colbourn, C.J., Fan, B.: Locating one pairwise interaction: three recursive constructions. J. Algebra Comb. Discrete Struct. Appl. **3**(3), 127–134 (2016)
5. Colbourn, C.J., Fan, B., Horsley, D.: Disjoint spread systems and fault location. SIAM J. Discrete Math. **30**(4), 2011–2026 (2016)
6. Colbourn, C.J., McClary, D.W.: Locating and detecting arrays for interaction faults. J. Comb. Optim. **15**(1), 17–48 (2008)
7. Colbourn, C.J., Syrotiuk, V.R.: Detecting arrays for main effects. In: Algebraic Informatics. LNCS, pp. 112–123. Springer International Publishing, Cham (2019)
8. Hartman, A.: Software and hardware testing using combinatorial covering suites. In: Graph Theory. Combinatorics and Algorithms, pp. 237–266. Springer, New York (2005)
9. Hartman, A., Raskin, L.: Problems and algorithms for covering arrays. Discrete Math. **284**(1), 149–156 (2004)
10. Hedayat, A.S., Sloane, N.J.A., Stufken, J.: Orthogonal Arrays. Springer Series in Statistics. Springer, New York (1999). Theory and Applications, With foreword by C. R. Rao
11. Konishi, T., Kojima, H., Nakagawa, H., Tsuchiya, T.: Finding minimum locating arrays using a SAT solver. In: Proceedings of the IEEE International Conference on Software Testing, Verification and Validation Workshops (ICSTW), Tokyo, Japan, pp. 276–277 (2017)
12. Konishi, T., Kojima, H., Nakagawa, H., Tsuchiya, T.: Finding minimum locating arrays using a CSP solver. Fundamenta Informaticae **174**(1), 27–42 (2020)
13. Konishi, T., Kojima, H., Nakagawa, H., Tsuchiya, T.: Using simulated annealing for locating array construction. Inf. Soft. Tech. **126**, 106346 (2020)
14. Kuhn, D.R., Kacker, R.N., Lei, Y.: Introduction to Combinatorial Testing, 1st edn. Chapman and Hall/CRC (2013)
15. Kuhn, D.R., Wallace, D.R., Gallo, A.M.: Software fault interactions and implications for software testing. IEEE Trans. Softw. Eng. **30**(6), 418–421 (2004)
16. Kuliamin, V.V., Petukhov, A.A.: A survey of methods for constructing covering arrays. Progr. Comput. Softw. **37**(3), 121–146 (2011)
17. Lanus, E., Colbourn, C.J., Montgomery, D.C.: Partitioned search with column resampling for locating array construction. In: 2019 IEEE International Conference on Software Testing, Verification and Validation Workshops (ICSTW), pp. 214–223 (2019)
18. Martínez, C., Moura, L., Panario, D., Stevens, B.: Locating errors using ELAs, covering arrays, and adaptive testing algorithms. SIAM J. Discrete Math. **23**(4), 1776–1799 (2009/10)
19. Poljak, S., Pultr, A., Rödl, V.: On qualitatively independent partitions and related problems. Discrete Appl. Math. **6**(2), 193–205 (1983)
20. Poljak, S., Tuza, Z.: On the maximum number of qualitatively independent partitions. J. Comb. Theory Ser. A **51**(1), 111–116 (1989)
21. Seidel, S.A., Sarkar, K., Colbourn, C.J., Syrotiuk, V.R.: Separating interaction effects using locating and detecting arrays. In: C. Iliopoulos, H.W. Leong, W.-K. Sung (eds.) Combinatorial Algorithms, pp. 349–360. Springer International Publishing, Cham (2018)
22. Shi, C., Tang, Y., Yin, J.: The equivalence between optimal detecting arrays and super-simple OAs. Designs Codes Cryptogr. **62**(2), 131–142 (2012)
23. Shi, C., Tang, Y., Yin, J.: Optimum mixed level detecting arrays. Ann. Stat. **42**(4), 1546–1563 (2014)
24. Shi, C., Jin, H., Tsuchiya, T.: Locating arrays with mixed alphabet sizes. Mathematics **8**(5), 128 (2020)
25. Tang, Y., Colbourn, C.J., Yin, J.: Optimality and constructions of locating arrays. J. Stat. Theory Pract. **6**(1), 20–29 (2012)

26. Tang, Y., Yin, J.X.: Detecting arrays and their optimality. Acta mathematica Sinica, English series **27**(12), 2309–2318 (2011)
27. Tatsumi, K.: Test case design support system. In: Proceedings of the International Conference on Quality Control (ICQC'87), pp. 615–620 (1987)
28. Torres-Jimenez, J., Izquierdo-Marquez, I.: Survey of covering arrays. In: 15th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, pp. 20–27 (2013)
29. Shi, C., Wang, C.M.: Optimum detecting arrays for independent interaction faults. Acta mathematica Sinica. English series **32**(2), 199–212 (2016)

Resolutions for an Infinite Family of Bose Triple Systems

Dylan Lusi and Charles J. Colbourn

Abstract A classical construction of Bose produces a Steiner triple system of order $3n$ from a symmetric, idempotent latin square of order n whenever n is odd. In an application to access-balancing in storage systems, these Bose triple systems play a central role. A natural question arises: For which orders n does there exist a resolvable Bose triple system? In this paper, a resolution of the Bose-averaging triple system of order $3n$ is determined whenever $n = 3p$ and $p \geq 5$ is prime.

Keywords Bose triple system · Resolvable triple system · Latin square

1 Introduction

A *Steiner system* with parameters t, k, and v, or $S(t, k, v)$, is a pair $(V, \mathcal{B})$, where V is a set of v points and $\mathcal{B}$ is a set of k-subsets (*blocks*) of V such that every t-subset of V occurs in exactly one block. A Steiner system $S(2, 3, v)$ is a *Steiner triple system* of *order* v, or $\mathrm{STS}(v)$; see [5] for background on STSs. A necessary and sufficient condition for the existence of an $\mathrm{STS}(v)$ is that $v \equiv 1, 3 \pmod 6$ [10].

A *parallel class* of an $S(t, k, v)$, $D = (V, \mathcal{B})$, is a subset of disjoint blocks from $\mathcal{B}$ whose union is V. A partition of $\mathcal{B}$ into parallel classes is a *resolution*, and a Steiner system is said to be *resolvable* if it admits a resolution. The study of resolvable block designs is one of the central pursuits of design theory [9]. An $\mathrm{S}(2, 3, v)$ with a resolution is a *Kirkman triple system* of *order* v, or $\mathrm{KTS}(v)$, named after Reverend Kirkman [11]. A $\mathrm{KTS}(v)$ exists if and only if $v \equiv 3 \pmod 6$ [12, 14].

Our goal is to establish that a large class of Steiner triple systems, which we call Bose-averaging triple systems (see Sect. 2), are resolvable. The motivation arises out of the demand for access balancing in storage systems. Given their balance prop-

D. Lusi · C. J. Colbourn (✉)
Arizona State University, Tempe, AZ 85281, USA
e-mail: colbourn@asu.edu

D. Lusi
e-mail: dlusi@asu.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_5

erties, block designs have served as a basis for many systems, such as distributed storage systems [7, 16], systems for batch coding [17], and multiserver private information retrieval systems [8]. Dau and Milenkovic [6] propose ranking the popularity of the files making up such a storage system to improve its access balancing: Formally this equips an $S(t, k, v)$ $(V, \mathcal{B})$ with a *point labelling*, that is, a bijection $\mathsf{rk} : V \to \{0, \ldots, v-1\}$. A Steiner system together with a point labelling is a *point-labelled Steiner system*. Given a point-labelled $S(t, k, v)$ $(D = (V, \mathcal{B}), \mathsf{rk})$, the *block sum* $\text{sum}(B)$ is $\sum_{x \in B} \mathsf{rk}(x)$ when $B \in \mathcal{B}$. Metrics for D to address access-balancing are defined in [6] and investigated in [3]. One metric is the *minimum sum* or *MinSum*, $\text{MinSum}(D, \mathsf{rk}) = \min_{\mathsf{B} \in \mathcal{B}} \text{sum}(\mathsf{B})$. Maximizing the MinSum is a nontrivial problem. In [6] the Bose-averaging triple system of order $v \equiv 3 \pmod 6$ is point-labelled to achieve MinSum v, the largest MinSum of any point-labelled Steiner triple system. Producing $S(2, 4, v)$s with large MinSum appears to be more challenging. There one might hope to use the $3v + 1$ construction [15] that combines an ingredient $S(2, 4, v)$ and a $\text{KTS}(2v + 1)$ to produce an $S(2, 4, 3v + 1)$. For this purpose one wants a resolvable MinSum-optimal $\text{STS}(2v + 1)$, hence our interest in *resolvable* Bose-averaging triple systems. Brummond [2], also motivated by access balancing in storage systems, established that every Bose-averaging triple system of order 3^k, $k \geq 1$, is resolvable. Here we show that the Bose-averaging triple system of order $9p$, with $p \geq 5$ an odd prime, is resolvable. The techniques developed are crucial in later establishing that some Bose triple system of order $3n$ is resolvable if and only if $n \equiv 3 \pmod 6$.

2 Preliminaries

2.1 Latin Squares and Quasigroups

A *latin square* of *order* n is an $n \times n$ array $L = (L_{x,y})$ in which every row is a permutation of an n-set S (the *symbol set* of L) and every column is a permutation of S. Although rows and columns could be indexed by different sets of size n, henceforth we take the index sets for rows, columns, and symbols to be the symbol set S. An ordered triple $(i, j, L_{i,j})$ is a *cell* of L whose row is i, column is j, and *entry* is $L_{i,j}$. If $c = (x, y, z)$ is a cell of L, denote by $c_{\{\}}$ the set $\{x, y, z\}$; if C is a set of cells of L, define $C_{\{\}} = \{c_{\{\}} : c \in C\}$.

Let Q be a finite set of size n, and let $\circ$ be a binary operation on Q. The pair $(Q, \circ)$ is a *quasigroup of order n* provided that it satisfies (1) For every $x, y \in Q$, the equation $x \circ z = y$ has a unique solution $z \in Q$, and (2) For every $x, y \in Q$, the equation $z \circ x = y$ has a unique solution $z \in Q$. The *operation table* of $(Q, \circ)$ is the $|Q| \times |Q|$ array $A = (A_{x,y})$, where $A_{x,y} = x \circ y$. A quasigroup $(Q, \circ)$ is *idempotent* if $x \circ x = x$ for all $x \in Q$, and *symmetric* if $x \circ y = y \circ x$ for all $x, y \in Q$. A symmetric and idempotent quasigroup of order n exists if and only if n is odd [18]. Quasigroups and latin squares are related [4], in that $(Q, \circ)$ is a quasigroup if and only if its operation table is a latin square.

Let L be an idempotent, symmetric latin square L of order n with symbol set S so that $n = |S| \equiv 0 \pmod 3$. A *partial latin square parallel class* (PLSPC) of L is a set $\mathcal{P}$ of cells of L so that whenever $c, c' \in \mathcal{P}$ and $c \neq c'$, we have $c_{\{\}} \cap c'_{\{\}} = \emptyset$. A *latin square parallel class* (LSPC) of L is a PLSPC $\mathcal{P}$ with $|\mathcal{P}| = (n/3)$; for an LSPC $\mathcal{P}$, $\bigcup_{c \in \mathcal{P}} c_{\{\}} = S$. An *upper triangular resolution* of L is a partition of the set of all cells above the main diagonal into LSPCs. When L has order 3ℓ, such an upper triangular resolution consists of $\frac{3}{2}(3\ell - 1)$ LSPCs.

Let L be a latin square with symbol set $S = \{0, \ldots, n-1\}$. For each $k \in S$, the *k-diagonal* of L is the set of cells $\{(i, i+k, L_{i,i+k}) : 0 \leq i < n-k\}$. When $0 < k \leq \lfloor n/2 \rfloor]$, the *k-diagonal pair* of L is the union of its k- and $(n-k)$-diagonals. $C_k(L)$ denotes the set of cells of the k-diagonal pair of L; if the latin square L is clear from the context we abbreviate this to C_k.

The *averaging latin square* $B = (B_{i,j})$ of order n, n odd, is the latin square with symbol set $S = \{0, \ldots, n-1\}$ such that for all $x, y \in S$, $B_{x,y} = \frac{n+1}{2}(x+y) \pmod n$; that is, B is the operation table of the quasigroup $Q_B = ([0, n-1], \circ)$, where $x \circ y = \frac{n+1}{2}(x+y) \pmod n$, and Q_B is both symmetric and idempotent [18]. A *Bose resolution* of B is either (1) an upper triangular resolution of B or (2) a partition of the set of cells of B above the main diagonal into LSPCs and a set $P = \{P_1, \ldots, P_p\}$ of $p \leq n$ proper PLSPCs such that there exists a partition $\pi = \{\pi_1, \ldots, \pi_q\}$ of $[0, n-1]$ into q sets and a bijection $\rho : P \to \pi$ such that $P_{i_{\{\}}} \cup \rho(P_i)$ is a partition of $[0, n-1]$ for all $i \in [1, p]$.

2.2 The Bose Construction

The Bose triple systems are built via the *Bose construction* [1]; we follow the presentation in [18]. Let $L = (L_{i,j})$ be the operation table of a symmetric idempotent quasigroup of order n having symbol set $S = [0, n-1]$, and put $V = S \times \mathbb{Z}_3$. For every $x \in [0, n-1]$ define the block $A_x = \{(x, 0), (x, 1), (x, 2)\}$. Then for every pair of distinct elements $x, y \in [0, n-1]$, define a block $C_{x,y,i} = \{(x, i), (y, i), (L_{x,y}, i + 1 \pmod{*3})\}$. Then define

$$\mathcal{B} = \{A_x : x \in [0, n-1]\} \cup \{C_{x,y,i} : x, y \in S, x < y, i \in \mathbb{Z}_3\}.$$

Then $(V, \mathcal{B})$ is an STS$(3n)$, a *Bose triple system* of order $3n$. Because a symmetric idempotent quasigroup of order n exists precisely when n is odd, a Bose triple system of order v exists precisely when $v \equiv 3 \pmod 6$. If L is the averaging latin square, $(V, \mathcal{B})$ is the *Bose-averaging triple system* of order $3n$. Now we establish that Bose resolutions of the averaging latin square of order n yield resolutions of the Bose-averaging triple system.

Theorem 1 *Let B be the averaging latin square of order n and D the Bose-averaging triple system of order $3n$ constructed from B. Then any Bose resolution of B induces a resolution of D.*

Proof Let $C_{x,y,i}$ and A_x denote the two block types of D. Suppose that the Bose resolution is an upper triangular resolution with LSPCs $\{\mathcal{L}_1, \ldots, \mathcal{L}_{(3n-3)/2}\}$. Then

$$\mathcal{P}_j = \bigcup_{(x,y,z)\in\mathcal{L}_j, i\in\mathbb{Z}_3} C_{x,y,i}$$

is a parallel class of D for all $j \in [1, (3n-3)/2]$. The set $\mathcal{A} = \bigcup_{x\in[0,n-1]} A_x$ is a parallel class of D, and thus the set $\{\mathcal{P}_1, \ldots, \mathcal{P}_{(3n-3)/2}, \mathcal{A}\}$ is a resolution of D.

Now suppose that the Bose resolution of B is of the second kind, and let $\mathcal{L} = \{\mathcal{L}_1, \ldots, \mathcal{L}_p\}$ denote its set of LSPCs and $P = \{P_1, \ldots, P_q\}$ its set of all proper PLSPCs having a partition $\pi = \{\pi_1, \ldots, \pi_q\}$ of $[0, n-1]$ and a bijection $\rho : P \to \pi$ such that $P_{i_{\{\}}} \cup \rho(P_i)$ is a partition of $[0, n-1]$ for all $i \in [1, q]$. Then both

$$\mathcal{P}_j = \bigcup_{(x,y,z)\in\mathcal{L}_j, i\in\mathbb{Z}_3} C_{x,y,i}$$

and

$$\mathcal{P}'_k = \left(\bigcup_{(x,y,z)\in P_j, i\in\mathbb{Z}_3} C_{x,y,i}\right) \bigcup \left(\bigcup_{x\in\rho(P_j)} A_x\right)$$

are parallel classes of D for all $j \in [1, p]$ and all $k \in [1, q]$ and thus

$$\mathcal{R} = \left(\bigcup_{j\in[1,p]} \mathcal{P}_j\right) \bigcup \left(\bigcup_{k\in[1,q]} \mathcal{P}'_k\right)$$

is a resolution of D. □

3 Constructing Bose Resolutions

3.1 Bose Square Properties

To construct Bose resolutions, we require several properties of the averaging latin square B. Henceforth, unless otherwise stated, n denotes the order of the averaging latin square B and C_k denotes the cells of the k-diagonal pair of B.

Property 1 The set of symbols of the cells of $C_k(B)$ is equal to $[0, n-1]$.

Proof Suppose that $c = (0, a, b) \in C_k(B)$. Then $b \equiv a \cdot \frac{n+1}{2} \pmod n$. For all $i \in [0, n-1]$, $\frac{n+1}{2}(i + (a+i)) \equiv \frac{n+1}{2}(a+2i) \pmod n \equiv a\frac{n+1}{2} + (n+1)i \pmod n \equiv b + i \pmod n$, and hence $(i, a+i, b+i) \pmod n \in C_k(B)$ if $i \in [0, n-k]$ and $(a+i, i, b+i) \pmod n \in C_k(B)$ if $i \in [n-k+1, n-1]$. That is, $C_k(B)$ is obtained by additively developing c over $\mathbb{Z}_n$ (and permuting the first two coordinates of $c + i \pmod n$ for all $i \in [n-k+1, n-1]$). □

Let $\{i, j\} = \{k, n-k\}$, and consider a cell $c = (x, y, z) \in C_k(B)$ belonging to the j-diagonal of B. The *next adjacent cell of* c, denoted $c \oplus 1$, is the cell $(x+1, y+1, z+1) \pmod n$ if $y < n-1$ or the cell $(0, x+1, z+1) \pmod n = (0, i, z+1) \pmod n$ if $y = n-1$. In plain English, the next adjacent cell after the bottom-most cell of the j-diagonal is the topmost cell of the i-diagonal. Extend $\oplus$ to all $\alpha \in \mathbb{N}$, defining $c \oplus 0 = c$ and $c \oplus \alpha = (c \oplus 1) \oplus (\alpha - 1)$ if $\alpha \geq 2$.

A triple $T = \{a, b, c\} \subset \mathbb{Z}_n$ is a *d-regular triple* if the elements of T can be permuted such that the second element minus the first element is equivalent to the third element minus the second element modulo n. For example, if $b - a \equiv c - b \equiv d \pmod n$, then arranging T as (a, b, c) certifies that T is d-regular. Any d-regular triple $\{a, b, c\}$ is also $(n-d)$-regular, since if $b - a \equiv c - b \equiv d \pmod n$, then $a - b \equiv b - c \equiv n - d \pmod n$, so that permuting T to get (c, b, a) certifies that T is $(n-d)$-regular.

Property 2 For each $d \in [1, n-1]$, define

$$\mathcal{T}_d = \{T_d : T_d \in \binom{[0, n-1]}{3} \text{ and } T_d \text{ is } d\text{-regular}\}.$$

Then for all $k \in [1, (n-1)/2]$, $\mathcal{T}_{k - k\frac{n+1}{2}} = \{c_{\{\}} : c \in C_k(B)\}$.

Proof Because $(0, k, \ell) \in C_k(B)$ when $\ell \equiv k \cdot (\frac{n+1}{2}) \pmod n$,

$$2\ell \equiv k(n+1) \pmod n \iff 2\ell \equiv k \pmod n \iff \ell \equiv k - \ell \pmod n. \tag{1}$$

Hence $\{0, \ell, k\}$ is ℓ-regular; developing it over $\mathbb{Z}_n$ gives $\mathcal{T}_\ell$ and

$$\begin{aligned} C_\ell(B) = &\{(i, k+i, \ell+i) : i \in [0, n-k-1]\} \\ &\cup \{(k+j, j, \ell+j) : j \in [n-k, n-1]\}, \end{aligned} \tag{2}$$

where addition is performed modulo n. □

Property 3 Let $C_k(B)$ denote an arbitrary diagonal pair with $c_0 = (0, k, \ell) \in C_k(B)$. Then for all $c \in C_k(B)$, $c_{\{\}}$ is d-regular only if $d \equiv \pm\ell \pmod n$.

Proof Suppose to the contrary that for some $d \in [1, n-1] - \{\ell, -\ell\} \pmod n$ and some $c \in C_k(B)$, $c_{\{\}}$ is d-regular. Write $c_{\{\}} = \{x, x+d, x+2d\} \pmod n$ for some $x \in [0, n-1]$. By Property 2, some $c' \in C_k(B)$ has $c'_{\{\}} = \{0, d, 2d\}$. Because cells in a diagonal pair occur above the main diagonal of B, there are three possible permutations of $c'_{\{\}}$, namely $(0, d, 2d)$, $(0, 2d, d)$, and $(d, 2d, 0)$. However, the first is not feasible, because $d(n+1)/2 \equiv 2d \pmod{*}n \iff d \equiv 0 \pmod{*}n$. The third is not feasible, because $3d(n+1)/2 \equiv 0 \pmod{*}n \iff d \equiv 0 \pmod{*}n$. Finally, the second is not feasible, since otherwise $d = \ell$. □

For each $k \in [1, (n-1)/2]$ define $\delta(C_k(B)) = \min\{\ell \pmod{n}, n - \ell \pmod{n}\}$, with $(0, k, \ell) \in C_k(B)$, so that $\delta(C_k(B))$ is the unique element of $[1, (n-1)/2]$ such that for all $c \in C_k(B)$, $c_{\{\}}$ is $\delta(C_k(B))$-regular.

Property 4 For distinct $k_1, k_2 \in [1, (n-1)/2]$, $\delta(C_{k_1}(B)) \neq \delta(C_{k_2}(B))$.

Proof Suppose to the contrary that $d = \delta(C_{k_1}(B)) = \delta(C_{k_2}(B)) \pmod{n}$. Then by Property 2, some permutation of $T = \{0, d, 2d\}$ is a cell of C_{k_1} and some (other) permutation of T is a cell of C_{k_2}. As the cells of both diagonal pairs occur above the main diagonal of B, there are three possible permutations of T that yield a cell (in C_{k_1} or C_{k_2}), namely $(0, d, 2d)$, $(0, 2d, d)$, and $(d, 2d, 0)$. As discussed before, the first and third are not feasible. Hence, the second must occur in two distinct diagonal pairs of B, which is impossible. □

Property 5 For all $d \in [0, (n-1)/2]$ and any $T_d \in \binom{[0,n-1]}{3}$ that is d-regular, there exists exactly one cell c above the main diagonal of B such that $c_{\{\}} = T_d$.

Proof This follows from Properties 3 and 4. □

Property 6 Let $n \equiv 0 \pmod{3}$ and $c \in C_k(B)$. Then if $k \equiv 0 \pmod{3}$, the elements of c are pairwise equivalent modulo 3; otherwise, the elements of c are pairwise inequivalent modulo 3.

Proof Consider $c = (0, k, \ell) \in C_k$. It suffices to prove that c has the desired property, because C_k is obtained by additively developing c over $\mathbb{Z}_n$. The result follows by (1) of Property 2. □

3.2 Constructions

Lemma 1 *If $n = 3p$, p an odd prime, and $k \not\equiv 0 \pmod{3}$, C_k can be partitioned into three LSPCs.*

Proof Put $c = (0, k, B_{0,k})$. Partition the cells of the k-diagonal pair of B into three sets, $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$, where

$$\mathcal{L}_h = \bigcup_{i \in [0,n-1] \text{ and } i \equiv 0 \pmod{*}3} c \oplus (h+i) \tag{3}$$

for all $h \in \{0, 1, 2\}$. Each $\mathcal{L}_h$ is an LSPC by Property 6. □

Before handling other diagonal pairs in general, we provide a Bose resolution for the averaging latin square of order 15.

Theorem 2 *The averaging latin square B of order 15 admits a Bose resolution.*

Proof We give a Bose resolution of the second kind. First form the LSPCs:

$$\begin{gathered}
\{(0,3,9),(1,7,4),(2,5,11),(12,14,13),(6,10,8)\}\\
\{(6,9,0),(7,13,10),(8,11,2),(3,5,4),(1,12,14)\}\\
\{(0,12,6),(2,14,8),(4,13,1),(9,11,10),(3,7,5)\}\\
\{(3,6,12),(4,10,7),(5,8,14),(0,2,1),(9,13,11)\}\\
\{(1,10,13),(9,12,3),(11,14,5),(6,8,7),(0,4,2)\}\\
\{(1,3,2),(4,6,5),(7,9,8),(10,12,11),(0,13,14)\}\\
\{(2,4,3),(5,7,6),(8,10,9),(11,13,12),(1,14,0)\},\ \text{and}\\
\{(3,6,4),(5,9,7),(8,12,10),(0,11,13),(3,14,1)\}.
\end{gathered}$$

Partition each of $C_1(B)$, $C_5(B)$, and $C_7(B)$ into LSPCs using Lemma 1 to get the remaining LSPCs. The set of cells of B above the main diagonal that do not occur in any of the LSPCs partitions into five PLSPCs, each of size four. For each PLSPC, we adjoin the 3-set of the points not covered by the PLSPC, and note that these 3-sets partition [0, 14]:

$$\begin{gathered}
\{(3,9,6),(4,7,13),(5,11,8),(10,14,12)\}\ \text{and}\ \{0,1,2\};\\
\{(6,12,9),(7,10,1),(8,14,11),(2,13,0)\}\ \text{and}\ \{3,4,5\};\\
\{(0,9,12),(2,11,14),(10,13,4),(1,5,3)\}\ \text{and}\ \{6,7,8\};\\
\{(1,13,7),(3,12,0),(5,14,2),(4,8,6)\}\ \text{and}\ \{9,10,11\};\\
\{(0,6,3),(1,4,10),(2,8,5),(7,11,9)\}\ \text{and}\ \{12,13,14\}.
\end{gathered}$$

□

Organizing diagonal pairs $C_k(B)$ for which $k \equiv 0 \pmod 3$ into LSPCs requires more elaborate methods.

Lemma 2 *Suppose that $n = 3p$ with $p > 5$ prime and that $C_k(B)$ is a diagonal pair with cell $c = (0, k, \ell)$ where $k \equiv 0 \pmod 3$. Set $m' \equiv \ell/3\lfloor n/9\rfloor^{-1} \pmod p$. Then for $m \equiv 3m' \pmod n$, any $d \not\equiv 0 \pmod 3$ and $j \in \mathbb{Z}_3$,*

$$P_{k,d,j} = \bigcup_{h\in[0,\lfloor n/9\rfloor-1],\, i\in\mathbb{Z}_3} c \oplus (hm + id + j\ell) \tag{4}$$

is a PLSPC of size $3\lfloor n/9\rfloor$ such that for distinct $j, j' \in \mathbb{Z}_3$, $P_{k,d,j} \cap P_{k,d,j'} = \emptyset$.

Proof Because $d \not\equiv 0 \pmod 3$, by Property 6, $P_{k,d,j}$ is a PLSPC provided that $P' \subset P_{k,d,j}$, given by

$$P' = \bigcup_{h \in [0, \lfloor n/9 \rfloor - 1]} c \oplus (hm + j\ell),$$

is a PLSPC. To establish this, consider a solution m' to the system of equations

$$\begin{cases} \ell/3 \equiv m' \lfloor n/9 \rfloor \pmod{p} & \text{(5a)} \\ k/3 \equiv 2m' \lfloor n/9 \rfloor \pmod{p}. & \text{(5b)} \end{cases}$$

Putting $m \equiv 3m' \pmod{n}$, the multiples of m modulo n are pairwise distinct (because m' generates $\mathbb{Z}_p$), and thus

$$S_\ell = \{\lfloor n/9 \rfloor m = \ell, (\lfloor n/9 \rfloor + 1)m, \ldots, (2\lfloor n/9 \rfloor - 1)m\}$$

gives the set of symbols of the cells of P', while the remaining multiples of m modulo n give the rows and columns. The system (5) appears to be overdetermined, but subtracting the first equation from the second, and noting that $\ell \equiv k - \ell \pmod{n}$ by (1) of Property 2, $\ell/3 \equiv k/3 - \ell/3 \equiv m' \lfloor n/9 \rfloor \pmod{p}$, so that $m' \equiv \ell/3 \lfloor n/9 \rfloor^{-1} \pmod{p}$ satisfies (5).

Finally, suppose that there exists some $(x_1, x_2, x_3) \in P_{k,d,j} \cap P_{k,d,j'}$, with $j, j' \in \{0, 1, 2\}$ distinct, and suppose without loss of generality that $x_3 \equiv 0 \pmod{3}$. Then there exist $(y_1, y_2, y_3), (z_1, z_2, z_3) \in P'$ (and hence $y_3, z_3 \in S_\ell$) such that $x_3 \equiv y_3 + j\ell \equiv z_3 + j'\ell \pmod{n}$. Supposing that $j > j'$, $(j - j')\ell \equiv z_3 - y_3 \pmod{n}$, and thus either

$$z_3 - y_3 \equiv \lfloor n/9 \rfloor m \quad (\text{mod } *)n, \text{ or}$$
$$z_3 - y_3 \equiv 2\lfloor n/9 \rfloor m \quad (\text{mod } *)n.$$

But neither $\lfloor n/9 \rfloor m$ nor $2\lfloor n/9 \rfloor m$ is a possible difference (modulo n) of any two elements of S_ℓ. □

Given $n = 3p$, $p > 5$ prime, and $k \equiv 0 \pmod{3}$, a PLSPC $P \subset C_k$ is *β-completable*, $\beta \in [1, (n-1)/2]$, if there exists some subset $S \subset C_\beta$, $|S| = n/3 - |P|$, such that $P' = P \cup S$ is an LSPC. Such a set S is a *β-completing set* for P.

Theorem 3 *Suppose that $n = 3p$, $p > 5$ prime, $k \equiv 0 \pmod{3}$, and $c_0 = (0, k, \ell) \in C_k$. For $m' \equiv \ell/3 \lfloor n/9 \rfloor^{-1} \pmod{p}$, $m \equiv 3m' \pmod{n}$, $d \not\equiv 0 \pmod{3}$ and $j \in \mathbb{Z}_3$, form the PLSPC $P_{k,d,j}$ as in (4). Let $N = C_k \setminus \bigcup_{j \in \mathbb{Z}_3} P_{k,d,j}$ and $\beta = \min\{2d \pmod{n}, n - 2d \pmod{n}\}$. Then there exist four sets S_0, S_1, S_2, and S_N such that S_i is a β-completing set for $P_{k,d,i}$ for $i \in \mathbb{Z}_3$, S_N is a β-completing set for N, and $(S_0 \cup S_1 \cup S_2 \cup S_N) \subset C_\beta$ is an LSPC of type (3).*

Proof For $j \in \{0, 1, 2\}$ define $Y_j = \bigcup_{c \in P_{k,d,j}} c_{\{\}}$. Then $N_j = [0, n-1] \setminus Y_j$ partitions into three sets $N_{j,0}$, $N_{j,1}$, and $N_{j,2}$ such that $|N_{j,0}| = |N_{j,1}| = |N_{j,2}|$ and $N_{j,i}$ is the set of all elements of N_j congruent to i modulo 3. Suppose that $x \in N_{j,0}$; then we claim that $x + hd \pmod{n} \in N_{j,hd\ (\text{mod } *)3}$ for all $h \in \{1, 2\}$. Suppose instead that $x + hd$

(mod n) $\in Y_j$ for some $h \in \{1, 2\}$; then there exists some $i \in [0, \lfloor n/9 \rfloor - 1]$ such that the cell $c_0 \oplus (hd + im + j\ell)$ has $x + hd \pmod n$ as one of its coordinates. But then the cell $c = c_0 \oplus (im + j\ell)$ satisfies $c \in P_{k,d,j}$ and has x as one of its coordinates, implying that $x \in Y_j$, a contradiction. Thus, by (2) of Property 2,

$$\begin{aligned} S_j =&\{(x, x + 2d, x + d) : x \in N_{j,0} \text{ and } x \in [0, n - \beta - 1]\} \\ &\cup\{(x + 2d, x, x + d) : x \in N_{j,0} \text{ and } x \in [n - \beta, n - 1]\} \end{aligned}$$

is a β-completing set for $P_{k,d,j}$.

Next we derive a β-completing set S_N for $N = C_k \setminus \bigcup_{j\in\mathbb{Z}_3} P_{k,d,j}$. We treat two cases:

$n \equiv 3 \pmod{18}$: Then $p \equiv 1 \pmod 6$, and so $|Y| = 9\lfloor n/9 \rfloor = 9(p - 1)/3 = n - 3$. Thus, $|N| = 3$. Moreover, since $d \not\equiv 0 \pmod 3$ and $m \equiv 0 \pmod 3$, $N = \{c_0, c_1, c_2\}$ such that the row, column, and symbol of cell c_i are equivalent to i modulo 3 for $i \in \{0, 1, 2\}$. Hence, N is a PLSPC. Further, we claim that $c_i = c_0 \oplus id$ for all $i \in \{1, 2\}$. For suppose to the contrary that $c_0 \oplus i'd \in Y$ for some $i' \in \{1, 2\}$. Then there must exist some $h \in [0, \lfloor n/9 \rfloor - 1]$ and $j \in \mathbb{Z}_3$ such that $c_0 \oplus i'd = c \oplus (hm + i'd + j\ell)$. But then $c_0 \in Y$, since $c_0 = c \oplus (hm + j\ell)$, a contradiction.

$n \equiv 15 \pmod{18}$: Then $p \equiv 5 \pmod 6$, and so $|Y| = 9\lfloor n/9 \rfloor = 9(n/3 - 2)/3 = n - 6$. Thus, $|N| = 6$. Since, $d \not\equiv 0 \pmod 3$ and $m \equiv 0 \pmod 3$, $N = \{c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}\}$ such that the row, column, and symbol of the cell $c_{i,j}$ are equivalent to i modulo 3 for all $i \in \mathbb{Z}_3$ and $j \in \{0, 1\}$. Using the same argument as in the previous case, without loss of generality $c_{i,j} = c_{0,j} \oplus id$ for all $i \in \{1, 2\}$, $j \in \{0, 1\}$. Now, using (2) of Property 2,

$$\begin{aligned} \mathcal{L} =&\{(x, x + 2d, x + d) : x \in [0, n - \beta - 1] \text{ and } x \equiv 0 \quad (\text{mod } *)3\} \\ &\cup\{(x + 2d, x, x + d) : x \in [n - \beta, n - 1] \text{ and } x \equiv 0 \quad (\text{mod } *)3\}, \end{aligned}$$

with addition performed modulo n, is an LSPC such that $\mathcal{L} \subset C_\beta$.

Hence, whether $n \equiv 3 \pmod{18}$ or $n \equiv 15 \pmod{18}$, $S_N = N \setminus \mathcal{L}$ is a β-completing set for N.

Next we show that $\mathcal{S} = S_0 \cup S_1 \cup S_2 \cup S_N$ is an LSPC. Define $Y' = \bigcup_{c\in N} c_{\{\}}$, $N_j = [0, n - 1] \setminus Y_j$, for $j \in \mathbb{Z}_3$, and $N' = [0, n - 1] \setminus Y'$. By Lemma 2, $|Y_j| = 9\lfloor n/9 \rfloor$ and if $n \equiv 3 \pmod{18}$, $|Y'| = 9$; on the other hand, if $n \equiv 15 \pmod{18}$, $|Y'| = 18$. In either case, $|Y_j| + |Y'| > n$ and thus by the pigeonhole principle, $[0, n - 1] = Y_j \cup Y'$, so that $N_j \cap N' = \emptyset$ for all $j \in \mathbb{Z}_3$. Now suppose there exists some $x \in [0, n - 1]$ such that $x \notin N_0 \cup N_1 \cup N_2 \cup N'$. Then $x \in Y_0 \cap Y_1 \cap Y_2 \cap Y'$, which is impossible since by Property 1, x can only occur in at most three cells of C_k. Hence, $[0, n - 1]$ admits a partition into sets N_0, N_1, N_2, and N', implying that $\mathcal{S}$ is an LSPC. Moreover, the symbol of every cell of $\mathcal{S}$ is congruent to $d \pmod 3$ and thus $\mathcal{S}$ is of the form (3), as desired. □

Example 1 (*for Theorem* 3) Suppose that $n = 21$ and $k = 3$, so that $(0, 3, 12) \in C_3$. Then $m' = 2$ and $m = 6$ (see Lemma 2). Putting $d = 1$, for each $j \in \mathbb{Z}_3$ we get a PLSPC $P_{3,1,j} \subset C_3$ of type (4):

$$P_{3,1,0} = \{(0, 3, 12), (1, 4, 13), (2, 5, 14), (6, 9, 18), (7, 10, 19), (8, 11, 20)\},$$
$$P_{3,1,1} = \{(12, 15, 3), (13, 16, 4), (14, 17, 5), (0, 18, 9), (1, 19, 10), (2, 20, 11)\}, \text{ and}$$
$$P_{3,1,2} = \{(3, 6, 15), (4, 7, 16), (5, 8, 17), (9, 12, 0), (10, 13, 1), (11, 14, 2)\}.$$

The remaining set of cells of C_3 is $N = \{(15, 18, 6), (16, 19, 7), (17, 20, 8)\}$. Compute $\beta = 2$. The 2-completing sets for the $P_{3,1,j}$s and N are

$$\begin{aligned} S_0 &= \{(15, 17, 16)\} \\ S_1 &= \{(6, 8, 7)\} \\ S_2 &= \{(18, 20, 19)\}, \text{ and} \\ S_N &= \{(0, 2, 1), (3, 5, 4), (9, 11, 10), (12, 14, 13)\}. \end{aligned}$$

Then $\mathcal{S} = (S_0 \cup S_1 \cup S_2 \cup S_N) \subset C_2$ is an LSPC of type (3), as desired.

Theorem 4 *For all $n = 3p$, $p > 5$ prime, the averaging latin square B of order n admits a Bose resolution.*

Proof Define $K = \{k \in [1, (n-1)/2] : k \equiv 0 \pmod 3\}$, and let $f : K \to \{d : d \in [1, (n-1)/2], d \not\equiv 0 \pmod 3\}$ be an arbitrary injective function. For each $k \in K$ and $j \in \mathbb{Z}_3$ let $P_{k,f(k),j}$ be a PLSPC of type (4). Then putting $\beta_k = \min\{2f(k) \pmod n, n - 2f(k) \pmod n\}$, apply Theorem 3 to obtain an LSPC $\mathcal{S}_{k,f(k)} = S_{k,0} \cup S_{k,1} \cup S_{k,2} \cup S_{N_k}$, where for $j \in \mathbb{Z}_3$ $S_{k,j}$ is a β_k-completing set for $P_{k,f(k),j}$ and S_{N_k} is a β_k-completing set for $N_k = C_k \setminus \bigcup_{j \in \mathbb{Z}_3} P_{k,f(k),j}$. Then for each $k \in K$ and $j \in \mathbb{Z}_3$, the two sets $L_{k,j} = P_{k,f(k),j} \cup S_{k,j}$ and $L_{N_k} = N_k \cup S_{N_k}$ are each LSPCs. Moreover, if $c = (0, \beta_k, B_{0,\beta_k})$, then for each $i \in \mathbb{Z}_3$, define i' to be the least positive residue of $(f(k) + i)$ modulo 3, so that

$$\mathcal{S}'_{k,i'} = \bigcup_{y \in [0,n-1] \text{ and } y \equiv 0 \pmod{*} 3} c \oplus (i' + y)$$

(note that $\mathcal{S}'_{k,0} = \mathcal{S}_{k,f(k)}$) are LSPCs of type (3) that partition $C_{\beta_k}(B)$. Finally, for all $d' \not\equiv 0 \pmod 3$ *not* in the range $f(K)$ of f, put $\beta_{d'} = \min\{2d' \pmod n, n - 2d' \pmod n\}$ and applying Lemma 1, partition $C_{\beta_{d'}}(B)$ into three LSPCs $L'_{\beta_{d'},0}$, $L'_{\beta_{d'},1}$ and $L'_{\beta_{d'},2}$. The Bose resolution $\mathcal{R}$ is

$$\left[\bigcup_{k \in K} \{L_{k,0}, L_{k,1}, L_{k,2}, L_{N_k}, \mathcal{S}'_{k,1}, \mathcal{S}'_{k,2}\}\right] \bigcup \left[\bigcup_{\substack{d' \not\equiv 0 \pmod{*} 3 \\ \text{and } d' \notin f(K)}} \{L'_{\beta_{d'},0}, L'_{\beta_{d'},1}, L'_{\beta_{d'},2}\}\right].$$

□

Example 2 (*for Theorem* 4) For the averaging latin square B of order 21, $K = \{3, 6, 9\}$, and we choose the injection $f : K \to \{1, 2, 4, 5, 7, 8, 10\}$ as $f(3) = 1$, $f(6) = 2$, and $f(9) = 4$. For $(0, 6, 3) \in C_6$, $m' = 4$ and $m = 12$, so that

$$P_{6,f(6),0} = \{(0, 6, 3), (2, 8, 5), (4, 10, 7), (9, 15, 12), (11, 17, 14), (13, 19, 16)\},$$
$$P_{6,f(6),1} = \{(3, 9, 6), (5, 11, 8), (7, 13, 10), (12, 18, 15), (14, 20, 17), (1, 16, 19)\}, \text{ and}$$
$$P_{6,f(6),2} = \{(6, 12, 9), (8, 14, 11), (10, 16, 13), (0, 15, 18), (2, 17, 20), (4, 19, 1)\}.$$

The remaining set of cells of C_6 is $N_6 = \{(1, 7, 4), (3, 18, 0), (5, 20, 2)\}$. Now $\beta_6 = 4$, and the corresponding 4-completing sets for the $P_{6,f(6),j}$'s and N_6 are

$$\begin{aligned}
S_{6,0} &= \{(1, 18, 20)\}\\
S_{6,1} &= \{(0, 4, 2)\}\\
S_{6,2} &= \{(3, 7, 5)\}, \text{ and}\\
S_{N_6} &= \{(6, 10, 8), (9, 13, 11), (12, 16, 14), (15, 19, 17)\},
\end{aligned}$$

Form the LSPCs $L_{6,j} = P_{6,f(6),j} \cup S_{6,j}$ for $j \in \mathbb{Z}_3$, and apply Lemma 1 to partition $C_6 \setminus (S_{6,0} \cup S_{6,1} \cup S_{6,2} \cup S_{N_6})$ into two LSPCs $\mathcal{S}'_{6,1}$ and $\mathcal{S}'_{6,2}$. For $(0, 9, 15) \in C_9$, $m' = 6$ and $m = 18$, so that

$$P_{9,f(9),0} = \{(0, 9, 15), (4, 13, 19), (8, 17, 2), (6, 18, 12), (1, 10, 16), (5, 14, 20)\},$$
$$P_{9,f(9),1} = \{(3, 15, 9), (7, 19, 13), (2, 11, 17), (0, 12, 6), (4, 16, 10), (8, 20, 14)\}, \text{ and}$$
$$P_{9,f(9),2} = \{(9, 18, 3), (1, 13, 7), (5, 17, 11), (6, 15, 0), (10, 19, 4), (2, 14, 8)\}.$$

The remaining set of cells of C_9 is $N_9 = \{(3, 12, 18), (7, 16, 1), (11, 20, 5)\}$. $\beta_9 = 8$, and the corresponding 8-completing sets for the $P_{9,f(9),j}$'s and N_9 are

$$\begin{aligned}
S_{9,0} &= \{(3, 11, 7)\},\\
S_{9,1} &= \{(5, 18, 1)\},\\
S_{9,2} &= \{(12, 20, 16)\}, \text{ and}\\
S_{N_9} &= \{(0, 8, 4), (6, 14, 10), (9, 17, 13), (2, 15, 19)\}.
\end{aligned}$$

Form the LSPCs $L_{9,j} = P_{9,f(9),j} \cup S_{9,j}$ for $j \in \mathbb{Z}_3$, and apply Lemma 1 to partition $C_9 \setminus (S_{9,0} \cup S_{9,1} \cup S_{9,2} \cup S_{N_9})$ into two LSPCs $\mathcal{S}'_{9,1}$ and $\mathcal{S}'_{9,2}$. Treat C_3 as in Example 1 to obtain three LSPCs $L_{3,0}$, $L_{3,1}$, and $L_{3,2}$ and the two remaining type (3) LSPCs $\mathcal{S}'_{3,1}, \mathcal{S}'_{3,2}$ of C_2. Finally, partition diagonal pairs $C_{k'}$ for $k' \in \{1, 5, 7, 10\}$ into LSPCs $L'_{k',0}$, $L'_{k',1}$, and $L'_{k',2}$ by applying Lemma 1 to each. The Bose resolution is

$$\mathcal{R} = \left[\bigcup_{k \in K} \{L_{k,0}, L_{k,1}, L_{k,2}, L_{N_k}, \mathcal{S}'_{k,1}, \mathcal{S}'_{k,2}\}\right] \bigcup \left[\bigcup_{k' \in \{1,5,7,10\}} \{L'_{k',0}, L'_{k',1}, L'_{k',2}\}\right].$$

3.3 Main Result

Suppose that $n = 3p$ with $p \geq 5$ a prime. Applying Theorem 2 when $p = 5$ and Theorem 4 otherwise, we obtain a Bose resolution to which we apply Theorem 1 to obtain a resolution of the corresponding Bose-averaging triple system, thus establishing our main result.

Theorem 5 *Any Bose-averaging triple system of order $3n$, with $n = 3p$ such that $p \geq 5$ is a prime, is resolvable.*

4 Concluding Remarks

Choices in the proof of Theorem 4 lead to many distinct Bose resolutions for a square of order $3p$ with $p > 5$ prime. Indeed, each of the $\frac{(2\lceil n/6\rceil)!}{\lceil n/6\rceil!}$ distinct injections $f : K \to \{d : d \in [1, (n-1)/2], d \not\equiv 0 \pmod 3\}$ produces a distinct Bose resolution of the square. Via Theorem 1, each Bose resolution produces a distinct (but perhaps isomorphic) resolution of the corresponding Bose-averaging triple system of order $9p$. This flexibility may prove useful in selecting a Kirkman triple system that has further desired properties.

Due to length restrictions, we have focused on averaging latin square s of order $3p$ for $p \geq 5$ a prime, and hence on Bose-averaging triple system s of order $9p$. In a longer version [13], we complete the characterization by showing that (1) no Bose triple system of order $3n$, Bose-averaging or otherwise, can be resolvable unless $n \equiv 3 \pmod 6$, and (2) the Bose-averaging triple system of order $3n$ is resolvable whenever $n \equiv 3 \pmod 6$.

Acknowledgements This work is supported by NSF grant CCF 1816913 (CJC).

References

1. Bose, R.C.: On the construction of balanced incomplete block designs. Ann. Eugen. **9**(4), 353–399 (1939)
2. Brummond, W.M.: Kirkman triple systems of order n with minimum block sum equal to n, for access balancing in distributed storage. Ph.D. thesis, University of Missouri-Saint Louis (2019)
3. Chee, Y.M., Colbourn, C.J., Dau, H., Gabrys, R., Ling, A.C.H., Lusi, D., Milenkovic, O.: Access balancing in storage systems by labeling partial Steiner systems. Designs Codes Cryptogr. **88**(11), 2361–2376 (2020)
4. Colbourn, C.J., Dinitz, J.H.: Handbook of Combinatorial Designs. CRC press (2006)
5. Colbourn, C.J., Rosa, A.: Triple Systems. Oxford University Press (1999)
6. Dau, H., Milenkovic, O.: MaxMinSum Steiner systems for access balancing in distributed storage. SIAM J. Discrete Math. **32**(3), 1644–1671 (2018)

7. El Rouayheb, S., Ramchandran, K.: Fractional repetition codes for repair in distributed storage systems. In: 2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1510–1517. IEEE (2010)
8. Fazeli, A., Vardy, A., Yaakobi, E.: Codes for distributed PIR with low storage overhead. In: 2015 IEEE International Symposium on Information Theory (ISIT), pp. 2852–2856. IEEE (2015)
9. Furino, S., Miao, Y., Yin, J.: Frames and Resolvable Designs: Uses, Constructions and Existence, vol. 3. CRC Press (1996)
10. Kirkman, T.P.: On a problem in combinations. Cambr. Dublin Math. J. **2**, 191–204 (1847)
11. Kirkman, T.P.: Query VI on "fifteen young ladies....". Lady's and Gentleman's Diary (148), 48 (1851)
12. Lu, J.: Collected Works of Lu Jiaxi on Combinatorial Designs. Inner Mongolia People's Press (1990)
13. Lusi, D., Colbourn, C.J.: The spectrum of resolvable Bose triple systems. Discrete Math. **346**(7), Paper No. 113,396, 9 (2023)
14. Ray-Chaudhuri, D.K., Wilson, R.M.: Solution of Kirkman's schoolgirl problem. In: Proceedings of Symposia in Pure Mathematics, vol. 19, pp. 187–203 (1971)
15. Reid, C., Rosa, A.: Steiner systems $s(2, 4, v)$-a survey. Electr. J. Comb. DS18–Feb (2012)
16. Silberstein, N., Etzion, T.: Optimal fractional repetition codes based on graphs and designs. IEEE Trans. Inf. Theory **61**(8), 4164–4180 (2015)
17. Silberstein, N., Gál, A.: Optimal combinatorial batch codes based on block designs. Designs Codes Cryptogr. **78**(2), 409–424 (2016)
18. Stinson, D.R.: Combinatorial Designs: Constructions and Analysis. Springer Science & Business Media (2007)

Decomposition of the Johnson Graphs into Graph-Pairs of Order 4

Atif Abueida and Mike Daven

Abstract A *graph-pair of order t* is a pair of graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. The Johnson graph $J(v, n)$ is the graph whose vertices are the n-element subsets of a v-element set, and two vertices are adjacent if the intersection of the corresponding subsets contains $n - 1$ elements. We show necessary and sufficient conditions for $J(v, 2)$ to admit a decomposition into graph-pairs of order 4.

Keywords Decompositions · Graph designs · Johnson graph

1 Introduction

The graph decomposition problem has an interesting history; see [5, 6]. Among the earliest results concerning graph decomposition is the problem of partitioning the edges of the complete graph into cycles. For instance, when the cycles each have length 3, the resulting decomposition of K_m is known as a Steiner triple system of order m or $STS(m)$. Cycle systems with larger cycle length have been well studied. Other well-known problems include decomposing the complete graph into hamilton cycles and other factors. Some of the early work in this area can be found in [4, 7–9].

We are particularly interested in decomposition problems involving two or more non-isomorphic subgraphs. For instance, consider the following partition of K_6 into edge-disjoint copies of K_3 and $K_{1,3}$ (Fig. 1).

We will focus on decompositions which match the following definition.

Definition 1 A *graph-pair of order t* is a pair of graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$ (Figs. 2 and 3).

A. Abueida (✉)
University of Dayton, Dayton, OH 45469-2316, USA
e-mail: aabueida1@udayton.edu

M. Daven
Mount Saint Mary College, Newburgh, NY 12550, USA
e-mail: mike.daven@msmc.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_6

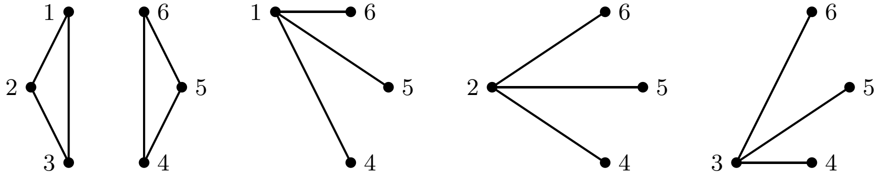


Fig. 1 K_6 can be partitioned into two copies of K_3 and three copies of $K_{1,3}$

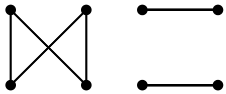


Fig. 2 It is clear that C_4 and $2K_2$ form the only graph-pair of order 4

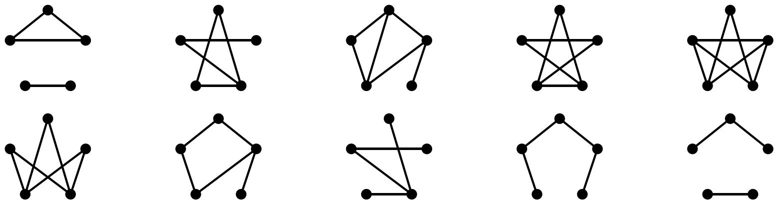


Fig. 3 There are five graph-pairs of order 5

In [1], the authors determined the complete spectrum for when the complete graph admits a decomposition into each of the graph-pairs of order 4 or 5. Consider the following:

Theorem 1 *There is a* $(C_4, 2K_2)$ *decomposition of* K_m *if and only if* $m = 0, 1 \bmod 4$ *with* $m \neq 5$. *Moreover, there is a* $(C_4, 2K_2)$*-packing (covering) of* K_m *with a leave (padding) consisting of a single edge whenever for* $m = 2, 3 \bmod 4$.

There has been a rich variety of work in the decomposition problem we have described. Other papers include decompositions of the complete graph, complete bipartite graph, or product graphs into some combination of paths, cycles, stars, Hamilton paths, or Hamilton cycles; see [2, 10, 11, 13–15, 17–19], for instance. In this paper, we begin to look at decompositions that begin with a different starting graph.

2 Johnson Graphs

Suppose v and n are integers satisfying $1 \leq n \leq v$. The Johnson graph $J(v, n)$ is the graph whose vertices are the n-element subsets of a v-element set, and two vertices are adjacent if the intersection of the corresponding subsets contains $n - 1$ elements.

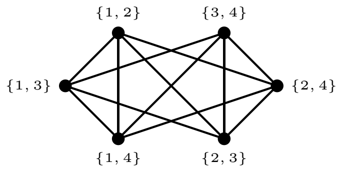


Fig. 4 Consider how the set $\{1, 2, 3, 4\}$ leads to the graph $J(4, 2)$

Note that $J(v, 2)$ has $\binom{v}{2}$ vertices, each of which has degree $2(v-2)$, so the number of edges is $(v-2)\binom{v}{2}$ (Fig. 4).

For other recent work involving the Johnson graphs, see [3, 12, 16]. In this paper, we aim to determine when $J(v, 2)$ admits a decomposition into copies of C_4 and $2K_2$.

3 The Main Result

Theorem 2 *There is a* $(C_4, 2K_2)$*-decomposition of* $J(v, 2)$ *when* $v \equiv 0, 1, 2$ *(mod 4) and* $v \geq 4$*. Moreover, for* $v \equiv 3$ *(mod 4) and* $v \geq 7$*, there is a* $(C_4, 2K_2)$*-packing (covering) of* $J(v, 2)$ *with a leave (padding) consisting of a single edge.*

Proof Let the vertices of $J(v, 2)$ be the 2-element subsets of the set $\{1, 2, \ldots, v\}$. Consider the small case $J(4, 2)$. This graph can be partitioned into two copies of C_4 and two copies of $2K_2$, as depicted below (Fig. 5).

Our intent is to use a recursive approach to construct a $(C_4, 2K_2)$-decomposition of $J(v, 2)$ using a $(C_4, 2K_2)$-decomposition of $J(v-1, 2)$. For $v \geq 5$, we will view $J(v, 2)$ as the union of three subgraphs.

- $L \cong K_{v-1}$ on the vertices $\{v, t\}$ for $t \in \{1, 2, \ldots, v-1\}$
- $R \cong J(v-1, 2)$ on the vertices $\{i, j\}$ such that $1 \leq i \neq j \leq v-1$

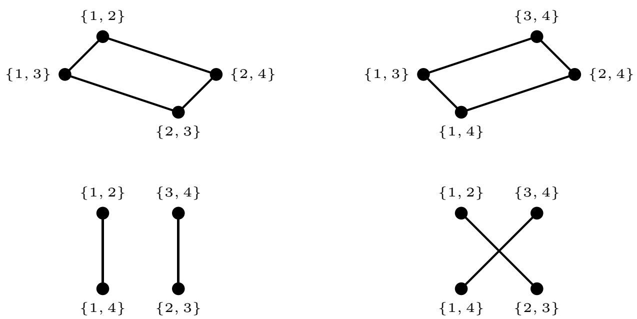


Fig. 5 The small case $J(4, 2)$

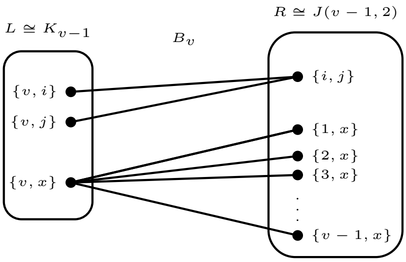


Fig. 6 $J(v, 2)$ can be viewed as the union of the three subgraphs L, B_v, and R

- The bipartite graph B_v with edges joining the vertex $\{i, j\} \in V(R)$ to the vertices $\{v, i\}$ and $\{v, j\} \in V(L)$ for $1 \leq x \neq y \leq v-1$.

These three subgraphs of $J(v, 2)$ are depicted in Fig. 6.

For $J(5, 2)$, consider each of the three subgraphs L, B_5 and R. The subgraph $L \cong K_4$ consists of one copy of C_4 and one copy of $2K_2$. The subgraph R consists of two copies of C_4 and two copies of $2K_2$. The subgraph B_5 can be partitioned into six copies of $2K_2$; here is one such partition:

$$\big\{\{1,2\},\{1,5\}\big\} \cup \big\{\{2,3\},\{2,5\}\big\} \qquad \big\{\{1,2\},\{2,5\}\big\} \cup \big\{\{2,4\},\{4,5\}\big\}$$

$$\big\{\{1,3\},\{1,5\}\big\} \cup \big\{\{2,3\},\{3,5\}\big\} \qquad \big\{\{1,3\},\{3,5\}\big\} \cup \big\{\{1,4\},\{4,5\}\big\}$$

$$\big\{\{1,4\},\{1,5\}\big\} \cup \big\{\{3,4\},\{4,5\}\big\} \qquad \big\{\{2,4\},\{2,5\}\big\} \cup \big\{\{3,4\},\{3,5\}\big\}$$

For $J(6, 2)$, we again consider each of the three subgraphs L, B_6 and R. The subgraph $L \cong K_5$ can easily be partitioned into five copies of $2K_2$. The subgraph R is isomorphic to the $J(5, 2)$ design listed above. The subgraph B_6 can be partitioned into ten copies of $2K_2$; here is one such partition:

$$\big\{\{1,2\},\{1,6\}\big\} \cup \big\{\{2,3\},\{2,6\}\big\} \qquad \big\{\{1,2\},\{2,6\}\big\} \cup \big\{\{3,5\},\{3,6\}\big\}$$

$$\big\{\{1,3\},\{1,6\}\big\} \cup \big\{\{3,4\},\{3,6\}\big\} \qquad \big\{\{1,3\},\{3,6\}\big\} \cup \big\{\{2,4\},\{4,6\}\big\}$$

$$\big\{\{1,4\},\{1,6\}\big\} \cup \big\{\{4,5\},\{4,6\}\big\} \qquad \big\{\{1,4\},\{4,6\}\big\} \cup \big\{\{2,4\},\{2,6\}\big\}$$

$$\big\{\{1,5\},\{1,6\}\big\} \cup \big\{\{2,5\},\{5,6\}\big\} \qquad \big\{\{1,5\},\{5,6\}\big\} \cup \big\{\{2,5\},\{2,6\}\big\}$$

$$\big\{\{2,3\},\{3,6\}\big\} \cup \big\{\{3,5\},\{5,6\}\big\} \qquad \big\{\{3,4\},\{4,6\}\big\} \cup \big\{\{4,5\},\{5,6\}\big\}$$

The number of edges in $J(v, 2)$ will give us some insight as to whether these edges can be partitioned into copies of C_4 and $2K_2$. Note that $J(v, 2)$ has an even number of edges when $v \equiv 0, 1, 2 \pmod 4$, so a decomposition into copies of C_4 and $2K_2$ is feasible. On the other hand, $J(v, 2)$ has an odd number of edges when

$v \equiv 3 \pmod 4$. In this case, the edges of $J(v, 2)$ cannot be partitioned into copies of C_4 and $2K_2$. Next, we will outline a recursive construction.

Consider what Theorem 1 tells us about $L \cong K_{v-1}$. If $v \equiv 1, 2 \pmod 4$, then $v - 1 \equiv 0, 1 \pmod 4$, and we can find a $(C_4, 2K_2)$-decomposition of $L \cong K_{v-1}$. If $v \equiv 2, 3 \pmod 4$, then $v - 1 \equiv 1, 2 \pmod 4$, and there is a $(C_4, 2K_2)$-packing (covering) of $L \cong K_{v-1}$ with a leave (padding) consisting of a single edge.

Now, consider $R \cong J(v - 1, 2)$. For $v \equiv 1, 2, 3 \pmod 4$, i.e. $v - 1 \equiv 0, 1, 2 \pmod 4$, we can recursively find a $(C_4, 2K_2)$-decomposition of $R \cong J(v - 1, 2)$. For $v \equiv 0 \pmod 4$, i.e. $v - 1 \equiv 3 \pmod 4$, there is a $(C_4, 2K_2)$-packing (covering) of $R \cong K_{v-1}$ with a leave (padding) consisting of a single edge. In the latter case ($v \equiv 0 \pmod 4$), the leaves (paddings) of the partitions of L and R each contain a single edge. These two edges can make a copy of $2K_2$.

It remains to show how we can partition the edges in B_v. We note that, in B_v, for $1 \leq i \neq j \leq v - 1$, every vertex $\{v, i\}$ in $L \cong K_{v-1}$ is adjacent to vertices $\{i, j\}$ in $R \cong J(v - 1, 2)$. Therefore, in B_v, the degree of every vertex $\{v, i\}$ in $L \cong K_{v-1}$ is $v - 2$. Meanwhile, in B_v, the degree of every vertex $\{i, j\}$ in $R \cong J(v - 1, 2)$ is 2 as that vertex is adjacent to vertices $\{v, i\}$ and $\{v, j\}$ in $L \cong K_{v-1}$. Hence, $|E(B_v)|$ is even.

Next, we greedily remove the maximum number of copies of $2K_2$ from the edges of B_v. As we are removing the maximal number, the only possible unused edges are either:

(a) two edges incident to a single vertex in $R \cong J(v - 1, 2)$, or
(b) an even number of edges (at least two edges) that are incident to a single vertex in $L \cong K_{v-1}$.

Otherwise, we can remove more copies of $2K_2$.

In either case, we can dismantle enough copies of $2K_2$'s in $R \cong J(v - 1, 2)$ and match these edges with the remaining unused edges of B_v to create copies of $2K_2$.

Therefore, for $v \geq 4$, we have partitioned the subgraphs $L \cong K_{v-1}$, B_v, and $R \cong J(v - 1, 2)$ into edge disjoint copies of C_4 and $2K_2$. In the case of $v \equiv 3 \pmod 4$, the leave (padding) is a single edge. □

4 Extensions and Future Directions

The Johnson graph has many interesting properties, leading to a number of relatively simple conclusions. For instance, $J(v, v - 2)$ is isomorphic to $J(v, 2)$ and the following result is a natural corollary of Theorem 2:

Corollary 1 *There is a $(C_4, 2K_2)$-decomposition of $J(v, v - 2)$ when $v \equiv 0, 1, 2$ (mod 4) and $v \geq 4$. Moreover, for $v \equiv 3$ (mod 4) and $v \geq 7$, there is a $(C_4, 2K_2)$-packing (covering) of $J(v, v - 2)$ with a leave (padding) consisting of a single edge.*

It would be interesting to explore the feasibility of $(C_4, 2K_2)$-decompositions of $J(v, k)$ for $3 \leq k \leq v - 3$.

The Johnson graphs are associated with another larger class of graphs. If v and n are integers satisfying $1 \leq n \leq v$, the Kneser graph $K(v, n)$ is the graph whose vertices correspond to the n-element subsets of a set of v elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. It follows that $J(v, 2)$ is isomorphic to $K^c(v, 2)$, the complement of the Kneser graph $K(v, 2)$, so we have the following:

Corollary 2 *There is a* $(C_4, 2K_2)$*-decomposition of* $K^c(v, 2)$ *when* $v \equiv 0, 1, 2$ *(mod 4) and* $v \geq 4$. *Moreover, for* $v \equiv 3$ *(mod 4) and* $v \geq 7$*, there is a* $(C_4, 2K_2)$*-packing (covering) of* $K^c(v, 2)$ *with a leave (padding) consisting of a single edge.*

It would also be interesting to see whether other variations of the Kneser graph would admit a $(C_4, 2K_2)$-decomposition.

In general, $J(v, 2)$ is the line graph of K_v. For instance, $J(5, 2)$ is also the line graph of K_5, and may also be viewed as the complement of the Peterson graph. Again referring to Theorem 2, we have:

Corollary 3 *There is a* $(C_4, 2K_2)$*-decomposition of* $L(K_v)$ *when* $v \equiv 0, 1, 2$ *(mod 4) and* $v \geq 4$. *Moreover, for* $v \equiv 3$ *(mod 4) and* $v \geq 7$*, there is a* $(C_4, 2K_2)$*-packing (covering) of* $L(K_v)$ *with a leave (padding) consisting of a single edge.*

We believe it would be worthwhile to explore $(C_4, 2K_2)$-decompositions of other line graphs.

References

1. Abueida, A., Daven, M.: Multidesigns for graphs-pairs on 4 and 5 vertices. Graphs & Combin. **19**(4), 433–447 (2003)
2. Abueida, A., Daven, M.: Multidecompositions of several graph products. Graphs & Combin. **29**(3), 315–326 (2013)
3. Alaeiyan, M., Abedi, A., Alaeiyan, M.: Perfect 3-colorings of the Johnson graph $J(6, 3)$. Bull. Iranian Math. Soc. **46**(6), 1603–1612 (2020)
4. Andersen, L.D.: Factorizations of graphs. In: Colbourn, C., Dinitz, J. (eds.) The CRC Handbook of Combinatorial Designs. CRC Press, Boca Raton (1996)
5. Bosák, J.: Decomposition of Graphs. Kluwer Academic Publishers, Boston (1990)
6. Lindner, C.C., Rodger, C.A.: Design Theory. CRC Press, Boca Raton (1997)
7. Lindner, C.C., Rodger, C.A.: Decompositions into cycles II: cycles systems. In: Dinitz, J., Stinson, D. (Eds.) Contemporary Design Theory: A Collection of Surveys, pp. 325–369. Wiley, New York (1992)
8. Hilton, A.J.W.: Hamilton decompositions of complete graphs. J. Combin. Theory (B) **36**(2), 125–134 (1984)
9. Hoffman, D.G., Rodger, C.A., Rosa, A.: Maximal sets of 2-factors and hamilton cycles. J. Combin. Theory (B) **57**, 69–76 (1993)
10. Jeevadoss, S., Muthusamy, A.: Decomposition of complete bipartite graphs into paths and cycles. Discrete Math. **331**, 98–108 (2014)
11. Jeevadoss, S., Muthusamy, A.: Decomposition of product graphs into paths and cycles of length four. Graphs & Combin. **32**(1), 199–223 (2016)
12. Koshelev, M.: New lower bound on the modularity of Johnson graphs. Mosc. J. Comb. Number Theory **10**(1), 77–82 (2021)

13. Lee, H.-C., Lin, J.-J.: Decomposition of the complete bipartite graph with a 1-factor removed into cycles and stars. Discrete Math. **313**(20), 2354–2358 (2013)
14. Lee, H.-C.: Packing and covering the balanced complete bipartite multigraph with cycles and stars. Discrete Math. Theor. Comput. Sci. **16**(3), 189–202 (2014)
15. Lee, H.-C., Chen, Z.-C.: Decomposing the complete graph into Hamiltonian paths (cycles) and 3-stars. Discuss. Math. Graph Theory **40**(3), 823–839 (2020)
16. Raigorodskii, A.M., Koshelev, M.M.: New bounds on clique-chromatic numbers of Johnson graphs. Discrete Appl. Math. **283**, 724–729 (2020)
17. Shyu, T.W.: Decomposition of complete graphs into paths and stars. Discrete Math. **310**(15–16), 2164–2169 (2010)
18. Shyu, T.W.: Decompositions of complete graphs into paths and cycles. Ars Combin. **97**, 257–270 (2010)
19. Shyu, T.W.: Decomposition of complete graphs into cycles and stars. Graphs & Combin. **29**(2), 301–313 (2013)

Some New Strongly Regular Graphs from Quadrics

Liz Lane-Harvard and Tim Penttila

Abstract We construct new strongly regular graphs with parameters (85, 20, 3, 5) and (4369, 272, 15, 17). These parameters are the same as those arising from the generalized quadrangle $Q(4, 4^h)$ for $h = 1, 2$.

Keywords Strongly regular graphs · Quadrics · Generalized quadrangle

1 Introduction

The study of strongly regular graphs can be categorized in three ways – (1) to show no strongly regular graphs exist for a given set of (unknown) parameters; (2) to construct new strongly regular graphs with previously unknown parameters; (3) to construct new strongly regular graphs with previously known parameters. The focus of this paper is on the third case. That is, we construct new strongly regular graphs with previously known parameters. In order to do this, we rely on quadrics. More specifically, we rely on the nonsingular quadric Q of projective index 1 of the projective space $PG(4, 4^h)$ for $h = 1, 2$.

Section 1 provides definitions and background information. In particular, we begin by defining a quadric in $PG(d, q)$ and proceed to discuss connections to generalized quadrangles and strongly regular graphs. The main result, in which the construction of two new strongly regular graphs with previously known parameters is presented in Sect. 2.

L. Lane-Harvard (✉)
Department of Mathematics and Statistics, University of Central Oklahoma, Edmond, OK 73034, USA
e-mail: elaneharvard@uco.edu

T. Penttila
School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
e-mail: penttila86@msn.com

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_7

1.1 Quadrics

Consider the projective space PG(d, q), whose one-dimensional subspaces are called points, two-dimensional subspaces are called lines, and d-dimensional subspaces are called hyperplanes. A **quadric** is the set of points $(x_0, x_1, \ldots, x_d)$ of PG(d, q) satisfying a homogeneous polynomial ϕ in $x_0, \ldots, x_d$ of degree two such that

$$\phi = \sum_{j \geq i} a_{ij} x_i x_j, \ (0 \leq i \leq j \leq d).$$

There are numerous properties and adjectives for quadrics; however, we will only consider a few of them. In particular, a quadric Q is *singular* if it contains at least one point P such that every line through P intersects Q doubly there. When $d = 2$, singular quadrics are reducible. However, when $d \geq 3$, there are singular quadrics which are irreducible.

If we consider q instead of d, additional properties come to light. For example, if $q = 2^n$ for some $n \in \mathbb{Z}^+$, the *nucleus* of a quadric Q of PG$(d, 2^n)$, with equation $\phi = 0$, is any point P of PG$(d, 2^n)$ which satisfies $\partial\phi/\partial x_i = 0$ for all i. Furthermore, we can pair this definition with that of a singular quadric to obtain the following result.

Theorem 1 *Let Q be a quadric of* PG$(d, 2^n)$ *for some $n \in \mathbb{Z}^+$ with equation $\phi = 0$.*

1. *If there exists two points of* PG$(d, 2^n)$ *which satisfy $\partial\phi/\partial x_i = 0$ for all i, then Q is singular.*
2. *If Q is nonsingular, then it has 0 or 1 nucleus, depending on whether d is odd or even, respectively.*

For the scope of this paper, $d = 4$ and $q = 4^h$ for $h = 1, 2$; thus, any nonsingular quadric will have exactly one nucleus.

1.2 Generalized Quadrangles

Generalized quadrangles play a prominent role in finite geometry, as they are connected to various other combinatorial structures, sometimes in multiple ways. Here we will display the relationship between generalized quadrangles and quadrics.

A (finite) **generalized quadrangle** (GQ) is an incidence structure $S = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ such that $\mathcal{P}$ is a non-empty set of points, $\mathcal{B}$ is a non-empty set of lines, and I is a symmetric point-line incidence relation that satisfies the following axioms:

(i) Each point is incident with $1 + t$ $(t \geq 1)$ lines and two distinct points are incident with at most one line.
(ii) Each line is incident with $1 + s$ $(s \geq 1)$ points and two distinct lines are incident with at most one point.
(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

The parameters of a generalized quadrangle are integers s and t, and S is said to have *order* (s, t). If $s = t$, we say that S has order s. The number of points of a GQ of order (s, t) is $(s + 1)(st + 1)$ and the number of lines is $(t + 1)(st + 1)$. Additional information on finite generalized quadrangles can be found in [4].

There are three known families of classical generalized quadrangles, all associated with classical groups and all of which embed in $\mathrm{PG}(d, q)$ for $3 \le d \le 5$. Of particular interest for this paper is the first classical generalized quadrangle, which can be described as follows.

Consider a nonsingular quadric Q of projective index 1 of the projective space $\mathrm{PG}(d, q)$, with $d = 3, 4$, or 5. Then the points of Q together with the lines of Q, which are the subspaces of maximal dimension on Q, form a GQ $Q(d, q)$ with parameters

- $s = q, t = 1, |\mathcal{P}| = (q + 1)^2, |\mathcal{B}| = 2(q + 1)$, when $d = 3$,
- $s = t = 1, |\mathcal{P}| = |\mathcal{B}| = (q + 1)(q^2 + 1)$, when $d = 4$, and
- $s = q, t = q^2, |\mathcal{P}| = (q + 1)(q^3 + 1), |\mathcal{B}| = (q^2 + 1)(q^3 + 1)$, when $d = 5$.

The following are the canonical equations for the quadric Q:

- $x_0x_1 + x_2x_3 = 0$, when $d = 3$,
- $x_0^2 + x_1x_2 + x_3x_4 = 0$, when $d = 4$, and
- $f(x_0, x_1) + x_2x_3 + x_4x_5 = 0$, where f is an irreducible binary quadratic form when $d = 5$.

Thus, when $d = 4$ and $q = 4^h$ for $h = 1, 2$, the canonical equation for both quadrics is $x_0^2 + x_1x_2 + x_3x_4 = 0$, and the number of points (and lines) of the GQ equals $(4 + 1)(4^2 + 1) = 85$, if $h = 1$, or $(16 + 1)(16^2 + 1) = 4369$, if $h = 2$.

1.3 Strongly Regular Graphs

One object generalized quadrangles are connected to is strongly regular graphs, which were introduced by Bose in 1963 [1]. A graph Γ, that is simple, undirected, and loopless, of order v is a **strongly regular graph** with parameters v, k, λ, and μ whenever Γ is not complete or edgeless and

(i) each vertex is adjacent to k vertices,
(ii) for each pair of adjacent vertices, there are λ vertices adjacent to both, and
(iii) for each pair of non-adjacent vertices, there are μ vertices adjacent to both.

The parameters of a strongly regular graph are denoted as (v, k, λ, μ).

Let S be a GQ of order (s, t). The *point graph* of S is a graph whose vertices are the points of S, where two vertices are adjacent if they are collinear as points of S. The point graph of S is a strongly regular graph with parameters.
$((s + 1)(st + 1), s(t + 1), s - 1, t + 1)$. Hence, the point graph of $Q(4, 4)$ is a

strongly regular graph with parameters (85, 20, 3, 5), and the point graph of $Q(4, 16)$ is a strongly regular graph with parameters (4369, 272, 15, 17).

Two simple, well-known examples of strongly regular graphs include the pentagon, whose parameters are (5, 2, 0, 1) and the Petersen graph, whose parameters are (10, 3, 0, 1). Interestingly enough, each of these graphs are uniquely determined by their parameters. However, that is not always the case.

The automorphism group of a graph is a distinguishing characteristic of seemingly like graphs. An *automorphism* of a graph $\Gamma = (V, E)$ is a permutation σ of the vertex set V such that the pair of vertices (u, v) form an edge of Γ if and only if $(\sigma(u), \sigma(v))$ also forms an edge. Less formally, an automorphism of a graph Γ is a graph isomorphism from Γ to itself. The *automorphism group* of a graph is the set of automorphisms of that graph under composition. Computing automorphism groups, no matter the type of graph, can be NP hard. However, knowing what the automorphism group of a strongly regular graph is can be extremely beneficial, if not necessary, to determine whether the graph is new. The group of $Q(4, 4^h)$ for $h = 1, 2$ is $\mathrm{PGO}(4, 4^h)$. For a complete list of strongly regular graphs and their automorphism groups, see [3].

2 Main Result

The following result relies on the point graph of $Q(4, 4^h)$ for $h = 1, 2$. Note that, by definition, $Q(4, 4^h)$ is a GQ of order $(4^h, 4^h)$. Hence, the point graph of $Q(4, 4^h)$ is a strongly regular graph with parameters $((4^h + 1)(4^{2h} + 1), 4^h(4^h + 1), 4^h - 1, 4^h + 1)$.

Theorem 2 *Fix a point P and a line l of $Q(4, 4^h)$ for $h = 1, 2$. Let Q be a point on the line joining P and the nucleus N of $Q(4, 4^h)$. Let Γ be the point graph of $Q(4, 4^h)$, and let Γ' be the graph obtained by deleting all edges $\{R, S\}$ of Γ where RS is skew to l and P is not collinear with R and S, and adding edges $\{R', S'\}$ whenever R' is not collinear with P, S' is not collinear with P if and only if there exists a $Q(4, 2^h)$ subquadrangle containing R', S', P, Q and meeting l in $2^h + 1$ points. Then Γ' is a strongly regular graph with parameters* (85, 20, 3, 5) *and* (4369, 272, 15, 17) *whose automorphism groups have orders 1536 and 3000, respectively.*

Proof This is a straightforward computation in Magma [2]. □

While these parameters are not new, the construction is. Further investigation is necessary to determine if this construction leads to an infinite family.

References

1. Bose, R.C.: Strongly regular graphs, partial geometries and partially balanced designs. Pacific J. Math. **13**, 389–419 (1963)
2. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system I: the user language. J. Symb. Comput. **24**, 235–265 (1997)
3. Brouwer, A.E.: Parameters of strongly regular graphs. http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html
4. Payne, S.E., Thas, J.A.: Program Finite Generalized Quadrangles. Pitman Advanced Publication, Boston (1984)

Nonexistence of a Subfamily of a Family of Edge-Regular Graphs

Robert McNellis, Tabitha Parker, and Kenneth Roblee

Abstract A simple, non-edgeless, regular graph is said to be *edge-regular* if the cardinality of the intersection of the neighborhoods of every pair of adjacent vertices is a fixed integer λ. Edge-regular graphs necessarily have a constant number p of common non-neighbors for every adjacent vertex pair. Much work has been done to determine the extremal edge-regular graphs for the inequality $n \leq 3(\lambda + p)$, as well as the edge-regular graphs satisfying $n = 3(\lambda + p) - 2$ with added structural conditions, where n is the number of vertices and $\lambda > 0$. We consider the case where $n = 3(\lambda + p) - 4$, with the added condition that the common neighbor set of every pair of adjacent vertices induces $\frac{\lambda}{2}K_2$, a disjoint union of edges.

Keywords Regular · Edge-regular

1 Introduction

We follow the notation in recent papers (see [1]). An *edge-regular* graph is a simple, regular graph $G = (V, E)$ with positive degree such that there is a nonnegative integer λ such that for every adjacent vertex pair in V, the intersection of their neighborhoods has size λ. If $uv \in E$, then we denote the common neighborhood by $\Lambda(u, v)$, or sometimes just Λ when the context is clear (the letter T has been used in other works instead, as it indicates the number of triangles of which each edge is a member). More specifically, we have $\Lambda(u, v) = N(u) \cap N(v)$, where $N(x)$

R. McNellis
Holy Family University, Philadelphia, PA 19114, USA
e-mail: rmcnellis@holyfamily.edu

T. Parker
Motlow State Community College, Fayetteville, TN 37334, USA
e-mail: tparker@mscc.edu

K. Roblee (✉)
Troy University, Troy, AL 36802, USA
e-mail: kroblee@troy.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_8

denotes the set of vertices adjacent to a vertex x; thus, $|\Lambda(u, v)| = \lambda$ for all adjacent vertices $u, v \in V$. The set of all edge-regular graphs is a superset of all the much-studied strongly regular graphs (which are edge-regular graphs for which there exists an integer μ such that every pair of non-adjacent vertices have exactly μ common neighbors). Thus, edge-regular graphs are interesting in their own right and numerous enough that we don't expect a complete classification of them.

If G is edge-regular, then there necessarily exists a parameter we denote by p (see [2]) that measures the size of the common non-neighbor set of any adjacent vertex-pair. Thus if $|V| = n$, then $p = n - |N(u) \cup N(v)|$ for all adjacent vertices $u,\ v$. To give some recent results involving classification of edge-regular graphs by their parameters, it was shown that the family of all edge-regular graphs with $p = 0$ consists precisely of the regular Turán graphs, i.e., the complete multipartite graphs with parts of equal size (see [3]). Hoffman and Johnson showed in [4] that the edge-regular graphs with $p = 1$ are strongly regular. It turns out they are the complements of the famous Moore graphs, which may be an infinite family, but only three are known for sure to exist: C_5, the complement of the Petersen graph, and the Hoffman-Singleton graph. For $p = 2$, it was shown (among other things) in [5, 6] that for $\lambda > 0$, the edge-regular graphs satisfy the inequalities $\lambda + 8 \leq n \leq 3(\lambda + 2)$. The extremal graphs for the upper bound turn out to be precisely $K_{\lambda+2,\lambda+2,\lambda+2}$ minus the edges of a perfect matching consisting of triangles. The upper bound was extended in [7] so that if G is an edge-regular graph on n vertices, parameter $\lambda > 0$ and $p \geq 0$, then $n \leq 3(\lambda + p)$. The extremal graphs for this inequality have, for the most part, been characterized. One of the interesting properties of these extremal graphs is that the common neighbor sets of every adjacent vertex pair is an independent set. For the "next" case where $n = 3(\lambda + p) - 2$, the authors in [8] proved the nonexistence of such edge-regular graphs, where the common neighbor set of every adjacent vertex pair is an independent set. In [2], the case where $n = 3(\lambda + p) - 2$ was also studied extensively; in particular, one result in that paper was that the subfamily of these edge-regular graphs where, for each $uv \in E$, the set $\Lambda = \Lambda(u, v)$ induces a union of disjoint edges, consists of precisely of four graphs, K_4 and three strongly regular graphs. Alternatively, we denote this structure requirement by writing $\langle\Lambda\rangle \cong \frac{\lambda}{2}K_2$; we use the notation $\langle\cdot\rangle$ to denote the subgraph of G induced by $\cdot$. Here, we consider this structural requirement, but with $n = 3(\lambda + p) - 4$, where $\lambda > 0$ and $p \geq 0$.

2 Main Results and Proofs

As mentioned above, we are concerned with the classification of edge-regular graphs with $n = 3(\lambda + p) - 4$ $(\lambda > 0,\ p \geq 0)$, where each $\langle\Lambda\rangle \cong \frac{\lambda}{2}K_2$. Note that this structural requirement on $\langle\Lambda\rangle$ thus requires λ to be even. Before we dive too much into this, it is worthwhile to observe that K_5 is edge-regular with $\lambda = 3,\ p = 0$, and $n = 3(\lambda + p) - 4$; this graph does not have the restricted structure required, but it is comforting to know that graphs with these parameters exist.

Another interesting observation we make in passing. Alexander Brassington of Troy University has noted that the complement of the grid graph $G_{4,5}$ is edge-regular with parameters $n = 20$, $\lambda = 6$, $p = 2$ and satisfying $n = 3(\lambda + p) - 4$. However, we leave it to the reader to check that the common neighbor set of any pair of adjacent vertices also does not induce $3K_2$ with which this paper is concerned.

We lay out some basic facts about edge-regular graphs in general, and some facts specific to our study. A simple counting argument shows that, for any edge-regular graph, we have that the degree d of each vertex in V satisfies

$$d = \frac{n - p + \lambda}{2}.$$

Since $n = 3(\lambda + p) - 4$, this translates to

$$d = 2\lambda + p - 2.$$

Furthermore, by the structural condition imposed on Λ, we note that λ must be even. It is also useful to observe that, by the structure of $\langle\Lambda\rangle$, there can be no induced P_3s in $\langle\Lambda\rangle$.

If we let $uv \in E$, then we let $A = A(u, v)$ denoted the set of neighbors of u that are not in $\Lambda \cup \{v\}$; we let $B = B(u, v)$ denote the set of neighbors of v that are not in $\Lambda \cup \{u\}$. We let $C = C(u, v)$ denote the set $V - (A \cup B \cup \Lambda \cup \{u, v\})$; in words, C is the set of common "non-neighbors" of u and v. Observe that $|A| = |B| = \lambda + p - 3$ and $|C| = p$.

We note a few passing facts first before we state and prove some lemmas. By the λ requirement, we note that each vertex $w \in \Lambda$ is adjacent to exactly $\lambda - 2$ vertices in A and to $\lambda - 2$ vertices in B. In attempting to find edge-regular graphs with these (admittedly harsh) requirements, we will need the following lemmas.

Lemma 1 *For every adjacent vertex-pair $w_1, w_2 \in \Lambda$, their common neighbor set in A (similarly, in B) is empty.*

Proof By contradiction, suppose that some adjacent pair $w_1, w_2 \in \Lambda$ have a common neighbor, namely $x \in A$. Now observe that $x, u, v \in \Lambda(w_1, w_2)$ with x adjacent to u and with u adjacent to v. In other words, this is an induced P_3 in $\Lambda(w_1, w_2)$, which is contrary to the $\frac{\lambda}{2}K_2$ structure of all the Λs. □

Let us also observe that the Λ-set structure requirement implies that the neighbors in A of such $w_1 \in \Lambda$ induce a $\frac{\lambda-2}{2}K_2$. This is because w_1 and u, being adjacent, must have λ common neighbors, two of which are v and some $w_2 \in A$, and must satisfy the Λ set structure requirement. By symmetry, the same holds for the neighbors of w_1 in B.

Furthermore, let us note that such a w_1 is adjacent to exactly $d - (3 + 2(\lambda - 2)) = p - 1$ vertices in C, as it is adjacent "so far" to u, v, w_2 and to $\lambda - 2$ vertices in each of A and B. This provides a pleasing relationship between λ and p, as indicated in the following lemma.

Lemma 2 $\lambda = p$

Proof As in the paragraph preceding the statement of this lemma, the number of neighbors in C of $w_1 \in \Lambda$ is $p - 1$; the same is, of course, true for $w_2 \in \Lambda$ (w_1's neighbor in Λ). As there are p vertices in C, and w_1 and w_2 must have all their $\lambda - 2$ common neighbors other than u and v in C (by the previous lemma), then, since each is adjacent to one more vertex outside the $\lambda - 2$ they are commonly adjacent to, it follows that either w_1 and w_2 are both non-adjacent to the same vertex (in which case it follows that $\lambda - 2 = p - 1$), or the vertex in C that w_1 is non-adjacent to is different than the vertex in C that w_2 is non-adjacent to (in which case it must be that $\lambda - 2 = p - 2$). The former case implies that, since λ is even, then p is odd, and so $n = 3(\lambda + p) - 4$ is odd. Coupling this then with $d = 2\lambda + p - 2$ is also odd shows, by the degree-sum formula, that this former case is impossible. The only alternative is the latter case, as claimed. □

Therefore, by lemma 2, we have $n = 3(\lambda + p) - 4 = 6\lambda - 4$, which implies that $|A| = |B| = 2\lambda - 3$. As w_1, w_2 are each adjacent to exactly $\lambda - 2$ vertices in A (and in B), and that these neighbor sets in A (and in B) are disjoint, then the union of their neighbor sets in A (and in B) have exactly $2\lambda - 4$ vertices, which is all but one vertex in A. Let us denote this one ("lone") vertex in A by a.

Lemma 3 *Vertex a is adjacent to at most $\lambda - 2$ vertices in A.*

Proof To the contrary, suppose a is adjacent to more than $\lambda - 2$ vertices in A. As mentioned before, the remaining vertices in A are neighbors of either w_1 or of w_2. Those that are neighbors of w_1 induce a $\frac{\lambda-2}{2}K_2$ (similarly for the neighbors of w_2 in A). With a adjacent to more than $\lambda - 2$ vertices in A, we have that a is adjacent to the endpoints of an edge in $\langle A \rangle$. Denote two such endpoints by a_1 and a_2. Without loss of generality, let us further suppose that a_1 and a_2 are both adjacent to (say) w_1. Observe then that the vertices a, u, w_1 induce a P_3 in $\langle \Lambda(a_1, a_2) \rangle$, contrary to the Λ structure requirement. □

As a consequence of this lemma and of the requirement that a and u must have exactly λ common neighbors, we have that a is adjacent to at least 2 vertices in Λ, say x and y, and these two vertices are non-adjacent by Lemma 1. Consider x: Its neighbor set (let us denote by $N_A(x)$) in A induces a $\frac{\lambda-2}{2}K_2$, and which contains vertex a. Thus, a is adjacent to precisely one vertex $a_1 \in N_A(x)$.

For ease of notation, and as in previous work (see [2], for instance), we will use the notation $ER(n, \lambda, p)$ to denote the family of edge-regular graphs with the indicated parameters. From earlier remarks, we do have that K_5 is an edge-regular graph with $n = 3(\lambda + p) - 4$ for $\lambda = 3$ and $p = 0$, and so $K_5 \in ER(5, 3, 0)$. In light of our interest in the structure induced by the common neighbor set of pairs of adjacent vertices, it is interesting to note that the common neighbor set of any two vertices in K_5 induces a K_3. And we also noted (using this "ER" notation) that the complement G of the grid graph $G_{4,5}$ is in $ER(20, 6, 2)$. Here, we observe in passing that the common neighbor set of any pair of adjacent vertices in G induces a C_6. We arrive at our theorem.

Theorem 1 *For $\lambda > 0$ and with $p \geq 0$, and with each $\langle \Lambda \rangle \cong \frac{\lambda}{2} K_2$, we have $ER(3(\lambda + p) - 4, \lambda, p) = \emptyset$.*

We first prove the following lemma, which establishes the result for $\lambda \geq 8$.

Lemma 4 *The theorem is true when $\lambda \geq 8$.*

Proof We proceed by contradiction, assuming there is some $G \in ER(n, \lambda, p)$ with $n = 3(\lambda + p) - 4$, where $\lambda > 0$, $p \geq 0$, and each $\Lambda \cong \frac{\lambda}{2} K_2$. As established in Lemma 1, the adjacent vertices w_1, w_2 have disjoint neighbor sets in A. Let us denote these neighbor sets as A_1, A_2, respectively; as noted earlier, both of these sets induce a copy of $\frac{\lambda-2}{2} K_2$.

Now, let us consider another adjacent pair $w_3, w_4 \in \Lambda$. We observe that w_3 must be adjacent to at most half of the vertices in A_1 and at most half of those in A_2 (otherwise, say if w_3 is adjacent to more than half in A_1, then it would be adjacent to both ends of an edge $xy \in \langle A_1 \rangle$, resulting in a P_3 in $\langle \Lambda(x, y) \rangle$). The same is true for w_4. We let A_3, A_4 denote the neighbors in A of w_3, w_4, respectively. With this notation, we have $|A_i \cap A_j| \leq (\lambda - 2)/2$ for $i = 1, 2$ and $j = 3, 4$, and $A_3 \cap A_4 = \emptyset$. It follows that A_3 must include exactly half of the vertices either in A_1 or A_2, possibly both (if not both, then all but one of the other in addition to vertex a). The same can be said for A_4. Since one of A_3, A_4 includes vertex a, then the other does not include a, and thus includes exactly half of A_1 and exactly half of A_2. Without loss of generality, we assume that A_3 is the set that includes exactly half of the vertices of A_1 and exactly half of A_2. As A_1, A_2 both induce $\frac{\lambda-2}{2} K_2$, so does A_3. Furthermore, each edge in $\langle A_3 \rangle$ must have one end in A_1 and the other in A_2, but cannot have both ends of an edge in A_1 nor both ends of an edge in A_2. With $|A| = 2\lambda - 4$, this gives a maximum of $2\lambda - 4 - 2(\frac{\lambda-2}{2}) = \lambda - 2$ possible vertices in A that can be adjacent to a vertex in A_3.

Next, we consider two more adjacent vertices $w_5, w_6 \in \Lambda$; their neighbor sets in A are, respectively, A_5, A_6, with the same general properties as $A_1, \ldots, A_4$ (size, induced structure, disjoint). As in the previous paragraph, at least one of A_5, A_6 must have exactly half its vertices in A_1 and exactly half in A_2. Without loss of generality, we suppose that this is A_5. Arguing as in the previous paragraph, the vertices in A_5 cannot be adjacent to both ends of an edge in $\langle A_1 \rangle$, $\langle A_2 \rangle$, $\langle A_3 \rangle$, nor in $\langle A_4 \rangle$. So then the maximum number of possible vertices in A that can be adjacent to a vertex in A_5 is $(2\lambda - 4) - 2(\frac{\lambda-2}{2}) - 2(\frac{\lambda-2}{4}) + 1 = \frac{\lambda}{2}$ (the $+1$ is since $a \in A_4$). Note that this is a sufficient number of neighbors for the vertices in A_5.

At the next step, we introduce adjacent vertices $w_7, w_8 \in \Lambda$ with their associated sets of neighbors in A, which are denoted by A_7, A_8, respectively. As established before, then at least one of w_7, w_8 is adjacent to exactly half of both A_1 vertices and to exactly half of A_2 vertices–let us suppose that w_7 is the vertex with this property. With similar (but extending the arguments to this case) reasoning as the previous two paragraphs, we can conclude that the maximum possible number of vertices in A that are adjacent to a vertex in A_7 would be $(2\lambda - 4) - 2(\frac{\lambda-2}{2}) - 2(\frac{\lambda-2}{4}) - 2(\frac{\lambda-2}{8}) + 2 = \frac{\lambda}{4} + \frac{3}{2}$. Clearly, this is not a sufficient number of neighbors in A for the vertices in A_7. Thus, a contradiction. □

As a consequence, the only possible edge-regular graphs under consideration would be $\lambda \leq 6$, which, with λ being even and positive, we consider $\lambda = 2, 4, 6$ in turn. The case where $\lambda = 2$ would be $n = 6\lambda - 4 = 8$. This possibility is ruled out by noting that if such a graph existed, then u's neighbor that is a non-neighbor of v could not meet the $\lambda = 2$ requirement with u. The $\lambda = 4$ graphs can be ruled out by a similar argument.

Lemma 5 *The statement of the theorem is true when* $\lambda = 6$.

Proof Let $G = (V, E)$ be an edge-regular graph with these parameters, and let $w_1, w_2 \in \Lambda$, neither of which is adjacent to vertex $a \in A$ as before; thus, we call a the "lone vertex" not in $N_A(w_1) \cup N_A(w_2)$. The other two adjacent pairs w_3, w_4 and w_5, w_6 in Λ each have their own "lone" vertices (say) b and c which are not in the union of their neighborhoods in A. By Lemma 3, these lone vertices a, b, and c are distinct. Each of these is adjacent to two vertices in Λ and thus four vertices in A (to meet their λ obligations with vertex u). Moreover, observe that each of a, b, and c are pairwise adjacent. Since each of these has degree of four in $\langle A \rangle$, then each of these must be adjacent to precisely two more vertices in A. Thus, by letting the other vertices in A be denoted by $a_1, a_2, \ldots, a_6$, the fact that each of these has degree of three in $\langle A \rangle$ implies that they induce a C_6. Without loss of generality, we suppose that the edges of the cycle are $a_1a_2, a_2a_3, \ldots a_5a_6$, and a_6a_1. Then to get the correct degrees for all vertices in A, each of a, b, and c would have to be adjacent to two vertices from $a_1, a_2, \ldots, a_6$. More precisely, let us say that a is adjacent to a_1 and a_2; b is adjacent to a_3 and a_4, and c is adjacent to a_5 and a_6. We note that the edges in C_6 must have endpoints that have a common neighbor in Λ. So consider the adjacent pair a_1 and a_2. Then we have that $\Lambda(a_1, a_2)$ consists of at least the vertices a, u, and their common neighbor (say) $w \in \Lambda(u, v)$, which induce a P_3, and is thus contrary to the structure of the graph. □

Further questions persist, such as what other edge-regular graphs (if any) are there with $n = 3(\lambda + p) - 4$. also, how can the proof in the theorem be modified for further families of edge-regular graphs with the structural condition, e.g., $n = 3(\lambda + p) - 6$?

References

1. Roblee, K.J., McNellis, R.F.: On a family of non-bipartite, triangle-free edge-regular graphs. Congr. Numer. **215**, 181–189 (2013)
2. Coolsaet, K., Johnson, P.D., Jr., Roblee, K.J., Smotzer, T.D.: Some extremal problems for edge-regular graphs. Ars Comb., CV (2012)
3. Johnson, J.D., Jr., Tuza, Z.: Another extremal property of some Turań graphs. Eur. J. Comb. **8**, 27–28 (1987)
4. Hoffman, D.G., Johnson, P.D., Jr.: An extremal family of strongly regular graphs. Congr. Numer. **78**, 119–122 (1990)
5. Johnson, P.D., Jr., Roblee, K.J.: More extremal graphs for a maximum-joint-neighborhood, average-triangles-per-edge inequality. Congr. Numer. **140**, 87–95 (1999)

6. Johnson, P.D., Jr., Roblee, K.J.: On an extremal subfamily of an extremal family of nearly strongly regular graphs. Australas. J. Comb. **25**, 279–284 (2002)
7. Roblee, K.J., Smotzer, T.D.: Some extremal families of edge-regular graphs. Eur. J. Comb. **25**(7), 927–933 (2004)
8. Johnson, P.D., Jr., Roblee, K.J.: Non-existence of a nearly extremal family of edge-regular graphs. In: Proceedings of the Forty-First Southeastern International Conference on Combinatorics, Graph Theory and Computing Congressus Numerantium, vol. 203, pp. 161–166 (2010)
9. Brouwer, A.E.: Strongly regular graphs, Chapter VI.5. In: Colbourn, C.J., Dinitz, J.H. (eds.) The CRC Handbook of Combinatorial Designs, pp. 667–685. CRC Press, New York (1996)
10. Johnson, Peter, Myrvold, Wendy, Roblee, Kenneth: More extremal problems for edge-regular graphs. Util. Math. **73**, 159–168 (2007)
11. Johnson, P.D., Jr., Roblee, K.J., Smotzer, T.D.: Another extremal family of edge-regular graphs. Congr. Numer. **185**, 61–64 (2007)

On a Convex Geometric Connection to Threshold Logic

M. R. Emamy-K. and Gustavo A. Meléndez Ríos

Abstract A convex geometric connection to Threshold Logic will be reviewed. We have presented necessary and sufficient conditions to recognize cut-complexes with 2 or 3 maximal faces from the class of all cubical complexes. This recognition of cut-complexes is closely related to an old proposal on cubical lattices by N. Metropolis and G. C. Rota. They proposed cubical lattices may also be used for synthesis of Boolean functions parallel to the conventional Boolean algebraic methods. The characterization will be applied to recognize several cut-complexes in the 4-dimensional cube. The cut-complexes of the 4-cube are used to define a new poset that happens to be a distributive lattice.

Keywords Threshold logic · Convex polytopes · Cubical lattices

1 Preliminaries

More than 40 years Metropolis-Rota [15, 16] initiated a dimension-free axiomatic approach to characterize the cubical lattice (or the face lattice) of the n-cube. They also proposed that their axioms for cubical lattices could be applied as an alternative tool in the synthesis problems of Boolean functions, parallel to the classical Boolean methods. This line of research has been followed in algebraic combinatorics by other mathematicians, for instance see [4, 5]. There is a pretty and natural phenomena behind their notable proposal: In the Euclidean space R^n for $n \geq 5$, there are only three regular solids, the n-dimensional simplex, the hypercube, and the hyperoctahedron. The last two are dual to each other. Therefore, the face lattices of these regular convex polytopes reveal only two distinct order structures: First the face lattice of the n-simplex, and second the face lattice of the n-cube. The former is the universally called Boolean lattice and second one is the cubical lattice.

M. R. Emamy-K. (✉) · G. A. Meléndez Ríos
Department of Mathematics, University of Puerto Rico, Box 23355, San Juan, PR 00931, USA
e-mail: mreza.emamy@upr.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_9

Our cubical study of cut-complexes here is closely related to the cubical lattices and the synthesis problems of Boolean functions, aimed by Metropolis-Rota [15, 16]. A cut-complex, roughly speaking, is a subgraph of the n-cube whose vertices are strictly separable from the rest of the vertices of the n-cube by a hyperplane of R^n, that is, basically the subgraph induced by the ON vertices of a threshold Boolean function. For more on threshold Boolean functions and threshold logic See [21]. By applying the cubical lattice techniques, simple necessary and sufficient conditions to recognize the cut-complexes with 2 or 3 maximal faces has been obtained in [10]. The latter characterization result is exactly a geometric recognition, presenting the synthesis of threshold Boolean functions in the class of all Boolean functions.

On the other hand, geometric recognition of nonisomorphic cut-complexes, over the n-cube for arbitrary $n \geq 3$, is a challenging fundamental problem and has a root in the paper of [13], see also [12]. For more on cut-complexes see [7–10, 20, 22]. A comprehensive exposition of the n-cube problems closely related to cut-complexes can be found in [18]. For more on this, see [14, 20, 22].

To begin with an overview of the recognition results on the cut-complexes with 2 or 3 maximal faces from [10] in Sects. 4 and 5, we develop basic facts of cubical complexes in Sect. 3. Then, we present a new application for the latter result to improve the characterization of the 4-cube cut-complexes, in Sect. 6. We then introduce a new distributive lattice on this cut-complexes in Sect. 7. The cubical lattice approach of [10] also applies a geometric flavor from convex polytopes. Evidently, the proofs via cubical lattice will be harder, for a lattice that is not even modular comparing to the distributive Boolean lattices. However, the cubical lattice enjoys certain distributive properties that will be part of the tools here, for instance see Propositions 5 and 6.

2 Terminologies

We begin with some basic terminology from convex polytopes. The n-th power of the unit interval $[0,\ 1]^n$ will be denoted by C^n. This set is an n-dimensional polytope in R^n whose vertex set is the Boolean n-cube $B^n = \{0,\ 1\}^n$ and its graph is the geometric unit n-cube Q_n. Here throughout the paper, by the n-cube we mean the Boolean cube B^n, unless otherwise stated. A cubical complex (or simply a complex) C is a nonempty collection of faces of the n-cube B^n such that for any two faces $F_1 \subset F_2,\ F_2 \in C \Longrightarrow F_1 \in C$. A cut-complex is a cubical complex whose vertices are strictly separable from the rest of the vertices of the n-cube by a hyperplane of R^n.

For the given Boolean n-cube B^n, a k-face is a Boolean k-subcube of the n-cube where $0 \leq k \leq n$. For instance, 0-face (or vertex) stands for the elements of the n-cube, and edges are the 1-dimensional faces. Let C be a given complex, the subcubes in C are often called faces of C. A face F is maximal in C if $F \in C$ and there is no face $F' \in C$ such that $F \subset F',\ F \neq F'$. The dimension of C is the maximum dimension among all the faces. The set of vertices, edges, and the dimension of a face or a complex X will be denoted by $vert(X)$, $E(X)$ and $dim\ X$ respectively.

These vertices and edges form the usual graph of X. Of course, in the special case when X is a face of the n-cube, the two sets $vert(X)$ and X coincide.

The complement $\widetilde{X}$ of a complex X of the n-cube B^n is defined to be the complex induced by $B^n - vert(X)$, that is; $F \in \widetilde{X}$ if and only if $vert(F) \subset B^n - vert(X)$. Two faces F and G are said to be incident if either $F \subset G$ or $G \subset F$, and they are parallel if one of them is a translation of the other. For a $0 - 1$ valued integer a, let us define the complement of a, by $\overline{a} := 1 - a$. Two vertices of B^n are said to be opposite if all the n coordinates of one of them are complements of those of the other one. For a set S of vertices in B^n the cube hull of S, denoted by $cub\ S$, is the smallest face of B^n containing S. Similarly $conv\ S$ and $aff\ S$ stand for the convex hull and the affine hull of S respectively.

A k-face F of the n-cube can be treated as a complex. To clarify this confusion, the k-dimensional complex whose faces are faces of F will be denoted by $< F >$. Thus, $< F >$ is the complex induced by vertices of F. In general when there is a sequence of faces $F_i (i = 1, 2, \ldots, N)$, $< F_i >_{i=1}^{N}$ denotes the complex induced by $\bigcup_{1 \le i \le N} F_i$. That is to say, $F \in < F_i >_{i=1}^{N}$ if $F \subset \bigcup_{1 \le i \le N} F_i$, in particular $F_i \in < F_i >_{i=1}^{N}\ \forall i$. For a face T of the n-cube, the restriction of the cut-complex C to T is denoted by $C_T := \{ F \cap T \mid F \in C\}$. For a face or a complex T, $ext\ T$ or the set of external edges connecting to T, denotes the set of all edges of the n-cube which have only one vertex in T. Clearly $(ext\ T) \cap E(T) = \emptyset$.

3 Cubical Complexes

In this section, we will develop basic tools from cubical complexes that form a basis for the proof of the main Theorem 10. Let F be a k-dimensional face of the n-cube and $I = \{1, 2, \ldots, n\}$. A typical vertex in F has two type of coordinates, those that have fixed 0–1 values for all vertices in F, and the others that take both values 0 or 1. The set $f_F \subset I$ is defined to be all of those $i \in I$ for which the i-th coordinate, denoted by $F(i)$, is fixed for all of the vertices in F, and define $\upsilon_F = I - f_F$. Clearly $| \upsilon_F | = k$ and $| f_F | = n - k$.

Note that $cub(F, G) = B^n$ if and only if either $f_F \cap f_G = \emptyset$ or $f_F \cap f_G \neq \emptyset$ and $F(i) \neq G(i)\ \forall i \in f_F \cap f_G$. Generally, $f_{cub(F, G)}$ is the maximal subset T of $f_F \cap f_G$ for which $F(i) = G(i)$ holds for all $i \in T$.

Proposition 1 *For any 3 proper faces F, G and K of the n-cube, if $F \cap G \neq \emptyset$, $F \cap K \neq \emptyset$ and $(F \cap G) \cap (F \cap K) = \emptyset$, then $G \cap K = \emptyset$.*

The opposite vertices of faces in the n-cube play a central role in our separability arguments. The properties of these vertices that will be useful to us are described in the next two propositions.

Proposition 2 *Let F and G be proper faces of the n-cube for which $F(i) \neq G(i)$ $\forall i \in f_F \cap f_G$. Then there exist opposite vertices υ and ω of the n-cube such that $\upsilon \in F$ and $\omega \in G$.*

Observe that Proposition 2 is false for arbitrary disjoint faces F and G. We however have a close conclusion.

Corollary 3 *Let F and G be disjoint nonempty faces of the n-cube. Then there exist opposite vertices υ and ω of the $cub(F, G)$ such that $\upsilon \in F$ and $\omega \in G$.*

Proposition 4 *Let G and K be two intersecting faces of a given k-face F in the $n - cube$, $2 \leq k \leq n$. A necessary and sufficient condition for $G \cup K$ to cover at least one vertex from each pair of opposite vertices of F is that; $max(dim\, G, dim\, K) \geq k - 1$ or, equivalently; at least one of the faces G or K be a facet of F.*

The cubical lattice is not modular and so not distributive. However, there are notable distributive properties that are useful and in need of much more attention. A couple of these properties that are also very similar are given in the next two propositions.

Proposition 5 *Let F be a face of the n-cube, and $S \subset ext\, F$. For any face G of $cub(F, S)$ such that $S \subset E(G)$, then $G = cub(G \cap F, S)$.*

Proposition 6 *Let F and G be any two proper faces of the n-cube with $F \cap G \neq \emptyset$ and S any set of edges in $ext\, F$ that are incident to a vertex $u \in F \cap G$, then $G \cap cub(F, S) = cub(G \cap F, S \cap E(G))$.*

Proof The inclusion $\supset$ is easy. For the equality case we show that both sides of the identity are cubes of the same dimension. Suppose k and r are the number of edges of $F \cap G$ and $G \cap S$ that are incident to u respectively. Then, it is easy to verify that $k + r$ is the dimension of the cubes that represent both sides. $\square$

Recall that the proper faces F, G, and K are assumed to be all the distinct maximal faces of the induced complex $< F, G, K >$.

4 Cut-Complexes with 2 Maximal Cubes

In this section, we proceed to study the easy case of the desired characterization, where we have only two maximal faces. Here is a crucial definition to state our main result. Let F and G be proper nonincident faces of the n-cube, such that $F \cap G \neq \emptyset$ and $dim\, F \geq dim\, G$. Then F is said to supprt G, denoted by $F \vdash G$, if $dim(F \cap G) = (dim\, G) - 1$. It is easy to show that $F \vdash G \iff dim(cub(F, G)) = (dim\, F) + 1 \iff$ for each vertex $\omega \in F \cap G$, $\exists$ a unique $e \in (ext\, F) \cap E(G)$ with $\omega \in e$.

Reviewing the terminology of Sect. 2, let F and G be two proper faces of the n-cube. By $< F,\ G >$ we mean an induced complex in which F and G are all the distinct maximal faces. In the next theorem, a simple necessary and sufficient condition for $< F,\ G >$ to be a cut-complex will be described. Recall that a cubical complex is said to be a cut-complex if its vertices are strictly separated from the rest of the vertices of the n-cube by a hyperplane of R^n. Let C be a cut-complex and T a face of the n-cube. Clearly $\widetilde{C}$ is a cut-complex of B^n, also $C_T \subset T$, the restriction of C to T, is a cut-complex, and so is the complex $\widetilde{C_T} = (\widetilde{C})_T$. In the sequel, the center of the face T will be denoted by $O(T)$.

Theorem 7 *Let proper faces F, G, and the induced complex $C := < F,\ G >$ be given over the n-cube with $dim\ F \geq dim\ G$. Then C is a cut-complex if and only if $F \vdash G$.*

Proof First suppose that $F \vdash G$. Choose a vertex $\upsilon \in F \cap G$ and the unique edge $e \in (ext\ F) \cap E(G)$ with $\upsilon \in e$. Bearing in mind having the identity $cub(F,\ G) = cub(F,\ e)$, where we consider $e \in ext\ F$ as a vector outward F. Thus $S = F + e$ and $K = F \cap G + e$ define two faces of $cub(F,\ e)$ parallel to F and $F \cap G$ respectively. In the cube S, K is a face and so $\widetilde{K_S}$ is a cut-complex of S. Hence $\widetilde{K_S}$ is also a cut-complex of $cub(F,\ e)$. However in the $cub(F,\ e)$ we have $\widetilde{\widetilde{K_S}} = C$; that is, C is a cut-complex. Conversely, suppose $F \nvdash G$ and $k = dim\ F \geq dim\ G$. If $F \cap G = \emptyset$, then $< F,\ G >$ must be disconnected, otherwise it would contain a maximal face different from F and G. Thus, it is not a cut-complex. Now, we may assume that $F \cap G \neq \emptyset$ and $dim(F \cap G) \leq (dim\ G) - 2$. Hence, for $\upsilon \in F \cap G$ there exist two edges $e_1,\ e_2 \in (ext\ F) \cap E(G)$, $\upsilon \in e_1 \cap e_2$. Defining $T = cub(F,\ e_1,\ e_2)$, one obtains $dim\ T = (dim\ F) + 2 = k + 2$. We claim that C_T is not a cut-complex (and so neither is C). Observing that $cub(e_1,\ e_2) \in C_T$, let u and w be the vertices opposite to v in $cub\ (e_1,\ e_2)$ and in F respectively. Since $dim\ T = k + 2$, $u,\ w \in C_T$ are opposite vertices of T, and hence $O(T) \in conv\ C_T$. On the other hand, $dim\ G_T \leq dim\ G \leq dim\ F < k + 1 \leq (dim\ T) - 1$. So Proposition 4 concludes that there exists a pair of opposite vertices of T such that none of them are in C_T, which yields $O(T) \in conv\ \widetilde{C_T}$, contradicting C_T to be a cut-complex. □

5 Cut-Complexes with 3 Maximal Cubes

Let F, G and K be three proper faces of the n- cube. From now on, by $< F,\ G,\ K >$ we mean an induced cubical complex in which F, G and K are all the distinct maximal faces. A rather simple necessary and sufficient condition for $< F,\ G,\ K >$ to be a cut-complex will be the main result of this section. We begin with necessary and sufficient conditions for characterization of important special cases, then these results will be combined to prove the main Theorem 10. For the proof of the next theorem and complete proof of Theorem 10, see [10].

Finally, let F_1 and F_2 be two parallel facets of a k-face F of the n-cube, say $F_2 = F_1 + e$, where $e \in (ext\ F_1) \cap E(F)$ is considered as a vector from F_1 toward F_2. Then for a face G of F_1, evidently $G + e \subset F_2$ is the orthogonal projection of G into F_2.

Theorem 8 *Let nonempty proper faces F, G, K, and the induced complex $C := < F, G, K >$ be given over the $n - cube$, with $F \vdash G$, and $F \cap G \cap K \neq \emptyset$. Suppose that for each vertex $u \in F \cap G$ there is at most a unique edge $e \in ext(F \cap G) \cap E(K)$ with $u \in e$. Then $< F, G, K >$ is a cut-complex.*

The condition on the vertices u of $F \cap G$ in Theorem 8 clearly implies $F \vdash K$ and $G \vdash K$. Indeed, Theorem 8 can be modified to achieve a stronger result.

Theorem 9 *Let proper nonempty faces F, G, K, and the induced complex $C := < F, G, K >$ be given over the $n - cube$, with $F \vdash G$, $F \vdash K$, and $G \vdash K$. Then $< F, G, K >$ is a cut-complex.*

Proof The definition of $\vdash$ evidently shows that $k := \dim F \geq \dim G \geq \dim K$, and also $F \cap G \cap K \neq \emptyset$, otherwise according to Proposition 1, two of the 3 faces must be disjoint which is impossible by the hypothesis. For a vertex $\upsilon \in F \cap G \cap K$, a typical edge $e \in ext(F \cap G \cap K) \cap E(K)$, and $\upsilon \in e$ there are 3 possible conditions:

(1) $e \in E(F) - E(G)$,
(2) $e \in E(G) - E(F)$,
(3) $e \notin E(F) \cup E(G)$.

If there exists at least one e of type (3), then this edge must be unique and there would be no edge of type (2) or (1) by the hypothesis of the theorem. In this case, clearly, $G \cap K \subset F \cap G$, $F \cap K \subset F \cap G$, and thus Theorem 8 concludes that $< F, G, K >$ is a cut-complex. Second, there is no edge of type (3), only one edge of type (1) and only one edge of type (2). Here, $K \subset cub(F, G)$, $dim(cub(F, G)) = dim(cub(F, G, K)) = k + 1$, and so F is a facet of $cub(F, G) = cub(F, G, K)$. Let $\overline{F}$ be the facet of $cub(F, G)$ parallel to F. Thus,

$$G \vdash K \Longrightarrow G \cap F \vdash K \cap F \Longrightarrow G \cap \overline{F} \vdash K \cap \overline{F}.$$

The latter is valid because $K \cap \overline{F}$ and $G \cap \overline{F}$ are the orthogonal projections of $K \cap F$ and $G \cap F$ into $\overline{F}$ respectively. Hence, $D := < K \cap \overline{F}, G \cap \overline{F} >$ is a cut-complex of $\overline{F}$ by Theorem 7, so is $\widetilde{D}_{\overline{F}}$. However, defining $C := < F, G, K >$, it is apparent that $\widetilde{D}_{\overline{F}} = \widetilde{C}_{cub(F, G, K)}$; that is, $< F, G, K >$ is a cut-complex of $cub(F, G, K)$. □

Finally, we close this section by the main Theorem 10 that can be concluded from the previous result by putting together different pieces from the past sections but with much more work needed to be done, see [10] for the complete proof.

Theorem 10 *Let F, G and K be all the distinct maximal faces of the induced complex $< F, G, K >$ over the n-cube with $dim\ F \geq dim\ G \geq dim\ K$. Then $< F, G, K >$ is a cut-complex if and only if $F \vdash G$, $G \vdash K$ and $F \vdash K$.*

6 Cut-Complexes of the 4-Cube

The 4-cube cut-complexes were introduced and characterized in [7] by an algorithm that is based on Lemma 11. However, the resulting complexes has been confirmed to be cut-complexes only by computational methods. In this section, we apply Theorems 7 and 10 to confirm the 14 complexes of Fig. 1 are cut-complexes of the 4-cube (up to complements), a short cut to the old result of [7]. First, here is the basic result from [7].

Lemma 11 ([7]) *Suppose M_1 and M_2 are two finite sets of points in R^n that are strictly separated by a hyperplane H such that given the open halfspaces of H, H^- and H^+, $M_1 \subset H^-$ and $M_2 \subset H^+$. Then there exists $x \in M_1$ such that for some relocation of H, say H_1, we have that $M_2 \cup \{x\} \subset H_1^+$ and $M_1 - \{x\} \subset H_1^-$.*

Theorem 12 *The following 14 complexes of C^4 are cut-complexes: C_1, C_2, C_3, C_4, C_4', C_5, C_5', C_6, C_6', C_7, C_7', C_8, C_8', C_8''.*

Proof We will show that each of the listed complexes are cut-complexes, by applying Theorems 7 and 10 as much as possible. However, there will be some complexes where, it will not work. Therefore, we divide the 14 complexes into 3 cases.

Case 1: C_1, C_2, C_4, C_8. Since they are all faces of the 4-cube, we can take their corresponding supporting hyperplanes and push them slightly inside to obtain the hyperplane strictly separating their vertices from the remaining vertices of the 4-cube. Case 2: C_3, C_4', C_5, C_6, C_6', C_7, C_7', and C_8'. All of them are induced connected complexes with only two or three maximal faces. More precisely; C_3, C_5, C_6,

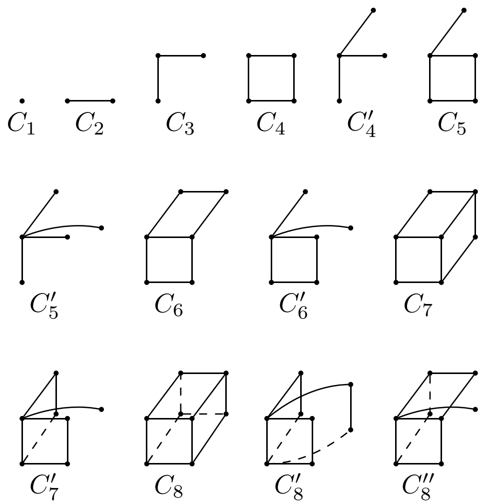


Fig. 1 Cut-complexes of 4-cube

are formed by two maximal faces, that Theorem 7 apples. And the rest of them, that is; C'_4, C'_6, C_7, C'_7, and C'_8 all are formed by three maximal faces, and so Theorem 10 works. Case 3: C'_5, C''_8 To obtain the desired hyperplane to strictly separate the vertices of C'_5 from all other vertices of C^4, begin with a hyperplane H perpendicular to the diagonal from 0000 to 1111 and push it towards the point 0000. Eventually, H will strictly separate 0000, 1000, 0100, 0010, and 0001 (isomorphic to C'_5 once the incident edges are added in) from all other vertices of C^4. Finally, by applying Lemma 11 three consecutive times starting with C'_5 and its complement, we can obtain a hyperplane strictly separating C''_8 from its complement. □

7 Cut-Complex Poset

We now explore forming a poset on the set of all cut-complexes of the 4-cube. We consider the set of cut-complexes up to isomorphism. For instance, C_2 may be any edge of the 4-cube but it is still only one element of poset. We present our proposed order relation for this poset. Note that given a cut-complex C, $vert(C)$ and $\widetilde{C}$ denotes its vertex set and its graph complement respectively.

Definition 13 (*Poset of Cut-Complexes of* C^4) Let $\mathcal{C}c(C^4)$ be the set of all non-isomorphic cut-complexes of the 4-cube C^4. We define an order $\leq$ on $\mathcal{C}c(C^4)$ as follows. Let C and C' be cut-complexes of C^4.

1. covering relation: We say that C covers C' if $vert(C) = vert(C') \cup \{x\}$ where x is obtained by applying Lemma 11 to $vert(C)$ and $vert(\widetilde{C})$.
2. order relation: We say that $C \geq C'$ if there is a finite sequence of cut-complexes $C = C_1, C_2, \ldots, C_k = C'$ such that C_i covers C_{i+1} for each i.

It is straightforward to see that the relation $\leq$ defined in Definition 13 is reflexive, antisymmetric, and transitive. The Hasse diagram of the resulting poset is shown in Fig. 2 (see Fig. 1 for the notation used to identify the cut-complexes). Note that for any $i \geq 8$, $C_i = \widetilde{C}_{16-i}$ and $C'_i = \widetilde{C}'_{16-i}$. We conclude this discussion by proving that $\langle \mathcal{C}c(C^4), \leq \rangle$ is a distributive lattice.

Theorem 14 *The poset* $\mathcal{C}c(C^4)$ *is a distributive lattice.*

Proof We show that $\mathcal{C}c(C^4)$ is a sublattice of $\mathbf{8} \times \mathbf{8}$. Since $\mathbf{8} \times \mathbf{8}$ is distributive (product of chains), it follows that $\mathcal{C}c(C^4)$ is distributive. By observing Fig. 3, we can see that $\mathcal{C}c(C^4)$ is a non-empty subset of $\mathbf{8} \times \mathbf{8}$. It then suffices to show that it is closed under the join and meet operations of $\mathbf{8} \times \mathbf{8}$. This is trivial for any comparable pair of elements. Thus, we need only to consider the 2-element antichains of $\mathcal{C}c(C^4)$.

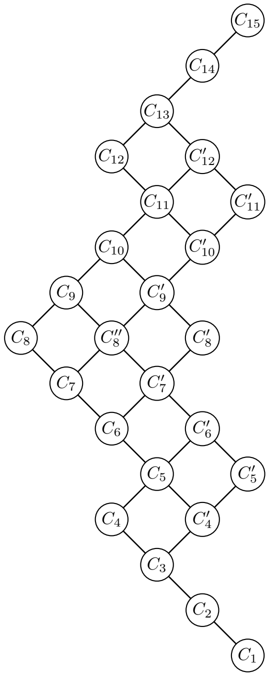


Fig. 2 Cut-complex poset of 4-cube

Table 1 lists all such antichains along with their joins and meets. It is constructed by examining Fig. 3. First, we must identify the antichains. Clearly, C_1, C_2, C_3, C_{13}, C_{14}, and C_{15} cannot form an antichain because they are comparable to all other cut-complexes in poset. Hence, we need only to consider C_i and C'_j for $4 \le i, j \le 12$ and C''_8. Observe also that all non-prime cut complexes are comparable to each other, that is, $C_i \le C_j$ or $C_i \ge C_j$ for all i and j. The same holds for the primes (i.e. $C'_i \le C'_j$ or $C'_i \ge C'_j$). As a result, we must consider only pairs of a non-prime with a prime or those with C''_8. It is worth noting that $\{C_i, C'_i\}$ is always an antichain for $4 \le i \le 12$.

We make the table by considering all non-primes in increasing order of index and checking which primes (or C''_8) form an antichain with it. For instance, when we consider C_4, we note that only C'_4 and C'_5 form an antichain with it, leading to the first two columns of Table 1. Once we are done with this we must only consider whether each prime can be paired with C''_8, which results in the last column of the table. With all the 2-element antichains listed, we find their joins and meets also using Fig. 3 (See the third and fourth rows of Table 1). It can be observed that they are all contained in $\mathcal{C}c(C^4)$. Therefore, $\mathcal{C}c(C^4)$ is a sublattice of $\mathbf{8} \times \mathbf{8}$ and we are done.

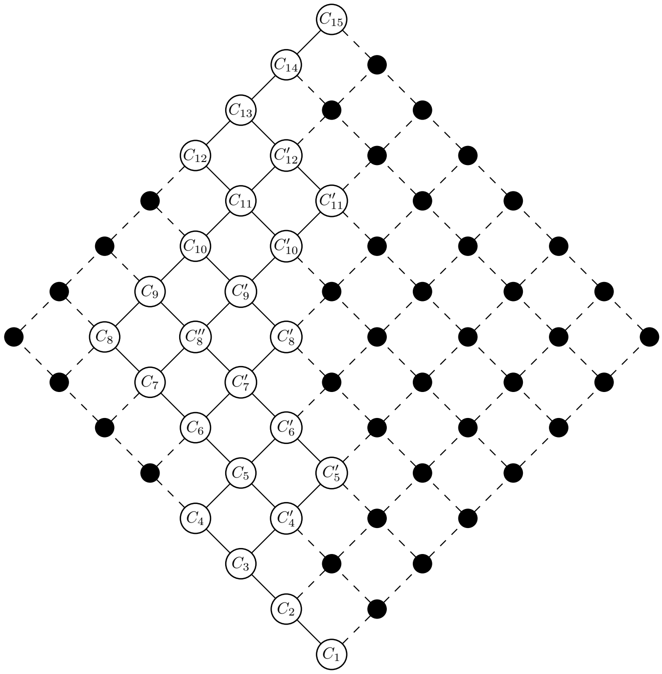


Fig. 3 $\mathcal{C}c(C^4)$ as a subset of $\mathbf{8} \times \mathbf{8}$: Labeled white vertices and solid lines correspond to $\mathcal{C}c(C^4)$

Table 1 Computations for the proof of Theorem 14

x	C_4	C_4	C_5	C_6	C_6	C_7	C_7	C_7	C_8	C_8	C_8	C_8	C_8
y	C'_4	C'_5	C'_5	C'_5	C'_6	C'_5	C'_6	C'_7	C'_5	C'_6	C'_7	C'_8	C''_8
$x \wedge y$	C_3	C_3	C'_4	C'_4	C_5	C'_4	C_5	C_6	C'_4	C_5	C_6	C_6	C_7
$x \vee y$	C_5	C'_6	C'_6	C'_7	C'_7	C''_8	C''_8	C''_8	C_9	C_9	C_9	C_{10}	C_9
x	C_8	C_8	C_8	C_9	C_9	C_9	C_9	C_{10}	C_{10}	C_{11}	C_{12}	C_{12}	C'_8
y	C'_9	C'_{10}	C'_{11}	C'_8	C'_9	C'_{10}	C'_{11}	C'_{10}	C'_{11}	C'_{11}	C'_{11}	C'_{12}	C''_8
$x \wedge y$	C_7	C_7	C_7	C'_7	C''_8	C''_8	C''_8	C'_9	C'_9	C'_{10}	C'_{10}	C_{11}	C'_7
$x \vee y$	C_{10}	C_{11}	C'_{12}	C_{10}	C_{10}	C_{11}	C'_{12}	C_{11}	C'_{12}	C'_{12}	C_{13}	C_{13}	C'_9

References

1. Boros, E., Gurvich, V., Hammer, P.L., Ibaraki, T., Kogan, A.: Decompositions of partially defined Boolean functions. Discret. Appl. Math. **62**, 51–75 (1995)
2. Boros, E., Hammer, P.L., Ibaraki, T., Kawakami, K.: Polynomial time recognition of 2-monotonic positive Boolean functions given by an oracle. SIAM J. Comput. **26**, 93–109 (1997)
3. Cepek, O., Kronus, D., Kucera, P.: Recognition of interval Boolean functions. Technical Report 15-06, RUTCOR Research Report RRR, Rutgers University, New Brunswick, NJ (2006). Accepted for publication in Annals of Mathematics and Artificial Intelligence
4. Chen, W.Y.C., Stanley, R.P.: Derangements on the n-cube. Discret. Math. **115**, 65–75 (1993)

5. Ehrenborg, R., Readdy, M.: The R-cubical lattice and a generalization of the CD-index. Eur. J. Comb. **17**, 709–725 (1996)
6. Elgot, C.C.: Truth functions realizable by single threshold organs. In: AIEE Conference Paper, 60–1311, pp. 225–245 (1960); also SCTLD (1961)
7. Emamy-K, M.R.: On the cuts and cut-number of the 4-cube. J. Combin. Theory, Ser. A **41**(2), 221–227 (1986)
8. Emamy-K, M.R.: On the covering cuts of $C^d,\ d \leq 5$. Disc. Math. **68**, 191–196 (1988)
9. Emamy-K, M.R.: Geometry of cut-complexes and threshold logic. J. Geom. **65**, 91–100 (1999)
10. Emamy-K, M.R.: A geometric connection to threshold logic via cubical lattices. Anal-OR **188**, 141–153 (2011)
11. Davey, B.A., Priestley, H.A.: Introduction to Lattice and Order. Cambridge University Press (2002)
12. Grünbaum, B.: Convex Polytopes. In: Kaibel, V., Klee, V., Ziegler, G.M. (eds.) Springer, Berlin (2003)
13. Grünbaum, B.: Polytopal graph. MAA Studies Math. **12**, 201–224 (1975)
14. Klee, V.: Shapes of the future: some unresolved problems in high-dimensional intuitive geometry. In: Proceedings of the 11th Canadian Conference on Computational Geometry, p. 17 (1999)
15. Metropolis, N., Rota, G.-C.: On the lattice of faces of the n-cube bull. Am. Math. Soc. **84**(2), 284–286 (1978)
16. Metropolis, N., Rota, G.-C.: Combinatorial structure of the faces of the n-cube. SIAM J. Appl. Math. **35**(4), 689–694 (1978)
17. Peled, U., Simeone, B.: A O(nm)-time algorithm for computing the dual of a regular Boolean function. Discret. Appl. Math. **49**(1–3), 309–323 (1994)
18. Saks, M.E.: Slicing the hypercube. In: Surveys in Combinatorics. Cambridge University Press (1993), pp. 211–255
19. Stanley, R.P.: Enumerative Combinatorics, vol. 1. Cambridge University Press (2002)
20. Sohler, C., Ziegler, M.: Computing cut numbers. In: Proceedings of the 12th CCCG, p. 17 (2000)
21. Muroga, S.: Threshold Logic and its Applications. Wiley Interscience, Toronto (1971)
22. Ziegler, M.: http://www.uni-paderborn.de/cs/cubecuts
23. Ziegler, G.M.: Lectures on Polytopes. Springer, Berlin (1994)

Multicore Graphs: Characterization and Properties

Lilian Markenzon and Newton Paciornik

Abstract In this paper we define the *multicore graphs*, a subclass of chordal graphs that extends the gen core-satellite graphs, defined by Estrada and Benzi (Linear Algebra App. 517:30–52 (2017)) [5]. We prove that the multicore graphs are the $(P_5, gem, dart)$-free chordal graphs and we present a characterization of the class which provides a simple linear time recognition algorithm. We also show its interrelation with other subclasses of chordal graphs: the clique-corona graphs and the starlike graphs.

Keywords Muticore graphs · Core-satellite graphs · Chordal graphs · Recognition algorithm

1 Introduction

Some subclasses of chordal graphs are well known in graphs literature [2]. The *windmill graph* $Wd(k, \ell)$, $k \geq 2$ and $\ell \geq 2$, is a graph constructed by joining ℓ copies of a complete graph K_k at a shared universal vertex; the *split-complete graph* is the join of a complete graph K_k and a set of $n - k$ independent vertices and the *star graph* is the particular case of the split-complete graph when $k = 1$. A class that generalises these classes, the *core-satellite graphs* was defined by Estrada and Benzi [5]; informally, they are the graphs consisting of η copies of the complete graph K_s (the satellites) meeting in a complete graph K_c (the core). In the same paper, the *generalized core-satellite graphs* were defined, relaxing the size of the satellites in the previous definition.

L. Markenzon (✉)
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
e-mail: markenzon@nce.ufrj.br

N. Paciornik
Independent Researcher, Rio de Janeiro, Brazil
e-mail: npaciornik@gmail.com

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_10

In this paper we extend the definition of the generalized core-satellite graphs, presenting the *multicore graphs* and showing its equivalence to (P_5, gem, dart)-free chordal graphs. A characterization that leads to a linear recognition is presented. We also show its interrelation with other subclasses of chordal graphs: the clique-corona graphs and the starlike graphs.

2 Basic Notions

Let $G = (V, E)$ be a graph, with $|E| = m$, $|V| = n > 0$. The *neighborhood* of a vertex $v \in V$ is denoted by $N(v) = \{w \in V \mid \{v, w\} \in E\}$ and its *closed neighborhood* by $N[v] = N(v) \cup \{v\}$. Two vertices u and v are *true twins* in G if $N[u] = N[v]$ and they are *false twins* in G if $N(u) = N(v)$. A clique in G is a subset of vertices such that every two distinct vertices in the clique are adjacent. A vertex v is said to be *simplicial* in G if $N(v)$ is a clique in G.

Basic concepts about chordal graphs (graphs containing no chordless cycles) are assumed to be known and can be found in Blair and Peyton [1] and Golumbic [8]. Let $G = (V, E)$ be a chordal graph. A subset $S \subset V$ is a *vertex separator* for non-adjacent vertices u and v (a uv-separator) if the removal of S from the graph separates u and v into distinct connected components. If no proper subset of S is a uv-separator then S is a *minimal uv-separator*. When the pair of vertices remains unspecified, we refer to S as a *minimal vertex separator* (*mvs*).

A *clique-tree* of G is defined as a tree T whose vertices are the maximal cliques of G such that for every two maximal cliques Q and Q' each clique in the path from Q to Q' in T contains $Q \cap Q'$. The set of maximal cliques of G is denoted by $\mathbb{Q}$. Blair and Peyton [1] proved that, for a clique-tree $T = (V_T, E_T)$, a set $S \subset V$ is a minimal vertex separator of G if and only if $S = Q' \cap Q''$ for some edge $\{Q', Q''\} \in E_T$. Moreover, the multiset $\mathbb{M}$ of the minimal vertex separators of G is the same for every clique-tree of G. We define the *multiplicity* of the minimal vertex separator S, denoted by $\mu(S)$, as the number of times that S appears in $\mathbb{M}$. The algorithm presented in Markenzon and Pereira [15] computes the set of minimal vertex separators, $\mathbb{S}$, of a chordal graph G and their multiplicities in linear time.

It is worth mentioning special kinds of cliques, presented by Hara and Takemura [10]. A *simplicial clique* is a maximal clique containing at least one simplicial vertex. A simplicial clique Q is called a *boundary clique* if there exists a maximal clique Q' such that $Q \cap Q'$ is the set of non-simplicial vertices of Q.

The *join* of two graphs G_1 and G_2, with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2, denoted by $G_1 \nabla G_2$, is the graph G obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2.

Some classes of graphs are mentioned in this paper. A graph is a *block graph* [11] if and only if all its blocks (maximal 2-connected components) are complete subgraphs. *Strictly chordal graphs*, also defined in [12, 13], were firstly introduced as *block duplicate graphs* by Golumbic and Peled [9]. A strictly chordal graph is a graph obtained by adding zero or more true twins to each vertex of a block graph

Fig. 1 The gem and the dart graphs

G. Strictly chordal graphs were proved to be chordal, gem-free and dart-free [9, 12] (Fig. 1 shows the gem and the dart graphs) and they can also be characterized in terms of the structure of their minimal vertex separators, as follows:

Theorem 1 ([16]) *Let $G = (V, E)$ be a chordal graph. The following statements are equivalent:*

1. *G is a strictly chordal graph.*
2. *For any distinct $S, S' \in \mathbb{S}$, $S \cap S' = \emptyset$.*
3. *G is a $\{$gem, dart$\}$-free graph.*

Based on Theorem 1, a linear time recognition algorithm for strictly chordal graphs relies only in the determination of $\mathbb{S}$.

Property 1 A boundary clique of a strictly chordal graph is a maximal clique that contains only one minimal vertex separator.

3 The Multicore Graphs

The generalized core-satellite (*gen core-satellite*) graphs [5] are defined as follows. Let $t \geq 1$ and consider a graph consisting of a core clique with c nodes and $\eta = \eta_1 + \eta_2 + \cdots + \eta_t$ satellite cliques where η_1 cliques have s_1 nodes, η_2 cliques have s_2 nodes, ..., and η_t cliques have s_t nodes, where $s_i \neq s_j$ for $i \neq j$. Here $s_i \geq 1$ and $\eta_i \geq 1$ for all $i = 1, \ldots, t$. Each node in every satellite clique is connected to each node in the core clique. The following notation was introduced in the same paper: letting $\mathbf{s} = (s_1, s_2, \ldots, s_t)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_t)$, $\Theta(c, \mathbf{s}, \boldsymbol{\eta})$ denotes the generalized core-satellite graph just described.

Observe that a complete graph is a gen core-satellite graph; however the same complete graph can be denoted as $\Theta(n-1, (1), (1))$ or $\Theta(n-2, (1), (2))$ and so on, but always with at least one satellite. We here extend this definition, considering also the existence of zero satellites. This graph is denoted by $\Theta(c, \emptyset, \emptyset)$.

The gen core-satellite graphs are obviously chordal graphs and they can be characterized as a chordal graph that have at most one minimal vertex separator (the

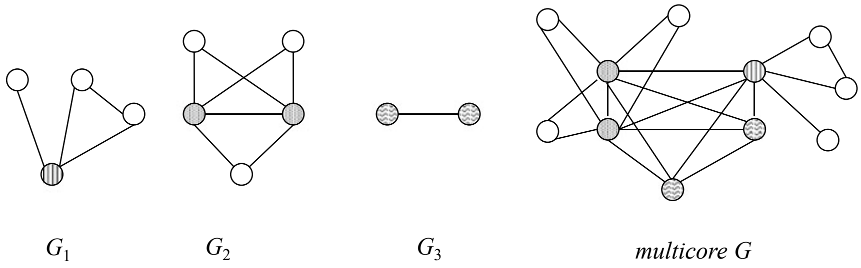


Fig. 2 Construction of a multicore graph

core). Some special cases are the windmills that have all maximal cliques with the same cardinality and the minimal vertex separator of cardinality 1; the split-complete graphs that have all maximal cliques with cardinality $k+1$ and the minimal vertex separator of cardinality k; the core-satellite graphs that have all maximal cliques with the same cardinality.

The *multicore graphs* extend the definition of gen core-satellite graphs. Let $G_1, G_2, \ldots, G_k$, $k \geq 1$, be distinct gen core-satellite graphs; the graph obtained by the join of the cores and the union of the satellites of G_i, $1 \leq i \leq k$, is a *multicore graph*. The clique resulting from the join operation is called the *core clique*.

Figure 2 shows an example of a multicore graph G. The gen core-satellite graphs that form G are: $G_1 = \Theta(1, (1, 2), (1, 1))$, $G_2 = \Theta(2, (1), (3))$ and $G_3 = \Theta(2, \emptyset, \emptyset)$.

Lemma 1 *A multicore graph is a strictly chordal graph.*

Proof Let G be a multicore graph. By the definition, G is obtained by the join of the cores of the gen core-satellite graphs $G_1, G_2, \ldots, G_k$, $k \geq 1$, and the union of their satellites. Each gen core-satellite is formed by maximal cliques sharing at most one minimal vertex separator. It is a strictly chordal graph since it obeys item (2) of Theorem 1. Hence for $k = 1$ the result is straightforward.

Let $k > 1$. G is a connected graph. The set of minimal vertex separators of G consists of the cores of G_i, $1 \leq i \leq k$, that have at least one satellite. As these mvs are disjoint in G by construction, they obey item (2) of Theorem 1. Hence, G is a strictly chordal graph. □

The next theorem shows a characterization of the new class, by forbidden subgraphs and by means of its structure.

Theorem 2 *Let $G = (V, E)$ be a non-complete connected chordal graph and $\mathbb{S}$ its set of minimal vertex separators. The following statements are equivalent:*

1. *G is a multicore graph.*
2. *G is a $(P_5, gem, dart)$-free graph.*
3. *$\forall S_i, S_j \in \mathbb{S}$, $S_i \cap S_j = \emptyset$ and at most one maximal clique is not a boundary clique.*

Proof *(1)* $\rightarrow$ *(2)*:

By Lemma 1, G is strictly chordal graph and, by Theorem 1, it is a (gem,dart)-free graph. If $|\mathbb{S}| = 1$, all maximal cliques of G are boundary cliques. Any induced path between two simplicial vertices have 2 or 3 vertices and G is P_5-free. If $|\mathbb{S}| > 1$, let consider $S, S' \in \mathbb{S}$, $S \neq S'$. By the definition of a multicore graph, S and S' belong to the same maximal clique. Let consider two non-adjacent simplicial vertices x and y. If $S \subset N(x)$ and $S \subset N(y)$ any induced path $\langle x, \ldots, y\rangle$ has 3 vertices; if x or y belongs to the core clique any induced path $\langle x, \ldots, y\rangle$ has also 3 vertices; otherwise any induced path $\langle x, \ldots, y\rangle$ has 4 vertices. In any case, G has not an induced P_5.

(2) $\rightarrow$ *(3)*:

By Theorem 1, if G is (gem,dart)-free then $S_i \cap S_j = \emptyset, \forall S_i, S_j \in \mathbb{S}$ (the graph is strictly chordal). By hypothesis, G is also P_5-free. Suppose that there are two maximal cliques Q_1 and Q_2 that are not boundary cliques. They have, at least, two *mvs* each. As G is a connected graph, without loss of generality, let $Q_1 \cap Q_2 = S \neq \emptyset$, $S \in \mathbb{S}$. There is a *mvs* $S_1 \subset Q_1$ such that $S_1 \not\subset Q_2$ and another *mvs* $S_2 \subset Q_2$ such that $S_2 \not\subset Q_1$. Let $v_1 \in S_1$, $v_2 \in S$, $v_3 \in S_2$. The path $\langle v_1, v_2, v_3\rangle$ is an induced P_3 in G. Consider a path $\langle x, v_1, v_2, v_3, y\rangle$ in G joining two non-adjacent vertices, x and y, such that $x \notin Q_1$ and $N(x) \supset S_1$ and $y \notin Q_2$ and $N(y) \supset S_2$. As G is a strictly chordal graph, $x \notin N(v_2)$, $x \notin N(v_3)$, $y \notin N(v_1)$ and $y \notin N(v_2)$. Hence, the path $< x, v_1, v_2, v_3, y >$ would be an induced P_5. Contradiction.

(3) $\rightarrow$ *(1)*:

Two cases must be analysed:

case 1: all maximal cliques are boundary cliques.

All maximal cliques contain only one mvs. As $\forall S_i, S_j \in \mathbb{S}$, $S_i \cap S_j = \emptyset$ and G is connected, all maximal cliques contain the same mvs. Then G is a gen core-satellite graph and, by definition, a multicore graph.

case 2: exactly one maximal clique is not a boundary clique.

Let Q be this max clique and let Q_i and Q_j be two boundary cliques, $Q_i \neq Q_j$. Suppose that $S_i \subset Q_i$ and $S_j \subset Q_j$, $S_i \not\subset Q_j$ and $S_j \not\subset Q_i$. As G is connected, Q contains the join of S_i and S_j; the remaining vertices of Q_i and Q_j are simplicial vertices of G, forming the satellites. So, G is a multicore graph. □

Based on item(3) of Theorem 2, a linear time recognition algorithm can be outlined. The recognition of a strictly chordal graph ($\forall S_i, S_j \in \mathbb{S}$, $S_i \cap S_j = \emptyset$) is already known to take linear time complexity [16]. It is well known that, in a chordal graph, $\sum_{Q \in \mathbb{Q}} |Q|$ is $O(n + m)$. So, the search for all maximal cliques establishing their condition as boundary cliques, i.e. recognizing if each maximal clique contains only one *mvs* as seen in Property 1, takes also linear time complexity.

4 Relation with Other Classes

4.1 Clique Corona Graphs

Since its original definition by Frucht and Harary [6], the graph resulting from the corona product has attracted attention from several fields of graph theory, as surveyed by Levit and Mandrescu [14].

Let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of graphs indexed by the vertex set of G. The *corona* $G \circ \mathcal{H}$ is the disjoint union of G and $\mathcal{H}$ with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v. If $H_v = H$ for every $v \in V(G)$, then we write $G \circ H$ instead of $G \circ \mathcal{H}$ [6].

If all H_v are complete graphs then $G \circ \mathcal{H}$ is a *clique corona graph* [14].

Theorem 3 *Let $G' = G \circ \mathcal{H}$ be a clique corona graph. If G is a complete graph then G' is a multicore graph.*

Proof For all $v \in V(G)$, the graph H_v is a complete graph. After the corona product each H_v together with vertex v_i of G establish a core-satellite graph $\Theta(1, |V(H_i)|, 1)$, where the core is the singleton graph composed by v_i and the satellite is the graph H_i. The join of the cores result in a complete graph (graph G) and the disjoint union of satellites is the same in both definitions. Hence, G' is a multicore graph. □

Theorem 4 *Let G be a non-complete multicore graph with $|\mathbb{S}| \geq 2$. If $|S| = 1$ and $\mu(S) = 1$, $\forall S \in \mathbb{S}$, and all simplicial vertices of G do not belong to the core clique of G then G is a clique corona graph.*

The proof is immediate.

Figure 3a is a clique corona that is not a multicore; Fig. 3b is a clique corona that is a multicore; Fig. 3c is a multicore that is not a clique corona.

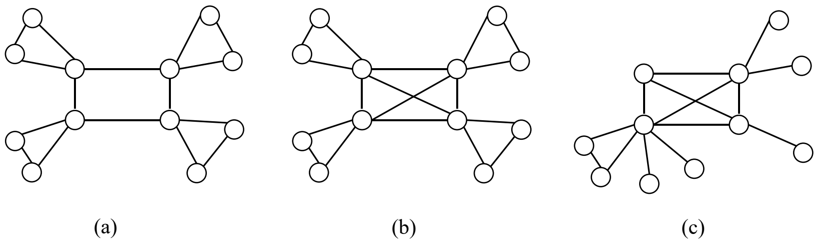


Fig. 3 Clique corona graphs

4.2 Starlike Graphs

The starlike graphs have been defined by Gustedt in 1993 [7]:

1. A chordal graph G is called *starlike* iff it has a tree decomposition $(T, \{X_0, \ldots, X_r\})$ so that every set induces a maximal clique and the tree T is a star.
2. A starlike graph G is called *primitive starlike* (*prim starlike*) if
   (a) $X_i \cap X_j = \emptyset, \forall i, j \neq 0$,
   (b) $X_0 \subseteq \bigcup_{i \neq 0} X_i$.
3. For a fixed tree decomposition $(T, \{X_0, \ldots, X_r\})$ of a starlike graph G put $\alpha_i = |X_i \cap X_0|$ and $\beta_i = |X_i \setminus X_0|$.
4. A starlike graph is called a *split graph* iff for every $i \in \{1, \ldots, r\}$ $\beta_i = 1$.

In 2006, Cerioli and Szwarcfiter [3] presented an alternative definition as follows: A graph G is *starlike* when there exists a partition $C, D_1, \ldots, D_s$ of its vertices, such that C is a maximal clique and, for all $u \in D_i, v \in D_j, i \neq j$ implies $uv \notin E(G)$, while if $i = j$, then $N[u] = N[v]$. In this case, $C, D_1, \ldots, D_s$ is called a starlike partition of G. It follows that each D_i is a clique contained in exactly one maximal clique C_i, and $Di = C_i \setminus C$.

Figure 4a presents a starlike graph and Fig. 4b, a primitive starlike.

Theorem 5 ([4]) *A graph G is starlike if and only if G does not contain any of the six graphs of Fig. 5 as an induced subgraph.*

An interesting result for strictly chordal graphs presented below helps to establish the relationship between the starlike graphs and the multicore graphs.

Theorem 6 *Let $G = (V, E)$ be a strictly chordal graph. If G is P_5-free then it is $2P_3$-free.*

Proof Suppose that $< v_1, v_2, v_3 >$ and $< w_1, w_2, w_3 >$ are two induced P_3 in G. Vertices v_1 and v_3 belong to different maximal cliques because the edge $(v_1, v_3) \notin E$;

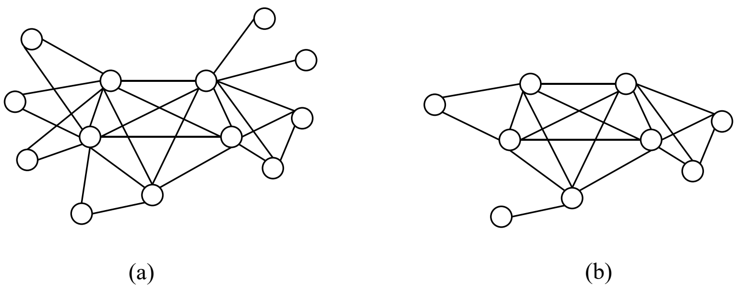


Fig. 4 Starlike graph and primitive starlike graph

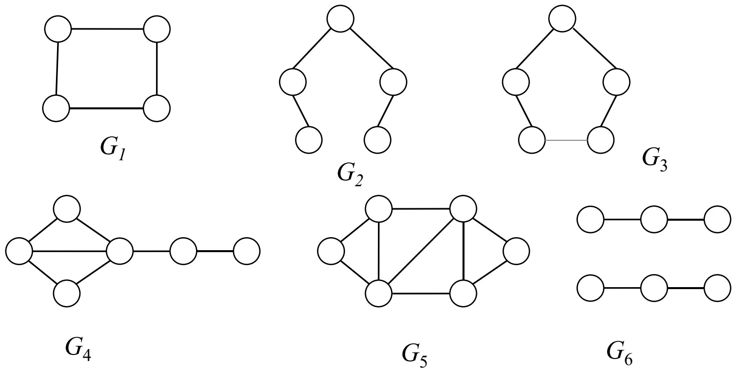


Fig. 5 Forbidden subgraphs

hence vertex v_2 belongs to a mvs of G. The same is true for vertices w_1, w_2 and w_3. Let S_v and S_w be these mvs.

As G is connected there is a path joining v_2 and w_2. Hence, there is at least another mvs S_k separating v_2 and w_2. As graph G is strictly chordal, by Theorem 1, $S_v \cap S_k = S_v \cap S_w = S_w \cap S_k = \emptyset$; there are at least four maximal cliques in G. Let Q_1, Q_2, Q_3 and Q_4 be these cliques. Without loss of generality, let $Q_2 \cap Q_3 = S_k$; there is at least a vertex $x \in S_k$ such that $x \in \langle v_2, \ldots, x, \ldots, w_2 \rangle$. Vertex $v_2 \in Q_1 \cap Q_2$ and $w_2 \in Q_3 \cap Q_4$. Either v_1 or v_3 belongs to Q_1 and either w_1 or w_3 belongs to Q_4 (but only one of them). Hence, $\langle v_1, v_2, \ldots, x, \ldots, w_2, w_1 \rangle$ is at least an induced P_5. □

Obviously, the reverse is not true: P_5 itself is a counter-example.

Proposition 1 *Let G be a multicore graph. Then G is a starlike graph.*

Proof A multicore graph is a chordal graph; it does not contain graphs G_1 and G_3 of Fig. 5. Graph G_2 is a P_5, also forbidden in a multicore graph. Graph G_4 contains a dart and graph G_5 contains a gem, as seen in Fig. 6. By Theorem 6, as G is a strictly chordal graph which is P_5 free it is also $2P_3$ free. Hence a multicore graph is a starlike graph. □

Theorem 7 *Let G be a primitive starlike. Then G is a multicore graph.*

Proof By definition, X_i, $1 \le i \le r$ are maximal cliques of G such that $X_i \cap X_j = \emptyset, \forall i, j \neq 0$, and $X_0 \subseteq \bigcup_{i \neq 0} X_i$.

As $X_i \cap X_j = \emptyset, \forall i, j \neq 0$, and the graph is connected, it is immediate that $X_i \cap X_0 = S_i$, $S_i \subset \mathbb{S}$, $\mu(S_i) = 1$ and $S_i \cap S_j = \emptyset$. Hence, all maximal cliques X_i, $i \neq 0$, are boundary cliques and by Theorem 2, G is a multicore graph. □

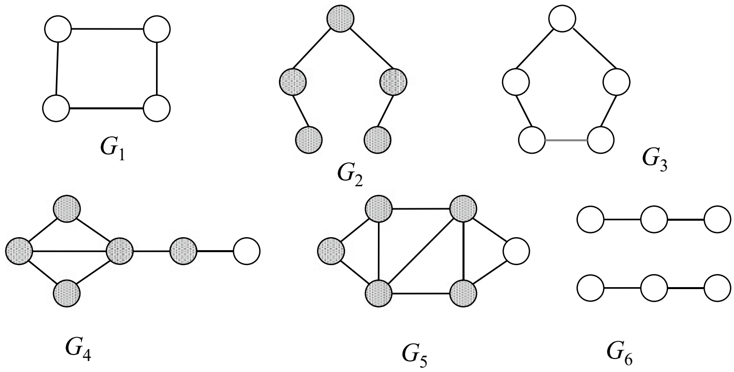


Fig. 6 Forbidden subgraphs

5 Conclusion

In this paper, we introduced the multicore graphs. Figure 7 presents the relationship between the classes mentioned.

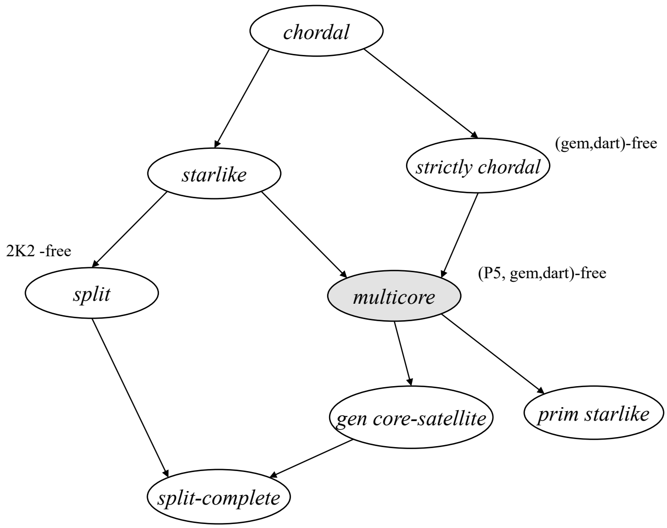


Fig. 7 Hierarchy of classes

References

1. Blair, J.R.S., Peyton, B.: An introduction to chordal graphs and clique trees. In: George, J.A., Gilbert, J.R., Liu, J.W.H. (eds.) Graph Theory and Sparse Matrix Computation, pp. 1–30. Springer Verlag, IMA 56 (1993)
2. Brandstädt, A., Le, V.B., Spinrad, J.P.: Graph classes: a survey. In: SIAM Monographs on Discrete Mathematics and Applications (1999)
3. Cerioli, M.R., Szwarcfiter, J.L.: Edge clique graphs and some classes of chordal graphs. Discrete Math. **242**, 31–39 (2002)
4. Cerioli, M.R., Szwarcfiter, J.L.: Characterizing intersection graphs of substars of a star. Ars Comb. **79**, 21–31 (2006)
5. Estrada, E., Benzi, M.: Core-satellite graphs: clustering, assortativity and spectral properties. Linear Algebra Appl. **517**, 30–52 (2017)
6. Frucht, R., Harary, F.: On the corona of two graphs. Aequationes Math. **4**, 322–324 (1970)
7. Gustedt, J.: On the pathwidth of chordal graphs. Discrete Appl. Math. **45**, 233–248 (1993)
8. Golumbic, M.C.: Algorithmic Graph Theory and Perfect Graphs, 2nd edn. Academic Press, New York (2004)
9. Golumbic, M.C., Peled, U.N.: Block duplicate graphs and a hierarchy of chordal graphs. Discrete Appl. Math. **124**, 67–71 (2002)
10. Hara, H., Takemura, A.: Boundary cliques, clique trees and perfect sequences of maximal cliques of a chordal graph, METR 2006–41. Department of Mathematical Informatics, University of Tokyo (2006)
11. Harary, F.: A characterization of block graphs. Canad. Math. Bull. **6**, 1–6 (1963)
12. Kennedy, W.: Strictly chordal graphs and phylogenetic roots, Master Thesis, University of Alberta (2005)
13. Kennedy, W., Lin, G., Yan, G.: Strictly chordal graphs are leaf powers. J. Discrete Algorithms **4**, 511–525 (2006)
14. Levit, V.E., Mandrescu, E.: On clique corona graphs—a short survey. Matemática Contemporânea **45**, 68–76 (2017)
15. Markenzon, L., Pereira, P.R.C.: One-phase algorithm for the determination of minimal vertex separators of chordal graphs. Int. Trans. Oper. Res. **17**, 683–690 (2010)
16. Markenzon, L., Waga, C.F.E.M.: New results on ptolemaic graphs. Discrete Appl. Math. **196**, 135–140 (2015)

Cospanning Characterizations of Violator and Co-violator Spaces

Yulia Kempner and Vadim E. Levit

Abstract Given a finite set E and an operator $\sigma : 2^E \longrightarrow 2^E$, two subsets $X, Y \subseteq E$ are *cospanning* if $\sigma(X) = \sigma(Y)$ (Korte, Lovász, Schrader; 1991). We investigate cospanning relations on violator spaces. A notion of a violator space was introduced in (Gärtner, Matoušek, Rüst, Škovroňby; 2008) as a combinatorial framework that encompasses linear programming and other geometric optimization problems. Violator spaces are defined by violator operators. We introduce *co-violator spaces* based on contracting operators known also as choice functions. Let $\alpha, \beta : 2^E \longrightarrow 2^E$ be a violator operator and a co-violator operator, respectively. Cospanning characterizations of violator spaces allow us to obtain some new properties of violator operators and co-violator operators, emphasizing their interconnections. In particular, we show that uniquely generated violator spaces satisfy so-called Krein-Milman properties, i.e., $\alpha(\beta(X)) = \alpha(X)$ and $\beta(\alpha(X)) = \beta(X)$ for every $X \subseteq E$.

Keywords Cospanning relation · Uniquely generated violator space · Co-violator space

1 Introduction

Each set operator determines the partition of sets to equivalence classes with equal value of the operator. Let us have some set operator α. Following [9] we call two sets X, Y *cospanning* if $\alpha(X) = \alpha(Y)$. Thus each set operator generates the cospanning equivalence relation on the family of feasible subsets of a ground set. Our goal is to investigate cospanning relations on violator spaces. These spaces were introduced in order to develop a combinatorial framework encompassing linear programming and

Y. Kempner (✉)
Department of Computer Science, Holon Institute of Technology, Holon, Israel
e-mail: yuliak@hit.ac.il

V. E. Levit
Department of Mathematics, Ariel University, Ariel, Israel
e-mail: levitv@ariel.ac.il

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_11

other geometric optimization problems [6]. Violator spaces are defined by violator operators, which generalize closure operators [8]. We also pay special attention to violator spaces with unique bases.

In Sect. 2, we introduce co-violator spaces based on contracting operators known also as choice functions.

In Sect. 3, we characterize the cospanning relation with regards to violator spaces and describe the equivalence classes of the relation for violator and co-violator spaces. Further, interconnections between the violator and co-violator operators bring about new insights. In particular, we show that uniquely generated violator spaces enjoy so-called Krein-Milman properties.

1.1 Violator Spaces

Violator spaces are arisen as a generalization of Linear Programming problems. LP-type problems have been introduced and analyzed by Matoušek, Sharir and Welzl [11, 14] as a combinatorial framework that encompasses linear programming and other geometric optimization problems. Further, Matoušek et al. [6] define a simpler framework: violator spaces, which constitute a proper generalization of LP-type problems. Intuitively, a violator space is an LP-type problem without an objective function. Hence, every LP-type problem can be naturally embedded into a corresponding violator space. On the other hand, not every violator space can be presented as an LP-type problem [6].

Originally, violator spaces were defined for a set of constraints E, where each subset of constraints $G \subseteq E$ was associated with $\nu(G)$—the set of all constraints violating G.

The classic example of an LP-type problem is the problem of computing the smallest enclosing ball of a finite set of points in $\mathbb{R}^d$. Here E is a finite set of points in $\mathbb{R}^d$, and the violated constraints of some subset of the points G are exactly the points lying outside the smallest enclosing ball of G. Moreover, violator spaces provide an abstract framework allowing to formulate and design efficient algorithms handling many types of optimization problems in the fields of computational geometry [13], computational algebra [10] and machine learning [5].

Definition 1.1 ([6]) A *violator space* is a pair (E, ν), where E is a finite set and ν is a mapping $2^E \to 2^E$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

Consistency: $X \cap \nu(X) = \emptyset$,
Locality: $(X \subseteq Y \text{ and } Y \cap \nu(X) = \emptyset) \Rightarrow \nu(X) = \nu(Y)$.

Let (E, ν) be a violator space. Define $\varphi(X) = E - \nu(X)$. It easy to see that two operators ν and φ determine the same cospanning equivalence relation on sets. So, in what follows, if (E, ν) is a violator space and $\varphi(X) = E - \nu(X)$, then (E, φ) will be called a violator space as well.

Definition 1.2 ([8]) A *violator space* is a pair (E, φ), where E is a finite set and φ is an operator $2^E \to 2^E$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:
V1: $X \subseteq \varphi(X)$ (extensivity),
V2: $(X \subseteq Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y)$ (self-convexity).

Each violator operator φ is idempotent. Indeed, extensivity implies $X \subseteq \varphi(X) \subseteq \varphi(X)$. Then, by self-convexity, we conclude with $\varphi(\varphi(X)) = \varphi(X)$.

Lemma 1.3 ([8]) *Let (E, φ) be a violator space. Then*

$$\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cup Y) = \varphi(X) = \varphi(Y) \tag{1}$$

and

$$(X \subseteq Y \subseteq Z) \wedge (\varphi(X) = \varphi(Z)) \Rightarrow \varphi(X) = \varphi(Y) = \varphi(Z) \tag{2}$$

for every $X, Y, Z \subseteq E$.

Since the second property deals with all sets lying between two given sets, following [12] we call the property *convexity*.

1.2 Uniquely Generated Violator Spaces

Let (E, α) be an arbitrary space with the operator $\alpha : 2^E \to 2^E$. $B \subseteq E$ is a *generator* of $X \subseteq E$ if $\alpha(B) = \alpha(X)$. For $X \subseteq E$, a *basis* (minimal generator) of X is a inclusion-minimal set $B \subseteq E$ (not necessarily included in X) with $\alpha(B) = \alpha(X)$. A space (E, α) is *uniquely generated* if every set $X \subseteq E$ has a unique basis.

Proposition 1.4 ([8]) *A violator space (E, φ) is uniquely generated if and only if for every $X, Y \subseteq E$*

$$\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cap Y) = \varphi(X) = \varphi(Y) \tag{3}$$

We can rewrite the property (3) as follows: for every set $X \subseteq E$ of a uniquely generated violator space (E, φ), the basis B of X is the intersection of all generators of X:

$$B = \bigcap \{Y \subseteq E : \varphi(Y) = \varphi(X)\}. \tag{4}$$

One of the known examples of a not uniquely generated violator space is the violator space associated with the smallest enclosing ball problem. A basis of a set of points is a minimal subset with the same enclosing ball. In particular, all points of the basis are located on the ball's boundary. For $\mathbb{R}^2$ the set X of the four corners of a square has two bases: the two pairs of diagonally opposite points. Moreover, one of these pairs is a basis of the second pair. Thus the equality (4) does not hold.

For each arbitrary space (E, α) with the operator $\alpha : 2^E \to 2^E$, an element x of a subset $X \subseteq E$ is *an extreme point* of X if $x \notin \alpha(X - x)$. The set of extreme points of X is denoted by $ex(X)$.

Proposition 1.5 ([8]) *Let (E, φ) be a violator space. Then*

$$ex(X) = \bigcap \{B \subseteq X : \varphi(B) = \varphi(X)\}.$$

Proposition 1.6 ([8]) *Let (E, φ) be a violator space. Then*

$$ex(\varphi(X)) \subseteq ex(X).$$

Theorem 1.7 ([8]) *Let (E, φ) be a violator space. Then (E, φ) is uniquely generated if and only if for every set $X \subseteq E$, $\varphi(X) = \varphi(ex(X))$.*

Corollary 1.8 ([8]) *Let (E, φ) be a uniquely generated violator space. Then for every $X \subseteq E$ the set $ex(X)$ is the unique basis of X.*

2 Co-Violator Spaces

Theorem 1.7 and Proposition 1.6 show that there is some duality between extensive $(X \subseteq \varphi(X))$ and contracting $(ex(X) \subseteq X)$ operators. To study this connection we introduce a new type of spaces.

Definition 2.1 A *co-violator space* is a pair (E, c), where E is a finite set and c is an operator $2^E \to 2^E$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

CV1: $c(X) \subseteq X$,
CV2: $(c(X) \subseteq Y \subseteq X) \Rightarrow c(X) = c(Y)$.

Operators satisfying the property **CV1** are called contracting operators.

In social sciences, contracting operators are called choice functions, usually adding a requirement that $c(X) \neq \emptyset$ for every $X \neq \emptyset$. The property **CV2** is called the *outcast property* or the *Aizerman property* [12].

The properties of co-violator spaces correspond to the corresponding ("mirrored") properties of violator spaces. For instance, every co-violator operator c is idempotent. Indeed, since c is contracting $c(X) \subseteq c(X) \subseteq X$. Then, **CV2** implies $c(c(X)) = c(X)$.

Lemma 1.3 is converted to the following.

Lemma 2.2 *Let (E, c) be a co-violator space. Then*

$$c(X) = c(Y) \Rightarrow c(X \cap Y) = c(X) = c(Y) \tag{5}$$

and

$$(X \subseteq Y \subseteq Z) \wedge (c(X) = c(Z)) \Rightarrow c(X) = c(Y) = c(Z) \tag{6}$$

for every $X, Y, Z \subseteq E$.

Proof Prove (5). Let $c(X) = c(Y)$. **CV1** implies that $c(X) \subseteq X$ and $c(Y) = c(X) \subseteq Y$. Then $c(X) \subseteq X \cap Y \subseteq X$, that gives (by **CV2**) $c(X \cap Y) = c(X)$.

To prove (6) let $(X \subseteq Y \subseteq Z) \wedge (c(X) = c(Z))$. **CV1** yields $c(Z) = c(X) \subseteq X \subseteq Y$. Then outcast property allows us to get $c(Z) \subseteq Y \subseteq Z \Rightarrow c(Y) = c(X) = c(Z)$. ■

It is easy to see that all the properties of violator spaces hold in their dual interpretation for co-violator spaces. Since a co-violator operator is a choice function with outcast properties, the connection between these two types of spaces may result in better understanding of two theories and in new findings in each of them.

Connections between contracting and extensive operators were studied in many works, while most of them were dedicated to connections between choice functions and closure operators [1, 3, 12]. Naturally, extreme point operators were considered as choice functions. But, as we will see in Theorem 3.9, the extreme point operator of a violator space satisfies the outcast property, and so it forms a co-violator space, if and only if the violator space is uniquely generated. We also consider choice functions investigated in [4]. The *interior operator* (well-known in topology) is dual to a closure operator. Given an extensive operator $\varphi : 2^E \to 2^E$, one can get a contracting operator c: $c(X) = E - \varphi(E - X)$ or $\overline{c(X)} = \varphi(\overline{X})$.

Proposition 2.3 *(E, φ) is a violator space if and only if (E, c) is a co-violator space, where $c(X) = \overline{\varphi(\overline{X})}$.*

Proof It is easy to see that φ is an extensive operator if and only if c is a contracting operator. To prove that c satisfies the outcast property if and only if φ is self-convex one has just to pay attention that:

$c(X) \subseteq Y \subseteq X \Leftrightarrow \overline{X} \subseteq \overline{Y} \subseteq \overline{c(X)} \Leftrightarrow \overline{X} \subseteq \overline{Y} \subseteq \varphi(\overline{X}) \Rightarrow \varphi(\overline{X}) = \varphi(\overline{Y}) \Leftrightarrow c(X) = c(Y)$. The opposite direction is proved completely analogously. ■

3 Cospanning Relations of Violator and Co-Violator Spaces

Let $E = \{x_1, x_2, \ldots, x_d\}$. The graph $H(E)$ is defined as follows. The vertices are the finite subsets of E, two vertices A and B are adjacent if and only if they differ in exactly one element. Actually, $H(E)$ is *the hypercube* on E of dimension d, since the hypercube is known to be equivalently considered as the graph on the Boolean space $\{0, 1\}^d$ in which two vertices form an edge if and only if they differ in exactly one position.

Let (E, φ) be a violator space. The two sets X and Y are *equivalent* (or *cospanning*) if $\varphi(X) = \varphi(Y)$. In what follows, $\mathcal{P}$ denotes a partition of $H(E)$ (or 2^E) into equivalence classes with regard to this relation, and $[A]_\varphi := \{X \subseteq E : \varphi(X) = \varphi(A)\}$.

Remark 3.1 Note, that the cospanning relation associated with a violator operator φ coincides with the cospanning relation associated with an original violator mapping ν.

The following theorem characterizes cospanning relations in violator spaces.

Theorem 3.2 *Let E be a finite set and $R \subseteq 2^E \times 2^E$ be an equivalence relation on 2^E. Then R is the cospanning relation of a violator space if and only if the following properties hold for every $X, Y, Z \subseteq E$:*
R1*: if $(X, Y) \in R$, then $(X, X \cup Y) \in R$*
R2*: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.*

Proof Necessity follows immediately from Lemma 1.3.

Let us define an operator φ and prove that it satisfies extensivity and self-convexity. Since R is an equivalence relation, it defines a partition of 2^E. Then, for each $X \subseteq E$ there is only one class containing X. Thus for every set X, we define $\varphi(X)$ as a maximal element in the class $[X]_R$. Notice, that the property **R1** implies that each equivalence class has a unique maximal element, so the partition is well-defined. Hence, we obtain that $X \subseteq \varphi(X)$ and $\varphi(\varphi(X)) = \varphi(X)$. Then the self-convexity follows immediately from **R2**. It is easy to see that the cospanning relation w.r.t. φ coincides with R. ■

In conclusion, each equivalence class of the cospanning relation of a violator space is closed under union (**R1**) and convex (**R2**).

The following theorem characterizes equivalence classes of co-violator spaces.

Theorem 3.3 *Let E be a finite set and $R \subseteq 2^E \times 2^E$ be an equivalence relation on 2^E. Then R is the cospanning relation of a co-violator space if and only if the following properties hold for every $X, Y, Z \subseteq E$:*
R3*: if $(X, Y) \in R$, then $(X, X \cap Y) \in R$*
R2*: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.*

Proof Necessity follows immediately from Lemma 2.2. By analogy with the proof of Theorem 3.2 we define $c(X)$ to be a minimal element in the class $[X]_R$. Since each class is closed under intersection (**R3**), the partition is well-defined. It is easy to see that operator c is contracting, satisfies the outcast property, and its cospanning relation coincides with R. ■

Consider now both a violator operator φ and a co-violator operator $c(X) = \overline{\varphi(\overline{X})}$.

Proposition 3.4 *There is a one-to-one correspondence between an equivalence class $[X]_\varphi$ of X of the cospanning relation associated with a violator operator φ and an equivalence class $[\overline{X}]_c$ w.r.t. a co-violator operator c, i.e., $A \in [X]_\varphi$ if and only if $\overline{A} \in [\overline{X}]_c$.*

Proof Indeed, $A \in [X]_\varphi \Leftrightarrow \varphi(X) = \varphi(A) \Leftrightarrow \overline{c(\overline{X})} = \overline{c(\overline{A})} \Leftrightarrow c(\overline{X}) = c(\overline{A}) \Leftrightarrow \overline{A} \in [\overline{X}]_c$. ■

A uniquely generated violator space defines a cospanning relation with additional property **R3** (see Proposition 1.4).

All in all, every uniquely generated violator space is a co-violator space as well. Each equivalence class of the cospanning relation of a uniquely generated violator space has an unique minimal element and an unique maximal element. More precisely, for the sets $A \subseteq B \subseteq E$, let us define the *interval* $[A, B]$ as $\{C \subseteq E : A \subseteq C \subseteq B\}$. Then each equivalence class of an uniquely generated violator space is an interval. We call a partition of $H(E)$ into disjoint intervals a *hypercube partition*. The following Theorem follows immediately from Theorem 3.2 and Proposition 1.4.

Theorem 3.5 ([2]) *(i) If (E, φ) is a uniquely generated violator space, then $\mathcal{P}$ is a hypercube partition of $H(E)$.*

(ii) Every hypercube partition is the partition $\mathcal{P}$ of $H(E)$ into equivalence classes of a uniquely generated violator space.

More specifically [8], $[A]_\varphi = [ex(A), \varphi(A)]$ for every set $A \subseteq E$.

Let us consider now a uniquely generated violator space (E, φ) and the operator ex. Since each equivalence class $[A]_\varphi$ w.r.t. operator φ is an interval $[ex(A), \varphi(A)]$, we can see that for each $X \in [ex(A), \varphi(A)]$ not only $\varphi(X) = \varphi(A)$, but $ex(X) = ex(A)$ as well. Since $\mathcal{P}$ is a hypercube partition of $H(E)$ we conclude with $[X]_\varphi = [X]_{ex}$. Thus the cospanning partition (quotient set) associated with an operator φ coincides with the cospanning partition associated with a contracting operator ex. Since $ex(X)$ is a minimal element of $[X]$ we immediately obtain the following

Theorem 3.6 *If (E, φ) is a uniquely generated violator space, then operator ex satisfies the following properties:*

***X1**: $ex(ex(X)) = ex(X)$*

***X2**: $ex(X) = ex(Y) \Rightarrow ex(X \cup Y) = ex(X) = ex(Y)$*

***X3**: $(X \subseteq Y \subseteq Z) \wedge (ex(X) = ex(Z)) \Rightarrow ex(X) = ex(Y) = ex(Z)$*

***X4**: $ex(X) = ex(Y) \Rightarrow ex(X \cap Y) = ex(X) = ex(Y)$*

If (E, φ) is not a uniquely generated violator space, then the operator ex may or may not satisfy the properties **X1**-**X4**. Consider the two following examples.

Example 3.7 Let $E = \{1, 2, 3\}$. Define $\varphi(X) = X$ for each $X \subseteq E$ except $\varphi(\{2\}) = \varphi(\{3\}) = \{2, 3\}$ and $\varphi(\{1, 2\}) = \varphi(\{1, 3\}) = \{1, 2, 3\}$. It is easy to check that (E, φ) is a violator space and the operator ex satisfies **X1**,**X2**, and **X4**, but while $ex(\{1\}) = ex(\{1, 2, 3\}) = \{1\}$, $ex(\{1\}) \neq ex(\{1, 2\})$, i.e., the operator ex is not convex.

Example 3.8 Let $E = \{1, 2, 3, 4, 5, 6\}$. Define $\varphi(X) = X$ for each $X \subseteq E$ except $\varphi(\{1\}) = \{1, 2\}$, $\varphi(\{1, 2, 3\}) = \varphi(\{1, 2, 4\}) = \{1, 2, 3, 4\}$ and $\varphi(\{1, 2, 5\}) = \varphi(\{1, 2, 6\}) = \{1, 2, 5, 6\}$. It is easy to check that (E, φ) is a violator space. In addition, $ex(\{1, 2, 3, 4\}) = ex(\{1, 2, 5, 6\}) = \{1, 2\}$, while $ex(\{1, 2\}) = \{1\}$. Hence, ex is not idempotent (**X1**) and does not satisfy **X4**. Since $ex(\{1, 2, 3, 4, 5, 6\}) = \{1, 2, 3, 4, 5, 6\}$ the operator ex does not satisfy **X2** as well, but, compared to the previous example, ex is convex.

Theorem 3.9 *Let (E, φ) be a violator space. The following assertions are equivalent:*

(i) (E, φ) is uniquely generated

*(ii) **X5**: $(ex(X) \subseteq Y \subseteq X) \Rightarrow ex(X) = ex(Y)$ (the outcast property)*

*(iii) **X6**: $\varphi(ex(X)) = \varphi(X)$*

*(iv) **X7**: $ex(\varphi(X)) = ex(X)$*

Proof If (E, φ) is a uniquely generated violator space, then operator ex satisfies **X5**,**X6** and **X7**, since $[X]_\varphi = [X]_{ex} = [ex(X), \varphi(X)]$.

Before we continue with the proof, it is important to mention that from the definition of the operator ex it follows that $ex(B) = B$ for each basis B.

Further we prove that if a violator space (E, φ) satisfies the property **X5**, then it is uniquely generated. Suppose that there is a set $X \subseteq E$ with two bases B_1 and B_2. Then $\varphi(X) = \varphi(B_1) = \varphi(B_2) = \varphi(B_1 \cup B_2)$. Thus Proposition 1.5 implies $ex(B_1 \cup B_2) \subseteq B_1 \cap B_2$. Then we have $ex(B_1 \cup B_2) \subseteq B_1 \subseteq B_1 \cup B_2$ and $ex(B_1 \cup B_2) \subseteq B_2 \subseteq B_1 \cup B_2$, but $ex(B_1) = B_1 \neq ex(B_2) = B_2$. In other words, we see that ex does not satisfy the outcast property.

$(iii) \Rightarrow (i)$ follows from Theorem 1.7.

Now, it is only left to prove that if a violator space (E, φ) satisfies the property **X7**, then it is uniquely generated. Suppose there is a set $X \subseteq E$ with two bases $B_1 \neq B_2$. Then $\varphi(X) = \varphi(B_1) = \varphi(B_2)$, and so $ex(\varphi(B_1)) = ex(\varphi(B_2))$. Since $ex(B_1) = B_1 \neq ex(B_2) = B_2$, we conclude that the property **X7** does not hold. ■

It is worth reminding that **X6** and **X7** are called Krein-Milman properties. In other words, every uniquely generated violator space is a Krein-Milman space [7].

4 Conclusion

Many combinatorial structures are described with the help of some operators defined on their ground sets. For instance, closure spaces are defined by closure operators, and violator spaces are described by violator operators. In this paper, we introduce co-violator spaces based on contracting operators known also as choice functions. Cospanning characterizations of violator spaces allow us to obtain a number of new properties of violator operators and co-violator operators. In addition, we reveal interconnections between violator and co-violator worlds. In further research, our intent is to extend the "cospanning" approach to a wider spectrum of combinatorial structures including closure spaces, convex geometries, antimatroids, etc.

References

1. Ando, K.: Extreme point axioms for closure spaces. Discret. Math. **306**(24), 3181–3188 (2006)
2. Brise, Y., Gärtner, B.: Clarkson's algorithm for violator spaces. Comput. Geom. **44**, 70–81 (2011)
3. Danilov, V., Koshevoy, G.: Choice functions and extensive operators. Order **26**, 69–94 (2009)
4. Demetrovics, J., Hencsey, G., Libkin, L., Muchnik, I.: On the interaction between closure operations and choice functions with applications to relational databases. Acta Cybernet. **10**(3), 129–139 (1992)
5. Esposito, G., Martin, M.: A Randomized algorithm for the exact solution of transductive support vector machines. Appl. Artif. Intell. **29**, 459–479 (2015)
6. Gärtner, B., Matoušek, J., Rüst, L., Škovroň, P.: Violator spaces: structure and algorithms. Discret. Appl. Math. **156**, 2124–2141 (2008)
7. Kempner, Y., Levit, V.E.: Krein-Milman spaces. Electron. Notes Discret. Math. **80**, 203–213 (2019)
8. Kempner, Y., Levit, V.E.: Violator spaces versus closure spaces. Eur. J. Comb. **68**, 281–286 (2018)
9. Korte, B., Lovász, L., Schrader, R.: Greedoids. Springer, New York/Berlin (1991)
10. De Loera, J.A., Petrovic, S., Stasi, D.: Random sampling in computational algebra: helly numbers and violator spaces. J. Symb. Comput. **77**, 1–15 (2016)
11. Matoušek, J., Sharir, M., Welzl, E.: A subexponential bound for linear programming. Algorithmica **16**, 498–516 (1996)
12. Monjardet, B., Raderanirina, V.: The duality between the anti-exchange closure operators and the path independent choice operators on a finite set. Math. Soc. Sci. **41**, 131–150 (2001)
13. Har-Peled, S.: Shortest path in a polygon using sublinear space. J. Symb. Comput. **7**, 19–45 (2016)
14. Sharir, M., Welzl, E.: A combinatorial bound for linear programming and related problems. Lect. Notes Comput. Sci. **577**, 569–579 (1992)

(2, 3)-Cordial Trees and Paths

Manuel Santana, Jonathan Mousley, Dave Brown, and LeRoy B. Beasley

Abstract Recently L. B. Beasley introduced (2, 3)-cordial labelings of directed graphs in [1]. He conjectured that every orientation of a path of length at least five is (2, 3)-cordial, and that every tree of max degree $n = 3$ has a cordial orientation. In this paper we formally define (2, 3)-cordiality from the viewpoint of *quasigroup* cordiality. We show both conjectures to be false, discuss the (2, 3)-cordiality of orientations of the Petersen graph, and establish an upper bound for the number of edges a graph can have and still be (2, 3)-orientable.

Keywords Orientation of an undirected graph · Graph labeling · Cordial labeling · (2,3)-cordial digraph

AMS Classification number 05C20 · 05C38 · 05C78

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, a convention we will use throughout this paper. A (0, 1)-labelling of the vertex set is a mapping $f : V \to \{0, 1\}$ and is said to be *friendly* if approximately one half of the vertices are labelled 0 and the others labelled 1. An induced labelling of the edge set is a mapping $g : E \to \{0, 1\}$ where for an edge uv, $g(uv) = \hat{g}(f(u), f(v))$ for some $\hat{g} : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ and is said to be cordial if f is friendly and about one half the edges of G are labelled 0. A graph, G, is called *cordial* if there exists a cordial induced labelling of the edge set of G.

In this article we investigate a cordial labelling of directed graphs that is not merely a cordial labelling of the underlying undirected graph. This labeling was introduced

M. Santana (✉) · J. Mousley · D. Brown · L. B. Beasley
Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322-3900, USA
e-mail: manuelarturosantana@gmail.com

D. Brown
e-mail: david.e.brown@usu.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_12

by Beasley in [1]. Let $D = (V, A)$ be a directed graph with vertex set V and arc set A, with a (0, 1) vertex set mapping $f : V \to \{0, 1\}$. Let $g : A \to \{-1, 0, 1\}$ be the induced labeling of the arcs of D such that for any $\overrightarrow{uv}$, by which we mean an arc going from u to v, $g(\overrightarrow{uv}) = f(v) - f(u)$. The digraph D is said to be (2, 3)-cordial if there exists a friendly labeling on D with this induced labeling on the arc set such that approximately one third of the arcs receive each labeling. Applications of balanced graph labelings can be found in the introduction of [7].

In this paper we will formally define (2, 3)-cordiality starting from the view of quasi-groups, resolve two conjecture posed in [1], both in the negative, and discuss the (2, 3)-cordiality of the Petersen graph and complete graphs.

2 Preliminaries

In [4] Hovey introduced A cordial graphs, with vertex labeling of an abelian group A. We use the short hand that for any $a \in A$ that V_a, E_a is the number of vertices or edges labeled a respectively We repeat the definition here, and then proceed to generalize it to the directed graph case.

Definition 1 A labeling function f is said to be balanced if it is surjective, and if for all a and b in the image of f, $||f^{-1}(a)| - |f^{-1}(b)|| \leq 1$.

Definition 2 Let A be an abelian group. A graph G is is $A-$cordial if there is a balanced labeling $f : V \to A$ that induces an edge labeling $f(a, b) = f(a) + f(b)$ such that the vertex labeling and edge labeling are balanced.

In [8] Penchenik and Wise generalized this idea by introducing *quasi-group cordiality*. A *quasi-group* Q is a set with binary operation $\cdot$ such that for all $a, b \in Q$ there exist unique $c, d \in Q$ such that $a \cdot c = b$ and $d \cdot a = b$. Particularly all (non-abelian) groups are quasi-groups. They offer the following definition for quasigroup cordiality.

Definition 3 Let Q be a quasi-group and G a directed graph. A labeling $f : V \to Q$ induces a labeling of the arcs in the following way. If (a, b) is an arc with head a, then $f(a, b) = f(a) \cdot f(b)$. If there is a balanced vertex labeling of G that induces a balanced edge labeling of G, then we say that G is $Q - cordial$.

We extend this now to a different form of quasi-group cordiality defined as thus.

Definition 4 Let Q be a quasi-group with subset $\mathbb{Q}$ and G a directed graph. A labeling $f : V \to \mathbb{Q}$ induces an arc labeling as in Definition 3. If there is a balanced vertex labeling of G that induces a balanced arc labeling of G, then we say G is $(\mathbb{Q}, Q) - cordial$.

In this paper consider only the simplest case $(\mathbb{Z}_2, \mathbb{Z}_3^-) - cordial$ defined by Beasley as $(2, 3)-$cordial in [1]. We will offer the formal definition of $(2, 3)-$cordial here.

Definition 5 A labeling $f : V \to \{0, 1\}$ is said to be friendly if it is balanced.

Definition 6 Let $\mathcal{D}_n$ be the set of all digraphs on n vertices. We will define $\mathcal{T}_n$ as the subset of $\mathcal{D}_n$ that consists of all digon-free digraphs, where a digon is a two cycle on a digraph.

Definition 7 Let $D \in \mathcal{T}_n$ with $D = (V, A)$. Let $f : V \to \{0, 1\}$ be a friendly labeling of the vertex set V of D. Let $g : A \to \{-1, 0, 1\}$ be an induced labeling of the arcs of D such that for any $i, j \in \{-1, 0, 1\}$, $-1 \leq |g^{-1}(i)| - |g^{-1}(j)| \leq 1$. Such a labeling is called a (2, 3)-*cordial* labeling, and a digraph $D \in \mathcal{T}_n$ that can possess a (2, 3)-cordial labeling will be called a (2, 3)-*cordial* digraph.

Definition 8 Let $D = (V, A)$ be a digraph with vertex labelling $f : V \to \{0, 1\}$ and with induced arc labelling $g : A \to \{-1, 0, 1, \}$. Define $\Gamma_{f,g}$ to be the real triple $\Gamma_{f,g}(D) = (\alpha, \beta, \gamma)$ where $\alpha = |g^{-1}(1)|$, $\beta = |g^{-1}(-1)|$, and $\gamma = |g^{-1}(0)|$.

Let $D \in \mathcal{T}_n$ and let D^r be the digraph such that every arc of D is reversed, so that $\overrightarrow{uv}$ is an arc in D^r if and only if $\overrightarrow{vu}$ is an arc in D. Let f be a (0, 1)-labeling of the vertices of D and let $g(\overrightarrow{uv}) = f(v) - f(u)$ so that g is a (−1, 0, 1)-labeling of the arcs of D. Let $\overline{f}$ be the complementary (0, 1)-labeling of the vertices of D, so that $\overline{f}(v) = 0$ if and only if $f(v) = 1$. Let $\overline{g}$ be the corresponding induced arc labeling of D, $\overline{g}(\overrightarrow{uv}) = \overline{f}(v) - \overline{f}(u)$.

Lemma 1 *Let $D \in \mathcal{T}_n$ with vertex labeling f and induced arc labeling g. Let $\Gamma_{f,g}(D) = (\alpha, \beta, \gamma)$. Then*

1. $\Gamma_{f,g}(D^r) = (\beta, \alpha, \gamma)$.
2. $\Gamma_{\overline{f},\overline{g}}(D) = (\beta, \alpha, \gamma)$, *and*
3. $\Gamma_{\overline{f},\overline{g}}(D^R) = \Gamma_{f,g}(D)$.

Proof If an arc is labeled 1, −1, 0 respectively then reversing the labeling of the incident vertices gives a labeling of −1, 1, 0 respectively, If an arc $\overrightarrow{uv}$ is labeled 1, −1, 0 respectively, then $\overrightarrow{vu}$ would be labeled −1, 1, 0 respectively. ■

Also in this article we will study when undirected graphs can have their arcs given an orientation such that the resulting graph is (2, 3)-cordial. We finish this section with a couple of definitions for that.

Definition 9 Define $\mathcal{G}_n$ to be the set of all simple, undirected, connected graphs. We say $G \in \mathcal{G}_n$ has vertex set V and edge set E and denote it by $G = (V, E)$.

Definition 10 Let $G \in \mathcal{G}_n$. An *orientation* of G is a digraph $D(G)$ whose vertex set is the same as the vertex set of G and whose arc set consists of the same number of arcs as the number of edges of G such that given an edge $\{u, v\}$ of G, either $\overrightarrow{uv}$ or $\overrightarrow{vu}$ is an arc of $D(G)$ but not both, so that $D(G)$ is digon free. A graph G is said to be (2, 3)-*orientable* if there exists and orientation of G, $D(G)$, that is (2, 3)-cordial.

3 Resolution of Two Conjectures

We begin with the first conjecture.

Conjecture 1 ([1, Conjecture 4.1]) Every orientation of every path is (2, 3) cordial except for a path with four vertices.

The orientation in Fig. 1 of the ten path has no cordial labeling.

We used the following brute force algorithm to test to see if a certain arc orientation is cordial on a friendly labeling of a graph.

Data: Arcs, Verticies
Result: Determine if an orientation is cordial on a path
for *arc in Arcs* **do**
  current = first vertex
  next = second vertex
  if *arc is left* **then**
    edgeLabel = current - next
  end
  else
    edgeLabel = current - next
  end
  store edge label
  current = next
  next = next vertex
end
if *edge labels are cordial* **then**
  return it is cordial
end

In investigating the conjecture we had to test every possible friendly labeling and arc orientation on ten vertices. If we let n denote the number of vertices on the path, then checking every possible friendly labeling against every arc set has complexity of $O(2^k)$. As a slight optimization by Lemma 1 with out loss of generality we can fix the first arc and the first label and still account for all cases up to isomorphism. That means there will be 2^{n-2} arc orientations to test. In calculating all arc orientations of the ten path we found the only orientation that is not (2, 3)-coridal is the one in Fig. 1. The next known case of a non (2, 3)-orientable path is one on 22 vertices with the same alternating arc structure.

Conjecture 2 ([1, Conjecture 2.3]) Every tree of max degree 3 is (2, 3)-orientable.

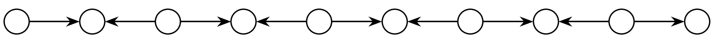


Fig. 1 The ten path with no cordial labeling

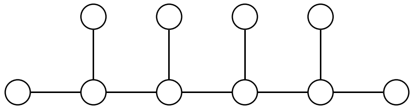


Fig. 2 A tree of max degree 3 that is not (2, 3)-orientable 15 Sketch 1: The neighborhood of 0101 on H 15 Sketch 2: The portion of the replacement of H by C that replaces node 0101 15 Sketch 3: The portion of $H \uparrow C$ that replaces node 010101

Fig. 3 Left and right subgraph

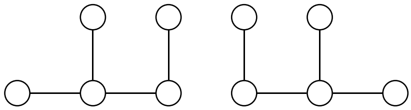


Figure 2 is a counter example. Though easily proved with a computer by a similar argument as on the one above, we will prove by cases.

Proof First note that the value of γ in a vertex labeling of the graph does not depend on arc orientation. Thus we show that there does not exist a friendly labeling on the vertex set of the graph that induces an edge labeling of $\gamma = 3$, by only considering vertex labelings on the undirected graph. We will separate our graph into a right and left subgraph and then connect them (Fig. 3).

Let x be the number of vertices labeled 1 on one subgraph in a friendly labeling of the whole graph. In order for the graph to be cordial x cannot be 4 or 5 since this would make $\gamma > 3$ on the entire graph. Therefore x also cannot be 0 or 1 since that would mean x would be too large on the other sub graph. Thus we must have $x = 2$ on one subgraph $x = 3$ on the other subgraph.

Given these constraints it is not possible to have a sub graph such that $\gamma = 0$, since there is no way to label two of the vertices on a subgraph without having at least two vertices of the other label connected. This means we do not need to account for the case when $\gamma = 3$ on a subgraph, since the other subgraph cannot have $\gamma = 0$. Now we will consider two cases (Figs. 4 and 5).

Case 1. With out loss of generality let the left subgraph have $\gamma = 1$, $x = 2$ and the right subgraph have $\gamma = 2$, $x = 3$. This would mean we would need to connect the subgraphs such that the connecting edge will not be labeled 0. Considering all cases we see that there is no way to connect the two subgraphs to make the full graph without $\gamma = 4$ on the full graph. **Case 2.** With out loss of generality let the left subgraph have $\gamma = 1$, $x = 2$, and the right subgraph have $\gamma = 1$, $x = 3$. This would

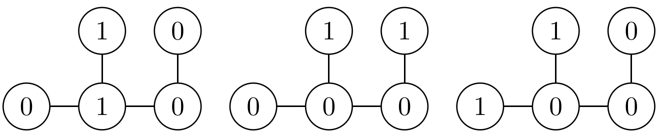


Fig. 4 All $\gamma = 2$ subgraph labelings

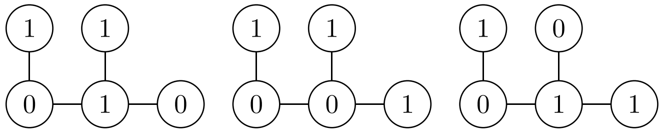


Fig. 5 All $\gamma = 1$ subgraph labelings

mean we need to find a way to connect the two subgraphs such that the connecting edge is labeled 0. Considering all cases shows that this is not possible.

The structure of the proof leads to the following theorem.

Theorem 1 *Let $G \in \mathcal{G}_n$. We define $\Lambda(G)$ to be the number of edges, uv such that vertices u and v have the same label for a given friendly labeling on G. G is (2, 3)-orientable if and only if there exsits a friendly vertex labling on G such that $\Lambda(G) = \lceil \frac{1}{3}|E| \rceil$ or $\Lambda(G) = \lfloor \frac{1}{3}|E| \rfloor$, where $|E|$ is the cardinality of the edge set of G*

Proof Suppose G satisfies $\Lambda(G) = \lceil \frac{1}{3}|E| \rceil$ or $\Lambda(G) = \lfloor \frac{1}{3}|E| \rfloor$. This would mean about $\frac{2}{3}$ of the edges are connected by vertices of different labels, and therefore arcs may be assigned such that G is (2, 3)-cordial. If G is (2, 3)-cordial, then clearly G has a friendly labeling that satisfies the above conditions.

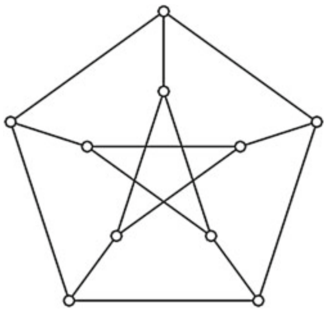
Though fairly intuitive we will now use Theorem 1 to show two more results that stem from it in the following section.

4 Theorem 3.1 Applied

Theorem 2 *The Petersen Graph, Fig. 6, is not (2, 3) -orientable.*

Proof By Theorem 1 only need to show that there is no friendly labeling such that one third of the edges are connected with the same label of the vertex. Again let x be the number of vertices labeled 1. We will start by dividing the Petersen graph into two subgraphs and then connect them (Fig. 7).

Fig. 6 The petersen graph



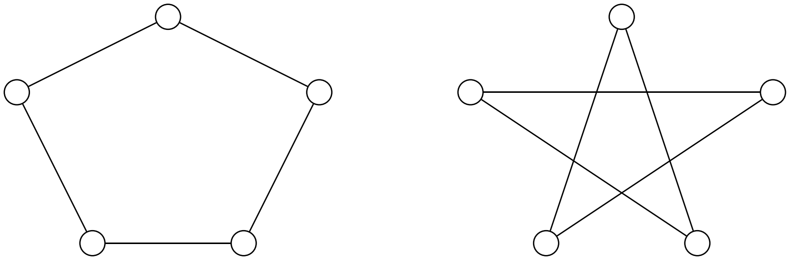


Fig. 7 Two subgraphs of the petersen graph

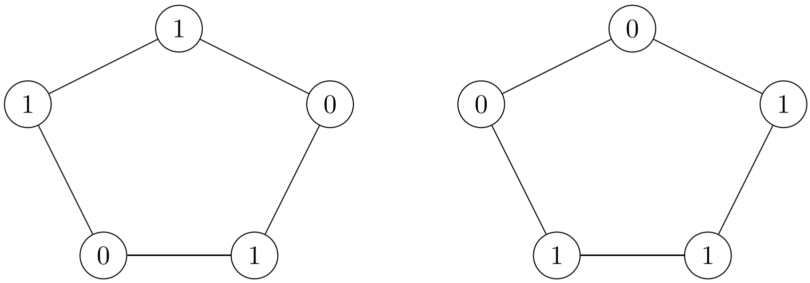


Fig. 8 $\gamma = 1, \gamma = 3$

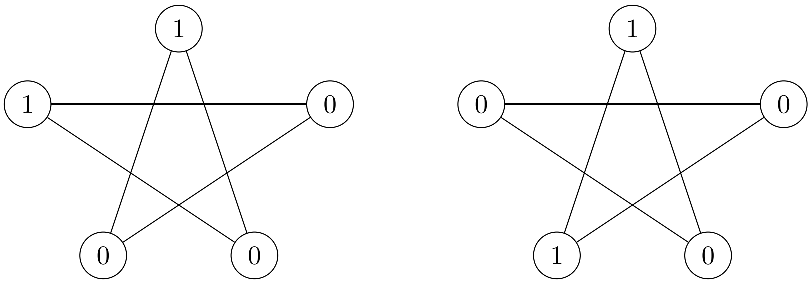


Fig. 9 $\gamma = 1, \gamma = 3$

With out loss of generality the only way for our friendly label to be cordial we must have one sub graph have $x = 2$, and have $x = 3$ on the other. The case when either sub graph has $x \geq 4$ of a certain label would result in γ being too large. Let us first consider the outer subgraph letting $x = 3$ for this subgraph.

There are two cases as shown in Fig. 8, up to isomorphism. $\gamma = 2$ is not possible since there is no way to label two 0's and two 1's without the third 1 connecting to another vertex labeled one. This means we need to connect the star sub graph such exactly two or exactly four more edges with zero are produced.

Figure 9 represents all labelings of the inner subgraph up to isomorphism, with $\gamma = 1$ or $\gamma = 3$ possible on the subgraph. Upon inspection there is no possible way

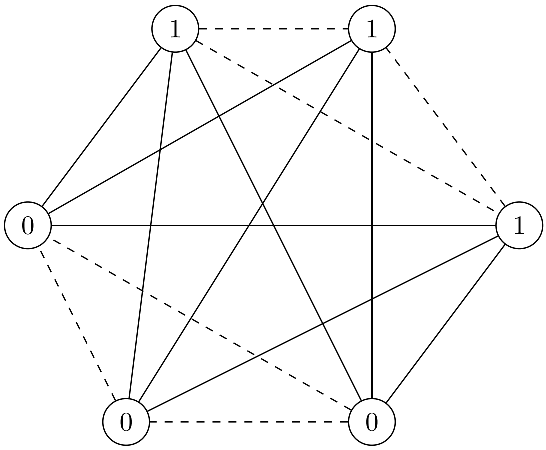


Fig. 10 A complete graph. Dashed lines represent edges labeled zero regardless of arc orientation

to connect any rotation of the inner sub graph with either of the outer sub graphs such that $\gamma = 5$ for the resulting Petersen graph. Therefore by Theorem 2 the Petersen graph is not (2, 3) cordial.

One more example shows how this theorem proved useful in proving an upper bound on the size of the edge set for any graph to be (2, 3)-orientable. For work on the upper bounds for other graph labelings see [6] (Fig. 10).

Theorem 3 *Given a directed graph $G = (V, E)$ with vertex set V and $n = |V|$ with $n \geq 6$, and edge set E. The maximum size of E such that G is* (2, 3) *orientable for any given n is*

$$|E|_{max} = \binom{n}{2} - Z + \left\lceil \frac{1}{2}\left(\binom{n}{2} - Z\right)\right\rceil \tag{1}$$

$$Z = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}. \tag{2}$$

Proof It can be shown, see [2], that any tournament with $n \leq 5$ vertices is (2, 3)-cordial, save for the case when $n = 4$. Thus we begin with a complete graph with $n \geq 6$. Recall that the number of vertices on a complete graph is $\binom{n}{2}$. Thus Z is the number of edges in two cliques comprised of the the subgraph of the vertices labeled 0 and the vertices labeled 1. In this way Z counts the number of edges labeled zero regardless of arc orientation. If n is even that will mean that $Z = 2\binom{\frac{n}{2}}{2}$. If n is odd then Z is as above. This also implies there are $\binom{n}{2} - Z$ edges that cannot be labeled zero.

For every complete graph with $n \geq 6$ vertices $Z > 1/3\binom{n}{2}$. By Theorem 1 this is too many edges to be $(2, 3)-$orientable. The number edges labeled 0 must be balanced with the number of edges labeled 1 and -1, and the number of edges labeled 1 or -1 is $1/2(\binom{n}{2} - Z)$. Thus the upper bound on the number of edges a graph on n vertices can have and be $(2, 3)-$orientable is as in Eq. 2.

References

1. Beasley, L.B.: Cordial digraphs. J. Comb. Math. Comb. Comput. **117**, 1–23 (2022)
2. Beasley, L.B., Santana, M.A., Mousley, J.M., Brown, D.E.: Cordiality of digraphs. J. Algebra Comb. Discret. Struct. Appl. **10**, 1–13 (2023)
3. Cahit, I.: Cordial graphs: a weaker version of graceful and harmonious graphs. Ars Comb. **23**, 201–208 (1987)
4. Hovey, M.: A-cordial graphs. Discret. Math. **93**, 183–194 (1991)
5. Salehi, E.: PC-labelling of a graph and its PC-set. Bull. Inst. Comb. Appl. **58**, 112–121 (2010)
6. Seoud, M.A., Salim, M.A.: Upper bounds of four types of graph labelings. Ars Comb. **127**, 271–278 (2016)
7. Deepa, S., Jaspreet, K.: Full friendly index set-I. Discret. Appl. Math. **161**, 1262–1274 (2013)
8. Penchenik, O., Wise, J.: Generalized graph cordiality. Discussiones Mathematicae Graph Theory **32**, 557–567 (2012)

(2, 3)-Cordial Oriented Hypercubes

Jonathan M. Mousley, LeRoy B. Beasley, Manuel A. Santana, and David E. Brown

Abstract In this article we investigate the existence of (2, 3)-cordial labelings of oriented hypercubes. In this investigation, we determine that there exists a (2, 3)-cordial oriented hypercube for any dimension divisible by 3. Next, we provide examples of (2, 3)-cordial oriented hypercubes of dimension not divisible by 3 and state a conjecture on existence for dimension $3k + 1$. We close by presenting the only 3D oriented hypercubes up to isomorphism that are not (2, 3)-cordial.

Keywords Labelings · Oriented graphs · Hypercube · Quasigroup cordiality

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, a convention we will use throughout this paper. A (0, 1)-labeling of the vertex set is a mapping $f : V \to \{0, 1\}$ and is said to be *friendly* if approximately one half of the vertices are labeled 0 and the others labeled 1. An induced labeling of the edge set is a mapping $g : E \to \{0, 1\}$ where for an edge uv, $g(uv) = \hat{g}(f(u), f(v))$ for some $\hat{g} : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ and is said to be cordial if f is friendly and about one half the edges of G are labeled 0. A graph, G, is called *cordial* if there exists a cordial induced labeling of the edge set of G [4].

In this article we investigate a labeling of directed graphs that is not simply a cordial labeling of the underlying undirected graph. The labeling scheme we investigate here was introduced by Beasley in [2]. Let $D = (V, A)$ be a directed graph with vertex set V and arc set A with a friendly vertex set mapping f. Let $g : A \to \{-1, 0, 1\}$

J. M. Mousley (✉) · L. B. Beasley · M. A. Santana · D. E. Brown
Department of Mathematics and Statistics, Utah State University, Logan, UT 84322, USA
e-mail: jonathanmousley@gmail.com

L. B. Beasley
e-mail: leroy.b.beasley@aggiemail.usu.edu

D. E. Brown
e-mail: david.e.brown@usu.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_13

be the induced labeling of the arcs of D such that for any arc initiating at u and terminating at v, $\overrightarrow{uv}$, $g(\overrightarrow{uv}) = f(v) - f(u)$. D is said to be (2, 3)-cordial if there exists a friendly vertex set mapping f such that g labels approximately one third of arcs 0, approximately one third of arcs 1, and approximately one third of arcs -1. Applications of balanced graph labelings can be found in the introduction of [5].

In [3], (2, 3)-cordial labelings are investigated on oriented trees, oriented paths, orientations of the Petersen graph, and complete graphs. In this article we consider (2, 3)-cordial labelings on oriented hypercubes. We confirm the existence of (2, 3)-cordial oriented hypercubes for every dimension $3k$ for $k \in \mathbb{N}$. Additionally, we provide examples of (2, 3)-cordial oriented hypercubes for dimensions 4 and 7 and conjecture that there exists a (2, 3)-oriented hypercube of dimension $3k + 1$ for every $k \in \mathbb{N}$. We close by presenting the only 3D oriented hypercubes up to isomorphism that are not (2, 3)-cordial, that is we present the only two 3D oriented hypercube up to isomorphism that do not admit a (2, 3)-cordial labeling.

2 Preliminaries

Definition 1 Let Z be a finite set and $f : Z \to \{0, 1\}$ be a mapping. The mapping f is a called a (0, 1)-labeling of the set Z. If $-1 \le |f^{-1}(0)| - |f^{-1}(1)| \le 1$, that is, the number of elements of Z labeled 0 and the number of elements of Z labeled 1 are about equal, then we say that the labeling f is *friendly*.

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Let $f : V \to \{0, 1\}$ be a labeling of V. An induced labeling of the edge set is a mapping $g : E \to \{0, 1\}$ where for an edge uv, $g(uv) = \hat{g}(f(u), f(v))$ for some $\hat{g} : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ and is said to be cordial if f and g are both friendly labelings. A graph G is *cordial* if there exists a cordial induced labeling of the edge set of G. In this article, as in [2], we define a cordial labeling of directed graphs that is not simply a cordial labeling of the underlying undirected graph.

Definition 2 Let $D = (V, A)$ be a directed graph with vertex set V and arc set A. Let $f : V \to \{0, 1\}$ be a friendly vertex labeling and let g be the induced labeling of the arc set, $g : A \to \{0, 1, -1\}$ where for an arc $\overrightarrow{uv}$, $g(\overrightarrow{uv}) = f(v) - f(u)$. The labelings f and g are (2, 3)-*cordial* if f is friendly and about one third the arcs of D are labeled 1, one third are labeled -1 and one third labeled 0, that is, for any $i, j \in \{0, 1, -1\}$, $-1 \le |g^{-1}(i)| - |g^{-1}(j)| \le 1$. A digraph, D, is called (2, 3)-*cordial* if there exists (2, 3)-cordial labelings f of the vertex set and g of the arc set of D. An undirected graph G is said to be *(2, 3)-orientable* if there is an orientation of the edges of G which is a (2, 3)-cordial digraph.

See [3] for an equivalent definition of (2, 3)-cordiality and (2, 3)-orientability beginning from the view of quasi-groups and *quasi-group cordiality* introduced in [7].

Definition 3 Let $\mathcal{D}_n$ be the set of all digraphs on n vertices. We will define $\mathcal{T}_n$ as the subset of $\mathcal{D}_n$ that consists of all digon-free digraphs, where a digon is a 2 cycle on a digraph.

Definition 4 Let $D = (V, A)$ be a digraph with vertex labeling $f : V \to \{0, 1\}$ and with induced arc labeling $g : A \to \{0, 1, -1\}$. Define $\Lambda_{f,g} : \mathcal{D}_n \to \mathbb{N}^3$ by $\Lambda_{f,g}(D) = (\alpha, \beta, \gamma)$ where $\alpha = |g^{-1}(1)|$, $\beta = |g^{-1}(-1)|$, and $\gamma = |g^{-1}(0)|$.

Let $D \in \mathcal{T}_n$ and let D^R be the digraph such that every arc of D is reversed, so that $\overrightarrow{uv}$ is an arc in D^R if and only if $\overrightarrow{vu}$ is an arc in D. Let f be a (0, 1)-labeling of the vertices of D and let $g(\overrightarrow{uv}) = f(v) - f(u)$ so that g is a $(1, -1, 0)$-labeling of the arcs of D. Let $\overline{f}$ be the complementary (0, 1)-labeling of the vertices of D, so that $\overline{f}(v) = 0$ if and only if $f(v) = 1$. Let $\overline{g}$ be the corresponding induced arc labeling of D, $\overline{g}(\overrightarrow{uv}) = \overline{f}(v) - \overline{f}(u)$.

Lemma 1 *Let $D \in \mathcal{T}_n$ with vertex labeling f and induced arc labeling g. Let $\Lambda_{f,g}(D) = (\alpha, \beta, \gamma)$. Then*

1. $\Lambda_{f,g}(D^R) = (\beta, \alpha, \gamma)$.
2. $\Lambda_{\overline{f},\overline{g}}(D) = (\beta, \alpha, \gamma)$, *and*
3. $\Lambda_{\overline{f},\overline{g}}(D^R) = \Lambda_{f,g}(D)$.

Proof If an arc is labeled 1, –1, 0 respectively then reversing the labeling of the incident vertices gives a labeling of –1, 1, 0 respectively, If an arc $\overrightarrow{uv}$ is labeled 1, –1, 0 respectively, then $\overrightarrow{vu}$ would be labeled –1, 1, 0 respectively.

Example 1 Now, consider a graph, III_n in $\mathcal{G}_n$ consisting of three parallel edges and n-6 isolated vertices. Is III_n (2, 3)-orientable? If $n = 6$, the answer is no, since any friendly labeling of the six vertices would have either no arcs labeled 0 or two arcs labeled 0. In either case, the orientation would never be (2, 3)-cordial. That is III_6 is not (2, 3)-orientable, however with additional vertices like III_7 the graph is (2, 3)-orientable.

Thus, for our investigation here, we will use the convention that a graph, G, is (2, 3)-orientable/(2, 3)-cordial if and only if the subgraph of G induced by its non-isolated vertices, $\hat{G}$, is (2, 3)-orientable/(2, 3)-cordial.

3 Existence

We begin with examples of (2, 3)-cordial oriented hypercubes for dimensions less than and equal to 3.

Example 2 (*Dimension* 1) Given in Fig. 1a is a 1-dimensional oriented hypercube C_1 that is (2, 3)-cordial as by the friendly vertex labeling f shown, $\Lambda_{f,g}(C_1) = (1, 0, 0)$.

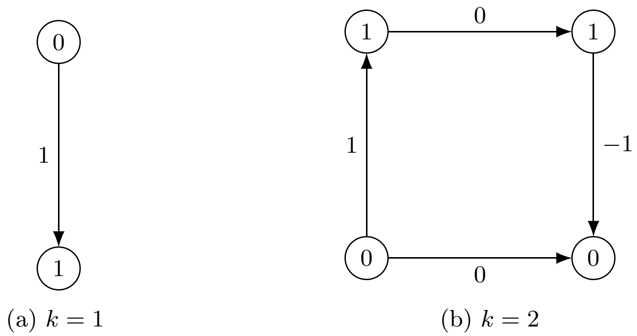
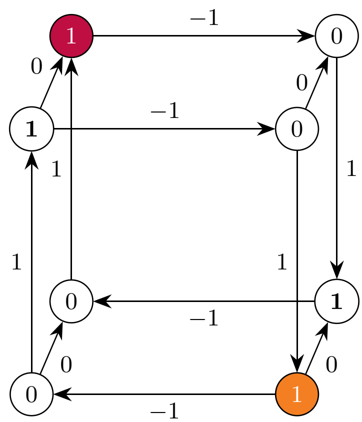


Fig. 1 (2, 3)-cordial k-dimensional oriented hypercubes

Fig. 2 (2, 3)-cordial 3D oriented hypercube



Example 3 (*Dimension* 2) Given in Fig. 1b is a 2-dimensional oriented hypercube C_2 that is (2, 3)-cordial as by the friendly vertex labeling f shown, $\Lambda_{f,g}(C_2) = (1, 1, 2)$.

Example 4 (*Dimension* 3) Given in Fig. 2 is a 3-dimensional oriented hypercube C_3 that is (2, 3)-cordial as by the friendly vertex labeling f shown, $\Lambda_{f,g}(C_3) = (4, 4, 4)$.

In Examples 2, 3, and 4, we see for dimension less than or equal to 3, there exist (2, 3)-cordial oriented hypercubes. The question of existence remains unanswered for dimension greater than 3. In the following theorem, this question is answered for the case in which dimension is a multiple of 3.

Theorem 1 *Let n be a multiple of* 3*, then there exists an n-dimensional oriented hypercube C_n that is (2, 3)-cordial.*

Proof We proceed by induction on the dimension n in multiples of 3. Example 4 serves as a base case for $n = 3$. Suppose the claim is true for some k that is a multiple of 3. Then there exists some oriented hypercube $Q_k = (V_k, A_k)$ of dimension k that is (2, 3)-cordial. That is, there exists a friendly labeling $f : V_k \to \{0, 1\}$ such that

$$\Lambda_{f,g} = \left(\frac{1}{3}|A_k|, \frac{1}{3}|A_k|, \frac{1}{3}|A_k|\right)$$

where g is defined as in Definition 2. We aim to construct an oriented hypercube $Q_{k+3} = (V_{k+3}, A_{k+3})$ of dimension $k + 3$ that is (2, 3)-cordial. We begin by constructing an oriented hypercube $Q_{k+1} = (V_{k+1}, A_{k+1})$ of dimension $k + 1$. Let L_k denote the digraph Q_k with vertex labeling f and induced arc labeling g applied and $\overline{L_k}$ denote the digraph Q_k with vertex labeling $\overline{f}$ and induced arc labeling $\overline{g}$. Now, let us draw arcs from L_k to $\overline{L_k}$ according to the trivial digraph isomorphism. That is, define an arc initiating at vertex x in L_k to vertex y in $\overline{L_k}$ if and only if $x = y$. We then label each of these arcs as $\overline{f}(x) - f(x)$. The result is a labeled digraph, call it L_{k+1}. By construction, the underlying digraph of L_{k+1} is an oriented hypercube of dimension $k + 1$, call it Q_{k+1}. Define f_{k+1} and g_{k+1} to be vertex and arc labelings of Q_{k+1} respectively such that Q_{k+1} with labelings f_{k+1} and g_{k+1} applied is the labeled oriented hypercube L_{k+1}. As f_{k+1} applies friendly labelings f and $\overline{f}$ to complementary subgraphs of Q_{k+1}, f_{k+1} is a friendly labeling. Further, g_{k+1} applies g and $\overline{g}$ to complementary subgraphs of Q_{k+1} and labels each arc $\overrightarrow{xx}$ from L_k to $\overline{L_k}$, $\overline{f}(x) - f(x)$. Then, as f and $\overline{f}$ are friendly,

$$\Lambda_{f_{k+1},g_{k+1}} = \left(\frac{2}{3}|A_k| + 2^{k-1}, \frac{2}{3}|A_k| + 2^{k-1}, \frac{2}{3}|A_k|\right).$$

Now, we repeat our procedure, constructing an oriented hypercube $Q_{k+2} = (V_{k+2}, A_{k+2})$ of dimension $k + 2$. We draw arcs from L_{k+1} and $\overline{L_{k+1}}$. Just as in the previous case, we define an arc from a vertex x in L_{k+1} to vertex y in $\overline{L_{k+1}}$ if and only if $x = y$, and we label this arc $\overline{f_{k+1}}(x) - f_{k+1}(x)$. The result, as in the previous step, is a labeled digraph, call it L_{k+2}. The underlying digraph of L_{k+2} is again an oriented hypercube, now of dimension $k + 2$, call it Q_{k+2}. As before, define f_{k+2} and g_{k+2} to be vertex and arc labelings of Q_{k+2} respectively such that when applied to Q_{k+2} yield the labeled oriented hypercube L_{k+2}. As before, f_{k+2} applies friendly labelings f_{k+1} and $\overline{f_{k+1}}$ to complementary subgraphs, thus f_{k+2} is friendly. Also, g_{k+2} applies g_{k+1} and $\overline{g_{k+1}}$ to complementary subgraphs of Q_{k+2} and labels each arc $\overrightarrow{xx}$ from L_{k+1} to $\overline{L_{k+1}}$, $\overline{f_{k+1}}(x) - f_{k+1}(x)$ and $\overline{f}$. As f_{k+1} and $\overline{f_{k+1}}$ are friendly,

$$\Lambda_{f_{k+2},g_{k+2}} \left(\left(\frac{4}{3}|A_k| + 2^k\right) + 2^k, \left(\frac{4}{3}|A_k| + 2^k\right) + 2^k, \frac{4}{3}|A_k|\right).$$

In our final step, we construct an oriented hypercube $Q_{k+3} = (V_{k+3}, A_{k+3})$ of dimension $k + 3$ by drawing edges between two identically labeled cubes L_{k+2}. We draw an arc from vertex x in the first L_{k+2} to vertex y in the second L_{k+2} if and only if

$x = y$ and we label this arc $f_{k+2}(x) - f_{k+2}(x) = 0$. The result is a labeled digraph, call it L_{k+3}. The underlying digraph of L_{k+3} is an oriented hypercube of dimension $k + 3$, call it Q_{k+3}. Finally, we define f_{k+3} and g_{k+3} to be vertex and arc labelings of Q_{k+3} respectively such that when applied to Q_{k+3} yield the labeled oriented hypercube L_{k+3}. Then f_{k+3} simply labels each complementary subgraph Q_{k+2} according to f_{k+2} and g_{k+3} labels each complementary subgraph Q_{k+2} according to g_{k+2} and the newly drawn 2^{k+2} edges are labeled 0. Let $\omega = \frac{4}{3}|A_k| + 2^{k+1}$. Then

$$\Lambda_{f_{k+3},g_{k+3}}(Q_{k+3}) = \left(2\omega, 2\omega, \frac{8}{3}|A_k| + 2^{k+2}\right).$$

Simplifying, we have

$$\Lambda_{f_{k+3},g_{k+3}}(Q_{k+3}) = \left(\frac{1}{3}(k+3)2^{k+2}, \frac{1}{3}(k+3)2^{k+2}, \frac{1}{3}(k+3)2^{k+2}\right).$$

As f_{k+3} is constructed to be a friendly labeling, the above implies Q_{k+3} is $(2, 3)$-cordial.

3.1 A Conjecture on Existence for Dimension $3k + 1$.

We have now answered the question of existence of $(2, 3)$-cordial oriented hypercubes for dimension less than and equal to 3 and all dimensions which are a multiple of 3. In this section, we now consider the existence of $(2, 3)$-cordial oriented hypercubes with dimension $3k + 1$ for $k \in \mathbb{N}$.

Example 5 (*Tesseract, Dimension* 4) Given in Fig. 3a and b are two 3D oriented hypercubes, A and B, that are $(2, 3)$-cordial as demonstrated by the friendly vertex labelings and induced arc labelings shown. In Fig. 4, edges are drawn between the vertices of the oriented cube B (outer) of Fig. 3b and the vertices of oriented cube A (inner) of Fig. 3a. By the induced arc labeling scheme g, 2 of these 8 edges (red) receive an induced label of 0 regardless of their orientation, and the remaining 6 edges (dashed) can be oriented such that 3 receive label 1 and 3 receive label -1, yielding a 4D oriented hypercube. As the outer and inner cubes of Fig. 4 have $(2, 3)$-cordial labelings applied, this is to say the dashed arcs in Fig. 4 can be oriented such that the result is a $(2, 3)$-cordial 4D oriented hypercube.

Definition 5 Let D_1 and D_2 be directed graphs with same sized vertex sets and friendly vertex labelings f_1 and f_2 respectively. Let $\beta\colon V(D_1) \to V(D_2)$ be a bijection on the vertex sets of D_1 and D_2 respectively. Then, let $h\colon V(D_1) \to \{0, 1\}$ such that $h(v_1) = |f_1(v) - f_2(\beta(v))|$ for all $v \in V(D_1)$. Then define $\Phi_\beta(D_1, D_2) = |h^{-1}(0)|$. In contexts where the bijection β is clear, we write $\Phi(D_1, D_2)$.

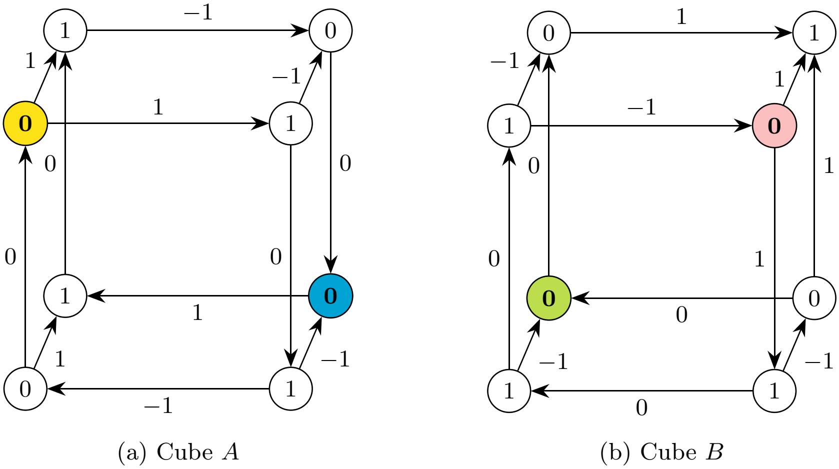


Fig. 3 (2, 3)-cordial 3D oriented hypercubes, A and B

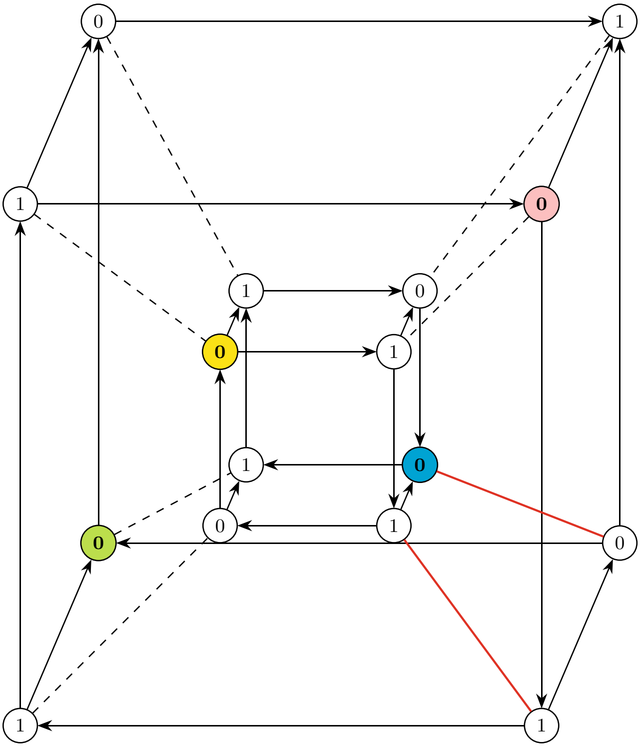


Fig. 4 4D (2, 3)-cordial oriented hypercube constructed from cubes A and B

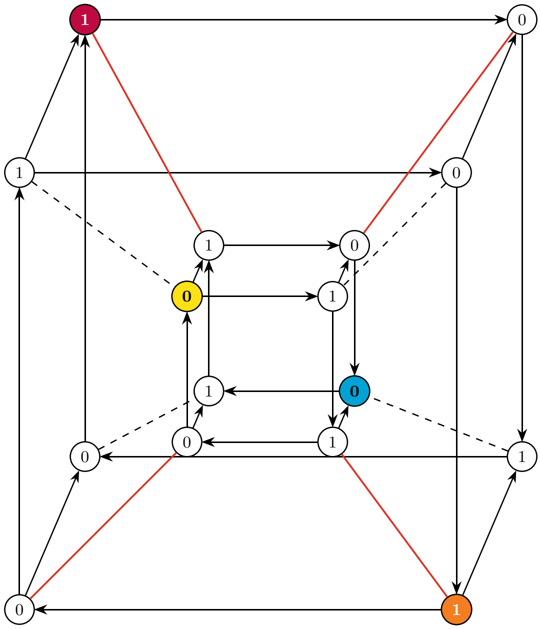
Remark 1 In the context of the previous definition, given arcs are drawn between vertices of digraphs D_1 and D_2 according to the bijection β, $\Phi(D_1, D_2)$ is simply the count of arcs shared by D_1 and D_2 that receive induced label 0 by g. In the following example, we work within such a context, and therefore, interpret $\Phi(D_1, D_2)$ this way.

Example 6 *(Dimension* 7*)* We have introduced 3 3D oriented hypercubes in Fig. 2a and b each with a (2, 3)-cordial labeling. Let us denote the labeled oriented cube in Fig. 2 as C. For this example, we adopt the convention that A, B, and C refer to labeled digraphs rather than the underlying unlabeled digraphs. We seek to construct a (2, 3)-cordial 7D oriented hypercube from these three cubes, A, B, and C. As given in Fig. 4, cubes A and B can be combined to form a 4D oriented cube such that only 2 of the arcs they share receive label 0. That is by the bijection between $V(A)$ and $V(B)$ defined by the edges drawn in Fig. 4, $\Phi(A, B) = 2$. In Fig. 5, we construct 2 individual 4D oriented cubes, 1 from cubes A and C, and 1 from cubes B and C. As in Fig. 4, arcs drawn between distinct cubes define bijections between distinct vertex sets. With respect to these bijections, in Fig. 5, we see $\Phi(A, C) = \Phi(B, C) = 4$.

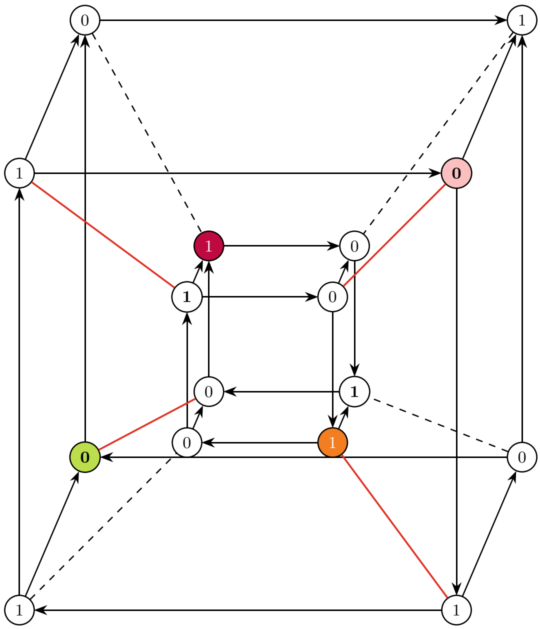
Now, for $D \in \{A, B, C\}$, given f is the friendly vertex labeling of D and g is the induced arc labeling of D, define $\overline{D}$ to be the underlying digraph D labeled instead by $\overline{f}$ and $\overline{g}$. Recall, such a labeling is (2, 3)-cordial by Lemma 1. Then, for all $D_1, D_2 \in \{A, B, C\}$, $\Phi(\overline{D_1}, \overline{D_2}) = \Phi(D_1, D_2)$ and $\Phi(\overline{D_1}, D_2) = \Phi(D_1, \overline{D_2}) = 8 - \Phi(D_1, D_2)$. Then, for all $D_1 \neq D_2$, taking $\Phi(D_1, \overline{D_2})$ to be with respect to the appropriate bijection between $V(D_1)$ and $V(D_2)$ defined in either Fig. 4, 5a, or b, we have $\Phi(\overline{A}, B) = 6$ and $\Phi(\overline{B}, C) = \Phi(\overline{A}, C) = 4$. Lastly, note we can construct a 4D oriented hypercube between 2 identical cubes D by drawing arcs between like vertices. According to such a bijection, $\Phi(D, D) = 8$. Now, define $\gamma = \{\overline{A}, A, \overline{B}, B, \overline{C}, C\}$. Then for all $Q_1, Q_2 \in \gamma$, $\Phi(Q_1, Q_2)$ with reference to the appropriate aforementioned bijections between $V(Q_1)$ and $V(Q_2)$ are given below in Table 1. As Φ is commutative by definition, the lower diagonal of Table 1 is left empty.

Now, we construct a (2, 3)-cordial 7D oriented hypercube by drawing edges between cubes in the set γ according to the previously defined vertex set bijections. Given in Fig. 6 are 2 $6D$ oriented hypercubes constructed from cubes in γ where for all $Q_1, Q_2 \in \gamma$, an edge between cube Q_1 and Q_2 signifies 8 edges between cubes Q_1 and Q_2 drawn according to the appropriate bijection between $V(Q_1)$ and $V(Q_2)$. Note, in Fig. 6, an edge between Q_1 and Q_2 is labeled $\Phi(Q_1, Q_2)$. Observe for each cube in Fig. 6, the edge label sum is equal to 32. By Remark 1, this is to say a total of 32 edges shared by distinct cubes in γ receive induced label 0 by g regardless of orientation. As each 6D cube in Fig. 6 has a total of $12 \cdot 8 = 96$ edges drawn between cubes in γ, and $32 = 96/3$, the remaining edges not labeled 0 in each 6D cube can be oriented such that half are labeled 1 and half are labeled -1 making each 6D cube (2, 3)-cordial. Now, in Fig. 7 a 7D cube is constructed from these (2, 3)-cordial 6D oriented cubes. In Fig. 7 as in Fig. 6, an edge between cube Q_1 and Q_2 signifies 8 edges between cubes Q_1 and Q_2 drawn according to the appropriate

Fig. 5 4D oriented cubes constructed from cubes A, B, C



(a) A (inner) and C (outer)



(b) B (outer) and C (inner)

Table 1 $\Phi(Q_1, Q_2)$ for all $Q_1, Q_2 \in \gamma$

Φ	$\overline{A}$	A	$\overline{B}$	B	$\overline{C}$	C
$\overline{A}$	8	0	2	6	4	4
A		8	6	2	4	4
$\overline{B}$			8	0	4	4
B				8	4	4
$\overline{C}$					8	0
C						8

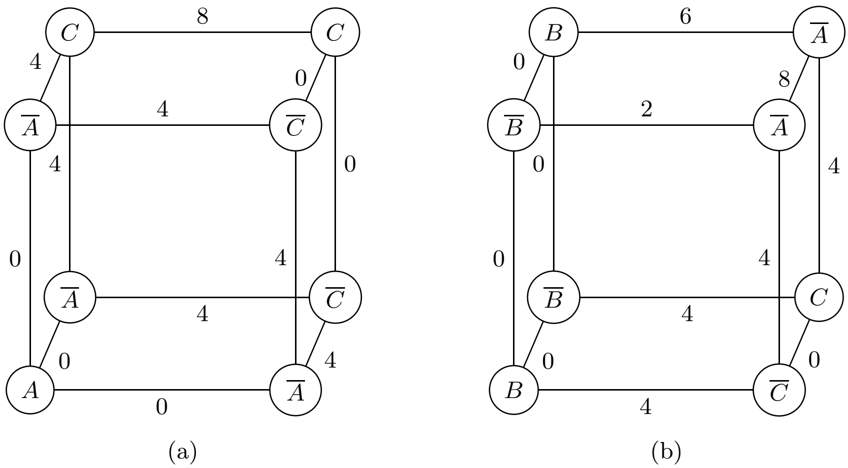


Fig. 6 (2, 3)-cordial 6D oriented hypercubes

bijection between $V(Q_1)$ and $V(Q_2)$, and each edge between Q_1 and Q_2 is labeled $\Phi(Q_1, Q_2)$.

In Fig. 7, the edge label sum is 22. Similar to before, by Remark 1, this is to say a total of 22 of the 64 edges drawn between vertices of the inner 6D cube and the outer 6D cube receive label 0. The remaining 42 edges can be oriented such that 21 receive a label of 1 and 21 receive a label of -1 by g. Because each 6D cube is (2, 3)-cordial, such a choice yields a (2, 3)-cordial 7D oriented hypercube.

In the previous two examples we have confirmed there exist (2, 3)-cordial oriented hypercubes of dimension $3k + 1$ for $k = 1, 2$. We now state the following conjecture.

Conjecture 1 Let n be a multiple of 3, then there exists an $(n + 1)$-dimensional oriented hypercube C_{n+1} that is (2, 3)-cordial.

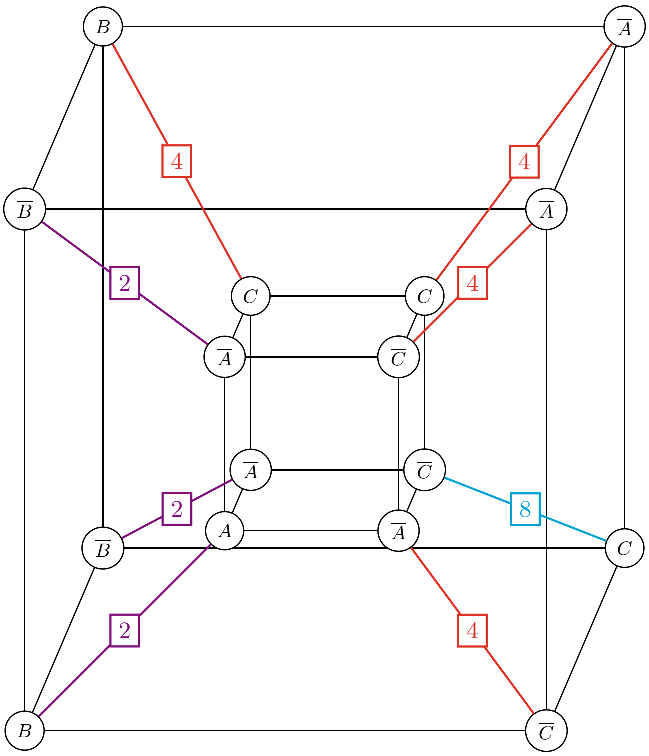


Fig. 7 7D (2, 3)-cordial oriented hypercube constructed from γ cubes

4 Non-(2, 3)-Cordial Oriented Cubes

In the previous section, we demonstrated the existence of (2, 3)-cordial oriented hypercubes of varying dimension including dimension 3. Now, we demonstrate the existence of oriented cubes that are not (2, 3)-cordial, that is, we demonstrate there exist oriented cubes that do not admit (2, 3)-cordial labelings.

Theorem 2 *The oriented cube V in Fig. 8 is not* (2, 3)*-cordial.*

Proof There are $\binom{8}{4}$ possible friendly vertex labelings for the oriented cube V. By a brute force algorithm, it can be shown that none of these vertex labelings induces a (2, 3)-cordial labeling.

Corollary 1 *The oriented cube V^R for V in Fig. 8 is not* (2, 3)*-cordial.*

Proof By Lemma 1.1, $\Lambda_{f,g}(V) = \Lambda_{f,g}(V^R)$ for any vertex-arc labeling f, g. Thus, given V does not admit a (2, 3)-cordial labeling by Theorem 2, neither does V^R. Equivalently, V^R is not (2, 3)-cordial.

Theorem 3 *The cubes V and V^R are the only oriented cubes up to isomorphism that are not* (2, 3)*-cordial.*

Proof This can be shown by a simple brute force algorithm.

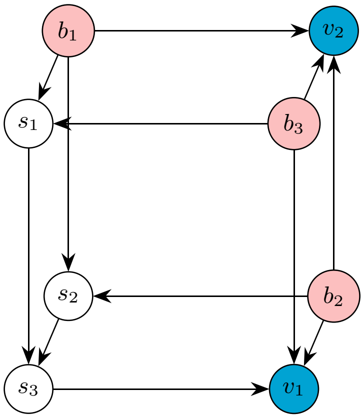


Fig. 8 Oriented cube V, 3 vertices of out-degree 3 (labeled b_i), and 2 vertices of in-degree 3 (labeled v_i)

References

1. Beasley, L.B., Santana, M.A., Mousley, J.M., Brown, D.E.: Cordiality of digraphs. J. Algebr. Comb. Discret. Struct. Appl. **10**, 1–13 (2023)
2. Beasley, L.B.: Cordial digraphs. J. Comb. Math. Comb. Comput. **117**, 1–23 (2022)
3. Beasley, L.B., Santana, M.A., Mousley, J.M., Brown, D.E.: (2, 3)-cordial trees and paths. In Press
4. Cahit, I.: Cordial graphs: a weaker version of graceful and harmonious graphs. Ars Combin. **23**, 201–208 (1987)
5. Deepa, Sinah, Jaspreet, Kuar: Full friendly index set -I. Discret. Appl. Math. **161**, 1262–1274 (2013)
6. Hovey, M.: A-cordial graphs. Discret. Math. **93**, 183–194 (1991)
7. Penchenik, O., Wise, J.: Generalized Graph Cordiality. Discussiones Mathematicae Graph Theory **32**, 557–567 (2012)

Structural Properties of m-Ary n-Dimensional Hypercubes

Saranya Anantapantula, Eddie Cheng, and László Lipták

Abstract The m-ary n-dimensional hypercube is a generalization of the hypercube when $m = 2$. In this paper, we study the structural properties of m-ary n-dimensional hypercube by considering the structure of the resulting graph when up to approximately $3n(m-1)$ vertices are deleted from it.

Keywords Connectivity · Interconnection networks · m-ary n-dimensional hypercube

1 Introduction

With recent advances in technology, multiprocessor systems with increasing number of interconnected computing nodes are becoming a reality. The underlying topology of such a system is an interconnection network. Computing nodes can be processors in which the resulting system is a multiprocessor supercomputer, or they can be computers in which the resulting system is a computer network. This interconnection network can be represented by a graph. Using the example of a multiprocessor supercomputer, one represents vertices as processors and edges as links between two processors. Since processors may fail, it is important to come up with fault resiliency measurements. One of the most fundamental measurements is the vertex connectivity of a graph. The *vertex connectivity* of a connected non-complete graph is the minimum number of vertices whose deletion disconnects the graph. (The vertex connectivity of a complete graph on n vertices is defined to be $n-1$.) Moreover, a connected non-complete graph is *k-vertex-connected* if with at most

S. Anantapantula
Spring-Ford Area High School, Royersford, PA 19468, USA

E. Cheng (✉) · L. Lipták
Oakland University, Rochester, MI 48309, USA
e-mail: echeng@oakland.edu

L. Lipták
e-mail: liptak@oakland.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_14

$k-1$ vertices being deleted, the resulting graph is connected, that is, the vertex connectivity is at least k. We remark that some researchers in computer engineering prefer using the term $(k-1)$-*vertex-fault tolerant* for k-vertex-connected. However, vertex connectivity may be too simplistic to be useful as a vulnerability/resiliency parameter. Researchers have introduced other enhancements by imposing additional requirements resulting in a number of "conditional/restricted" types of connectivity. One such example is the following: A set of vertices T in a connected non-complete graph G is called a *good-neighbor vertex-cut of order* m if $G-T$ is disconnected and every vertex in $G-T$ has degree at least m. The *good-neighbor vertex connectivity of order* m is the size of a smallest good-neighbor vertex-cut of order m. Thus a good-neighbor vertex-cut of order 0 is a vertex-cut and the good-neighbor vertex connectivity of order 0 is the vertex connectivity.

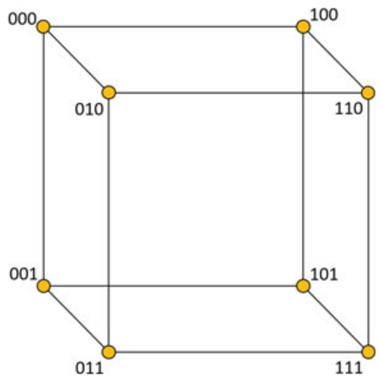
There is another way to measure vulnerability. At first glance, this measurement is not as refined and somewhat raw. On the other hand, this means it has more flexibility. A graph G is *super* m*-vertex-connected of order* q if with at most m vertices being deleted, the resulting graph is either connected or it has one large component and the small components collectively have at most q vertices in total. That is, the resulting graph has a component of size at least $|V(G-T)|-q$, where T is the set of deleted vertices. This parameter is related to many other parameters that enhance vertex connectivity. See [1] for details and see [2] for a brief history on the topic of connectivity type measures in interconnection networks and an extensive list of references.

The class of hypercubes is one of the most fundamental classes of interconnection networks. In a sequence of papers [3–5], the authors gave precise results of the structure of H_n-T where T as set of vertices with $|T|=f(n)$. Here H_n is the n-dimensional hypercube and $f(n)$ is in the order of kn, with some restrictions on the constant k. (We remark that H_n is n-regular.) This is what some authors called "deleting linearly many faults." In this paper, we consider the structure of HC_n^m-T, where HC_n^m is the m-ary n-dimensional hypercube. The ultimate goal is to obtain results when $|T|$ is in the order of kr, with some restrictions on the constant k, where r is the regularity of HC_n^m. Given that it took three papers to solve this problem for the hypercube and the structure of the hypercube is much simpler than the m-ary n-dimensional hypercube, one can expect higher complexity of the solution to achieve this goal. We start the process of this investigation by considering $k=2$ and $k=3$ in this paper.

2 Definitions and Preliminaries

Throughout the paper, we use standard graph theory terminology. Let $G=(V(G), E(G))$ be a simple graph with a vertex set $V(G)$ and edge set $E(G)$. Graph G is r-*regular* if the degree of every vertex in $V(G)$ is r. If $K\subset V(G)$ is a subset of vertices in G, the graph obtained by deleting the elements in K from G will be denoted by $G-K$. A r-regular r-connected graph G is called *maximally connected*.

Fig. 1 HC_3^2



A set of edges is *independent* is no two of its distinct edges are incident to a common vertex.

The *m-ary n-dimensional hypercube*, denoted as HC_n^m, is defined for positive integers m and n such that $m \geq 2$ and $n \geq 1$. The vertex set of the graph consists of all m^n strings of n elements of the set $\{0, 1, \ldots, m-1\}$ with repetitions allowed. A vertex can be represented as $a_0a_1 \ldots a_{n-1}$, where $a_i \in \{0, 1, \ldots, m-1\}$ for all $i = 0, 1, \ldots, n-1$. Two vertices $a = a_0a_1 \ldots a_{n-1}$ and $b = b_0b_1 \ldots b_{n-1}$ are adjacent if and only if an integer $i \in \{0, 1, \ldots, n-1\}$ exists such that $a_i \neq b_i$ and $a_j = b_j$ for all $j \in \{0, 1, \ldots, n-1\} \setminus \{i\}$. We remark that the m-ary n-dimensional hypercube is also known as the *generalized hypercube* obtained from the Cartesian product of n copies of the complete graph K_m.

The figures below are some examples Fig. 1 represents HC_3^2, which is also the 3-dimensional cube, and Fig. 2 represents HC_2^3.

HC_n^m can be partitioned into m disjoint subgraphs called subcubes. Each subcube H_i is isomorphic to HC_{n-1}^m for $i = 0, 1, \ldots, m-1$. Each of these subcubes contains the set of vertices from HC_n^m whose n^{th} element is i for a fixed $i \in \{0, 1, \ldots, m-1\}$. The graph induced by these vertices is denoted by H_i. In this paper we examine the connectivity properties of HC_n^m. We will denote the set of vertices to be deleted by F. The set of vertices deleted in a specific subcube H_i is defined by $F_i = F \cap V(H_i)$. This notation will be used throughout the paper.

Let us note some preliminary facts about HC_n^m, which all follow immediately from its definition.

(1) HC_n^m is $n(m-1)$-regular and has m^n vertices. Each H_i has m^{n-1} vertices.
(2) Each vertex in H_i has a unique neighbor in H_j for each $j \neq i$, these vertices are called its *outside neighbors*. There are exactly m^{n-1} independent edges between H_i and H_j for every $i \neq j$.

It is well known that the vertex connectivity of HC_n^m is $n(m-1)$, so it is maximally connected (Fig. 3).

The ultimate goal is to prove the following: if F is a set of vertices of HC_n^m with $|F| \leq f(r, n, m)$, where f is a function of r, n, and m, then $HC_n^m - F$ is either

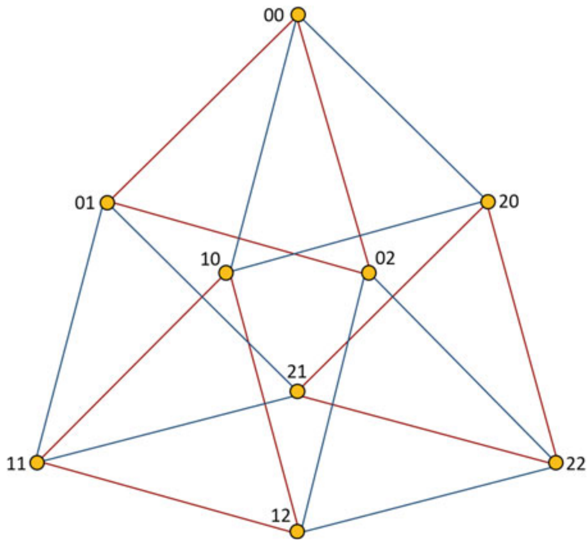


Fig. 2 HC_2^3

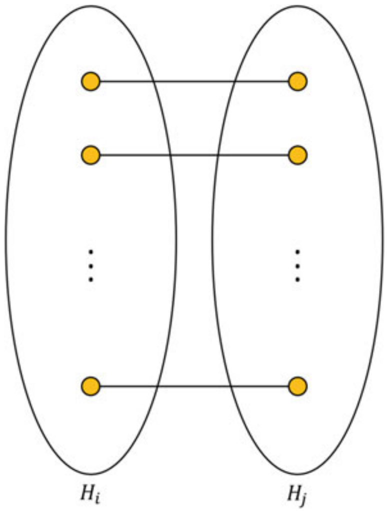


Fig. 3 m^{n-1} independent edges between 2 subcubes: H_i and H_j

connected or has a large component and small components with at most $r-1$ vertices in total. Based on the complexity of the proofs for the known results for hypercubes, we expect this to be a difficult process. In this paper, we solve the problem for $r=2$ and $r=3$.

3 Precise Results When $r = 2$ and $r = 3$

In this section we provide precise bounds on how many vertices we can delete in HC_n^m and have up to two vertices in small components.

Theorem 1 *Let $n \geq 1$ and $m \geq 2$. If F is a subset of the vertices of HC_n^m such that $|F| \leq 2n(m-1) - m - 1$, then $HC_n^m - F$ is either connected or has two components, one of which is a singleton.*

Proof Note that for $m = 2$ the graph HC_n^m is simply the n-dimensional hypercube, for which it is known that deleting up to $2n - 3$ vertices leaves either a connected graph or a graph with two components, one of which is a singleton [3]. So we can assume $m \geq 3$. Define H_i and F_i as usual. We proceed by induction on n. For the base case we show the claim for $n = 1$ and $n = 2$.

For $n = 1$, we have $|F| \leq m - 3$, and since HC_1^m is a complete graph on m vertices, $HC_1^m - F$ is also a complete graph. So $HC_1^m - F$ is connected and the statement is true for $n = 1$.

Next assume $n = 2$, so we are deleting up to $4(m-1) - m - 1 = 3m - 5$ vertices in HC_2^m. Note that HC_2^m consists of m subgraphs $H_0, \ldots, H_{m-1}$, each of which is isomorphic to the complete graph K_m. Without loss of generality we may assume that $|F_0| \leq |F_i|$ for all $i = 1, 2, \ldots, n-1$. Since we are deleting up to $3m - 5$ vertices, $|F_0| < 3$. Note that H_i is a complete graph, so $H_i - F_i$ is always connected. So if $|F_0| = 0$, then since every vertex has a neighbor in H_0, the graph $HC_2^m - F$ is connected. If $|F_0| = 1$, then apart from the one deleted vertex in each H_i, we have at most $2m - 5$ more deleted vertices. So there can be at most one i for which $|F_i| \geq m - 1$, and for all $j \neq i$ we have $|F_j| \leq m - 2$. For each $j \neq i$ there is at least one edge remaining between $H_0 - F_0$ and $H_j - F_j$, so all of those subgraphs will belong to a component of $HC_2^m - F$. If there is no i such that $|F_i| = m - 1$, then this implies that $HC_2^m - F$ is connected, while if there is such an i, then at most the one vertex in $H_i - F_i$ is not in the big component of $HC_2^m - F$, proving the claim. Finally, if $|F_0| = 2$, then there are at most $m - 5$ deleted vertices in addition to the two deleted vertices in each H_i, so $|F_i| \leq m - 3$ for each $i > 0$ (since in this case we are deleting at least $2m$ vertices, $2m \leq 3m - 5$ implies that we must have $m \geq 5$). Thus at least one edge remains between $H_0 - F_0$ and $H_i - F_i$ for each $i > 0$, so $HC_2^m - F$ is connected.

Now for the induction step suppose that the statement holds for n where $n \geq 2$. We will show that by deleting $2(n+1)(m-1) - m - 1$ vertices from HC_{n+1}^m, the resulting graph has at most one vertex separated from the large component. Without loss of generality we may assume that $|F_0| \geq |F_i|$ for all $i = 1, 2, \ldots, n-1$ and consider three cases depending on the size of F_0.

Case 1: $|F_0| \leq n(m-1) - 1$.

Then $|F_i| \leq n(m-1) - 1$ as well for all $i = 1, 2, \ldots, n-1$. Since the vertex connectivity of HC_n^m is $n(m-1)$, each subgraph $H_i - F_i$ is connected. Also, there are m^n independent edges between any two subcubes H_i and H_j $(i \neq j)$. We will show that at least one of these edges remains by showing that $m^n > 2(n(m-1) - 1)$

when $n \geq 2$ and $m \geq 3$, where $2(n(m-1)-1)$ is the maximum number of deleted vertices in H_i and H_j combined. This is easy to check for $n = 2$ (the inequality is equivalent to $(m-2)^2 + 2 > 0$), while for $n \geq 3$ we have $m^{n-1} \geq 3^{n-1} \geq 2n$, so $m^n = m^{n-1} \cdot m \geq 2nm > 2(n(m-1)-1)$. Thus at least one edge must remain between any pair of subcubes, and since each of them is connected, $HC^m_{n+1} - F$ is also connected.

Case 2: $n(m-1) \leq |F_0| \leq 2n(m-1) - m - 1$.

Then $|F_i| \leq (2(n+1)(m-1) - m - 1) - n(m-1) = n(m-1) + (m-3)$ for each $i > 0$. Thus there can be at most one more $i > 0$ such that $|F_i| \geq n(m-1)$. For notational simplicity, assume that $i = 1$ if there is such an i, and for all $j > 1$ we have $|F_j| \leq n(m-1) - 1$.

Since $m \geq 3$, the subgraph $H_j - F_j$ is connected for all $j > 1$, and they all belong to a large, connected component C in $HC^m_{n+1} - F$ as shown in Case 1. Since $|F_0|, |F_1| \leq 2n(m-1) - m - 1$, the induction hypothesis implies that each of $H_0 - F_0$ and $H_1 - F_1$ is either connected or has two components, one of which is a singleton. First we show that the large components all connect to C. Consider $H_0 - F_0$. There are m^n independent edges between H_0 and H_j for $j > 1$, and there can be at most $2(n+1)(m-1) - m - 1$ deleted vertices in H_0 and H_j, so it is enough to show $m^n > 1 + 2(n+1)(m-1) - m - 1 = 2(n+1)(m-1) - m$, where the extra 1 accounts for the potential singleton in $H_0 - F_0$.

When $n \geq 4$, we have $3^{n-1} \geq 2(n+1)$, so $m^n \geq m^{n-1} \cdot m \geq 2(n+1)m > 2(n+1)(m-1) - m$. When $n = 3$, we get $m^3 \geq 9m > 8(m-1) - m$. Finally, when $n = 2$, the inequality simplifies to $(m-2)(m-3) > 0$, which is clearly true for $m > 3$. The only remaining case to check is when $m = 3$, $n = 2$, and all the faults are in H_0 and H_2. This means that $H_1 - F_1 = H_1$ is connected and is part of C, so repeating the argument with $H_0 - F_0$ and H_1 proves that the large component of $H_0 - F_0$ is part of C and there can be at most one isolated vertex that is not part of C.

Now we can repeat the argument with $H_1 - F_1$ and get that the large components of all subgraphs are part of C, so at most two isolated vertices from $H_0 - F_0$ and $H_1 - F_1$ (one from each) may remain separated from C. If there was only one singleton to begin with or if at least one of these singletons belongs to C, we are done. If both singletons remain separated from C, then the neighborhood of these two vertices must be part of F, and the size of the neighborhood of any two vertices in HC^m_{n+1} is at least $2(n+1)(m-1) - m$, contradicting the assumption about the size of F.

Case 3: $2n(m-1) - m \leq |F_0|$.

Then for all $i > 0$ we have $|F_i| \leq (2(n+1)(m-1) - m - 1) - (2n(m-1) - m) = 2m - 3 < 2(m-1) \leq n(m-1)$ for all $i > 0$, so $H_i - F_i$ is connected. As in Case 1, all of these subgraphs belong to a component C in $HC^m_{n+1} - F$. Since each vertex of $H_0 - F_0$ has $m - 1$ outside neighbors, different vertices of $H_0 - F_0$ cannot have the same outside neighbor, and there are less than $2(m-1)$ faulty vertices outside H_0, at most one of the vertices of H_0 may remain isolated from C.

This finishes all cases, and the proof is complete.

Note that the bound is tight for Theorem 1. To see this consider any two adjacent vertices u and v in two different subcubes in HC_n^m, with $n \geq 1$ and $m \geq 2$. Each vertex has degree $n(m-1)$ and is adjacent to $n(m-1)-1$ vertices other than each other. Their common neighbors can only be vertices with a different n^{th} digit. There are only $m-2$ possibilities of this, so u and v have $m-2$ common neighbors. The common neighborhood of these vertices has size $2n(m-1)-2-(m-2) = 2n(m-1)-m$. Hence it is possible to delete $2n(m-1)-m$ vertices and get two vertices separated from the rest of the graph, proving that the bound of $2n(m-1)-m-1$ in Theorem 1 is best possible.

Next we look at how many vertices we can delete and still only have at most two vertices separated from the largest component of the remaining graph. Unfortunately there are some exceptional small cases, so the result is more complicated than Theorem 1.

Theorem 2 *Let $m \geq 3$ and F be a subset of the vertices of HC_n^m.*

1. *When $n = 2$, if $|F| \leq 4m-9$, then $HC_2^m - F$ is either connected or has a large component and small components with at most two vertices in total. If $|F| = 4m-8$, then in addition there is the possibility that $HC_2^m - F$ has exactly two components, one of which having size 4, and if $|F| = 4m-7$, then $HC_2^m - F$ can also have exactly two components, one of which having size 3.*
2. *If $n \geq 3$ and $|F| \leq 3n(m-1)-2m-2$, then $HC_n^m - F$ is either connected or has a large component and small components with at most two vertices in total.*

Proof We note that when $m = 2$ and $n \geq 4$, the graph HC_n^2 is an n-dimensional hypercube for which it is known that deleting up to $3n-6$ vertices results in at most two vertices separated from the large component C of $HC_n^2 - F$ [3]. Thus we assume $m \geq 3$. Define H_i and F_i as usual. We proceed by induction on n. We need a slightly stronger result for the case $n = 2$, so as the base case we show the claim for both $n = 2$ and $n = 3$.

Assume $n = 2$ and that we are deleting up to $4m-9$ vertices in HC_2^m. Each of the m subcubes $H_0, H_1, \ldots, H_{m-1}$ in HC_2^m is isomorphic to the complete graph K_m. Without loss of generality we may assume that $|F_0| \leq |F_i|$ for all $i \neq 0$. Since we are deleting up to $4m-9$ vertices, $|F_0| \leq 3$. Additionally, each $H_i - F_i$ is always connected because H_i is complete.

If $|F_0| = 0$, then since every vertex has a neighbor in H_0, the graph $HC_2^m - F$ is connected.

If $|F_0| = 1$, then apart from the one deleted vertex in each H_i, there are at most $3m-9$ additional deleted vertices. So, there can be at most two subgraphs $H_1 - F_1$ and $H_2 - F_2$ where $|F_1|, |F_2| \geq m-1$ (we can choose these subcubes for notational simplicity). Then for all $j \neq 1, 2$, we have $|F_j| \leq m-2$ so there is at least one edge remaining between $H_0 - F_0$ and $H_j - F_j$. Thus all these subgraphs belong to a component C in $HC_2^m - F$. If $|F_1|, |F_2| < m-1$, then $HC_2^m - F$ is connected by a similar argument, while if either is of size at least $m-1$, then at most one vertex will be separated from C for each of $H_1 - F_1$ and $H_2 - F_2$. Thus, at most two vertices are not in the large component of $HC_2^m - F$. The same argument applies if we delete up to $4m - 7$ vertices.

If $|F_0| = 2$, then there are at most $2m - 9$ more deleted vertices apart from the two deleted vertices in each H_i and we must have $m \geq 5$ (in order for $2m \leq 4m - 9$). So there can be at most one i for which $|F_i| \geq m - 2$, and for all $j \neq 0, i$ we have $|F_j| \leq m - 3$. For each $j \neq 0, i$ such that $|F_j| \leq m - 3$, there is at least one edge remaining between $H_0 - F_0$ and $H_j - F_j$, so all these subgraphs belong to one connected component of $HC_2^m - F$. If there is no i such that $|F_i| \geq m - 1$, then $HC_2^m - F$ is connected. If there is such an i, then at most two vertices in $H_i - F_i$ are not in the large component of $HC_2^m - F$. In this subcase we also need to carefully consider if we delete one or two additional vertices. First, if $|F| = 4m - 8$, then $m \geq 4$ (so that $2m \leq 4m - 8$) and there is an additional possibility that there are exactly two subcubes with $m - 2$ deleted vertices, and each of the other subcubes has two deleted vertices. For notational simplicity let $|F_1| = |F_2| = m - 2$. When $m \geq 5$, all the subcubes with two deleted vertices have an edge remaining between them, so they are part of a component C in $HC_2^m - F$. If either $H_1 - F_1$ or $H_2 - F_2$ is part of C, then $HC_2^m - F$ has at most two vertices separated from C. If neither is part of C, then the two vertices of $H_1 - F_1$ and $H_2 - F_2$ must be adjacent due to the structure of HC_2^m, F contains exactly their outside neighbors, and $HC_2^m - F$ has two components, the smaller of which having size 4. If $m = 4$, then each subcube has two deleted vertices, forming four K_2's. If one of these K_2's joins none of the others, then due to the structure of HC_2^m the other three must form a component C. So either at least three of them join to a component, getting at most two vertices separated from it, or we get two components of size 4 each. The last possibility we need to consider is when $|F| = 4m - 7$ (so we still have $m \geq 4$). The only difference in this case is that one of the K_2's will be a single vertex. Again due to the structure of HC_2^m, this single vertex will always join either component C or one of the K_2's. Thus either $HC_2^m - F$ has at most two vertices separated from C or has exactly two components, one of which having size 3.

Finally, if $|F_0| = 3$, then there are at most $m - 9$ deleted vertices in addition to the three deleted vertices in each H_i, so $m \geq 9$ and $|F_i| \leq m - 6$ for each i. Thus, at least one edge remains between $H_0 - F_0$ and $H_i - F_i$ for each $i > 0$, so $HC_2^m - F$ is connected. The same argument applies if we delete up to $4m - 7$ vertices. This finishes the proof of the base case $n = 2$.

Now, suppose that the statement holds for n where $n \geq 2$. We will now show that if we delete at most $3(n + 1)(m - 1) - 2m - 2$ vertices from HC_{n+1}^m, the resulting graph has at most two vertices separated from the large component. Without loss of generality, we may assume that $|F_0| \geq |F_i|$ for all $i = 1, 2, \ldots, n - 1$ and consider four cases depending on the size of F_0.

Case 1: $|F_0| \leq n(m - 1) - 1$.

Then $|F_i| \leq n(m - 1) - 1$ for all $i = 1, 2, \ldots, n - 1$. Since the vertex connectivity of HC_n^m is $n(m - 1)$, each subgraph $H_i - F_i$ is connected. Moreover, as shown in Case 1 in Theorem 1, $m^n > 2(n(m - 1) - 1)$. Thus, there is at least one edge remaining between any two subcubes. Hence, the remaining graph $HC_{n+1}^m - F$ is connected.

Case 2: $n(m-1) \leq |F_0| \leq 2n(m-1) - m - 1$**.**

Then $|F_i| \leq (3(n+1)(m-1) - 2m - 2) - n(m-1) = 2n(m-1) + (m-5)$ for each $i > 0$. Thus there can be at most two more $i > 0$ such that $n(m-1) \leq |F_i| \leq 2n(m-1) - m - 1$. Without loss of generality we may assume $|F_{m-1}| \leq |F_i|$ for each $i < m-1$, and then we have $|F_{m-1}| \leq \frac{2n(m-1)+(m-5)}{m-1} = 2n + 1 - \frac{4}{m-1} < 2n + 1$, so $|F_{m-1}| \leq 2n$. Clearly $2n < n(m-1)$ for $m \geq 4$, and for $m = 3$ we actually get $|F_2| \leq 2n - 1 = n(m-1) - 1$. So $H_{m-1} - F_{m-1}$ is connected and belongs to a component C of $HC^m_{n+1} - F$.

As $|F_0|, |F_1|, |F_2| \leq 2n(m-1) - m - 1$, Theorem 1 implies that the subgraphs $H_0 - F_0$, $H_1 - F_1$, and $H_2 - F_2$ are either connected or each has two components, one of which is a singleton. Since $|F_j| \leq n(m-1)$ for each $j > 2$, each $H_j - F_j$ is connected. We will show that the large components of each of these subcubes joins to C. Consider $H_j - F_j$ and $H_{m-1} - F_{m-1}$ for $j < m-1$. We delete at most $2n(m-1) - m - 1 + 2n = 2nm - m - 1$ vertices in these subcubes combined. So it is enough to show that $m^n > 1 + 2nm - m - 1 = 2nm - m = (2n-1)m$, where the extra 1 represents the possible vertex separated from the large component in $H_j - F_j$. This inequality is equivalent to $m^{n-1} > 2n - 1$, and it is easy to see that this inequality holds for $m \geq 3$ and $n \geq 3$. So the only remaining case is when $m = 3$ and $n = 2$, for which $|F_0| = 4$ and $|F_2| \leq 3$, so $1 + 4 + 3 = 8 < 9 = 3^2$, and there must be an edge remaining between $H_2 - F_2$ and the large components of $H_0 - F_0$ and $H_1 - F_1$.

Thus there are at most three isolated vertices in the subgraphs $H_0 - F_0$, $H_1 - F_1$, and $H_2 - F_2$, that can remain separated from C. If there are at most two such vertices, or at least one out of the three belongs to C, we are done. If all three vertices remain separated from C, then the neighborhood of these vertices must be part of F, which consists of at least $3n(m-1)$ neighbors within $H_0 - F_0$, $H_1 - F_1$, and $H_2 - F_2$, and at least $m - 3$ outside neighbors. Together this gives $|F| \geq 3n(m-1) + m - 3 = 3(n+1)(m-1) - 2m$, which contradicts the assumption about the size of F.

Case 3: $2n(m-1) - m \leq |F_0| \leq 3n(m-1) - 2m - 2$**.**

Then $|F_i| \leq 3(n+1)(m-1) - 2m - 2 - (2n(m-1) - m) = n(m-1) + (2m-5)$ for each $i > 0$. Since $2m - 5 \leq n(m-1) - m - 1$ is equivalent to $3 - \frac{1}{m-1} \leq n$, which clearly holds when $n \geq 3$, there are no subcubes H_i where $i > 0$ such that $2n(m-1) - m \leq |F_i| \leq 3n(m-1) - 2m - 1$, and there is at most one subcube H_i where $i > 0$ such that $n(m-1) \leq |F_i|$. Since $|F_0| \leq 3n(m-1) - 2m - 2$, the induction hypothesis implies that $H_0 - F_0$ is either connected, has at most two vertices separated from its large component C, or $n = 2$ and $H_0 - F_0$ has two components, one of which has size 4. As in Case 2, without loss of generality we may assume $|F_{m-1}| \leq |F_i|$ for each $i < m-1$, and we have $|F_{m-1}| \leq \frac{n(m-1)+(2m-5)}{m-1} = n + 2 - \frac{3}{m-1} < n + 1$, so $|F_{m-1}| \leq n$. Clearly $n < n(m-1)$ for $m \geq 3$, so $H_{m-1} - F_{m-1}$ is connected and belongs to a component C of $HC^m_{n+1} - F$. We show that the large components of each other subcube also belong to C. This can be shown similarly for $H_i - F_i$ for $i > 0$: We delete at most $n(m-1) + (2m-5) + n = (n+2)m - 5$ vertices in these two subcubes, and there can be at most one singleton in $H_i - F_i$, so it is enough to show $1 + (n+2)m - 5 < m^n$. This is clearly true for $m \geq 3$ and

$n \geq 3$, so the only remaining case is when $n = 2$. Then the condition on the size of F_0 simplifies to $4 \leq m$, and the required inequality simplifies to $4m - 4 < m^2$, which holds for $m \geq 4$. Thus the large components of each $H_i - F_i$ belong to C for each $i \geq 1$.

Assume $|F_1| \geq |F_i|$ where $i > 0$. We can have two subcases depending on the size of F_1.

Subcase 3a: $|F_1| \leq n(m-1) - 1$.

Then, every subgraph $H_i - F_i$, where $i > 0$, is connected, and they all belong to C, as shown in Case 1. If $H_0 - F_0$ has at most two vertices separated from a large, connected component, then this large component belongs to C as shown in Case 2. Thus, $HC^m_{n+1} - F$ has at most two vertices separated from a large, connected component. The other possibility is that $n = 2$ and $H_0 - F_0$ has two components, one of which has size 4, $|F| = 7m - 11$ and $|F_0| = 4m - 8$. The four vertices in $H_0 - F_0$ have at least $4(m-1)$ neighbors in the other subcubes, and since $4m - 8 + 4(m - 1) > 7m - 11$, both components of $H_0 - F_0$ must be part of C, so $HC^m_{n+1} - F$ is connected.

Subcase 3b: $n(m-1) \leq |F_1|$.

Then, $|F_j| \leq n(m-1) + (2m-5) - n(m-1) = 2m - 5$ where $i > 1$. Without loss of generality we may assume $|F_{m-1}| \leq |F_i|$ for each $i < m - 1$, and we have $|F_{m-1}| \leq \frac{2m-5}{m-2} < 2$. So $|F_{m-1}| \leq 1$, and so at most one vertex each in $H_0 - F_0$ and $H_1 - F_1$ can be separated from C. Thus, $HC^m_{n+1} - F$ is either connected or has a large, connected component and smaller components with at most two vertices in total.

Case 4: $3n(m-1) - 2m - 1 \leq |F_0|$.

Then for all $i > 0$, we have $|F_i| \leq (3(n+1)(m-1) - 2m - 2) - (3n(m-1) - 2m - 1) = 3m - 4 < 3(m-1) \leq n(m-1)$, so $H_i - F_i$ is connected for each $i > 0$. As in previous cases we can show that each $H_i - F_i$ belongs to a component C of $HC^m_{n+1} - F$. Since each vertex in $H_0 - F_0$ has $m - 1$ outside neighbors, and different vertices in H_0 cannot have the same outside neighbor, and there are less than $3(m-1)$ faulty vertices outside H_0, the resulting graph consists of at most two vertices separated from C, so $HC^m_{n+1} - F$ is either connected or has a large, connected component and smaller components with at most two vertices in total.

This finishes all cases, and the proof is complete.

Note that the bounds are tight for Theorem 2. To see this, for $n = 2$ and $m \geq 3$ consider two vertices in a subcube in HC^m_2 and two of their outside neighbors in the same subcube. These vertices have $m - 2$ neighbors each in their own subcubes and each corresponding pair has $m - 2$ outside neighbors in the other subcubes. So their neighborhood has $4(m-2) = 4m - 8$ vertices, and if we delete those, we get a large connected component and a second component containing four vertices.

For $n \geq 3$ and $m \geq 3$, consider the three vertices $u = 00\ldots0$, $v = 10\ldots0$, and $w = 010\ldots0$. To calculate their combined neighborhood, note that each has $n(m-1)$ neighbors, and u is adjacent to both v and w. Vertices u and v have $m - 2$ common neighbors, and so have u and w. Finally, v and w have the common neighbor $110\ldots0$. So the size of their neighborhood is $3n(m-1) - 2 \cdot 2 - 2(m-2) - 1 =$

$3n(m-1)-2m-1$. So it is possible to delete $3n(m-1)-2m-1$ vertices and get three vertices separated from the rest of the graph, proving that the bound of $3n(m-1)-2m-2$ in Theorem 2 is best possible.

Acknowledgements We would like to thank the anonymous referee for a number of helpful comments and suggestions.

References

1. Cheng, E., Hao, R.-X., Qiu, K., Shen.: Structural properties and fault resiliency of interconnection networks. In: Akl, A., Sirakoulis (eds.) From Parallel to Emergent Computing. CRC Press, pp. 77–101 (2019)
2. Cheng, E., Qiu, K., Shen., Yang, W.: A brief account on the development and future research directions of connectivity properties of interconnection networks. Parallel Process. Lett. **30**, 2040009 (14 pages) (2020)
3. Yang, X., Evans, D.J., Chen, B., Megson, G.M., Lai, H.: On the maximal connected component of hypercube with faulty vertices. Int. J. Comput. Math. **81**(5), 515–525 (2004)
4. Yang, X., Evans, D.J., Megson, G.M.: On the maximal connected component of hypercube with faulty vertices (II). Int. J. Comput. Math. **81**(10), 1175–1185 (2004)
5. Yang, X., Evans, D.J., Megson, G.M.: On the maximal connected component of hypercube with faulty vertices III. Int. J. Comput. Math. **83**(1), 27–37 (2006)

Graph Constructions Derived from Interconnection Networks

Richard Draper

Abstract A class of interconnection networks for massively parallel processors are designed by taking copies of a building block network and wiring them together. For Dragonfly networks, the building block network is a complete graph and the wiring together is done by either a cycle or a complete graph. The process may be viewed as a way to construct a new graph from two component graphs.The resulting graph is known as a replacement graph. Furthermore, one of these constructions leads to a very large number of graphs, some of which are provably not isomorphic. The point is that the construction of a replacement in G by H requires that G be converted to a network. This paper explains the way the graph of an interconnection network is labeled and a table which is analogous to the adjacency matrix of a labeled graph. The table is used to demonstrate the nondeterminism of the concept of a replacement graph. The graph constructions are presented along with the motivating interconnection networks. The graph constructions can be generalized.

Keywords Graph constructions · Replacement graph · Parallel processor

1 Introduction

There are nine well-known graph products. Their definitions involve statements that can be made about the coordinates of the product in terms of equality or adjacency in the component graphs G and H. A tenth product, the replacement product, cannot be constructed using equality and adjacency. It requires enumeration. The enumeration effectively converts the graph G into an interconnection network. People who work on networks have a different set of interests than graph theorists. The purpose of this paper is to exhibit some of those interests and see what they mean for graph theorists.

R. Draper (✉)
Center for Computing Sciences, Institute for Defense Analyses, Bowie, MD, US
e-mail: rndrape@super.org; rndrape@gmail.com

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_15

Interconnection networks get implemented in expensive hardware, so they get studied in a different way than replacement graphs. In this paper, three replacement graphs derived from interconnection networks are presented. The first is very simple because it is defined in graph theoretic terms, but it is not a graph product. The second is the replacement product and it will be examined in some detail. The third generalizes the notion of a replacement graph.

This paper is organized as follows. The first section discusses a graph that is the topology of a network called the Swapped Dragonfly. This section also introduces a concept called the Structure Table of a network. The second section uses the Structure Table to study three interconnection networks whose topologies are replacement graphs. Two of these are complete graph networks that are not isomorphic but are emulations of each other. The third section generalizes the notion of a replacement graph. This generalization is motivated by the Dragonfly networks, which are the way many supercomputers are designed. The fourth section comments on the references and the fifth is the Conclusion.

2 Two Constructions

Given graphs G and H, there is a graph with vertices (g, h, k), $g \in V(G)$, h and $k \in V(H)$. Adjacency is defined by

$$\begin{gathered}(g, h, k) \sim (g', h', k') \iff \\ g = g' \ \wedge \ h = h' \ \wedge \ k \sim k' \\ \vee \\ g \sim g' \ \wedge \ h = k' \ \wedge \ k = h'.\end{gathered}$$

This is called a *swapped graph product* and is denoted $G\sigma H$. It is suggested by the construction of the Swapped Dragonfly [5] in which case G is an Abelian group of order d and H is a regular complete graph of order d. It is defined by a graph theoretic construction. It's use as an interconnection network comes after the graph construction.

The next graph construction is a result of the creation of a network, not the precursor for the design of a network. It is called a replacement product. Assume G is regular of degree d and H is of order d. The *replacement* graph of G by H is denoted $G \uparrow H$. It is defined using a network G_N and the graph H. Label the nodes of H, $h_1, \ldots, h_d$ and the ports of G_N, $p_1 \ldots, p_d$. On $G \uparrow H$

$$\begin{gathered}(g, h_i) \sim (g', h_j) \iff \\ g = g' \wedge \ h_i \sim h_j \\ \vee \\ (g, p_i) \leftrightarrow (g', p_j) \text{ on } G_N.\end{gathered}$$

Node (g, h_i) has port p_i. The network is called a *blow-up* of G_N by H and is denoted $G_N \uparrow H$. Unlike the swapped graph product, the replacement graph is not well-defined. Given G and H, there are many $G_N \uparrow H$'s and many $G \uparrow H$'s.

To study the range of possible replacement graphs that can be constructed from graphs G and H, a table called a *Structure Table* is used to represent the network G_N with topology of the graph G. G is regular of degree d, so the nodes of G_N have d ports. The structure table is $|G| \times d$ and the entries of the table are nodes. Entry $(g, p) = h$ means port p of node g is connected to node h, Node g must appear in row h at $(h, q)$for some q. This means there is a link in the network G_N connecting (g, p) and (h, q). The only constraint on a structure table is that row g contains every node but g. Permuting each row of a structure table creates a different network G_N.

The graph theoretic definition of a replacement graph of H in G enumerates the neighbors of each vertex of G. The vertex is replaced by a copy of H. This is called a cloud. In Interconnection networks it is called a group. The replacement graph has $|G||H|$ nodes (g, h). A connection is made between (g, h) and (g', h') if there is a link (g, g') such that g' is the hth neighbor of g and g is the h'th neighbor of g'. Compare this to the structure table procedure. The graph G determines the set of entries of each row of the structure table. The entries of row g are the neighbors of g in G. It is automatic that the row g contains g' and row g' contains g.If g is in column h and g' is in column h', the link on the the replacement graph connects (g, h) to (g', h'). The structure of the graph H is not exhibited by the structure table of G.

3 Examples of the Replacement Graph

The remainder of the paper is devoted to replacement products. It is shown that the structure table of the graph G is an effective way to define a replacement product. This method reveals the multiplicity of replacement graphs $G \uparrow H$ for a given pair of graphs H and G. It gives an example in which two forms of $G \uparrow H$ are provably not isomorphic but it shows that there is only one emulation class for $G \uparrow H$.

Let H denote the Boolean 4-cube and C denote a 4-cycle. Here is part of a structure table for $H_N \uparrow C$. Each node has four attached links.

G/p	0	1	2	3
0101	0100	0111	1101	0001
0100	0110	0101	1100	0000
0111	1111	0110	0011	0101
0000	0100	1000	0001	0010

From the table, we see that on $H_N \uparrow C$

$$(0101, 1) \leftrightarrow (0111, 3) \text{ and } (0101, 0) \leftrightarrow (0100, 2).$$

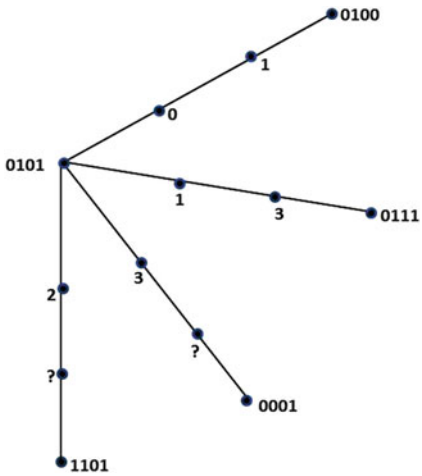


Fig. 1 The neighborhood of 0101 on H

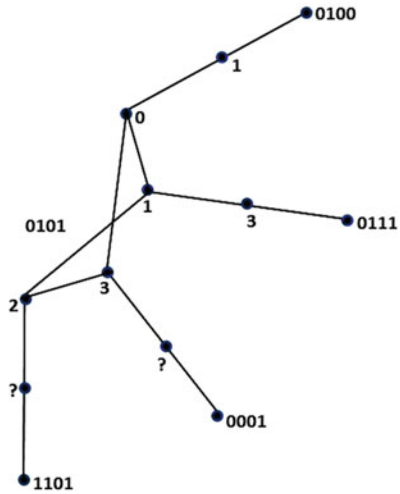


Fig. 2 The portion of the replacement of H by C that replaces node 0101

Sketch 1 shows the neighborhood of 0101 on H. Ports are moved onto the appropriate link so that they can be distinguished. In Sketch 2, the portion of the replacement of H by C that replaces node 0101 is shown. The blow-up network $H_N \uparrow C$ would also show ports at the nodes. For example, node $(0101, 1)$ has three ports; ± 1 on the cycle links and 1 on the hypercube link.

This graph has a curious appearance. Although there are many ways to fill out the structure table of $H_N \uparrow C$, there is only one way that reflects the additive structure of H. The next table reflects that structure by letting the node c of the cycle denote swapping bits in the nodes of H. Sketch 3 shows the portion of $H \uparrow C$ that replaces node 0101 using this structure table.

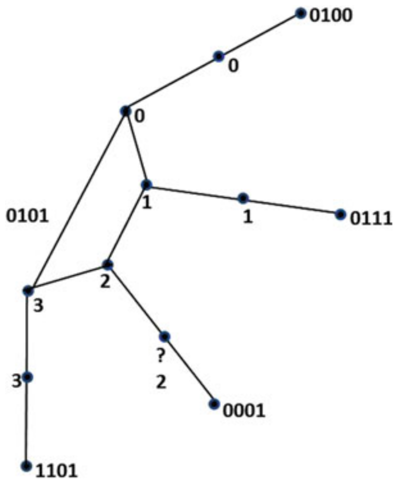


Fig. 3 The portion of $H \uparrow C$ that replaces node 010101

G/p	0	1	2	3
0101	0100	0111	0001	1101
0100	0101	0110	0001	1100
0111	0110	0101	0011	1111
0000	0001	0010	0100	1000

If graph G has an algebraic or arithmetic method of determining adjacency this may be used to determine a unique structure table. This is likely to be viewed as *the* replacement graph. If a graph does not have this property there are a very large number of structure tables determining distinct replacement graphs.

There are two interconnection networks that exhibit the structure $G \uparrow H$. The first is the Cube-Connected Cycle [10] where G is a Boolean n-cube and H is an n-cycle. Sketch 3 is part of a CCC. The CCC is a dilation $n/2$, degree three emulation of the n-cube. The objective of the concept was to reduce the degree of nodes from n to 3. This was designed during the era of the first Connection Machine [7], which had a Boolean hypercube topology. Switch size limited the machine size but the Cube-Connected Cycle was a way to resolve that problem because all nodes are of degree three.

The next example of a replacement graph is motivated by the Dragonfly topology [9] for interconnection networks. The definition of a Dragonfly network is somewhat vague. A collection of M routers connected as a complete graph is a building block, called a *group*. Each router has K ports which are used to connect to other identical groups. These connections are called global connections. The objective of the design is that passage between two nodes should require at most one global hop. A group has KM ports so the largest Dragonfly has $KM + 1$ groups connected in a complete graph. That Dragonfly is denoted $MDF(K, M)$.

Unlike the hypercube, a complete graph has no algebraic structure that leads to a way to fill out a structure table. Originally, the physical problem of connecting the ports of a large number of groups with fibre optic cables lead to two very different structure tables for a complete graph with $KM + 1$ nodes.

Using $KM + 1 = 13$, the table called Standard and the table called Graceful are examples of these two structure tables. Ignore the top two lines of both tables for the moment. What remains are two very different structure tables for a network on K_{13}. For example, port 6 on the Standard Table induces a permutation of the nodes of K_{13}. On the Graceful Table it connects all the nodes only to nodes 5 or 6. One can remove port 12 and node 12 from the Graceful Table. The result is a Graceful network on K_{12}. Removing port 12 and node 12 from the Standard Table results in something that is not a structure table on K_{12}. For example, 11 appears in row 10 but 10 does not appear in row 11 because port 12 has been removed. Therefore the two networks are not isomorphic. They are, however, dilation one emulations of each other. To see this, form a 13×13 table parametrized by routers. Entry (r, s) is the port p on r that connects r to s. This can be found in the structure table. This table is called a *difference table*.

Consider row 4 from the difference table for the Standard and Graceful networks.

routers	1	2	3	4	5	6	7	8	9	10	11
Standard	5	6	7	8	9	10	11	0	1	2	3
Graceful	0	1	2	3	5	6	7	8	9	10	11

For Graceful to emulate Standard, if Standard uses port 0 of router 4, Graceful uses port 8. In this way every port on Standard is emulated by Graceful. This example shows that two versions of $G \uparrow H$ can be very different graphs. However, there is only one emulation class.

4 A Generalization

At this point, interconnection network theory generalizes the notion of a replacement graph. The top two lines convert the Tables 1 and 2 to the global structure obtained by blowing up the network with a group consisting of four routers with three global ports. Using these two structure tables, two versions of the Dragonfly network $MDF(3, 4)$ may be constructed as the blow-up of the table by a graph H with 4 routers, each having 3 global ports. This blow-up network has coordinates $(g, r; p)$. For example, using the Standard Table

$$\begin{array}{l} (5, 1; 1) \leftrightarrow (9, 2; 3) \\ (5, 1; 2) \leftrightarrow (10, 2; 2) \quad \text{and} \\ (5, 1; 3) \leftrightarrow (11, 2; 1) \end{array}$$

Table 1 Standard dragonfly

r	0			1			2			3		
p	1	2	3	1	2	3	1	2	3	1	2	3
g\p	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

Table 2 Gracefully scalable dragonfly

r	0			1			2			3		
p	0	1	2	0	1	2	0	1	2	0	1	2
g\p	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	2	3	4	5	6	7	8	9	10	11	12
2	0	1	3	4	5	6	7	8	9	10	11	12
3	0	1	2	4	5	6	7	8	9	10	11	12
4	0	1	2	3	5	6	7	8	9	10	11	12
5	0	1	2	3	4	6	7	8	9	10	11	12
6	0	1	2	3	4	5	7	8	9	10	11	12
7	0	1	2	3	4	5	6	8	9	10	11	12
8	0	1	2	3	4	5	6	7	9	10	11	12
9	0	1	2	3	4	5	6	7	8	10	11	12
10	0	1	2	3	4	5	6	7	8	9	11	12
11	0	1	2	3	4	5	6	7	8	9	10	12
12	0	1	2	3	4	5	6	7	8	9	10	11

This is on the network $K_{13,N} \uparrow H$ using the Standard structure table for K_{13}. On the replacement graph $K_{13,N} \uparrow H$ $(5, 1) \sim (9, 2)$, $(10, 2)$ and $(11, 2)$. Using the Graceful structure table for K_{13} the result is

$$(5, 1) \sim (3, 1), (4, 1) \text{ and } (6, 1),$$

What is important is that the two networks and the two replacement graphs are not isomorphic. To see this, observe that router three and nodes 10, 11 and 12 can be removed from the Graceful structure. Let $H' = 0, 1, 2$. What remains is a Graceful $K_{10N} \uparrow H'$. Doing the same thing to the Standard structure leaves a network in which some groups have routers which are connected to nothing. Furthermore, the two networks are not dilation one emulations of each other.

In an emulation of a Standard Dragonfly by the Graceful Dragonfly, router 1 must be assigned to a router of each group in the Standard Dragonfly. In the Graceful Dragonfly router 1 connects to only four groups. Every router in the Standard Dragonfly connects to every group three times. After the first three assignments, router 1 has been assigned in such a way that it is connected to at least five groups. Assignment to a different router in one of these rows would connect router 1 to even more distinct groups. Therefore, the Standard Dragonfly cannot be emulated by the Graceful Dragonfly with dilation one. These are truly different networks. Furthermore, stripping the ports of their numbers shows that the replacement graphs do not emulate each other.

This result is important for the following reason. In [5] it was shown that source-vector routing is valuable in creating parallel algorithms. Actual computers have been built using the Graceful structure. The Graceful structure does not support source-vector routing, but the Standard structure does[1] If the Graceful structure could emulate the Standard structure, source-vector routing could be performed on existing Dragonfly machines. We have shown that this is not possible.

To summarize this procedure as a source of graphs, the graph G needs to be regular of degree d and the graph H needs to be of order d. It need not be connected. As far as this author knows, it is necessary to convert to networks by assigning labels to nodes and port labels to the links of G in order to generate a $G \uparrow H$. One must leave the domain of graph theory to construct $G \uparrow H$.

The IBM PERCS© machine uses a complete graph on 22 nodes with a fine structure like that in the second table [3]. The IBM PERCS© machine uses a Graceful Structure on 22 ports and CRAY XC© [1] series uses a Graceful structure on 16 ports.This structure was chosen because it allows a machine with KM routers to be expanded to $(K + 1)M$ routers without rewiring. The Swapped Dragonfly [5] demonstrates the usefulness of source-vector routing which is possible because it is based on the Standard table. If the $MDF(K, M)$ with Graceful structure could emulate $MDF(K, M)$ with Standard structure, source-vector routing would be possible on both PERCS and XC machines. This paper shows that is not possible.

The structure of a maximal Dragonfly has been examined in three papers,[2, 6], and [4]. They did not use a structure table but defined different structures using formulas to generate the connections. They found the two structures shown here and three or four others.

[1] If global ports do not permute the groups, a source vector using a global port that sends two groups to one group can lead to a local link collision. This can lead to algorithms which generate hot spots.

It follows that the term Dragonfly may lead to networks which are in different emulation classes. For graph theorists it means the replacement product can produce an essentially unbounded source of distinct graphs.

5 Conclusion

The paper presents three graph theory constructions derived from interconnection network theory. The first is not a graph theory product because it induces a graph on $G \times H \times H$ given graphs G and H. It is derived from the theory of the Swapped Dragonfly.

The second and third constructions are both replacement products. There are two reasons for treating them as two cases. First, they are derived from completely different network models. Second, they demonstrate the difference between a replacement product $G \uparrow H$ when G is algebraic and when it is not. These examples are not isomorphic replacement graphs but they emulate each other.

Lastly, a generalization of replacement graph is introduced. It is derived from Dragonfly networks. Each node of H is connected to K vertices of G for K dividing d.The networks motivating this are constructed from routers having K global ports and are denoted $MDF(K, M)$ where $M = |H|$ and H is a complete graph. Using the structure tables 1 and 2 produces two versions of $MDF(3, 4)$. Even though the original networks G_N are emulations of each other, the two Dragonflies are not emulations of each other. Consequently, the generalized replacement graphs are not emulations of each other.

The structure table is original with this paper as far as the author knows. It is a perfect encoding of the network structure and is, in fact, a crude schematic of the network. All local connections are horizontal and all global connections may be realized as doglegs.

In addition to showing that two replacement graphs do not emulate one another, this paper shows that source-vector routing is not possible on existing Dragonfly networks.

6 Comments on the References

The paper[8] defined the cube-connected cycle. This was during the era of the Connection Machine [7] which had a Boolean hypercube topology. Nodes were of degree $\log N$ which imposed a limit on N. The goal of the CCC was to reduce node degree to 3 with an emulation of the hypercube of dilation $\log N/2$.

The Dragonfly network was defined in [9]. It has to be viewed as one of the most influential network papers of last twenty years. Every manufacture has used some version of the Dragonfly network. The paper [3] describes the Dragonfly used for the IBM PERCS© architecture. the paper [1] describes the XC series of CRAY. Both used the Graceful structure for the complete graph.

The paper [4] was the first to examine the two different ways to define a Dragonfly. The author also designed another deterministic way to structure the complete graph.

The papers [6, 9] are the result of student-faculty projects at Knox College in Illinois. They have designed several other methods for connecting the complete graph. They studied the question of effect on performance. They did a study of bisection bandwidth and found that it was the same for the examples they had. However, increasing the bandwidth of global links lead to a divergence of bisection bandwidth across the various dragonflies.

In paper [8] the replacement product is referred to as well-known. In lieu of defining ports on links, the authors enumerated the neighbors of each node in the graph. The nodes of H were enumerated and the replacement graph was defined. It is easy to see that the authors could have filled out a structure table with neighbor numbers as ports. Their goal was a product graph with $\deg(G, H) = 1 + \deg(H)$. Therefore, they did not generate graphs like $MDF(3, 4)$ for which $\deg(G, H) = 3 + \deg(H)$.

The paper [5] studies an incomplete Dragonfly with KM groups rather that $KM + 1$ groups. It has many useful properties, chief of which is the fact that a Swapped Dragonfly with groups having M routers with K global ports contains multiple Swapped Dragonflies with $K' < K$ and $M' < M$.

References

1. Alverson, B., Froese, E., Kaplan, L., Roweth, D.: Cray XC Series. www.cray.com
2. Belka, M. et.al.: New link arrangement for dragonfly networks. In: 2017 IEEE 3rd International Workshop on High Performance Interconnection Networks in the Exascale and Big Data Era
3. Chakavavarthy, V.T. et.al.: Mapping strategies for the PERCS architecture. In: 2012 19th International Conference on High Performance Computing. IEEE
4. Comanero, C. et.al.: topological characterization of hamming and dragonfly networks and it implication for routing. ACM Trans. Architecture. Code Optim. **11** (2014)
5. Draper, R.: The Swapped Dragonfly, arXiv.org/abs/2202.01843. February (2022)
6. Hastings, E. et.al.: Comparing global link arrangements for dragonfly networks. Proc. IEEE Cluster 365–370 (2015)
7. Hillis, W.R.: The Connection Machine. MIT Press. ISBN 0262081571 (1986)
8. Hoory, S. et. al.: Expander Graphs and their Applications. Bulletin, AMS (2006)
9. Kim, J., Dally, W.J., Scott, S., Abts, D.: Technology-driven, highly-scalable dragonfly topology. In: International Symposium on Computer Architecture, pp. 77–88 (2008)
10. Preparata, F., Vullemin, K.: The Cube-Connected Cycle. Commun. ACM. **24**(5), 300–309 (1981)

On the Target Pebbling Conjecture

Glenn Hurlbert and Essak Seddiq

Abstract Graph pebbling is a network optimization model for satisfying vertex demands with vertex supplies (called pebbles), with partial loss of pebbles in transit. The pebbling number of a demand in a graph is the smallest number for which every placement of that many supply pebbles satisfies the demand. The Target Conjecture (Herscovici-Hester-Hurlbert, 2009) posits that the largest pebbling number of a demand of fixed size t occurs when the demand is entirely stacked on one vertex. This truth of this conjecture could be useful for attacking many open problems in graph pebbling, including the famous conjecture of Graham (1989) involving graph products. It has been verified for complete graphs, cycles, cubes, and trees. In this paper we prove the conjecture for 2-paths and Kneser graphs over pairs.

Keywords Graph pebbling · Target conjecture · Pebbling configuration · Target distribution · 2-path · Kneser graph

2010 MSC 05C57 (05C35 · 90B06)

1 Introduction

Graph pebbling of the type we study here began as a method for proving a number-theoretic conjecture of Erdős and Lemke (see [6]), which was further applied to prove a group-theoretic conjecture of Kleitman and Lemke (see [8]), as well as to prove a result in 2-adic analysis (see [18, 19]). It has since grown into a network optimization model for satisfying vertex demands with vertex supplies (called pebbles), with partial loss of pebbles in transit. While there are a number of different pebbling

G. Hurlbert (✉) · E. Seddiq
Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, 1015 Floyd Ave, Richmond, VA 23220, USA
e-mail: ghurlbert@vcu.edu

E. Seddiq
e-mail: seddiqe@mymail.vcu.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_16

games on graphs, with a wide range of rules and applications (see [9, 10, 14, 16, 17, 20, 22, 23, 25, 27]), this version is defined as follows.

For a finite, connected graph $G = (V, E)$, a *configuration* (or *supply*) C is a non-negative integer-valued function on V, with *size* $|C| = \sum_{v \in V} C(v)$. That is, $C(v)$ represents the number of pebbles on the vertex v, while $|C|$ denotes the total number of pebbles on V. For a vertex v such that $C(v) > 0$, we define the configuration $C - v$ by $(C - v)(v) = C(v) - 1$ and $(C - v)(u) = C(u)$ for all $u \neq v$. Similarly, a *target distribution* (or *demand*) D is a non-negative integer-valued function on V, with *size* $|D| = \sum_{v \in V} D(v)$. That is, $D(v)$ equals the number of pebbles required to eventually place on the vertex v, while $|D|$ denotes the total demand of pebbles on V. The notation $D - v$ is defined similarly: $(D - v)(v) = D(v) - 1$ and $(D_v)(u) = D(u)$ for all $u \neq v$. For a target D, define $\dot{D}$ to be the multiset $\{v^{D(v)}\}_{v \in V}$. ($D(v)$ is the *multiplicity* of v in $\dot{D}$.)

For a configuration C, a *pebbling step* from u to an adjacent vertex v removes two pebbles from u and places one of those pebbles on v; the resulting configuration C' is defined by $C'(u) = C(u) - 2$, $C'(v) = C(v) + 1$, and $C'(w) = C(w)$ otherwise. We say that C is *D-solvable* (or that C *solves* D) if C can be converted via pebbling steps to a configuration C^* such that $C^*(v) \geq D(v)$ for all $v \in V$; C is *D-unsolvable* otherwise. A sequence of such pebbling steps is called a *D-solution* (*r-solution* in the case that $\dot{D} = \{r\}$). Suppose that r has $D(r) > 0$ and that σ is an r-solution given by the sequence of pebbling steps $\sigma_1, \ldots, \sigma_k$. For each $1 \leq i \leq k$, denote by $C^{(i)}$ the configuration resulting from $C^{(i-1)}$ via the pebbling step σ_i (where $C^{(0)} = C$). Then we define the configuration $C - \sigma = C^{(k)} - r$. That is, $C - \sigma$ is the configuration resulting from removing all pebbles involved in the r-solution σ. Thus, if $C - \sigma$ solves $D - r$, then C solves D.

The *pebbling number of a demand D in a graph G* is denoted $\pi(G, D)$ and defined to be the smallest m such that every configuration of size m is D-solvable. In the case that $|D| = 1 = D(r)$, we simply write $\pi(G, r)$. When $|D| = t = D(r)$ we say that D is *stacked* (on r); in this case we may write that C *t-fold solves* r instead of that C solves D, and use the notation $\pi_t(G, r) = \pi(G, D)$. The *t-fold pebbling number of G* is defined as $\pi_t(G) = \max_{r \in V} \pi_t(G, r)$; if $t = 1$ we omit the subscript and avoid writing “1-fold”. As with many fractional analogues of graph theoretical invariants (chromatic number, clique number, matching number, etc.—see [24]), the *fractional pebbling number* is defined to be $\hat{\pi}(G) = \liminf_{n \to \infty} \pi_t(G)/t$. It was proved in [12, 13] that $\hat{\pi}(G) = 2^{\mathsf{diam}(G)}$ for every graph G.

The original application of graph pebbling only involved the case $t = 1$. However, in [6] the problem was immediately generalized so that the parameter t could be used in an inductive manner to prove results on trees. Similarly, the problem was further expanded to more general D[1] in [12], wherein is found the following Target Conjecture.

Conjecture 1 (Target Conjecture) [12] Every graph G satisfies $\pi(G, D) \leq \pi_{|D|}(G)$ for every target distribution D.

The authors of [12] verified this conjecture for trees, cycles, complete graphs, and cubes. The generalization to D has proved useful in obtaining results inductively on powers of paths (see [4]). The hope is that it may be a powerful tool more generally, for example on chordal graphs, for which it has been conjectured that the pebbling numbers of chordal graphs of a certain type can be calculated in polynomial time (see [1]). Furthermore, one might suspect that the use of general targets could be helpful in attacking the famous conjecture of Graham (see [6]) that $\pi(G\square H) \leq \pi(G)\pi(H)$, where $\square$ denotes the cartesian product of graphs. Herscovici, et al. [11], generalize this to conjecture that $\pi(G_1\square G_2, D_1 \times D_2) \leq \pi(G, D_1)\pi(H, D_2)$. The truth of Conjecture 1 may prove to be a useful tool in this direction.

In this paper we verify Conjecture 1 in Theorem 9 for the family of 2-paths, defined in Sect. 2, and in Theorem 20 for the family of Kneser graphs $K(m, 2)$, defined in Sect. 3.

1.1 Preliminaries

Before beginning, we introduce a few key concepts used in the proofs. For a fixed vertex r denote by $V_i(r)$ the set of all vertices at distance i from r. For a configuration C we define a vertex v to be a *zero* of C if $C(v) = 0$, and denote the number of zeros of C by $z(C)$. In addition, we define the *support* of C ($\mathsf{supp}(C)$) to be the set of vertices v with $C(v) > 0$, and denote $s(C) = |\mathsf{supp}(C)|$. Note that $|V| = s(C) + z(C)$ since every vertex either has at least one pebble or none at all. Furthermore, we define the *potential* of C: $\mathsf{pot}(C) = \sum_{v\in V} \lfloor C(v)/2\rfloor$. This equals the number of pairwise disjoint pairs of pebbles with pebbles of the same pair sitting on the same vertex (each such pair is called *a potential*); in other words, it is the total number of initial pebbling steps that can be made from C (with pebbles on the same vertex being indistinguishable).

Lemma 2 (Potential Lemma) *Let C be a configuration on a graph G with potential* $\mathsf{pot}(C)$. *Then the following hold.*

[1] We note that the D-pebbling number was first introduced in [7] for the case $D(v) \geq 1$ for all $v \in V(G)$, and was called the *cover pebbling number*. In that paper, the authors prove for trees that the largest D-unsolvable configuration is obtained by stacking the entire configuration on a single vertex—the result for all graphs was proved in [26]. It is then easy to calculate the size of such a configuration for each vertex, discern which vertex has the largest stack, and therefore derive the cover pebbling number.

1. $\mathsf{pot}(C) \geq \left\lceil \frac{|C|-|V|+z(C)}{2} \right\rceil$.
2. *If C solves a distribution D and* $\mathsf{supp}(C) \cap \mathsf{supp}(D) = \emptyset$ *then* $\mathsf{pot}(C) \geq |D|$.

Proof Part (1) is a simple consequence of the relation $|C| \leq 2\mathsf{pot}(C) + s(C)$. Part (2) holds since placing a pebble on a target requires a potential to make a pebbling step. □

Finally, for a configuration C, a path $v_1, \ldots, v_k$ is called a *slide* if $C(v_1) \geq 2$ and $C(v_i) \geq 1$ for all $1 < i < k$. In addition, for a single target r, we define the *cost*, $\mathsf{cost}(\sigma)$, of an r-solution σ to be the number of pebbles discarded by σ, including the pebble placed on r; thus it equals one more than the number of pebbling steps of σ. For example, for $r = v_k$ in the slide above, the cost of the solution that moves a pebble from v_1 to r equals k. Consequently, the resulting configuration C' of pebbles remaining to use to solve other targets has size $|C'| = |C| - \mathsf{cost}(\sigma)$. We define an r-solution to be *cheap* if its cost is at most $2^{\mathsf{ecc}(r)}$; a configuration is *r-cheap* if it has a cheap r-solution. A *cost-k* solution is simply a solution of cost exactly k. We say that a graph G is *r-(semi)greedy* if every configuration of size at least $\pi(G, r)$ has a (semi)greedy r-solution; that is, every pebbling step in the solution decreases (does not increase) the distance of the moved pebble to r. It is known, for example (see [4]), that trees are greedy and that chordal graphs are semi-greedy.

Lemma 3 (Cheap Lemma) *[3] Given the graph G with target r let H be an r-greedy spanning subgraph of G preserving distances to r. Then any configuration of G of size at least $\pi(H, r)$ is r-cheap.*

In particular, a breadth-first-search spanning tree can play the role of H in the Cheap Lemma. In the case of 2-paths, below, we use a caterpillar as the best choice of such a tree. A *caterpillar* is a tree that contains a path P such that every vertex not on P is adjacent to some vertex on P.

2 2-Paths

2.1 Definition and Notation

A *simplicial* vertex is a vertex whose set of neighbors forms a complete graph. A *chordal* graph is a graph with no induced cycle of length 4 or more. A *k-path* is either a complete graph K_k or K_{k+1}, or a graph G with exactly two simplicial vertices u and v such that the neighborhood of v is K_k and $G - v$ is a k-path.

Let G be a chordal graph with two simplicial vertices, which we denote r and s, with the shortest path $P = (x_0, x_1, \ldots, x_d)$ between them, where $d = \mathsf{dist}(r, s) = \mathsf{diam}(G)$ ($r = x_0, s = x_d$). We call this path the *spine* of G. We define a *fan* F to be a subgraph of G which consists of a path $Q = a, v_1, \ldots, v_k, c$, and an additional vertex b which is adjacent to every vertex of Q and where a, b, c form a subpath of P (e.g. $a = x_{i-1}, b = x_i$, and $c = x_{i+1}$ for some $0 < i < d$). We say the F is an

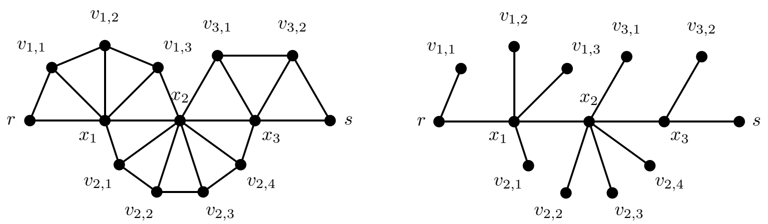


Fig. 1 An overlapping fan graph of diameter 4 on the left. The BFS tree rooted at r on the right

ac-fan with fan vertices $F' = \{v_1, \ldots, v_k\}$. Then G is an *overlapping fan graph* if, for every $0 < i < d$, there is an $x_{i-1}x_{i+1}$-fan F_i in G, every vertex of G is in some fan of G, and $|F_i' \cap F_{i+1}'| \leq 1$. An example of a fan graph is seen in left diagram of Fig. 1, where $d = 4$, $k_1 = 4$, $k_2 = 3$ and $k_3 = 3$.

By Lemma 2.1 of [2], every 2-path is an overlapping fan graph (and vice-versa). In [2] we find the following result.

Theorem 4 *If G is a 2-path with* $\mathsf{diam}(G) = d$ *then* $\pi_t(G) = t2^d + n - 2d$.

The point of this section is to prove this bound for all targets D of size t, rather than just t targets on the same vertex—see Theorem 9.

2.2 Preliminary Facts and Lemmas

We present some important preliminary facts and lemmas necessary for the proof of Theorem 9. These first two facts are used to determine the pebbling numbers of specially constructed trees which are essential to the proof. Suppose T is a tree with a root r. A *path partition* $\mathcal{P}$ of T is a set of pairwise edge-disjoint directed paths whose union is T. It is an *r-path partition* if r is an endpoint of the longest path of $\mathcal{P}$. A path partition is said to *majorize* ($\succ$) another if its non-increasing sequence of the path sizes majorizes that of the other—that is, $(a_1, a_2, \ldots, a_i) \succ (b_1, b_2, \ldots, b_t)$ if and only if $a_i > b_i$, where $i = \min\{j : a_j \neq b_j\}$. An r-path partition of a tree T is said to be *maximum* if it majorizes all other r-path partitions.

Fact 5 ([6]) *Let T be a tree rooted at r, and let $\mathcal{P} = \{P_1, ..., P_k\}$ be a maximum r-path partition of T. Denote by a_i the length of the path P_i, and suppose that the paths are labeled so that $a_i \geq a_{i+1}$ for all $1 \leq i < k$. (Note that $a_1 =$ $\mathsf{ecc}(r)$.) Then $\pi_t(T, r) = t2^{a_1} + \sum_{i=2}^{k} 2^{a_i} - k + 1$ for all $t \geq 1$.*

Let T be any Breadth First Search (BFS) spanning tree of a 2-path G, rooted at r. Then T preserves all distances from r; that is, $\mathsf{dist}_T(v, r) = \mathsf{dist}_G(v, r)$ for all vertices v. For a simplicial vertex r, we define T_r (the *spinal* tree of r) to be such a BFS tree chosen in a specific manner, namely, with the priority of choosing vertices of the spine P_1 (from the maximum r-path partition) before other vertices whenever

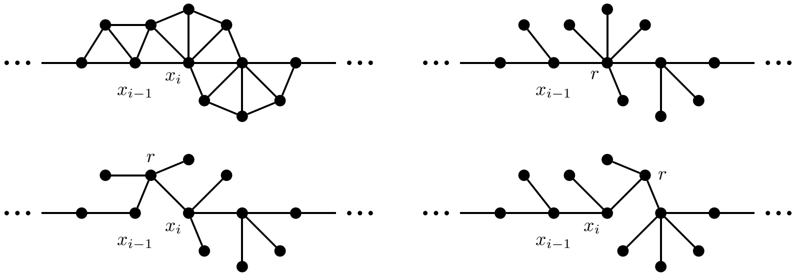


Fig. 2 A 2-path with rooted trees T_r for the three cases of internal roots r

possible. As we see from Fig. 1, this produces a caterpillar, which has a very simple pebbling number formula, according to Fact 5, because its path partition has only one long path. Let S be the set of two simplicial vertices of G.

For an internal, spinal root r, let F_i be the fan centered on r, and let A_r be the set of vertices of F_i that are in no other fan. For each $v \in A_r$ we have that the edge rv is a 2-path, which we denote by G_r^v. Then $G - A_r$ is the union of two 2-paths, which we denote by G_r^s for $s \in S$, that share only the vertex r. Note that, for each $u \in S \cup A_r$, we have that r and u are the simplicial vertices of G_r^u; we denote a spinal tree of r in G_r^u by $T_r(G_r^u)$. Now define $T_r = \cup_{u \in S \cup A_r} T_r(G_r^u)$.

For an internal, non-spinal root r, with r in two fans, let F_i and F_{i+1} be the two fans that share r, centered on x_i and x_{i+1} respectively. Then $G - x_i x_{i+1}$ is the union of two 2-paths G_r^s, for $s \in S$, that share only the vertex r. Note that, for each $s \in S$, we have that r and s are the simplicial vertices of G_r^s. We denote a spinal tree of r in G_r^s by $T_r(G_r^s)$, and so define $T_r = \cup_{s \in S} T_r(G_r^s)$.

For an internal, non-spinal root r, with r unique to one fan, let F_i be the fan, centered on x_i, that contains r. Then G is the union of two 2-paths G_r^s, for $s \in S$, that share the edge rx_i. Note that, for each $s \in S$, we have that r and s are the simplicial vertices of G_r^s. We denote a spinal tree of r in G_r^s by $T_r(G_r^s)$, and so define $T_r = \cup_{s \in S} T_r(G_r^s)$.

Observe that, in the first two cases, T_r is an edge-disjoint union of $\deg(r) - 2$ caterpillars, all sharing only the vertex r. In the third case, however, T_r is a union of two caterpillars, each containing the edge rx_i. See Fig. 2 for some examples. The next result follows from Fact 5.

Corollary 6 *Let G be a 2-path of diameter d on n vertices, and T_r be a spinal tree rooted at vertex r. Then $\pi(T_r, r) =$*

1. $2^{\mathsf{ecc}(r)} + 2^{d-\mathsf{ecc}(r)} + n - d - 2$ *if r is spinal for some representation, and*
2. $2^{\mathsf{ecc}(r)} + 2^{d+1-\mathsf{ecc}(r)} + n - d - 3$ *if r is non-spinal in every representation.*

In particular, $\pi(T_r, r) \le 2^d + n - d - 1$ for all r.

Proof Write $S = \{s_1, s_2\}$. If r is a spinal root then, for each $T_r(G_r^v)$, $v \in S \cup A_r$, r is a simplicial vertex. Label the trees so that $T_r(G_r^{s_1})$ has diameter $\mathsf{ecc}(r)$ and $T_r(G_r^{s_2})$ has diameter $d - \mathsf{ecc}(r)$. The remaining trees are single edges. Then, by Fact 5, we have

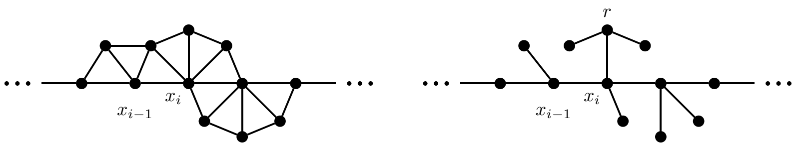


Fig. 3 A 2-path with rx_i in the spines of both trees $T_r(G_r^{s_1})$ and $T_r(G_r^{s_2})$

$$\begin{aligned}\pi(T_r, r) &= (2^{\mathsf{ecc}(r)} - 1) + (2^{d-\mathsf{ecc}(r)} - 1) + (n - d - 1)(2^1 - 1) + 1\\ &= 2^{\mathsf{ecc}(r)} + 2^{d-\mathsf{ecc}(r)} + n - d - 2.\end{aligned}$$

If r is non-spinal then r is simplicial in each $T_r(G_r^{s_i})$. Observe that if r is non-spinal in every representation then it is never in three fans; otherwise, if x_i is the center of the middle fan then we obtain a spinal representation for r by using the path $x_{i-1}rx_{i+1}$ in place of $x_{i-1}x_ix_{i+1}$ in the spine.

Suppose it is the case that the edge rx_i is not in the spine of $T_r(G_r^{s_j})$ for some j. We remark in this case that $\mathsf{diam}(T_r) = d + 1$, since we always gain one more edge in the spine of T_r than in the spine of G only. Label the trees so that $T_r(G_r^{s_1})$ has diameter $\mathsf{ecc}(r)$ and $T_r(G_r^{s_2})$ has diameter $d + 1 - \mathsf{ecc}(r)$. Because in each case (see also Fig. 3) we have $n - d - 2$ vertices not on either spine, Fact 5 implies that

$$\begin{aligned}\pi(T_r, r) &= (2^{\mathsf{ecc}(r)} - 1) + (2^{d+1-\mathsf{ecc}(r)} - 1) + (n - d - 2)(2^1 - 1) + 1\\ &= 2^{\mathsf{ecc}(r)} + 2^{d+1-\mathsf{ecc}(r)} + n - d - 3.\end{aligned}$$

All these formulas are maximized when $\mathsf{ecc}(r)$ is as large as possible. Hence we have

$$\begin{aligned}\max\{2^{d-1} + 2 + n - d - 2, 2^d + 2 + n - d - 3\} &= \max\{2^d + n - d - 2^{d-1}, 2^d + n - d - 1\}\\ &= 2^d + n - d - 1,\end{aligned}$$

as claimed. □

Lemma 7 *If r^* is a simplicial target of a 2-path G, with rooted spinal tree $T^* = T_{r^*}$, then $\pi(T_r, r) \le \pi(T^*, r^*)$.*

Proof Corollary 6 shows that the pebbling numbers for each type of r in G are bounded above by $2^d + n - d - 1$, which equals the pebbling number at r^* by Fact 5. □

Lemma 8 *Let G be a 2-path and r be any vertex. Then $\pi_2(G) \ge \pi(T^*, r^*)$. Thus, any configuration of size at least $\pi_2(G)$ is r-cheap.*

Proof Note that for any d, $2^d > d$. Then by Theorem 4 we have:

$$\begin{aligned}\pi(T^*, r^*) &= 2^d + n - d - 1\\ &\le 2^d + n - d - 1 + (2^d - d + 1)\\ &= 2^{d+1} + n - 2d\\ &= \pi_2(G).\end{aligned}$$

The existence of a cheap solution follows from the Cheap Lemma 3. □

2.3 Verification of the Target Conjecture

Theorem 9 *Let G be a 2-path and D be a target distribution of size t. Then $\pi_t(G) \ge \pi(G, D)$.*

Proof Let $|C| = \pi_t(G)$. We proceed by induction. If $t = 1$ then $\pi_1(G) \ge \pi(G, r)$ for any r, so C solves any D of size 1.

Now let $t \ge 2$ and recall from Theorem 4 that $\pi_t(G) = \pi_{t-1}(G) + 2^d$. By the induction hypothesis, we assume that $\pi(G, D') \le \pi_{t-1}(G)$ for all $|D'| = t - 1$. Let D be given. Choose any $r \in \dot{D}$ and let T_r denote a spinal tree of r. Then, since $|C| \ge \pi_2(G) \ge \pi(T_r, r)$, we know by the Cheap Lemma 8 that C has a cheap r-solution σ. Thus $|C - \sigma| = |C| - \mathsf{cost}(\sigma) \ge \pi_t(G, D) - 2^d = \pi_{t-1}(G, D - r)$. Therefore $C - \sigma$ solves $D - r$ by induction, and hence C solves D. □

3 Kneser Graphs

3.1 Definition

For $m \ge h \ge 1$, the *Kneser graph* $K(m, h)$ is the graph whose vertex set equals the set all subsets of $[m] = \{1, 2, \ldots, m\}$ of size h—so that $|V(K(m, h))| = \binom{m}{h}$—with two vertices adjacent if and only if they are disjoint. Observe that $K(m, 1)$ is the complete graph on m vertices and that $K(5, 2)$ is the Petersen graph. Also note that $K(m, h)$ is empty for $m < 2h$, is a matching for $m = 2h$, is connected for $m \ge 2h + 1$, has $\mathsf{diam} = 2$ for $m \ge 3h - 1$, and is vertex transitive. The following known theorems are used in the proofs below. Let $\kappa(G)$ denote the connectivity of a graph G.

Theorem 10 ([5]) *For $m \ge 2h + 1 \ge 3$, $K(m, h)$ is connected, edge transitive, and regular of degree $\binom{m-h}{h}$.*

Theorem 11 (Exercise 15(c) of Chap. 12 of *[21]*) *If a graph G is connected, edge transitive, and regular of degree δ, then $\kappa(G) = \delta$.*

Corollary 12 *For $m \geq 2h + 1 \geq 3$, $\kappa(K(m, h)) = \binom{m-h}{h}$.*

Proof Theorem 10 shows that $K(m, h)$ satisfies the hypothesis of Theorem 11, from which the result follows. □

We will use the following general form of Menger's Theorem (see Exercise 4.2.28 of [28]).

Theorem 13 *Let X and Y be disjoint sets of vertices in a k-connected graph G. Let $U(x)$, for $x \in X$, and $W(y)$, for $y \in Y$, be nonnegative integers such that $\sum_{x\in X} U(x) = \sum_{y\in Y} W(y) = k$. Then G has k pairwise internally disjoint X, Y-paths so that $U(x)$ of them start at x and $W(y)$ of them end at y for all $x \in X$ and $y \in Y$.*

The following pebbling result is also known.

Theorem 14 ([15]) *For all $m \geq 5$ we have $\pi(K(m, 2)) = \binom{m}{2}$.*

3.2 Results

Let $n = |V(K(m, 2))| = \binom{m}{2}$. For any vertex r, we define the configuration $C_{t,1}$ to have $2t - 1$ pebbles on one $x \in V_2(r)$, and one pebble on every other vertex except for r. Note that $|C_{t,1}| = 2t - 1 + n - 2 = n + 2t - 3$. In addition, we define the configuration $C_{t,2}$ to have $4t - 1$ pebbles on one $y \in V_2(r)$, and one pebble on every other $v \in V_2(r)$. Observe that $|C_{t,2}| = 4t - 1 + 2(m - 2) - 1 = 4t + 2(m - 2) - 2$. We prove in Lemma 15 that each $C_{t,i}$ is r-unsolvable.

Define $p_1(m, t) = |C_{t,1}| + 1 = n + 2t - 2$, $p_2(m, t) = |C_{t,2}| + 1 = 4t + 2m - 5$, $p(m, t) = \max_i p_i(m, t)$, and $t_0 = \deg(K(m, 2))/2 = \binom{m-2}{2}/2$. Then

$$p(m, t) = \begin{cases} p_1(m, t) \text{ for } t \leq t_0, \\ p_2(m, t) \text{ for } t \geq t_0. \end{cases}$$

Lemma 15 *Let $G = K(m, 2)$ for any $m \geq 5$. Then $\pi_t(G) \geq p(m, t)$.*

Proof Because $\mathsf{pot}(C_{t,1}) = t - 1$, the Potential Lemma 2(2) implies that $C_{t,1}$ is not t-fold r-solvable for some root r. Furthermore, we have $\mathsf{pot}(C_{t,2}) = 2t - 1$, with the only potential vertex u at distance two from r. Thus $C_{t,2}$ can place at most $2t - 1$ pebbles on $V_1(r)$, resulting in a configuration C' with potential at most $t - 1$. By the Potential Lemma 2 (2), C', and hence C, is not t-fold r-solvable. □

Lemma 16 *Let $G = K(5, 2)$ and $|D| = t$. Then $\pi(G, D) \leq p(5, t)$.*

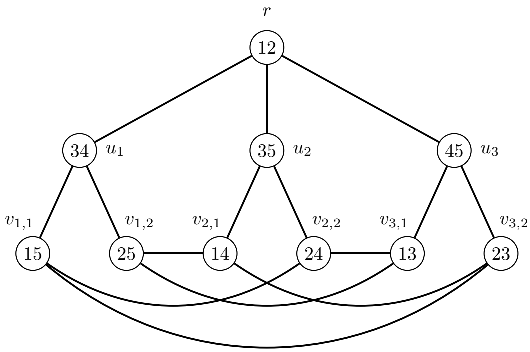


Fig. 4 The Petersen graph $P = K(5, 2)$, labeled as in the proof of Lemma 16

Proof Let r be any target in D. Label the vertices as follows (see Fig. 4): $V_1(r) = \{u_i \mid i \in [3]\}$ and $V_2(r) = \{v_{i,j} \mid i \in [3], j \in [2]\}$, with adjacencies $u_i \sim v_{i,j}$ for all i and j and $v_{i,j} \sim v_{i',j'}$ if and only if $i \neq i'$ and $j \neq j'$. Define the configuration J_r by $J_r(r) = 0$ and $J_r(x) = 1$ otherwise. Also, define $U_i = \{u_i, v_{i,1}, v_{i,2}\}$ and write $J_{r,i}$ for the restriction of J_r to U_i. For a configuration C, we say that a vertex v is *big* if $C(v) \geq 2$.

Claim A. If C is r-unsolvable of size 9 then $C = J_r$.

Proof Let C be r-unsolvable of size 9. Suppose that $\mathsf{pot}(C) > 0$. Since C does not solve r, no neighbor of r is big. If $C(U_i) \geq 5$ then C solves r, either by having two pebbles on u_i, one pebble on u_i and moving a pebble to u_i, or no pebbles on u_i and moving two pebbles to u_i. Thus we may assume that $C(U_i) \leq 4$ for all i.

If some $C(U_i) = 4$ then some $C(U_{i'}) \geq 3$. If $U_{i'}$ has a big vertex, it can move a fifth pebble into U_i, which can then solve r. Thus $C(U_{i'}) = J_{r,i'}$. However, this allows a big vertex from U_i to slide through $U_{i'}$ to r, which is a contradiction.

Hence $C(U_i) = 3$ for all i. If C has two potentials then two pebbles can be moved into a third U_i, which yields 5 pebbles that can solve r. Thus we may assume that C has exactly one potential, say in U_i. But then some i' has $C(U_{i'}) = J_{r,i'}$, and so the big vertex can slide through $U_{i'}$ to r, a contradiction.

Hence $\mathsf{pot}(C) = 0$; i.e. $C = J_r$. ◇

Claim B. If $|C| \geq 13$ then C is r-cheap.

Proof By the pigeonhole principle we have that some $C(U_i) \geq 5$. Thus U_i has a big vertex w. We have a cost-2 solution if $w = u_i$, so consider that $w = v_{i,j}$ for some j. We have a cost-3 solution if u_i has a pebble, so consider that u_i is empty. Then U_i has two potentials, which produces a cost-4 solution. ◇

Because $t_0 = \binom{3}{2}/2 = 1.5$, we have $p(5, t) = p_1(5, t)$ for $t = 1$ and $p(5, t) = p_2(5, t)$ for $t \geq 2$. The statement is true for $t = 1$ because we know from Theorem 14 that $\pi(K(5, 2)) = 10 = p_1(5, 1)$.

For $|D| = t = 2$ we have $|C| = p_2(5, 2) = 13$. Let r_1 and r_2 be the two (not necessarily distinct) vertices with $D(r_i) > 0$. Claim B implies that C has a cheap r_1-solution σ. If $C - \sigma$ has an r_2-solution then we're done, so assume otherwise. It must be then that $|C - \sigma| = 9$, and so Claim A implies that $C - \sigma = J_{r_2}$. Thus r_2 is the only empty vertex in C (it is the only empty vertex in $C - \sigma$, and if it is not empty in C then C solves D without moving), so any big vertex has a greedy slide; i.e. an r_2-solution σ' of cost at most 3. Then $C - \sigma'$ has size at least 10, which provides an r_1-solution. Thus C solves D.

For $|D| = t \geq 3$ we have $|C| = p_2(5, t) = 4t + 5$. Fix a target r with $D(r) > 0$. Then Claim B implies that C has a cheap r-solution σ, and so $|C - \sigma| \geq |C| - 4 = 4(t - 1) + 5 = p_2(5, t - 1)$. By induction, $C - \sigma$ is $(D - r)$-solvable. Hence C is D-solvable. □

Fact 17 *Let $G = K(m, 2)$ for some $m > 5$ and let r be any vertex. Then for every (not necessarily distinct) $u, v \in V_2(r)$ there exists a $w \in V_1(r)$ such that $w \in N(u) \cap N(v)$.*

Proof For every $x \in V_2(r)$, we have $|x \cap r| = 1$. Thus, for every $u, v \in V_2(r)$, we have $|(r \cup u \cup v)| \leq 4$. Because $m \geq 6$, there exists $y = \{c, d\} \subseteq [m] - (r \cup u \cup v)$. Thus $y \in N(r) \cap N(u) \cap N(v)$. □

Lemma 18 *Let $G = K(m, 2)$ for some $m > 5$ and let $|D| = t$. Then $\pi(G, D) \leq p(m, t)$.*

Proof We prove the upper bound using induction on t. Because of Theorem 14, the statement is true for $t = 1$, so we assume that $t \geq 2$. Let C be a configuration on G of size $p(m, t)$. If some vertex r has $C(r) > 0$ and $D(r) > 0$, then a pebble on r solves a target with cost 1. Since $|C - r| = |C| - 1 > p(m, t - 1)$, we know by induction that $C - r$ solves $D - r$. Hence we may assume that no pebbles of C already sit on any target of D.

From Fact 17, we see that, for any r, every pair of potentials yields an r-solution when $m > 5$. Indeed, if one of the potentials is in $V_1(r)$ then it solves r immediately. Otherwise both are in $V_2(r)$ and have a common neighbor to move two pebbles to, solving r subsequently. Notice that both of these solutions are greedy, and hence cheap.

Define $\dot{P}$ to be the multiset of vertices on which the $\mathsf{pot}(C)$ potentials of C sit, and Z to be the set of z zeroes. Note that $\mathsf{pot}(C) \geq t$ by Lemma 2 (1). Set $t' = \min\{t, 2t_0\}$, and let $\dot{P}'$ be any submultiset of $\dot{P}$ of size t', and $\dot{D}'$ be any submultiset of $\dot{D}$ of size t'. By Corollary 12 and Theorem 13, since $\kappa(G) = 2t_0$, G has t' internally disjoint paths $\mathcal{P}$ from $\dot{P}'$ to $\dot{D}'$.

Now define s to be the number of paths of $\mathcal{P}$ that are slides. If $s = t$ then of course we are done, so we assume that $s < t$. Then $(t' - s)$ of the paths of $\mathcal{P}$ are not slides, and therefore have zeroes on them. Let $z = |Z|$ and note that $z \geq 2t_0 - s + 1$. Indeed, let $Z' = Z - \dot{D}'$, $G' = G - Z$, and suppose that $|Z'| \leq 2t_0 - s - 1$. Then $\kappa(G') \geq \kappa(G) - |Z'| \geq 2t_0 - (2t_0 - s - 1) = s + 1$. Thus there are at least $s + 1$ paths from $\dot{P}'$ to $\dot{D}'$—these are all slides of C in G. Hence, if there are exactly s slides of C in G, then $|Z'| \geq 2t_0 - s$. Therefore $|Z| = |Z'| + s(D') \geq (2t_0 - s) + 1 = 2t_0 - s + 1$. We will show that the remaining potential after using s slides is large enough to solve the remaining $t - s$ solutions.

We first consider the case when $t \leq t_0$, which implies that $t' = t$. Furthermore, it forces $2 \leq t_0 = \binom{m-2}{2}/2$, which requires $m > 5$. In this case we have $|C| = p_1(m,t) = n + 2t - 2$. Then $s = \lfloor (s+1)/2 \rfloor + \lfloor s/2 \rfloor \geq \lfloor (s+1)/2 \rfloor$, and so

$$\begin{aligned}
\mathsf{pot}(C) - s &\geq \Big\lceil \big[(n + 2t - 2) - n + (2t_0 - s + 1)\big]/2 \Big\rceil - s \\
&= \big\lceil (2t + 2t_0 - s - 1)/2 \big\rceil - s \\
&\geq t + t_0 - s - \big\lfloor (s+1)/2 \big\rfloor \\
&\geq 2t - s - s \\
&\geq 2(t - s)\ .
\end{aligned}$$

Thus, after solving s slides, we have at least $2(t - s)$ potential. As noted above, for each remaining target, we can solve it from any pair of potentials, consequently solving all $t - s$ remaining targets from the at least $2(t - s)$ remaining potentials.

Second, we consider the case when $t_0 < t = \lceil t_0 \rceil$ (still we have $t' = t$). In this case we have $|C| = p_2(m,t) = 4t + 2m - 5$. Then because $n = |\{r\}| + |V_1(r)| + |V_2(r)| = 1 + \binom{m-2}{2} + 2(m-2)$ for any r, we have

$$\begin{aligned}
\mathsf{pot}(C) - s &\geq \Big\lceil \big[(4t + 2m - 5) - n + (2t_0 - s + 1)\big]/2 \Big\rceil - s \\
&= \Big\lceil \big[4t - \big(n - 1 - 2t_0 - 2(m-2)\big) - (s+1)\big]/2 \Big\rceil - s \\
&= 2t - \lfloor (s+1)/2 \rfloor - s \\
&\geq 2t - s - s \\
&\geq 2(t - s)\ .
\end{aligned}$$

Thus, after solving s slides, we have at least $2(t - s)$ potential, which gives us the $t - s$ remaining solutions, as before.

Finally, we consider the case when $t > \lceil t_0 \rceil$. In this case we have $|C| = p_2(m,t) = 4t + 2m - 5$. Then

$$\begin{aligned}
\mathsf{pot}(C) &\geq \Big\lceil \big([4t + 2m - 5] - n + 1\big)/2 \Big\rceil \\
&= \Big\lceil \big(4t - \big[n - 2(m-2)\big]\big)/2 \Big\rceil \\
&\geq \left\lceil \frac{1}{2}\left(4t - \left[\binom{m-2}{2} + 1\right]\right) \right\rceil \\
&\geq \left\lceil \frac{1}{2}\left(2t + 2t - \binom{m-2}{2} - 1\right) \right\rceil
\end{aligned}$$

$$\begin{aligned}
&\geq \left\lceil \frac{1}{2}\left(2t-1+\left[2t-\binom{m-2}{2}\right]\right)\right\rceil \\
&\geq \lceil (t-1/2)+(t-t_0)\rceil \\
&\geq \lceil (t+1/2)\rceil \\
&> t \\
&\geq 2.
\end{aligned}$$

As noted above, $\mathsf{pot}(C) \geq 2$ yields a cheap r-solution σ for any $r \in \dot{D}$. After using σ, the remaining configuration C' has size at least $|C| - 4 = p_2(m, t-1)$. Thus, we can use induction on t to get from C' the $t-1$ remaining solutions of $D - r$. □

3.3 Verification of the Target Conjecture

Corollary 19 *Let $G = K(m, 2)$ for $m \geq 5$. Then $\pi_t(G) = p(m, t)$.*

Proof For $m = 5$ this follows from Lemmas 15 and 16. For $m > 5$ this follows from Lemmas 15 and 18. □

Theorem 20 *Let $G = K(m, 2)$ and $|D| = t$. Then for all D, $\pi(G, D) \leq \pi_t(G)$.*

Proof This follows from Lemmas 16 and 18 and from Corollary 19. □

4 Remarks

A natural next step for verifying the Target Conjecture would be to consider k-paths for $k \geq 3$. However, the t-fold pebbling numbers for this family are not presently known. In [4] the subfamily of kth powers of paths is studied. Additional interesting families to investigate include diameter two graphs, Class 0 graphs, and chordal graphs, among others. Unfortunately, t-fold pebbling numbers are not known for such broad classes, so more specific subclasses need to be examined.

References

1. Alcón, L., Gutierrez, M., Hurlbert, G.: Pebbling in split graphs. SIAM J. Discret. Math. **28**(3), 1449–1466 (2014)
2. Alcón, L., Gutierrez, M., Hurlbert, G.: Pebbling in 2-paths. Elec. Notes Discret. Math. **150**, 145–150 (2015)
3. Alcón, L., Gutierrez, M., Hurlbert, G.: Pebbling in semi-2-trees. Discret. Math. **340**(7), 1467–1480 (2017)

4. Alcón, L., Hurlbert, G.: Pebbling in powers of paths. In preparation
5. Chen, B.L., Lih, K.W.: Hamiltonian uniform subset graphs. J. Combin. Theory Ser. B **42**, 257–263 (1987)
6. Chung, F.R.K.: Pebbling in hypercubes. SIAM J. Discret. Math. **2**(4), 467–472 (1989)
7. Crull, B., Cundiff, T., Feltman, P., Hurlbert, G., Pudwell, L., Szaniszlo, Z., Tuza, Z.: The cover pebbling number of graphs. Discret. Math. **296**(1), 15–23 (2005)
8. Elledge, S., Hurlbert, G.: An application of graph pebbling to zero-sum sequences in abelian groups. Integers: Elec. J. Combin. Number Th. **5**(1), #A17 (2005)
9. Gilbert, J., Lengauer, T., Tarjan, R.: The pebbling problem is complete in poly- nomial space. SIAM J. Comput. **9**, 513–525 (1980)
10. Gurevich, Y., Shelah, S.: On finite rigid structures. J. Symbol. Logic **61**(2), 549–562 (1996)
11. Herscovici, D., Hester, B., Hurlbert, G.: Generalizations of Graham's pebbling conjecture. Discret. Math. **312**(15), 2286–2293 (2012)
12. Herscovici, D., Hester, B., Hurlbert, G.: t-Pebbling and extensions. Graphs and Combin. **29**(4), 955–975 (2013)
13. M. Hoffmann, J. Matoušek, Y. Okamoto, and P. Zumstein, *The t-pebbling number is eventually linear in t.* Electron. J. Combin. **18** (2011), no. 2, Paper 153, 4 pp
14. J. Hopcroft, W. Paul, and L. Valiant, *On time versus space*, J. Assoc. Comput. Mach. **24** (1977), no. 2,
15. Hurlbert, G.: General graph pebbling. Discrete Appl. Math. **161**(9), 1221–1231 (2013)
16. Kirousis, L.M., Papadimitriou, C.H.: Searching and pebbling. Theoret. Comput. Sci. **47**(2), 205–218 (1986)
17. Klawe, M.M., The complexity of pebbling for two classes of graphs, in Graph theory with applications to algorithms and computer science (Kalamazoo, Mich.,: 475–487, p. 1985. Wiley-Intersci. Publ, Wiley, New York (1984)
18. Knapp, M.P.: Distance pebbling on directed cycle graphs. J. Combin. Math. Combin. Comput. **111**, 283–286 (2019)
19. M.P. Knapp, *2-adic zeros of diagonal forms*, preprint (2021)
20. Liu, J.W.H.: An application of generalized tree pebbling to sparse matrix factorization. SIAM J. Algebraic Discrete Methods **8**(3), 375–395 (1987)
21. Lovász, L.: Combinatorial Problems and Exercises. North Holland Pub, Amsterdam, New York, Oxford (1979)
22. T.D. Parsons, *Pursuit-evasion in a graph*, in Theory and Applications of Graphs (Y. Alani and D. R. Lick, eds.), 426–441. Lecture Notes in Math **642**, Springer, Berlin, 1976
23. M.S. Paterson and C.E. Hewitt, *Comparative schematology*, in Proj. MAC Conf. on Concurrent Systems and Parallel Computation, June 2–5, 1970, Woods Hole, Mass., 119–127
24. Scheinerman, E.R., Ullman, D.H.: Fractional graph theory: A rational approach to the theory of graphs. John Wiley & Sons Inc, New York (1997)
25. Sethi, R.: Complete register allocation problems. SIAM J. Comput. **4**(3), 226–248 (1975)
26. J. Sjöstrand, *The cover pebbling theorem*, Electron. J. Combin. **12** (2005), Note 22, 5 pp
27. Streinu, I., Theran, L.: Sparse hypergraphs and pebble game algorithms. European J. Combin. **30**(8), 1944–1964 (2009)
28. West, D.: Introduction to Graph Theory, 2nd edn. Pearson, London (2000)

C_4-Face-Magic Labelings on Even Order Projective Grid Graphs

Stephen J. Curran and Stephen C. Locke

Abstract For a graph $G = (V, E)$ embedded in the projective plane, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic projective graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. We call $\{x_v : v \in V(G)\}$ a C_n-face-magic projective labeling on G. We consider the $m \times n$ grid graph, denoted by $\mathcal{P}_{m,n}$, embedded in the projective plane in the natural way. It is known that, for $m, n \geq 2$, $\mathcal{P}_{m,n}$ admits a C_4-face-magic projective labeling if and only if m and n have the same parity. When m and n are even, a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ has C_4-face-magic value $2mn + 2$. We show that there are 144 distinct C_4-face-magic projective labelings on the 4×4 projective grid graph $\mathcal{P}_{4,4}$ (up to symmetries on the projective plane).

Keywords C_4-face-magic graphs · Polyomino · Projective grid graphs

1 Introduction

Graph labelings were formally introduced in the 1970s by Kotzig and Rosa [15]. Graph labelings have been applied to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader should consult J. A. Gallian's comprehensive dynamic survey on graph labelings [11] for further investigation.

We refer the reader to Chartrand, Lesniak, and Zhang [5] for concepts and notation not explicitly defined in this paper. All graphs in this paper are connected multigraphs. The concept of a C_4-face-magic labeling was first applied to planar graphs. For a

S. J. Curran (✉)
University of Pittsburgh at Johnstown, Johnstown, PA 15904, USA
e-mail: sjcurran@pitt.edu

S. C. Locke
Florida Atlantic University, Boca Raton, FL 33431, USA
e-mail: lockes@fau.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_17

planar (projective) graph $G = (V, E)$ embedded in the plane (projective plane), let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a *C_n-face-magic planar (projective)* graph if there exists a bijection $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. Here, the constant S is called a *C_n-face-magic value* of G. More generally, C_n-face-magic planar graph labelings are a special case of (a, b, c)-magic labeling introduced by Lih [16]. For assorted values of a, b and c, Baca and others [1–3, 12–14, 16] have analyzed the problem for various classes of graphs. Wang [17] showed that the toroidal grid graphs $C_m \times C_n$ are antimagic for all integers $m, n \geq 3$. Butt et al. [4] investigated face antimagic labelings on toroidal and Klein bottle grid graphs. Curran, Low and Locke [6, 7] investigated C_4-face-magic toroidal labelings on $C_m \times C_n$ and C_4-face-magic cylindrical labelings on $P_m \times C_n$. Also, Curran, Low and Locke [8] investigated C_4-face-magic Klein bottle labelings on an $m \times n$ Klein bottle grid graph.

Curran [9, 10] investigated C_4-face-magic projective labelings on an $m \times n$ projective grid graph, denoted by $\mathcal{P}_{m,n}$. It is known that $\mathcal{P}_{m,n}$ admits a C_4-face-magic labeling if and only if m and n have the same parity. Suppose $m \geq 3$ and $n \geq 3$ are odd. Then the C_4-face-magic value of a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ is either $2mn + 1$, $2mn + 2$, or $2mn + 3$. The C_4-face-magic labelings on the $m \times n$ projective grid graph with C_4-face-magic value $2mn + 2$ are characterized in [9]. Also, a category of C_4-face-magic labelings on the $m \times n$ projective grid graph with C_4-face-magic value $2mn + 1$ or $2mn + 3$ are determined in [10].

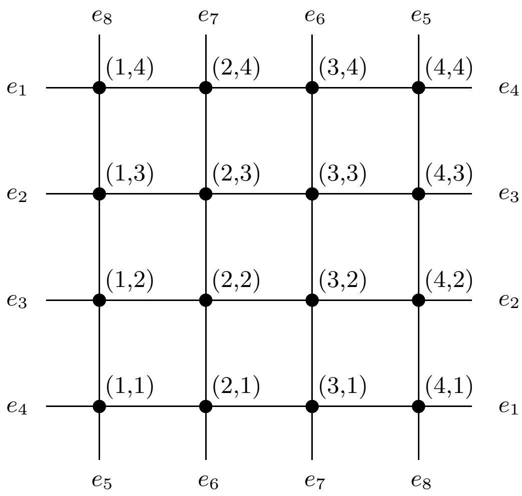
In this paper, we study C_4-face-magic labeling on $\mathcal{P}_{m,n}$ when m and n are even. If $m \geq 2$ and $n \geq 2$ are even integers, then the C_4-face-magic value of a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ is $2mn + 2$. We show that there are 144 distinct C_4-face-magic projective labelings on the 4×4 projective grid graph $\mathcal{P}_{4,4}$ (up to symmetries on the projective plane). We show that these labelings can be categorized into two specific types of labelings called centrally balanced labelings and vertically pairwise balanced labelings.

2 Preliminaries

Note 1 Let m and n be even positive integers. For convenience, let m_0 and n_0 be the positive integers such that $m = 2m_0$ and $n = 2n_0$.

Definition 1 For a graph $G = (V, E)$ embedded on the projective plane (plane or torus or Klein bottle), let $\mathcal{F}(G)$ denote the set of faces of G. Then G is called a *C_n-face-magic projective (planar or toroidal or Klein bottle)* graph if there exists a bijection $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. We call S the *C_n-face-magic value*.

Fig. 1 4×4 projective grid graph $\mathcal{P}_{4,4}$



Definition 2 Let m and n be integers such that $m, n \geq 2$. The $m \times n$ *projective grid graph*, denoted by $\mathcal{P}_{m,n}$, is the graph whose vertex set is

$$V\left(\mathcal{P}_{m,n}\right) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\},$$

and whose edge set consists of the following edges:

– there is an edge from (i, j) to $(i, j+1)$, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$,
– there is an edge from (i, n) to $(m+1-i, 1)$, for $1 \leq i \leq m$,
– there is an edge from (i, j) to $(i+1, j)$, for $1 \leq i \leq m-1$ and $1 \leq j \leq n$, and
– there is an edge from (m, j) to $(1, n+1-j)$, for $1 \leq j \leq n$.

The graph $\mathcal{P}_{m,n}$ has a natural embedding on the projective plane.

Example 1 The 4×4 projective grid graph $\mathcal{P}_{4,4}$ is illustrated in Fig. 1. Due to the orientation of the vertices in $\mathcal{P}_{m,n}$, we refer to the vertices $\{(i, j) : 1 \leqslant j \leqslant n\}$ as column i of $V(\mathcal{P}_{m,n})$ and $\{(i, j) : 1 \leqslant i \leqslant m\}$ as row j of $V(\mathcal{P}_{m,n})$. This is a multigraph since there are double edges on the pair of vertices at opposite corners.

The values of m and n for which $\mathcal{P}_{m,n}$ admits a C_4-face-magic labeling is given by the following theorem.

Theorem 1 ([9]) *Let m and n be integers such that $m, n \geq 2$. Then $\mathcal{P}_{m,n}$ has a C_4-face-magic projective labeling if and only if m and n have the same parity.*

When m and n are even, the next lemma states the C_4-face-magic value of a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ is $2mn + 2$.

Lemma 1 ([9]) *Suppose $m \geqslant 2$ and $n \geqslant 2$ are even integers. Let $\{x_{i,j} : (i, j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Then $S = 2mn + 2$.*

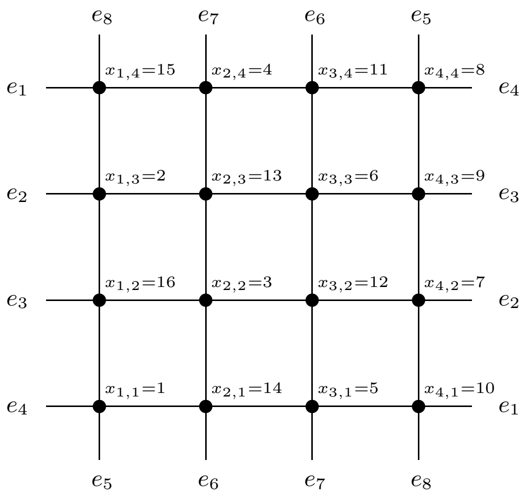


Fig. 2 C_4-face-magic projective labeling on $\mathcal{P}_{4,4}$ having C_4-face-magic value 34

For completeness, we include the proof of Lemma 1.

Proof Let m_0 and n_0 be positive integers such that $m = 2m_0$ and $n = 2n_0$. Consider the sum

$$\tfrac{1}{4}mnS = m_0n_0S = \sum_{i=1}^{m_0}\sum_{j=1}^{n_0}(x_{2i-1,2j-1} + x_{2i-1,2j} + x_{2i,2j-1} + x_{2i,2j})$$
$$= \left(\sum_{k=1}^{mn} k\right) = \tfrac{1}{2}(mn)(mn+1).$$

Thus, $S = 2mn + 2$. □

Example 2 We illustrate an example of a C_4-face-magic labeling on the 4×4 projective grid graph $\mathcal{P}_{4,4}$ with C_4-face-magic value 34 in Fig. 2.

We need to distinguish between the various types of C_4-face-magic labelings on $\mathcal{P}_{m,n}$. To this end we introduce the following definitions.

Definition 3 Let $m \geq 2$ and $n \geq 2$ be even integers. Suppose $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{m,n})\}$ is a C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Let $S = 2mn + 2$ be the C_4-face-magic value of X.

1. We say that X is *centrally balanced* if, for all $(i, j) \in V(\mathcal{P}_{m,n})$, we have

$$x_{i,j} + x_{m+1-i,n+1-j} = \tfrac{1}{2}S = mn + 1.$$

2. We say that X is *vertically pairwise balanced* if, for all $1 \leq i \leq m$ and $1 \leq j \leq \frac{1}{2}n$, we have

$$x_{i,2j-1} + x_{i,2j} = \tfrac{1}{2}S = mn + 1.$$

3. We say that X is *horizontally pairwise balanced* if, for all $1 \leq i \leq \frac{1}{2}m$ and $1 \leq j \leq n$, we have

$$x_{2i-1,j} + x_{2i,j} = \frac{1}{2}S = mn + 1.$$

Note 2 Since the corner vertices of $\mathcal{P}_{m,n}$ are the only vertices connected by double edges, any homeomorphism of the projective plane that induces a graph automorphism of $\mathcal{P}_{m,n}$ must send a corner vertex to another corner vertex.

The graph automorphisms of $\mathcal{P}_{m,n}$ that are induced by homeomorphisms of the projective plane are described in terms of the center of $\mathcal{P}_{m,n}$. We let R_θ denote the rotation by θ degrees in the counter-clockwise direction about the center of $\mathcal{P}_{m,n}$. The symmetry H (V) is the reflection about the horizontal (vertical) axis passing through the center of $\mathcal{P}_{m,n}$. Thus, for distinct integers m and n, the set of symmetries on $\mathcal{P}_{m,n}$ is $\{R_0, R_{180}, H, V\}$. We let D_+ (D_-) denote the reflection about the diagonal with positive (negative) slope passing through the center of $\mathcal{P}_{m,n}$. When $m = n$, the set of symmetries on $\mathcal{P}_{m,m}$ is $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D_+, D_-\}$.

Let γ be a graph isomorphism on $\mathcal{P}_{m,n}$ induced by a homeomorphism of the projective plane. Then γ is induced by one of the homeomorphisms in $\{R_0, R_{180}, H, V\}$ if $m \neq n$, or by one of the homeomorphisms in D_4 if $m = n$.

Definition 4 Let $X = \{x_{i,j} : (i, j) \in \mathcal{P}_{m,n}\}$ and $Y = \{y_{i,j} : (i, j) \in \mathcal{P}_{m,n}\}$ be C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$. We say that X is *projective labeling equivalent* to Y if there exists a graph isomorphism γ induced by a homeomorphism on the projective plane such that $x_{i,j} = y_{\gamma(i,j)}$ for all $(i, j) \in V(\mathcal{P}_{m,n})$.

3 C_4-Face-Magic Labelings on 4 × 4 Projective Grid Graph

We first determine the structure of a C_4-face-magic projective labeling on $\mathcal{P}_{4,4}$. This structure is dependent on the sum of the labels on the vertices of each edge of $\mathcal{P}_{4,4}$.

Definition 5 Let $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Let u and v be the vertices incident to edge e of $\mathcal{P}_{m,n}$. The *edge sum* of e with respect to X is given by $x_u + x_v$.

The pattern of edge-sum values is determined in Proposition 1.

Proposition 1 *Suppose $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{4,4})\}$ is a C_4-face-magic labeling on $\mathcal{P}_{4,4}$. Let $S = 34$ be the C_4-face-magic value of X. Let $a_i = x_{i,1} + x_{i+1,1}$ for all $1 \leq i \leq 2$, $b_j = x_{1,j} + x_{1,j+1}$ for all $1 \leq j \leq 2$, and $a_3 = x_{1,1} + x_{4,4}$. Then for all $1 \leq i \leq 2$ and $1 \leq j \leq 2$, we have*

$$x_{2i-1,2j-1} + x_{2i,2j-1} = a_1, \qquad x_{2i-1,2j} + x_{2i,2j} = S - a_1,$$
$$x_{2i-1,2j-1} + x_{2i-1,2j} = b_1, \quad \textit{and} \qquad x_{2i,2j-1} + x_{2i,2j} = S - b_1.$$

Also, for all integers $1 \le i \le 2$*, we have*

$$\begin{aligned} x_{2i-1,2} + x_{2i-1,3} &= b_2, & x_{2i,2} + x_{2i,3} &= S - b_2, \\ x_{2i-1,1} + x_{6-2i,4} &= a_3, \quad \textit{and} & x_{2i,1} + x_{5-2i,4} &= S - a_3, \end{aligned}$$

and, for all integers $1 \le j \le 2$*, we have*

$$\begin{aligned} x_{2,2j-1} + x_{3,2j-1} &= a_2, & x_{2,2j} + x_{3,2j} &= S - a_2, \\ x_{1,2j-1} + x_{4,6-2j} &= a_3, \quad \textit{and} & x_{1,2j} + x_{4,5-2j} &= S - a_3. \end{aligned}$$

Proof For $1 \le k \le 2$, we have

$$x_{k,2} + x_{k+1,2} = S - (x_{k,1} + x_{k+1,1}) = S - a_k.$$

Setting the C_4-face sums below equal

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i+1,j} + x_{i+1,j+1} + x_{i+2,j} + x_{i+2,j+1}$$

yields

$$x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}. \tag{1}$$

Thus, for all $1 \le j \le 2$ and $1 \le k \le 2$,

$$x_{k,2j-1} + x_{k+1,2j-1} = a_k \text{ and } x_{k,2j} + x_{k+1,2j} = S - a_k.$$

We observe that

$$x_{3,1} + x_{4,1} = S - (x_{1,4} + x_{2,4}) = a_1 \text{ and } x_{3,2} + x_{4,2} = S - (x_{3,1} + x_{4,1}) = S - a_1.$$

A similar argument using (1) shows that, for all $1 \le j \le 2$,

$$x_{3,2j-1} + x_{4,2j-1} = a_1 \text{ and } x_{3,2j} + x_{4,2j} = S - a_1.$$

A similar argument shows that, for all $1 \le i \le 2$ and $1 \le j \le 2$, we have

$$\begin{aligned} x_{2i-1,2j-1} + x_{2i-1,2j} &= b_1, & x_{2i,2j-1} + x_{2i,2j} &= S - b_1, \\ x_{2i-1,2} + x_{2i-1,3} &= b_2, \quad \text{and} & x_{2i,2} + x_{2i,3} &= S - b_2. \end{aligned}$$

Setting the C_4-face sums below equal

$$x_{i,1} + x_{i+1,1} + x_{5-i,4} + x_{4-i,4} = S = x_{i+1,1} + x_{i+2,1} + x_{4-i,4} + x_{3-i,4}$$

yields

$$x_{i,1} + x_{5-i,4} = x_{i+2,1} + x_{3-i,4}. \tag{2}$$

Since $x_{2,1} + x_{3,4} = S - (x_{1,1} + x_{4,4}) = S - a_3$, we have, for all $1 \leq i \leq 2$,

$$x_{2i-1,1} + x_{6-2i,4} = a_3 \quad \text{and} \quad x_{2i,1} + x_{5-2i,4} = S - a_3.$$

A similar argument shows that, for all $1 \leq j \leq 2$,

$$x_{1,2j-1} + x_{4,6-2j} = a_3 \quad \text{and} \quad x_{1,2j} + x_{4,5-2j} = S - a_3.$$

□

Lemma 2 *Suppose $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{4,4})\}$ is a C_4-face-magic labeling on $\mathcal{P}_{4,4}$. Let $S = 34$ be the C_4-face-magic value of X. Let $a_1 = x_{1,1} + x_{2,1}$ and $a_3 = x_{1,1} + x_{4,4}$. Then there exists a symmetry of the projective plane that induces a graph isomorphism on X such that $a_1 \leq \frac{1}{2}S = 17$ and $a_3 \geq \frac{1}{2}S = 17$.*

Proof Since $a_k + (S - a_k) = 34$ for $k = 1$ and 3, either $a_1 \leq 17$ or $S - a_1 \leq 17$, and either $a_3 \geq 17$ or $S - a_3 \geq 17$. We observe that the graph isomorphism induced by H replace a_1 with $S - a_1$ and a_3 with $S - a_3$, the graph isomorphism induced by V replaces a_3 with $S - a_3$, and the graph isomorphism induced by R_{180} replaces a_1 with $S - a_1$. See Note 2. If X does not satisfy $a_1 \leq 17$ and $a_3 \geq 17$, we may apply one of these graph isomorphisms to X to replace it with a labeling that satisfies $a_1 \leq 17$ and $a_3 \geq 17$. □

We show in the next proposition that if the value of the edge sum $a_3 = x_{1,1} + x_{4,4}$ is 17 for a C_4-face-magic projective labeling X, then X is centrally balanced.

Proposition 2 *Suppose $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{4,4})\}$ is a C_4-face-magic labeling on $\mathcal{P}_{4,4}$. Let $S = 34$ be the C_4-face-magic value of X. Let $a_3 = x_{1,1} + x_{4,4}$. If $a_3 = \frac{1}{2}S = 17$, then X is centrally balanced.*

Proof By Proposition 1, for all integers $1 \leq i \leq 2$, we have

$$x_{2i-1,1} + x_{6-2i,4} = a_3, \quad \text{and} \quad x_{2i,1} + x_{5-2i,4} = S - a_3,$$

and for all integers $1 \leq j \leq 2$, we have

$$x_{1,2j-1} + x_{4,6-2j} = a_3, \quad \text{and} \quad x_{1,2j} + x_{4,5-2j} = S - a_3.$$

Since $a_3 = 17 = \frac{1}{2}S$, we have, for all integers $1 \leq i \leq 4$, we have

$$x_{i,1} + x_{5-i,4} = \tfrac{1}{2}S,$$

and, for all integers $1 \leq j \leq 4$, we have

$$x_{1,j} + x_{4,5-j} = \tfrac{1}{2}S.$$

Adding the face-sum equations below together

$$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} = S \text{ and}$$
$$x_{3,3} + x_{3,4} + x_{4,3} + x_{4,4} = S$$

yields

$$(x_{1,1} + x_{4,4)} + (x_{1,2} + x_{4,3}) + (x_{2,1} + x_{3,4)} + (x_{2,2} + x_{3,3}) = 2S.$$

Since $x_{i,j} + x_{5-i,5-j} = \frac{1}{2}S$ for $(i, j) = (1, 1), (1, 2)$, and $(2, 1)$, we have $x_{2,2} + x_{3,3} = \frac{1}{2}S$. A similar argument shows that, for all $(i, j) \in V(\mathcal{P}_{4,4})$,

$$x_{i,j} + x_{5-i,5-j} = \tfrac{1}{2}S.$$

□

In the next proposition, we show that if a C_4-face-magic projective labeling is not centrally balanced, then it is projective labeling equivalent to a vertically pairwise balanced labeling.

Proposition 3 *Suppose $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{4,4})\}$ is a C_4-face-magic labeling on $\mathcal{P}_{4,4}$ that is not centrally balanced. Then X is projective labeling equivalent to a vertically pairwise balanced labeling.*

Proof Let $a_i = x_{i,1} + x_{i+1,1}$ for all $1 \le i \le 2$, $b_j = x_{1,j} + x_{1,j+1}$ for all $1 \le j \le 2$, $a_3 = x_{1,1} + x_{4,4}$, and $a_4 = x_{4,1} + x_{1,4}$. By Lemma 1, $S = 34$ is the C_4-face-magic value of X. By Lemma 2, we may assume $a_1 \le 17$ and $a_3 \ge 17$. Since X is not centrally balanced, by Proposition 2, $a_3 > 17$. Thus, $a_4 = S - a_3 < 17$. If $a_1 = 17$, we may apply the graph isomorphism on $\mathcal{P}_{4,4}$ induced by R_{180} to transform X into a vertically pairwise balanced labeling. So, we may assume $a_1 < 17$.

By Proposition 1, for all $1 \le i \le 2$ and $1 \le j \le 2$, we have

$$\begin{aligned} x_{2i-1,2j-1} + x_{2i,2j-1} &= a_1, & x_{2i-1,2j} + x_{2i,2j} &= S - a_1,\\ x_{2i-1,2j-1} + x_{2i-1,2j} &= b_1, \quad \text{and} & x_{2i,2j-1} + x_{2i,2j} &= S - b_1. \end{aligned}$$

Also, for all integers $1 \le i \le 2$, we have

$$\begin{aligned} x_{2i-1,2} + x_{2i-1,3} &= b_2, & x_{2i,2} + x_{2i,3} &= S - b_2,\\ x_{2i-1,1} + x_{6-2i,4} &= a_3, \quad \text{and} & x_{2i,1} + x_{5-2i,4} &= S - a_3, \end{aligned}$$

and for all integers $1 \le j \le 2$, we have

$$\begin{aligned} x_{2,2j-1} + x_{3,2j-1} &= a_2, & x_{2,2j} + x_{3,2j} &= S - a_2,\\ x_{1,2j-1} + x_{4,6-2j} &= a_3, \quad \text{and} & x_{1,2j} + x_{4,5-2j} &= S - a_3. \end{aligned}$$

We need to show that $b_1 = 17$. Then, for all integers $1 \le i \le 4$ and $1 \le j \le 2$, we have

$$x_{i,2j-1} + x_{i,2j} = \tfrac{1}{2}S = 17. \tag{3}$$

Thus, X is vertically pairwise balanced.

From the equations $x_{1,1} + x_{2,1} = a_1$, $x_{2,1} + x_{3,1} = a_2$, $x_{3,1} + x_{4,1} = a_1$, $x_{4,1} + x_{1,4} = a_4$, and $x_{1,4} + x_{1,3} = b_1$, we have

$$b_1 = x_{1,1} + x_{1,3} + a_2 + a_4 - 2a_1. \tag{4}$$

By Proposition 1, the set

$$\{\{x_{i,2j-1}, x_{i,2j}\} : 1 \le i \le 4 \text{ and } 1 \le j \le 2\} \tag{5}$$

is a partition of the set $\{1, 2, \ldots, 16\}$ into eight 2-element sets such that

– the sum of the elements of a set is a_1 for four of the 2-element sets, and
– the sum of the elements of a set is $S - a_1$ for the other four 2-element sets.

The smallest positive integer a_1 such that a_1 can be written as a sum of two positive integers such that all terms are distinct is $a_1 = 9$ since $9 = 1 + 8 = 2 + 7 = 3 + 6 = 4 + 5$. We will show that the only possible values of a_1 with $a_1 < 17$ are 9, 13, 15, and 16.

Case 1 Suppose $a_1 = 9$. There is only one way to partition the set $\{1, 2, \ldots, 16\}$ into eight 2-element sets such that

– the sum of the elements of a set is $a_1 = 9$ for four of the 2-element sets, and
– the sum of the elements of a set is $S - a_1 = 25$ for the other four 2-element sets.

This unique partition of the set $\{1, 2, \ldots, 16\}$ is given by the four sets,

$$\{1, 8\}, \{2, 7\}, \{3, 6\}, \text{ and } \{4, 5\},$$

whose sum of elements is $a_1 = 9$, and by the four sets,

$$\{9, 16\}, \{10, 15\}, \{11, 14\}, \text{ and } \{12, 13\},$$

whose sum of elements is $S - a_1 = 25$.

By (5), the elements in each 2-element set form the labels on the vertices $(i, 2j - 1)$ and $(i, 2j)$ for some $1 \le i \le 4$ and $1 \le j \le 2$. See Figs. 3 and 4. We adjoin the vertices from two pairs of edges in Fig. 3 to form two paths on four vertices such that the edge sum on each new edge is a_2.

Simultaneously, we adjoin the vertices from two pairs of edges in Fig. 4 to form two paths on four vertices such that the edge sum on each new edge is $S - a_2$.

For example, for $a_2 = 5$, we have only one way of adjoining pairs of edges in Fig. 3 to produce two paths on four vertices with edge-sum sequence $(a_1, a_2, a_1) =$

Fig. 3 Edges with edge sum $a_1 = 9$

Fig. 4 Edges with edge sum $S - a_1 = 25$

Fig. 5 Paths with edge-sum sequence $(a_1, a_2, a_1) = (9, 5, 9)$

Fig. 6 Paths with edge-sum sequence $(S - a_1, S - a_2, S - a_1) = (25, 29, 25)$

$(9, 5, 9)$. See Fig. 5. Also, there is only one way to adjoin pairs of edges in Fig. 4 to produce two paths on four vertices with edge-sum sequence $(S - a_1, S - a_2, S - a_1) = (25, 29, 25)$. See Fig. 6.

Next, we adjoin each path in Fig. 5 with a path from Fig. 6 by an edge whose edge sum is $a_4 < 17$. There are two possible ways of doing this. The first way is illustrated in Fig. 7 with $a_4 = 15$. The edge-sum sequence is $(9, 5, 9, 15, 25, 29, 25)$. The second way is illustrated in Fig. 8 with $a_4 = 16$. The edge-sum sequence is $(9, 5, 9, 16, 25, 29, 25)$. The labeled paths in Fig. 7 yield two distinct C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ up to symmetries on the projective plane. The two possibilities depend on whether one chooses either $x_{1,1} = 7$ and $x_{1,3} = 8$, or $x_{1,1} = 8$ and $x_{1,3} = 7$. These two labelings are illustrated in Table 1. We encapsulate the relevant information for these two C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ with the sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4) = (\{7, 8\}, 9, 5, 15)$ We observe that a C_4-face-magic projective labeling on $\mathcal{P}_{4,4}$ is uniquely determined by the values in the sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$.

The second way is illustrated in Fig. 8 with $a_3 = 16$. The edge-sum sequence is $(9, 5, 9, 16, 25, 29, 25)$. The labeled paths in Fig. 8 yield two distinct C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ up to symmetries on the projective plane. The two possibilities depend on whether one chooses either $x_{1,1} = 6$ and $x_{1,3} = 8$, or $x_{1,1} = 8$ and $x_{1,3} = 6$. These two labelings are illustrated in Table 2. We encapsulate the relevant information for these two C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ with the sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4) = (\{6, 8\}, 9, 5, 16)$.

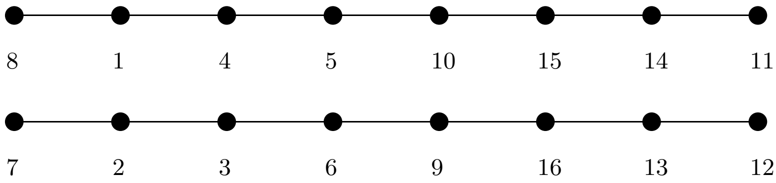


Fig. 7 Paths with edge-sum sequence (9, 5, 9, 15, 25, 29, 25)

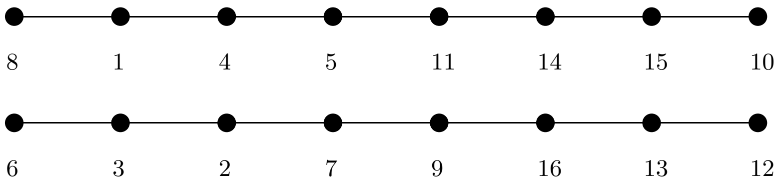


Fig. 8 Paths with edge-sum sequence (9, 5, 9, 16, 25, 29, 25)

Table 1 The C_4-face-magic labelings on $\mathcal{P}_{4,4}$ that arise from the two paths in Fig. 7. For convenience, we represent $\mathcal{P}_{4,4}$ as a 4×4 projective checkerboard. The first labeling uses $x_{1,1} = 7$ and $x_{1,3} = 8$. The second labeling uses $x_{1,1} = 8$ and $x_{1,3} = 7$

First Labeling				Second Labeling			
9	16	13	12	10	15	14	11
8	1	4	5	7	2	3	6
10	15	14	11	9	16	13	12
7	2	3	6	8	1	4	5

Table 2 The C_4-face-magic labelings on $\mathcal{P}_{4,4}$ that arise from the two paths in Fig. 8. For convenience, we represent $\mathcal{P}_{4,4}$ as a 4×4 projective checkerboard. The first labeling uses $x_{1,1} = 6$ and $x_{1,3} = 8$. The second labeling uses $x_{1,1} = 8$ and $x_{1,3} = 6$

First Labeling				Second Labeling			
9	16	13	12	11	14	15	10
8	1	4	5	6	3	2	7
11	14	15	10	9	16	13	12
6	3	2	7	8	1	4	5

Table 3 The possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ with $a_1 = 9$ and $a_4 < 17$

$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$	$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$
$(\{7, 8\}, 9, 5, 15)$	$(\{5, 7\}, 9, 10, 13)$
$(\{6, 8\}, 9, 5, 16)$	$(\{3, 7\}, 9, 10, 15)$
$(\{7, 8\}, 9, 7, 13)$	$(\{5, 6\}, 9, 11, 13)$
$(\{4, 8\}, 9, 7, 16)$	$(\{2, 6\}, 9, 11, 16)$
$(\{6, 8\}, 9, 8, 13)$	$(\{3, 4\}, 9, 13, 15)$
$(\{4, 8\}, 9, 8, 15)$	$(\{2, 4\}, 9, 13, 16)$

A similar analysis shows that there are two possible values of $a_4 < 17$ for each possible value a_2 given by 7, 8, 10, 11, and 13. The sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ for each of the possible values of $a_1 = 9$ and $a_4 < 17$ are given in Table 3.

By (4), each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ in Table 3 yields $b_1 = 17$. Thus, the C_4-face-magic labelings corresponding to each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ are vertically pairwise balanced.

Case 2 We will show that $a_1 \neq 14$. For the purposes of contradiction, assume $a_1 = 14$. We observe that the 2-element subsets of $\{1, 2, \ldots, 16\}$ such that the sum of the elements in the subset equals $a_1 = 14$ are

$$\{1, 13\}, \{2, 12\}, \{3, 11\}, \{4, 10\}, \{5, 9\}, \text{ and } \{6, 8\}, \tag{6}$$

and the 2-element subsets of $\{1, 2, \ldots, 16\}$ such that the sum of the elements in the subset equals $S - a_1 = 20$ are

$$\{4, 16\}, \{5, 15\}, \{6, 14\}, \{7, 13\}, \{8, 12\}, \text{ and } \{9, 11\} \tag{7}$$

By (5), we need to choose four 2-elements from (6) and four 2-elements from (7) that form a partition of $\{1, 2, \ldots, 16\}$. Since $\{1, 13\}$ is the only subset with the element 1 and $\{7, 13\}$ is the only subset with the element 7, we need to include both subsets in the partition of $\{1, 2, \ldots, 16\}$. However, $\{1, 13\} \cap \{7, 13\} = \{13\} \neq \emptyset$. This contradicts that there is a partition with the necessary requirements. Hence, $a_1 \neq 14$. . A similar analysis shows that $a_1 \neq 10$, $a_1 \neq 11$, and $a_1 \neq 12$.

Case 3 Suppose $a_1 = 13$. There is only one way to partition the set $\{1, 2, \ldots, 16\}$ into eight 2-element sets such that

- the sum of the elements of a set is $a_1 = 13$ for four of the 2-element sets, and
- the sum of the elements of a set is $S - a_1 = 21$ for the other four 2-element sets.

This unique partition of the set $\{1, 2, \ldots, 16\}$ is given by the four sets,

$$\{1, 12\}, \{2, 11\}, \{3, 10\}, \text{ and } \{4, 9\},$$

Fig. 9 Edges with edge sum $a_1 = 13$

Fig. 10 Edges with edge sum $S - a_1 = 21$

Table 4 The possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ with $a_1 = 13$ and $a_4 < 17$

$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$	$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$
({11, 12}, 13, 5, 15)	({9, 11}, 13, 14, 9)
({10, 12}, 13, 5, 16)	({3, 11}, 13, 14, 15)
({11, 12}, 13, 11, 9)	({9, 10}, 13, 15, 9)
({4, 12}, 13, 11, 16)	({2, 10}, 13, 15, 16)
({10, 12}, 13, 12, 9)	({3, 4}, 13, 21, 15)
({4, 12}, 13, 12, 15)	({2, 4}, 13, 21, 16)

whose sum of elements is $a_1 = 13$, and by the four sets,

$$\{5, 16\}, \{6, 15\}, \{7, 14\}, \text{ and } \{8, 13\},$$

whose sum of elements is $S - a_1 = 21$.

By (5), the elements in each 2-element set form the labels on the vertices $(i, 2j - 1)$ and $(i, 2j)$ for some $1 \le i \le 4$ and $1 \le j \le 2$. We adjoin two pairs of vertices from the edges in Fig. 9 with a new edge to form two paths on four vertices so that the edge sum on each new edge is a_2. Simultaneously, we must adjoin two pairs of vertices from the edges in Fig. 10 by a new edge to form two paths on four vertices so that the edge sum on each new edge is $S - a_2$. Then we need to adjoin an end vertex of a path with edge-sum sequence (a_1, a_2, a_1) with the end vertex of a path with edge-sum sequence $(S - a_1, S - a_2, S - a_1)$ by a new edge whose edge sum is $a_4 < 17$. The result will be two paths each having edge-sum sequence $(a_1, a_2, a_1, a_4, S - a_1, S - a_2, S - a_1)$.

An analysis similar to that in Case 1 shows that the possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ corresponding to a C_4-face-magic projective labeling X on $\mathcal{P}_{4,4}$ with $a_1 = 13$ and $a_4 < 17$ are given in Table 4.

By (4), each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ in Table 4 yields $b_1 = 17$. Thus, the C_4-face-magic labelings corresponding to each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ are vertically pairwise balanced.

dummy

Fig. 11 Edges with edge sum $a_1 = 15$

Fig. 12 Edges with edge sum $S - a_1 = 19$

Case 4 Suppose $a_1 = 15$. There is only one way to partition the set $\{1, 2, \ldots, 16\}$ into eight 2-element sets such that

– the sum of the elements of a set is $a_1 = 15$ for four of the 2-element sets, and
– the sum of the elements of a set is $S - a_1 = 19$ for the other four 2-element sets.

This unique partition of the set $\{1, 2, \ldots, 16\}$ is given by the four sets,

$$\{1, 14\}, \{2, 13\}, \{5, 10\}, \text{ and } \{6, 9\},$$

whose sum of elements is $a_1 = 15$, and by the four sets,

$$\{3, 16\}, \{4, 15\}, \{7, 12\}, \text{ and } \{8, 11\},$$

whose sum of elements is $S - a_1 = 19$.

By (5), the elements in each 2-element set form the labels on the vertices $(i, 2j - 1)$ and $(i, 2j)$ for some $1 \le i \le 4$ and $1 \le j \le 2$. We adjoin two pairs of vertices from the edges in Fig. 11 with a new edge to form two paths on four vertices so that the edge sum on each new edge is a_2.

Simultaneously, we must adjoin two pairs of vertices from the edges in Fig. 12 by a new edge to form two paths on four vertices so that the edge sum on each new edge is $S - a_2$. Then we need to adjoin an end vertex of a path with edge-sum sequence (a_1, a_2, a_1) with the end vertex of a path with edge-sum sequence $(S - a_1, S - a_2, S - a_1)$ by a new edge whose edge sum is $a_4 < 17$. The result will be two paths each having edge-sum sequence $(a_1, a_2, a_1, a_4, S - a_1, S - a_2, S - a_1)$.

An analysis similar to that in Case 1 shows that the possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ corresponding to a C_4-face-magic projective labeling X on $\mathcal{P}_{4,4}$ with $a_1 = 15$ and $a_4 < 17$ are given in Table 5.

By (4), each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ in Table 5 yields $b_1 = 17$. Thus, the C_4-face-magic labelings corresponding to each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ are vertically pairwise balanced.

Table 5 The possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ with $a_1 = 15$ and $a_4 < 17$

$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$	$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$
({13, 14}, 15, 7, 13)	({9, 13}, 15, 16, 9)
({10, 14}, 15, 7, 16)	({5, 13}, 15, 16, 13)
({13, 14}, 15, 11, 9)	({9, 10}, 15, 19, 9)
({6, 14}, 15, 11, 16)	({2, 10}, 15, 19, 16)
({10, 14}, 15, 14, 9)	({5, 6}, 15, 23, 13)
({6, 14}, 15, 14, 13)	({2, 6}, 15, 23, 16)

Case 5 Suppose $a_1 = 16$. There is only one way to partition the set $\{1, 2, \ldots, 16\}$ into eight 2-element sets such that

- the sum of the elements of a set is $a_1 = 16$ for four of the 2-element sets, and
- the sum of the elements of a set is $S - a_1 = 18$ for the other four 2-element sets.

This unique partition of the set $\{1, 2, \ldots, 16\}$ is given by the four sets,

$$\{1, 15\}, \{3, 13\}, \{5, 11\}, \text{ and } \{7, 9\},$$

whose sum of elements is $a_1 = 16$, and by the four sets,

$$\{2, 16\}, \{4, 14\}, \{6, 12\}, \text{ and } \{8, 10\},$$

whose sum of elements is $S - a_1 = 18$.

By (5), the elements in each 2-element set form the labels on the vertices $(i, 2j - 1)$ and $(i, 2j)$ for some $1 \le i \le 4$ and $1 \le j \le 2$. We adjoin two pairs of vertices from the edges in Fig. 13 with a new edge to form two paths on four vertices so that the edge sum on each new edge is a_2.

Simultaneously, we must adjoin two pairs of vertices from the edges in Fig. 14 with a new edge to form two paths on four vertices so that the edge sum on each new edge is $S - a_2$. Then we need to adjoin an end vertex of a path with edge-sum sequence (a_1, a_2, a_1) with the end vertex of a path with edge-sum sequence $(S - a_1, S - a_2, S - a_1)$ by a new edge whose edge sum is $a_4 < 17$. The result will be two paths each having edge-sum sequence $(a_1, a_2, a_1, a_4, S - a_1, S - a_2, S - a_1)$.

An analysis similar to that in Case 1 shows that the possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ corresponding to a C_4-face-magic projective labeling X on $\mathcal{P}_{4,4}$ with $a_1 = 16$ and $a_4 < 17$ are given in Table 6.

By (4), each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ in Table 6 yields $b_1 = 17$. Thus, the C_4-face-magic labelings corresponding to each sequence $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ are vertically pairwise balanced. □

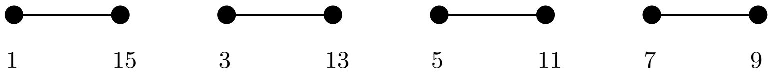


Fig. 13 Edges with edge sum $a_1 = 16$

Fig. 14 Edges with edge sum $S - a_1 = 18$

Table 6 The possible sequences $(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$ with $a_1 = 16$ and $a_4 < 17$

$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$	$(\{x_{1,1}, x_{1,3}\}, a_1, a_2, a_4)$
({13, 15}, 16, 8, 13)	({9, 13}, 16, 18, 9)
({11, 15}, 16, 8, 15)	({5, 13}, 16, 18, 13)
({13, 15}, 16, 12, 9)	({9, 11}, 16, 20, 9)
({7, 15}, 16, 12, 15)	({3, 11}, 16, 20, 15)
({11, 15}, 16, 14, 9)	({5, 7}, 16, 24, 13)
({7, 15}, 16, 14, 13)	({3, 7}, 16, 24, 15)

Definition 6 Let $W_{16} = \{\{i, j\} : i, j \in \mathbb{Z},\ i, j \geq 1, \text{ and } i + j = \frac{1}{2}S = 17\}$. Then $W_{16} = \{\{1, 16\}, \{2, 15\}, \{3, 14\}, \{4, 13\}, \{5, 12\}, \{6, 11\}, \{7, 10\}, \{8, 9\}\}$. Let d be an integer such that $1 \leq d \leq 8$. We define a symmetric relation $\sim_d$ on W_{16} by $\{y_1, y_2\} \sim_d \{z_1, z_2\}$ if and only if $\{y_1, y_2\} \neq \{z_1, z_2\}$, and either $y_1 = z_1 + d$ and $y_2 = z_2 - d$, or $y_1 = z_1 - d$ and $y_2 = z_2 + d$.

Note 3 In Definition 6 it should be observed that $\{y_1, y_2\} \sim_d \{z_1, z_2\}$ is the same as $\{y_1, y_2\} \sim_d \{z_2, z_1\}$.

Note 4 Let $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{4,4})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{4,4}$. Then $W_{16} = \{\{x_{i,j}, x_{5-i,5-j}\} : i = 1, 2, 3, 4 \text{ and } j = 1, 2\}$.

Let $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{4,4})\}$ be a vertically pairwise balanced C_4-face-magic labeling on $\mathcal{P}_{4,4}$. Then $W_{16} = \{\{x_{i,j}, x_{i,j+1}\} : i = 1, 2, 3, 4 \text{ and } j = 1, 3\}$.

Definition 7 Let d be an integer such that $1 \leq d \leq 8$. The *17-complements with difference d graph*, denoted by G_d, has vertex set $V(G_d) = W_{16}$, and there is an edge from vertex $\{y_1, y_2\}$ to vertex $\{z_1, z_2\}$ if and only if $\{y_1, y_2\} \sim_d \{z_1, z_2\}$.

Note 5 We have $\{1, 16\} \sim_1 \{2, 15\}$, $\{3, 14\} \sim_1 \{4, 13\}$, $\{5, 12\} \sim_1 \{6, 11\}$ and $\{7, 10\} \sim_1 \{8, 9\}$. Also, we observe that $\{1, 16\} \sim_2 \{3, 14\}$, $\{2, 15\} \sim_2 \{4, 13\}$, $\{5, 12\} \sim_2 \{7, 10\}$ and $\{6, 11\} \sim_2 \{8, 9\}$. Next, we have $\{1, 16\} \sim_4 \{5, 12\}$, $\{2, 15\} \sim_4 \{6, 11\}$, $\{3, 14\} \sim_4 \{7, 10\}$ and $\{4, 13\} \sim_4 \{8, 9\}$. Lastly, we observe that $\{1, 16\} \sim_8 \{8, 9\}$, $\{2, 15\} \sim_8 \{7, 10\}$, $\{3, 14\} \sim_8 \{6, 11\}$ and $\{4, 13\} \sim_8 \{5, 12\}$.

We show that there is a perfect matching in the graphs G_1, G_2, G_4, and G_8.

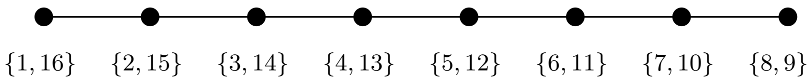


Fig. 15 Graph G_1 has a perfect matching

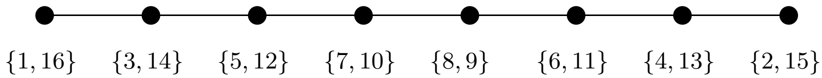


Fig. 16 Graph G_2 has a perfect matching

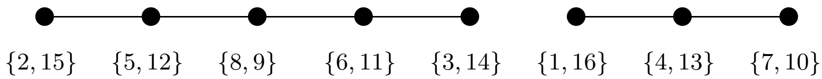


Fig. 17 Graph G_3 has no perfect matching

Lemma 3 *Let d be an integer such that $1 \le d \le 8$. Then the 17-complements with difference d graph G_d has a perfect matching if and only if $d \in \{1, 2, 4, 8\}$. Furthermore, the perfect matching in G_d is unique, and it is given by the relations in Note 5.*

Proof For all $1 \le d \le 8$, the 17-complements with difference d graph G_d is a disjoint union of paths. See Figs. 15, 16, 17, 18, 19, 20, 21 and 22. A disjoint union of paths has a perfect matching if and only if each component is a path on an even number of vertices. We observe that G_3, G_5, G_6, and G_7 each have a component that is a path on 3 vertices. So, they do not have perfect matchings.

However, G_1 and G_2 are paths on 8 vertices, G_4 is a disjoint union of two path on 4 vertices, and G_8 is the disjoint union of 4 paths on 2 vertices. Thus, they each have a unique perfect matching.

The only perfect matching in the graph G_1 is given by the relations $\{1, 16\} \sim_1 \{2, 15\}$, $\{3, 14\} \sim_1 \{4, 13\}$, $\{5, 12\} \sim_1 \{6, 11\}$ and $\{7, 10\} \sim_1 \{8, 9\}$. See Fig. 15. The only perfect matching in the graph G_2 is given by the relations $\{1, 16\} \sim_2 \{3, 14\}$, $\{2, 15\} \sim_2 \{4, 13\}$, $\{5, 12\} \sim_2 \{7, 10\}$ and $\{6, 11\} \sim_2 \{8, 9\}$. See Fig. 16. The only perfect matching in the graph G_4 is given by the relations $\{1, 16\} \sim_4 \{5, 12\}$, $\{2, 15\} \sim_4 \{6, 11\}$, $\{3, 14\} \sim_4 \{7, 10\}$ and $\{4, 13\} \sim_4 \{8, 9\}$. See Fig. 18. The only perfect matching in the graph G_8 is given by the relations $\{1, 16\} \sim_8 \{8, 9\}$, $\{2, 15\} \sim_8 \{7, 10\}$, $\{3, 14\} \sim_8 \{6, 11\}$ and $\{4, 13\} \sim_8 \{5, 12\}$. See Fig. 22. □

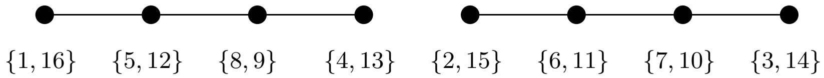


Fig. 18 Graph G_4 has a perfect matching

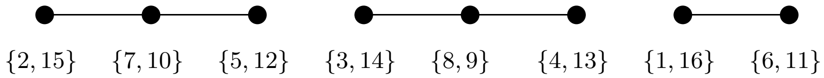


Fig. 19 Graph G_5 has no perfect matching

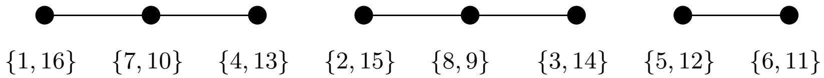


Fig. 20 Graph G_6 has no perfect matching

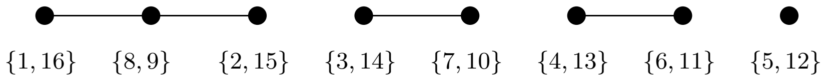


Fig. 21 Graph G_7 has no perfect matching

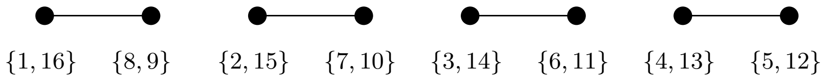


Fig. 22 Graph G_8 has a perfect matching

Theorem 2 *There are 48 distinct centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ (up to symmetries on the projective plane). Furthermore, these 48 C_4-face-magic projective labelings are given by*

$$
\begin{aligned}
x_{1,1} &= 1 + c_1 d_1 + c_2 d_2, & x_{1,3} &= x_{1,1} + d_3,\\
x_{3,1} &= x_{1,1} + d_4, & x_{3,3} &= x_{1,1} + d_3 + d_4,\\
x_{2,2} &= 1 + (1-c_1)d_1 + (1-c_2)d_2, & x_{2,4} &= x_{2,2} + d_3,\\
x_{4,2} &= x_{2,2} + d_4, & x_{4,4} &= x_{2,2} + d_3 + d_4,\\
x_{3,4} &= 1 + e\big(c_1 d_1 + (1-c_2)d_2\big) & &\\
&\quad + (1-e)\big((1-c_1)d_1 + c_2 d_2\big), & x_{3,2} &= x_{3,4} + d_3,\\
x_{1,4} &= x_{3,4} + d_4, & x_{1,2} &= x_{3,4} + d_3 + d_4,\\
x_{4,3} &= 1 + (1-e)\big(c_1 d_1 + (1-c_2)d_2\big) & &\\
&\quad + e\big((1-c_1)d_1 + c_2 d_2\big), & x_{4,1} &= x_{4,3} + d_3,\\
x_{2,3} &= x_{4,3} + d_4, \quad \textit{and} & x_{2,1} &= x_{4,3} + d_3 + d_4.
\end{aligned}
$$

where (d_1, d_2, d_3, d_4) *is a permutation on the set* $\{1, 2, 4, 8\}$ *such that* $d_1 < d_2$ *and* $d_3 < d_4$, $c_1 \in \{0, 1\}$, $c_2 \in \{0, 1\}$, *and* $e \in \{0, 1\}$. *See Fig. 23.*

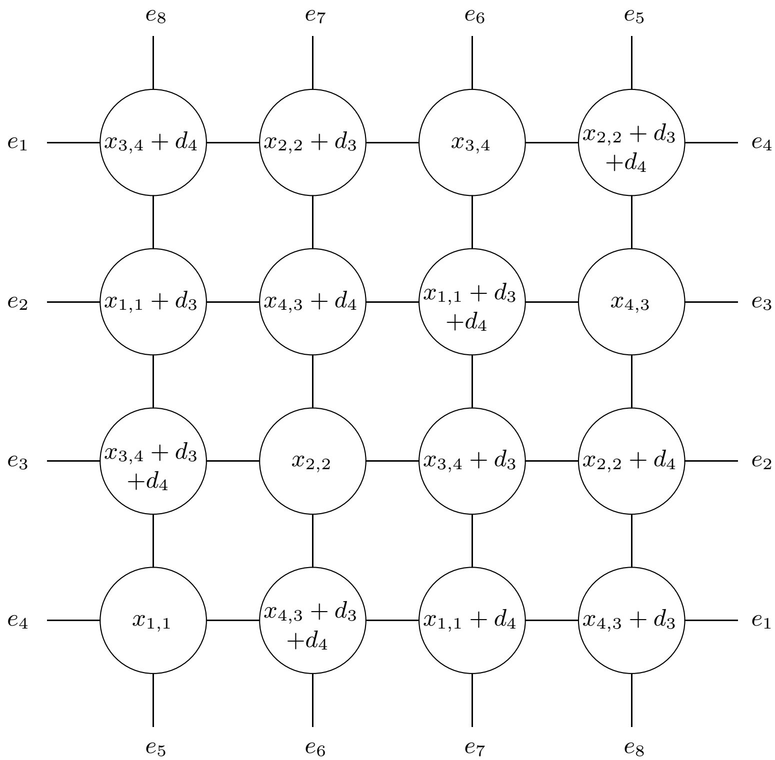


Fig. 23 Centrally balanced C_4-face-magic labelings on a 4×4 projective grid graph

Proof Let $d_3 = x_{1,3} - x_{1,1}$ and $d_4 = x_{3,1} - x_{1,1}$. By Proposition 1, $x_{i,1} + x_{i+1,1} = x_{i,3} + x_{i+1,3}$ for $i = 1, 2, 3$. Thus, for all $1 \leq i \leq 4$, we have

$$x_{i,3} = x_{i,1} + (-1)^{i+1} d_3. \tag{8}$$

By Proposition 1, $x_{i,1} + x_{i,2} = x_{i,3} + x_{i,4}$ for $i = 1, 2, 3, 4$. By (8), for all $1 \leq i \leq 4$, we have

$$x_{i,4} = x_{i,2} + (-1)^{i+2} d_3.$$

Hence, for all $1 \leq i \leq 4$ and $1 \leq j \leq 2$, we have

$$x_{i,j+2} = x_{i,j} + (-1)^{i+j} d_3. \tag{9}$$

A similar argument shows that, for all $1 \leq i \leq 2$ and $1 \leq j \leq 4$, we have

$$x_{i+2,j} = x_{i,j} + (-1)^{i+j} d_4. \tag{10}$$

By (9) and (10), we have

$$\begin{aligned} x_{1,2} &= x_{1,4} + d_3, & x_{3,4} &= x_{1,4} - d_4, \\ x_{4,3} &= x_{4,1} - d_3, & x_{2,1} &= x_{4,1} + d_4, \\ x_{4,2} &= x_{4,4} - d_3, \text{ and} & x_{2,4} &= x_{4,4} - d_4. \end{aligned}$$

So, we may apply a graph isomorphism induced by H, V, or R_{180} to ensure that $d_3 > 0$ and $d_4 > 0$.

Since X is centrally balanced, $W_{16} = \{\{x_{i,j}, x_{5-i,5-j}\} : 1 \leq i \leq 4 \text{ and } 1 \leq j \leq 2\}$. By (9) and Lemma 3, we have $d_3 \in \{1, 2, 4, 8\}$. Also, $W_{16} = \{\{x_{i,j}, x_{5-i,5-j} : 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 4\}$. By (10) and Lemma 3, we have $d_4 \in \{1, 2, 4, 8\}$. In addition, we may apply the graph isomorphism induced by D_+ to ensure that $d_3 < d_4$. Let d_1 and d_2 be the remaining two elements in $\{1, 2, 4, 8\} \setminus \{d_3, d_4\}$ such that $d_1 < d_2$. Observe that each of the labels in $\{1, 2, \ldots, 16\}$ can be written uniquely as $1 + c_1 d_1 + c_2 d_2 + c_3 d_3 + c_4 d_4$ for some $(c_1, c_2, c_3, c_4) \in \{0, 1\}^4$. By (9) and (10), we have

$$\begin{aligned} x_{i,j+2} &= x_{i,j} + d_3, \\ x_{i+2,j} &= x_{i,j} + d_4, \text{ and} \\ x_{i+2,j+2} &= x_{i,j} + d_3 + d_4, \end{aligned}$$

for $(i, j) = (1, 1)$ and $(2, 2)$, and

$$\begin{aligned} x_{i,j-2} &= x_{i,j} + d_3, \\ x_{i-2,j} &= x_{i,j} + d_4, \text{ and} \\ x_{i-2,j-2} &= x_{i,j} + d_3 + d_4, \end{aligned}$$

for $(i, j) = (3, 4)$ and $(4, 3)$. So the labels $x_{1,1}$, $x_{2,2}$, $x_{3,4}$, and $x_{4,3}$ are the values 1, $1 + d_1$, $1 + d_2$, and $1 + d_1 + d_2$. We set

$$x_{1,1} = 1 + c_1 d_1 + c_2 d_2$$

for some $c_1 \in \{0, 1\}$ and $c_2 \in \{0, 1\}$. Since $x_{3,3} = x_{1,1} + d_3 + d_4$ and $x_{2,2} + x_{3,3} = 2 + d_1 + d_2 + d_3 + d_4$, we have $x_{1,1} + x_{2,2} = 2 + d_1 + d_2$. Thus,

$$x_{2,2} = 1 + (1 - c_1)d_1 + (1 - c_2)d_2.$$

Hence, $x_{3,4}$ is one of the remaining two values $1 + c_1 d_1 + (1 - c_2)d_2$ or $1 + (1 - c_1)d_1 + c_2 d_2$ We set

$$x_{3,4} = 1 + e\big(c_1 d_1 + (1 - c_2)d_2\big) + (1 - e)\big((1 - c_1)d_1 + c_2 d_2\big)$$

for some $e \in \{0, 1\}$. Since $x_{1,1} + x_{4,4} = 2 + d_1 + d_2 + d_3 + d_4$, $x_{2,2} + x_{3,3} = 2 + d_1 + d_2 + d_3 + d_4$, and $x_{1,1} + x_{2,2} = 2 + d_1 + d_2$, we have $x_{3,3} + x_{4,4} = 2 + d_1 + d_2 + 2d_3 + 2d_4$. Since $x_{3,3} + x_{3,4} + x_{4,3} + x_{4,4} = 4 + 2d_1 + 2d_2 + 2d_3 + 2d_4$, we have $x_{3,4} + x_{4,3} = 2 + d_1 + d_2$. Thus,

$$x_{4,3} = 1 + (1 - e)\big(c_1 d_1 + (1 - c_2)d_2\big) + e\big((1 - c_1)d_1 + c_2 d_2\big).$$

Since we may (possibly) use a graph isomorphism induced by either H, V, or R_{180} to ensure that $d_3 > 0$ and $d_4 > 0$, and a graph isomorphism D_+ to ensure that $d_3 < d_4$, no two labelings among those listed in this theorem are projective labeling equivalent. $\square$

Theorem 3 *There are 96 distinct vertically pairwise balanced C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ (up to symmetries on the projective plane). Furthermore, these 96 C_4-face-magic projective labelings are given by*

$$\begin{aligned}
x_{1,1} &= 1 + c_1 d_1 + c_2 d_2, & x_{1,3} &= x_{1,1} + d_3,\\
x_{4,3} &= x_{1,1} + d_4, & x_{4,1} &= x_{1,1} + d_3 + d_4,\\
x_{4,2} &= 1 + (1 - c_1)d_1 + (1 - c_2)d_2, & x_{4,4} &= x_{4,2} + d_3,\\
x_{1,4} &= x_{4,2} + d_4, & x_{1,2} &= x_{4,2} + d_3 + d_4,\\
x_{2,2} &= 1 + e\big(c_1 d_1 + (1 - c_2)d_2\big) & &\\
&\quad + (1 - e)\big((1 - c_1)d_1 + c_2 d_2\big), & x_{2,4} &= x_{2,2} + d_3,\\
x_{3,4} &= x_{2,2} + d_4, & x_{3,2} &= x_{2,2} + d_3 + d_4,\\
x_{3,1} &= 1 + (1 - e)\big(c_1 d_1 + (1 - c_2)d_2\big) & &\\
&\quad + e\big((1 - c_1)d_1 + c_2 d_2\big), & x_{3,3} &= x_{3,1} + d_3,\\
x_{2,3} &= x_{3,1} + d_4, \quad \textit{and} & x_{2,1} &= x_{3,1} + d_3 + d_4.
\end{aligned}$$

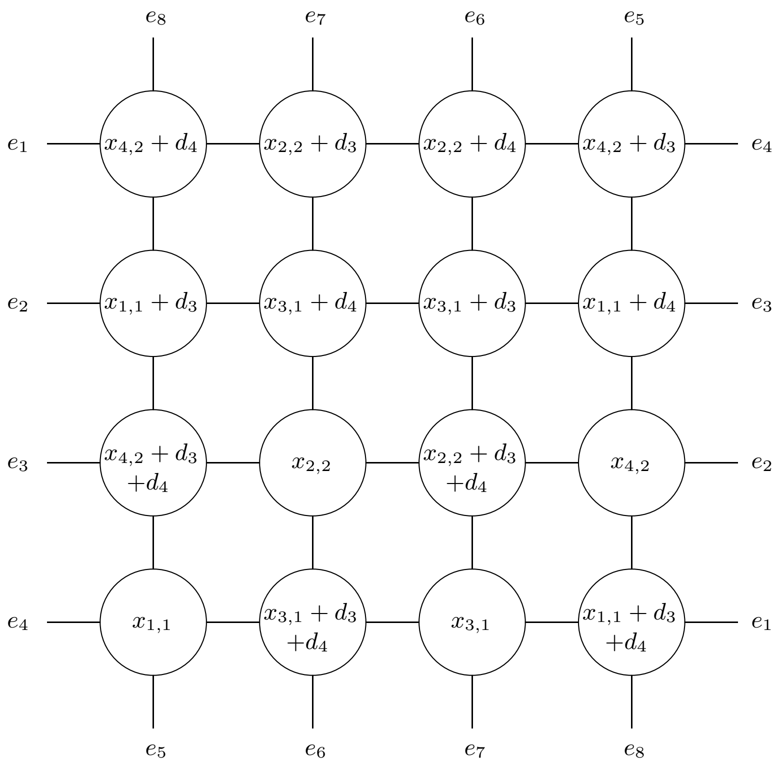


Fig. 24 Vertically pairwise balanced C_4-face-magic labelings on a 4×4 projective grid graph

where (d_1, d_2, d_3, d_4) *is a permutation on the set* $\{1, 2, 4, 8\}$ *such that* $d_1 < d_2$*,* $c_1 \in \{0, 1\}$*,* $c_2 \in \{0, 1\}$*, and* $e \in \{0, 1\}$. *See Fig. 24.*

Proof Let $d_3 = x_{1,3} - x_{1,1}$. By Proposition 1, $x_{i,1} + x_{i+1,1} = x_{i,3} + x_{i+1,3}$ for $i = 1, 2, 3$. Thus, for all $1 \le i \le 4$, we have

$$x_{i,3} = x_{i,1} + (-1)^{i+1} d_3. \tag{11}$$

By Proposition 1, $x_{i,1} + x_{i,2} = x_{i,3} + x_{i,4}$ for $i = 1, 2, 3, 4$. By (11), for all $1 \le i \le 4$, we have

$$x_{i,4} = x_{i,2} + (-1)^{i+2} d_3.$$

Hence, for all $1 \le i \le 4$ and $1 \le j \le 2$, we have

$$x_{i,j+2} = x_{i,j} + (-1)^{i+j} d_3. \tag{12}$$

Let $d_4 = x_{4,3} - x_{1,1}$. By Proposition 1, $x_{i,1} + x_{i+1,1} = x_{5-i,3} + x_{4-i,3}$ for $i = 1, 2, 3$. Thus, for all $1 \leq i \leq 4$, we have

$$x_{5-i,3} = x_{i,1} + (-1)^{i+1} d_4. \tag{13}$$

Since X is vertically pairwise balanced, $x_{i,1} + x_{i,2} = \frac{1}{2}S = x_{5-i,3} + x_{5-i,4}$ for $i = 1, 2, 3, 4$. By (13), for all $1 \leq i \leq 4$, we have

$$x_{5-i,4} = x_{i,2} + (-1)^{i+2} d_4.$$

Hence, for all $1 \leq i \leq 4$ and $1 \leq j \leq 2$, we have

$$x_{5-i,j+2} = x_{i,j} + (-1)^{i+j} d_4. \tag{14}$$

By (12) and (14), we have

$$\begin{aligned} x_{1,2} &= x_{1,4} + d_3, & x_{4,2} &= x_{1,4} - d_4, \\ x_{4,2} &= x_{4,4} - d_3, & x_{1,2} &= x_{4,4} + d_4, \\ x_{4,3} &= x_{4,1} - d_3, \text{ and} & x_{1,3} &= x_{4,1} - d_4. \end{aligned}$$

So, we may apply a graph isomorphism induced by H, V, or R_{180} to ensure that $d_3 > 0$ and $d_4 > 0$.

Since X is vertically pairwise balanced, $W_{16} = \{\{x_{i,2j-1}, x_{i,2j}\} : 1 \leq i \leq 4 \text{ and } 1 \leq j \leq 2\}$. By (12) and Lemma 3, we have $d_3 \in \{1, 2, 4, 8\}$. Also, $W_{16} = \{\{x_{i,1}, x_{i,2}\}, \{x_{5-i,3}, x_{5-i,4}\} : 1 \leq i \leq 4\}$. By (14) and Lemma 3, we have $d_4 \in \{1, 2, 4, 8\}$. Let d_1 and d_2 be the remaining two elements in $\{1, 2, 4, 8\} \setminus \{d_3, d_4\}$ such that $d_1 < d_2$. Observe that each of the labels in $\{1, 2, \ldots, 16\}$ can be written uniquely as $1 + c_1 d_1 + c_2 d_2 + c_3 d_3 + c_4 d_4$ for some $(c_1, c_2, c_3, c_4) \in \{0, 1\}^4$. By (12) and (14), we have

$$\begin{aligned} x_{i,j+2} &= x_{i,j} + d_3, \\ x_{5-i,j+2} &= x_{i,j} + d_4, \text{ and} \\ x_{5-i,j} &= x_{i,j} + d_3 + d_4, \end{aligned}$$

for $(i, j) = (1, 1), (2, 2), (3,1)$, and $(4, 2)$. So the labels $x_{1,1}$, $x_{2,2}$, $x_{3,1}$, and $x_{4,2}$ are the values 1, $1 + d_1$, $1 + d_2$, and $1 + d_1 + d_2$. We set

$$x_{1,1} = 1 + c_1 d_1 + c_2 d_2$$

for some $c_1 \in \{0, 1\}$ and $c_2 \in \{0, 1\}$. Since $x_{4,1} = x_{1,1} + d_3 + d_4$ and $x_{4,1} + x_{4,2} = 2 + d_1 + d_2 + d_3 + d_4$, we have $x_{1,1} + x_{4,2} = 2 + d_1 + d_2$. Thus,

$$x_{4,2} = 1 + (1 - c_1) d_1 + (1 - c_2) d_2.$$

Hence, $x_{2,2}$ is one of the remaining two values $1 + c_1d_1 + (1 - c_2)d_2$ or $1 + (1 - c_1)d_1 + c_2d_2$ We set

$$x_{2,2} = 1 + e\big(c_1d_1 + (1 - c_2)d_2\big) + (1 - e)\big((1 - c_1)d_1 + c_2d_2\big)$$

for some $e \in \{0, 1\}$. Since $x_{3,1} + x_{3,2} = 2 + d_1 + d_2 + d_3 + d_4$ and $x_{3,2} = x_{2,2} + d_3 + d_4$, we have $x_{2,2} + x_{3,1} = 2 + d_1 + d_2$. Thus,

$$x_{3,1} = 1 + (1 - e)\big(c_1d_1 + (1 - c_2)d_2\big) + e\big((1 - c_1)d_1 + c_2d_2\big).$$

First, a graph isomorphism induced by either H, V, or R_{180} may possibly be used to ensure that $d_3 > 0$ and $d_4 > 0$. In addition, we may possibly use the graph isomorphism induced by D_+ to transform a horizontally pairwise balanced labeling to a vertically pairwise balanced labeling. Therefore, no two labelings among those listed in this theorem are projective labeling equivalent. □

We can state the main theorem of this manuscript.

Theorem 4 *There are 144 distinct C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ (up to symmetry on the projective plane). Of these 144 labelings, 48 labelings are centrally balanced. They are listed in Theorem 2. The other 96 labelings are projective labeling equivalent to a vertically pairwise balanced labeling. They are listed in Theorem 3.*

Proof Let X be a C_4-face-magic projective labeling on $\mathcal{P}_{4,4}$. Suppose X is centrally balanced. By Theorem 2, X is projective labeling equivalent to one of the 48 labelings stated in that theorem. Suppose X is not centrally balanced. By Proposition 3, X is projective labeling equivalent to a vertically pairwise balanced labeling on $\mathcal{P}_{4,4}$. By Theorem 3, X is projective labeling equivalent to one of the 96 labelings stated in that theorem. □

References

1. Baca, M.: On magic labelings of Mobius ladders. J. Franklin Inst. **326**, 885–888 (1989). https://doi.org/10.1016/0016-0032(89)90010-0
2. Baca, M.: On magic labelings of honeycomb. Discrete Math. **105**, 305–311 (1992). https://doi.org/10.1016/0012-365X(92)90153-7
3. Baca, M.: On magic labelings of grid graphs. Ars Combin. **33**, 295–299 (1992)
4. Butt, S.I., Numan, M., Ali, S., Semaničová-Feňovčíková, A.: Face antimagic labelings of toroidal and Klein bottle grid graphs. AKCE Int. J. Graphs Comb. **17**, 109–117 (2020). https://doi.org/10.1016/j.akcej.2018.09.005
5. Chartrand, G., Lesniak, L., Zhang, P.: Graphs & Digraphs, 6th edn. CRC Press, Boca Raton (2016)
6. Curran, S.J., Low, R.M.: C_4-face-magic torus labelings on $C_4 \times C_4$. Cong. Numer. **233**, 79–94 (2019)
7. Curran, S.J., Low, R.M., and Locke, S.C.: C_4-face-magic toroidal labelings on $C_m \times C_n$. Art Discrete Appl. Math. **4**, #P1.04, 33 pp (2021). https://doi.org/10.26493/2590-9770.1368.f37

8. Curran, S.J., Low, R.M., Locke, S.C.: C_4-face-magic labelings on Klein bottle grid graphs. Manuscript
9. Curran, S.J.: C_4-face-magic labelings on odd order projective grid graphs. Manuscript
10. Curran, S.J.: Odd order C_4-face-magic projective grid graphs. Manuscript
11. Gallian, J.A.: A dynamic survey of graph labeling. Electron. J. Combin. **20**, #DS6 (2017). https://doi.org/10.37236/27
12. Hsieh, J., Lee, S-M., Liang, P-J., & Low, R.M.: On C_4 face-$(1, 0, 0)$ magic polyomino graphs and their related graphs. Manuscript
13. Kasif, A., Muhammad, H., Ali, A., Miller, M.: Magic labeling of type (a, b, c) of families of wheels. Math. Comput. Sci. **7**, 315–319 (2013)
14. Kathiresan, K., Gokulakrishnan, S.: On magic labelings of type $(1, 1, 1)$ for the special classes of plane graphs. Util. Math. **63**, 25–32 (2003)
15. Kotzig, A., Rosa, A.: Magic valuations of finite graphs. Canad. Math. Bull. **13**, 451–461 (1970). https://doi.org/10.4153/CMB-1970-084-1
16. Lih, K.-W.: On magic and consecutive labelings of plane graphs. Util. Math. **24**, 165–197 (1983)
17. Wang, T.-M.: Toroidal grids are anti-magic. In: Wang, L. (Ed.) Computing and Combinatorics. COCOON 2005. Lecture Notes in Computer Science, vol. 3595, pp. 671–679 . Springer, Berlin, Heidelberg (2005). https://doi.org/10.1007/11533719_68

On the Locating Rainbow Connection Number of the Comb Product with Complete Graphs or Trees

Mahmud Imrona, A. N. M. Salman, Saladin Uttunggadewa, and Pritta Etriana Putri

Abstract This paper investigates the locating rainbow connection number ($\text{rvcl}(G)$) of comb products of graphs. We introduce the concept of a rainbow-vertex ℓ-coloring and define the locating rainbow connection number within this framework. Our main results establish tight upper and lower bounds for $\text{rvcl}(G)$ in the context of comb products. Additionally, we determine the locating rainbow connection number for the comb product of an arbitrary graph with a complete graph or a tree.

Keywords Comb product · Locating rainbow connection number · Partition dimension · Rainbow code · Rainbow vertex coloring

1 Preliminary

All the graphs discussed in this paper are finite and simple. For $m \geq 1$, the symbol $[1, m]$ represents the set $\{1, 2, \ldots, m\}$. We write $|G|$ to represent the order of the graph G. The symbol $\text{cutv}(G)$ indicates the number of cut vertices on the graph G. The symbol $\text{diam}(G)$ represents the diameter of G. The symbol $\deg(u)$ represents

M. Imrona (✉)
Doctoral Program in Mathematics, Faculty of Mathematics and Natural Sciences, Bandung, Indonesia
e-mail: 30118003@mahasiswa.itb.ac.id

A. N. M. Salman · S. Uttunggadewa · P. E. Putri
Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl Ganesa 10, Bandung 40132, Indonesia
e-mail: msalman@itb.ac.id

S. Uttunggadewa
e-mail: s_uttunggadewa@itb.ac.id

P. E. Putri
e-mail: etrianaputri@itb.ac.id

M. Imrona
School of Computing, Telkom University, Bandung, Indonesia

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_18

the degree of $u \in V(G)$. A leaf vertex of a connected graph G is a vertex of degree one. A leaf edge of a connected graph G is an edge incident with the leaf vertex. Graph G and H are connected graphs with $|G| = m$ and $|H| = n$ for $m, n \geq 2$.

The locating chromatic coloring (Chartrand et al. [3]) induced the concept of locating rainbow coloring. The locating chromatic coloring combines the chromatic coloring concepts, and the partition dimension [4]. Meanwhile, the locating rainbow coloring concept combines the rainbow-vertex coloring [6] and the partition dimension [4]. The locating rainbow coloring was introduced by Ariestha et al. in 2020 [2]. They determined the locating rainbow connection number of a complete graph K_n, a star graph S_n, and a path graph P_n.

Let $\ell \in \mathbb{N}$ and G be a simple and finite graph. A *rainbow-vertex ℓ-coloring* of G is a function $f : V(G) \to [1, \ell]$ such that every pair of distinct vertices in G is connected by a rainbow-vertex path. A *rainbow-vertex path* is a path whose internal vertices have distinct colors. For a vertex $v \in V(G)$ and a subset $R \subseteq V(G)$, the *distance* between v and R is $\mathrm{d}(v, R) = \min\{\mathrm{d}(v, x) | x \in R\}$. For $i \in \{1, 2, \ldots, \ell\}$, let R_i be the set of vertices with color i and $\Pi = \{R_1, R_2, \ldots, R_\ell\}$ be an ordered partition of $V(G)$. The *rainbow code* of a vertex v with respect to Π is defined as the ℓ-tuple

$$\mathrm{rc}_\Pi(v) = (\mathrm{d}(v, R_1), \mathrm{d}(v, R_2), \ldots, \mathrm{d}(v, R_\ell)).$$

If $\mathrm{rc}_\Pi(v) \neq \mathrm{rc}_\Pi(w)$ for every two distinct vertices $v, w \in V(G)$, then f is called a *locating rainbow ℓ*-coloring of G. The smallest positive integer ℓ such that G has a locating rainbow ℓ-coloring is called the *locating rainbow connection number* of G, denoted by $\mathrm{rvcl}(G)$.

The comb product of graphs G and H with a contact vertex $o \in H$, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th copy of H at the vertex o with the i-th vertex of G. By the definition of comb product, we can say that $V(G \rhd_o H) = \{(x, v) | x \in V(G), v \in V(H)\}$ and $(x, v)(y, w) \in E(G \rhd_o H)$ whenever $x = y$ and $vw \in E(H)$, or $xy \in E(G)$ and $v = w = o$. In that case, G and H are called a *backbone* and a *finger*, respectively. We refer to the book [5] and the paper [7] for an overview of the comb product. The comb product can describe a communication network system with a grafting vertex [1].

Lemmas 1, 2, 3, and 4 are used to build the lower bound of the locating rainbow connection number of a connected graph G.

Lemma 1 ([2]) *Let G be a connected graph of order n, then*

$$\mathrm{rvc}(G) \leq \mathrm{rvcl}(G) \leq n.$$

Lemma 2 ([2]) *Let f be a locating rainbow coloring function of G and $w, y \in V(G)$ with $w \neq y$. If* $\mathrm{d}(w, z) = \mathrm{d}(y, z)$ *for every $z \in V(G) - \{w, y\}$, then* $f(w) \neq f(y)$.

Lemma 3 ([2]) *Let G be a connected graph containing a vertex adjacent to t leaves of G, then* $\mathrm{rvcl}(G) \geq t$.

Lemma 4 ([6]) *Let G be a connected graph, then*

1. $\mathrm{rvc}(G) \geq \mathrm{cutv}(G)$.
2. $\mathrm{rvc}(G) \geq \mathrm{diam}(G) - 1$.

2 Main Results

2.1 The Bounds of the Locating Rainbow Connection Number of Comb Product of Graphs

By Lemma 1 and 4, we have Corollary 1.

Corollary 1 *If G is a connected graph, then*

1. $\mathrm{rvcl}(G) \geq \mathrm{cutv}(G)$.
2. $\mathrm{rvcl}(G) \geq \mathrm{diam}(G) - 1$.

Definition 1 Let G be a connected graph. The number of leaf vertices adjacent to a vertex $u \in V(G)$ is denoted by $\mathrm{leaf}(u)$. Define $\mathrm{leaf}(G) = \max\{\mathrm{leaf}(u) | u \in V(G)\}$. A maximum leaf-vertex set is a leaf-vertex set with a cardinality of leaf(G).

Based on the Definition 1, Lemma 3 can be rewritten as follows.

Lemma 5 *If G is a connected graph, then*

$$\mathrm{rvcl}(G) \geq \mathrm{leaf}(G).$$

The naming of vertices in $V(G \rhd_o H)$ follows the rules.

- Each vertex in $V(G)$ is named u_i for $i \in [1, |G|]$.
- Each vertex on finger attached to vertex u_i is named u_i^k with $k \in [1, |H| - 1]$.

Definition 2 Let H_i be a sub graph of $G \rhd_o H$ and $\gamma_i \in V(H_i)$ where $\mathrm{d}(\gamma_i, u_i) = \mathrm{diam}(H_i)$ for each $i \in [1, |G|]$.

Observation 1 Let G and H be two connected graphs. The $G \rhd_o H$ has facts:

1. $|G \rhd_o H| = |G||H|$,
2. $\mathrm{cutv}(G \rhd_o H) = |G|\mathrm{cutv}(H)$ if $o \in V(H)$ is a cut-vertex,
3. $\mathrm{cutv}(G \rhd_o H) = |G|(\mathrm{cutv}(H) + 1)$ if $o \in V(H)$ is not a cut-vertex,
4. $G \rhd_o K_1 = G$,
5. $K_1 \rhd_o H = H$.

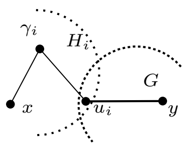


Fig. 1 Situations of vertices x and y do not have rainbow-vertex paths

Theorem 1 *Let G and H be two connected graphs. If* $|G| \geq 2$ *and* $|H| \geq 2$, *then*

$$|G| \leq \text{rvcl}(G \rhd_o H) \leq |G|(|H| - 1).$$

Proof We consider two cases.
Case 1. $|G| \leq \text{rvcl}(G \rhd_o H)$.

By Corollary 1, Observation 1 no. 2, no. 3, and $\text{cutv}(H) \geq 0$, we have

$$\text{rvcl}(G \rhd_o H) \geq |G|.$$

Case 2. $\text{rvcl}(G \rhd_o H) \leq |G|(|H| - 1)$.

Let $f : V(G \rhd_o H) \to [1, |G|(|H| - 1)]$ be a vertex-coloring function that satisfies the following condition.

- $f(u_i) = f(\gamma_i) = i$, for $i \in [1, |G|]$,
- $f(V(G \rhd_o H) - \{u_i, \gamma_i | 1 \leq i \leq |G|\}) = [|G| + 1, |G|(|H| - 1)]$.

It is easy to see that as many functions satisfy this condition as $|G|((|H| - 1)!)$. Therefore, the function f is defined, and f is bijective.

We will show that f satisfies a locating rainbow coloring function. However, first, we will show that f satisfies every two distinct vertices of $G \rhd_o H$ connected by a rainbow-vertex path. Suppose two distinct vertices x and y do not have rainbow-vertex paths, then $d(x, y) \geq 3$. Let $xw_1 \ldots w_k y$ be a shortest path connecting x and y, then $xw_1 \ldots w_k y = x \ldots \gamma_i \ldots u_i \ldots y$ for an i with $x, y \notin \{\gamma_i, u_i\}$ or $xw_1 \ldots w_k y = x \ldots u_j \ldots \gamma_j \ldots y$ for a j with $x, y \notin \{\gamma_j, u_j\}$. Without loss of generality, let's say $xw_1 \ldots w_k y = x \ldots \gamma_i \ldots u_i \ldots y$ for an i with $x, y \notin \{\gamma_i, u_i\}$. As a result, x is in $V(H_i)$. Consequently, $x \ldots \gamma_i \ldots u_i$ is the shortest path connecting x and u_i. As a result, $d(x, u_i) = d(x, \gamma_i) + d(\gamma_i, u_i) \geq 1 + \text{diam}(H_i)$. The contradiction is that $d(x, u_i) \leq \text{diam}(H_i)$. Thus, there is always a rainbow-vertex path between two distinct vertices. Figure 1 illustrates this situation.

Next, we will show that each vertex on $G \rhd_o H$ has a different rainbow code. Review the vertices that have the same color. If x and y are two different vertices with the same color, then $\{x, y\} = \{u_i, \gamma_i\}$ for an i. Without loss of generality, let us say $x = u_i$ and $y = \gamma_i$. Since $|H| \geq 2$ and G is a connected graph, there is $u_j \in V(G \rhd_o H)$ with $i \neq j$ and $u_i u_j \in E(G \rhd_o H)$. Based on f, it's easy to see $d(y, R_j) > d(x, R_j)$. Consequently, x and y have different rainbow codes. □

An example of $\text{rvcl}(G \rhd_o H) = |G|$ occurs in Theorem 2 namely $H = K_n$ and $|G| \geq |H|$. Another example of a graph $\text{rvcl}(G \rhd_o H) = |G|$ is $H = S_n$ with o is the center vertex of S_n and $|G| \geq |H|$ (see Theorem 3). An example of a graph that satisfies $\mathrm{rvcl}(G \rhd_o H) = |G|(|H| - 1)$ is $H = P_n$ with o is one of the leaf vertices of P_n (see Theorem 4).

2.2 The Locating Rainbow Connection Number of Comb Product of Any Graph with a Complete

In this subsection, the naming of vertices in $G \rhd_o K_n$ follows the rules.

- The vertices on the graph G are named u_i with $i \in [1, |G|]$.
- Finger K_n containing the graft vertex u_i is denoted K_n^i with $i \in [1, |G|]$. The vertices on the finger K_n^i other than the vertex u_i are named u_i^k with $i \in [1, |G|]$ and $k \in [1, |K_n| - 1]$.

Based on the vertex names above, the set $\{u_1, u_2, \ldots, u_m\}$ will induce the sub graph G and the set $\{u_i\} \cup \{u_i^1, u_i^2, \ldots, u_i^{n-1}\}$ will induce a sub graph K_n for every $i \in [1, |G|]$.

Observation 2 The $G \rhd_o K_n$ has facts:

1. $|G \rhd_o K_n| = |G||K_n|$,
2. $\mathrm{cutv}(G \rhd_o K_n) = |G|$,
3. $\mathrm{diam}(G \rhd_o K_n) = \mathrm{diam}(G) + 2$,
4. $G \rhd_o K_n$ has no leaf-vertex,
5. if G and K_n are non-trivial graphs, then $G \rhd_o K_n$ is not a complete graph.

Now, we will give the locating rainbow connection number of $G \rhd_o K_n$.

Theorem 2 *Let G be a connected graph. If $|G| \geq 2$ and $|K_n| \geq 2$, then*

$$\mathrm{rvcl}(G \rhd_o K_n) = \max\{|G|, |K_n|\}.$$

Proof We consider two cases.

Case 1. $|G| \geq |K_n|$.

Based on Corollary 1 and Observation 2 no. 2, we have

$$\mathrm{rvcl}(G \rhd_o K_n) \geq |G|. \tag{1}$$

Next, we will show $\mathrm{rvcl}(G \rhd_o K_n) \leq |G|$. We define a vertex-coloring $f: V(G \rhd_o K_n) \to [1, |G|]$ as follows. For $i \in [1, |G|]$ and $j \in [1, |K_n| - 1]$,

- $f(u_i) = i$, for $i \in [1, |G|]$;
- $f(u_i^j) = i - 1 + j$, for $i + j - |G| \leq 1$;
- $f(u_i^j) = (i - 1 + j) \mod |G|$, otherwise.

We will show that every two distinct vertices on $V(G \rhd_o K_n)$ are connected by a rainbow-vertex path. We only need to examine two vertices with a minimum distance of 3. Let $x, y \in V(G \rhd_o K_n)$ with $x \neq y$ and $\mathrm{d}(x, y) \geq 3$, then there is a path $xw_1 \ldots w_k y$ with $w_1, w_2, \ldots, w_k \in \{u_i\}_{i=1}^{m}$. Based on f coloring, each vertex u_i for $i \in [1, |G|]$ has a different color. Therefore, every two vertices on $V(G \rhd_o K_n)$ are always connected by a rainbow-vertex path.

Next, we will show that each vertex in $V(G \rhd_o K_n)$ has a different rainbow code. Finally, we examine two vertices in $V(G \rhd_o K_n)$ that are the same color. Let $x, y \in V(G \rhd_o K_n)$ with $x \neq y$ and $f(x) = f(y)$. There are two sub-cases.

Subcase 1.1. $x \in K_n^i$ and $y = u_i$. Because $|G| \geq |K_n| - 1$, there is an unused color in the set $f(K_n^i)$, let's say that color is z. There is only path $xy - f^{-1}(z)$ so $\mathrm{d}(x, R_z) = 1 + \mathrm{d}(y, R_z)$. Therefore, $\mathrm{rc}_\Pi(x) \neq \mathrm{rc}_\Pi(y)$.

Subcase 1.2. $x \in K_n^i$ and $y \in K_n^j$ with $i \neq j$, then x adjacent u_i means $\mathrm{d}(x, R_{f(u_i)}) = 1$ but x is not a adjacency of u_j means $\mathrm{d}(x, R_{f(u_j)}) \neq 1$. While y is a adjacency of u_j means $\mathrm{d}(y, R_{f(u_j)}) = 1$ but y is not a adjacency of u_i means $\mathrm{d}(y, R_{f(u_i)}) \neq 1$. As a result, $\mathrm{rc}_\Pi(x) \neq \mathrm{rc}_\Pi(y)$.

Therefore,

$$\mathrm{rvcl}(G \rhd_o K_n) \leq |G|. \tag{2}$$

Since (1) and (2), it applies

$$\mathrm{rvcl}(G \rhd_o K_n) = |G|.$$

Case 2. $|G| < |K_n|$.

Suppose $\mathrm{rvcl}(G \rhd_o K_n)=n-1$, then there are $x, y \in K_n^i$ with $f(x)=f(y)$. Since all the colors $[1, |K_n| - 1]$ are all used up in K_n^i, then $\forall x \in K_n^i$, $\mathrm{d}(x, R_i) = 1$, $\forall i \in [1, |K_n| - 1]$ $\forall i \in [1, |K_n| - 1]$. As a result, $\mathrm{rc}_\Pi(x) = \mathrm{rc}_\Pi(y)$. We get a contradiction. Hence,

$$\mathrm{rvcl}(G \rhd_o K_n) \geq |K_n|. \tag{3}$$

Next, we will show $\mathrm{rvcl}(G \rhd K_n) \leq |K_n|$. We define a vertex coloring $f : V(G \rhd_o K_n) \to [1, |K_n|]$ as follows. For $i \in [1, |G|]$ and $j \in [1, |K_n| - 1]$,

- $f(u_i) = i$, for $i \in [1, |G|]$;
- $f(u_i^j) = i - 1 + j$, for $i + j - |K_n| \leq 1$;
- $f(u_i^j) = (i - 1 + j) \mod |K_n|$, otherwise.

By a similar argument to Case 1, we obtain

$$\mathrm{rvcl}(G \rhd_o K_n) \leq |K_n|. \tag{4}$$

Since (3) and (4), we obtain

$$\mathrm{rvcl}(G \rhd_o K_n) = |K_n|.$$

□

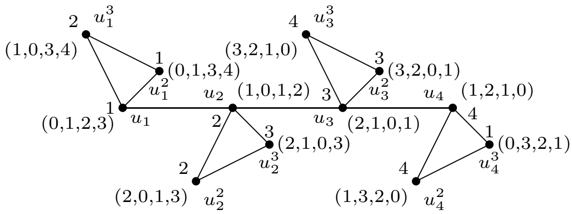


Fig. 2 rvcl($P_4 \rhd_o K_3$) = 4

We provide an example for Theorem 2. For the case $|G| \geq |K_n|$, it is illustrated by Fig. 2. Because of diam($P_4 \rhd_o K_3$) − 1=4, and cutv($P_4 \rhd_o K_3$)=4, we obtain rvcl($P_4 \rhd_o K_3$) ≥ cutv($P_4 \rhd_o K_3$) = 4. Using the coloring function in Theorem 2 in Case 1, we get rvcl($P_4 \rhd_o K_3$) = 4. Figure 3 shows for the case orde(C_m) ≤ $|K_n|$ where rvcl($C_4 \rhd K_6$) = 6.

2.3 The Locating Rainbow Connection Number of Comb Product of Any Graph with a Tree

In this subsection, the naming of vertices in $G \rhd_o T$ follows the rules. Every vertex on $V(G \rhd_o T)$ is given three indices, namely $u_i^{p,q}$. Index i represents the vertex-ith of $V(G)$ with $i \in [1, |G|]$. Index p represents the cut-vertex of the sub-graph T. While index q represents the leaf vertex adjacent to the cut-vertex p with $p \in [1, \text{cutv}(T)]$ and $q \in [0, \text{leaf}(u_i^{p,0})]$.

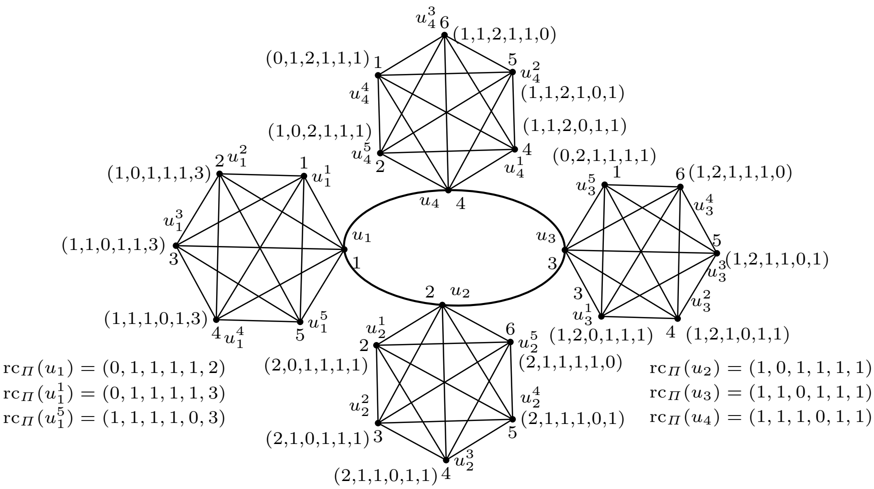
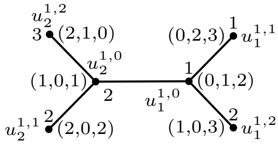


Fig. 3 rvcl($C_4 \rhd_o K_6$) = 6

Fig. 4 rvcl($P_2 \rhd_c S_2$) = 3



Observation 3 Let T be a tree with $\text{cutv}(T) \geq 1$ and c be a cut-vertex of T. The $G \rhd_c T$ has facts.

1. $\text{cutv}(G \rhd_c T) = |G|\text{cutv}(T)$,
2. $\text{leaf}(G \rhd_c T) = \text{leaf}(T)$.

Theorem 3 *Let G be a connected graph with $|G| \geq 2$, T be a tree with $\text{cutv}(T) \geq 1$, and c be a cut-vertex of T. Then*

$$\text{rvcl}(G \rhd_c T) = \begin{cases} 3, & \textit{if } G = P_2 \textit{and } T = S_2, \\ \max\{|G|\text{cutv}(T), \text{leaf}(T)\}, & \textit{otherwise.} \end{cases}$$

Proof We consider two cases.

Case 1. $G = P_2$ and $T = S_2$.

We have $\text{cutv}(P_2 \rhd_c S_2) = 2$, $\text{diam}(P_2 \rhd_c S_2) = 3$, and $\text{leaf}(P_2 \rhd_c S_2) = 2$. But by using two colors, we only get four rainbow codes whereas $\text{orde}(P_2 \rhd_c S_2) = 6$. Therefore, it should be $\text{rvcl}(P2 \rhd_c S_2) = 3$. Figure 4 shows proof of this.

Case 2. otherwise G and T.

We consider two subcases.

Subcase 2.1. $|G|\text{cutv}(T) \geq \text{leaf}(T)$. Based on Corollary 1, we get

$$\text{rvcl}(G \rhd_c T) \geq |G|\text{cutv}(T). \tag{5}$$

We will show that $\text{rvcl}(G \rhd_c T) \leq |G|\text{cutv}(T)$. For $i \in [1, |G|]$, $p \in [1, \text{cutv}(T)]$, and $q \in [0, \text{leaf}(u_i^{p,0}]$, define the vertex-coloring function $f:(G \rhd_c T) \rightarrow [1, |G|\text{cutv}(T)]$ as follows.

- $f(u_i^{p,q}) = (i-1)\text{cutv}(T) + p$, for $q \in [0, 1]$;
- $f(u_i^{p,q}) = (i-1)\text{cutv}(T) + p + q - 1$, for $p + q - 1 \leq (1 - i + |G|)\text{cutv}(T)$;
- $f(u_i^{p,q}) = ((i-1)\text{cutv}(T) + p + q - 1) \mod (|G|\text{cutv}(T))$, otherwise.

Let $x, y \in V(G \rhd_c T)$ be two distinct vertices. Let $xv_1v_2 \ldots v_n y$ be a path with end vertices x and y. Based on the coloring f, $v_1, v_2, \ldots, v_n$ are cut vertices, so these vertices have different colors. Therefore, the two vertices are connected by a rainbow-vertex path.

Let $x, y \in V(G \rhd_c T)$ be two distinct vertices. We consider two sub-cases.

Subcase 2.1.1. x and y are two leaf vertices. The vertex x is adjacent to the cut vertex of w, and y is adjacent to the cut vertex of u. Based on Lemma 2, must be $f(w) \neq f(u)$. Furthermore, $\mathrm{d}(x, R_{f(w)}) = 1$ and $\mathrm{d}(y, R_{f(w)}) \neq 1$, so $\mathrm{rc}_\Pi(x) \neq \mathrm{rc}_\Pi(y)$.

Subcase 2.1.2. x is a leaf vertex and y is a cut vertex. Because $\deg(x) = 1$ then component is worth one of $\mathrm{rc}_\Pi(x)$ only one, while $\deg(y) \geq 2$ component is worth one of $\mathrm{rc}_\Pi(y)$ at least two. Therefore, so $\mathrm{rc}_\Pi(x) \neq \mathrm{rc}_\Pi(y)$.

Thus, the function f is a locating rainbow coloring function, and we have $\mathrm{rvcl}(G \rhd_c T) \leq |G|\mathrm{cutv}(T)$. Together with (5), we have

$$\mathrm{rvcl}(G \rhd_c T) = |G|\mathrm{cutv}(T). \tag{6}$$

Subcase 2.2. $|G|\mathrm{cutv}(T) < \mathrm{leaf}(T)$

Based on Lemma 5, we have

$$\mathrm{rvcl}(G \rhd_c T) \geq \mathrm{leaf}(T). \tag{7}$$

Similarly with Subcase 2.1, for $i \in [1, |G|]$, $p \in [1, \mathrm{cutv}(T)]$, and $q \in [2, \mathrm{leaf}(u_i^{p,0})]$, define a coloring function $f : V(G \rhd_c) \rightarrow [1, \mathrm{leaf}(T)]$ as follows:

- $f(u_i^{p,q}) = (i-1)\mathrm{cutv}(T) + p$, for $q \in [0, 1]$;
- $f(u_i^{p,q}) = (i-1)\mathrm{cutv}(T) + p + q - 1$, for $p + q - 1 \leq (1-i)\mathrm{cutv}(T)\mathrm{leaf}(T)$;
- $f(u_i^{p,q}) = ((i-1)\mathrm{cutv}(T) + p + q - 1) \mod \mathrm{leaf}(T)$, otherwise.

For a similar reason as Subcase 2.1, we obtain

$$\mathrm{rvcl}(G \rhd_c T) \leq \mathrm{leaf}(T). \tag{8}$$

Since (7) and (8), we obtain

$$\mathrm{rvcl}(G \rhd_c T) = \mathrm{leaf}(T).$$

Thus, $\mathrm{rvcl}(G \rhd_c T) = \max\{|G|\mathrm{cutv}(T), \mathrm{leaf}(T)\}$. □

The grafting vertex in the vertex-comb product plays an important role. If we consider the graft vertex as the leaf vertex, we get Theorem 4. When we choose the grafting vertex as a leaf-vertex, the cut-vertex of T will increase by one, and the leaf-vertex of T will decrease by one.

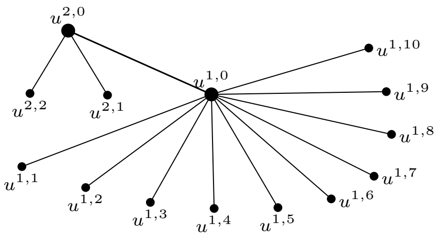
Definition 3 Let G be a connected graph with cut vertices. A maximum leaf-vertex set of graph G, denoted by $\mathrm{Leaf}(G)$, is the set of leaves adjacent to $v_{\max}$ with $\mathrm{leaf}(v_{\max}) = \max\{\mathrm{leaf}(v^{i,0}) | i \in [1, \mathrm{cutv}(G)]\}$. The single maximum leaf-vertex set of graph G, denoted by $\mathrm{Leaf}^*(G)$, is the only maximum leaf-vertex set of graph G.

Theorem 4 *Let G be a connected graph with $|G| \geq 2$, $|T| \geq 3$, and l is a leaf vertex of T. Then* $\mathrm{rvcl}(G \rhd_l T) =$
$$\begin{cases} \max\{|G|(\mathrm{cutv}(T)+1), \mathrm{leaf}(T)-1\}, & \textit{if } l \in \mathrm{Leaf}^*(T); \\ \max\{|G|(\mathrm{cutv}(T)+1), \mathrm{leaf}(T)\}, & \textit{if } l \notin \mathrm{Leaf}^*(T). \end{cases}$$

Proof Similar to the argument on Theorem 3's proof regarding the cut-vertex and the number of leaf-vertex, we derive this theorem's conclusion. □

Figure 5 is a tree T with $|T|$=14, cutv(T)=2, and leaf(T)=10. Illustration of the Theorems 3 and 4 using Fig. 5. Figure 6 illustrates Theorem 3 for leaf(T) $\geq$ $|G|$cutv(T) with the grafting vertex being the cut-vertex $u^{2,0}$, namely $\mathrm{rvcl}(C_3 \rhd_{u^{2,0}} T) = 10$.

Fig. 5 Tree T with $|T| = 14$, cutv$(T) = 2$, and leaf$(T) = 10$



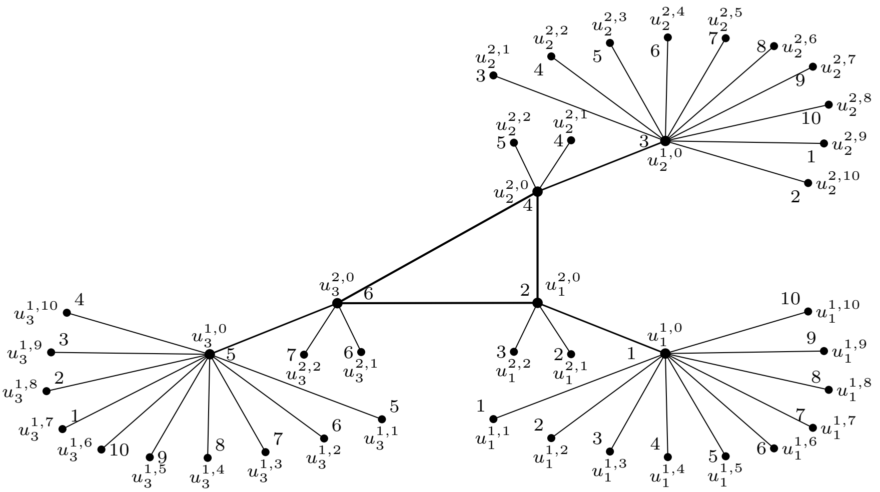


Fig. 6 $\mathrm{rvcl}(C_3 \rhd_c T) = 10$

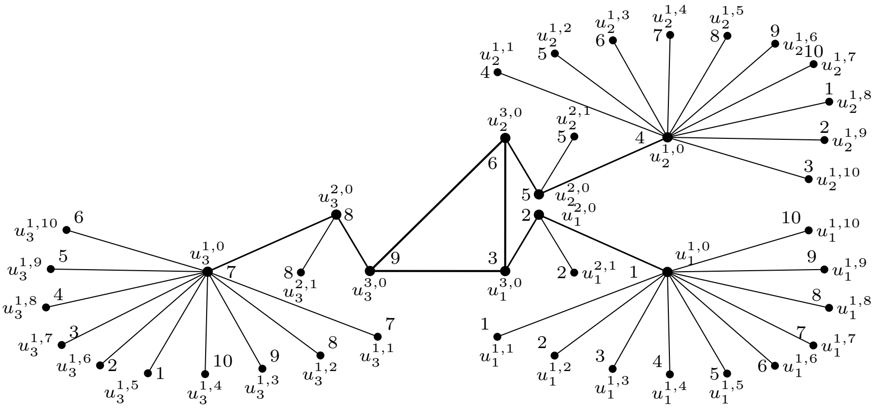


Fig. 7 rvcl($C_3 \rhd_l T$) = 10

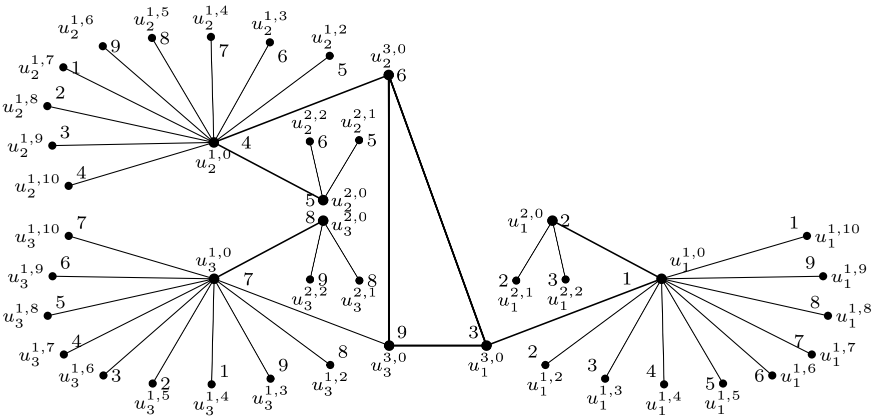


Fig. 8 rvcl($C_3 \rhd_{lmax} T$) = 9

Figure 7 illustrates Theorem 4 with $o = u^{2,2}$ for leaf(T) $\geq |G|$cutv(T), namely rvcl($C_3 \rhd_{u^{2,2}} T$) = 10. Figure 8 illustrates Theorem 4 with member grafting vertices of a single maximum leaf-vertex set $u^{1,1}$, namely rvcl($C_3 \rhd_{u^{1,1}} T$) = 9. Because there are too many vertices, the rainbow code cannot be listed.

Acknowledgements This work is partially supported by Telkom University under grant 043/SDM12/BSDM/2019.

References

1. Awanis, Z.Y., Salman, A.N.M., Saputro, S.W.: the strong 3-rainbow index of comb product of a tree and a connected graph. J. Inf. Process. **28**, 865–875 (2020)
2. Bustan, A.W., Salman, A.N.M., Putri, P.E.: On the locating rainbow connection number of a graph. J. Phys.: Conf. Ser. **1764**, 012057 (2021)

3. Chartrand, G., Erwin, D., Henning, M., Slater, P., Zhang, P.: The locating-chromatic number of a graph. Bull. Inst. Combin. Appl. **36**(01), 89–101 (2002)
4. Chartrand, G., Salehi, E., Zhang, P.: The partition dimension of a graph. Aequationes Mathematicae, Springer **59**, 45–54 (2000)
5. Hora, A., Obata, N.: Quantum Probability and Spectral Analysis of Graphs. Springer Science & Business Media (2007)
6. Krivelevich, M., Yuster, R.: The rainbow connection of a graph is (at most) reciprocal to its minimum degree. J. Graph Theory, Wiley Online Library **63**(3), 185–191 (2010)
7. Saputro, S.W., Mardiana, N., Purwasih, I.A.: The metric dimension of comb product graphs. MatematickiVesnik, **10**, 4, 248–258

Cycle-Compelling Colorings of Graphs

Anna Bachstein, Wayne Goddard, and John Xue

Abstract We define a cycle-compelling coloring of a graph as a proper coloring of the vertices such that every subgraph induced by one vertex of each color contains a cycle. The cycle-compelling number is defined to be the minimum k such that some k-coloring is cycle-compelling. We provide some general bounds and algorithmic results on this and related parameters. We also investigate the value in specific graph families including cubic graphs, disjoint union of cliques, and outerplanar graphs.

Keywords Proper vertex coloring · Cycle-compelling coloring · Coloring algorithm · Graph families

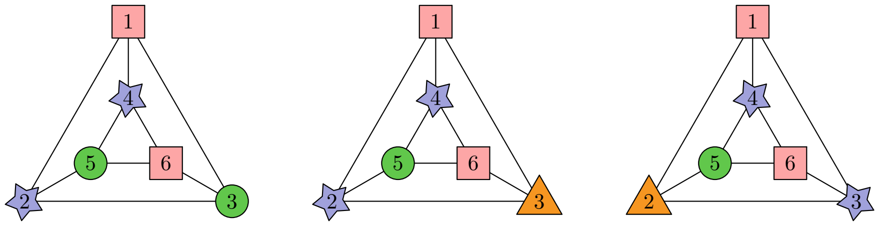
1 Introduction

Consider a graph G where the vertices are colored. A **rainbow committee** (RC) is a set consisting of one vertex of each color. We define a proper coloring of graph G as **cycle-compelling** if the subgraph induced by every RC of G contains a cycle. (Such a coloring exists if and only if G contains a cycle.) Then the **cycle-compelling number** of graph G, denoted $CCN(G)$, is the minimum number of colors needed for the existence of a proper coloring of G that is cycle-compelling. Further, the **guaranteed cycle-compelling number**, denoted $\mathcal{G}CCN(G)$, is the minimum number of colors such that every proper coloring with that many colors is cycle-compelling.

For example, let P be the prism of K_3. Below are three colorings of P: the first two are not cycle-compelling and the rightmost one is cycle-compelling. The 3-coloring is unique up to isomorphism. And it is easy to see that every 5 vertices contain a cycle. Thus, $CCN(P) = 4$ and $\mathcal{G}CCN(P) = 5$.

A. Bachstein · W. Goddard (✉) · J. Xue
School of Mathematical and Statistical Sciences, Clemson University, Clemson, USA
e-mail: goddard@clemson.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_19



The cycle-compelling number is a special case of the compelling number from [1]. Let $\mathcal{P}$ denote some property of subsets of vertices. Then a proper coloring of G **compels** $\mathcal{P}$ if every RC of G has property $\mathcal{P}$. Such a coloring might not exist, but if it does, then the $\mathcal{P}$ **-compelling chromatic number** of the graph G is the minimum number of colors in a proper coloring that compels property $\mathcal{P}$. For example we showed that the dominator chromatic number and total chromatic number correspond to $\mathcal{P}$-compelling chromatic numbers for suitable choice of property $\mathcal{P}$. (For definitions of these parameters, see the recent survey [2].)

In Sect. 2 we provide some general bounds and facts, including characterizing the graphs where the cycle-compelling number is 3. Then in Sect. 3 we look at the parameters in graph families including joins, disjoint union of cliques, cubic graphs, and maximal outerplanar graphs.

2 Some General Bounds and Facts

It is trivial that CCN of the complete graph K_n is n, as it is for the cycle C_n. Likewise $\mathcal{G}CCN$ is n for these two graphs. Note too that if there exists a cycle-compelling coloring with k colors and k is less than the order, then there also exists one with $k+1$ colors: simply choose any vertex that does not have a unique color and recolor it with a new hue. However, it is unclear whether all k-colorings being cycle-compelling implies all $(k+1)$-colorings are too.

As another example consider the complete bipartite graph $K_{r,s}$ with $r \geq s \geq 2$. The CCN is 4: For a suitable coloring, color one partite set with red and green, and color the other partite set with blue and orange. But $\mathcal{G}CCN = r + 2$. For a coloring with $r + 1$ colors that is not cycle-compelling, give the same color to all vertices in the smaller partite set and give a unique color to each vertex in the bigger partite set: every RC induces a star.

Recall that the **girth** $g(G)$ is the length of the shortest cycle of the graph G. The following result is a special case of Proposition 1 from [1].

Proposition 1 *Let G be a graph.*
(a) $CCN(G) \geq \max\{\, g(G), \chi(G) \,\}$.
(b) $CCN(G) \leq g(G) + \chi(G)$.

(The chromatic number is a lower bound since one needs a proper coloring; the girth is a lower bound since one needs the RC to contain a cycle. The chromatic number plus the girth is an upper bound since one can pick a shortest cycle and give each vertex therein a unique color and then color the remainder of the graph properly with new colors.)

These bounds can be achieved. For example, both the cycle and the complete graph achieve the lower bound; many disjoint union of cliques achieve the upper bound (see Theorem 11).

It follows that the smallest cycle-compelling number is 3. We determine next the graphs that achieve this:

Theorem 2 *For graph G, it holds that $CCN(G) = 3$ if and only if G is a complete tripartite graph.*

Proof Assume that $CCN(G) = 3$. Form an RC by picking one vertex of each color. The three vertices must form a cycle. It follows that vertices of different colors are always adjacent. That is, G is a complete tripartite graph.

On the other hand, there is no simple characterization of the case that $CCN(G) = 5$. Consider the join of K_2 with a graph H. In such a graph every proper coloring is cycle-compelling. So, since testing H for 3-colorability is NP-hard, it follows that in general testing whether $CCN(G) = 5$ is NP-hard. It is unclear what happens for $CCN(G)$ equal to 4.

We now determine when the cycle-compelling number equals the girth:

Theorem 3 *If graph G has girth 4, then $CCN(G) = 4$ if and only if G is a complete bipartite graph.*

Proof We noted earlier that a cyclic complete bipartite graph has $CCN = 4$.

Assume $CCN(G) = 4$. Pick a vertex of each color. By the girth constraint, this RC forms a chordless 4-cycle, say $abcda$. Say a has color A, b has color B, c has color C and d has color D. Let a' be any other vertex; say it has color A. Then consider the RC $\{a', b, c, d\}$. This subgraph must contain a cycle, and by the girth condition it must be a 4-cycle. Vertex b is not adjacent to d, so b must be adjacent to a'. Similarly vertex d must be adjacent to a'. And by the girth, vertex a' is not adjacent to c.

Now consider any other vertex b' of color B. Start with the RC $\{a, b', c, d\}$. By the above reasoning, b' is adjacent to a and c but not d. So consider the RC $\{a', b', c, d\}$. By the same reasoning, a' is adjacent to b' and to d. That is, every vertex of color A is adjacent to every vertex of color B. Similarly this holds for every consecutive pair of colors on the original cycle. Further, a vertex of color B is not adjacent to any vertex of color D, while a vertex of color A is not adjacent to any vertex of color C. It follows that the graph is complete bipartite with bipartition $[A \cup C, B \cup D]$.

Theorem 4 *If graph G has girth at least 5, then $CCN(G) = g(G)$ if and only if G is a cycle.*

Proof We noted earlier that the cycle has CCN equal to its order.

Assume $CCN(G) = g(G)$. Pick a vertex of each color. This RC forms a g-cycle, say $abcd \dots za$, necessarily chordless. Suppose there exists another vertex; say a' has the same color as a. Then consider the g-tuple $\{a', b, c, d, \dots, z\}$. This subgraph must contain a cycle. By the girth condition, this must in fact form a chordless g-cycle. Vertex b is not adjacent to anything else; so it must be adjacent to a'. Similarly, vertex z must be adjacent to a'. Thus edges ab, az, $a'b$, and $a'z$ exist: but these edges form a 4-cycle, which is a contradiction. That is, vertex a' does not exist.

Turning to the guaranteed cycle-compelling number, let $\mathrm{LIF}(G)$ denote the largest order of an acyclic subgraph, also known as the **largest induced forest**. (See for example [4].) If one uses more colors than this parameter, an RC has more vertices than this, and is therefore cyclic. Thus we obtain the following bound:

Proposition 5 *For all graphs G it holds that* $\mathcal{G}CCN(G) \le \max\{\chi(G), \mathrm{LIF}(G) + 1\}$.

A useful case of equality is the following result (using $\Delta(G)$ for maximum degree):

Observation 6 If $\Delta(G) < \mathrm{LIF}(G)$, then $\mathcal{G}CCN(G) = \mathrm{LIF}(G) + 1$.

Proof We know from Proposition 5 (and Brooks' Theorem) that $\mathcal{G}CCN(G) \le \mathrm{LIF}(G) + 1$. Let J be a largest induced forest. Give each vertex of J a different color. Then extend to a coloring of G using these colors (which can be done greedily because every vertex has fewer neighbors than available colors). The resultant coloring uses $\mathrm{LIF}(G)$ colors and the set J is an RC that is acyclic.

In contrast, an example of a graph where the bound of Proposition 5 is not attained is a complete tripartite graph.

One can also consider graphs with large values of the chromatic number. For example:

Theorem 7 *If graph G of order n has chromatic number at least $(n + 3)/2$, then $CCN(G) = \mathcal{G}CCN(G) = \chi(G)$.*

Proof Consider any proper coloring using the minimum number of colors. By the pigeonhole principle there must be at least three colors that are used only once. Since each of these colors is necessary, these vertices must be mutually adjacent. That is, every RC contains a triangle. QED

3 Graph Families and Operations

3.1 Join

We next consider the join of two graphs. As in [1], we define a proper coloring of a graph as **edge-compelling** if every RC contains an edge, and the **edge-compelling number** as the minimum number of colors in an edge-compelling coloring. We start with the join of a graph with a single vertex:

Theorem 8 *For any graph H, it holds that $CCN(H \vee K_1)$ is 1 more than the edge-compelling number of H.*

Proof Every proper coloring of $H \vee K_1$ gives the universal vertex v a unique color and H a proper coloring. It follows that every RC J of $H \vee K_1$ contains v, and so J induces a cyclic subgraph if and only if $J - \{v\}$ is not independent.

A similar result holds for the guaranteed cycle-compelling number. On the other hand, consider the case that graphs G and H both contain edges. Then every proper coloring of the join is cycle-compelling, since every RC contains (at least) two vertices from G and (at least) two vertices from H and these four vertices contain a 4-cycle. Thus:

Observation 9 For any graphs G and H that contain at least one edge, it holds that $CCN(G \vee H) = \mathcal{G}CCN(G \vee H) = \chi(G) + \chi(H)$.

3.2 Disjoint Union of Cliques

Turning to disjoint union, one can write bounds for CCN of the union in terms of the parameters of the components, but these are mostly no better than the bounds given in Proposition 1. One improvement that does arise is that $CCN(G \cup H) \leq CCN(G) + CCN(H) - 1$, since an RC is still compelling if two components share one color.

For specific graphs we consider the case where the components are complete. Assume graph G is the disjoint union of cliques with maximum clique order m. By Proposition 1, it holds that $m \leq CCN(G) \leq m + 3$. Here are two instances where the lower bound is attained:

Observation 10 Let G be the disjoint union of s cliques with the largest clique of order m and the second-largest clique of order ℓ.
(a) If $m \geq \ell + 3$, then $CCN(G) = m$.
(b) If $m > 2s$, then $CCN(G) = \mathcal{G}CCN(G) = m$.

Proof (a) Consider a proper $(\ell + 3)$-coloring such that all but the largest clique are colored using the same palette of ℓ colors. Then there are (at least) three colors that appear only on the biggest clique, and so the corresponding vertices form a triangle in every RC.

(b) Consider any proper coloring. By the pigeonhole principle, every RC must contain at least 3 vertices from some clique.

First we resolve the case of two cliques:

Theorem 11 *Consider the disjoint union $K_\ell \cup K_m$ with $3 \leq \ell \leq m$. Then $CCN = \mathcal{G}CCN = \max\{m, 5\}$.*

Proof If $m \geq 5$, then the result follows from part b of Observation 10. So assume $m \leq 4$. Every 5-coloring works by the pigeonhole principle.

Consider a coloring of the graph with four colors. Then we claim there exists an RC J consisting of two vertices from each clique and therefore acyclic. Begin constructing J by taking up to two vertices with unique colors from each clique; then try to complete to the claimed RC. Suppose one gets stuck. Since each clique has at least three colors, we must have two vertices from one clique and one vertex from the other, and the fourth color exists only in the first clique. But then by the way the RC was started, the first clique contains three vertices of unique colors, which does not leave enough colors for the second clique to be colored, a contradiction.

Second we resolve the case where the cliques have the same order: To show that a coloring is not compelling, we use the well-known weighted version of Hall's theorem (see for example [3]).

Theorem 12 *Consider a collection $\mathcal{X} = (X_1, \ldots, X_s)$ of sets where each X_i has a value b_i. There exist disjoint sets $B_i \subseteq X_i$ with $|B_i| = b_i$ if and only if for all subsets D of $\{1, \ldots, s\}$ it holds that $\left|\bigcup_{i \in D} X_i\right| \geq \sum_{i \in D} b_i$.*

In our application, we write the values b_i as a vector $\mathbf{b}$.

Theorem 13 *Consider the disjoint union sK_m for $m \geq 3$. The cycle compelling number is:*

$$\begin{array}{ll} m & \text{if } s < \lceil m/2 \rceil, \\ 2s+1 & \text{if } s = \lceil m/2 \rceil, \text{ and} \\ m+3 & \text{if } s > \lceil m/2 \rceil. \end{array}$$

Proof For the upper bound: $m + 3$ is always an upper bound, as was observed above. Similarly, any coloring using at least $2s + 1$ colors is automatically compelling by the pigeonhole principle.

For the lower bound: m is always a lower bound on the chromatic number. Assume $s > \lceil m/2 \rceil$ and consider a coloring with $m + 2$ colors. We show that there exists an acyclic RC. Create family $\mathcal{X}$ of size $\lceil m/2 \rceil + 1$ as follows: use as sets (the colors in) the first $\lceil m/2 \rceil$ cliques and then combine the colors in the remaining cliques together as one set. Assume first that m is even, and let $\mathbf{b} = (2, 2, \ldots, 2)$. Consider a sub-collection $\mathcal{T}$ of $\mathcal{X}$ containing t sets. Then $\mathcal{T}$ has at least m colors, which is at least $2t$ except if $t > m/2$. That is, we may assume that $t = m/2 + 1$; but that means $\mathcal{T} = \mathcal{X}$ and so all $m + 2$ colors are present. Hence by Theorem 12 there is an RC with two vertices from each clique.

So assume that m is odd, and let $\mathbf{b} = (2, 2, \ldots, 2, 1)$. Again the condition on $\mathcal{T}$ is immediate except for $t > m/2$. If $t = (m + 3)/2$, then $\mathcal{T}$ is all of $\mathcal{X}$ and has all $m + 2 = 2t - 1$ colors. So assume $t = (m + 1)/2$. Then the condition on collection $\mathcal{T}$ is satisfied unless $\mathcal{T}$ is the first $(m + 1)/2$ cliques and all of these cliques have the same $m = 2t - 1$ colors. But since more than m colors are used in total, we can go back and index $\mathcal{X}$ such that the first and second clique do not have exactly the same colors. Hence by Theorem 12 there is an acyclic RC.

Assume now that $s = \lceil m/2 \rceil$. Then consider a coloring with $2s$ colors. One can in a very similar way apply the above theorem with $(2, 2, \ldots, 2)$. We omit the details. QED

By the above result and Observation 6 one can deduce the following:

Theorem 14 *For $m \geq 3$ it holds that $\mathcal{G}CCN(sK_m) = m$ if $s < \lceil m/2 \rceil$ and $\mathcal{G}CCN$ $(sK_m) = 2s + 1$ otherwise.*

3.3 Cubic Graphs

By Theorem 2, an r-regular graph G with $CCN(G) = 3$ is a complete tripartite graph with equal number of vertices in each partite set. So it follows that r is even. That is, there is no cubic graph G with $CCN(G) = 3$.

There are a few cubic graphs with CCN equal to 4. Namely, K_4, $K_{3,3}$ and P, the prism-of-K_3, discussed earlier. These are all the examples, as we now show.

Theorem 15 *There is no cubic graph G of order 8 or more with $CCN(G) = 4$.*

Proof By Theorem 3, the only cubic example with girth 4 is $K_{3,3}$. So we may assume G has girth 3 and order at least 8.

Consider a triangle T of G with vertices a, b, c. Let d be any vertex of the fourth color. Suppose vertex d is not adjacent to T. Consider any neighbor e of d. One can build an RC with d and e and two vertices of T. The only way this RC can contain a cycle is that e is adjacent to the two vertices taken from T. But this has to be true for each choice of e, and thus there are six edges going into T, but only three coming out, a contradiction. Thus vertex d is adjacent to T.

Consider some vertex f that is not adjacent to T. (Such a vertex exists since G has order at least 8.) We may assume that f has the same color as a. We can build an RC with vertices d, f, b, and c. The only way this RC can contain a cycle is that d is adjacent to b and c. There can only be one vertex with two neighbors in T; thus d is the unique vertex of the fourth color.

By the same argument applied to the triangle $\{b, c, d\}$, it follows that a is the unique vertex of its color. This contradicts the fact that a has the same color as f. That is, there is no such G.

In contrast, there are infinitely many cubic graphs G with $CCN(G) = 5$. For example, take a bipartite cubic graph and perform a Δ–Y operation at one vertex v: that is, replace vertex v with a triangle $v_1v_2v_3$ with the same neighbors. Then a cycle-compelling coloring is obtained by using three colors on the triangle and two colors on the remainder (which is bipartite).

It is easy to see that every cubic graph G of order at least 6 has an induced forest of order at least 4. Thus by Observation 6 for these it holds that $\mathcal{G}CCN(G) = \text{LIF}(G) + 1$.

3.4 Maximal Outerplanar Graphs

Recall that a **maximal outerplanar graph** is obtained from a cycle drawn in the plane by adding chords inside to make a triangulation. For a maximal outerplanar graph G, it follows from Proposition 1 that $CCN(G) \leq 6$, since the girth and chromatic number are both 3. According to computer, the snake P_9^2 has $CCN = 6$. From this it can be shown that all longer snakes have $CCN = 6$ also. On the other hand, by Theorem 8 and the value of the edge-compelling number of the path given in [1], the fan $F_n = K_1 \vee P_{n-1}$ has $CCN = 5$ for $n \geq 8$.

Finally we consider the guaranteed cycle-compelling number. There are many cases where the parameter equals the upper bound from Proposition 5. If the degree is small, such as in the snake P_n^2, this follows from Observation 6. But even if the degree is large, such as the fan F_n, the result can be shown. We omit the details.

4 Future Thoughts

Open questions already mentioned include resolving the value of CCN for all disjoint union of cliques. One intriguing question is whether one can duplicate a vertex and have the cycle-compelling number go down. Other questions include considering the parameters in graph families such as products. In another direction, one could also consider colorings where there exists an RC that contains a cycle.

Acknowledgements Thanks to the audience members at the talk for their questions that, in particular, suggested considering the guaranteed version.

References

1. Bachstein, A., Goddard, W., Henning, M.A., Xue, J.: Compelling colorings: A generalization of the dominator chromatic number, submitted. Appl. Math. Comput. **428**, 127193 (2022)
2. Henning, M.A.: Dominator and total dominator colorings in graphs. In: Haynes, T.W., Hedetniemi, S.T., Henning, M.A. (eds) Structures of Domination in Graphs, pp. 101–134. Springer (2021). https://doi.org/10.1007/978-3-030-58892-2
3. Mirsky, L., Perfect, H.: Systems of representatives. J. Math. Anal. Appl. **15**, 520–568 (1966)
4. Staton, W.: Induced forests in cubic graphs. Discrete Math. **49**, 175–178 (1984)

Harmonious Colorings of Graphs

Alexis Byers, Alyssa Adams, Erica Bajo Calderon, Olivia Bindas, Madeline Cope, Andrew Summers, and Rabin Thapa

Abstract A harmonious labeling of a graph G of order n and size m is an injective function $f : V(G) \to \mathbb{Z}_m$ that induces an injective function $f' : E(G) \to \mathbb{Z}_m$ defined by $f'(uv) = f(u) + f(v) \pmod{m}$. When G is a tree, then we allow f to repeat one vertex label. A proper vertex coloring $c : V(G) \to \mathbb{Z}_k$ is called a harmonious k-coloring if the induced edge coloring $c' : E(G) \to \mathbb{Z}_k$ defined by $c'(uv) = c(u) + c(v) \pmod{k}$ is also proper. The minimum positive integer k for which G has a harmonious k-coloring is called the harmonious chromatic number of G, $\chi_h(G)$. The harmonious chromatic number of all trees, cycles, grids, and graphs of diameter at most two are determined, and connections are made to existing graph labelings and colorings.

Keywords Graph theory · Graph labeling · Harmonious labeling · Vertex coloring · Edge coloring · Harmonious chromatic number

1 Introduction

In 1967, Alexander Rosa presented his paper on graph valuations, or graph labelings as we now call them today, to a symposium in Rome [6]. In that presentation, he described what has become the most popular graph labeling and has inspired hundreds of related concepts. Rosa called this labeling an “ -valuation,” and we now call it a “graceful labeling,” thanks to Solomon Golomb [4].

A *graceful labeling* of a graph G of order n and size m is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$ that induces an injective function $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$. A graph that admits a graceful labeling is a graceful graph.

A. Byers (✉) · A. Adams · E. B. Calderon · O. Bindas · M. Cope · A. Summers · R. Thapa
Youngstown State University, Youngstown, OH 44555, USA
e-mail: abyers@ysu.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_20

Graceful labelings can alternatively be defined in terms of colorings. A *rainbow vertex coloring* of a graph G of size m is an assignment f of distinct colors to the vertices of G. If the colors are chosen from the set $\{0, 1, \ldots, m\}$, resulting in each edge uv of G being colored $f'(uv) = |f(u) - f(v)|$ such that the colors assigned to the edges of G are also distinct, then this rainbow vertex coloring results in a *rainbow edge coloring*, $f' : E(G) \to \{1, 2, \ldots, m\}$. Such a rainbow vertex coloring is therefore equivalent to a graceful labeling of G.

The most popular type of graph colorings are *proper vertex colorings* and *proper edge colorings*, of which rainbow colorings are a special case. In a proper vertex or edge coloring, we require only that adjacent vertices or adjacent edges are assigned different colors. The minimum number of colors needed to produce a proper vertex coloring of G is the *chromatic number*, denoted $\chi(G)$, and the minimum number of colors needed to produce a proper edge coloring of G is the *chromatic index*, denoted $\chi'(G)$.

Combining the two popular areas of graceful labelings and proper colorings, the concept of *graceful colorings* was introduced in [1]. A *graceful k-coloring* of a nonempty graph G is a proper vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$, with $k \geq 2$, that induces a proper edge coloring $c' : E(G) \to \{1, 2, \ldots, k-1\}$ defined by $c'(uv) = |c(u) - c(v)|$. A vertex coloring c of a graph G is a *graceful coloring* if c is a graceful k-coloring for some $k \in \mathbb{N}$. The minimum k for which G has a graceful k-coloring is called the *graceful chromatic number* of G, denoted by $\chi_g(G)$. If G is a nonempty graph of order n, then it has been shown that $\chi_g(G)$ exists and $\max\{\chi(G), \chi'(G)\} \leq \chi_g(G)$. This concept was studied in detail in [1, 2].

Introduced by Ronald Graham and Neil Sloane in [5], the graph labeling called a *harmonious labeling* induces the edge labelings by taking the sum of the labels of the incident vertices, modulo m, rather than the positive difference as with graceful labelings. Thus, for a connected graph G of size m, a *harmonious labeling* of G can be defined as an injective function $f : V(G) \to \{0, 1, \ldots, m-1\}$ that induces an injective function $f' : E(G) \to \{1, 2, \ldots, m-1\}$ defined by $f'(uv) = f(u) + f(v) \pmod{m}$. Such a vertex labeling is not possible if G is a tree; in this case some element of $\mathbb{Z}_m$ is assigned to two vertices of G while others are used exactly once. A graph that admits a harmonious labeling is called a *harmonious graph*.

Both the concepts of harmonious labelings and graceful labelings have been studied extensively, and there exists a large survey on these labelings and other variations kept up by Joseph Gallian on the Electronic Journal of Combinatorics [3].

2 The Harmonious Chromatic Number

Just as was done for graceful labelings, we can view harmonious labelings through the lens of graph colorings. A harmonious labeling on a connected graph G that is not a tree can be viewed as a rainbow vertex coloring that induces a rainbow edge coloring where the colors come from the set $\mathbb{Z}_m$ of integers modulo m and the

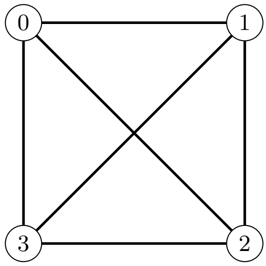


Fig. 1 Harmonious 4-coloring of K_4

resulting edge coloring assigns an edge uv of G with the color corresponding to $f(u) + f(v) \pmod{m}$.

We can naturally extend the idea of proper vertex colorings and proper edge colorings to the concept of harmonious labelings as it was to graceful labelings in [1]. We define a *harmonious k-coloring* of a nonempty graph G to be a proper vertex coloring $c : V(G) \to \mathbb{Z}_k$ that induces a proper edge coloring $c' : E(G) \to \mathbb{Z}_k$ defined by $c'(uv) = c(u) + c(v) \pmod{k}$. The minimum positive integer k for which G has a harmonious k-coloring is called the *harmonious chromatic number* of G, denoted $\chi_h(G)$.

A harmonious 4-coloring of K_4 is given in (Fig. 1). In fact, $\chi_h(K_4) = 4$.

As Graham and Sloane proved in [5], almost all graphs are not harmonious. However, our first observation is that every graph does have a harmonious k-coloring for some positive integer k. In particular, every graph of order n has a harmonious n-coloring since the graph can be given a rainbow coloring from the set of colors 0 through $n - 1$, which will always induce a proper edge coloring.

Observation 1 If G is a nonempty graph G of order n, then $\chi_h(G)$ exists, and

$$\chi_h(G) \leq n.$$

If we have a harmonious k-coloring of a graph G, then its restriction to a subgraph of G will also be a harmonious k-coloring. Therefore, we have the following observation.

Observation 2 For a subgraph H of G, we have $\chi_h(H) \leq \chi_h(G)$.

If a graph has a harmonious k-coloring, then it is necessary that it has a proper vertex k-coloring and a proper edge k-coloring. Thus, the chromatic number and chromatic index provide lower bounds for the harmonious chromatic number.

Observation 3 For any nontrivial graph G,

$$\chi_h(G) \geq \max\{\chi(G), \chi'(G)\}.$$

As seen in Observation 1, all graphs of order n have harmonious chromatic number at most n. On the other hand, all graphs of diameter two or less must have harmonious chromatic number *at least* n. Indeed, consider any two vertices of a graph G that are distance two apart from each other, say x and y, with (x, v, y) being a shortest $x - y$ path. If $c(x) = c(y)$ for some harmonious k-coloring c of G, then this implies the following:

$$c'(xv) = c(x) + c(v) \pmod k = c(y) + c(v) \pmod k = c'(yv)$$

However, this contradicts the fact that c' is a proper edge coloring. Therefore, in a graph of diameter two, a harmonious coloring must assign distinct colors to each vertex. This establishes the following observation.

Observation 4 Let G be a graph of diameter at most 2. Then $\chi(G) = n$.

Moreover, note that for any harmonious k-coloring c of a graph G and any vertex v in $V(G)$, we must have $c(x) \neq c(y)$ for all $x, y \in N_G[v]$. In other words, all vertices in the closed neighborhood of a vertex v must have distinct colors in order to have a proper coloring of the edges and vertices. This implies the following observation.

Observation 5 For any graph G with maximum degree Δ,

$$\chi_h(G) \geq \Delta + 1.$$

This bound is sharp, as will be shown in the next section.

3 Harmonious Colorings of Trees

Of particular interest in the field of graph labelings is the class of trees. Famously, we have the Graceful Tree Conjecture and the Harmonious Tree Conjecture that all trees are graceful and that all trees are harmonious, respectively. In [1], it was determined that caterpillars had graceful chromatic number either Δ or $\Delta + 1$ depending on the placement of the vertices of maximum degree along the spine. Otherwise, only an upper bound is known in general for the graceful chromatic number of a general tree T:

$$\chi_g(T) \leq \left\lceil \frac{5\Delta(T)}{3} \right\rceil$$

Here we show that the harmonious chromatic number for all trees is in fact the lower bound given in Observation 5, namely $\Delta(T) + 1$.

To show this, we consider a *rooted tree*. For each integer $k \geq 2$, let $T_{k,1} = K_{1,k}$ be the star of order $k + 1$. Then construct each tree $T_{k,h}$ from $T_{k,h-1}$, for $h \geq 2$, by making each end vertex of $T_{k,h-1}$ be the central vertex of the star $K_{1,k}$. The resulting graph $T_{k,h}$, is the rooted tree of height h and maximum degree k. Note that any tree T is a subgraph of some $T_{k,h}$.

Theorem 1 *If T is a nontrivial tree with maximum degree Δ, then $\chi_h(T) = \Delta + 1$.*

Proof It suffices to show that the tree $T_{\Delta,h}$ has a harmonious $(\Delta + 1)$-coloring for all $h \geq 1$. We will construct such a coloring through induction on h.

We begin with $T_{\Delta,1}$, a star of order $\Delta + 1$. Consider any proper vertex coloring of $T_{\Delta,1}$ using elements from the set $\mathbb{Z}_{\Delta+1}$. Suppose that the central vertex is colored $c_0 \in \mathbb{Z}_{\Delta+1}$ and all leaves of the star are then colored $c_i \in \mathbb{Z}_{\Delta+1}$ for $1 \leq i \leq \Delta$. Thus, the colors of the edges of $T_{\Delta,1}$ will be $c_i + c_0 \bmod (\Delta + 1)$ for $1 \leq i \leq \Delta$. Since the c_i are unique modulo $\Delta + 1$, it follows that the $c_i + c_0$ are also unique modulo $\Delta + 1$ since c_0 is fixed. Thus, this is a harmonious $(\Delta + 1)$-coloring of $T_{\Delta,1}$.

Suppose that $\chi_h(T_{\Delta,k-1}) = \Delta + 1$ for $k \geq 2$. Consider the rooted tree $T = T_{\Delta,k}$. The subgraph of T n which all end vertices have been removed is a rooted tree of height $k - 1$. Let this subgraph be T'. By the induction hypothesis, $T' = T_{\Delta,k-1}$ has a harmonious $(\Delta + 1)$-coloring. Let $c : V(G) \to \mathbb{Z}_{\Delta+1}$ be a harmonious coloring of T', and label all vertices of T' according to c. Choose an end vertex v of T', and let u be the neighbor of v in T'. Therefore, there are still $\Delta - 1$ colors available in the set $\mathbb{Z}_{\Delta+1} - \{c(v), c(u)\}$ to color the end vertices in T that are adjacent to v. As detailed above, all of these labels for the neighbors of v will result in distinct edge labels for all edges incident to v. We can continue this process for all end vertices of T' and extend the harmonious $(\Delta + 1)$-coloring to T. □

In particular, this gives us the harmonious chromatic number of all paths.

Corollary 1 *For all* $n \geq 3$, $\chi_h(P_n) = 3$.

4 Harmonious Colorings of Cycles

We will now present harmonious colorings of cycles. However, the colorings themselves, as well as the harmonious chromatic numbers, will depend on the order of the cycle modulo three. For each of the three cases considered, we will describe the cycle of order n as $C_n = (v_1, \ldots, v_n, v_1)$.

Note that by Observation 2 and Corollary 1, we have $\chi_h(C_n) \geq 3$ for all $n \geq 3$. In the first case we consider, this lower bound is achieved.

Proposition 1 *If* $n \geq 3$ *and* $n \equiv 0 \pmod 3$, *then* $\chi_h(C_n) = 3$.

Proof It suffices to define a harmonious 3-coloring. Let $c : V(C_n) \to \mathbb{Z}_3$ be defined as follows.

$$c(v_k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 3 \\ 1 & \text{if } k \equiv 2 \pmod 3 \\ 2 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

This then induces the following edge coloring, $c' : E(C_n) \to \mathbb{Z}_3$, where $e_k = v_k v_{k+1}$.

$$c'(e_k) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod 3 \\ 0 & \text{if } k \equiv 2 \pmod 3 \\ 2 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

□

However, for cycles of order n, not congruent to 0 modulo three, we show next that 3 colors will not suffice.

Proposition 2 *If $n \geq 4$ and $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$, then $\chi_h(C_n) > 3$.*

Proof Consider the cycle C_n such that $n \geq 4$ and n is not divisible by 3. Assume to the contrary that there exists a harmonious 3-coloring of C_n. Let $c : V(C_n) \to \mathbb{Z}_3$ be such a coloring.

We know that all vertices in the closed neighborhood $N[v_1] = \{v_0, v_1, v_2\}$ must be assigned different colors by c. Let $c(v_0) = a$, $c(v_1) = b$, $c(v_2) = c$. Therefore $a, b,$ and c are distinct elements of $\mathbb{Z}_3$. Moreover, we must have that all vertices in the closed neighborhood $N[v_i] = \{v_{i-1}, v_i, v_{i+1}\}$ of each vertex v_i for $1 \leq i \leq n-1$ must be assigned different colors by c. This implies that c must provide the following assignments for $0 \leq k \leq n$.

$$c(v_k) = \begin{cases} c & \text{if } k \equiv 2 \pmod 3 \\ a & \text{if } k \equiv 0 \pmod 3 \\ b & \text{if } k \equiv 1 \pmod 3 \end{cases}$$

However, if $n \equiv 1 \pmod 3$, then this means that $c(v_{n-1}) = a = c(v_0)$. On the other hand, if $n \equiv 2 \pmod 3$, then we would have $c(v_{n-1}) = b = c(v_1)$. Both cases are contradictory to the assumption that c is a harmonious 3-coloring. Thus, such a coloring is not possible, and we have that $\chi_h(C_n) > 3$ when $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. □

This implies that $\chi_h(C_n) \geq 4$ when $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. We show next that we have equality in both cases.

Proposition 3 *If $n \geq 4$ and $n \equiv 1 \pmod 3$, then $\chi_h(C_n) = 4$.*

Proof Due to Proposition 2, it suffices to provide a harmonious 3-coloring. Define $c : V(C_n) \to \mathbb{Z}_4$ as follows. For $1 \leq k \leq n-1$, let

$$c(v_k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 3 \\ 1 & \text{if } k \equiv 2 \pmod 3 \\ 2 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

Let $c(v_n) = 3$.

This then induces the edge coloring $c' : E(C_n) \to \mathbb{Z}_4$, where $e_k = v_k v_{k+1}$, as follows. For $1 \le k \le n-1$,:

$$c'(e_k) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod 3 \\ 3 & \text{if } k \equiv 2 \pmod 3 \\ 2 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

Additionally, $c'(e_n) = 3$. □

The labeling described in the proof of Proposition 3 is illustrated in (Fig. 2).

For the case where $n \equiv 2 \pmod 3$, note that we exclude C_5 from consideration since this cycle has diameter two and hence has harmonious chromatic number equal to its order as discussed in Observation 4.

Proposition 4 *If $n \ge 8$ and $n \equiv 2 \pmod 3$, then $\chi_h(C_n) = 4$.*

Proof We define a harmonious 3-coloring $c : V(C_n) \to \mathbb{Z}_4$. For $1 \le k \le n-5$:

$$c(v_k) = \begin{cases} 3 & \text{if } k \equiv 1 \pmod 3 \\ 2 & \text{if } k \equiv 2 \pmod 3 \\ 1 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

Then, label the remaining vertices as follows: $c(v_{n-4}) = 0, c(v_{n-3}) = 3, c(v_{n-2}) = 2$, $c(v_{n-1}) = 1$, and $c(v_n) = 0$.

This then induces the edge coloring $c' : E(C_n) \to \mathbb{Z}_4$, where $e_k = v_k v_{k+1}$, as follows. For $1 \le k \le n-6$:

$$c'(e_k) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod 3 \\ 3 & \text{if } k \equiv 2 \pmod 3 \\ 2 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

Additionally, we get the following induced edge labels: $c'(e_{n-5}) = 1$, $c'(e_{n-4}) = 3$, $c'(e_{n-3}) = 1$, $c'(e_{n-2}) = 3$, $c'(e_{n-1}) = 1$, $c'(e_n) = 3$. □

We summarize the above results in the following theorem.

Theorem 2 *For all $n \ge 3$,*

$$\chi_h(C_n) = \begin{cases} 3 & \textit{if } \; n \equiv 0 \pmod 3 \\ 4 & \textit{if } \; n \equiv 1, 2 \pmod 3, n \ne 5 \\ 5 & \textit{if } \; n = 5 \end{cases}$$

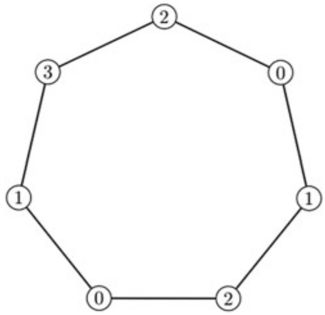


Fig. 2 Harmonious 4-coloring of C_7

5 Harmonious Colorings of Grids

Next, we consider grid graphs $P_n \Box P_m$, and show that they will have harmonious chromatic number equal to the lower bound of $\Delta + 1$ when at least one of m and n is greater than 2. Note that if $m = n = 2$, then $K_2 \Box K_2 = C_4$ has harmonious chromatic number 4 as seen in Theorem 2. There are two cases to consider: when $\Delta = 3$ and $\Delta = 4$. The only grid graphs which have maximum degree 3 are ladders, $L_n = P_n \Box K_2$ for $n \geq 3$. We consider those first.

Proposition 5 *The ladder $L_n = P_n \Box K_2$ for $n \geq 3$ has harmonious chromatic number $\chi_h(L) = 4$.*

Proof Label the ladder L_n in the following way. Let the vertices in the first copy of P_n be labeled $(v_1, \dots, v_n)$ and the vertices of the second copy of P_n be labeled $(u_1, \dots, u_n)$. Also, label $e_k = v_k v_{k+1}$ and $f_k = v_k v_{k+1}$ for $0 \leq k \leq n-2$. See Fig. 3.

Since we already have $\chi_h(L_n) \geq \Delta(L_n) + 1 = 4$, it suffices to provide a harmonious 4-coloring of L_n.

We define a harmonious 4-coloring $c : V(L_n) \to \mathbb{Z}_4$ in the following way. For $1 \leq k \leq n$:

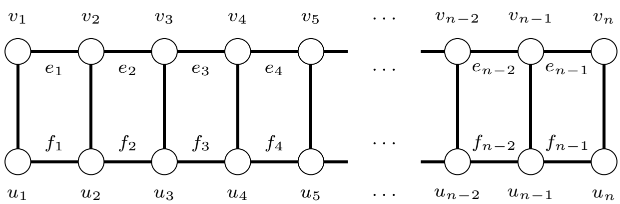


Fig. 3 Labeling of the ladder L_n in Theorem 5

$$c(v_k) = \begin{cases} 2 & \text{if } k \equiv 1 \pmod 4 \\ 0 & \text{if } k \equiv 2 \pmod 4 \\ 1 & \text{if } k \equiv 3 \pmod 4 \\ 3 & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

$$c(u_k) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod 4 \\ 3 & \text{if } k \equiv 2 \pmod 4 \\ 2 & \text{if } k \equiv 3 \pmod 4 \\ 0 & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

Let $c' : E(L_n) \to \mathbb{Z}_4$ be the induced edge coloring. The vertex coloring c from above forces the rungs of the ladder to all be labeled three; hence, $c'(v_i u_i) = 3$ for $1 \leq n \leq n$. Moreover, it guarantees no edges other edges will be labeled three.

The rest of the induced edge coloring will be as follows.

$$c'(e_k) = \begin{cases} 2 & \text{if } k \equiv 1 \pmod 4 \\ 1 & \text{if } k \equiv 2 \pmod 4 \\ 0 & \text{if } k \equiv 3 \pmod 4 \\ 1 & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

$$c'(f_k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 4 \\ 1 & \text{if } k \equiv 2 \pmod 4 \\ 2 & \text{if } k \equiv 3 \pmod 4 \\ 1 & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

□

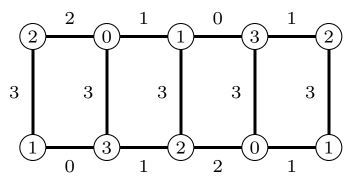
The labeling described in the proof of Proposition 5 is illustrated in (Fig. 4).

Next, we are going to show that all other grid graphs with maximum degree 4 have harmonious chromatic number 5.

Theorem 3 *The grid graph* $G = P_n \square P_m$ *has harmonious chromatic number* $\chi_h(G) = 5$ *when* $n, m \geq 3$.

Proof Label the grid $G = P_n \square P_m$ in the following way. Let $v_{i,j}$ represent the vertex in G that is in the ith row and jth column for $1 \leq i \leq n$ and $1 \leq j \leq m$. The edges of in the rows of G are labeled as $e_{i,j} = v_{i,j} v_{i,j+1}$ and the edges in the columns of G are labeled as $f_{i,j} = v_{i,j} v_{i+1,j}$. See Fig. 5.

Fig. 4 Harmonious 4-coloring of the ladder L_5



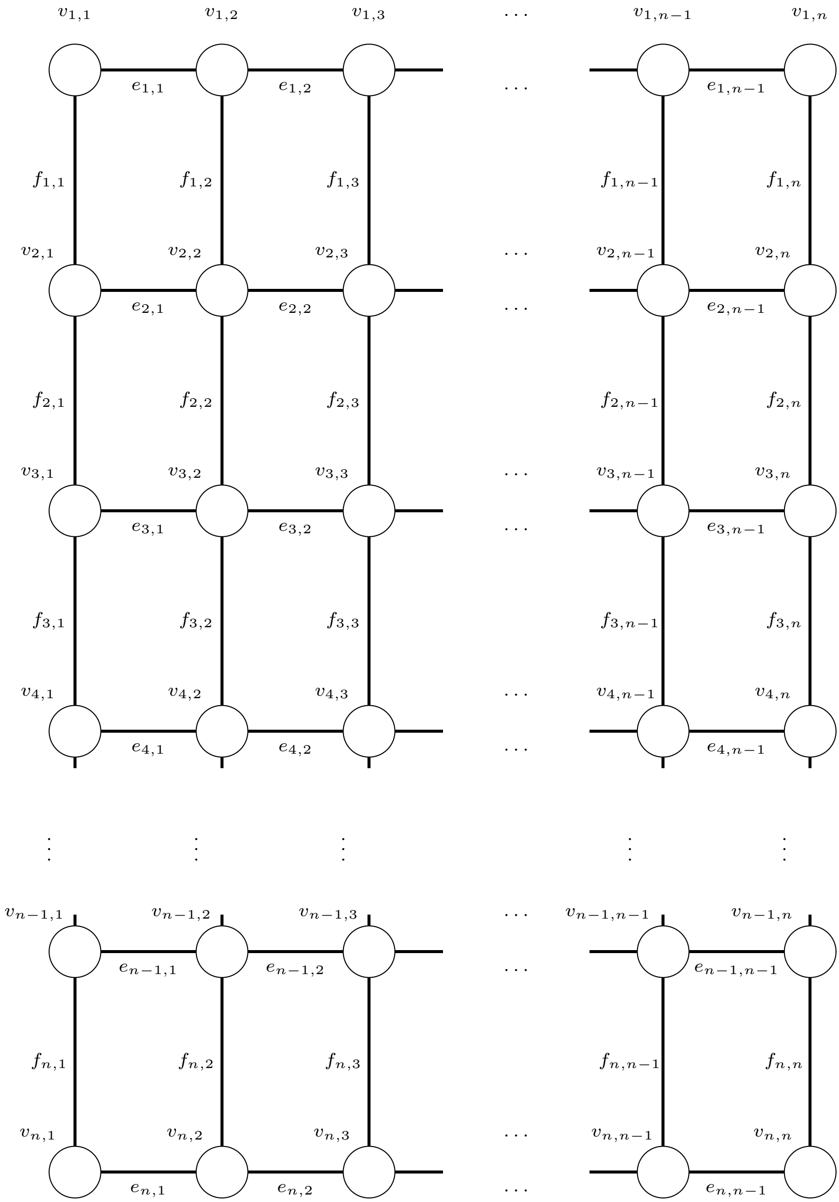


Fig. 5 Labeling of vertices and edges of $G = P_n \Box P_m$ in the proof of Theorem 3

To show that $\chi_h(G) = 5 = \Delta(G) + 1$ it suffices to provide a harmonious 5-coloring of G. Define the vertex coloring $c : V(G) \to \mathbb{Z}_5$ as follows.

$$c(v_{i,j}) = \begin{cases} (j-1) \pmod 5 & \text{if } i \equiv 1 \pmod 5 \\ (j+1) \pmod 5 & \text{if } i \equiv 2 \pmod 5 \\ (j-2) \pmod 5 & \text{if } i \equiv 3 \pmod 5 \\ j \pmod 5 & \text{if } i \equiv 4 \pmod 5 \\ (j+2) \pmod 5 & \text{if } i \equiv 0 \pmod 5 \end{cases}$$

This will result in a proper coloring of the vertices of G and induce the following proper edge coloring of G as well.

$$c'(e_{i,j}) = \begin{cases} (3-j) \pmod 5 & \text{if } i \equiv 1 \pmod 5 \\ -j \pmod 5 & \text{if } i \equiv 2 \pmod 5 \\ (2-j) \pmod 5 & \text{if } i \equiv 3 \pmod 5 \\ (4-j) \pmod 5 & \text{if } i \equiv 4 \pmod 5 \\ (1-j) \pmod 5 & \text{if } i \equiv 0 \pmod 5 \end{cases}$$

$$c'(f_{i,j}) = \begin{cases} (2-i) \bmod 5 & \text{if } j \equiv 1 \pmod 5 \\ (4-i) \bmod 5 & \text{if } j \equiv 2 \pmod 5 \\ (1-i) \bmod 5 & \text{if } j \equiv 3 \pmod 5 \\ (3-i) \bmod 5 & \text{if } j \equiv 4 \pmod 5 \\ -j \bmod 5 & \text{if } j \equiv 0 \pmod 5 \end{cases}$$

$\square$

The labeling described in the proof of Theorem 3 is illustrated in (Fig. 6).

We can summarize the results of Proposition 5 and Theorem 3 in the following theorem.

Theorem 4 *The grid graph $G = P_n \square P_m$ has harmonious chromatic number $\chi_h(G) = \Delta(G) + 1$ when at least one of n and m is greater than 2.*

6 Future Work

We have seen infinitely many graphs which achieve the upper bound of n, in particular all graphs of diameter at most two. We have also seen infinitely many graphs that achieve the lower bound of $\Delta + 1$, notably trees and grid graphs.

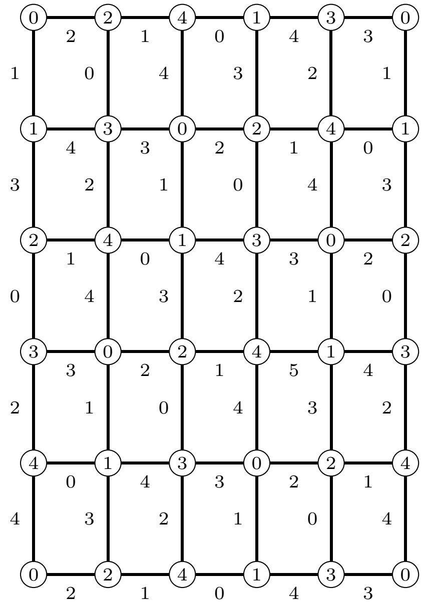


Fig. 6 Harmonious 5-coloring of $P_6 \square P_6$ as described in the proof of Theorem 3

The cycle of order $n \equiv 1, 2 \pmod 3$ (and $n \neq 5$) was the first graph for which we saw a harmonious chromatic number that was strictly between these bounds. In Propositions 3 and 4, we proved that the harmonious chromatic number of these graphs was $\Delta + 2$. An interesting question to investigate along these lines would be: For which integers $c > 1$ is there a graph G such that $\Delta(G) + 1 < \chi_h(G) = \Delta + 1 + c < n$?

References

1. Bi, Z., Byers, A., English, S., Laforge, E., Zhang, P.: Graceful colorings of graphs. J. Combin. Math. Combin. Comput. **101**, 101–119 (2017)
2. Bi, Z., Byers, A., Zhang, P.: Revisiting graceful labelings of graphs. J. Combin. Math. Combin. Comput. **102**, 141–158 (2017)
3. Gallian, J.: A dynamic survey of graph labeling. Electronic J. Combinat. (2019)
4. Golomb, S.W.: How to number a graph. In: Graph Theory and Computing, pp. 23–37. Academic, New York (1972)
5. Graham, R.L., Sloane, N.J.A.: On addition bases and harmonious graphs. SIAM J. Alg. Disc. Math. **1**, 382–404 (1980)
6. Rosa, A.: On certain valuations of the vertices of a graph. In: Theory of Graphs. (International Symposium, Rome, July 1966), pp. 349–355 (1967)

A Note on the Immersion Number of Generalized Mycielski Graphs

Karen L. Collins, Megan E. Heenehan, and Jessica McDonald

Abstract The immersion number of a graph G, denoted $\mathrm{im}(G)$, is the largest t such that G has a K_t-immersion. In this note we are interested in determining the immersion number of the m-Mycielskian of G, denoted $\mu_m(G)$. Given the immersion number of G we provide a lower bound for $\mathrm{im}(\mu_m(G))$. To do this we introduce the "distinct neighbor property" of immersions. We also include examples of classes of graphs where $\mathrm{im}(\mu_m(G))$ exceeds the lower bound. We conclude with a conjecture about $\mathrm{im}(\mu_m(K_t))$.

Keywords Immersion · Mycielski graphs · Generalized Mycielski construction · Cliques · m-Mycielskian

1 Introduction

A pair of adjacent edges uv and vw in a graph are *split off* (or *lifted*) from their common vertex v by deleting the edges uv and vw, and adding the edge uw. Given graphs G and H, we say that G has a H*-immersion* if a graph isomorphic to H can be obtained from a subgraph of G by splitting off pairs of edges, and removing isolated vertices. Immersions have gained considerable interest in the last number of years (see e.g. [3, 4, 14]). In [2] we define the immersion number of G, denoted $\mathrm{im}(G)$, to

K. L. Collins
Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459, USA
e-mail: kcollins@wesleyan.edu

M. E. Heenehan (✉)
Department of Mathematical Sciences, Eastern Connecticut State University, Willimantic, CT 06226, USA
e-mail: heenehanm@easternct.edu

J. McDonald
Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA
e-mail: mcdonald@auburn.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_21

be the largest t for which G has a K_t-immersion. Abu-Khzam and Langston [1] have conjectured that for a graph G, if the chromatic number of G is t, then $\mathrm{im}(G) \geq t$.

In this note we focus on the immersion number of Mycielski graphs and generalized Mycielski graphs. Mycielski [7] proved that if G is triangle-free and $\chi(G) = t$, then $\mu(G)$, the Mycielskian of G (whose definition appears in the next section), is triangle-free and $\chi(\mu(G)) = t + 1$. Hence Mycielski graphs provide examples of triangle-free graphs with arbitrarily high chromatic number. The Mycielski construction was generalized by Stiebitz [9] (see also [8]) and independently by Van Ngoc [11] (see also [12]). The general construction was also described as the "cone over G" by Tardif [10]. While the chromatic number of the generalized Mycielski graphs does not necessarily increase when the construction is applied [6], Stiebitz [9] proved the chromatic number does increase if the construction is applied to a specific class of graphs. We will show that the immersion number of Mycielski graphs and generalized Mycielski graphs increases by at least 1, showing that these graphs behave as expected with respect to the Abu-Khzam–Langston Conjecture.

We begin, in Sect. 2, by providing an alternate definition of immersion along with definitions for Mycielski graphs and generalized Mycielski graphs. In Sect. 3, we define the "distinct neighbor property" of immersions and prove that if a graph has a K_t-immersion, then it has a K_t-immersion with the distinct neighbor property. This is then used to prove that if the immersion number of G is t, then the immersion number of the Mycieskian (or generalized Mycielskian) of a graph is at least $t + 1$. In Sect. 4, we provide several examples of graphs where this increase is more than one. We conclude with a conjecture about the immersion number of the generalized Mycielskian of complete graphs.

2 Definitions

It will be useful to keep the following alternate definition of immersion in mind.

Definition 1 Given graphs G and H, G has an *H-immersion* if there is a one-to-one function $\phi : V(H) \to V(G)$ such that for each edge $uv \in E(H)$, there is a path P_{uv} in G joining vertices $\phi(u)$ and $\phi(v)$, and the paths P_{uv} are pairwise edge-disjoint for all $uv \in E(H)$. In this context we call the vertices of $\{\phi(v) : v \in V(H)\}$ the *terminals* of the H-immersion, and we call internal vertices of the paths $\{P_{uv} : uv \in E(H)\}$ the *pegs* of the H-immersion.

In this note, we are interested in clique immersions in Mycielski graphs.

Definition 2 [7] Given a graph G the *Mycielski graph* (or Mycielskian of a graph), denoted $\mu(G)$, is defined to be the graph with vertex set $(V(G) \times \{0, 1\}) \cup \{w\}$, and edges $(u, 0) - (v, 0)$ and $(u, 0) - (v, 1)$ for all $uv \in E(G)$, and edges $(u, 1) - w$ for all $u \in V(G)$.

The notation in Definition 2 follows [6]. Figure 1 shows $\mu(C_5)$, as an example.

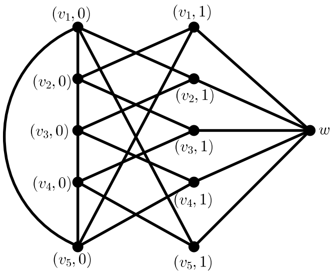


Fig. 1 The Mycielski graph $\mu(C_5) = \mu_2(C_5)$, where consecutive vertices in the C_5 have been labeled $v_1, v_2, \ldots, v_5$

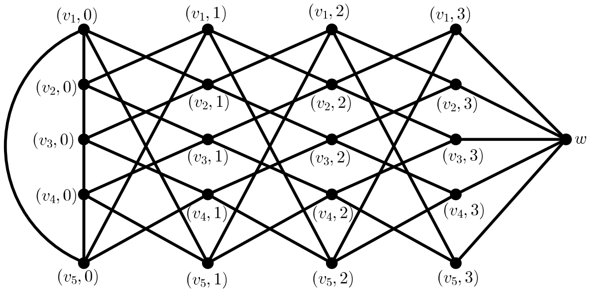


Fig. 2 The 4-Mycielskian of C_5, $\mu_4(C_5)$, where consecutive vertices in the C_5 have been labeled $v_1, v_2, \ldots, v_5$

The Mycielski construction can be generalized to the *m-Mycielskian of G*, denoted $\mu_m(G)$, as follows.

Definition 3 [8, 9, 11, 12] Given a graph G and an integer $m \geq 1$, the *m-Mycielskian of G*, denoted $\mu_m(G)$, is defined to be the graph with vertex set $(V(G) \times \{0, 1, \ldots, m-1\}) \cup \{w\}$, and edges $(u, 0) - (v, 0)$ and $(u, i) - (v, i+1)$ for all $uv \in E(G)$, and edges $(u, m-1) - w$ for all $u \in V(G)$.

The notation in Definition 3 again follows [6]. Notice that $\mu_1(G)$ is obtained from a copy of G with an additional vertex adjacent to all vertices in G and $\mu_2(G)$ is the Mycielskian of G, that is $\mu(G) = \mu_2(G)$. This construction is also known as the *cone over G* [10]. When described as the cone over G, we use a direct product of graphs. We let $\mathbb{P}_m$ be a path on $m+1$ vertices with a loop at one end. The *m-Mycielskian of G*, denoted $\mu_m(G)$, is obtained from $G \times \mathbb{P}_m$ (where $\times$ is the direct product) by collapsing all vertices whose second coordinate is m to a single vertex labeled $(*, m)$ (w in Definition 3) [10]. Figure 2 shows $\mu_4(C_5)$, as an example.

Notice that $\chi(\mu_1(G)) = \chi(G) + 1$ and $\chi(\mu_2(G)) = \chi(G) + 1$ [7]. Stiebitz proved that when $M(3)$ is the set of all 3-color-critical graphs and for $k \geq 3$ and $M(k+1) = \{\mu_m(G) \mid G \in M(k) \text{ and } r \geq 1\}$, then $M(k)$ is k-color-critical for all $k \geq 3$ [9]. However, for all $t \geq 4$, there exists a graph G such that $\chi(G) = \chi(\mu_3(G)) = t$ [10]. The smallest example of this is the circulant graph formed by placing 7 vertices around a circle, equally spaced; each vertex is adjacent to its nearest two vertices in each direction around the circle. This graph has chromatic number 4 as does the 3-Mycielskian of the graph [10].

Our main result is Theorem 1, appearing in the next section, which states that if $\mathrm{im}(G) = t$ and $m \geq 1$, then $\mathrm{im}(\mu_m(G)) \geq t + 1$. This result confirms the Abu-Khzam–Langston Conjecture for this class of graphs. Moreover, given the special nature of generalized Mycielski graphs, as an immediate corollary we get that there exist graphs with fixed clique number and arbitrarily high chromatic number that satisfy the conjecture.

In the next section we prove Theorem 1. This is accomplished by first defining the "distinct neighbor property" of an immersion and proving every graph with a K_t-immersion contains a K_t-immersion with the distinct neighbor property.

3 Results

As the main element of proof of Theorem 1, we present a lemma about immersions with the "distinct neighbor property." Proposition 1 proves that if G has a K_t-immersion, then it has a K_t-immersion in which each terminal has a distinct neighbor. This may be of independent interest as immersions require edges on paths be distinct, but not vertices.

Definition 4 Let G be a graph having a K_t-immersion, with terminals $v_1, \ldots, v_t$. If, for each $i \in \{1, 2, \ldots, t\}$, it is possible to choose a distinct neighbor $v_{f(i)}$ of v_i in G (i.e. such that $\{v_{f(i)} : 1 \leq i \leq t\}$ is a set of t vertices), then we say that the immersion has the *distinct neighbor property*.

Proposition 1 *Let G be a graph that has a K_t-immersion, then G has a K_t-immersion with the distinct neighbor property.*

Proof Let $A = \{a_1 \ldots, a_t\}$ be the set of terminals of a K_t-immersion in G that does not have the distinct neighbor property. Let $B = \{b_1, b_2, \ldots\}$ be the set of all vertices in G which are neighbors of at least one vertex in A (this definition allows for $A \cap B \neq \emptyset$, although we will soon see that this does not actually occur). Define H to be the bipartite graph with bipartition (A, B) where $a_i b_j \in E(H)$ if and only if $a_i b_j \in E(G)$. Since our immersion does not have the distinct neighbor property, H does not have a matching saturating A. Hence by Hall's Theorem [5], there exists $S \subseteq A$ such that $|N_H(S)| < |S|$. Since A is the set of terminals of a K_t-immersion, and G is simple, each vertex in A has at least $t - 1$ distinct neighbors in B. Hence it must be the case that $N_H(a_i) = B$ for all $i \in \{1, 2, \ldots, t\}$ and $|B| = t - 1$. In particular, this means that $A \cap B = \emptyset$ and hence that H is a subgraph of G.

We now describe a new K_t-immersion in G with terminals

$$b_1, b_2, \ldots, b_{t-1}, a_t.$$

First note that a_t is adjacent to all of the other terminals in the list. For the other paths required, we will show that in fact $H \setminus a_t$ has a K_{t-1}-immersion using $b_1, \ldots, b_{t-1}$ as terminals.

Consider an auxiliary copy of K_{t-1} that has been properly edge-colored with at most $t-1$ colors (possible by Vizing's Theorem [13]). Label the vertices of the K_{t-1} with $b_1, \dots, b_{t-1}$ and label the colors of the edges with $a_1, a_2, \dots, a_{t-1}$. In the K_t immersion in G, we use the path $b_i - a_k - b_j$, where a_k is the color of the edge $b_i b_j$ in the auxiliary graph. Since the edge-coloring is proper, no edge of G gets used twice in this assignment of paths, and we have defined a K_t-immersion.

Using this new K_t-immersion in G we define a new H in the same way as at the beginning of the proof. We now see there is a matching saturating A and therefore this new K_t-immersion has the distinct neighbor property. For example, for each terminal b_i we choose as its distinct neighbor a_i (for $i \in \{1, 2, \dots, t-1\}$) and for a_t we choose b_1 as its distinct neighbor (really any b_i works since a_t is adjacent to b_i). □

We now use Proposition 1 to prove Theorem 1.

Theorem 1 *Let $m \geq 1$ be an integer and suppose that* $\mathrm{im}(G) = t$. *Then* $\mathrm{im}(\mu_m(G)) \geq t + 1$.

Proof By Proposition 1, we can choose a K_t-immersion in G with the distinct neighbor property. Let $v_1, \dots, v_t$ be the terminals of the immersion, and let $v_{f(1)}, \dots, v_{f(t)}$ be their distinct neighbors.

When $m = 1$ we use the immersion in G with the additional terminal w which is adjacent to all vertices in G.

When $m = 2$, to form a K_{t+1}-immersion in $\mu_2(G)$, we take $(v_1, 0), (v_2, 0), \dots, (v_t, 0)$ and w as our terminals. The first t of these have the required paths joining them in G, and the vertex w is joined to each $(v_i, 0)$ via the path $(v_i, 0) - (v_{f(i)}, 1) - w$. These paths are edge-disjoint since $f(i)$ is v_i's distinct neighbor.

For $m > 2$, we again take $(v_1, 0), (v_2, 0), \dots, (v_t, 0)$ and w as our terminals. We use the K_t-immersion in G to connect the first t terminals. To join w to the other terminals we alternate between v_i and its distinct neighbor $v_{f(i)}$ (that is $(v_i, 0) - (v_{f(i)}, 1) - (v_i, 2) - (v_{f(i)}, 3) - \dots$) until we come to the vertex with second coordinate $m - 1$ which is adjacent to w. This completes the K_{t+1}-immersion in $\mu_m(G)$. □

In the next section we provide examples of classes of graphs where the bound of Theorem 1 is best possible and examples of classes of graphs that exceed the bound of Theorem 1.

4 Examples

While chromatic number may not increase by more than one when the generalized Mycielski construction is applied, the immersion number of certain classes of graphs may increase more drastically. When m increases the paths in $\mu_m(G)$ get longer and the degree of many vertices doubles. In some cases this allows the immersion number to increase by more than one. We explore this idea by examining several classes of graphs.

To calculate the immersion numbers of specific Mycielski (and generalized Mycielski) graphs we must consider the degrees of the vertices in the graph and the number of vertices of each degree. In order to have an immersion of K_t there must be at least t vertices of degree at least $t-1$ and the immersion number of any graph must be less than or equal to the maximum degree plus one.

The bound of Theorem 1 is best possible when $m = 1$ since, when forming $\mu_1(G)$, one vertex is added that is adjacent to all vertices in G. The bound is also best possible when $m = 2$ and G is a complete graph.

Proposition 2 $\mathrm{im}(\mu_2(K_t)) = t + 1$

Proof The immersion number of K_t is t, thus by Theorem 1 $\mathrm{im}(\mu_2(K_t)) \geq t + 1$. The graph $\mu_2(K_t)$ has t vertices of degree $2t-2$ and $t+1$ vertices of degree t, therefore, $\mathrm{im}(\mu_2(K_t)) \leq t + 1$. Thus, $\mathrm{im}(\mu_2(K_t)) = t + 1$. □

The following provides examples where the immersion number is greater than the bound of Theorem 1. We begin by considering the path on n vertices, P_n, with the knowledge that $\mathrm{im}(P_n) = 2$.

Proposition 3 $\mathrm{im}(\mu_m(P_5)) = \begin{cases} 4 & \textit{for } m = 1, 2 \\ 5 & \textit{for } m \geq 3 \end{cases}$

Proof Label consecutive vertices in P_5 as $1, 2, \ldots, 5$ so that vertex 1 has degree 1 and consider $\mu_m(P_5)$. When $m = 1$ or 2 there are not enough vertices of degree 4 for $\mu_m(P_5)$ to have a K_5-immersion, therefore $\mathrm{im}(\mu_m(P_5)) \leq 4$. Immersions of K_4 can easily be found in these cases.

When $m \geq 3$, there are $2m$ vertices of degree 2, m vertices of degree 3, $3(m-1)$ vertices of degree 4, and one vertex of degree 5, therefore $\mathrm{im}(\mu_m(P_5)) \leq 5$. We now provide a construction for a K_5-immersion.

As terminals of our K_5-immersion we use vertices (2, 0), (3, 0), (4, 0), (2, 1), and (4, 1). Notice that (3, 0) is adjacent to all of the other terminals. To connect the remaining pairs of terminals we use the paths:

$$(2,0) - (3,1) - (4,0) \qquad (2,0) - (1,0) - (2,1)$$

$$(2,1) - (3,2) - (4,1) \qquad (4,0) - (5,0) - (4,1)$$

$$(2,0) - (1,1) - (2,2) - (1,3) - \ldots - w - \ldots - (5,4) - (4,3) - (5,2) - (4,1)$$

$$(4,0) - (5,1) - (4,2) - (5,3) - \ldots - w - \ldots - (1,4) - (2,3) - (1,2) - (2,1)$$

This completes a K_5-immersion in $\mu_m(P_5)$. □

This result can be generalized as follows.

Corollary 1 *If* $m \geq 3$ *and* $n \geq 5$, *then* $\mathrm{im}(\mu_m(P_n)) = 5$.

Proof Let $m \geq 3$ and $n \geq 5$. In $\mu_m(P_n)$ there are $2m$ vertices of degree 2, $n-2$ vertices of degree 3, $(m-1)(n-2)$ vertices of degree 4, and one vertex of degree n. Therefore, $\mathrm{im}(\mu_m(P_n)) \leq 5$. We see that $\mu_m(P_5)$ is a subgraph of $\mu_m(P_n)$, thus by Proposition 3 $\mathrm{im}(\mu_m(P_n)) \geq 5$. We have shown $\mathrm{im}(\mu_m(P_n)) = 5$. □

As another example where the immersion number exceeds the bound of Theorem 1 we consider the cycle on n vertices, C_n. Note that $\mathrm{im}(C_n) = 3$.

Proposition 4 *If* $n \geq 5$, *then* $\mathrm{im}(\mu_m(C_n)) = \begin{cases} 4 & \text{for } m = 1 \\ 5 & \text{for } m \geq 2 \end{cases}$

Proof Let $n \geq 5$.

In $\mu_1(C_n)$ there are n vertices of degree 3 and one vertex of degree n, therefore, $\mathrm{im}(\mu_1(C_n)) \leq 4$. Notice that $\mu_1(P_5)$ is a subgraph of $\mu_1(C_n)$ thus, by Proposition 3, $\mathrm{im}(\mu_1(C_n)) \geq 4$ and we get $\mathrm{im}(\mu_1(C_n)) = 4$.

When $m \geq 2$, $\mu_m(C_n)$ has n vertices of degree 3, $(m-1)n \geq 5$ vertices of degree 4, and one vertex of degree n, therefore, $\mathrm{im}(\mu_m(C_n)) \leq 5$.

When $m \geq 3$, we use the fact that $\mu_m(P_n)$ is a subgraph of $\mu_m(C_n)$ therefore, by Proposition 3 and Corollary 1, $\mathrm{im}(\mu_m(C_n)) \geq 5$. Thus for $n \geq 5$ and $m \geq 3$ $\mathrm{im}(\mu_m(C_n)) = 5$.

For the case when $m = 2$ we will construct a K_5-immersion. Label consecutive vertices in C_n $v_1, v_2, \ldots, v_n$. As the terminals of our immersion we use $(v_1, 0)$, $(v_2, 0)$, $(v_3, 0)$, $(v_4, 0)$ and $(v_5, 0)$. No matter the value of n we use edges $(v_1, 0) - (v_2, 0)$, $(v_2, 0) - (v_3, 0)$, $(v_3, 0) - (v_4, 0)$, and $(v_4, 0) - (v_5, 0)$ and the paths $(v_1, 0) - (v_2, 1) - (v_3, 0)$, $(v_2, 0) - (v_3, 1) - (v_4, 0)$, and $(v_3, 0) - (v_4, 1) - (v_5, 0)$ in our K_5-immersion. The remaining paths depend on the value of n.
Case 1: When $n = 5$ the remaining paths are

$$(v_1, 0) - (v_5, 1) - (v_4, 0),$$

$$(v_1, 0) - (v_5, 0),$$

$$(v_2, 0) - (v_1, 1) - (v_5, 0)$$

Case 2: When $n = 6$ the remaining paths are

$$(v_1, 0) - (v_6, 0) - (v_5, 1) - (v_4, 0),$$

$$(v_1, 0) - (v_6, 1) - (v_5, 0),$$

$$(v_2, 0) - (v_1, 1) - (v_6, 0) - (v_5, 0)$$

Case 3: When $n \geq 7$ the remaining paths are

$$(v_1, 0) - (v_n, 1) - w - (v_5, 1) - (v_4, 0),$$

$$(v_1, 0) - (v_n, 0) - (v_{n-1}, 0) - \cdots - (v_6, 0) - (v_5, 0),$$

$$(v_2, 0) - (v_1, 1) - w - (v_6, 1) - (v_5, 0)$$

This completes the K_5-immersion in $\mu_2(C_n)$. □

Some preliminary work leads us to believe that $\mathrm{im}(\mu_m(K_t)) > t + 1$ for a large enough m. For example, we can show $\mathrm{im}(\mu_3(K_4)) = 7$ and $\mathrm{im}(\mu_4(K_5)) = 9$ (see Appendix A for explicit immersions). In $\mu_m(K_t)$ there are $t + 1$ vertices of degree t and $(m - 1)t$ vertices of degree $2t - 2$. When $m > 2$ there are enough vertices to potentially have a K_{2t-1}-immersion in $\mathrm{im}(\mu_m(K_t))$ and for $t > 2$ we know $t + 1 < 2t - 1$. This means $t + 1 \leq \mathrm{im}(\mu_m(K_t)) \leq 2t - 1$ for $m > 2$ and $t > 2$. This leads us to make the following conjecture.

Conjecture 1 If $m \geq 3$, then $\mathrm{im}(\mu_m(K_{m+1})) = 2m + 1$.

A Appendix

Here we provide explicit immersions in $\mu_3(K_4)$ and $\mu_4(K_5)$.

Example 1 Label the vertices of a K_4 $v_1, v_2, \ldots, v_4$. As the terminals of our K_7-immersion in $\mu_3(K_4)$ we use vertices $(v_i, 0)$ for $i = 1, 2, 3, 4$ and $(v_j, 1)$ for $j = 1, 2, 3$. The terminals $(v_i, 0)$ form a K_4 and $(v_i, 0)$ is adjacent to $(v_j, 1)$ when $i \neq j$. We complete the K_7-immersion using the following paths.

$(v_1, 0) - (v_4, 1) - (v_2, 2) - (v_1, 1)$ $\quad$ $(v_1, 1) - (v_4, 2) - (v_3, 1)$

$(v_1, 1) - (v_3, 2) - w - (v_1, 2) - (v_2, 1)$ $\quad$ $(v_2, 0) - (v_4, 1) - (v_3, 2) - (v_2, 1)$

$(v_2, 1) - (v_4, 2) - w - (v_2, 2) - (v_3, 1)$ $\quad$ $(v_3, 0) - (v_4, 1) - (v_1, 2) - (v_3, 1)$

Example 2 Label the vertices of a K_5 $v_1, v_2, \ldots, v_5$. As the terminals of our K_9-immersion in $\mu_4(K_5)$ we use vertices $(v_i, 0)$ for $i = 1, 2, 3, 4, 5$ and $(v_j, 1)$ for $j = 1, 2, 3, 4$. The terminals $(v_i, 0)$ form a K_5 and $(v_i, 0)$ is adjacent to $(v_j, 1)$ when $i \neq j$. We complete the K_9-immersion using the following paths.

$(v_1, 0) - (v_5, 1) - (v_4, 2) - (v_1, 1)$ $\quad$ $(v_2, 0) - (v_5, 1) - (v_3, 2) - (v_2, 1)$

$(v_3, 0) - (v_5, 1) - (v_2, 2) - (v_3, 1)$ $\quad$ $(v_4, 0) - (v_5, 1) - (v_1, 2) - (v_4, 1)$

$(v_1, 1) - (v_3, 2) - (v_2, 3) - (v_4, 2) - (v_2, 1)$ $\quad$ $(v_1, 1) - (v_5, 2) - (v_3, 1)$

$(v_1, 1) - (v_2, 2) - (v_4, 1)$ $\quad$ $(v_2, 1) - (v_1, 2) - (v_3, 1)$

$(v_3, 1) - (v_4, 2) - (v_5, 3) - (v_3, 2) - (v_4, 1)$ $\quad$ $(v_2, 1) - (v_5, 2) - (v_4, 1)$

References

1. Abu-Khzam, F., Langston, M.: Graph coloring and the immersion order. Lect. Notes Comput. Sci. **2697**, 394–403 (2003)
2. Collins, K.L., Heenehan, M.E., McDonald, J.: Clique immersion in graph products. Discrete Math. 346(8) (2023)
3. Giannopoulou, A., Pilipczuk, M., Raymond, J., Thilikos, D., Wrochna, M.: Linear kernels for edge deletion problems to immersion-closed graph classes. SIAM J. Discrete Math. **35**(1), 105–151 (2021)
4. Guyer, M., McDonald, J.: On clique immersions in line graphs. Discrete Math. **343**(12), 7 (2020)
5. Hall, P.: On representatives of subsets. J. London Math. Soc. **10**, 26–30 (1935)
6. Müller, T., Stehlík, M.: Generalised Mycielski graphs and the Borsuk-Ulam theorem. Electron. J. Comb. **26**(4) (2019)
7. Mycielski, J.: Sur le coloriage des graphes. Colloq. Math. **3**, 161â-162 (1955)
8. Sachs, H., Stiebitz, M.: On constructive methods in the theory of colour-critical graphs. Discrete Math. **74**(1–2), 201–226 (1989)
9. Stiebitz, M.: Beiträge zur Theorie der färbungskritischen Graphen. Dissertationsschrift**(B) TH** Ilmenau (1985)
10. Tardif, C.: Fractional chromatic numbers of cones over graphs. J. Graph Th. **38**, 87–94 (2001)
11. Van Ngoc, N.: On graph colourings, Ph.D., thesis, Hungarian Academy of Sciences (1987)
12. Van Ngoc, N., Tuza, Z.S.: 4-chromatic graphs with large odd girth. Discrete Math. **138**(1–3), 387–392 (1995)
13. Vizing, V.: On an estimate of the chromatic class of a p-graph. Diskret. Analiz **3**, 25–30 (1964)
14. Wollan, P., Wood, D.: Nonrepetitive colorings of graphs excluding a fixed immersion or topological minor. J. Graph Th. **91**(3), 259–266 (2019)

Recent Developments of Star-Critical Ramsey Numbers

Jonelle Hook

Abstract The star-critical Ramsey number $r_*(G, H)$ is the smallest integer k such that every 2-coloring of the edges of $K_r - K_{1,r-k-1}$ contains either a red copy of G or a blue copy of H where $r = R(G, H)$ the graph Ramsey number. Since the introduction of star-critical Ramsey numbers in 2010, there have been a significant number of values discovered as well as numerous classifications of critical graphs. Some variants to the star-critical Ramsey number have been recently introduced in an attempt to further analyze when the Ramsey property is forced and to make progress on unknown Ramsey numbers. This paper will discuss recent developments of the star-critical Ramsey number and a survey will be provided for all known star-critical Ramsey numbers.

Keywords Star-critical Ramsey number · Critical graph · Ramsey number

1 Introduction

The *graph Ramsey number* $R(G, H)$ is the smallest integer r such that every 2-coloring of the edges of K_r contains either a red copy of G or a blue copy of H. This definition implies that there exists a 2-coloring of K_{r-1} that avoids both a red G and a blue H. Such a coloring is called a (G, H)*-free coloring* of K_{r-1} and is considered to be a *critical graph* for $R(G, H)$. In the language of arrowing [9], we say the Ramsey number $R(G, H)$ is the smallest r such that $K_r \longrightarrow (G, H)$ which implies $K_{r-1} \not\longrightarrow (G, H)$.

Star-critical Ramsey numbers were introduced with Isaak in 2010 [22, 23] and provide insight into understanding the behavior of Ramsey numbers. We find the largest star that can be removed from K_r so that the underlying graph must contain a red G or a blue H. Thus, this intermediate graph between K_{r-1} and K_r satisfies the Ramsey property. The *star-critical Ramsey number* $r_*(G, H)$ is the smallest integer

J. Hook (✉)
Mount St. Mary's University, Emmitsburg, MD 21727, USA
e-mail: jhook@msmary.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_22

k such that every 2-coloring of the edges of $K_r - K_{1,r-k-1}$ contains either a red G or a blue H i.e. $K_r - K_{1,r-k-1} \longrightarrow (G, H)$. Note that $K_r - K_{1,r-k-1}$ is the union of K_{r-1} and $K_{1,k}$ such that v is the vertex of $K_{1,k}$ with degree k whose adjacencies are vertices of K_{r-1}. The size of the star varies widely and in some cases all the edges to K_{r-1} are required to force the Ramsey property and no intermediate graph exists. Erdös and Faudree found this was true for various classes of graphs in [9, 10] and were interested in finding all (G, H) where this was the case. Zhang, Broersma, and Chen formally define this notion in [55] as follows: A pair of graphs (G, H) is called *Ramsey-full* if $K_r \longrightarrow (G, H)$ but $K_r - e \not\longrightarrow (G, H)$. Here, we have that $r_*(G, H) = R(G, H) - 1$.

There are other Ramsey numbers concerning edges and subgraphs. The *size Ramsey number* $\hat{r}(G, H)$ introduced in 1978 by Erdös et al. [11] is the smallest integer m such that there exists a graph with m edges that arrows (G, H). In 1991, Erdös and Faudree [9] defined size Ramsey functions. The *lower size Ramsey number* $l(G, H)$ is the smallest number of edges such that there exists a subgraph F of K_r with $l(G, H)$ edges where every 2-coloring must contain a red G or a blue H. The *upper size Ramsey number* $u(G, H)$ is the smallest number of edges such that if a subgraph F of K_r has at least $u(G, H)$ edges, then F contains a red G or blue H. So, for a graph F on r vertices and s edges, the lower size Ramsey number is the smallest s with no (G, H)-free coloring for some F and the upper size Ramsey number is the smallest s with no (G, H)-free coloring for every F. If (G, H) is Ramsey-full, then $l(G, H) = u(G, H) = \binom{r}{2}$. The relationship to the star-critical Ramsey number is $l(G, H) \leq \binom{r-1}{2} + r_*(G, H) \leq u(G, H)$.

Section 2 includes general lower bounds for the star-critical Ramsey number. It also aligns the star-critical Ramsey number with recent papers that discuss an alternate perspective keeping the literature consistent. In Sect. 3, a survey of known star-critical Ramsey numbers is provided. For comparison, the graph Ramsey numbers are included. The citations of the graph Ramsey numbers utilize the great work of Radziszowki and his dynamic survey on Small Ramsey numbers [44].

2 Recent Work

General lower bounds are presented in [18, 19, 55] that rely on the the work of Burr [4]. Let $\chi(G)$ be the chromatic number. The *chromatic surplus* $s(G)$ is the minimum number of vertices in a color class for any coloring of G that uses $\chi(G)$ colors. Burr proved that $R(G, H) \geq (\chi(G) - 1)(n - 1) + s(G)$ for any connected graph H with n vertices where $n \geq s(G)$. He further defined that H is G-good if equality holds.

Theorem 1 ([55]) *Let H be a connected graph with at least two vertices that is G-good. Let $\tau(G)$ be the minimum degree of some vertex in the smallest color class. If $s(G) = 1$, or $\delta(H) = 1$, or $\kappa(H) \geq 2$, then*

$$r_*(G, H) \geq (\chi(G) - 2)(|V(H) - 1) + s(G) + \delta(H) + \tau(G) - 2.$$

This lower bound improves the lower bound for the upper size Ramsey number in [9]. In [18], we have another lower bound stated below and a similar result in [19] when H contains no vertex cut or $\delta(H) = 1$.

Theorem 2 ([18]) *Let G be a graph with $\chi(G) \geq 2$, and let H be a connected graph of order $n \geq s(G)$ with minimum degree $\delta(H)$. If H is G-good, then*

$$r_*(G, H) \geq (\chi(G) - 2)(n - 1) + \min\{n, \delta(H) + \tau(G) - 1\}.$$

Theorem 3 ([19]) *Let G be a graph with $\chi(G) \geq 2$, and H a connected graph of order $n \geq s(G) + 1$ with minimum degree $\delta(H)$. If H contains no vertex cut or $\delta(H) = 1$, then*

$$r_*(G, H) \geq (\chi(G) - 2)(n - 1) + \min\{n, \delta(H) + \tau(G) - 1\} + s(G) - 1.$$

We now turn our attention to some alternate definitions of the star-critical Ramsey number.

Budden and Dejonge [5] define the k-deleted Ramsey number and the deleted edge number. For $k \in \mathbb{N}$, the *k-deleted Ramsey number $D_k(G, H)$* is the least natural number p such that every 2-coloring of the edges of $K_p - E(K_{1,k})$ contains a red G or blue H. The *deleted edge number $de(G, H)$* is the least $k \in \mathbb{N}$ such that $D_{k-1}(G, H) < D_k(G, H)$. So, $D_{k-1}(G, H) = R(G, H)$ and $D_k(G, H) = R(G, H) + 1$. There is a connection to the star-critical Ramsey number. The sum of the star-critical Ramsey number and the deleted edge number equals the Ramsey number. That is, $r_*(G, H) + de(G, H) = R(G, H)$. Therefore, the star-critical Ramsey number is the size of the star added to K_{r-1} that forces the Ramsey property and the deleted edge Ramsey number is the complementary approach capturing the size of the star removed from K_r.

Wang and Li [50] define the critical Ramsey number where the deleted graph is not always a star. For a class of graphs $\mathbb{G} = \{G_k, G_{k+1}, \ldots\}$ without isolated vertices and graphs H_1 and H_2 with $r = R(H_1, H_2)$, the *critical Ramsey number* denoted $R_{\mathbb{G}}(H_1, H_2)$ is the maximum n such that $K_r - G_n \to (H_1, H_2)$ where $G_n \in \mathbb{G}$ and $|V(G_n)| \leq r$. They refer to $R_{\mathbb{S}}(G, H)$ as the star-critical Ramsey number, but the relationship here to $r_*(G, H)$ is similar to that of the deleted edge number. Here we have $r_*(G, H) + R_{\mathbb{S}}(G, H) = r - 1$. The complete-critical Ramsey number $R_{\mathbb{K}}(G, H)$ is the largest p such that $K_r - K_p \to (G, H)$. [50] contains a general upper bound on the complete-critical Ramsey number and provides results for stars versus complete graphs, paths versus C_4, and fans versus K_3. The matching critical Ramsey number $R_{\mathbb{M}}(G, H)$ and path-critical Ramsey number $R_{\mathbb{P}(G,H)}$ are similarly defined. See [36] for results on complete-bipartite critical Ramsey numbers.

Table 1 Trees

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References	Notes
(T_n, K_m)	$(n-1)(m-1)+1$	[7]	$(n-1)(m-2)+1$	[23]	CG
$(K_{1,n}, K_{1,m})$	$n+m-1$	[20]	$n+m-2$	[9]	RF
(T_n, B_m)	$2n-1$	[12]	n	[51]	

3 Survey

This section enumerates all known star-critical Ramsey numbers. There are 7 tables of data: Trees, Complete graphs, Cycles versus complete graphs, Paths and cycles, Fans, Other, and For large n. Each table gives the pair of graphs (G, H), the graph Ramsey number with citation, and the star-critical Ramsey number with citation. In the notes column, CG means the critical graphs for $R(G, H)$ have been classified and RF means that (G, H) is Ramsey-full.

3.1 Trees

Let T_n be a tree on n vertices. In the introductory paper with Isaak [23], we show the critical graph for $R(T_n, K_m)$ is unique and, for any n, $r_*(T_n, K_m) = (n-1)(m-2)+1$. Erdös and Faudree [9] determined that $(K_{1,n}, K_{1,m})$ is Ramsey-full for even n and even m which implies that $r_*(K_{1,n}, K_{1,m}) = n$. Let $B_m = K_2 + mK_1$ be the book graph. The book-tree star-critical Ramsey number is found in [51] for $n \geq 3m$ to be $R_{\mathbb{S}}(T_n, B_m) = n-2$ which implies that $r_*(T_n, B_m) = n$. These results are summarized in Table 1.

3.2 Complete Graphs

Using an observation of Chvátal, we can see that (K_n, K_m) is Ramsey-full. Let u be any vertex of a (K_n, K_m)-free coloring of K_{s-1}. Add a copy of u, say v, and the graph $K_s - \{uv\}$ possesses no monochromatic cliques of size n or m. Thus, if $r(K_n, K_m) = s$, then $r_*(K_n, K_m) = s-1$.

The critical graph for $R(K_3 - e, K_{m+1} - e)$ is unique and $r_*(K_3 - e, K_{m+1} - e) = 2m-3$ for all $m \geq 2$ [22]. Multiple copies of K_2 and K_3 are discussed in [23]. The critical graphs for $R(nK_2, mK_2)$ are classified and $r_*(nK_2, mK_2) = m$ for $n \geq m \geq 1$. The pair (nK_3, mK_3) is Ramsey-full and $r_*(nK_3, mK_3) = 3n+2m-1$ for $n \geq m \geq 1$ and $n \geq 2$. Results on complete graphs have also been investigated in [34]. Li and Li determine that matchings versus K_n are Ramsey-full and char-

Table 2 Complete graphs

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References	Notes
(K_n, mK_2)	$n + 2m - 2$	[39]	$n + 2m - 3$	[34]	CG, RF
$(K_3 - e, K_{m+1} - e)$	$2m - 1$		$2m - 3$	[22]	CG
(nK_2, mK_2)	$2n + m - 1$	[8]	m	[23]	CG
(nK_3, mK_3)	$3n + 2m$	[2]	$3n + 2m - 1$	[23]	RF
(nK_4, mK_3), $n \geq m$	$4n + 2m + 1$	[40]	$4n + 2m$	[34]	RF
(nK_4, mK_3), $m \geq n$	$3n + 3m + 1$	[40]	$3n + 3m$	[34]	RF

acterize the critical graphs. For $m \geq 1$ and $n \geq 2$, $r_*(K_n, mK_2) = n + 2m - 3$. They also show that $r_*(nK_4, mK_3) = 4n + 2m$ for $n \geq m \geq 1$ with $n \geq 2$ and $r_*(nK_4, mK_3) = 3n + 3m$ for $m \geq n \geq 2$. In both cases, they are Ramsey-full. These results are summarized in Table 2.

3.3 Cycles Versus Complete Graphs

There has been great progress on cycles versus complete graphs. The star-critical Ramsey numbers for cycles versus K_3 and K_4 were initially studied in [22]. The critical graphs for $R(C_n, K_3)$ are classified in [45] and a more concise proof is located in [22]. For $n \geq 3$, $r_*(C_n, K_3) = n + 1$ [23]. The classification of critical graphs for $R(C_n, K_4)$ and $r_*(C_n, K_4) = 2n$ for $n \geq 5$ were proven in [22] and alternate proofs establishing the same results were recently published in [29].

Cycles versus K_5 have been studied independently by Jayawardene [30] and M. Ferreri et al. [13]. They have both classified the critical graphs for $R(C_n, K_5)$ and found the star-critical Ramsey number. For a 4-cycle, $r_*(C_4, K_5) = 13$. For $n \geq 5$, $r_*(C_n, K_5) = 3n - 1$. For $n \geq 15$, the critical graphs for $R(C_n, K_6)$ have been enumerated in [26] and $r_*(C_n, K_6) = 4n - 2$ is determined in [27]. The case for large cycles versus complete graphs can be found in [28] where $r_*(C_n, K_m) = (m - 2)(n - 1) + 2$ for $m \geq 7$ and $n \geq (m - 3)(m - 1)$.

All of the above results are summarized in Table 3.

3.4 Paths and Cycles

The path-path critical graphs are classified for all $n \geq m \geq 2$ and $r_*(P_n, P_m) = \lceil \frac{m}{2} \rceil$ for $n \geq m \geq 4$ [24]. For cycles of length 4: the $R(P_n, C_4)$ critical graphs and $r_*(P_n, C_4) = 3$ can be found in [23]; and the $R(C_4, C_n)$ critical graphs and

Table 3 Cycles versus complete graphs

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References	Notes
(C_n, K_3)	$2n-1$	[15]	$n+1$	[22]	CG
(C_n, K_4)	$3n-2$	[54]	$2n$	[22, 29]	CG
(C_n, K_5)	$4n-3$	[3]	$3n-1$	[13, 30]	CG
(C_n, K_6)	$5n-4$	[48]	$4n-2$	[27]	CG
(C_n, K_m)	$(n-1)(m-1)+1$	[41]	$(m-2)(n-1)+2$	[28]	

Table 4 Paths and cycles

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References	Notes
(P_n, P_m)	$n+\lfloor \frac{m}{2}+\rfloor-1$	[16]	$\lceil \frac{m}{2} \rceil$	[24]	CG
(P_n, C_4)	$n+1$	[1, 14]	3	[23]	CG
(C_4, C_n)	$n+1$	[32]	5	[53]	CG
(P_n, C_m), m even	$n+\frac{m}{s}-1$	[14]	$\frac{m}{2}+1$	[25]	CG
(P_n, C_m), m odd	$2n-1$	[14]	n	[25]	CG
(C_n, C_m), m odd	$2n-1$	[32]	$n+1$	[55]	

$r_*(C_4, C_n) = 5$ can be found in [53]. The path-cycle critical graphs are classified for all $n \geq m \geq 6$ and the star-critical Ramsey number has been found in [25]. For m even with $n \geq m \geq 6$, $r_*(P_n, C_m) = \frac{m}{2} + 1$. For m odd with $n \geq m \geq 3$, $r_*(P_n, C_m) = n$. The cycle-cycle star-critical Ramsey number is discussed in [55] where they establish a lower bound for special cases of $r_*(C_n, C_m)$ with m even, and for m odd with $n \geq m \geq 3$ (not both 3) they find $r_*(C_n, C_m) = n + 1$. See Table 4.

3.5 Fans

Let the n-fan be the join of a vertex with n copies of K_2, i.e. $F_n = K_1 + nK_2$. The critical graphs for fans versus triangles are classified in [34]. They show that, for $n \geq 2$, $r_*(F_n, K_3) = 2n + 2$. For $n \geq 4$, the $R(F_n, K_4)$ critical graphs are described in [17] and $r_*(F_n, K_4) = 4n + 2$. For $m \geq 2$ and $n \geq 2m + 1$, [52] proves that $R_{\mathbb{S}}(F_m, P_n) = n - m - 1$ which implies that $r_*(F_m, P_n) = n + m - 1$. A generalized fan $F_{t,n}$ is the graph $K_1 + nK_t$. In [18], they find the graph Ramsey number in addition to $r_*(K_3, F_{3,n}) = 3n + 3$. See Table 5.

Table 5 Fans

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References	Notes
(F_n, K_3)	$4n + 1$	[37]	$2n + 2$	[34]	CG
(F_n, K_4)	$6n + 1$	[21, 49]	$4n + 1$	[17]	CG
(F_m, P_n)	$2n - 1$	[47]	$n + m - 1$	[52]	
$(K_3, F_{3,n})$	$6n + 1$	[18]	$3n + 3$	[18]	

Table 6 Various classes of graphs

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References	Notes
(W_{n+1}, K_3)	$2n + 1$	[6]	$n + 3$	[22]	CG
$(K_{1,m}, P_n)$	n	[43]	m	[52]	
$(K_{1,m}, C_n)$	n	[33]	$m + 1$	[52]	
(B_m, P_n)	$2n - 1$	[46]	n	[52]	

3.6 Other

In this section, we list some known values for various classes of graphs including wheels $W_n = K_1 + C_{n-1}$, stars $K_{1,n}$, paths P_n, cycles C_n and books $B_m = K_2 + nK_1$. The classification of critical graphs for $R(W_{n+1}, K_3)$ can be found in [45]. And, $r_*(W_{n+1}, K_3) = n + 3$ for $n \geq 6$ is found in [22]. The following results are in [52]. For $n \geq 2m + 1$, $R_{\mathbb{S}}(K_{1,m}, P_n) = n - m - 1$ which implies $r_*(K_{1,m}, P_n) = m$. For $n \geq 2m + 2$, $R_{\mathbb{S}}(K_{1,m}, C_n) = n - m - 2$ which implies $r_*(K_{1,m}, C_n) = m + 1$. For $m \geq 4$ and $n \geq 2m + 1$, $R_{\mathbb{S}}(B_m, P_n) = n - 2$ which implies $r_*(B_m, P_n) = n$. See Table 6.

A multicolor result is discussed in [31] for a complete graph versus any number of matchings. They characterize the critical graphs for the graph Ramsey number and find the star-critical Ramsey number which is stated below.

Theorem 4 ([31]) *Let $n \geq 3$, $t \geq 2$, $n \geq t + 1$ and $m_1, \ldots, m_t$ be positive integers such that $m_1 = \max\{m_1, \ldots, m_t\}$. Then*

$$r_*(K_n, m_1K_2, \ldots, m_tK_2) = \begin{cases} \displaystyle\sum_{i=1}^{t-1}(2m_i - 1) + 1 & n = t + 1 \\ \displaystyle\sum_{i=1}^{t} 2(m_i - 1) + n - 1 & n \geq t + 2. \end{cases}$$

Table 7 For large n

(G, H)	$r(G, H)$	References	$r_*(G, H)$	References
$(K_1 + nK_t, K_{m+1})$	$mtn + 1$	[37]	$(m-1)tn + t$	[19]
$(K_p + nK_1, K_{m+1})$	$m(n + p - 1) + 1$	[42]	$(m - 1 + o(1))n$	[19]
$(C_{2t+1}, K_1 + nG)$	$2n\|G\| + 1$	[38]	$n\|G\| + \delta(G) - 1$	[35]
$(C_{2t+1}, K_s + nK_1)$	$2(n + s - 1) + 1$	[38]	$n + 2s - 1$	[35]

3.7 For Sufficiently Large Values

The star-critical Ramsey number for multiple copies of complete graphs where $k, l \geq 3$ and the values of m and n are large has been determined in [55] to be $r_*(mK_k, nK_l) = (k-1)m + (l-1)n + \max\{m, n\} + R(K_{k-1}, K_{l-1}) - 3$. It is also Ramsey-full.

For all sufficiently large n, [19, 35] prove results on generalized fans and books. Note that $K_1 + nK_t$ is the fan $F_{t,n}$ and $K_p + nK_1$ is $B_{p,n}$. Hao and Lin find the star-critical Ramsey number $r_*(K_1 + nK_t, K_{m+1}) = (m-1)tn + t$ where $m, t \geq 2$ and n is large. In addition, they prove for any fixed integers $p, m \geq 2$, $r_*(K_p + nK_1, K_{m+1}) = (m - 1 + o(1))n$ as $n \to \infty$. Li, Li, and Wang find $R_{\mathbb{S}}(C_{2t+1}, K_1 + nG) = n|G| - \delta(G) - 1$ and $R_{\mathbb{S}}(C_{2t+1}, K_s + nK_1) = n - 1$ for large n. These results are equivalent to $r_*(C_{2t+1}, K_1 + nG) = n|G| + \delta(G) - 1$ and $r_*(C_{2t+1}, K_s + nK_1) = n + 2s - 1$. See Table 7.

References

1. Burr, S.A., et al.: Some complete bipartite graph-tree Ramsey numbers. Ann. Discrete Math. **41**, 79–89 (1989)
2. Burr, S.A., Erdös, P., Spencer, J.H.: Ramsey theorems for multiple copies of graphs. Trans. Am. Math. Soc. **209**, 87–99 (1975)
3. Bollobäs, B., et al.: On a conjecture involving cycle-complete graph Ramsey numbers. Australasian J. Combinat. **22**, 63–71 (2000)
4. Burr, S.A.: Ramsey numbers involving graphs with long suspended paths. J. Lond. Math. Soc. **24**, 405–413 (1981)
5. Budden, M., DeJonge, E.: Destroying the Ramsey property by the removal of edges. Australasian J. Combinat. **74**(3), 476–485 (2019)
6. Burr, S.A., Erdös, P.: Generalizations of a Ramsey-theoretic result of Chvátal. J. Graph Theory **7**, 39–51 (1983)
7. Chvátal, V.: Tree-complete graph Ramsey numbers. J. Graph Theory **1**, 93 (1977)
8. Cockayne, E.J., Lorimer, P.J.: The Ramsey number for stripes. J. Australian Math. Soc., Ser. A **19**, 252–256 (1975)

9. Erdös, P., Faudree, R.J.: Size Ramsey functions. In: Sets, Graphs and Numbers, pp. 219–238 (1991)
10. Erdös, P., Faudree, R.J.: Size Ramsey numbers involving matchings. In: Finite and Infinite Sets, pp. 247–264 (1981)
11. Erdös, P., et al.: The size Ramsey number. Period. Math. Hungar. **9**, 145–161 (1978)
12. Erdös, P. et al.: The book-tree Ramsey numbers. In: Scientia. Series A: Mathematical Sciences, Valparaíso, Chile, vol. 1, pp. 111–117 (1988)
13. Ferreri, M., Liddy, J., Radziszowski, S.P.: All Ramsey-critical graphs and star-critical Ramsey numbers for cycles versus K_5. In: Preprint (2019). https://cs.rit.edu/~spr/PUBL/publ.html
14. Faudree, R.J., et al.: Path-cycle Ramsey numbers. Discrete Math. **10**, 269–277 (1974)
15. Faudree, R.J., Schelp, R.H.: All Ramsey numbers for cycles in graphs. Discrete Math. **8**, 313–329 (1974)
16. Gerencsér, L., Gyárfás, A.: On Ramsey-type problems. In: Annales Universitatis Scientiarum Budapestinensis, Eötvös Sect. Math. **10**, 167–170 (1967)
17. Haghi, Sh., Maimani, H.R., Seify, A.: Star-critical Ramsey number of F_n versus K_4. Discrete Appl. Math. **2017**, 203–209 (2017)
18. Hao, Y., Lin, Q.: Ramsey number of K_3 versus $F_{3,n}$. Discrete Appl. Math. **251**, 345–348 (2018)
19. Hao, Y., Lin, Q.: Star-critical Ramsey numbers for large generalized fans and books. Discrete Math. **341**, 3385–3393 (2018)
20. Harary, F.: Recent results on generalized ramsey theory for graphs. In: Alavi, Y., et al. (eds.) Graph Theory and Applications, pp. 125–138 (1972)
21. Hendry, G.R.T.: Ramsey numbers for graphs with five vertices. J. Graph Theory **13**, 245–248 (1989). https://doi.org/10.1002/jgt.3190130210
22. Hook, J.: The classification of critical graphs and star-critical Ramsey numbers. PhD thesis. Lehigh University (2010)
23. Hook, J., Issak, G.: Star-critical Ramsey numbers. Discrete Appl. Math. **159**, 328–334 (2011). https://doi.org/10.1016/j.dam.2010.11.007
24. Hook, J.: Critical graphs for $R(P_n, P_m)$ and the star-critical Ramsey number for paths. Discuss. Math. Graph Theory **35**, 689–701 (2015). https://doi.org/10.7151/dmgt.1827
25. Hook, J.: The path-cycle critical graphs and $r_*(P_n, C_m)$. In: Preprint (2021)
26. Jayawardene, C.J., Navaratna, W.C.W., Senadheera, J.N.: All Ramsey (C_n, K_6) critical graphs for large n. In: Preprint (2020). arXiv:1902.02646 [math.CO]
27. Jayawardene, C., Navaratna, W.C.W.: A note on star-critical Ramsey (C_n, K_6) numbers for large n. Ann. Pure Appl. Math. **20**, 9–12 (2019). https://doi.org/10.22457/apam.621v20n1a2
28. Jayawardene, C., Navaratna, W.C.W.: On star-critical Ramsey numbers related to large cycles versus complete graphs. In: Preprint (2020). arXiv:1903.10891 [math.CO]
29. Jayawardene, C.J., Narváez, D., Radziszowski, S.P.: Star-Critical Ramsey Numbers for Cycles versus K_4. Discuss. Math. Graph Theory **41**, 381–390 (2021)
30. Jayawardene, C.: Star-critical Ramsey numbers for cycles versus the complete graph on 5 vertices. In: Progress in Nonlinear Dynamics and Chaos, pp. 115–126 (2018). https://doi.org/10.22457/pindac.v6n2a6.
31. Kamranian, A., Raeisi, G.: Star-critical Ramsey numbers of matchings versus a complete graph. In: Proceedings of the 3rd International Conference on Combinatorics, Cryptography, and Computation (2018)
32. Károlyi, G., Rosta, V.: Generalized and geometric ramsey numbers for cycles. Theor. Comput. Sci. **263**, 87–98 (2001)
33. Lawrence, S.L.: Cycle-star Ramsey numbers. In: Notices of the American Mathematical Society, vol. 20, Abstract A-420 (1973)
34. Li, Z., Li, Y.: Some star-critical Ramsey numbers. Discrete Appl. Math. **181**, 301–305 (2015)
35. Li, Y., Li, Y., Wang, Y.: Minimal Ramsey graphs on deleting stars for generalized fans and books. In: Applied Mathematics and Computation 372 (2020). https://doi.org/10.1016/j.amc.2019.125006.
36. Li, Y., Li, Y., Wang, Y.: Complete bipartite graphs deleted in Ramsey graphs. Theoret. Comput. Sci. **840** (2020)

37. Li, Y., Rousseau, C.C.: Fan-complete graph Ramsey numbers. J. Graph Theory **23**, 413–420 (1996)
38. Liu, M., Li, Y.: Ramsey numbers of a fixed odd-cycle and gener- alized books and fans. Discrete Math. **339**, 2481–2489 (2016)
39. Lorimer, P.J.: The Ramsey number for stripes and one complete graph. J. Graph Theory **8**, 177–184 (1984)
40. Lorimer, P.J., Mullins, P.R.: Ramsey numbers for quadrangles and triangles. J. Combinat. Theory, Ser. B **23**, 262–265 (1977)
41. Nikiforov, V.: The cycle-complete graph Ramsey numbers. Combinator., Prob. Comput. **14**, 349–370 (2005)
42. Nikiforov, V., Rousseau, C.C.: Large generalized books are p-good. J. Combinat. Theory, Ser. B **92**, 85–97 (2004)
43. Parsons, T.D.: Path-star Ramsey numbers. J. Combinat. Theory. Ser. B **17**, 51–58 (1974)
44. Radziszowski, S.P.: Small Ramsey numbers. Electronic J. Combinat. Dynamic Surv. DS1 (2021)
45. Radziszowski, S.P., Jin, X.: Paths, cycles and wheels in graphs without antitriangles. Australasian J. Combinat. **9**, 221–232 (1994)
46. Rousseau, C.C., Sheehan, J.: A class of Ramsey problems involving trees. J. Lond. Math. Soc. **18**(2), 392–396 (1978)
47. Salman, A.N.M., Broersma, H.J.: Paths-fan Ramsey numbers. Discrete Appl. Math. **154**, 1429–1436 (2006)
48. Schiermeyer, I.: All cycle-complete graph Ramsey numbers $r(C_m, K_6)$. J. Graph Theory **44**, 251–260 (2003)
49. Baskoro Surahmat, E.T., Broersma, H.J.: The Ramsey numbers of fans versus K4. Bull. Inst. Combinat. Appl. **43**, 96–102 (2005)
50. Wang, Y., Li, Y.: Deleting edges from Ramsey graphs. Discrete Math. **343** (2020). https://doi.org/10.1016/j.disc.2019.111743
51. Wang, Y., Li, Y., Li, Y.: Maximum star deleted from Ramsey graphs of book and tree. Theoret. Comput. Sci. **836** (2020)
52. Wang, Y., Li, Y., Li, Y.: Star-critical Ramsey numbers involving graphs with long suspended paths. Discrete Math. **344** (2021)
53. Wu, Y., Sun, Y., Radziszowski, S.P.: Wheel and star-critical Ramsey numbers for quadrilaterals. Discrete Appl. Math. **186**, 260–271 (2015)
54. Sheng, Y.J., Huang, Y.R., Zhang, K.M.: The value of the Ramsey number $R(C_n, K_4)$ is $3(n-1)+1 (n \geq 4)$. Australasian J. Combinat. **20**, 205–206 (1999)
55. Zhang, Y., Broersma, H., Chen, Y.: On star-critical and up- per size Ramsey numbers. Discrete Appl. Math. **202**, 174–180 (2016)

The Zero Forcing Span of a Graph

Bonnie Jacob

Abstract In zero forcing, the focus is typically on finding the minimum cardinality of any zero forcing set in the graph; however, the number of cardinalities between 0 and the number of vertices in the graph for which there are both zero forcing sets and sets that fail to be zero forcing sets is not well known. In this paper, we introduce the zero forcing span of a graph, which is the number of distinct cardinalities for which there are sets that are zero forcing sets and sets that are not. We introduce the span within the context of standard zero forcing and skew zero forcing as well as for standard zero forcing on directed graphs. We characterize graphs with high span and low span of each type, and also investigate graphs with special zero forcing polynomials.

Keywords Zero forcing · Failed zero forcing · Zero forcing polynomial

MSC2020: 05C50

1 Introduction

Throughout this paper, we use G to denote a finite, simple graph on $n = |V(G)|$ vertices with edge set $E(G)$. We use D to denote a directed graph, or digraph, with vertex set $V(D)$ and arc set $E(D)$. Like in our undirected graphs, hereafter simply "graphs," we do not allow loops or multiple arcs in our digraphs, though between a pair of vertices there may be an arc in each direction (from vertex u to vertex v and from vertex v to vertex u, for example). When the graph is understood, we use V in place of $V(D)$ or $V(G)$. Unless otherwise stated, we use $n = |V(G)|$ and call $|V(G)|$ the *order* of the graph.

For any $v \in V(G)$, the *open neighborhood* of v denoted $N(v)$ is the set of vertices adjacent to v. In a digraph, the *open in-neighborhood* of v denoted $N^-(v)$ is the set

B. Jacob (✉)
National Technical Institute for the Deaf, Rochester Institute of Technology, Rochester, NY, USA
e-mail: bcjntm@rit.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_23

of vertices from which there is an arc to v, and the *open out-neighborhood* denoted $N^+(v)$ the set of vertices to which there is an arc from v. A *neighbor* of v is a vertex in the open neighborhood of v, with analogous definitions for in- and out-neighbor.

Zero forcing is a process that consists of designating a subset $S \subseteq V(G)$ as blue, and the remaining vertices as white. A color change rule of varying forms is then applied. If repeated applications of the color change rule results in all vertices eventually turning blue, the original set is called a *zero forcing set*.

In this paper, we use three different color change rules, and define those here.

1. The *standard color change rule* (standard zero forcing): if any blue vertex has exactly one white neighbor, then the white neighbor becomes blue.
2. The *skew color change rule* (skew zero forcing): if any vertex (blue or white) has exactly one white neighbor, then the white neighbor becomes blue.
3. The *(standard) digraph color change rule* (standard zero forcing on digraphs): if any blue vertex has exactly one white out-neighbor, then the white out-neighbor becomes blue.

Under each of these rules, the minimum cardinality of any starting set of blue vertices that eventually results in the entire graph turning blue is called the *zero forcing number*, first formally introduced in [2] (the *skew zero forcing number* introduced in [9], or the *digraph zero forcing number* introduced in [4] respectively) and is denoted $\mathrm{Z}(G)$ (or $\mathrm{Z}^-(G)$, or $\mathrm{Z}(D)$). The maximum cardinality of any starting set of blue vertices that never results in the entire graph turning blue is called the *failed zero forcing number* introduced in [7] (the *failed skew zero forcing number* introduced in [3], or the *digraph failed zero forcing number* introduced in [1] respectively) and is denoted $\mathrm{F}(G)$ (or $\mathrm{F}^-(G)$, or $\mathrm{F}(D)$).

Given a graph G, the *zero forcing span* $\lambda(G)$ is the number of distinct cardinalities, $k_1, k_2, \ldots k_{\lambda(G)}$ such that for each i, $1 \leq i \leq \lambda(G)$, there exist sets $Z, F \subseteq V(G)$ with $|Z| = |F| = k_i$, where Z is a zero forcing set and F is not. We define $\lambda^-(G)$ analogously, but where Z is a skew zero forcing set and F is not, and $\lambda(D)$ as well, but where D is a digraph and the digraph color change rule is applied.

Note that $\lambda(G)$ denotes how much bigger than $\mathrm{Z}(G)$ a set must be to guarantee that it must be a zero forcing set, without regard to which vertices are in the set.

2 Motivation

The concept of zero forcing span has connections with several other problems in the literature. First, we look at the connection to linear algebra. With G we associate a set of symmetric matrices denoted $\mathcal{S}(G)$. Number the vertices $1, 2, \ldots, n$ and define $\mathcal{S}(G)$ as follows.

$$\mathcal{S}(G) = \left\{A \in \mathbb{R}^{n\times n} \ : \ A^T = A \ \text{ and for } i \neq j, a_{ij} \neq 0 \text{ if and only if } ij \notin E(G)\right\}.$$

Note that the diagonal is unconstrained.

By applying [2, Proposition 2.3], we find that the zero forcing span $\lambda(G)$ is the number of values k such that both of the following conditions hold.

1. There exists a set $S \subseteq \{1, 2, \ldots, |V(G)|\}$ with $|S| = k$ such that for any $A \in \mathcal{S}(G)$, if $v \in \ker(A)$ with $v_i = 0$ for all $i \in S$, then $v = 0$, but
2. there exists $A \in \mathcal{S}(G)$ with $v \in \ker(A)$, $v \neq 0$ and some k entries of v that are all 0.

We can also relate the zero forcing span to the *zero forcing polynomial* $\mathcal{Z}(G; x) = \sum_{i=1}^{n} z(G; i)x^i$, introduced in [5], where $z(G; i)$ is the number of zero forcing sets of cardinality i. Then $\lambda(G)$ gives the number of terms where $0 < z(G; i) < \binom{n}{i}$, that is, the number of terms in the polynomial that are not simply 0 or $\binom{n}{i}$ (the minimum or maximum possible for each coefficient in the polynomial).

While not yet explicitly defined in the literature, we can define analogous zero forcing polynomials for variants of zero forcing. Let the *skew zero forcing polynomial* be $\mathcal{Z}^-(G; x) = \sum_{i=0}^{n} z^-(G; i)x^i$ where $z^-(G; i)$ is the number of skew zero forcing sets of cardinality i, and the *directed zero forcing polynomial* be $\mathcal{Z}^{\mathcal{D}}(D; x) = \sum_{i=1}^{n} z^D(D; i)x^i$ where $z^D(D; i)$ is the number of zero forcing sets of cardinality i in a digraph D. Note that, unlike standard zero forcing, the skew zero forcing polynomial may have $z(G; 0) > 0$.

3 Results

3.1 Extreme Values of $\lambda(G)$

We note the following formula for $\lambda(G)$ in terms of $\mathrm{F}(G)$ and $\mathrm{Z}(G)$. The equivalent statement holds for each type of zero forcing.

Observation 1 $\lambda(G) = \mathrm{F}(G) - \mathrm{Z}(G) + 1$.

Recall that $\mathrm{Z}(G) \geq 1$ for any undirected graph G. Also recall that $\mathrm{F}(G) \leq n - 1$ and that $\mathrm{F}(G) \geq \mathrm{Z}(G) - 1$. The same statements hold for a digraph D, which give us the following trivial bounds.

Observation 2 For a graph G and digraph D, $0 \leq \lambda(G) \leq n - 1$ and $0 \leq \lambda(D) \leq n - 1$.

In the skew case, there exist graphs for which $\mathrm{Z}^-(G) = 0$, implying the following observation.

Observation 3 $0 \leq \lambda^-(G) \leq n$

Characterizations of Graphs with Zero Forcing Span of 0

Lemma 4 *The following are equivalent:*

1. $\lambda(G) = 0$
2. $\mathrm{F}(G) < \mathrm{Z}(G)$
3. $\mathrm{Z}(G) = \mathrm{F}(G) + 1$

The same equivalence holds if the graph G is replaced by a digraph D.

Proof By Observation 1, $\lambda(G) = 0$ if and only if $\mathrm{Z}(G) = \mathrm{F}(G) + 1$. Since $\mathrm{F}(G) \geq \mathrm{Z}(G) - 1$ (as noted in [7] or by observing that any set of vertices must be a failed zero forcing set or a zero forcing set), the equivalence holds. □

For skew zero forcing, since $\mathrm{F}^-(G)$ is not always defined, we have the following list of equivalences. The proof is identical to that of Lemma 4 but with the addition of the possibility that we may have $\mathrm{Z}^-(G) = 0$, which is equivalent to $\mathrm{F}^-(G)$ being undefined.

Lemma 5 *The following are equivalent:*

1. $\lambda^-(G) = 0$
2. $F^-(G) < Z^-(G)$ *or* $Z^-(G) = 0$.
3. $Z^-(G) = F^-(G) + 1$ *or* $F^-(G)$ *is undefined.*

In [7], it was established that $\mathrm{F}(G) < \mathrm{Z}(G)$ if and only if $G = K_n$ or $G = \overline{K_n}$, leading to the following characterization of graphs with $\lambda(G) = 0$.

Theorem 6 $\lambda(G) = 0$ *if and only if* $G = K_n$ *or* $G = \overline{K_n}$

Proof We have that $\mathrm{F}(G) = \mathrm{Z}(G) - 1$ if and only if $G = K_n$ or $G = \overline{K_n}$ [7]. By Lemma 4, the result follows. □

For skew zero forcing, we need a few definitions to characterize graphs with $\lambda^-(G) = 0$. In [3], we established that $\mathrm{F}^-(G) < \mathrm{Z}^-(G)$ if and only if G is an odd cycle or nonempty set of cycles intersecting in a single vertex, or a doubly extended bouquet-dipole, pictured in Fig. 1 on the left.

Definition 7 We call a graph G a *doubly extended bouquet-dipole* if it consists of vertices u and v that are each on a nonempty set of odd cycles, where all other vertices on the cycles have degree two, and u, v are joined by a path of even order that alternates between single even order paths whose internal vertices all have degree two, and multiple even order paths whose internal vertices all have degree two.

In addition, there exist graphs that have $\mathrm{Z}^-(G) = 0$ and therefore $\mathrm{F}^-(G)$ is undefined, specifically two-set perfectly orderable graphs.

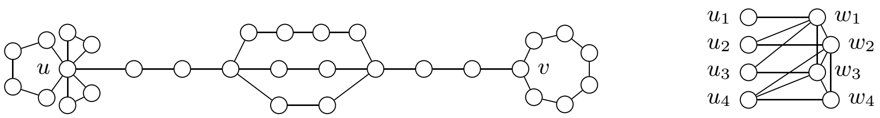


Fig. 1 Two graphs that have $\lambda^-(G) = 0$: a doubly extended bouquet dipole graph on the left and a two-set perfectly orderable graph on the right

Definition 8 We say that a graph G is *two-set perfectly orderable* if

1. $V(G)$ can be partitioned into ordered sets, $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_m\}$ such that $u_i v \in E(G)$ only if $v = w_j$ where $j \leq i$, and
2. $u_i w_i \in E(G)$ for all $i \in \{1, 2, \ldots, m\}$.

This gives us the following characterization of graphs with $\lambda^-(G) = 0$.

Theorem 9 $\lambda^-(G) = 0$ *if and only if G is one of the following graphs.*

1. K_n
2. $\overline{K_n}$
3. *a doubly extended bouquet-dipole graph.*
4. *a collection of one or more odd cycles that intersect in exactly one vertex.*
5. *a two-set perfectly orderable graph.*

Proof In [3] the graphs with $\mathrm{Z}^-(G) < \mathrm{F}^-(G)$ were characterized and are precisely Graphs 1–4. Graphs with $\mathrm{Z}^-(G) = 0$ and $\mathrm{F}^-(G)$ undefined were characterized in the same paper as Graph 5. By Lemma 5, the results holds. □

Since we've seen that two-set perfectly orderable graphs not only have $\lambda^-(G) = 0$, but they also are precisely the graphs with $\mathrm{Z}^-(G) = 0$, we have the following statement about their skew zero forcing polynomials.

Corollary 10 *The only graphs with skew zero forcing polynomial* $\mathcal{Z}^-(G; x) = \sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n$ *are two-set perfectly orderable graphs.*

Finally, we characterize digraphs with $\lambda(D) = 0$.

Theorem 11 *A digraph D has* $\lambda(D) = 0$ *if and only if D is one of the following.*

1. *a directed cycle.*
2. *a regular tournament on 5 vertices.*
3. *a digraph obtained from* K_n *by removing the arcs of*
   (a) *a collection of vertex-disjoint directed cycles each of length at least 3 that span V* ($n \geq 3$)*,*
   (b) *a collection of vertex-disjoint directed cycles each of length at least 3 that span* $V \backslash \{v\}$ *for some* $v \in V$ ($n \geq 4$)*, or*
   (c) vu *for some* $u, v \in V$ *and a collection of vertex-disjoint directed cycles each of length at least 3 that span* $V \backslash \{v\}$ ($n \geq 4$)*.*

4. *a digraph obtained from* $K_{n-1}\overrightarrow{\vee}\{v\}$ *by removing the arcs of a collection of vertex-disjoint directed cycles each of length at least 3 that span* K_{n-1} $(n \geq 4)$.
5. $K_j\overrightarrow{\vee}\overline{K_\ell}$ *where* $j \geq 2$ *and* $\ell \geq 0$.
6. $\overline{K_n}$.

Proof In [1], the list of graphs in the statement of this theorem were shown to be the graphs that have $\mathrm{Z}(D) < \mathrm{F}(D)$. We then apply Lemma 4. □

Comments on Graphs with Standard Zero Forcing Span of 1

We provide here a list of graphs that have $\lambda(G) = 1$. We have neither a characterization, nor any reason to believe the list of graphs is complete. First, we note an immediate but notable property of graphs with $\lambda(G) = 1$, and introduce a few definitions.

Observation 12 $\lambda(G) = 1$ if and only if $\mathrm{Z}(G) = \mathrm{F}(G)$.

Recall that a *module* $S \subseteq V(G)$ is a set of vertices such that for any vertex $u \notin S$, either $uv \in E(G)$ for all $v \in S$, or $uv \notin E(G)$ for any $v \in S$. In [7] it was shown that $\mathrm{F}(G) = n - 2$ if and only if G has a module of order 2. The *union* of graphs G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \vee H$ is simply $G \cup H$ with the addition of an edge between v_G and v_H for every $v_G \in V(G)$ and every $v_H \in V(H)$.

Proposition 13 *The following graphs have* $\lambda(G) = 1$.

1. *a complete multipartite graph* $K_{n_1,n_2,\dots n_k}$ *where* $n_1 \geq 2$.
2. $K_m \cup K_n$ *where* $m, n \geq 2$.
3. $K_n \backslash M$ *where* $n \geq 3$ *and* M *is a nonempty matching.*
4. $K_n \vee \overline{K_m}$ *where* $m \geq 2$.
5. *a path on* 4 *vertices.*

Proof For the complete multipartite graph, Item 1, note that any pair of vertices in a single partite set forms a module of order 2, so $F(G) = n - 2$. For any $S \subseteq V(G)$ with $|S| \leq n - 3$, S is a failed zero forcing set since if two of the vertices in $V\backslash S$ are in a single partite set, then S is a failed zero forcing set, and otherwise, each vertex of $V\backslash S$ is in a distinct partite set, and each vertex in S is adjacent to at least two vertices in $V\backslash S$. Take $u \in V(P_1)$ where P_1 is a partite set with at least two vertices, and $v \in P_2$ where P_2 is any other partite set. Then $V\backslash\{u, v\}$ is a zero forcing set, giving us that $\mathrm{F}(G) = \mathrm{Z}(G) = n - 2$, and $\lambda(G) = 1$.

For $K_m \cup K_n$, Item 2, note that any pair of vertices in the same component form a module of order 2. Thus, $F(G) = n + m - 2$. For S to be a zero forcing set, it must contain $m - 1$ vertices of K_m and $n - 1$ vertices of K_n, giving us that $\mathrm{Z}(G) = n + m - 2$ as well.

For Item 3, $G = K_n\backslash M$ where M is a matching, note that the endpoints of any edge in the matching form a module of order 2, so $\mathrm{F}(G) = n - 2$. However, $\mathrm{Z}(G) = n - 2$

as well, since taking $S = V \backslash \{u, v\}$ for any $u, v \in V$ where uv is not an edge in the matching forms a zero forcing set, and any set of S' with $|S'| \leq n - 3$ is not a zero forcing set since for any $v \in S'$, v is adjacent to at least two vertices in $V \backslash S'$.

For Item 4, note that any pair of vertices in either K_n or $\overline{K_m}$ forms a module of order two, giving us $\mathrm{F}(G) = n - 2$. Since any set $S \subseteq V(K_n \vee \overline{K_m})$ with $|S| \leq n - 3$ has at least two vertices missing from K_n or $\overline{K_m}$, we have that $\mathrm{Z}(G) \geq n - 2$. By picking a set S' with $S' = V(K_n \vee \overline{K_m}) \backslash \{u, v\}$ where $u \in K_n$ and $v \in \overline{K_m}$ we see $\mathrm{Z}(G) = n - 2 = \mathrm{F}(G)$.

For Item 5, note that $\mathrm{Z}(P_4) = \mathrm{F}(P_4) = 1$. $\square$

Characterizations of Graphs with High Zero Forcing Spans

We now characterize graphs and digraphs with high zero forcing spans. Specifically, we characterize graphs that have $\lambda(G) \geq n - 3$ and digraphs that have $\lambda(D) \geq n - 2$. For skew zero forcing, we show that $\lambda^-(G) \neq n$ for any graph G and characterize graphs that have $\lambda^-(G) = n - 1$.

Theorem 14 *Graphs with $\lambda(G) = n - 1$ or $\lambda(G) = n - 2$ can be characterized as follows.*

$$\lambda(G) = \begin{cases} n - 1 \textit{ if and only if } G = K_1 \\ n - 2 \textit{ if and only if } G = P_{n-1} \cup K_1 \textit{ or } G = P_3 \end{cases}$$

Proof For $\lambda(G) = n - 1$, we must have $\mathrm{F}(G) = n - 1$ and $\mathrm{Z}(G) = 1$. From [7], $\mathrm{F}(G) = n - 1$ gives us that G has an isolated vertex. It is well known that $\mathrm{Z}(G) = 1$ if and only if G is a path. Thus, $G = K_1$.

If $\lambda(G) = n - 2$, then either $\mathrm{F}(G) = n - 1$ and $\mathrm{Z}(G) = 2$, or $\mathrm{F}(G) = n - 2$ and $\mathrm{Z}(G) = 1$. In the first case, $\mathrm{F}(G) = n - 1$ implies that G has an isolated vertex. Since G has an isolated vertex with $\mathrm{Z}(G) = 2$, we must have that the other component of G is a path, giving us a path and a single isolated vertex. Note that we can construct a failed zero forcing set of G with $n - 1$ vertices by taking all vertices but the isolated vertex, and a zero forcing set with 2 vertices by taking one end vertex of the path along with the isolated vertex.

If $\mathrm{F}(G) = n - 2$ and $\mathrm{Z}(G) = 1$, we have that G must be a path, but from [7] that there are two pendant vertices since G is a tree with $\mathrm{F}(G) = n - 2$, giving us that $G = P_3$. Note the middle vertex forms a failed zero forcing set of maximum order, and the end vertex a zero forcing set of minimum order. $\square$

We pause here to recall definitions that are essential in some of the characterizations below.

Definition 15 We say that G is a graph of *two parallel paths* if G itself is not a path, and $V(G)$ can be partitioned into subsets V_1 and V_2 such that the subgraphs induced by V_1 and V_2 are paths, and G can be drawn in the plane so that the paths induced by V_1 and V_2 are parallel line segments, and edges between V_1 and V_2 can be drawn as straight line segments that do not cross. Such a drawing is known as a *standard drawing*.

Lemma 16 *A graph G that consists of two parallel paths has a module of order 2 if and only if any standard drawing of G consists of one of the following:*

1. $P_1 \cup P_2$ *or* $P_1 \cup P_1$.
2. $P_1 = \{x\}$ *and* P_k *where* $k \geq 3$, *and*
   (a) *for some* u, v, w *that form a subpath of* P_k, $N(x) = \{u, w\}$ *or* $N(x) = \{u, v, w\}$, *or*
   (b) *for an end vertex* v *of* P_k *with neighbor* w, $N(x) = \{w\}$ *or* $N(x) = \{v, w\}$.
3. $P_2 = uv$ *and* P_k *with* $N(u) \cap V(P_k) = N(v) \cap V(P_k)$.
4. $P_3 = uvw$ *and* P_k *where* $N(u) = N(w) = \{v\}$ *or* $N(u) = N(w) = \{v, x\}$ *for some* x *on* P_k.
5. $P_k = v_1v_2v_3v_4 \cdots v_k$ *and* $P_j = w_1w_2w_3 \cdots w_j$ *where* $k, j \geq 2$, *and the edges between* P_k *and* P_j *are one of* $\{v_{k-1}w_1, v_kw_2\}$, $\{v_{k-1}w_1, v_kw_2, v_{k-1}w_2\}$, *or* $\{v_{k-1}w_1, v_kw_2, v_kw_1\}$.

Proof Note that in Item 1, if $G = P_1 \cup P_2$, then $V(P_2)$ forms a module of order 2; if $G = P_1 \cup P_1$ then $V(G)$ itself is a module of order 2. In Item 2, $\{v, x\}$ forms a module of order 2. For Item 3, $\{u, v\}$ forms a module of order 2. For Item 4, $\{u, w\}$ forms a module of order 2. For Item 5, $\{v_k, w_1\}$ forms a module of order 2.

Now assume that G is two parallel paths and has a module of order 2. We will show that G is one of the graphs described in Items 1 through 5. Consider a standard drawing of G. Call the two paths P_k and P_j.

First, assume that $k, j \geq 4$, and that $\{u, v\}$ is a module of order 2. Note that we cannot have that $u, v \in V(P_k)$ (without loss of generality) because then u, v will have different neighbors along P_k. Hence, we must have that $u \in V(P_k)$ and $v \in V(P_j)$. If u is an interval vertex in P_k, then v is adjacent to both vertices that are adjacent to u along P_k, and there is also an edge between u and the vertex (or vertices) adjacent to v; this edge will cross one of the edges from v to the neighbors of u which contradicts the definition of parallel paths. Thus we must have that u is an end vertex of P_k and v is an end vertex of P_j.

Recall that we're considering a standard drawing, $P_k = v_1v_2v_3v_4 \cdots v_k$ and $P_j = w_1w_2w_3 \cdots w_j$. Note that we cannot have that $u = v_1$ and $v = w_1$, or $u = v_k$ and $v = w_j$, since then the edge from u to the neighbor of v will cross the edge from v to the neighbor of u. Thus, without loss of generality, we have $u = v_k$ and $v = w_1$. To satisfy $\{u, v\}$ being a module of order 2, we then have that $v_{k-1}v \in E(G)$ and $w_2u \in E(G)$. That is, v_{k-1}, v, w_2, u form a C_4. Note then that we may have $uv \in E(G)$ or $v_{k-1}w_2 \in E(G)$, but not both, since the edges would cross, giving us Item 5.

We now consider $j = 3$, so $P_j = uvw$. Note that along P_j, $N(u) = N(w) = \{v\}$. For $\{u, w\}$ to form a module of order 2, we must have that u and w have the same neighbors in P_k as well. Note that if u and w have more than one neighbor in P_k, then we will have crossed edges between P_k and P_j. Hence, u and w have at most one neighbor in P_k, giving us Item 4. By the same arguments we made for the case when $j \geq 4$, the only other possibility for $j = 3$ is if the graph satisfies Item 5.

If $j = 2$, let $P_2 = uv$. For the case $k = 1$, let $V(P_1) = \{x\}$. Note either $ux, vx \in E(G)$ or $ux, vx \notin E(G)$ satisfying Item 1 or 3. If $k \geq 2$, note that for $\{u, v\}$ to be a module of order 2, then they must have the same neighborhood in $V(P_k)$, satisfying Item 2. Note that by the definition of parallel paths, $|N(u) \cap V(P_k)| = |N(v) \cap V(P_k)| \in \{0, 1\}$, else edges from u and v to their neighbors in P_k would cross. If u, v do not form a module, then without loss of generality $\{u, w\}$ form a module of order 2 for some $w \in V(P_j)$. By the same arguments for the cases $k, j \geq 4$, G must satisfy Item 5.

Finally, suppose $j = 1$. If $k = 1$, we have Item 1. If $k = 2$, we have the same situation just described for $j = 2$ and $k = 1$. If $k = 3$, we must have Item 4 by the arguments for the case $j = 3$. If $k = 4$, note that no two vertices of P_k can form a module of order 2. Thus we must have that $\{x, v\}$ form a module of order 2 where $\{x\} = V(P_j)$ and $v \in V(P_k)$. Then x must be adjacent to $N(v)$, and may be adjacent to v as well, giving us Item 2. $\square$

If $S \subseteq V(G)$, then we denote the subgraph of G induced by S by $G[S]$. We now characterize graphs with zero forcing span of $n - 3$.

Theorem 17 *$\lambda(G) = n - 3$ if and only if G is one of the following graphs.*

1. *two parallel paths with an additional K_1 component.*
2. *P_4 or P_5.*
3. *two parallel paths such that any standard drawing has one of the following forms:*
   - (a) *$P_1 = \{x\}$ and P_k where $k \geq 3$, and*
     - i. *for some u, v, w that form a subpath of P_k, $N(x) = \{u, w\}$ or $N(x) = \{u, v, w\}$, or*
     - ii. *for an end vertex v with neighbor w, $N(x) = \{w\}$ or $N(x) = \{v, w\}$.*
   - (b) *$P_2 = uv$ and P_k with $N(u) \cap V(P_k) = N(v) \cap V(P_k)$, and if $k = 1$, then $\{ux, vx\} \in E(G)$ where $x = V(P_1)$.*
   - (c) *$P_3 = uvw$ and P_k where $N(u) = N(w) = \{v\}$ or $N(u) = N(w) = \{v, x\}$ for some x on P_k.*
   - (d) *$P_k = v_1v_2v_3v_4 \cdots v_k$ and $P_j = w_1w_2w_3 \cdots w_j$ where $k, j \geq 2$, and the edges between P_k and P_j are one of $\{v_{k-1}w_1, v_kw_2\}$, $\{v_{k-1}w_1, v_kw_2, v_{k-1}w_2\}$, or $\{v_{k-1}w_1, v_kw_2, v_kw_1\}$.*

Proof Using Observation 1, we know that $\lambda(G) = n - 3$ implies that (I) $\mathrm{F}(G) = n - 1$ and $\mathrm{Z}(G) = 3$, (II) $\mathrm{F}(G) = n - 2$ and $\mathrm{Z}(G) = 2$, or (III) $\mathrm{F}(G) = n - 3$ and $\mathrm{Z}(G) = 1$.

For (I) we know that $\mathrm{F}(G) = n - 1$ if and only if G contains an isolated vertex [7], giving us that (I) is satisfied if and only if G has an isolated vertex v and $\mathrm{Z}(G[V(G)\backslash\{v\}]) = 2$. By [6], $\mathrm{Z}(G[V(G)\backslash\{v\}]) = 2$ if and only if $G[V(G)\backslash\{v\}]$ is two parallel paths. Hence, we have that $\mathrm{F}(G) = n - 1$ and $\mathrm{Z}(G) = 3$ if and only if Item 1 holds.

For (II), we know that $\mathrm{F}(G) = n - 2$ if and only if G has a module of order 2 [7] and no isolated vertices, and $\mathrm{Z}(G) = 2$ if and only if G is two parallel paths. Hence, (II) holds if and only if G is two parallel paths with a module of order 2 and no isolated vertices. By Lemma 16, then (II) holds if and only if G satisfies Item 3.

For (III), $\mathrm{Z}(G) = 1$ if and only if G is a path. From [7], $\mathrm{F}(P_n) = n - 3$ if and only if $n = 4$ or $n = 5$. Hence, we have that $\mathrm{F}(G) = n - 3$ and $\mathrm{Z}(G) = 1$ if and only if Item 2 holds. □

We now characterize graphs of highest possible skew zero forcing span, and improve the trivial bound from Observation 3 to a tight bound.

Theorem 18 $\lambda^-(G) = n - 1$ *if and only if G consists of a (possibly empty) two-set perfectly orderable graph and an isolated vertex. Also,* $\lambda^-(G) \leq n - 1$.

Proof By Observation 1, $\lambda^-(G) = n - 1$ implies that $\mathrm{F}^-(G) = n - 1$ and $\mathrm{Z}^-(G) = 1$, or $\mathrm{F}^-(G) = n - 2$ and $\mathrm{Z}^-(G) = 0$. If $\mathrm{Z}^-(G) = 0$, then $\mathrm{F}^-(G)$ is undefined, meaning that the only feasible case is that $\mathrm{F}^-(G) = n - 1$ and $\mathrm{Z}^-(G) = 1$. From [3], $\mathrm{F}^-(G) = n - 1$ if and only if G has an isolated vertex, v. Note that one possibility is that $V(G) = \{v\}$. Otherwise, since $\mathrm{Z}^-(G) = 1$, and any zero forcing set must contain v, we have that $\mathrm{Z}^-(H) = 0$ where H is the subgraph induced by $V(G)\backslash\{v\}$. Thus, H must be a two-set perfectly orderable graph.

We noted above that $\lambda^-(G) \leq n$. If $\lambda^-(G) = n$, then $\mathrm{F}^-(G) = n - 1$ and $\mathrm{Z}^-(G) = 0$. This gives us that G is a two-set perfectly orderable graph with an isolated vertex, which is a contradiction since no two-set perfectly orderable graph has an isolated vertex. Hence $\lambda^-(G) \leq n - 1$. □

Finally, we turn to high zero forcing spans of digraphs. First, we recall some definitions.

A *source* $v \in V(D)$ has $N^-(v) = \varnothing$. A path $P = (v_1, v_2, \ldots, v_k)$ is *Hessenberg* if $E(D)$ does not contain any arc of the form (v_i, v_j) with $j > i + 1$. A simple digraph D is a *digraph of two parallel Hessenberg paths* if D is not a Hessenberg path, $V(D) = \{u_1, \ldots u_r, v_1, \ldots, v_s\}$ where $(u_1, \ldots, u_r)$ and $(v_1, \ldots v_s)$ are nonempty Hessenberg paths, and there do not exist, i, j, k, ℓ with $i < j$ and $k < \ell$ such that $\big\{(u_k, v_j), (v_i, k_\ell)\big\} \subseteq E(D)$. That is, there are no pairs of forward crossing arcs.

Theorem 19 $\lambda(D) = n - 1$ *if and only if D is a Hessenberg path* $(v_1, v_2, \ldots, v_n)$ *such that v_1 is a source.*

Proof From [8], $\mathrm{Z}(D) = 1$ if and only if D is a Hessenberg path. From [1], $\mathrm{F}(D) = n - 1$ if and only if D has a source. Thus, $\lambda(D) = n - 1$ if and only if D is a Hessenberg path with a source. The only vertex in a Hessenberg path $(v_1, \ldots, v_n)$ that can be a source is v_1, since $v_{i-1} \in N^-(v_i)$ for any v_i with $i > 1$. Taking any Hessenberg path such that v_1 has no in-neighbors gives us a Hessenberg path with a source, v_1. □

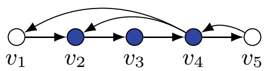


Fig. 2 A digraph with $\lambda(D) = n - 2$

Theorem 20 $\lambda(D) = n - 2$ *if and only if* D *is one of*

1. *two parallel Hessenberg paths* $(u_1, u_2, \ldots, u_k)$ *and* $(v_1, v_2, \ldots, v_j)$ *such that* u_1 *or* v_1 *is a source, or*
2. *a single Hessenberg path* $(v_1, \ldots, v_n)$ *with* $N^-(v_1) = N^-(v_{i+1}) = \{v_i\}$ *for some* i*,* $1 < i < n$*.*

Proof Note that from Observation 1, $\lambda(D) = n - 2$ if and only if either $Z(D) = 1$ and $F(D) = n - 2$, or $Z(D) = 2$ and $F(D) = n - 1$. From [4], $Z(D) = 1$ if and only if D is a Hessenberg path, and $Z(D) = 2$ if and only if D consists of two parallel Hessenberg paths. From [1], $F(D) = n - 1$ if and only if D has a source, and $F(D) = n - 2$ if and only if D has no source and there exist $u, v \in V(D)$ with $N^-(u) \backslash \{v\} = N^-(v) \backslash \{u\}$.

Noting that only the first vertex in a Hessenberg path can be a source, we have $\lambda(D) = n - 2$ if and only if either Item 1 holds, or D is a Hessenberg path with no source and there exist $u, v \in V(D)$ with $N^-(u) \backslash \{v\} = N^-(v) \backslash \{u\}$. By the definition of a Hessenberg path, $N^-(u) \backslash \{v\} = N^-(v) \backslash \{u\}$ is true if and only if $u = v_1$ and $v = v_{i+1}$ where $1 < i < n$, and $N^-(v_1) = N^-(v_{i+1}) = \{v_i\}$, Item 2. □

The graph shown in Fig. 2 is a Hessenberg path with $\lambda(D) = n - 2$, since the blue vertices represent a failed zero forcing set S with $|S| = n - 2$, and $\{v_1\}$ forms a zero forcing set of order 1.

4 Zero Forcing Span Characteristics for Some Graphs

In this section, we look at the spans of trees, graphs with two or more components, and Cartesian products.

Proposition 21 *For a tree* T *on four or more vertices,* $1 \leq \lambda(T) \leq n - 3$ *and these bounds are sharp.*

Proof For any tree T, by Theorems 6 and 14, $\lambda(T) \in \{0, n - 2, n - 1\}$ if and only if T is K_1, K_2, or P_3. Thus, if $|V(T)| \geq 4$, we have $1 \leq \lambda(T) \leq n - 3$.

From Proposition 13, for the star $T = K_{m,1}$ with $m \geq 2$, we see that $\lambda(K_{m,1}) = 1$, showing sharpness of the lower bound.

From Theorem 17, if T consists of a path on at least two vertices with two pendant vertices on one of its end vertices, we see that $\lambda(T) = n - 3$. Hence the bounds are sharp. □

For a disconnected graph, we display a formula for its span in terms of the zero forcing numbers, orders, and failed zero forcing numbers of its components.

Proposition 22 *If G is a graph with at least two components: $G_1, G_2, \ldots, G_k$, then*

$$\lambda(G) = |V(G)| + \max_{1 \le i \le k} (\mathrm{F}(G_i) - |V(G_i)|) - \left(\sum_{i=1}^{k} \mathrm{Z}(G_i)\right) + 1$$

Proof From [7], $\mathrm{F}(G) = |V(G)| + \max_{1 \le i \le k} (\mathrm{F}(G_i) - |V(G_i)|)$. For $S \subseteq V(G)$ to be a zero forcing set, $S \cap V(G_i)$ must be a zero forcing set for each i, $1 \le i \le k$, giving us that $\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i)$. Applying the formula from Observation 1 completes the result. □

We provide a bound on the zero forcing span of the Cartesian product of graphs G and H, $G \square H$, in terms of the orders of G and H and their zero forcing and failed zero forcing numbers.

Proposition 23 *Let $G \square H$ denote the Cartesian product of graphs G and H. Then*

$$\lambda(G \square H) \ge \max\{\mathrm{F}(G)|V(H)|, F(H)|V(G)|\} - \min\{\mathrm{Z}(G)|V(H)|, \mathrm{Z}(H)|V(G)|\} + 1$$

Proof This follows from the bound on the failed zero forcing number of a Cartesian product [7] and the bound on the zero forcing number of a Cartesian product [2], together with Observation 1. □

5 Conclusion

In this paper, we introduced the idea of zero forcing span, and characterized graphs with high and low values in the context of standard zero forcing for both undirected and directed graphs, as well as in skew zero forcing. Since the zero forcing span is intimately related to linear algebra and to zero forcing polynomials, further investigation of these relationships is a compelling direction, and could include further investigation of parameters studied here, or other variants including, for example, positive semidefinite zero forcing or power domination.

References

1. Adams, A., Jacob, B.: Failed zero forcing and critical sets on directed graphs. Australasian J. Combinat. **81**(3), 367–387 (2021)
2. AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. Linear Algebra Its Appl. **428**(7), 1628–1648 (2008). AIM Minimum Rank-Special Graphs Work Group members: Barioli, F., Barrett, W., Butler, S., Cioabă, S.M., Cvetković, D., Fallat, S.M., Godsil, C., Haemers, W., Hogben, L., Mikkelson, R., Narayan, S., Pryporova, O., Sciriha, I., So, W., Stevanović, D., van der Holst, H., Vander Meulen, K., Wehe, A.W.
3. Ansill, T., Jacob, B., Penzellna, J., Saavedra, D.: Failed skew zero forcing on a graph. Linear Algebra Appl. **509**, 40–63 (2016)

4. Berliner, A., Catral, M., Hogben, L., Huynh, M., Lied, K., Young, M.: Minimum rank, maximum nullity, and zero forcing number of simple digraphs. Electron. J. Linear Algebra **26**, 762 (2013)
5. Boyer, K., Brimkov, B., English, S., Ferrero, D., Keller, A., Kirsch, R., Phillips, M., Reinhart, C.: The zero forcing polynomial of a graph. Discret. Appl. Math. **258**, 35–48 (2019)
6. Ferrero, D., Flagg, M., Hall, H.T., Hogben, L., Lin, J.C.H., Meyer, S., Nasserasr, S., Shader, B.: Rigid linkages and partial zero forcing. Electron. J. Combinat. **26**(2) (2019)
7. Fetcie, K., Jacob, B., Saavedra, D.: The failed zero forcing number of a graph involve. J. Math. **8**(1), 99–117 (2015)
8. Hogben, L.: Minimum rank problems. Linear Algebra Appl. **432**(8), 1961–1974 (2010)
9. IMA-ISU research group on minimum rank. Minimum rank of skew-symmetric matrices described by a graph. Linear Algebra Appl., 432(10):2457–2472, 2010. IMA-ISU research group members: Allison, M., Bodine, E., DeAlba, L.M., Debnath, J., DeLoss, L., Garnett, C., Grout, J., Hogben, L., Im, B., Kim, H., Nair, R., Pryporova, O., Savage, K., Shader, B., Wehe, A.W.

DNA Self-assembly: Complete Tripartite Graphs and Cocktail Party Graphs

Leyda Almodóvar, Jane HyoJin Lee, MeiRose Neal, Heiko Todt, and Jessica Williams

Abstract Based on the tile method for DNA self-assembly, which involves branched junction molecules whose flexible k-arms are double strands of DNA, we design a collection of tiles that will construct a nanostructure shaped like a target graph G. We find the minimum number of tile and bond-edge types required to construct complete tripartite graphs and cocktail party graphs in three different scenarios representing distinct levels of laboratory constraints.

Keywords DNA self-assembly · Branched junction molecules · Multipartite graphs

1 Introduction

DNA self-assembly is a rapidly advancing field for which good overviews can be found in [5, 6]. While there are several methods for the construction of synthetic DNA molecules [1, 3], we use the approach of the flexible-tile model introduced in [4] and we expand on the results presented in [2].

In this paper, we focus on the graph-theoretical aspect of designing the tiles that will construct complete tripartite graphs and cocktail party graphs. The tiles are star-shaped molecules whose flexible k-arms are double strands of DNA. A tile with k-arms is represented by a vertex of degree k in a graph. The arms of each tile are labeled using letters, called *bond-edge types*, with complementary Watson-Crick bases represented by hatted and unhatted letters, respectively. For example, given the bond-edge type b, its complementary sequence of bases is represented by $\hat{b}$. A collection of tiles, called a *pot*, realizes a graph G, if the collection can be assembled

L. Almodóvar · J. H. Lee · M. Neal · H. Todt (✉) · J. Williams
Stonehill College, North Easton, MA 02357, USA
e-mail: htodt@stonehill.edu

L. Almodóvar
e-mail: lalmodovarvel@stonehill.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_24

to construct G without any arms remaining unmatched. The *tile type* is the multiset of letters corresponding to bond-edge types for the tile.

Our goal is to find the minimum number of tile and bond-edge types needed in order to construct a target graph G, where G represents either a complete tripartite graph or a cocktail graph.

We consider these minimum numbers under three different scenarios corresponding to three laboratory constraints:

- Scenario 1: The incidental construction of a graph smaller than G is allowed.
- Scenario 2: The incidental construction of a graph smaller than G is not allowed but the construction of a non-isomorphic graph of the same size as G is allowed.
- Scenario 3: Any non-isomorphic graph incidentally constructed must be larger than G.

We let $T_i(G_n)$ for $i = 1, 2, 3$ be the minimum number of tiles required to construct a complex in each of the scenarios above. Similarly, $B_i(G_n)$ denotes the minimum number of bond-edge types needed for each scenario.

We illustrate how these scenarios differ in Example 1.

Example 1 Consider the complete tripartite graph $K_{1,1,2}$.

The pot $P = \{\{a, \hat{a}\}, \{a^2, \hat{a}\}, \{a, \hat{a}^2\}\}$ realizes $K_{1,1,2}$ as can be seen in Fig. 1. However, this pot does not satisfy scenarios 2 or 3 as the tile $\{a, \hat{a}\}$ can also realize the graph with a single a vertex and a loop by connecting the ends a and $\hat{a}$. It can be shown that in this case, $B_1(K_{1,1,2}) = 1$ and $T_1(K_{1,1,2}) = 3$.

The pot $P = \{\{a, b\}, \{\hat{a}^2\}, \{a^2, \hat{a}\}, \{a, \hat{a}, \hat{b}\}\}$ realizes $K_{1,1,2}$ as can be seen in Fig. 2. It is not possible to realize a graph with less than four vertices with this pot. However, it is possible to construct the graph in Fig. 3, which is not isomorphic to $K_{1,1,2}$. Therefore, this pot satisfies scenario 2, but not scenario 3. It can be shown that $B_2(K_{1,1,2}) = 2$ and $T_2(K_{1,1,2}) = 3$.

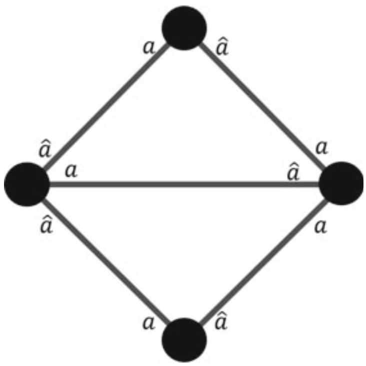


Fig. 1 A construction of $K_{1,1,2}$ that satisfies scenario 1

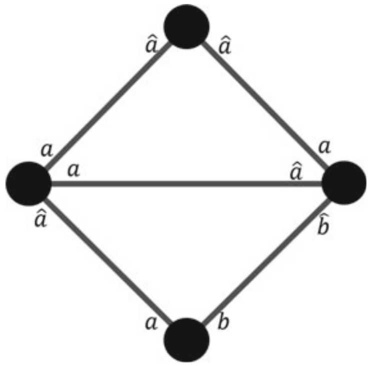


Fig. 2 A construction of $K_{1,1,2}$ that satisfies scenario 2

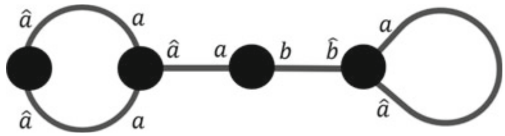


Fig. 3 A graph realized by the same pot as for $K_{1,1,2}$ in scenario 2

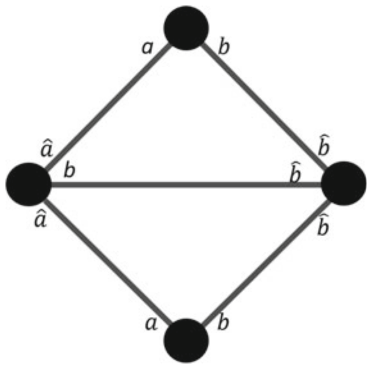


Fig. 4 A construction of $K_{1,1,2}$ that satisfies scenario 3

Finally, the pot $P = \{\{a, b\}, \{\hat{a}^2, b\}, \{\hat{b}^3\}\}$ realizes $K_{1,1,2}$ as can be seen in Fig. 4. It can be checked that any graph that can be realized by this pot has at least four vertices and if it has four vertices it is isomorphic to $K_{1,1,2}$. Therefore, this pot satisfies scenario 3. It can be shown that in this case, $B_3(K_{1,1,2}) = 2$ and $T_3(K_{1,1,2}) = 3$.

2 Definitions and Prior Results

In this section we include definitions and results from [2] that will be used throughout our paper. We have divided these according to the scenarios where they are needed.

2.1 Scenario 1

Corollary 1 $B_1(G) = 1$ *for all G.*

Theorem 1 $av(G) \leq T_1(G) \leq ev(G) + 2ov(G)$*, where* $av(G)$*,* $ev(G)$*, and* $ov(G)$ *are the number of different degrees, different even degrees, and different odd degrees of the graph G, respectively.*

2.2 Scenario 2

Definition 1 Let P be a pot with p tile types labeled $t_1, \ldots, t_p$, let A_{ij} be the number of cohesive ends of type a_i on tile t_j, let $\hat{A}_{i,j}$ be the number of cohesive ends of type $\hat{a}_i$, and let $z_{i,j} = A_{i,j} - \hat{A}_{i,j}$.

The *construction matrix of* P, denoted $M(P)$ is given by

$$M(P) = \begin{bmatrix} z_{1,1} & z_{1,2} & \ldots & z_{1,p} & 0 \\ \vdots & \vdots & & \vdots & \\ z_{m,1} & z_{m,2} & \ldots & z_{m,p} & 0 \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix}$$

and it captures the requirements of a complete complex. That is, the sum of the proportion of tile types needs to add up to 1, and the total number of hatted cohesive end types must equal the total number of unhatted cohesive-end types.

Proposition 1 *Let P be a pot with p tile types labeled $t_1, \ldots, t_p$. Let r_i to be the proportion of tile type t_i used in the assembly process. If $\langle r_1, \ldots, r_p \rangle$ is a solution of the construction matrix $M(P)$, and there is a positive integer n such that $nr_j \in \mathbb{Z}_{\geq 0}$ for all j, then there is a graph of size n that may be constructed from P using nr_j tiles of type t_j.*

The construction matrix of a given pot P along with Proposition 1 allows us to determine the smallest sized graph that may be constructed from P. This is particularly useful in scenario 2 since the realization of smaller graphs than a target graph G are not allowed.

Theorem 2 $B_2(G) + 1 \leq T_2(G)$.

2.3 Scenario 3

Definition 2 Given a pot P, the set of graphs of minimum size that may be constructed from P is denoted $C_{\min}(P)$.

Lemma 1 *If P is a pot such that $\{G\} = C_{\min}(P)$ and G has no loops, then no tile type $T \in P$ used in the construction of G may have both a hatted and an unhatted cohesive end of the same type.*

Lemma 2 *If P is a pot such that $\{G\} = C_{\min}(P)$ and G has no loops, then no tile type $T \in P$ used in the construction of G may be used for two adjacent vertices in G.*

Lemma 3 *If P is a pot such that $\{G\} = C_{\min}(P)$, and two nonadjacent edges $\{u, v\}$ and $\{s, t\}$ of $G = \{V, E\}$ use the same bond-edge type, then G is isomorphic to $G' = \{V, E'\}$, where $E' = E - \{\{u, v\}, \{s, t\}\} \cup \{\{u, t\}, \{s, v\}\}$.*

The following Lemma has been modified from [2] to extend it to a complete k-partite graph.

Lemma 4 *In any pot P such that $\{G\} = C_{min}(P)$ where G is a complete k-partite graph, the following must be true:*

1. *Any two edges formed from the same bond-edge type must be incident with the same vertex, and, moreover, that vertex must have the two corresponding cohesive ends either both hatted or both unhatted.*
2. *Two tiles corresponding to vertices in the same partition of G (whether of the same or different tile types) cannot both have two cohesive ends of the same kind.*

Proof The first item follows from Lemma 3. Note that the two unhatted cohesive ends could end up in the same partition or in different partitions. Regardless of the placement of both unhatted cohesive ends, multiple edges between vertices could result. Both cohesive ends must be hatted or unhatted by Lemma 1, since a complete k-partite graph has no loops. The second item follows from the first item, which implies that all four of the edges involved must be pairwise adjacent. This cannot occur in a complete k-partite graph which has no multiple edges. □

3 Complete Tripartite Graphs

We consider the complete tripartite graphs $K_{p,q,r}$ and determine their minimum number of bond-edge types and tile types in the three scenarios. A complete tripartite graph is a graph on $p + q + r$ vertices that are partitioned into three disjoint sets of order p, q, r, respectively, such that each vertex is adjacent to every vertex in the other sets, but not adjacent to any of the other vertices in the same set. Figure 5 shows $K_{2,2,3}$ as an example of a complete tripartite graph. The usual convention is that $p \le q \le r$, but for the statements of our theorems we do not assume this unless specified, as it means that we need to describe fewer cases for some of the theorems.

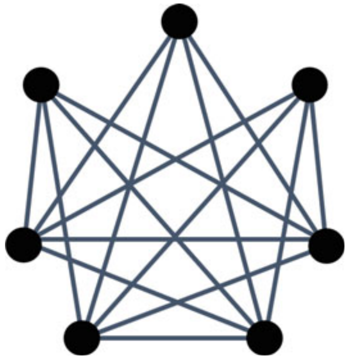


Fig. 5 The complete tripartite graph $K_{2,2,3}$

3.1 Scenario 1

While the minimum number of bond-edge types is one by Corollary 1, there are several cases for the minimum number of tile types.

Theorem 3 *For $p, q, r \in \mathbb{N}$, we have $B_1(K_{p,q,r}) = 1$ and*

$$T_1(K_{p,q,r}) = \begin{cases} 1 & \text{if } p = q = r, \\ 2 & \text{if } p = q \neq r \text{ and } p \text{ is even}, \\ 2 & \text{if } p = q \neq r \text{ and } p \text{ and } r \text{ are odd}, \\ 3 & \text{if } p = q \neq r \text{ and } p \text{ is odd and } r \text{ is even}, \\ 3 & \text{if } p \neq q, q \neq r, \text{ and } p \neq r. \end{cases}$$

Proof We prove the individual cases for $T_1(K_{p,q,r})$.

Case 1: $p = q = r$.
Since all vertices have degree $2p$, we have $av(K_{p,p,p}) = 1$, $ev(K_{p,p,p}) = 1$, and $ov(K_{p,p,p}) = 0$. Thus $T_1(K_{p,p,p}) = 1$ by Theorem 1.
Alternatively, the pot $\{t_1 = \{a^p, \hat{a}^p\}\}$ realizes $K_{p,p,p}$.

Case 2: $p = q \neq r$ and p is even.
All vertices have degree either $2p$ or $p + r$. Therefore, $av(K_{p,p,r}) = 2$, and $T_1(K_{p,p,r}) \geq 2$ by Theorem 1. The pot $\{t_1 = \{a^{2p}\}, t_2 = \{a^{p/2}, \hat{a}^{r+p/2}\}\}$ realizes $K_{p,p,r}$ with two tiles. Figure 6 shows an example for this case.

Case 3: $p = q \neq r$ and p and r are odd.
Again, all vertices have degree either $2p$ or $p + r$, and here $p + r$ is even. Thus, $av(K_{p,p,r}) = 2$, $ev(K_{p,p,r}) = 2$, and $ov(K_{p,p,r}) = 0$, and therefore $T_1(K_{p,p,r}) = 2$ by Theorem 1.
Alternatively, the pot $\{t_1 = \{a^p, \hat{a}^p\}, t_2 = \{a^{(p+r)/2}, \hat{a}^{(p+r)/2}\}\}$ realizes $K_{p,p,r}$ with two tiles.

Case 4: $p = q \neq r$ and p is odd and r is even.
Again, all vertices have degree either $2p$ or $p + r$. Therefore, $av(K_{p,p,r}) = 2$, and $T_1(K_{p,p,r}) \geq 2$ by Theorem 1. We show that $T_1(K_{p,p,r}) \geq 3$ by contradiction. Suppose that $T_1(K_{p,p,r}) = 2$. Then there exists a pot with tile t_1

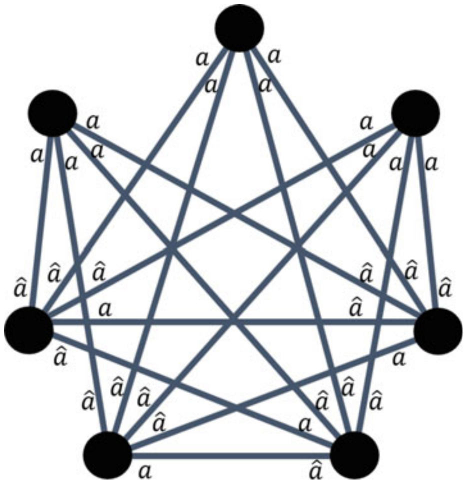


Fig. 6 A construction of $K_{2,2,3}$ that satisfies scenario 1 in case 2

of degree $2p$ and tile t_2 of degree $p+r$ that realizes $K_{p,p,r}$. We can assume that all cohesive ends are either a or $\hat{a}$. Let d_1 and d_2 denote the difference between the numbers of cohesive ends of a and $\hat{a}$ in t_1 and t_2, respectively. Then d_1 is even, since t_1 has an even degree. Similarly, d_2 is odd. Since the pot realizes $K_{p,p,r}$, we have $r \cdot d_1 + 2p \cdot d_2 = 0$. Note that r and d_1 are both even, so $r \cdot d_1$ is divisible by 4. However, p and d_2 are odd, so $2p \cdot d_2$ is not divisible by 4. Therefore, $r \cdot d_1 + 2p \cdot d_2 = 0$ cannot be true. We conclude that $T_1(K_{p,p,r}) \geq 3$.

The pot $\{t_1 = \{a^p, \hat{a}^p\}, t_2 = \{a^p, \hat{a}^r\}, t_3 = \{a^r, \hat{a}^p\}\}$ realizes $K_{p,p,r}$ with three tiles, so $T_1(K_{p,p,r}) = 3$. Figure 7 shows an example for this case.

Case 5: $p \neq q$, $q \neq r$, and $r \neq p$.

All vertices have degree $p+q$, $p+r$ or $q+r$, which are all distinct. Therefore $T_1(K_{p,q,r}) \geq 3$ by Theorem 1. The pot $\{t_1 = \{a^p, \hat{a}^q\}, t_2 = \{a^q, \hat{a}^r\}, t_3 = \{a^r, \hat{a}^p\}\}$ realizes $K_{p,q,r}$ with three tiles. Figure 8 shows an example for this case. □

3.2 Scenario 2

The results in scenario 2 are rather complex, as the following two examples illustrate.

Example 2 $B_2(K_{1,2,4}) = 1$ and $T_2(K_{1,2,4}) = 3$.

Proof The pot $\{t_1 = \{a^6\}, t_2 = \{a, \hat{a}^2\}, t_3 = \{\{a^2, \hat{a}^3\}\}$ realizes $K_{1,2,4}$ but no smaller graphs. Therefore, $B_2(K_{1,2,4}) = 1$ and $T_2(K_{1,2,4}) \leq 3$. Since $K_{1,2,4}$ has vertices of three different degrees, $T_2(K_{1,2,4}) \geq 3$, and thus $T_2(K_{1,2,4}) = 3$. Figure 9 illustrates this construction. □

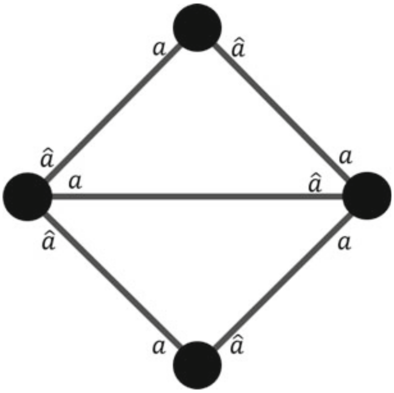


Fig. 7 A construction of $K_{1,1,2}$ that satisfies scenario 1 in case 4

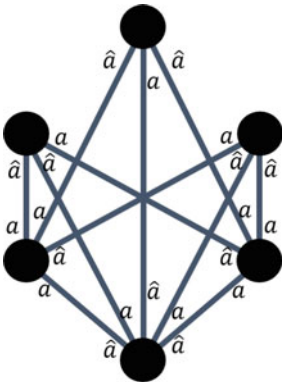


Fig. 8 A construction of $K_{1,2,3}$ that satisfies scenario 1 in case 5

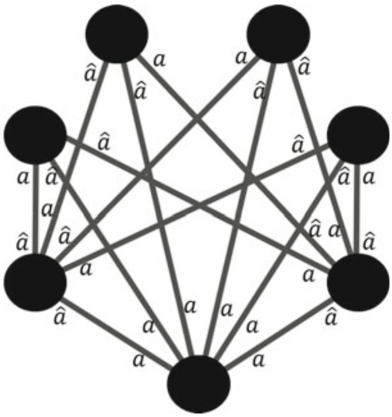


Fig. 9 A construction of $K_{1,2,4}$ that satisfies scenario 2

The next example seems like it should be similar to $K_{1,2,4}$ as it also has odd p and even q and r such that $r = 2q$, however, the result is different.

Example 3 $B_2(K_{3,4,8}) = 2$ and $T_2(K_{3,4,8}) = 3$.

Proof Note that the pot $\{t_1 = \{a^4, b^8\}, t_2 = \{\hat{a}^3, b^8\}, t_3 = \{\hat{b}^7\}\}$ realizes $K_{3,4,8}$ but no smaller graphs, since $\gcd(3, 4) = 1$. Therefore, $B_2(K_{3,4,8}) \leq 2$ and $T_2(K_{3,4,8}) \leq 3$, and since $K_{3,4,8}$ has vertices of three different degrees we have $T_2(K_{3,4,8}) = 3$.

We use a brute-force argument to show that $B_2(K_{3,4,8}) > 1$. Any pot that realizes $K_{3,4,8}$ must have at least three tile types, namely at least one for degrees 7, 11, and 12. Since a larger number of tiles increases the likelihood of creating smaller graphs from a pot, we first consider whether it is possible to have a pot with exactly three tiles using only one bond-edge type that realizes $K_{3,4,8}$ but no smaller graph. Since only one bond-edge type is used, each tile has the form $\{a^m, \hat{a}^n\}$ for some m, n. Let d_1 be the difference between the number of hatted and unhatted cohesive ends for the tile of degree 7, and d_2 and d_3 the differences for the tiles of degree 11 and 12, respectively. d_1 has to be an odd number such that $|d_1| \leq 7$. Similarly, d_2 is odd such that $|d_2| \leq 11$ and d_3 is even such that $|d_3| \leq 12$.

Since the pot needs to realize $K_{3,4,8}$, we have $8d_1 + 4d_2 + 3d_3 = 0$. Note that this is equivalent to $4(2d_1 + d_2) + 3d_3 = 0$. This implies that d_3 is divisible by 4, and so $d_3 \in \{-12, -8, -4, 0, 4, 8, 12\}$. We do not have to consider $d_3 = 0$, since in that case t_3 creates a graph of order 1 with loops. We now consider all possible cases with negative values for d_3, namely all triples (d_1, d_2, d_3) such that $8d_1 + 4d_2 + 3d_3 = 0$ and $d_3 < 0$. This describes all possible cases, as the cases for positive values for d_3 are similar, just with switched signs.

Table 1 contains all possible triples and for each triple there is a way to construct a smaller graph than $K_{3,4,8}$. The columns labeled t_1, t_2, t_3 refer to how many times that tile type is used in the construction of a smaller graph. The size of the smaller

Table 1 Different tiles for $K_{3,4,8}$ and how to construct a smaller graph

d_1	d_2	d_3	t_1	t_2	t_3	Size of smaller graph
5	−7	−4	3	1	2	6
3	−3	−4	1	1	0	2
1	1	−4	4	0	1	5
−1	5	−4	1	1	1	3
−3	9	−4	3	1	0	4
7	−5	−12	5	7	0	12
5	−1	−12	1	5	0	6
3	3	−12	4	0	1	5
1	7	−12	5	1	1	7
−1	11	−12	11	1	0	12

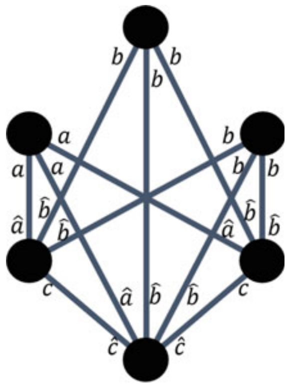


Fig. 10 A construction of $K_{1,2,3}$ that satisfies scenario 2

graph is then given by the sum of the entries in these columns. Note that all of the cases allow the construction of a graph of order less than $3 + 4 + 8 = 15$.

So far, we only considered a pot with exactly three tiles, but this is not sufficient to show that $B_2(K_{3,4,8}) = 2$. There may be a pot with four or more tiles using only one bond-edge type that realizes $K_{3,4,8}$ but no smaller graph. Considering pots with four or more tiles creates even more cases, but all of these can be ruled out with the help of a computer, rather than checking all of them by hand. □

It should be pointed out that the difference between Examples 2 and 3 is not related to $p = 1$ in Example 2. For example, it can be shown that $B_2(K_{5,8,16}) = 1$.

Rather than trying to distinguish between all of the different cases that can arise for $K_{p,q,r}$ in scenario 2, we prove several general theorems that provide upper bounds for B_2 and T_2.

The first theorem establishes a general upper bound for all complete tripartite graphs.

Theorem 4 *$B_2(K_{p,q,r}) \leq 3$ and $T_2(K_{p,q,r}) \leq 4$ for all $p \leq q \leq r$.*

Proof If $r = 1$, then the resulting graph only has three vertices and three edges, so the statement is clearly true. So consider $r > 1$.

The pot $\{t_1 = \{a^{p+q}\}, t_2 = \{b^{p+q}\}, t_3 = \{\hat{a}, \hat{b}^{r-1}, c^p\}, t_4 = \{\hat{a}, \hat{b}^{r-1}, \hat{c}^q\}\}$ realizes $K_{p,q,r}$ but no smaller graphs. Figure 10 illustrates the pot with an example. □

The following two theorems establish lower upper bounds than the ones in Theorem 4 for certain conditions on p, q, and r.

Theorem 5 *$B_2(K_{p,q,r}) \leq 2$ and $T_2(K_{p,q,r}) \leq 3$ whenever $\gcd(p, q) = 1$ or $\gcd(p, r) = 1$ or $\gcd(q, r) = 1$.*

Proof We consider the case where $\gcd(p, q) = 1$. The pot $\{t_1 = \{a^q, b^r\}, t_2 = \{\hat{a}^p, b^r\}, t_3 = \{\hat{b}^{p+q}\}\}$ realizes $K_{p,q,r}$ but no smaller graph since $\gcd(p, q) = 1$. Since the pot uses two bond-edge types and has three tiles, this establishes the upper bounds for this case. The other two cases are similar. Figure 11 illustrates the pot with an example. □

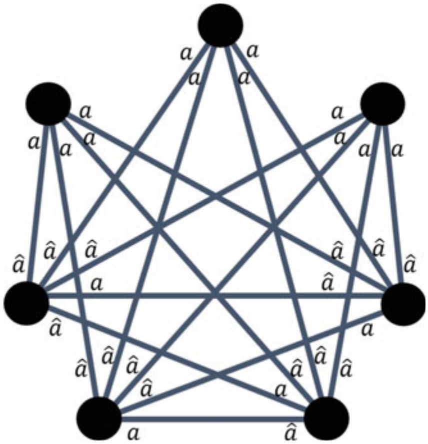


Fig. 11 A construction of $K_{2,3,3}$ that satisfies scenario 2

Theorem 6 *If* $\gcd(p+q, r) = 1$ *or* $\gcd(p+r, q) = 1$ *or* $\gcd(q+r, p) = 1$*, then* $B_2(K_{p,q,r}) \leq 2$ *and* $T_2(K_{p,q,r}) \leq 3$.

Proof We consider the case where $\gcd(p+q, r) = 1$. The pot $\{t_1 = \{a^{p+q}\}, t_2 = \{\hat{a}^r, b^q\}, t_3 = \{\hat{a}^r, \hat{b}^p\}\}$ realizes $K_{p,q,r}$ but no smaller graph since $\gcd(p+q, r) = 1$. This establishes the upper bounds. The other two cases are similar. □

These two theorems cover many different complete tripartite graphs. For example, both theorems would apply to $K_{3,4,8}$ in Example 3. Establishing lower bounds is much more complicated as Examples 2 and 3 illustrate. However, we are able to prove a general theorem that applies to more than just complete tripartite graphs. This theorem establishes lower bounds for complete tripartite graphs where either p, q, r are all even or all odd.

Theorem 7 *If G is a graph of order $n > 2$ such that all vertices have even degrees and each degree is less than n, then $B_2(G) \geq 2$.*

Proof Let G be a graph of order $n > 2$ such that all vertices have even degrees and each degree is less than n. Suppose there is a pot P with p tiles using just one bond-edge type that realizes G. The construction matrix of P must then have the form

$$M(P) = \begin{bmatrix} z_{1,1} & z_{1,2} & \dots & z_{1,p} & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

Since every degree is even, each tile type has an even number of cohesive ends and so $z_{1,j}$ is even for all j. We also have $|z_{1,j}| < n$ for all j, as each degree is less than n. Furthermore, $z_{1,j} \neq 0$ for all j, since a tile with an equal number of hatted and unhatted cohesive ends could form a graph with just one vertex using loops. At least one of the $z_{1,j}$ has to be positive, and at least one has to be negative. Through reordering, if necessary, assume $z_{1,1} > 0$ and $z_{1,2} < 0$. Then $M(P)$ is row equivalent to

$$\begin{bmatrix} 1 & 0 & -\frac{-z_{1,3}+z_{1,2}}{z_{1,1}-z_{1,2}} & \dots & -\frac{-z_{1,p}+z_{1,2}}{z_{1,1}-z_{1,2}} & -\frac{z_{1,2}}{z_{1,1}-z_{1,2}} \\ 0 & 1 & \frac{z_{1,1}-z_{1,3}}{z_{1,1}-z_{1,2}} & \dots & \frac{z_{1,1}-z_{1,p}}{z_{1,1}-z_{1,2}} & \frac{z_{1,1}}{z_{1,1}-z_{1,2}} \end{bmatrix}.$$

Therefore, there exists a solution of the form $\langle -z_{1,2}/(z_{1,1}-z_{1,2}), z_{1,1}/(z_{1,1}-z_{1,2}), 0, \dots, 0\rangle$. Note that $z_{1,1} - z_{1,2} < 2n$. Also, $z_{1,1}$ and $z_{1,2}$ are even, so $z_{1,1} - z_{1,2}$ is even as well, and the numerators and denominators in the solution have a common factor of 2. Hence the solution has the form $\langle a/m, b/m, 0, \dots, 0\rangle$ for some integer $m < n$, and so P realizes some graph of size $m < n$. This is a contradiction under scenario 2, so there cannot exist a pot using just one bond-edge type that realizes G. We conclude that $B_2(G) \geq 2$. □

We then have the following corollary.

Corollary 2 *If p,q,r are either all even or all odd, then* $B_2(K_{p,q,r}) \geq 2$ *and* $T_2(K_{p,q,r}) \geq 3$.

Proof If p,q,r are either all even or all odd, then the degrees of all vertices in $K_{p,q,r}$ are even. Clearly, all degrees are less than $p+q+r$. Thus, $B_2(K_{p,q,r}) \geq 2$ by Theorem 7 and then $T_2(K_{p,q,r}) \geq 3$ by Theorem 2. □

We close this section with one last result where the previous two theorems do not apply but may be of interest as it is a special case.

Theorem 8 *If p is even, then* $B_2(K_{p,p,p}) = 2$ *and* $T_2(K_{p,p,p}) = 3$.

Proof Since all vertices in $K_{p,p,p}$ are of degree $2p$ if p is even, we have $B_2(K_{p,p,p}) \geq 2$ and $T_2(K_{p,p,p}) \geq 3$ by Theorem 7.

The pot $\{t_1 = \{a^{2p}\}, t_2 = \{b^{2p}\}, t_3 = \{\hat{a}, b^{p/2}, \hat{b}^{3p/2-1}\}\}$ realizes $K_{p,p,p}$ but no smaller graph. Note that t_1 is needed at least once, so t_3 is needed at least $2p$ times. There is a difference of $p-1$ between the hatted and unhatted cohesive ends for b in t_3, so using t_3 at least $2p$ times means that t_2 has to be used at least $p-1$ times. Therefore, the size of any graph constructed by this pot is $1+2p+p-1 = 3p$, which is the number of vertices of $K_{p,p,p}$. This establishes that $B_2(K_{p,p,p}) = 2$ and $T_2(K_{p,p,p}) = 3$. □

3.3 Scenario 3

We are able to prove complete results in scenario 3. The first theorem describes the minimum number of bond-edge types.

Theorem 9 $B_3(K_{p,q,r}) = p + q$ *for all* $p \leq q \leq r$.

Proof Let P denote the set with p elements, Q the set with q elements, and R the set with r elements. The vertices in P have degree $q+r$, the vertices in Q have degree $p+r$, and the vertices in R have degree $p+q$, where $p \leq q \leq r$. By Lemma 4, if a bond-edge of type a appears on a vertex, all edges of that type must be incident with the same vertex. We use a greedy algorithm to create tiles for the vertices. We minimize the bond-edge types by always assigning each bond-edge type to the most edges possible. We start by assigning tiles to the vertices in P since they have higher degrees. The first tile is $t_1 = \{a_1^{q+r}\}$. By Lemma 4, a_1 cannot be used in any other tile. As the second tile we use $t_2 = \{a_2^{q+r}\}$. Following this procedure, the tiles corresponding to vertices in P are $t_i = \{a_i^{q+r}\}$ for $i \in \{1, \dots, p\}$. Note that $a_1, \dots, a_p$ cannot be used in any other tile. Any further tiles now already need $\hat{a}_1, \dots, \hat{a}_p$. Now, the most efficient way is to assign tiles to the vertices in Q. Following the same procedure, let $t_j = \{\hat{a}_1, \dots, \hat{a}_p, a_j^r\}$ for $j \in \{p+1, \dots, q\}$. Finally, the last tile is $t_{p+q+1} = \{\hat{a}_1, \hat{a}_2, \dots, \widehat{a_{p+q}}\}$ and this is the only tile that can be repeated. This pot realizes $K_{p,q,r}$ and realizes no smaller graph or non-isomorphic graph of the same size. This shows $B_3(K_{p,q,r}) \leq p + q$. Since this process maximizes the number of times each bond-edge type is used for the edges, the resulting pot uses the minimum number of bond-edge types. Hence, $B_3(K_{p,q,r}) = p + q$. □

The pot created in the proof of Theorem 9 also is used to determine the minimum number of tiles, which results in the following theorem.

Theorem 10 $T_3(K_{p,q,r}) = p + q + 1$ *for all* $p \leq q \leq r$.

Proof The pot $P = \{\{t_i = \{a_i^{q+r}\}, i = 1, \dots, p\}, \{t_j = \{\hat{a}_1, \dots, \hat{a}_p, a_j^r\}, j = p + 1, \dots, q\}, t_{p+q+1} = \{\hat{a}_1, \hat{a}_2, \dots, \widehat{a_{p+q}}\}\}$ realizes $K_{p,q,r}$ with $p + q + 1$ tiles, where the last tile is used r times on the vertices in R. By Lemma 2, a tile may only be repeated on vertices in the same set. Suppose by way of contradiction that $T_3(K_{p,q,r}) < p + q + 1$. Then there exists two vertices x_1 and x_2 using the same tile type t_* in either set P or set Q. Without loss of generality, assume $x_1, x_2 \in P$. By Lemma 4, any two edges from the same bond-edge type must be incident with the same vertex. Hence, if bond-edge type a appears in t_*, then all instances of $\hat{a}$ must appear on a tile t_y corresponding to a vertex y in Q or R that may not be repeated. Additionally, the bond-edge type a cannot appear more than once in t_* because that could create a double edge between x_1 and y. This implies that the repeated tile t_* has $q + r$ distinct bond-edge types. Note that this results in distinct tile types used on vertices in R. Hence, $T_3(K_{p,q,r}) \geq p + q + 1$. □

4 Cocktail Party Graphs

Due to the number of cases in scenario 1 and the difficult nature of scenario 2 for complete tripartite graphs, one would expect this to be even more complicated for complete k-partite graphs where $k \geq 4$. Rather than looking at all complete k-partite graphs, we focus on a specific class of complete k-partite graphs to illustrate that it is possible to find results for some complete k-partite graphs where $k \geq 4$.

Let CP_n denote the cocktail party graph with n pairs that connect to all other vertices except to the other vertex in the pair. Each edge in a cocktail party graph

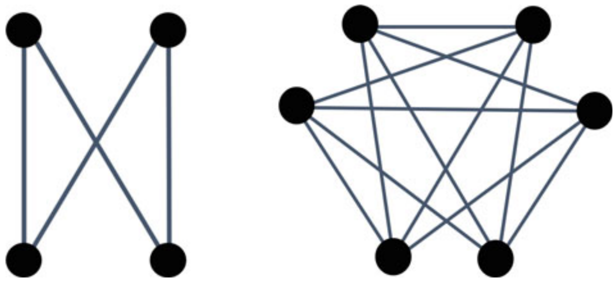


Fig. 12 Cocktail party graphs CP_2 and CP_3

may represent a handshake at a party where each guest arrives as part of a pair and shakes the hands of all guests except their partner. CP_n has $2n$ vertices, all of which have degree $2n - 2$. All cocktail party graphs are complete k-partite graphs. For example, $CP_2 = K_{2,2}$, $CP_3 = K_{2,2,2}$, and $CP_4 = K_{2,2,2,2}$. Figure 12 shows two examples of cocktail party graphs.

4.1 Scenario 1

We obtain the following result for cocktail party graphs in scenario 1.

Theorem 11 $B_1(CP_n) = T_1(CP_n) = 1$ *for all* $n \geq 2$.

Proof The pot $P = \{t_1 = \{a^{n-1}, \hat{a}^{n-1}\}\}$ realizes CP_n. Figure 13 illustrates this construction. $\square$

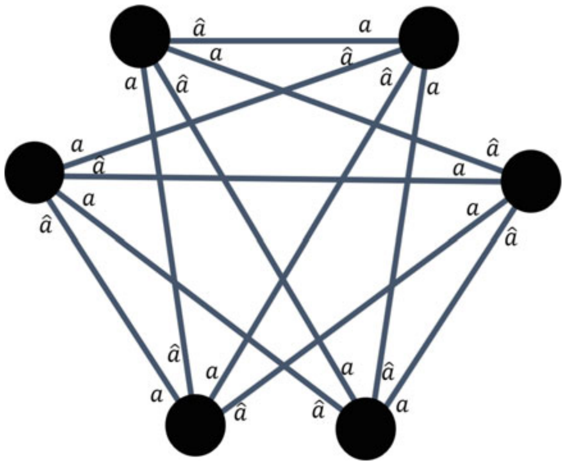


Fig. 13 A construction of CP_3 that satisfies scenario 1

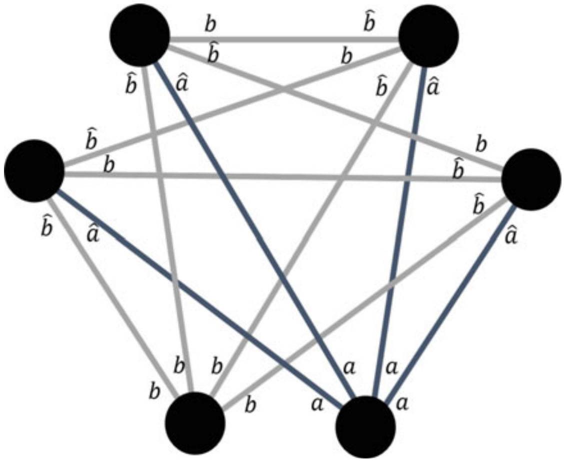


Fig. 14 A construction of CP_3 that satisfies scenario 2

4.2 Scenario 2

We can also apply Theorem 7 to cocktail party graphs, since each vertex is of even degree. The minimum number of bond-edge types and tile types for cocktail party graphs in scenario 2 can then be easily obtained.

Theorem 12 $B_2(CP_n) = 2$ *and* $T_2(CP_n) = 3$ *for all* $n > 2$.

Proof Since CP_n has only vertices of even degrees and all degrees are less than $2n$, we have $B_2(CP_n) \geq 2$ by Theorem 7.

The pot $P = \{t_1 = \{a^{2n-2}\}, t_2 = \{b^{2n-2}\}, t_3 = \{\hat{a}, b^{n-2}, \hat{b}^{n-1}\}\}$ realizes CP_n, so this shows that $B_2(CP_n) = 2$. By Theorem 2, $T_2(CP_n) \geq B_2(CP_n) + 1 = 3$. Since the pot P has exactly three tiles, we have $T_2(CP_n) = 3$. Figure 14 illustrates the construction. □

4.3 Scenario 3

The following theorem describes the minimum number of bond-edge types in scenario 3.

Theorem 13 $B_3(CP_n) = 2n - 2$ *for all* $n \geq 2$.

Proof We use a greedy algorithm to create tiles for the vertices. We minimize the bond-edge types by always assigning each bond-edge type to the most edges possible. The first tile is $t_1 = \{a_1^{2n-2}\}$. By Lemma 4, a_1 can now not be used in any other tile. As the second tile we use $t_2 = \{a_2^{2n-2}\}$. The first two tiles will create the two vertices in the first pair of vertices in CP_n. Any further tiles now already need $\hat{a}_1$ and $\hat{a}_2$. So now, the most efficient way is to use the third and fourth tiles as $t_3 = \{\hat{a}_1, \hat{a}_2, a_3^{2n-4}\}$ and $t_4 = \{\hat{a}_1, \hat{a}_2, a_4^{2n-4}\}$. These will be the tiles for the second pair of vertices. Proceeding in similar fashion, we create tiles for the vertices in all pairs except for the last one. The last two vertices then have to be the tile $t_{2n-1} = \{\hat{a}_1, \hat{a}_2, ..., \hat{a}_{2n-2}\}$, and this is the only tile that can be repeated. Since this process maximized the number of times each bond-edge type is used for the edges, the resulting pot uses the minimum number of bond-edge types.

The following pot is obtained by this process:

$$\{t_1 = \{a_1^{2n-2}\}, t_2 = \{a_2^{2n-2}\}, t_{2i+1} = \{\hat{a}_1, \ldots, \hat{a}_{2i}, \hat{a}_{2i+1}^{2n-2-2i}\},$$
$$t_{2i+2} = \{\hat{a}_1, \ldots, \hat{a}_{2i}, a_{2i+2}^{2n-2-2i}\}, t_{2n-1} = \{\hat{a}_1, \ldots, \hat{a}_{2n-2}\}\}$$

This pot realizes CP_n and realizes no smaller graph or non-isomorphic graph of the same size. □

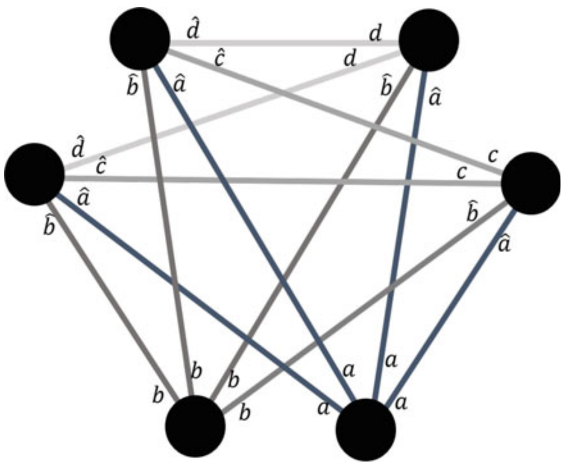
We are now able to determine the minimum number of tile types in scenario 3.

Theorem 14 $T_3(CP_n) = 2n - 1$ *for* $n \geq 2$.

Proof Note that the pot used to find $B_3(CP_n)$ has $2n - 1$ tiles, so $T_3(CP_n) \leq 2n - 1$. Suppose $T_3(CP_n) < 2n - 1$. Then at least two tiles must be reused. By Lemma 4, tiles can only be reused in the same pair, so there must be two pairs that each use the same two tiles.

From Lemma 4 it is also clear that for any bond-edge type there must be a unique central vertex that all edges with the same bond-edge type are incident to, with the only exception if a bond-edge type is used exactly once. Since we consider two pairs that each use the same two tiles, there has to be such central vertex for any bond-edge type used in any of these tiles. However, since in each pair the tile is reused, this

Fig. 15 A construction of CP_3 that satisfies scenario 3



implies that for any bond-edge type in such tile, the tile cannot create this central vertex. Now consider the bond-edge type that is used to connect two vertices that are in different pairs that use the same tiles. Neither one of these vertices can be the central vertex for that bond-edge type, but that is a contradiction. Thus, tiles can only be reused in at most one pair, and we have $T_3(CP_n) \geq 2n - 1$. Figure 15 illustrates this construction. $\square$

5 Conclusion

We obtained complete results for complete tripartite graphs in scenarios 1 and 3, and for cocktail party graphs in all scenarios. Further investigation into scenario 2 for complete tripartite graphs may reveal more theorems, but most likely there will be many different cases to consider.

Theorem 7 may be useful for other types of graphs, for example, it does apply to simple k-regular graphs where k is even.

Acknowledgements We would like to thank the Stonehill Undergraduate Research Experience program (SURE) for support and funding to make this paper possible.

References

1. Chen, J., Seeman, N.C: Synthesis from DNA of a molecule with the connectivity of a cube. Nature **350**(6319), 631–633. Springer (1991)
2. Ellis-Monaghan, J., Pangborn, G., Beaudin, L., Miller, D., Bruno, N., Hashimoto, A.: Minimal tile and bond-edge types for self-assembling DNA graphs. In: Discrete and Topological Models in Molecular Biology, pp. 241–270. Springer (2014)
3. Jonoska, N., Karl, S.A., Saito, M.: Three dimensional DNA structures in computing. BioSystems **52**, 143–153. Elsevier (1999)
4. Jonoska, N., McColm, G.L., Staninska, A.: Spectrum of a pot for DNA complexes. In: International Workshop on DNA-Based Computers, pp. 83–94. Springer (2006)
5. Pelesko, J.A: Self Assembly: The Science of Things That Put Themselves Together. Chapman and Hall/CRC (2007)
6. Seeman, N.C.: An overview of structural DNA nanotechnology. Molecul. Biotechnol. **37**(3), 246. Springer (2007)

Inverse of Hermitian Adjacency Matrix of Mixed Bipartite Graphs

Omar Alomari

Abstract Mixed graph D is a graph that can be obtained from a graph by orienting some of its edges. The Hermitian adjacency matrix of a mixed graph is defined to be the matrix $H = [h_{rs}]$ where $h_{rs} = i$ if $v_r v_s$ is an arc in D, $h_{rs} = -i$ if $v_s v_r$ is an arc in D, $h_{rs} = 1$ if $v_s v_r$ is a digon in D and $h_{rs} = 0$ otherwise. In this paper we investigate when the hermitian adjacency matrix of a bipartite graph is invertible and we prove for any tree mixed graph T with invertible hermitian adjacency matrix that H^{-1} is $\{0, \pm 1, \pm i\}$-matrix.

Keywords Adjacency matrix · Digraph · Hermitian · Inverse · Spectrum · Bipartite

1 Introduction

The literature of algebraic graph theory has grown immensely for the past several decades. The aim is to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about graphs. One of the most important fields in algebraic graph theory is spectrum of graphs where in spectral graph the eigenvalues and eigenvectors of matrices associated with graphs are studied. Traditionally, the adjacency matrix associated with undirected graph is commonly studied in literature, that is, the square matrix with an entry of 1 in uv position if there is an edge from vertex u to vertex v, otherwise the uv position is zero. This matrix is symmetric and all of its eigenvalues are real which makes investigating this matrix easy in terms of utilizing theorems from linear algebra that deal with symmetric matrices. On the other hand, dealing with directed graph (oriented graphs, digraphs and mixed graph), the traditional adjacency matrix where we set position uv to 1 if there is an arc from u to v and zero otherwise is utilized. This matrix is not symmetric and not always diagonalizable which makes its spectrum not always real.

O. Alomari (✉)
German Jordanian University, Amman, Jordan
e-mail: omar.alomari@gju.edu.jo

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_25

Consequently, the topic of investigating spectrum of adjacency matrix of digraph is not very common in the literature. Alternatively, researchers studied symmetric matrices that are related to the traditional adjacency matrix of digraph. For example, authors in [5, 6], studied the singular values of the traditional adjacency matrix of digraphs, in their work the authors observed that there is a strong relationship between singular values and the common out neighbors among vertices. One other such adjacency matrix and most interesting was proposed by Guo and Mohar in [1] where they defined the hermitian adjacency matrix of digraph as follows: For any digraph $D = (V(D), E(D))$ the hermitian adjacency matrix $H(D) = [h_{uv}]$ where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \text{ is digon in } D \\ i & \text{if } uv \in E \text{ and } vu \notin E, \\ -i & \text{if } uv \notin E \text{ and } vu \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The following matrix is the Hermitian adjacency matrix corresponding to the graph shown in (Fig. 1).

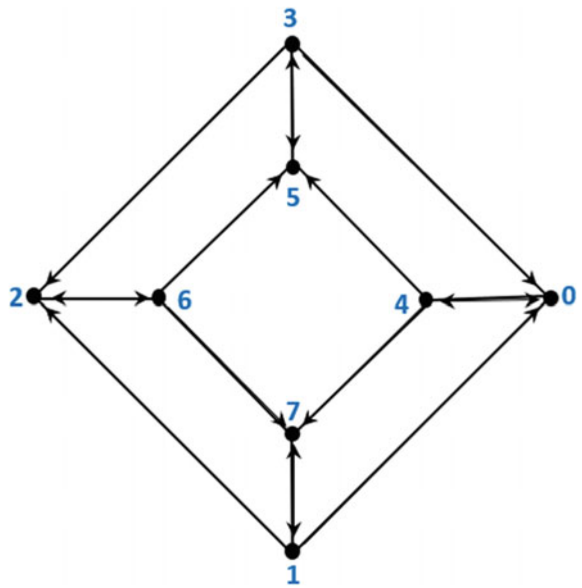


Fig. 1 Directed/mixed graph

Example 1

$$H(D) = \begin{pmatrix} 0 & -i & 0 & -i & 1 & 0 & 0 & 0 \\ i & 0 & i & 0 & 0 & 0 & 0 & 1 \\ 0 & -i & 0 & -i & 0 & 0 & 1 & 0 \\ i & 0 & i & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & i & 0 & i \\ 0 & 0 & 0 & 1 & -i & 0 & -i & 0 \\ 0 & 0 & 1 & 0 & 0 & i & 0 & i \\ 0 & 1 & 0 & 0 & -i & 0 & -i & 0 \end{pmatrix}$$

$$\sigma_H(D) = \{-\sqrt{5}, \sqrt{5}, -\sqrt{3}, \sqrt{3}, -1, -1, 1, 1\}$$

2 Main Results

Given a tree graph T, it is known that if T has invertible adjacency matrix $A(T)$, then $A^{-1}(T)$ is a $\{-1, 0, 1\}$-matrix, see [9]. As an extension of that, in this section we investigate when the hermitian adjacency matrix of a bipartite mixed graph is invertible, also we prove that for any tree mixed graph $\mathbb{T}$ with invertible hermitian adjacency matrix, H^{-1} is $\{0, \pm 1, \pm i\}$-matrix. It is well known that the hermitian adjacency matrix of a tree mixed graph $\mathbb{T}$ is similar to the adjacency matrix of the underlying graph $\Gamma(\mathbb{T})$. However, the inverse of the adjacency matrix of a tree graph T is an adjacency matrix of a graph if and only if $T = K_2$, yet it is $\{\pm 1\}$-diagonally similar to adjacency matrix of a graph T'. The case is different for tree mixed graphs. In fact, the inverse of the Hermitian adjacency matrix of a mixed tree $\mathbb{T}$ graph can be a Hermitian adjacency matrix of a mixed graph $\mathbb{T}'$. In general, the inverse of Hermitian adjacency matrix of a mixed graph X can be a Hermitian adjacency matrix of a mixed graph X' even when the inverse of the adjacency matrix of its underlying graph $\Gamma(X)$ is not an adjacency matrix of a mixed graph Y.

Example 2 In Fig. 2 one can easily check that the inverse of the adjacency matrix A of the graph $\Gamma(X)$ is given by

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

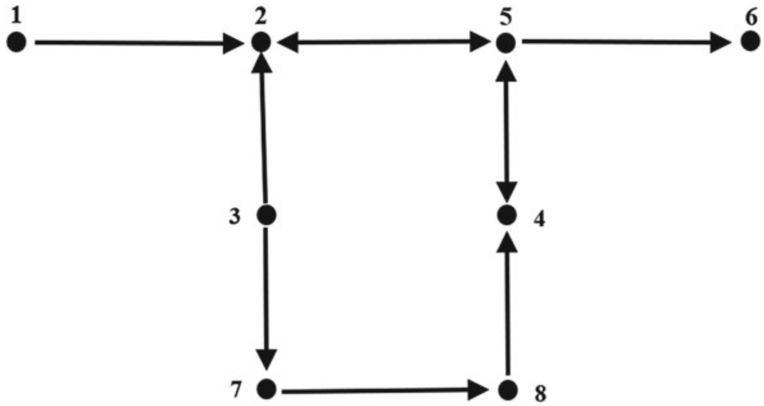


Fig. 2 Mixed graph X

While the inverse of the Hermitian adjacency matrix of the mixed graph X is given by

$$H^{-1} = -\begin{pmatrix} 0 & i & 0 & -i & 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 1 & i & 0 \\ i & 0 & -i & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 & -i & 0 & 0 & 1 \\ i & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Clearly, A^{-1} is not $\{0, \pm 1\}$-matrix, while H^{-1} is Hermitian adjacency matrix of the mixed graph X.

Let D be a bipartite mixed graph with bipartition sets U and V, since the vertices of each U and V are independent, the vertices of D can be relabeled so that the hermitian adjacency matrix of D can be written as

$$H(D) = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

where B^* is the transpose conjugate of B. Note that if $|U| \neq |V|$, then $H(D)$ does not have a full rank, therefore $H(D)$ is singular, moreover $\det(H(D)) = \det(-BB^*) = (-1)^n \det(B)\overline{\det(B)}$ this can be summarized in the following observation.

Observation 1 *Let D be a bipartite mixed graph with bipartition sets U and V and $H(D) = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ its hermitian matrix, then*

1. *If* $\det(H) \neq 0$ *then* $|U| = |V|$
2. $\det(H) \neq 0$ *if and only if* $\det(B) \neq 0$*, moreover,* $\det(H) = (-1)^n |\det(B)|^2$

Godsil in [10] proved that for a bipartite graph G with adjacency matrix $A = \begin{bmatrix} 0 & B(G) \\ B^*(G) & 0 \end{bmatrix}$, G has a unique perfect matching if and only if the vertices of G can be ordered such that $B(G)$ is lower triangle with diagonal entries ones, the following theorem is an extension of Godsil's work for mixed graph.

Theorem 1 *Let D be a bipartite mixed graph with bipartition sets U and V then $\Gamma(D)$ has a unique perfect matching if and only if the vertices in U and V can be ordered such that $H(D) = \begin{bmatrix} 0 & B(D) \\ B^*(D) & 0 \end{bmatrix}$, where $B(D)$ is the lower triangle matrix with all of its diagonal entries from the set $\{1, i, -i\}$.*

Proof Suppose that $\Gamma(D)$ has a perfect matching set M, suppose that $U = \{u_1, u_2, \dots, u_n\}$, using the perfect matching set M we may assume that $V = \{v_1, v_2, \dots, v_n\}$ where v_i is the unique vertex such that $u_i v_i \in M$. Since $\Gamma(D)$ is a bipartite graph and has exactly one perfect matching it is clear that U contains a pedant vertex say u_i, denote u_i and v_i by 1 and $1'$ respectively see Fig. 3. It is obvious that deleting the vertices 1 and $1'$ will leave us with a new bipartite mixed graph D', with bipartition sets U' and V'. Furthermore $\Gamma(D')$ has a unique perfect matching. Continuing in this process allows us to relabel U and V vertices. By construction and since we consider pedant vertex in each step, B under the new labeling of vertices should be lower triangle matrix with diagonal entries correspond to the perfect matching edges in $\Gamma(D)$ and values correspond to the directions of the arcs are from U to V.

On the other hand, suppose that $H(D) = \begin{bmatrix} 0 & B(D) \\ B^*(D) & 0 \end{bmatrix}$ and $B(D)$ is a lower triangle, then since any perfect matching of $\Gamma(D)$ determines a permutation matrix dominated by $B(D)$, $\Gamma(D)$ would have exactly one perfect matching. □

Example 3 In Fig. 3 and according to the red labeling the hermitian adjacency matrix of D is

$$H(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & i & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & i & 1 \\ -i & -i & 0 & i & 0 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

in this case the determinant of $B(D)$ can be easily calculated by multiplying the value of the arcs from U to V.

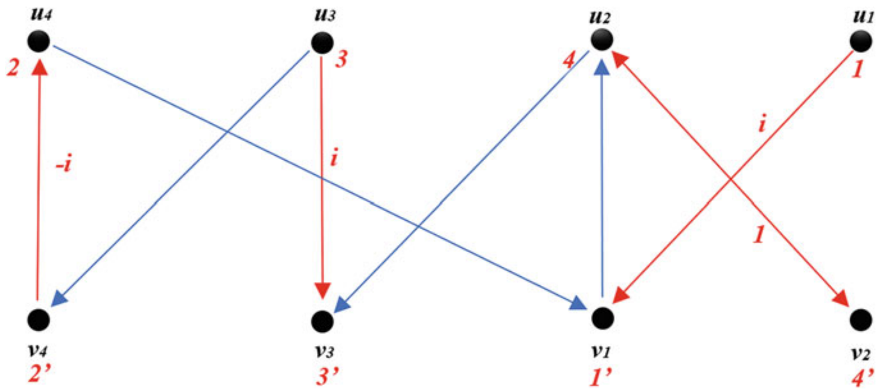


Fig. 3 The red unoriented edges form unique perfect matching for $\Gamma(D)$, according to red labeling $B(D)$ is lower triangle

Corollary 1 *Let D be a bipartite mixed graph with bipartition sets U and V and $H(D) = \begin{bmatrix} 0 & B(D) \\ B^*(D) & 0 \end{bmatrix}$, where $B(D)$ corresponds to the edges from U to V, if $\Gamma(D)$ has a unique perfect matching M then* $\det(B(D)) = \prod_{e \in M} f(e)$ *where*

$$f(e) = \begin{cases} 1 & \text{if } e \text{ is digon,} \\ i & \text{if } e \text{ is an arc from } U \text{ to } V, \\ -i & \text{if } e \text{ is an arc from } V \text{ to } U. \end{cases}$$

Moreover, $\det(B(D)) = i^{k-r}$ *where k is the number of arcs from U to V and r is the number of arcs from V to U.*

Theorem 2 *Let T be a tree mixed graph, if $\Gamma(T)$ has a perfect matching then $[H(T)]^{-1}$ is $\{0, 1, -1, i, -i\}$ matrix.*

Proof Let T be a tree mixed graph and $H(T) = \begin{bmatrix} 0 & B(T) \\ B^*(T) & 0 \end{bmatrix}$ its hermitian adjacency matrix, since $\Gamma(T)$ has a perfect matching, then H is invertible, moreover

$$(H(T))^{-1} = \begin{bmatrix} 0 & [B^*(T)]^{-1} \\ [B(T)]^{-1} & 0 \end{bmatrix}.$$

It is enough to show that every minor of the matrix $B(T)$ belongs to $\{0, 1, -1, i, -i\}$, let $[B(T)]_{rs}$ be the minor of $B(T)$ after removing r^{th} row and s^{th} column, then the hermitian adjacency matrix

$$H' = \begin{bmatrix} 0 & [B(T)]_{rs} \\ [B(T)]^*_{rs} & 0 \end{bmatrix}$$

is the hermitian adjacency matrix of the forest $T' = T - \{v_r, v_s\}$ (the induced subgraph over the vertex set $V(T) - \{v_r, v_s\}$) therefore, $\det([B(T)]_{rs}) \in \{0, \pm 1, \pm i\}$, and so, $(B(T))^{-1}$ is $\{0, \pm 1, \pm i\}$ matrix. □

Acknowledgements The authors wish to acknowledge the support by the Deanship of Scientific Research at German Jordanian University.

References

1. Guo, K., Mohar, B.: Hermitian adjacency matrix of digraphs and mixed graphs. J. Graph Theory **85**(1), 217–248 (2017)
2. Mohar, B.: A new kind of Hermitian matrices for digraphs. Linear Algebra Its Appl. **584**, 343–352 (2020)
3. Biggs, N., Biggs, N.L., Norman, B.: Algebraic Graph Theory, vol. 67. Cambridge University Press (1993)
4. Liu, J., Li, X.: Hermitian-adjacency matrices and Hermitian energies of mixed graphs. Linear Algebra its Appl. **466**, 182–207 (2015)
5. Alomari, O., Abudayah, M., Sander, T.: The non-negative spectrum of a digraph. Open Math. **18**(1), 22–35 (2020)
6. Abudayah, M., Alomari, O., Sander, T.: On the N-spectrum of oriented graphs. Open Math. **18**(1), 486–495 (2020)
7. Wilf, H.S.: The eigenvalues of a graph and its chromatic number. J. London Math. Soc. **1**(1), 330–332 (1967)
8. Chen, C., Huang, J., Li, S.: On the relation between the H-rank of a mixed graph and the matching number of its underlying graph. Linear Multilinear Algebra **66**(9), 1853–1869 (2018)
9. Cvetković, D.M., Gutman, I., Simić, S.K.: On self pseudo-inverse graphs. Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika 602/633 (1978): 111-117
10. Godsil, C.D.: Inverses of trees. Combinatorica **5**(1), 33–39 (1985)

Properties of Sierpinski Triangle Graphs

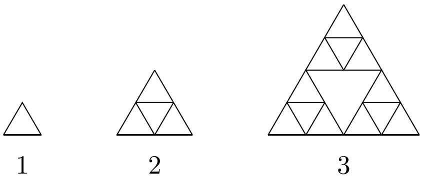
Allan Bickle

Abstract The Sierpinski triangle can be modeled using graphs in two different ways, resulting in classes of graphs called Sierpinski triangle graphs and Hanoi graphs. The latter are closely related to the Towers of Hanoi problem, Pascal's triangle, and Apollonian networks. Parameters of these graphs have been studied by several researchers. We determine the number of Eulerian circuits of Sierpinski triangle graphs and present a significantly shorter proof of their domination number. We also find the 2-tone chromatic number and the number of diameter paths for both classes, generalizing the classic Towers of Hanoi problem.

Keywords Sierpinski triangle graph · Hanoi graph · Eulerian circuit · Diameter · 2-tone coloring

1 Introduction

The Sierpinski triangle is a familiar fractal. One way to iteratively construct it is to start with a triangle (level 1). In each step, combine three copies of level k together to produce level $k + 1$.



There are two ways to model the Sierpinski triangle as a graph.

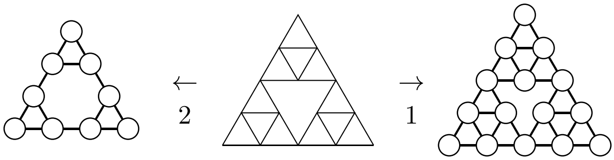
In Model 1, each intersection of lines is represented by a vertex, and each line segment between vertices is represented by an edge.

A. Bickle (✉)
Purdue University, West Lafayette, USA
e-mail: aebickle@purdue.edu
URL: https://allanbickle.wordpress.com/

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_26

In Model 2, each copy of level 1 of the fractal is represented by a vertex, and there are edges between vertices that have a point in common.

The graphs in Model 2 can be considered duals of sorts for the graphs in Model 1.



Undefined notation and terminology will generally follow [1].

2 Sierpinski Triangle Graphs

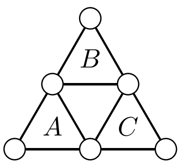
The graphs in Model 1 are known as **Sierpinski triangle graphs**. Denote the level k Sierpinski triangle graph ST_k. Thus $ST_1 = K_3$.

The recursive construction naturally leads to recurrence relations for many parameters. A recurrence for order is $n_1 = 3$, $n_{k+1} = 3n_k - 3$, with solution $n_k = \frac{3}{2}3^{k-1} + \frac{3}{2}$. The size of ST_k is clearly $m_k = 3^k$. Note that ST_k has 3 degree 2 vertices and $\frac{3}{2}3^{k-1} - \frac{3}{2}$ degree 4 vertices. Denote the degree 2 vertices of ST_k as **corners**, and the three vertices contained in two copies of ST_{k-1} as **middle vertices**.

Many researchers have determined properties of these graphs. Hinz et al. [4] have a survey of Sierpinski triangle graphs and related concepts. Bradley [3] found that for $k \geq 3$, There are $3^{\frac{3^{k-2}-3}{2}} 2^{3^{k-2}}$ Hamiltonian cycles in ST_k.

Since ST_k is connected with all even degrees, it is Eulerian. We can count the number of Eulerian circuits, considered as ordered lists of edges without regard to starting vertex or direction. This sequence begins 1, 16, 65536, 4503599627370496 ...

Theorem 1 *There are* $4^{3^{k-1}-1}$ *Eulerian circuits in* ST_k.



Proof Let E_k be the number of Eulerian circuits of ST_k. Clearly $E_1 = 1$. Consider ST_{k+1}, which is formed from three copies of ST_k, which we denote A, B, and C. The circuit can be split into segments based on which of A, B, or C contain each edge. We split it each time the circuit reaches a middle vertex. Then there are exactly two segments for each of A, B, and C. The segments may occur in a consistent order (e.g. $ABCABC$) or there may be a reversal (e.g. $ABCCBA$) where the circuit touches a middle vertex but continues in the same copy of ST_k. There cannot be more than one reversal, since then the circuit would not be Eulerian.

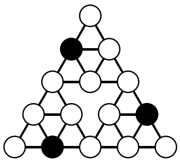
If there is a reversal, there are three middle vertices where it may occur. Without loss of generality, assume the circuit has the form $ABCCBA$. The segments of the circuit in A correspond to an Eulerian circuit of A. To construct an Eulerian circuit of G, we start with A and choose a trail to B in $2E_k$ ways. Similarly, there are $2E_k$ trails through B and C, after which the rest of the circuit is forced. There are two directions that could be taken, so divide by 2 to find $4\,(E_k)^3$ choices for each of the three reversals.

If there is no reversal, we similarly have $2E_k$ choices for each subgraph, and two directions, so there are $4\,(E_k)^3$ choices. Thus $E_{k+1} = 3 \cdot 4\,(E_k)^3 + 4\,(E_k)^3 = 16\,(E_k)^3$.

It is easily checked that $E_k = 4^{3^{k-1}-1}$ is the solution to this recurrence relation. □

A **dominating set** of a graph G is a set S of vertices so that every vertex not in S is adjacent to a vertex in S. The **domination number** $\gamma\,(G)$ is the minimum size of a dominating set of G. Teguia and Godbole [6] showed that the domination number $\gamma\,(ST_k) = 3^{k-2}$ for $k \geq 3$. Their proof is about 1.5 pages. A much shorter proof of this uses a discharging-type argument.

Proposition 1 ([6]) *We have* $\gamma\,(ST_k) = 3^{k-2}$ *for* $k \geq 3$.



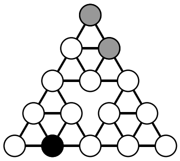
Proof Clearly $\gamma\,(ST_3) \leq 3$. Also, $\gamma\,(ST_{k+1}) \leq 3\gamma\,(ST_k)$, so $\gamma\,(ST_k) \leq 3^{k-2}$ for $k \geq 3$.

For a lower bound, consider the 3^{k-2} copies of ST_2 in ST_k. Any minimum dominating set S of ST_k contains at least one degree 4 vertex or at least two corners of every ST_2. Assign each copy of ST_2 a score of 1 for each degree 4 vertex in S and $\frac{1}{2}$ for each corner in S. Let t be the total score. Then each ST_2 gets a score of at least 1, so $\gamma\,(ST_k) = |S| \geq t \geq 3^{k-2}$. □

Teguia and Godbole [6] showed that $diam\,(ST_k) = 2^{k-1}$. We can determine which pairs of vertices achieve this maximum.

Proposition 2 *For* $k \geq 1$, $diam\,(ST_k) = 2^{k-1}$, *and for* $k \geq 2$, *the pairs of vertices at distance* 2^{k-1} *are those on different exterior sides of the graph such that their distances to the closest middle vertices sum to at least* 2^{k-2}.

For example, the vertices at maximum distance from the black vertex are colored gray.



Proof This is obvious for ST_1 and ST_2. For ST_k, let u, v be vertices at maximum distance. A geodesic between them must go through a middle vertex w. Now

$$d(u, v) \leq d(u, w) + d(w, v) \leq 2^{k-2} + 2^{k-2} = 2^{k-1}.$$

For this to be an equality, u and v must be on the exterior sides of their copies of ST_{k-1}. Thus they are on opposite exterior sides of ST_k. If the sum of their distances to the closest middle vertices is less than 2^{k-2}, there is a shorter path through the third copy of ST_{k-1}. □

Corollary 1 *For $k \geq 2$, there are $3\left(2^{k-2}+1\right)2^{k-2}$ pairs of vertices at distance 2^{k-1} in ST_k.*

Proof There are three pairs of corners and three pairs of corner and middle vertex. Consider a vertex u that is distance r from the nearest middle vertex. Then the other end v of a maximum distance $u - v$ path must be at least $2^{k-2} - r$ from its nearest middle vertex. Thus there are $r + 1$ choices for v. By symmetry, there are

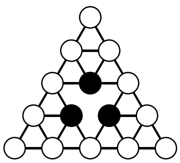
$$3 + 3 + 6 \sum_{r=1}^{2^{k-2}-1} (r+1) = 6 \sum_{r=1}^{2^{k-2}} r = 6\frac{1}{2}\left(2^{k-2}+1\right)2^{k-2} = 3\left(2^{k-2}+1\right)2^{k-2}$$

such pairs. □

We can also determine the radius of ST_k.

Proposition 3 *For $k \geq 3$, $rad(ST_k) = 3 \cdot 2^{k-3}$.*

Proof Let v be a vertex of ST_k, and u be a vertex at maximum distance from v. Then u is in a different copy of ST_{k-1} from v. Let w be a middle vertex contained in the copies of ST_{k-1} containing u and v. Then $d(v, u) = d(v, w) + d(w, u) \geq \frac{1}{2}2^{k-2} + 2^{k-2} = 3 \cdot 2^{k-3}$. All three vertices that are midpoints of the geodesic paths between the middle vertices of ST_k achieve this minimum. □



A graph is **uniquely 3-colorable** if its vertex set has a unique partition into 3 independent sets. Klavzar [5] proved that ST_k is uniquely 3-colorable. We include the proof of this result since it is essential to the corollary that follows it.

Proposition 4 ([5]) *For $k \geq 1$, ST_k is uniquely 3-colorable.*

Proof We use induction on the assumption that ST_k is uniquely 3-colorable and has three different colors on its corners. For ST_1, this is obvious; assume it holds for ST_{k-1}. We construct ST_k from three copies of ST_{k-1}. Thus the middle vertices of ST_k have three distinct colors. Thus the colorings of each copy of ST_{k-1} are forced, and there are three different colors on the corners of ST_k. □

In fact, ST_k is critical with respect to this property–deleting any edge results in a graph that is not uniquely 3-colorable.

Corollary 2 *For $k \geq 1$, ST_k is critical with respect to the property of being uniquely 3-colorable.*

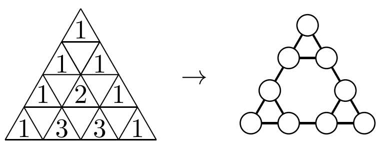
Proof For ST_1, deleting any edge e results in a 2-colorable graph. When constructing $ST_2 - e$, this allows a coloring with a common color on two corner vertices. Similarly, there is a coloring of $ST_k - e$ with a common color on two corner vertices. □

3 Hanoi Graphs

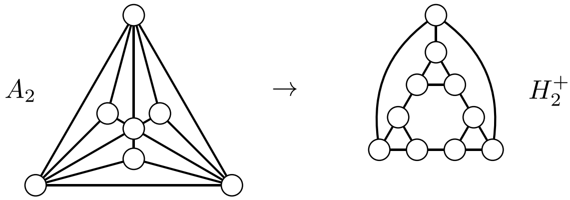
The graphs in Model 2 are known as **Hanoi graphs**. Denote the level k Hanoi graph H_k, so $H_1 = K_3$. The order of H_k is clearly $n_k = 3^k$. A recurrence for size is $m_1 = 3, m_{k+1} = 3m_k + 3$, with solution $m_k = \frac{3}{2}\left(3^k - 1\right)$. Note that H_k has 3 degree 2 vertices (**corners**) and $3^k - 3$ degree 3 vertices. Denote the three edges joining two copies of H_{k-1} as **middle edges**. Note that contracting every edge of H_k that is not on a triangle produces a Sierpinski triangle graph ST_{k-1}.

Hanoi graphs get their name from the Towers of Hanoi problem. In this problem, there are three pegs and k disks of different sizes, and a larger disk cannot be placed on a smaller disk. One disk at a time can be moved to another peg. This can be modeled with a graph. Label the pegs 0, 1, and 2 and assign each state a string indicating which peg contains the disks from smallest to largest. An edge joins two vertices when there is a valid move between the pegs. The result is a Hanoi graph.

Hanoi graphs also appear in Pascal's Triangle. Let vertices represent each binomial coefficient $\binom{r}{x}$ with $x < 2^k$ whose value is odd. Add edges between $\binom{r_1}{x_1}$ and $\binom{r_2}{x_2}$ if $r_1 = r_2$ or $r_1 = r_2 - 1$ and $|x_1 - x_2| \leq 1$. The result is the Hanoi graph H_k.



An **Apollonian network** is a planar 3-tree. Consider a particular class of Apollonian networks A_k, where $A_0 = K_3$, and A_{k+1} is formed by adding degree 3 vertices in all bounded triangular regions of A_k. Now the weak dual of A_k (excluding the outside region) is H_k. The dual graph is called the **extended Hanoi graph** H_k^+, which can be formed from H_k by adding a degree 3 vertex adjacent to the three corners. Thus H_k^+ is cubic.



There is a survey of Hanoi graphs and related concepts in [4]. A graph is **uniquely 3-edge-colorable** if its edge set has a unique partition into 3 independent sets. Klavzar [5] proved that H_k is uniquely 3-edge-colorable using a bijection with 3-colorings of ST_k. A direct (inductive) proof is also possible.

Proposition 5 ([5]) *For all $k \geq 1$, H_k is uniquely 3-edge-colorable.*

Proof We use induction on the hypothesis that H_k is uniquely 3-edge-colorable, and the colors not used on the three corners are distinct. For H_1, this is obvious. Assume this is true for H_k, and use three copies of H_k to construct H_{k+1}. Each pair of edges added to join these three graphs have distinct colors, so all three do. Thus the coloring of one copy of H_k forces the colorings of the other two copies of H_k. Also, the colors not used on the three corners are distinct. □

The solution to the classic Towers of Hanoi problem is that it takes $2^k - 1$ moves to transfer all k disks from one peg to another peg. In graph theory terms, this means that the distance between two corners of H_k is $2^k - 1$. More generally, it is easy to prove that $diam\,(H_k) = 2^k - 1$. We can determine which pairs of vertices achieve this maximum.

Theorem 2 *For $k \geq 1$, $diam\,(H_k) = 2^k - 1$, and for $k \geq 2$, the pairs of vertices at distance $2^k - 1$ are those on different exterior sides of the graph such that their distances to the closest corner vertices sum to at most 2^{k-1}.*

Proof This is obvious for H_1 and H_2. For H_k, let u, v be vertices at maximum distance. A geodesic between them must go through a middle edge $e = xy$. Now

$$d\,(u, v) \leq d\,(u, x) + 1 + d\,(y, v) \leq \left(2^{k-1} - 1\right) + 1 + \left(2^{k-1} - 1\right) = 2^k - 1.$$

For this to be an equality, u and v must be on the exterior sides of their copies of H_{k-1}. Thus they are on opposite exterior sides of H_k. If the sum of their distances to the closest corner vertices is more than 2^{k-1}, there is a path through the third copy of H_{k-1} with length at most

$$2\left(2^{k-1} - 1\right) - \left(2^{k-1} + 1\right) + 1 + \left(2^{k-1} - 1\right) + 1 = 2^k - 2,$$

which is shorter. □

Corollary 3 *For* $k \geq 1$, *there are* $3\left(2^{2k-2}+2^{k-1}-1\right)$ *pairs of vertices at distance* 2^k-1 *in* H_k.

Proof There are three pairs of corners. Consider a vertex u that is distance r from the nearest corner vertex. Then the other end v of a maximum distance $u-v$ path must be at most $2^{k-1}-r$ from its nearest corner vertex. Thus there are $2^{k-1}-r+1$ choices for v. By symmetry, the number P_k of such pairs is

$$\begin{aligned}
P_k =& 3+6\sum_{r=1}^{2^{k-1}-1}\left(2^{k-1}-r+1\right)\\
=& 3+6\left(\left(2^{k-1}-1\right)\left(2^{k-1}+1\right)-\frac{1}{2}\left(2^{k-1}-1\right)\left(2^{k-1}\right)\right)\\
=& 3+6\left(2^{k-1}-1\right)\left(2^{k-2}+1\right)\\
=& 3\left(2^{2k-2}+2^{k-1}-1\right)
\end{aligned}$$

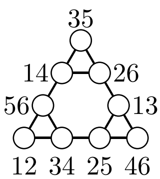
□

A **proper** 2**-tone coloring** of a graph G assigns two distinct colors to each vertex so that adjacent vertices have no common colors, and vertices at distance 2 have at most one common color. The 2**-tone chromatic number** of G, $\tau_2(G)$, is the smallest k for which G has a proper t-tone k-coloring.

If H is a subgraph of G then $\tau_2(H) \leq \tau_2(G)$. Since $\tau_2(K_n) = 2n$, we have $\tau_2(G) \geq 2\omega(G)$. See [2] for basic information on 2-tone coloring.

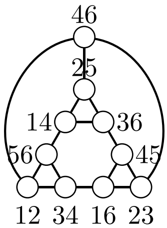
Proposition 6 *For* $k \geq 1$, $\tau_2(H_k) = 6$.

Proof Since $K_3 \subseteq H_k$, we have $\tau_2(H_k) \geq \tau_2(K_3) = 6$. A 2-tone 6-coloring can be found by piecing together copies of H_2 with the coloring below. □

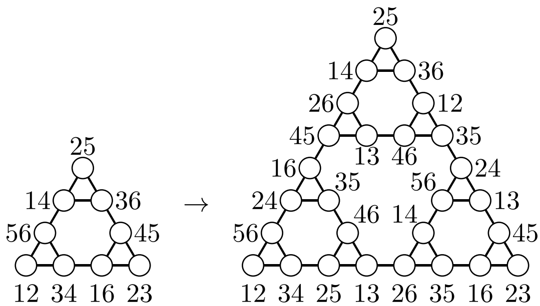


A somewhat more complicated argument is required to show that $\tau_2\left(H_k^+\right) = 6$ for all $k > 1$.

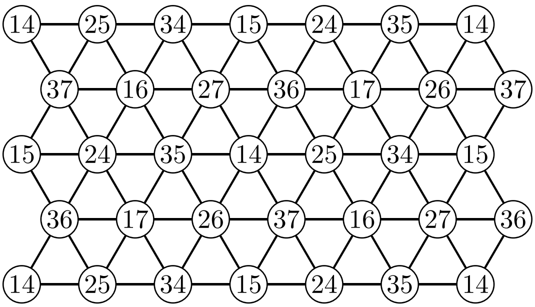
Theorem 3 *For* $k \geq 2$, $\tau_2\left(H_k^+\right) = 6$.



Proof For $k = 2$, the coloring above works. For larger values of k, we use three copies of this coloring, but permute the colors to avoid conflicts. Clearly, permuting colors within a copy of H_k cannot create a conflict. In each copy of H_k, swap the pairs of colors in the left and top of the K_3 containing the corner. This maintains the same labels on the K_3s containing the corners and does not create any conflicts between the three H_ks. This is illustrated for H_3 below. The label 46 can be added to the extra vertex. □



The 2-tone chromatic number of Sierpinski triangle graphs is 7 for $k \geq 2$. The infinite triangular grid has 2-tone chromatic number 7, and Sierpinski triangle graphs are subgraphs of it. The coloring below can be extended infinitely since the boundaries use the same colors.



References

1. Bickle, A.: Fundamentals of Graph Theory, AMS (2020)
2. Bickle, A., Phillips, B.: t-Tone colorings of graphs. Utilitas Math. **106**, 85–102 (2018)
3. Bradley, R.M.: Statistical mechanics of the travelling salesman on the Sierpinski gasket. J. Physique **47**, 9–14 (1986)
4. Hinz, A., Klavzar, S., Zemljic, S.: A survey and classification of Sierpinski-type graphs. Discrete Appl. Math. **217**(3), 565–600 (2017)
5. Klavzar, S.: Coloring Sierpinski graphs and Sierpinski gasket graphs. Taiwanese J. Math. **12**(2), 513–522 (2008)
6. Teguia, A., Godbole, A.: Sierpinski Gasket graphs and some of their properties. Australasian J. Combin. **35**, 181–192 (2006)

Counting Vertices in Iterated Line Graphs

Zack King, Liz Lane-Harvard, and Thomas Milligan

Abstract We introduce new methods of studying properties of iterated line graphs and demonstrate the use of these methods on a class of tree graphs. We also describe plans to generalize these methods further and derive identities for the number of vertices in the iterated line graphs of arbitrary graphs among other potential extensions of the methodology.

Keywords Line graphs · Tree graphs · Graph families · Enumeration

1 Introduction

This paper will cover the construction and development of a "P-Function" on graphs which will be used to count degrees of vertices in the iterated line graphs of arbitrary connected, simple graphs that are not necessarily finite. For a more detailed explanation of these terms, see [3].

In Sect. 1, we establish definitions and some general results which will be used later in the paper. Of particular interest is Theorem 1, which allows the order and the sum of the degrees of the vertices in iterated line graphs to be expressed in terms the P-Function. In Sect. 2 we construct a graph, Ω_δ, and derive results on the P-function centered around a particular vertex in this graph. Finally, in Sect. 3, we discuss plans for extensions of our methodology with the major goal of deriving general identities for the number of vertices in the iterated line graphs of arbitrary graphs.

Z. King · L. Lane-Harvard (✉) · T. Milligan
Department of Mathematics and Statistics, University of Central Oklahoma,
Edmond, OK 73034, USA
e-mail: elaneharvard@uco.edu

T. Milligan
e-mail: tmilligan1@uco.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_27

1.1 Basic Definitions

A graph $G = (V, E)$ is defined as an ordered pair such that V is a set of vertices and $E \subseteq \{\{x, y\} \mid x \in V, y \in V, x \neq y\}$ is a set of edges. Additionally, for any graph G, let $V(G) = V$ and $E(G) = E$.

We will say two vertices $x, y \in V(G)$ are **adjacent**, denoted $x \sim y$, if and only if there exists $e \in E(G)$ such that $x \in e, y \in e$, and $x \neq y$. Likewise, two edges, $e, f \in E(G)$ are adjacent if and only if there exists $x \in V(G)$ such that $x \in e, x \in f$, and $e \neq f$. An edge $e \in E(G)$ and a vertex $x \in V(G)$ are **incident** if and only if $x \in e$. If $x \in V(G)$, then the **degree** of x, denoted $\deg_G(x)$, will be the number of edges incident to x; similarly, if $e \in E(G)$, then $\deg_G(e)$ will be the number of edges adjacent to e.

A **finite graph** is a graph G where $|V(G)|$ and $|E(G)|$ are both finite, and a **locally finite graph** is a graph G with $\forall x \in V(G)$, $\deg_G(x)$ is finite. All graphs considered in this paper will be finite or locally finite.

The **distance** between two vertices $x, y \in V(G)$, denoted $d(x, y)$, is the number of edges in the shortest path between x and y.

Throughout this paper we will frequently refer to **edge sets**, which we will be defining as sets of doubletons or unordered pairs. Additionally, given an edge set S, the graph **induced** by S, denoted S^*, is given by $E(S^*) = S$ and $V(S^*) = \{x \mid x \in e \in S\}$. That is, if $G = (V, E)$ and $S \subset E$, then $S^* = (V(S), S)$, where $V(S) = \{x \in V : x \in e \in S\}$.

The **intersection product** of two edge sets A, B, denoted $A \triangle B$, is a new edge set defined by $A \triangle B = \{\{x, y\} \mid x \in A, y \in B, x \neq y, x \cap y \neq \emptyset\}$. Utilizing this definition, given a graph G, the **line graph** of G, denoted $L(G)$, will be a graph such that $V(L(G)) = E(G)$ and $E(L(G)) = E(G) \triangle E(G)$. Note that, by this definition, the $\triangle$ operator is commutative; that is, $A \triangle B = B \triangle A$.

We'll be looking at iterating the line graph operator, so the n^{th} **intersection exponent** of an edge set A, denoted $A^{\triangle n}$, is defined recursively by $A^{\triangle 0} = A$ and $A^{\triangle k+1} = A^{\triangle k} \triangle A^{\triangle k}$ for $k \geq 1$. Utilizing this notation, the n^{th} line graph of G, denoted $L^n(G)$, is a graph given by

$$L^n(G) = \begin{cases} G & \text{if } n = 0 \\ (E(G)^{\triangle(n-1)}, E(G)^{\triangle n}) & \text{if } n \geq 1. \end{cases}$$

This paper will focus on studying the properties of a P-function on iterated line graphs. If S is a set, the **P-function** on x of S, denoted $P_x(S)$ is defined recursively as follows:

$$P_x(S) = \begin{cases} 1 & \text{if } x \in S \text{ or } x = S \\ \sum_{y \in S} P_x(y) & \text{Otherwise} \end{cases}$$

where an empty sum is taken to be 0. Here, x is an indeterminate, which may be a set, vertex, edge, or other object, and we are intentionally maintaining this ambiguity for the sake of keeping this definition general. In the case when x is not being treated as a set, we will simply define, for another indeterminate y, $P_x(y) = 1$ if $y = x$ and $P_x(y) = 0$ otherwise. Furthermore, if G is a graph, let $P_x(G) := P_x(E(G))$. We will say that a graph G is **homogeneous** if for every $x, y \in V(G)$ with $y \neq x$, $P_x(y) = 0$. It is trivial to see that every inhomogeneous graph is isomorphic to a homogeneous graph by a simple relabeling of vertices. Furthermore, if G is homogeneous then we note that $P_x(G) = \deg_G(x)$ for every $x \in G$ since there will be exactly $\deg_G(x)$ edges in G of the form $\{x, y\}$ for some $y \in V(G)$.

We will say that an edge set E is **uniform** with respect to x if and only if there exists $c \in \mathbb{Z}$ such that $P_x(y) = c, \forall y \in E$. If E is uniform with respect to x, then we will define $\mu_x(E) = c$.

In Sect. 1.2 we will investigate propositions related to finite simple graphs. Furthermore, in Sects. 1.3 and 1.4, we look at two classes of graphs, defined below, with some well-known examples of the second class. In Sect. 2, we will be constructing a new type of graph and examining properties of the P-function centered at a particular vertex in this graph under iterations of the line graph operator. In Sect. 3, we will discuss some of the potential extensions of our methodology as well as our future goals.

A $k-$**regular graph** is a connected graph R such that $\deg_R(x) = k$ for some $k \in \mathbb{Z}$ and for all $x \in V(R)$.

A **bi-regular** graph, $\hat{B}$, is a graph that satisfies the following conditions:

- $V(\hat{B}) = A \cup B$ with $A \cap B = \emptyset$.
- Every edge $e \in E(\hat{B})$ is incident to exactly one vertex in A and exactly one vertex in B.
- There exists $d_a, d_b \in \mathbb{Z}$ such that for every $x \in A$, $\deg_{\hat{B}}(x) = d_a$, and, for every $y \in B$, $\deg_{\hat{B}}(y) = d_b$.

Furthermore, if $|A| = a$ and $|B| = b$, the bi-regular graph as described above will be associated with the notation

$$B_{a,b}(d_a, d_b).$$

It is important to stress that the above notation does not necessarily uniquely define a bi-regular graph, and for given values of a, b, d_a, and d_b there may be multiple graphs which are described by $B_{a,b}(d_a, d_b)$. However, for our purposes, one can think of $B_{a,b}(d_a, d_b)$ as identifying a class of graphs, and any results derived will apply to any graph within that class.

A **complete bipartite graph** is a bi-regular graph, with vertex partitions A and B, such that every vertex in A is adjacent to every vertex in B and vice versa. If $|A| = m$ and $|B| = n$ then we can denote this complete bipartite graph by $K_{m,n}$. Equivalently, $K_{m,n}$ is a bi-regular graph of the form $B_{m,n}(n, m)$.

Lastly, a **star graph**, S, is a graph with a vertex $x \in V(S)$, called the center vertex, such that every edge is incident to x and a distinct pendant vertex. The star graph with $n \geq 1$ vertices will be denoted S_n. Also, note that S_n is isomorphic to a bi-regular graph of the form $B_{n-1,1}(1, n-1)$ or a complete bipartite graph of the form $K_{1,n-1}$.

1.2 General Propositions

Proposition 1 is a well-known and famous result sometimes referred to as the "Handshaking lemma". We present it here without proof for use in the rest of the paper.

Proposition 1 *If G is a finite simple graph,*

$$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|.$$

Proposition 2 *If G is a simple graph and* $\{x, y\} \in E(G)$*, then* $\{x, y\} \in V(L(G))$ *and*

$$\deg_{L(G)}(\{x, y\}) = \deg_G(\{x, y\}) = \deg_G(x) + \deg_G(y) - 2$$

Proof If $x, y \in V(G)$, there are $\deg_G(x) - 1$ edges incident to x, excluding $\{x, y\}$ and $\deg_G(y) - 1$ edges incident to y excluding $\{x, y\}$. Since G is simple, $\{x, y\}$ is the only edge incident to both x and y, so, $\deg_{L(G)}(\{x, y\}) = \deg_G(\{x, y\}) = (\deg_G(x) - 1) + (\deg_G(y) - 1) = \deg_G(x) + \deg_G(y) - 2$. □

Theorem 1 *If G is a homogeneous finite simple graph and* $k \geq 1$*, then*

$$\sum_{v \in V(L^k(G))} \deg_{L^k(G)}(v) = \sum_{v \in V(G)} P_v(L^{k-1}(G)) \left(\deg_G(v) - 2 + 2^{1-k}\right).$$

Proof If $x \in V(L^k(G))$, then $x \in E(L^{k-1}(G))$, and $x = \{e, f\}$ for some $e, f \in E(L^{k-1}(G))$. Furthermore, by Proposition 2, we know $\deg_{L^k(G)}(x) = \deg_{L^{k-1}(G)}(e) + \deg_{L^{k-1}(G)}(f) - 2$. Since e and f are both elements of $V(L^{k-1}(G))$, they can also be expressed in terms of edges in $E(L^{k-2}(G))$; thus, Proposition 2 can be applied again. By applying the above argument k times, and observing that each iteration the number of distinct vertices being considered doubles, we see that

$$\begin{aligned}
\deg_{L^k(G)}(x) &= \left(\sum_{i=1}^{2^k} \deg_G(x_i)\right) - \sum_{i=1}^{k} 2^i \\
&= \left(\sum_{i=1}^{2^k} \deg_G(x_i)\right) - 2(2^k - 1) \\
&= \sum_{i=1}^{2^k} \left(\deg_G(x_i) - 2 + 2^{1-k}\right)
\end{aligned}$$

where $x_1, x_2, \cdots, x_{2^k} \in V(G)$. Now, recall, by definition, each vertex x will occur exactly $P_x(L^{k-1}(G))$ times in $E(L^{k-1}(G)) = V(L^k(G))$. It follows that,

$$\sum_{v \in V(L^k(G))} \deg_{L^k(G)}(v) = \sum_{v \in V(G)} P_v(L^{k-1}(G)) \left(\deg_G(v) - 2 + 2^{1-k}\right).$$

□

Corollary 1.1 *If G is a homogeneous finite simple graph and $k \geq 2$, then*

$$|L^k(G)| = |E(L^{k-1}(G))| = \frac{1}{2} \sum_{v \in V(G)} P_v(L^{k-2}(G)) \left(\deg_G(v) - 2 + 2^{2-k}\right).$$

Proof This follows directly from Proposition 1 and Theorem 1. □

Corollary 1.2 (Krausz's Lemma) *If G is a finite simple graph, then*

$$|L^2(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg_G(v) \left(\deg_G(v) - 1\right) = \sum_{v \in V(G)} \binom{\deg_G(v)}{2}.$$

Proof This follows directly from Corollary 1.1 and the observation that, given a homogeneous finite graph G and a vertex $x \in V(G)$, $P_x(G) = \deg_G(x)$. □

1.3 Propositions on Regular Graphs

Proposition 3 is a commonly-known result; see [1] for a proof.

Proposition 3 *For a given regular graph R of degree d, the kth line graph of R, $L^k(R)$, is a regular graph of degree $2^k(d-2)+2$.*

Proposition 4 *If $k \geq 0$ and R is a homogeneous $d-$regular graph with $d \geq 2$, then for every $v \in V(R)$,*

$$P_v(L^k(R)) = \prod_{j=0}^{k} \left(2^j(d-2)+2\right).$$

Proof We proceed by induction. For the base case, note that when $k = 0$, clearly, since R is homogeneous, $P_v(R) = d$ for each $v \in V(R)$ and

$$\prod_{j=0}^{0} \left(2^j(d-2)+2\right) = d.$$

Now, suppose that

$$P_v(L^i(R)) = \prod_{j=0}^{i} \left(2^j(d-2)+2\right)$$

for some $i \in \mathbb{Z}$. Each edge in $L^i(R)$ becomes a vertex in $L(L^i(R)) = L^{i+1}(R)$ and, by Proposition 3, we know that that each vertex in $L^{i+1}(R)$ is of degree $2^{i+1}(d-2)+2$.

Let $m = |V(L^{i+1}(R))|$. Label the vertices in $L^{i+1}(R)$ as $x_1, x_2, x_3, \cdots, x_m$ and let $p_j = P_v(x_i)$ for each $1 \leq j \leq m$. We know that each edge in $L^{i+1}(R)$ is of the form $\{x_j, x_q\}$ for $1 \leq j, q \leq m$ and $j \neq q$, furthermore, $P_v(\{x_j, x_q\}) = P_v(x_j) + P_v(x_q) = p_j + p_q$. For fixed j, there are precisely $2^{i+1}(d-2)+2$ edges of the form $\{x_j, x_q\}$; hence, when computing $P_v(L^{i+1}(R))$, p_j will be counted precisely $2^{i+1}(d-2)+2$ times for each $1 \leq j \leq m$. It follows that,

$$\begin{aligned} P_v(L^{i+1}(R)) &= P_v(L^i(R))(2^{i+1}(d-2)+2) \\ &= \left(\prod_{j=0}^{i} \left(2^j(d-2)+2\right)\right)(2^{i+1}(d-2)+2) \\ &= \prod_{j=0}^{i+1} \left(2^j(d-2)+2\right). \end{aligned}$$

□

Now, Corollary 4.1 illustrates how results about this P-function, such as Proposition 4, immediately yield results about the order and sum of the degrees of the vertices in iterated line graphs by application of Theorem 1.

Corollary 4.1 *If R is a regular graph of order n and degree $d \geq 2$, then,*

$$|L^k(R)| = n \prod_{j=0}^{k-1} (2^{j-1}(d-2)+1)$$

Proof Applying Corollary 1.1 to the formula from Proposition 4 yields

$$\begin{aligned} |L^k(R)| &= \frac{1}{2} \sum_{v \in R} \left(P_v(L^{k-2}(R))\left(\deg_R(v) - 2 + 2^{2-k}\right)\right) \\ &= \frac{1}{2} n \left(\prod_{j=0}^{k-2} \left(2^j(d-2)+2\right)\right)\left(d-2+2^{2-k}\right) \\ &= 2^{k-2} n \left(\prod_{j=0}^{k-2} \left(2^{j-1}(d-2)+1\right)\right)(d-2+2^{2-k}) \\ &= n \left(\prod_{j=0}^{k-2} \left(2^{j-1}(d-2)+1\right)\right)(2^{k-2}(d-2)+1) \\ &= n \prod_{j=0}^{k-1} \left(2^{j-1}(d-2)+1\right). \end{aligned}$$

□

1.4 Propositions on Bi-Regular Graphs

The following result, Proposition 5, is analogous to Proposition 4 but for bi-regular rather than regular graphs and can be extended via Corollary 1.1 by the same argument presented in the proof of Corollary 4.1 to derive a formula for $L^k(\hat{B})$.

Proposition 5 *If $\hat{B}$ is a homogeneous bi-regular graph with partitions A and B of the form $B_{a,b}(d_a, d_b)$, where $|A| = a$, $|B| = b$, every vertex in A is of degree d_a, every vertex in B is of degree d_b, $v_a \in A$, and $v_b \in B$, then*

$$P_{v_a}(L^k(\hat{B})) = d_a \prod_{j=0}^{k-1} \left(2^j(d_a + d_b - 4) + 2\right)$$

$$\textit{and } P_{v_b}(L^k(\hat{B})) = d_b \prod_{j=0}^{k-1} \left(2^j(d_a + d_b - 4) + 2\right).$$

Proof By Proposition 2, if $x \in A$, $y \in B$, and $\{x, y\} \in E(\hat{B})$, then

$$\deg_G(\{x, y\}) = \deg_{L(G)}(\{x, y\}) = \deg_G(x) + \deg_G(y) - 2 = d_a + d_b - 2.$$

Since this is true of every edge in $E(\hat{B})$, $L(\hat{B})$ is a regular graph of degree $d_a + d_b - 2$.

Now, given a vertex $v_a \in A$, there will be precisely d_a edges, $q_1, q_2, \cdots, q_{d_a} \in E(\hat{B}) = V(L(\hat{B}))$ such that $P_{v_a}(q_i) = 1$ for each $i \in \{1, \cdots, d_a\}$ and, for all other edges $e \in E(\hat{B}) = V(L(\hat{B}))$, $P_{v_a}(e) = 0$. Thus, if $i \in \{1, \cdots, d_a\}$, then, since $L(\hat{B})$ is a regular graph of degree $d_a + d_b - 2$, by Proposition 4,

$$P_{q_i}(L^{k+1}(\hat{B})) = P_{q_i}(L^k(L(\hat{B}))) = \prod_{j=0}^{k} \left(2^j(d_a + d_b - 4) + 2\right).$$

Thus, when $k \geq 1$

$$P_{q_i}(L^k(\hat{B})) = \prod_{j=0}^{k-1} \left(2^j(d_a + d_b - 4) + 2\right)$$

Since this is the same for all q_i for $i \in \{1, \cdots, d_a\}$, we have

$$P_{v_a}(L^k(\hat{B})) = d_a \prod_{j=0}^{k-1} \left(2^j(d_a + d_b - 4) + 2\right).$$

By a similar argument, if $v_b \in B$, then

$$P_{v_b}(L^k(\hat{B})) = d_b \prod_{j=0}^{k-1} \left(2^j(d_a + d_b - 4) + 2\right).$$

$\square$

Note that if the empty product is interpreted as being equal to 1, then the identities derived in Proposition 4.1 hold when $k = 0$. With this result, we can consider $S_n = K_{1,n-1}$ and more generally $K_{n,m}$, both of which are bi-regular graphs.

Corollary 5.1 *If $K_{m,n}$ is a complete bipartite graph with partitions A and B, then,*

$$P_x(L^k(K_{m,n})) = n \prod_{j=0}^{k-1} \left(2^j(m+n-4)+2\right) \qquad x \in A$$

$$\text{and } P_y(L^k(K_{m,n})) = m \prod_{j=0}^{k-1} \left(2^j(m+n-4)+2\right) \qquad y \in B.$$

Corollary 5.2 *If S_{n+1} is a star graph with $n+1$ vertices, then,*

$$P_x(L^k(S_{n+1})) = n \prod_{j=0}^{k-1} \left(2^j(n-3)+2\right) \qquad \text{for } x \text{ the central vertex and}$$

$$P_y(L^k(S_{n+1})) = \prod_{j=0}^{k-1} \left(2^j(n-3)+2\right) \qquad y \text{ a pendant vertex.}$$

2 Constructing Ω_δ

In Sect. 2 we will be constructing a class of locally finite tree graphs Ω_δ and proving results related to the partition counting function on Ω_δ of a particular vertex in $V(\Omega_\delta)$.

If i is a positive integer, let $\mathbb{Z}^{i+}$ be the set of ordered i-tuples of positive integers and let

$$\ell = \{\emptyset\} \cup \bigcup_{i=1}^{\infty} \mathbb{Z}^{i+}.$$

We refer to elements of ℓ as **multi-indices**. For a given multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, let $\alpha[i] = \alpha_i$ and let $|\alpha| = n$. For example, if $\alpha = (1, 3, 5, 2)$, then $|\alpha| = 4$ and $\alpha[3] = 5$. We treat $\emptyset$ as an empty tuple with $|\emptyset| = 0$.

If α and β are multi-indices then we say α is a **subindex** of β, denoted $\alpha \subset \beta$, if $|\beta| = |\alpha| + 1$ and $\beta[i] = \alpha[i]$ for each $1 \leq i \leq |\alpha|$. Likewise, in this case, β is a **superindex** of α. Every multi-index aside from $\emptyset$ has a unique subindex which can be found by removing the last element. For example, the unique subindex of $(1, 3, 5, 2)$ is $(1, 3, 5)$. If α is a multi-index and $n \in \mathbb{Z}^+$ then let $\alpha + n$ be a multi-index with n appended on to the end of α. For example, if $\alpha = (1, 3, 5, 2)$ then $\alpha + 4 = (1, 3, 5, 2, 4)$. Note that $\alpha \subset \alpha + n$ for any n. Also, if $\alpha = \emptyset$ then $\emptyset + n = (n)$.

A **degree function** is any function $\delta : \ell \to \mathbb{Z}_{\geq 0}$. Given a particular degree function δ, we define Ω_δ as follows:

- $V(\Omega_\delta) \subseteq \ell$.
- $\emptyset \in V(\Omega_\delta)$.
- If there exists $\beta \in V(\Omega_\delta)$ such that $\alpha = \beta + k$ for some positive integer $k \leq \delta(\beta)$, then $\alpha \in V(\Omega_\delta)$. Otherwise, $\alpha \notin V(\Omega_\delta)$.
- Two vertices $\alpha, \beta \in V(\Omega_\delta)$ are adjacent if and only if $\alpha \subset \beta$ or $\beta \subset \alpha$.

Lastly, we define a canonical partition ρ for $E(\Omega_\delta)$. Let $\rho = \{L_\alpha : \alpha \in V(\Omega_\delta)\}$ where an edge $\{x, y\} \in E(\Omega_\delta)$ is an element of L_α if and only if $x = \alpha$ and $\alpha \subset y$.

An illustration of Ω_δ for a particular instance of δ is given in Fig. 1. In this figure we have that, $\delta(\emptyset) = 4$, $\delta(\alpha) = 3$ for all $\alpha \in \ell$ such that $0 < |\alpha| \leq 2$, and $\delta(\beta) = 0$ for all other $\beta \in \ell$. The partitions defined by ρ are also highlighted with distinct colors in Fig. 1.

For the remainder of this paper, we will let δ represent any degree function and the results derived will apply regardless of the particular choice of degree function.

2.1 Properties of Ω_δ

Proposition 6 *Let* $\alpha \in V(\Omega_\delta)$. *If* $\alpha = \emptyset$, *then* $\deg_{\Omega_\delta}(\alpha) = \delta(\alpha)$; *otherwise* $\deg_{\Omega_\delta}(\alpha) = \delta(\alpha) + 1$.

Proof The only way a multi-index $\beta \in V(\Omega_\delta)$ could be adjacent to α is if $\beta \subset \alpha$ or $\alpha \subset \beta$. The superindices of α are all of the form $\alpha + n$ for some $n \leq \delta(\alpha)$ and there are precisely $\delta(\alpha)$ vertices in $V(\Omega_\delta)$ of this form. If $\alpha = \emptyset$ then it has no subindices so $\deg_{\Omega_\delta}(\alpha) = \delta(\alpha)$, otherwise, α has a unique subindex, so $\deg_{\Omega_\delta}(\alpha) = \delta(\alpha) + 1$. □

Proposition 7 Ω_δ *is locally finite.*

Proof By Proposition 6, if $\alpha \in V(\Omega_\delta)$ then $\deg_{\Omega_\delta}(\alpha) \leq \delta(\alpha) + 1$. Since δ is a degree function, $\delta(\alpha)$ is a nonnegative integer, implying $\deg_{\Omega_\delta}(\alpha)$ is finite. Since every vertex of Ω_δ is of finite degree, it is locally finite. □

Proposition 8 Ω_δ *is connected.*

Proof If $\alpha \neq \emptyset$ is in $V(\Omega_\delta)$ then its unique subindex is also in Ω_δ and is adjacent to α. This implies that that there exists a chain of connected vertices in Ω_δ of the form $\alpha \supset \alpha_1 \supset \alpha_2 \supset \alpha_3 \supset \cdots$ and we furthermore know that $|\alpha_n| = |\alpha| - n$ for each n. Since $|\alpha|$ is finite, this sequence must eventually terminate at $\emptyset$, implying there exists a path in Ω_δ from α to $\emptyset$. Thus, every vertex in Ω_δ is connected, via a path, to $\emptyset$. This implies that Ω_δ is connected. □

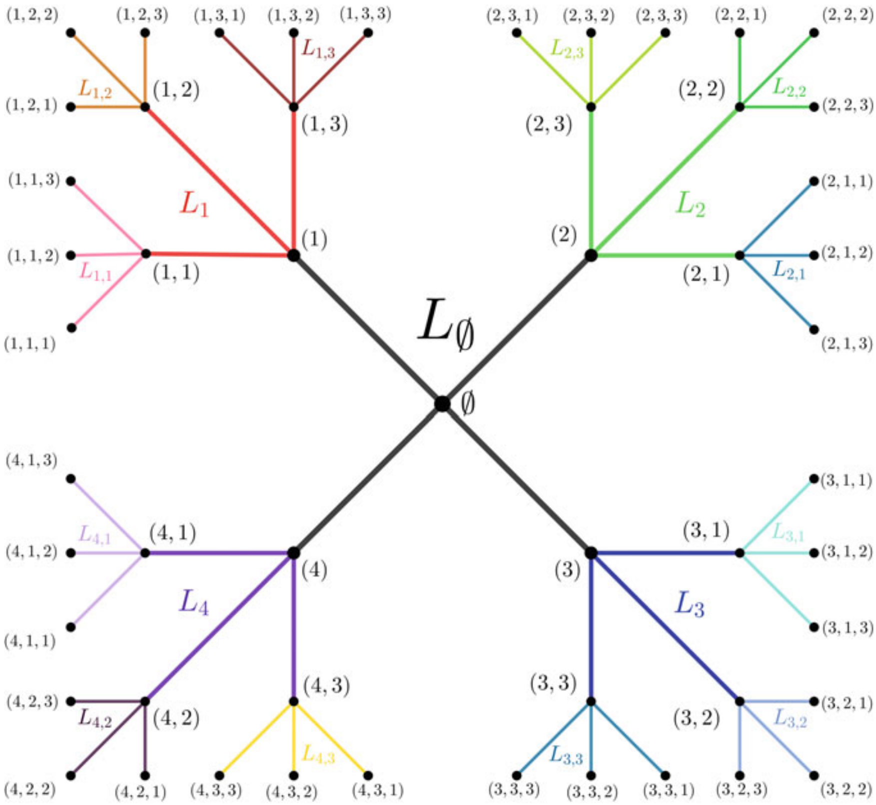


Fig. 1 Illustrated Example of Ω_δ

Proposition 9 *Ω_δ is acyclic.*

Proof Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n \in V(\Omega_\delta)$ and that they form a cycle. Since two vertices can only be adjacent if one is a subindex of the other, this implies, without loss of generality, that $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_n$ and $\alpha_n \subset \alpha_1$. Thus, $\alpha_1 \subset \alpha_n$ and $\alpha_n \subset \alpha_1$, which implies $|\alpha_n| = |\alpha_1| + 1$ and $|\alpha_1| = |\alpha_n| + 1$. Thus, $|\alpha_n| = |\alpha_n| + 1 + 1$ which is a contradiction. It follows that no such cycle can exist. $\square$

Corollary 9.1 *Ω_δ is a tree.*

Proof This follows directly from Propositions 8 and 9. $\square$

Proposition 10 *The set ρ is a disjoint partition of $E(\Omega_\delta)$. That is, if $\alpha, \beta \in V(\Omega_\delta)$ and $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \emptyset$.*

Proof Suppose α and β are distinct elements of $V(\Omega_\delta)$ and that $L_\alpha \cap L_\beta \neq \emptyset$. This implies that there exists at least one edge $\{x, y\} \in E(\Omega_\delta)$ such that $\{x, y\} \in L_\alpha$ and $\{x, y\} \in L_\beta$. Since $\{x, y\} \in L_\alpha$, either x or y must equal α. Suppose, without loss of generality, that $x = \alpha$. Then we must further have that $\alpha \subset y$. Now, since $\{x, y\} \in L_\beta$, $x = \alpha$, and $\alpha \neq \beta$, we must have that $y = \beta$ and $\beta \subset x = \alpha$. Thus, $\alpha \subset \beta$ and $\beta \subset \alpha$, which implies $|\alpha| = |\beta| + 1$ and $|\beta| = |\alpha| + 1$. Thus $|\alpha| = |\alpha| + 1 + 1$ which is a contradiction. It follows that if $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \emptyset$. □

Proposition 11 *If $\alpha \in V(\Omega_\delta)$ then the graph induced by L_α is isomorphic to the star graph $S_{\delta(\alpha)+1}$.*

Proof Every element of L_α is of the form $\{\alpha, y\}$ for some $y = \alpha + n$ with $n \leq \delta(\alpha)$. If L_α^* is the graph induced by L_α then every vertex, except for α, in L_α^* is only adjacent to α. Thus, L_α^* is isomorphic to a star graph with central vertex α. There are $\delta(\alpha)$ vertices in Ω_δ which are superindices of α, so $|V(L_\alpha^*)| = \delta(\alpha) + 1$, and thus, L_α^* is isomorphic to $S_{\delta(\alpha)+1}$. □

Proposition 12 *If $\alpha \in V(\Omega_\delta)$, then $d(\emptyset, \alpha) = |\alpha|$.*

Proof If $\alpha = \emptyset$ then, clearly, $|\alpha| = 0$ and $d(\emptyset, \alpha) = 0$.

Suppose $\alpha \neq \emptyset$. The multi-index α has a unique subindex α_1, and if $\alpha_1 \neq \emptyset$, then it has a unique subindex α_2, etc. Let α_k be the k^{th} subindex of α. Since $\alpha \in V(\Omega_\delta)$, $\alpha_k \in V(\Omega_\delta)$ for $1 \leq k \leq |\alpha|$, $\alpha_{|\alpha|} = \emptyset$, and $(\alpha, \alpha_1, \alpha_2, \cdots, \alpha_{|\alpha|})$ is a path of length $|\alpha|$ from α to $\emptyset$. Since Ω_δ is a tree by Corollary 9.1, there is a unique path connecting any two vertices in $V(\Omega_\delta)$. Hence, this is the only, and thus shortest, path connecting α to $\emptyset$. Therefore $d(\emptyset, \alpha) = |\alpha|$. □

2.2 Computing $P_\emptyset(L(\Omega_\delta))$

In Sect. 2.2 we express $P_\emptyset(L(\Omega_\delta))$ in terms of δ. Propositions 13, 14, 15, and 16 as well as Corollary 17.1 establish a number of results which are used in the proof of Theorem 2 where we give an identity for $P_\emptyset(L(\Omega_\delta))$ in terms of δ.

We note here that there is a potential ambiguity in the way $P_\emptyset(S)$ is computed if S is a multi-index. For instance, some texts define $(a, b) = \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$ (see [4]), in which case $P_\emptyset((a, b)) = 1$. For our purposes, we will remain agnostic toward any particular set-theoretic definition of multi-indices and will be treating them as atomic; that is, if x and y are multi-indices, then $P_x(y) = 1$ only if $x = y$ and $P_x(y) = 0$ otherwise. This makes Ω_δ a homogeneous graph.

Proposition 13 *If A, B, and C are sets of edges and $x \neq A \cup B$ is an indeterminate, then*

1. $(A \cup B) \triangle C = (A \triangle C) \cup (B \triangle C)$
2. $P_x(A \cup B) = P_x(A) + P_x(B) - P_x(A \cap B)$

Additionally, if $A \cap B = \emptyset$, then,

a. $P_x(A \cup B) = P_x(A) + P_x(B)$
b. $(A \triangle C) \cap (B \triangle C) = \emptyset$

Proof

1. Suppose $\{e, f\} \in (A \cup B) \triangle C$, this implies $e \in A \cup B$ and $f \in C$. If $e \in A$ then, $\{e, f\} \in A \triangle C$, and if $e \in B$, then $\{e, f\} \in B \triangle C$. Thus, $\{e, f\} \in (A \triangle C) \cup (B \triangle C)$. Hence, $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.
   Now, suppose $\{e, f\} \in (A \triangle C) \cup (B \triangle C)$. This means $\{e, f\} \in A \triangle C$ or $\{e, f\} \in B \triangle C$. If $\{e, f\} \in A \triangle C$ then $x \in A \subseteq A \cup B$ and $f \in C$. If $\{e, f\} \in B \triangle C$ then $e \in B \subseteq A \cup B$ and $f \in C$. Thus $\{e, f\} \in (A \cup B) \triangle C$. Hence, $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.
   Therefore, $(A \cup B) \triangle C = (A \triangle C) \cup (B \triangle C)$.
2. This can be thought of as a particular case of the inclusion-exclusion principle. If $x \neq A \cup B$ then it follows from the definition of $P_x(A \cup B)$,

$$
\begin{aligned}
P_x(A \cup B) &= \sum_{y \in A \cup B} P_x(y) \\
&= \sum_{y \in A} P_x(y) + \sum_{y \in B} P_x(y) - \sum_{y \in A \cap B} P_x(y) \\
&= P_x(A) + P_x(B) - P_x(A \cap B).
\end{aligned}
$$

a. If $A \cap B = \emptyset$ then $P_x(A \cap B) = 0$ and this result follows from 2.
b. Suppose $\{x, y\} \in A \triangle C$ with $x \in A$ and $y \in C$. Since $A \cap B = \emptyset$, $x \notin B$, and hence $\{x, y\} \notin B \triangle C$.

□

Proposition 14 utilizes the property of uniformity. Recall that an edge set E is uniform with respect to x if there exists $c \in \mathbb{Z}$ such that $P_x(y) = c$, $\forall y \in E$. And, we define $\mu_x(E) = c$.

Proposition 14 *If A and B are sets of edges which are both uniform with respect to an indeterminate x, $x \notin A \triangle B$, and $x \neq A \triangle B$, then $A \triangle B$ is uniform with respect to x. Furthermore, $\mu_x(A \triangle B) = \mu_x(A) + \mu_x(B)$, and*

$$P_x(A \triangle B) = (\mu_x(A) + \mu_x(B))|A \triangle B|.$$

Proof Every edge in $A \triangle B$ is of the form $\{a, b\}$ for some $a \in A$ and $b \in B$. Thus, since $x \neq \{a, b\}$, $P_x(\{a, b\}) = P_x(a) + P_x(b) = \mu_x(A) + \mu_x(B)$. Since this is true of every edge in $A \triangle B$, $A \triangle B$ is uniform with respect to x and we see that $\mu_x(A \triangle B) = \mu_x(A) + \mu_x(B)$. Furthermore, as $P_x(y) = \mu_x(A) + \mu_x(B)$ for every $y \in A \triangle B$ and there are $|A \triangle B|$ elements in $A \triangle B$, it follows that $P_x(A \triangle B)$ is of the prescribed form. □

Proposition 15 *If $\alpha \in V(\Omega_\delta)$ then L_α is uniform with respect to $\emptyset$, and $\mu_\emptyset(L_\alpha) = 1$ if and only if $\alpha = \emptyset$, otherwise, $\mu_\emptyset(L_\alpha) = 0$.*

Proof Every element of $L_\emptyset$ is of the form $\{\emptyset, y\}$ for some $y \neq \emptyset$ and clearly $P_\emptyset(\{\emptyset, y\}) = 1$, so $L_\emptyset$ is uniform with respect to $\emptyset$ and $\mu_\emptyset(L_\emptyset) = 1$.

If $\alpha \neq \emptyset$ then every element of L_α is of the form $\{\alpha, y\}$ for some $\alpha \subset y$ which implies that $|y| = |\alpha| + 1$, and since $\alpha \neq \emptyset$, $|\alpha| > 0$, so $|y| > 1$ which implies $y \neq \emptyset$. Since $\alpha \neq \emptyset$ and $y \neq \emptyset$, $P_\emptyset(\{\alpha, y\}) = 0$. Thus, L_α is uniform with respect to $\emptyset$ and $\mu_\emptyset(L_\alpha) = 0$. □

Proposition 16 *If $\alpha, \beta \in V(\Omega_\delta)$ with $|\alpha| < |\beta|$ then,*

1. *The graph induced by $L_\alpha \triangle L_\alpha$ is isomorphic to a regular graph of order $\delta(\alpha)$ and degree $\delta(\alpha) - 1$. Likewise, the graph induced by $L_\beta \triangle L_\beta$ is isomorphic to a regular graph of order $\delta(\beta)$ and degree $\delta(\beta) - 1$. Note, this implies, by Proposition 1, that $|L_\alpha \triangle L_\alpha| = \frac{1}{2}\delta(\alpha)(\delta(\alpha) - 1)$ and $|L_\beta \triangle L_\beta| = \frac{1}{2}\delta(\beta)(\delta(\beta) - 1)$.*
2. *If $\alpha \subset \beta$, then the graph induced by $L_\alpha \triangle L_\beta$ is isomorphic to the star graph $S_{\delta(\beta)+1}$. Note, this implies that $|L_\alpha \triangle L_\beta| = \delta(\beta)$.*
3. *Otherwise, $L_\alpha \triangle L_\beta = \emptyset$.*

Proof

1. Let L^*_α be the graph induced by L_α, by definition, $L(L^*_\alpha)$ is the graph induced by $L_\alpha \triangle L_\alpha$. By Proposition 11, L^*_α is isomorphic to $S_{\delta(\alpha)+1}$. So, firstly, $|L(L^*_\alpha)| = |L(S_{\delta(\alpha)+1})| = |E(S_{\delta(\alpha)+1})| = \delta(\alpha)$. Secondly, if $x \in E(S_{\delta(\alpha)+1})$ then x is adjacent to every edge in $S_{\delta(\alpha)+1}$ excluding itself, so $\deg_{L(L^*_\alpha)}(x) = \deg_{L(S_{\delta(\alpha)+1})}(x) = \deg_{S_{\delta(\alpha)+1}}(x) = \delta(\alpha) - 1$. Hence, $L(L^*_\alpha)$ is isomorphic to a regular graph of order $\delta(\alpha)$ and degree $\delta(\alpha) - 1$. The case for L_β follows symmetrically by the same argument.
2. If $\alpha \subset \beta$, then, since the subindex of β is unique, there is precisely one edge in L_α, $\{\alpha, \beta\}$, which is incident to β. Hence, every element of $L_\alpha \triangle L_\beta$ is of the form $\{\{\alpha, \beta\}, y\}$ with $y \in L_\beta$. It follows that the graph induced by $L_\alpha \triangle L_\beta$ is a star graph with $\{\alpha, \beta\}$ as the central vertex. Since β is adjacent to every vertex in the graph induced by L_β, the graph induced by $L_\alpha \triangle L_\beta$ is of order $|L_\beta| = |E(S_{\delta\beta+1})| = \delta(\beta)$ excluding $\{\alpha, \beta\}$ and $\delta(\beta) + 1$ including $\{\alpha, \beta\}$. Thus, the graph induced by $L_\alpha \triangle L_\beta$ is isomorphic to $S_{\delta(\beta)+1}$.
3. Every element of L_α is of the form $\{\alpha, x\}$ with $\alpha \subset x$, likewise, every element of L_β is of the form $\{\beta, y\}$ with $\beta \subset y$. If $\{\alpha, x\} \cap \{\beta, y\} \neq \emptyset$ then, since $\alpha \neq \beta$, we must have that $x = y$. Since $\beta \not\subset \alpha$ and $\alpha \not\subset \beta$, it cannot be the case that $x = \beta$ or $y = \alpha$. Thus, if $x = y = z$, we must have that $\alpha \subset z$ and $\beta \subset z$, but since z has a unique subindex this would imply $\beta = \alpha$. Hence, $a \cap b = \emptyset$ for all $a \in L_\alpha$ and $b \in L_\beta$, therefore $L_\alpha \triangle L_\beta = \emptyset$.

□

Proposition 17 *If G is a locally finite simple homogeneous graph, $x \in V(G)$, and $k \geq 1$, then*

$$P_x(L^k(G)) = P_x\left(\left(\bigcup_{\substack{\min\{d(x,y),d(x,z)\}\leq k \\ \{y,z\}\in E(G)}} \{\{y, z\}\}\right)^{\triangle k}\right).$$

Proof Given a vertex $z \in V(L^k(G))$, we can define $D(z) = \{y \in V(G) : P_y(z) \geq 1\}$. We will now establish, by induction, that, for all $k \geq 1$ and all $z \in V(L^k(G))$, $D(z)$ forms a connected subset of $V(G)$, $z = \{z_1, z_2\} \in E(L^{k-1}(G))$ with $D(z) = D(z_1) \cup \{q\}$ for some $q \in V(G)$ (or, equivalently, $D(z) = D(z_2) \cup \{q\}$ for some $q \in V(G)$), and $|D(z)| \leq k + 1$.

For the base case, let $z \in V(L(G))$, this implies that $z = \{z_1, z_2\} \in E(G)$, so $D(z) = \{z_1, z_2\}$ and $|D(z)| = 2 \leq 1 + 1$. In this case, since $z_1 \in V(G)$ and G is homogeneous, we can see that $D(z_1) = \{z_1\}$, and thus $D(z) = D(z_1) \cup \{z_2\}$, furthermore, since $z_1 \sim z_2$, this forms a connected subset of $V(G)$.

Now, suppose that for some $i \geq 1$ and any $z \in V(L^i(G))$, $D(z)$ forms a connected subset of $V(G)$, $z = \{z_1, z_2\} \in E(L^{i-1}(G))$ with $D(z) = D(z_1) \cup \{q\}$ for some $q \in V(G)$, and $|D(z)| \leq i + 1$. Now, let $z \in V(L^{i+1}(G))$, this implies $z = \{z_1, z_2\} \in E(L^i(G))$, and furthermore $z_1 = \{z_{1,1}, z_{1,2}\}$ and $z_2 = \{z_{2,1}, z_{2,2}\}$ with $z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2} \in V(L^{i-1}(G))$. Since $z_1 \sim z_2$, the definition of the line graph implies that $z_1 \cap z_2 \neq \emptyset$, so suppose without loss of generality that $z_{1,1} = z_{2,1} = a \in V(L^{i-1}(G))$, that is, $z_1 = \{a, z_{1,2}\}$ and $z_2 = \{a, z_{2,2}\}$.

By the induction hypothesis, $D(z_1) = D(a) \cup \{v\}$ and $D(z_2) = D(a) \cup \{q\}$ for some $v, q \in V(G)$. By definition, if $s \in V(G)$, then $P_s(z) = P_s(z_1) + P_s(z_2)$, it follows that $P_s(z) \geq 1$ if and only if $P_s(z_1) \geq 1$ or $P_s(z_2) \geq 1$, so $D(z) = D(z_1) \cup D(z_2)$. Thus,

$$\begin{aligned} D(z) &= D(z_1) \cup D(z_2) \\ &= (D(a) \cup \{v\}) \cup (D(a) \cup \{q\}) \\ &= D(a) \cup \{v\} \cup \{q\} \\ &= D(z_1) \cup \{q\}. \end{aligned}$$

Now, since $D(z) = D(z_1) \cup q$, $|D(z)| \leq |D(z_1)| + |\{q\}| \leq (i + 1) + 1 = i + 2$. Furthermore, by the induction hypothesis, $D(z_1) = D(a) \cup \{v\}$ and $D(z_2) = D(a) \cup \{q\}$ are both connected subsets of $V(G)$, this implies that for any vertex $r \in D(a)$, there exists a path in G connecting r to v likewise there exists a path connecting r to a, it follows that $D(z) = D(z_1) \cup D(z_2)$ is a connected subset of $V(G)$. This concludes the inductive step.

Let $z \in V(L^k(G))$ and let $x, y \in V(G)$. By definition, $P_x(z) \geq 1$ and $P_y(z) \geq 1$ if and only if $x \in D(z)$ and $y \in D(z)$ which, since $D(z)$ is a connected, implies there is a path from x to y in G with every vertex in $D(z)$. Since $|D(z)| \leq k + 1$, the length of this path must be less than or equal to k (counting the number of edges in the path). It follows that if $d(x, y) > k$ and $P_x(z) \geq 1$ we must have that $P_y(z) = 0$, and since this is true of any $z \in V(L^k(G))$, $P_x(V(L^k(G))) = P_x(V(L^k(G \setminus \{y\})))$ where $G \setminus \{y\}$ is the graph G with the vertex y and all its incident edges removed. Informally, this means we can remove any vertex outside a distance of k from x without changing the value of $P_x(V(L^k(G)))$. Now, since $V(L^{k+1}(G)) = E(L^k(G))$, $P_x(V(L^{k+1}(G))) = P_x(L^k(G))$, we can remove any vertex outside of a distance of $k + 1$ from x without changing the value of $P_x(L^k(G))$. What remains is to apply the definition of the P-function and the definition of the line graph,

$$
\begin{aligned}
P_x(L^k(G)) &= P_x\left(L^k\left(G \setminus \bigcup_{\substack{d(x,y)>k+1\\ y\in V(G)}} \{y\}\right)\right)\\
&= P_x\left(\left(E(G) \setminus \bigcup_{\substack{\min\{d(x,y),d(x,z)\}>k\\ \{y,z\}\in E(G)}} \{\{y,z\}\}\right)^{\triangle k}\right)\\
&= P_x\left(\left(\bigcup_{\substack{\min\{d(x,y),d(x,z)\}\le k\\ \{y,z\}\in E(G)}} \{\{y,z\}\}\right)^{\triangle k}\right).
\end{aligned}
$$

□

Corollary 17.1 *If $k \geq 1$, then*

$$
P_{\emptyset}(L^k(\Omega_\delta)) = P_{\emptyset}\left(\left(\bigcup_{\substack{|\alpha|\le k\\ \alpha\in V(\Omega_\delta)}} L_\alpha\right)^{\triangle k}\right).
$$

Proof By Proposition 12, if $\alpha \in V(\Omega_\delta)$ then $d(\emptyset, \alpha) = |\alpha|$. Now, suppose $z \in E(\Omega_\delta)$ and $z = \{\alpha, \beta\}$ with $\alpha, \beta \in V(\Omega_\delta)$ and $\alpha \subset \beta$. We know that $z \in L_\alpha$ by definition, and $\min\{d(\emptyset, \alpha), d(\emptyset, \beta)\} = d(\emptyset, \alpha) = |\alpha|$ since $|\alpha| < |\beta|$. Thus, applying Proposition 17 yields,

$$
\begin{aligned}
P_{\emptyset}(L^k(\Omega_\delta)) &= P_{\emptyset}\left(\left(\bigcup_{\substack{\min\{d(x,y),d(x,z)\}\le k\\ \{y,z\}\in E(\Omega_\delta)}} \{\{y,z\}\}\right)^{\triangle k}\right)\\
&= P_{\emptyset}\left(\left(\bigcup_{\substack{|\alpha|\le k\\ \alpha\in V(\Omega_\delta)}} L_\alpha\right)^{\triangle k}\right).
\end{aligned}
$$

□

Theorem 2 is a culmination of Propositions 13, 14, 15, and 16 as well as Corollary 17.1.

Theorem 2

$$
P_{\emptyset}(L(\Omega_\delta)) = \delta(\emptyset)^2 - \delta(\emptyset) + \sum_{\substack{|\alpha|=1\\ \alpha\in V(\Omega_\delta)}} \delta(\alpha).
$$

Proof In order to simplify the notation for this proof, if $n \geq 1$, define

$$\mathcal{L}_n = \bigcup_{\substack{|\alpha|=n \\ \alpha \in V(\Omega_\delta)}} L_\alpha.$$

Now, by Corollary 17.1,

$$\begin{aligned}
P_\emptyset(L(\Omega_\delta)) &= P_\emptyset\left(\left(\bigcup_{\substack{|\alpha|\leq 1 \\ \alpha \in V(\Omega_\delta)}} L_\alpha\right)^{\triangle 1}\right) \\
&= P_\emptyset\left((\mathcal{L}_0 \cup \mathcal{L}_1)^{\triangle 1}\right) \\
&= P_\emptyset((\mathcal{L}_0 \cup \mathcal{L}_1)\triangle(\mathcal{L}_0 \cup \mathcal{L}_1))
\end{aligned}$$

This can be simplified by application of Proposition 13, noting that $\mathcal{L}_0 \cap \mathcal{L}_1 = \emptyset$,

$$\begin{aligned}
P_\emptyset((\mathcal{L}_0 \cup \mathcal{L}_1)\triangle(\mathcal{L}_0 \cup \mathcal{L}_1)) &= P_\emptyset\big(((\mathcal{L}_0 \cup \mathcal{L}_1)\triangle\mathcal{L}_0) \cup ((\mathcal{L}_0 \cup \mathcal{L}_1)\triangle\mathcal{L}_1)\big) \\
&= P_\emptyset\big(((\mathcal{L}_0\triangle\mathcal{L}_0) \cup (\mathcal{L}_1\triangle\mathcal{L}_1)) \cup ((\mathcal{L}_0\triangle\mathcal{L}_1) \cup (\mathcal{L}_1\triangle\mathcal{L}_0))\big) \\
&= P_\emptyset\big(((\mathcal{L}_0\triangle\mathcal{L}_0) \cup (\mathcal{L}_1\triangle\mathcal{L}_1)) \cup ((\mathcal{L}_0\triangle\mathcal{L}_1) \cup (\mathcal{L}_0\triangle\mathcal{L}_1))\big) \\
&= P_\emptyset\big((\mathcal{L}_0\triangle\mathcal{L}_0) \cup (\mathcal{L}_1\triangle\mathcal{L}_1) \cup (\mathcal{L}_0\triangle\mathcal{L}_1)\big) \\
&= P_\emptyset(\mathcal{L}_0\triangle\mathcal{L}_0) + P_\emptyset(\mathcal{L}_1\triangle\mathcal{L}_1) + P_\emptyset(\mathcal{L}_0\triangle\mathcal{L}_1).
\end{aligned}$$

Now, $\mathcal{L}_0 = L_\emptyset$, which is uniform with $\mu_\emptyset(L_\emptyset) = 1$ by Proposition 15. By Proposition 16, the graph induced by $\mathcal{L}_0\triangle\mathcal{L}_0 = L_\emptyset\triangle L_\emptyset$ is isomorphic to a regular graph of order $\delta(\emptyset)$ and degree $\delta(\emptyset) - 1$ so, by Proposition 14,

$$\begin{aligned}
P_\emptyset(\mathcal{L}_0\triangle\mathcal{L}_0) &= P_\emptyset(L_\emptyset\triangle L_\emptyset) \\
&= (\mu_\emptyset(L_\emptyset) + \mu_\emptyset(L_\emptyset))|L_\emptyset\triangle L_\emptyset| \\
&= 2\frac{1}{2}\delta(\emptyset)(\delta(\emptyset) - 1) \\
&= \delta(\emptyset)^2 - \delta(\emptyset).
\end{aligned}$$

By Proposition 15, if $|\alpha| > 0$ then $\mu_\emptyset(L_\alpha) = 0$, this implies that $P_\emptyset(\mathcal{L}_1) = 0$ and $P_\emptyset(\mathcal{L}_1\triangle\mathcal{L}_1) = 0$.

Finally, using Propositions 15, 16, and 14, $P_\emptyset(\mathcal{L}_0\triangle\mathcal{L}_1)$ can be rewritten as

$$\begin{aligned}
P_\emptyset(\mathcal{L}_0 \triangle \mathcal{L}_1) &= P_\emptyset\left(L_\emptyset \triangle \bigcup_{\substack{|\alpha|=1 \\ \alpha \in V(\Omega_\delta)}} L_\alpha \right) \\
&= P_\emptyset\left(\bigcup_{\substack{|\alpha|=1 \\ \alpha \in V(\Omega_\delta)}} L_\emptyset \triangle L_\alpha \right) \\
&= \sum_{\substack{|\alpha|=1 \\ \alpha \in V(\Omega_\delta)}} P_\emptyset(L_\emptyset \triangle L_\alpha) \\
&= \sum_{\substack{|\alpha|=1 \\ \alpha \in V(\Omega_\delta)}} (\mu_\emptyset(L_\emptyset) + \mu_\emptyset(L_\alpha))|L_\emptyset \triangle L_\alpha| \\
&= \sum_{\substack{|\alpha|=1 \\ \alpha \in V(\Omega_\delta)}} \delta(\alpha).
\end{aligned}$$

Putting this all together shows that $P_\emptyset(L(\Omega_\delta))$ is of the prescribed form. □

3 Conjectures and Future Work

The most natural direction now is to consider whether it is possible to compute $P_\emptyset(L^n(\Omega_\delta))$ for $n \geq 2$. We have already made some progress in this direction. Conjecture 1 arises as a result of some informal calculations which we hope to make explicit in future publications.

Conjecture 1

$$\begin{aligned}
P_\emptyset(L^2(\Omega_\delta)) &= \delta(\emptyset)(\delta(\emptyset)-1)(2\delta(\emptyset)-4) \\
&\quad + \sum_{\substack{\emptyset \subset \alpha_1 \\ \alpha_1 \in V(\Omega_\delta)}} \left(\delta(\alpha_1)(3\delta(\emptyset) + 2\delta(\alpha_1) - 5) + \sum_{\substack{\alpha_1 \subset \alpha_2 \\ \alpha_2 \in V(\Omega_\delta)}} \delta(\alpha_2) \right).
\end{aligned}$$

It seems likely that the methods described in Sect. 2.2 will generalize and yield a procedural method of generating identities for $P_\emptyset(L^n(\Omega_\delta))$ for all $n \geq 1$. However, primarily due to the complexity of the calculations involved, we have so far been unable to determine identities for $P_\emptyset(L^n(\Omega_\delta))$ for any $n \geq 3$. Assuming these identities exist, whether they follow any patterns or there are more efficient methods of computing them is also a question of general interest.

The primary motivation for deriving these identities for $P_\emptyset(L^n(\Omega_\delta))$ is based on the observation that such identities yield similar results for arbitrary locally finite graphs.

We hope to elucidate exactly how results on Ω_δ transfer to arbitrary graphs in future publications. For now we state these generalized identities here as Conjectures 2 and 3. We hope to show, in future publications, that Conjecture 2 follows from Theorem 2 and Conjecture 3 follows from Conjecture 1.

Conjecture 2 If G is a locally finite homogeneous graph and $x \in V(G)$, then

$$P_x(L(G)) = \deg_G(x)^2 - 2\deg_G(x) + \sum_{u \sim x} \deg_G(u).$$

Conjecture 3 If G is a locally finite homogeneous graph and $x \in V(G)$, then

$$\begin{aligned} P_x(L^2(G)) &= \deg_G(x)(\deg_G(x) - 1)(2\deg_G(x) - 4) \\ &\quad + \sum_{u \sim x} \left((\deg_G(u) - 1)(3\deg_G(x) + 2\deg_G(u) - 7) + \sum_{\substack{w \sim u \\ w \neq x}} (\deg_G(w) - 1) \right). \end{aligned}$$

Applying Corollary 1.1 to Conjectures 2 and 3 immediately yields Conjectures 4 and 5.

Conjecture 4 If G is a finite graph, then

$$|L^3(G)| = \frac{1}{2} \sum_{v \in V(G)} \left(\deg_G(v)^2 - 2\deg_G(v) + \sum_{u \sim v} \deg_G(u) \right) \left(\deg_G(v) - \frac{3}{2} \right).$$

Conjecture 5 If G is a finite graph, then

$$\begin{aligned} |L^4(G)| = \frac{1}{2} \sum_{v \in G} \Bigg\{ \Bigg\{ & \deg_G(v)(\deg_G(v) - 1)(2\deg_G(v) - 4) \\ & + \sum_{u \sim v} \Bigg\{ (\deg_G(u) - 1)(3\deg_G(v) + 2\deg_G(u) - 7) \\ & + \sum_{\substack{w \sim u \\ w \neq v}} (\deg_G(w) - 1) \Bigg\} \Bigg\} \left(\deg_G(v) - \frac{7}{4} \right) \Bigg\}. \end{aligned}$$

The partition counting function, in combination with Proposition 17 and the correspondence between Ω_δ and other broad classes of graphs, effectively provide a way of reducing the global problem of tracking the change in the degrees of the vertices under iterations of the line graph operator to a local problem on restricted subsets of vertices in the base graph. With these techniques we derived identities for the number of vertices of iterated line graphs of arbitrary finite graphs. Similar techniques may potentially be applicable for studying other properties of the iterated line graphs, such as the minimal and maximal degrees of the vertices, distances and the diameter, properties of subgraphs, and problems related to graph coloring.

We only considered homogeneous graphs, but it may be possible to take advantage of an inhomogeneous labeling to track larger scale structures under iterated line graphs.

Finally, there's the possibility that our techniques may be applicable to the study of other operators which are similar to the line graph. For instance, we have already started using these techniques to study H-Line Graphs, defined in [5], and super-structure line graphs, defined in [2].

References

1. Balch, B.: Iterated line graphs on bi-regular graphs and trees. Rose-Hulman Undergrad. Math. J. **21**(1) , Article 2 (2020)
2. King, Z.: Iterated line graphs of graphs with regular and bi-regular partitions. Unpublished.
3. Merris, R.: Graph Theory. Wiley (2001)
4. Wiener, N.W.: A Simplification of the Logic of Relations (1912)
5. Kathiresan, K.M.: A Study on H-line graphs. Australian J. Combin. **58**(3), 358–364 (2014)

On 2-Factorizations of the Complete 3-Uniform Hypergraph of Order 12 Minus a 1-Factor

Peter Adams, Saad I. El-Zanati, Peter Florido, and William Turner

Abstract A k-factorization of the complete t-uniform hypergraph $K_v^{(t)}$ is an H-decomposition of $K_v^{(t)}$ where H is a k-regular spanning subhypergraph of $K_v^{(t)}$. It is known that seven of the eight non-isomorphic 3-uniform 2-regular hypergraphs of order $v \leq 9$ factorize $K_v^{(3)}$. We use nauty to generate the 2-regular spanning sub-hypergraphs of $K_{12}^{(3)}$ and show that they all factorize $K_{12}^{(3)} - I$, where I is a 1-factor.

Keywords Hypergraph · Decomposition · Nauty

1 Introduction

A commonly studied problem in combinatorics concerns decompositions (or factorizations) of graphs into edge-disjoint (spanning) subgraphs. The notions of decompositions and factorizations of graphs naturally extend to decompositions and factorizations of uniform hypergraphs. A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set E of nonempty subsets of V called *hyper-edges* or simply *edges*. If, for each $e \in E$, we have $|e| = t$, then H is said to be *t-uniform*. Thus graphs are 2-uniform hypergraphs. For integers $v \geq 1$ and $t \geq 2$, the *complete t-uniform hypergraph of order* v, denoted $K_v^{(t)}$, is the hypergraph with a vertex set V of size v and edge set the set of all t-element subsets of V. A *decom-position* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of K

P. Adams
University of Queensland, Queensland 4072, Australia

Saad I. El-Zanati (✉)
Illinois State University, Normal, IL 61790, USA
e-mail: saad@ilstu.edu

P. Florido
Normal Community High School, Normal, IL 61761, USA

W. Turner
Pekin Community High School, Pekin, IL 61554, USA

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_28

such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$ and $E(H_i) \cap E(H_j) = \varnothing$ for all $1 \le i < j \le s$. If each element H_i of Δ is isomorphic to a fixed hypergraph H, then H_i is called an *H-block*, and Δ is called an *H-decomposition* of K. If H is a spanning subhypergraph of K, then an H-decomposition of K is also an *H-factorization* of K and in this case we may say that *H factorizes K*. If in addition, H is k-regular, then an H-factorization of K is also a *k-factorization* of K.

It is trivial to see that for all positive integers m and $t \ge 2$, any two 1-factors in $K_{mt}^{(t)}$ are necessarily isomorphic. It has long been known that there exists a 1-factorization of $K_{2m}^{(2)}$ for all positive integers m. Such a 1-factorization corresponds to scheduling a round-robin tournament for $2m$ teams. In 1936 Peltesohn [13] proved that $K_{3m}^{(3)}$ has a 1-factorization. In 1975, Baranyai [7] proved the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m. To this date, Baranyai's result remains the only known general result on k-factorizations of $K_v^{(t)}$ for $t \ge 3$.

The investigation of 2-factorizations of $K_v^{(2)}$ originated with Walecki's solution of the problem of decomposing $K_v^{(2)}$, v odd, into Hamiltonian cycles in the 1890s (see Lucas [11] or the more recent article by Alspach [3]). Unlike with 1-factors, the number of non-isomorphic 2-factors in $K_v^{(2)}$ is strictly increasing for $v \ge 7$. It corresponds to the number of partitions of v into parts of size at least 3 (sequence A008483 in OEIS [16]). If F is a 2-factor in $K_v^{(2)}$, an F-factorization of $K_v^{(2)}$ is known as the *Oberwolfach problem* for v and F. Clearly, this is only possible when v is odd. If v is even, it is now customary to extend the Oberwolfach problem to F-factorizations of $K_v^{(2)} - I$, where I is a 1-factor. Though some progress on the Oberwolfach problem has been made (see for example, [17], [8], and [4]), the problem is far from settled. It is believed that an F-factorization of K_v, v odd, or of $K_v - I$, v even, exists if and only if $(v, F) \notin \{(9, C_4 \cup C_5), (11, C_3 \cup C_3 \cup C_5), (6, C_3 \cup C_3), (12, C_3 \cup C_3 \cup C_3 \cup C_3)\}$.

For $t \ge 3$, very little is known about 2-factorizations of $K_v^{(t)}$. It is simple to see that a 2-factor in $K_v^{(3)}$ can exist only if $v \equiv 0 \pmod 3$ and a 2-factorization of $K_v^{(3)}$ can only exist when $v \equiv 6$ or $9 \pmod{12}$. In the cases when $v \equiv 0$ or $3 \pmod{12}$, it is necessary to consider 2-factorizations of $K_v^{(3)} - I$, where I is a 1-factor. In [2], we showed that seven of the eight non-isomorphic 3-uniform 2-regular hypergraphs of order $v \le 9$ factorize $K_v^{(3)}$. We also showed that all but one of the 49 non-isomorphic 3-uniform 3-regular hypergraphs of order $v \le 8$ factorize $K_v^{(3)}$ or $K_v^{(3)} - I$. Moreover, we showed that at most two of the 148 non-isomorphic 3-uniform 3-regular hypergraphs of order 9 do not factorize $K_9^{(3)} - I$. In this note, we investigate 2-factorizations of $K_{12}^{(3)} - I$.

We use the nauty [12] function in SageMath [14] to generate the 2-regular 3-uniform hypergraphs on 12 vertices and investigate which of these hypergraphs factorize $K_{12}^{(3)} - I$ where I is a 1-factor. This work extends the results from [2] and mirrors some of the graph factorization results by Anderson [5] and by Adams, Bryant and Khodkar [1]. It also generalizes the Oberwolfach problem and expands on Baranyai's 1-factorization results [7].

If a and b are integers with $a \le b$, we define $[a, b]$ to be $\{a, a+1, \ldots, b\}$. Let $\mathbb{Z}_n$ denote the group of integers modulo n. We will describe our hypergraphs by giving their edge set only. Since the hypergraphs we consider will never contain isolated vertices, this is enough to uniquely define them. To save space, we will often list an edge $\{a, b, c\}$ as the string abc.

2 Main Results

Our main result involves $K_{12}^{(3)}$ and $K_{12}^{(3)} - I$. Before giving that result, we first restate our results from [2] and include their short proofs for completeness purposes.

Lemma 1 *There are two non-isomorphic 2-regular spanning subhypergraphs of $K_6^{(3)}$. One of them factorizes $K_6^{(3)}$ and the other does not.*

Proof For $k \in [1, 2]$, let $H_{2,k}[0, 1, 2, 3, 4, 5]$ denote the hypergraph $H_{2,k}$ with vertex set $[0, 5]$ and edge sets $E(H_{2,1}) = \{012, 234, 450, 135\}$ and $E(H_{2,2}) = \{012, 123, 345, 045\}$. It is easy to see that these are the only non-isomorphic 2-regular spanning subhypergraphs of $K_6^{(3)}$. It is is shown in [9] that $H_{2,1}$ does not factorize $K_6^{(3)}$. Let $V\left(K_6^{(3)}\right) = \mathbb{Z}_5 \cup \{\infty\}$ and let $B_{2,2} = \{H_{2,2}[2, 0, 1, 3, \infty, 4]\}$. Then an $H_{2,2}$-factorization of $K_6^{(3)}$ consists of the orbit of the $H_{2,2}$-block in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod 5$. □

Lemma 2 *There are six non-isomorphic 2-regular spanning subhypergraphs of $K_9^{(3)}$. All six factorize $K_9^{(3)}$.*

Proof For $k \in [3, 8]$, let $H_{2,k}[0, 1, 2, 3, 4, 5, 6, 7, 8]$ denote the hypergraph $H_{2,k}$ with vertex set $[0, 8]$ and edge set as defined in Table 1.
Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$
\begin{aligned}
B_{2,3} &= \{H_{2,3}[0, 1, 2, 3, 5, 4, 6, \infty_2, \infty_1], H_{2,3}[0, 1, 2, \infty_1, 4, 3, 6, \infty_2, 5]\},\\
B_{2,4} &= \{H_{2,4}[0, 1, 2, 3, 6, 5, \infty_1, 4, \infty_2], H_{2,4}[0, 1, 3, 5, 6, 4, 2, \infty_1, \infty_2]\},\\
B_{2,5} &= \{H_{2,5}[0, 1, 2, 3, \infty_1, 5, \infty_2, 6, 4], H_{2,5}[0, 1, 3, 2, 6, 4, \infty_1, \infty_2, 5]\},\\
B_{2,6} &= \{H_{2,6}[0, 1, 2, 3, 6, 5, \infty_1, 4, \infty_2], H_{2,6}[0, 1, 3, 2, \infty_1, 4, 6, 5, \infty_2]\},\\
B_{2,7} &= \{H_{2,7}[0, 1, 2, 5, 6, \infty_1, 4, 3, \infty_2], H_{2,7}[0, 1, 3, 4, 2, 6, \infty_1, 5, \infty_2]\},\\
B_{2,8} &= \{H_{2,8}[0, 1, 2, 3, 4, \infty_1, 6, \infty_2, 5], H_{2,8}[0, 1, 3, 2, 4, 5, \infty_1, 6, \infty_2]\}.
\end{aligned}
$$

Table 1 Edge sets for the six non-isomorphic 2-regular spanning subhypergraphs of $K_9^{(3)}$

k	Edge Set $E(H_{2,k})$
3	$\{014, 058, 123, 234, 567, 678\}$
4	$\{012, 067, 138, 234, 456, 578\}$
5	$\{012, 078, 168, 234, 345, 567\}$
6	$\{012, 078, 156, 234, 345, 678\}$
7	$\{012, 078, 123, 345, 456, 678\}$
8	$\{012, 036, 147, 258, 345, 678\}$

Table 2 Edge sets for the 23 non-isomorphic 2-regular spanning subhypergraphs of $K_{12}^{(3)}$

k	Edge set $E(H_{2,k})$
9	$\{012, 013, 456, 457, 89a, 89b, 26a, 37b\}$
10	$\{012, 034, 135, 678, 69a, 79b, 245, 8ab\}$
11	$\{012, 034, 156, 357, 89a, 89b, 46a, 27b\}$
12	$\{012, 034, 156, 378, 59a, 79b, 28a, 46b\}$
13	$\{012, 034, 567, 89a, 13b, 568, 24b, 79a\}$
14	$\{012, 034, 567, 89a, 15b, 268, 347, 9ab\}$
15	$\{012, 034, 567, 89a, 15b, 368, 279, 4ab\}$
16	$\{012, 034, 567, 89a, 15b, 289, 36a, 47b\}$
17	$\{012, 034, 567, 89a, 15b, 289, 34b, 67a\}$
18	$\{012, 034, 567, 89a, 15b, 289, 67b, 34a\}$
19	$\{012, 034, 567, 89a, 15b, 389, 46a, 27b\}$
20	$\{012, 034, 567, 89a, 56b, 789, 12a, 34b\}$
21	$\{012, 034, 567, 89a, 56b, 789, 13a, 24b\}$
22	$\{012, 034, 567, 89a, 58b, 123, 469, 7ab\}$
23	$\{012, 034, 567, 89a, 58b, 136, 249, 7ab\}$
24	$\{012, 034, 567, 89a, 58b, 136, 27b, 49a\}$
25	$\{012, 034, 567, 89a, 58b, 167, 234, 9ab\}$
26	$\{012, 034, 567, 89a, 58b, 169, 23b, 47a\}$
27	$\{012, 034, 567, 89a, 58b, 16b, 234, 79a\}$
28	$\{012, 034, 567, 89a, 58b, 16b, 379, 24a\}$
29	$\{012, 034, 567, 89a, 58b, 16b, 29a, 347\}$
30	$\{012, 345, 678, 9ab, 013, 679, 245, 8ab\}$
31	$\{012, 345, 678, 9ab, 036, 149, 278, 5ab\}$

Then for $k \in [3, 8]$, an $H_{2,k}$-factorization of $K_9^{(3)}$ consists of the orbit of the $H_{2,k}$-block in $B_{2,k}$, under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 7$. $\square$

We now include the new results on 2-factorizations of $K_{12}^{(3)} - I$.

Lemma 3 *There are* 23 *non-isomorphic* 2*-regular subhypergraphs of* $K_{12}^{(3)}$*. All of them factorize* $K_{12}^{(3)} - I$*, where* I *is a* 1*-factor.*

Proof For $k \in [9, 31]$, let $H_{2,k}[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b]$ denote the hypergraph $H_{2,k}$ with vertex set $[0, 9] \cup \{a, b\}$ and edge set as defined in Table 2.
Let $V\left(K_{12}^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3\}$ and let

$$B_{2,9} = \{H_{2,9}[0, 1, 2, 3, 4, 5, 8, \infty_1, 6, 7, \infty_2, \infty_3], H_{2,9}[0, 1, 5, 6, 2, 4, 7, 8, 3, \infty_2, \infty_1, \infty_3], H_{2,9}[0, 2, 3, 7, 1, 5, \infty_1, \infty_3, 8, \infty_2, 6, 4]\},$$

$$B_{2,10} = \{H_{2,10}[0, 1, 2, 3, 5, \infty_3, 4, 6, 7, 8, \infty_1, \infty_2], H_{2,10}[0, 1, 3, 2, 7\infty_2, 4, 5, 8, 6, \infty_1, \infty_3], H_{2,10}[0, 2, 5, 8, 4, \infty_1, 7, 6, 3, 1, \infty_2, \infty_3]\},$$

$B_{2,11} = \{H_{2,11}[0, 1, 2, 3, 4, 5, 6, 7, 8, \infty_1, \infty_2, \infty_3], H_{2,11}[0, 1, 3, 2, 5, 4, 6, \infty_1, 7, \infty_2, \infty_3, 8], H_{2,11}[0, 1, 4, 3, \infty_2, \infty_1, 2, 6, 5, 8, 7, \infty_3]\}$,

$B_{2,12} = \{H_{2,12}[0, 1, 2, 3, 4, 5, 6, 7, \infty_1, 8, \infty_2, \infty_3], H_{2,12}[0, 1, 3, 2, 4, 5, 7, 6, \infty_2, 8, \infty_3, \infty_1], H_{2,12}[0, 1, 4, 2, 3, 6, \infty_3, 7, 5, 8, \infty_2, \infty_1]\}$,

$B_{2,13} = \{H_{2,13}[0, 1, 2, 3, 4, 5, 6, \infty_1, \infty_2, 7, \infty_3, 8], H_{2,13}[0, 1, 3, 2, 5, 4, 7, 6, \infty_1, 8, \infty_2, \infty_3], H_{2,13}[0, 1, 4, 5, \infty_1, 2, 8, \infty_2, \infty_3, 3, 7, 6]\}$,

$B_{2,14} = \{H_{2,14}[0, 1, 2, 3, 4, 5, 6, 8, \infty_1, 7, \infty_2, \infty_3], H_{2,14}[0, 1, 4, 2, 3, 5, 7, \infty_1, \infty_3, 6, 8, \infty_2], H_{2,14}[0, 1, \infty_2, 2, 7, \infty_3, 5, 4, 8, 3, 6, \infty_1]\}$,

$B_{2,15} = \{H_{2,15}[0, 1, 2, 3, 4, 5, 6, 8, \infty_1, 7, \infty_2, \infty_3], H_{2,15}[0, 1, 5, 2, 3, 4, 6, 8, \infty_1, \infty_3, 7, \infty_2], H_{2,15}[0, 1, \infty_1, 6, 4, \infty_2, 3, 5, 8, 7, \infty_3, 2]\}$,

$B_{2,16} = \{H_{2,16}[0, 1, 2, 3, 4, 5, 6, \infty_1, 7, 8, \infty_2, \infty_3], H_{2,16}[0, 1, 3, 2, 4, 5, 6, \infty_3, 8, \infty_1, \infty_2, 7], H_{2,16}[0, 1, 5, 2, 3, \infty_2, \infty_3, 6, 7, \infty_1, 4, 8]\}$,

$B_{2,17} = \{H_{2,17}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,17}[0, 1, 4, 2, 3, 5, 6, \infty_1, 8, \infty_2, \infty_3, 7], H_{2,17}[0, 1, \infty_2, 2, 4, 7, 5, \infty_1, 3, 6, 8, \infty_3]\}$,

$B_{2,18} = \{H_{2,18}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,18}[0, 1, 4, 2, 3, 5, 6, \infty_1, 7, \infty_2, \infty_3, 8], H_{2,18}[0, 1, 5, 4, 6, \infty_1, 2, 8, 3, 7, \infty_2, \infty_3]\}$,

$B_{2,19} = \{H_{2,19}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,19}[0, 1, 4, 2, 3, 5, \infty_1, 6, 7, \infty_2, \infty_3, 8], H_{2,19}[0, 1, 5, 3, 7, 8, \infty_2, 2, 4, \infty_3, 6, \infty_1]\}$,

$B_{2,20} = \{H_{2,20}[0, 1, 2, 3, 4, 5, 6, \infty_1, 7, \infty_2, \infty_3, 8], H_{2,20}[0, 1, 4, 2, 3, 5, 7, \infty_1, 6, \infty_3, 8, \infty_2], H_{2,20}[0, 2, 6, 3, \infty_1, 1, \infty_2, 5, 8, \infty_3, 4, 7]\}$,

$B_{2,21} = \{H_{2,21}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,21}[0, 1, 4, 2, 5, 3, 7, \infty_3, 6, \infty_1, 8, \infty_2], H_{2,21}[0, 1, 5, 4, 7, 6, \infty_1, 2, 8, \infty_2, \infty_3, 3]\}$,

$B_{2,22} = \{H_{2,22}[0, 1, 2, 4, 3, 5, 6, \infty_1, 7, \infty_2, 8, \infty_3], H_{2,22}[0, 1, 4, 3, 5, 2, 6, 7, 8, \infty_3, \infty_2, \infty_1], H_{2,22}[0, 2, 7, \infty_3, 6, 1, \infty_1, 5, \infty_2, 4, 8, 3]\}$,

$B_{2,23} = \{H_{2,23}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,23}[0, 1, 4, 2, 3, 5, \infty_2, 7, \infty_1, 8, \infty_3, 6], H_{2,23}[0, 2, 7, 5, \infty_1, 3, \infty_2, 8, 6, 1, 4, \infty_3]\}$,

$B_{2,24} = \{H_{2,24}[0, 1, 2, 3, 4, 5, 6, 8, \infty_1, 7, \infty_2, \infty_3], H_{2,24}[0, 1, 4, 2, 3, 5, 6, \infty_1, \infty_2, 8, \infty_3, 7], H_{2,24}[0, 1, \infty_2, 2, 6, 5, \infty_3, 3, \infty_1, 4, 8, 7]\}$,

$B_{2,25} = \{H_{2,25}[0, 1, 2, 3, 5, 4, 6, 7, \infty_1, 8, \infty_2, \infty_3], H_{2,25}[0, 1, 5, 2, \infty_1, 3, 4, \infty_2, 6, 8, \infty_3, 7], H_{2,25}[0, 1, \infty_1, 3, 7, 2, 5, \infty_3, 4, 8, \infty_2, 6]\}$,

$B_{2,26} = \{H_{2,26}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,26}[0, 1, 4, 2, 3, 5, 6, \infty_1, \infty_2, 8, \infty_3, 7], H_{2,26}[0, 1, 5, 2, 6, 3, \infty_3, 8, 4, 7, \infty_1, \infty_2]\}$,

$B_{2,27} = \{H_{2,27}[0, 1, 2, 3, 5, 4, 6, 7, \infty_1, 8, \infty_2, \infty_3], H_{2,27}[0, 1, 4, 2, 7, 3, 5, \infty_2, 6, 8, \infty_3, \infty_1], H_{2,27}[0, 1, 5, 6, \infty_3, 3, 8, 2, 4, 7, \infty_2, \infty_1]\}$,

$B_{2,28} = \{H_{2,28}[0, 1, 2, 3, 4, 5, 6, 8, 7, \infty_1, \infty_2, \infty_3], H_{2,28}[0, 1, 4, 2, 3, 5, 6, \infty_2, \infty_1, 7, \infty_3, 8], H_{2,28}[0, 1, 5, 6, \infty_2, 4, 3, \infty_1, 2, 8, \infty_3, 7]\}$,

$B_{2,29} = \{H_{2,29}[0, 1, 2, 3, 4, 5, 6, 8, \infty_1, 7, \infty_2, \infty_3], H_{2,29}[0, 1, 4, 2, 3, 5, 7, \infty_1, \infty_2, 6, 8, \infty_3], H_{2,29}[0, 1, \infty_2, 3, 5, 6, \infty_3, 8, \infty_1, 4, 7, 2]\}$,

$$B_{2,30} = \{H_{2,30}[0, 1, 2, 3, 4, 7, 5, 6, \infty_1, \infty_2, 8, \infty_3], H_{2,30}[0, 1, 5, 6, 2, \infty_1, 3, 8, \infty_2, \infty_3, 4, 7], \\ H_{2,30}[0, 1, 7, \infty_3, 3, 5, 2, 4, 8, \infty_1, 6, \infty_2]\},$$

$$B_{2,31} = \{H_{2,31}[0, 1, 2, 3, 4, 7, 5, 6, \infty_1, \infty_2, 8, \infty_3], H_{2,31}[0, 1, 5, 6, 2, \infty_1, 3, 8, \infty_2, \infty_3, 4, 7], \\ H_{2,31}[0, 1, 7, \infty_3, 3, 5, 2, 4, 8, \infty_1, 6, \infty_2]\}.$$

Then for $k \in [9, 31]$, an $H_{2,k}$-factorization of $K_{12}^{(3)}$ consists of the orbit of the $H_{2,k}$-blocks in $B_{2,k}$, under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod 9$. □

Acknowledgements This research is supported in part by grant number A1659815 from the Division of Mathematical Sciences at the National Science Foundation. Part of this work was initiated while the third and fourth authors were teacher participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

References

1. Adams, P., Bryant, D., Khodkar, A.: Uniform 3-factorisations of K_{10}, Proceedings of the Twenty-eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing, Boca Raton, FL, 1997. Congr. Numer. **127**, 23–32 (1997)
2. Adams, P., El-Zanati, S.I., Florido, P., Turner, W.: On 2- and 3-factorizations of complete 3-uniform hypergraphs of order up to 9. Australas. J. Combin. **78**, 100–113 (2020)
3. Alspach, B.: The wonderful Walecki construction. Bull. Inst. Combin. Appl. **52**, 7–20 (2008)
4. Alspach, B., Bryant, D., Horsley, D., Maenhaut, B., Scharaschkin, V.: On factorisations of complete graphs into circulant graphs and the Oberwolfach problem. Ars Math. Contemp. **11**, 157–173 (2016)
5. Anderson, L.D.: Factorizations of graphs. In: Colbourn, C.J., Dinitz, J.H. (eds.), The CRC Handbook of Combinatorial Designs, 2nd ed., pp. 740–754. CRC Press, Boca Raton (2007)
6. Bailey, R.F., Stevens, B.: Hamilton decompositions of complete k-uniform hypergraphs. Discrete Math. **310**, 3088–3095 (2010)
7. Baranyai, Z.S.: On the factorization of the complete uniform hypergraph. In: Infinite and Finite Sets, Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam, vol. 10, pp. 91–108 (1975)
8. Bryant, D., Danziger, P.: On bipartite 2-factorizations of $K_n - I$ and the Oberwolfach problem. J. Graph Theory **68**, 22–37 (2011)
9. Bryant, D., Herke, S., Maenhaut, B., Wannasit, W.: Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs. Australas. J. Combin. **60**, 227–254 (2014)
10. Kuhl, J., Schroeder, M.W.: Hamilton cycle decompositions of k-uniform k-partite hypergraphs. Australas. J. Combin. **56**, 23–37 (2013)
11. Lucas, E.: Récreations Mathématiqués, Paris, vol. II (1892)
12. McKay, B.D., Piperno, A.: Practical graph isomorphism, II. J. Symb. Comput. **60**, 94–112 (2013)
13. Peltesohn, R.: Eine Lösung der beiden Heffterschen Differenzenprobleme. Compositio Math. **6**, 251–257 (1939)
14. SageMath, the Sage Mathematics Software System (Version 8.8), The Sage Developers (2019). https://www.sagemath.org
15. Schroeder, M.W.: On Hamilton cycle decompositions of r-uniform r-partite hypergraphs. Discrete Math. **315**, 1–8 (2014)

16. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences. http://oeis.org/A008483/
17. Traetta, T.: A complete solution to the two-table Oberwolfach problems. J. Combin. Theory Ser. A**120**, 984–997 (2013)

On Decompositions of Complete 3-Uniform Hypergraphs into a Linear Forest with 4 Edges

Ryan C. Bunge, Erin Dawson, Mary Donovan, Cody Hatzer, and Jacquelyn Maass

Abstract A 3-uniform linear forest is any hypergraph obtained by starting with a single 3-uniform edge and adding other 3-uniform edges sequentially such that each additional edge intersects with the previous hypergraph at no more than one vertex. There are nine such 3-uniform linear forests with four edges. In this paper we establish necessary and sufficient conditions for a decomposition of a complete 3-uniform hypergraph into isomorphic copies of a linear forest with four edges.

Keywords Hypergraphs · Linear forest · Decomposition

1 Introduction

A *hypergraph* H consists of a finite, nonempty set V of *vertices* and a finite set $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of V called *hyperedges* or simply *edges*. For a given hypergraph H, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set of H, respectively. We call $|V(H)|$ and $|E(H)|$ the *order* and *size* of H, respectively. If for each $e \in E(H)$ we have $|e| = t$, then H is said to be t*-uniform*. Thus t-uniform hypergraphs are generalizations of the concept of a graph (where $t = 2$). The hypergraph with vertex set V and edge set the set of all t-element subsets of V is called the *complete* t*-uniform hypergraph on* V and is denoted by $K_V^{(t)}$. If $v = |V|$, then we may also use $K_v^{(t)}$ to denote any hypergraph isomorphic to $K_V^{(t)}$.

R. C. Bunge (✉) · M. Donovan
Illinois State University, Normal, IL, USA
e-mail: rcbunge@ilstu.edu

E. Dawson
Colorado State University, Fort Collins, CO, USA

C. Hatzer
Grayslake Central High School, Grayslake, IL, USA

J. Maass
Niagara University, Niagara Falls, NY, USA

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*, Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_29

When $t = 2$, we may use K_v in place of $K_v^{(2)}$. If H' is a subhypergraph of H, then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H'. We may refer to $H \setminus H'$ as the hypergraph H with a *hole* H'. The vertices in H' may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs into edge-disjoint subgraphs. A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of subgraphs of K such that $\{E(G_1), E(G_2), \ldots, E(G_s)\}$ is a partition of $E(K)$. If each element of Δ is isomorphic to a fixed graph G, then Δ is called a *G-decomposition* of K. A G-decomposition of K_v is also known as a *G-design of order v*. The problem of determining all v for which there exists a G-design of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of K such that $\{E(H_1), E(H_2), \ldots, E(H_s)\}$ is a partition of $E(K)$. Any element of Δ isomorphic to a fixed hypergraph H is called an *H-block*. If all elements of Δ are H-blocks, then Δ is called an *H-decomposition* of K. An H-decomposition of $K_v^{(t)}$ is also called an *H-design of order v*. The problem of determining all v for which there exists an H-design of order v is called the *spectrum problem for H-designs*.

Keevash [12] has recently shown that for all t and k the obvious necessary conditions for the existence of a $K_k^{(t)}$-design of order v are sufficient for sufficiently large values of v. Similar results were obtained by Glock et al. [10, 11] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror the celebrated results of Wilson [17] for graphs. Although these asymptotic results assure the existence of H-designs for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on G-decompositions of K_v where G is a graph with a relatively small number of edges (see [1, 7] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [5], the spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the spectrum problem was settled in [6] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [6], they also settle the spectrum problem for the 3-uniform hypergraph of order 6 and size 4 whose edges form the lines of the Pasch configuration.

The best known general result on decompositions of complete t-uniform hypergraphs is Baranyai's result [4] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m. There are, however, several articles on decompositions of complete t-uniform hypergraphs (see [3, 15]) and of t-uniform t-partite hypergraphs (see [13, 16]) into variations on the concept of a Hamilton cycle.

In this paper we are interested in settling the spectrum problem for 3-uniform linear forests of size 4. A *t-uniform linear forest* is any hypergraph that can be obtained when you start with a single t-uniform hyperedge and sequentially add t-uniform hyperedges such that each added edge intersects the prior hypergraph at no

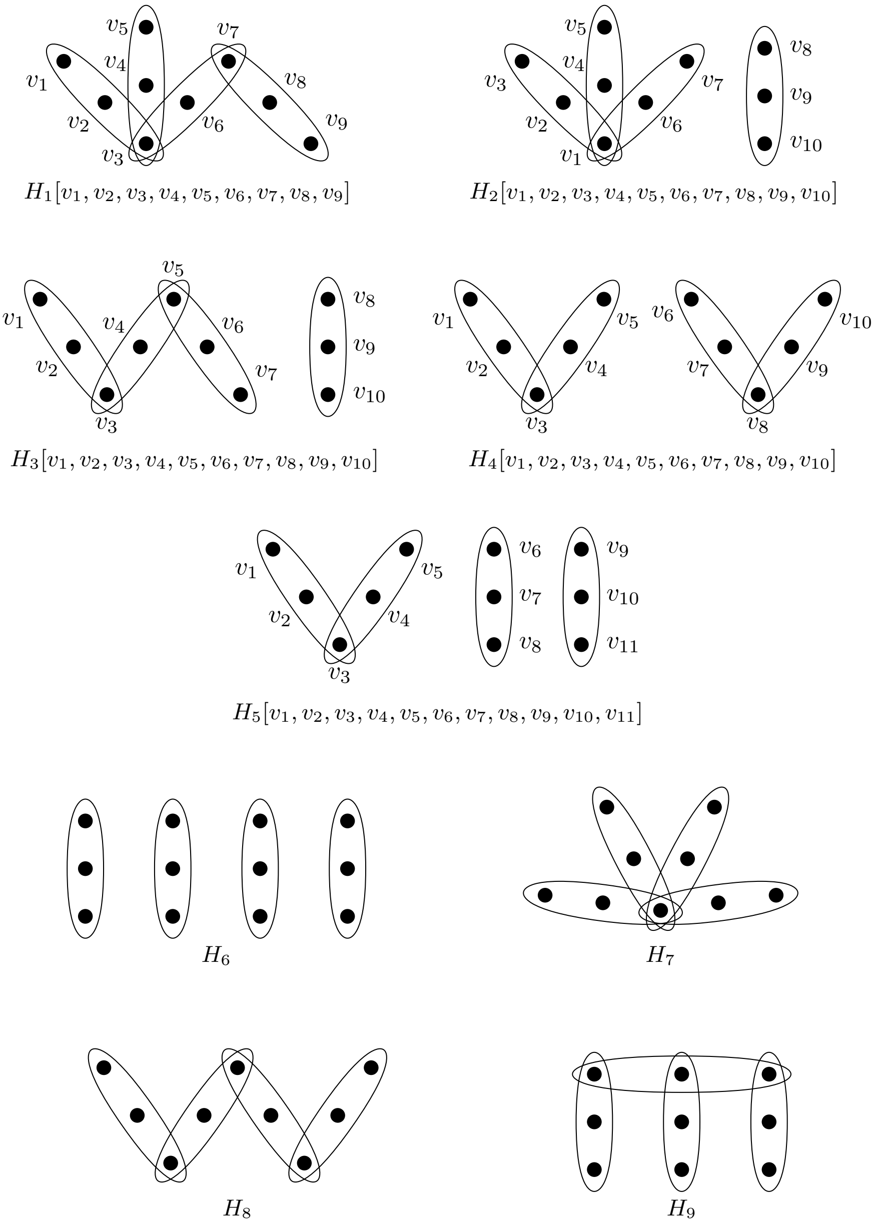


Fig. 1 All 3-uniform linear forests with four edges

more than one vertex. All nine possible 3-uniform linear forests of size 4 can be seen in Fig. 1. The spectrum problem for four of these hypergraphs is already known.

Before proceeding, we note that if there exists an H-decomposition of $K_v^{(3)}$ for some 3-uniform hypergraph H, then we must have that $|E(H)|$ divides $|E(K_v^{(3)})|$, and if the decomposition is nontrivial, then we must also have $v \geq |V(H)|$. Thus if H has 4 edges, then $4 \mid \binom{v}{3}$ and hence $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod 8$, and if $v > 2$, then $v \geq |V(H)|$. We summarize these observations in the following lemma.

Lemma 1 *Let H be a 3-uniform linear forest with 4 edges and let $v \geq 3$. If there exists an H-decomposition of $K_v^{(3)}$, then $v \equiv 0, 1, 2, 4$, or $6 \pmod 8$ and $v \geq |V(H)|$.*

The necessary conditions in Lemma 1 are known to be sufficient (with one exception) in the following cases. When H is a collection of 4 independent edges (H_6 in Fig. 1, i.e., a matching of size 4), then the result follows from Baranyai's work [4]. When H is a 4-star with a single vertex center (H_7 in Fig. 1), the result follows from Lonc's work [14], except in this case there is no H-decomposition of $K_9^{(3)}$. When H is a loose 4-path (H_8 in Fig. 1), the result is established in [8]. When H is the 4-edge symmetric triple-star (H_9 in Fig. 1), the result can be found in [2].

For the remaining five 3-uniform linear forests of size 4, we show that the necessary conditions in Lemma 1 are sufficient. We denote these five 3-uniform linear forests with H_1, H_2, H_3, H_4, and H_5 as seen in Fig. 1. Also seen in the figure is an implied notation for identifying the vertices of each of these linear forests. For example, $H_1[a, b, c, d, e, f, g, h, i]$ denotes that hypergraph with vertex set $\{a, b, c, d, e, f, g, h, i\}$ and edge set $\big\{\{a, b, c\}, \{c, d, e\}, \{c, f, g\}, \{g, h, i\}\big\}$. This expanded notation is used extensively in the examples found in Sect. 2 and Appendix A.

1.1 Additional Notation and Terminology

If a and b are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$. For any edge-disjoint hypergraphs G and H, we use $G \cup H$ to indicate the hypergraph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Similarly, if H is a hypergraph and r is a nonnegative integer, we let rH denoted the edge-disjoint union of r copies of H. We next define some notation for certain types of 3-uniform hypergraphs.

Let A, B, C be pairwise-disjoint sets. The hypergraph with vertex set $A \cup B \cup C$ and edge set consisting of all 3-element sets having exactly one vertex in each of A, B, C is denoted by $K_{A,B,C}^{(3)}$. The hypergraph with vertex set $A \cup B$ and edge set consisting of all 3-element sets having at most 2 vertices in each of A and B is denoted by $L_{A,B}^{(3)}$. If $|A| = a$, $|B| = b$, and $|C| = c$, we may use $K_{a,b,c}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{A,B,C}^{(3)}$ and $L_{a,b}^{(3)}$ to denote any hypergraph that is isomorphic to $L_{A,B}^{(3)}$. We use $K_{a,\overline{b},\overline{c}}^{(3)} \cup L_{\overline{b},\overline{c}}^{(3)}$ to denote any hypergraph isomorphic to $K_{A,B,C}^{(3)} \cup L_{B,C}^{(3)}$.

It is simple to observe that if A, B, B', and C are pairwise-disjoint, then $K^{(3)}_{A,B\cup B',C} = K^{(3)}_{A,B,C} \cup K^{(3)}_{A,B',C}$. Thus we have the following lemma.

Lemma 2 *If a, b, b', c, and z are positive integers, then* $K^{(3)}_{a,b+b',zc} = z\left(K^{(3)}_{a,b,c} \cup K^{(3)}_{a,b',c}\right)$.

2 Some Basic Examples

In this section, we give several examples of H_k-decompositions that are used in proving our main result. Additional examples are found in Appendix A. For the most part, these decompositions, as well as the ones found in the Appendix A, are either cyclic or r-pyramidal as defined in [9]. They were found either by hand or by computer searches.

Example 1 Let $V\left(K^{(3)}_9\right) = \mathbb{Z}_7 \cup \{f_1, f_2\}$ and let

$$B_1 = \{H_1[3, 4, 1, 0, 6, 2, f_1, f_2, 5],\ H_1[2, f_1, 0, 1, 4, 3, 5, 6, f_2],\ H_1[6, 2, 0, 3, f_1, 5, f_2, 1, 4]\}.$$

Then an H_1-decomposition of $K^{(3)}_9$ consists of the orbits of the H_1-blocks in B_1 under the action of the map $f_i \mapsto f_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod 7$ on the vertices.

Example 2 Let $V\left(K^{(3)}_{10}\right) = \mathbb{Z}_{10}$ and let

$$B_1 = \{H_1[7, 8, 0, 1, 2, 3, 9, 4, 5],\ H_1[0, 6, 1, 3, 9, 2, 8, 7, 5],\ H_1[4, 8, 0, 2, 5, 3, 6, 1, 9]\},$$

$$B_2 = \{H_2[0, 1, 3, 2, 7, 4, 8, 5, 6, 9],\ H_2[0, 1, 5, 3, 8, 7, 9, 2, 4, 6],\ H_2[0, 1, 2, 3, 6, 5, 9, 4, 7, 8]\},$$

$$B_3 = \{H_3[0, 7, 1, 6, 3, 4, 9, 2, 5, 8],\ H_3[1, 8, 0, 2, 7, 5, 9, 3, 4, 6],\ H_3[0, 2, 1, 7, 3, 4, 8, 5, 6, 9]\},$$

$$B_4 = \{H_4[0, 1, 2, 3, 5, 6, 8, 9, 4, 7],\ H_4[0, 3, 6, 7, 1, 2, 4, 8, 9, 5],\ H_4[0, 1, 4, 6, 9, 8, 2, 3, 5, 7]\}.$$

Then for $k \in [1, 4]$ an H_k-decomposition of $K^{(3)}_{10}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $j \mapsto j + 1 \pmod{10}$ on the vertices.

For the remaining examples, if H_k has r_k vertices, the H_k-blocks listed in sets B_k and B'_k are displayed as $[v_1, v_2, \ldots, v_{r_k}]$ rather than as $H_k[v_1, v_2, \ldots, v_{r_k}]$. This is done for compactness purposes.

Example 3 Let $V\left(L^{(3)}_{8,8}\right) = \mathbb{Z}_{16}$ with vertex partition $\big\{\{0, 2, \dots, 14\}, \{1, 3, \dots, 15\}\big\}$. Now, let

$$\begin{aligned}
B_1 = \big\{&[0, 1, 14, 9, 4, 8, 5, 12, 3],\ [10, 11, 0, 3, 15, 1, 2, 4, 7],\ [5, 4, 0, 15, 9, 1, 8, 10, 3],\\
&[10, 9, 0, 15, 4, 1, 3, 12, 14],\ [4, 3, 0, 15, 10, 1, 9, 11, 6],\ [7, 12, 0, 13, 4, 3, 8, 14, 11],\\
&[12, 3, 0, 13, 8, 6, 9, 14, 5]\big\},\\
B_2 = \big\{&[2, 0, 1, 5, 8, 7, 12, 10, 14, 15],\ [3, 0, 1, 6, 13, 7, 14, 9, 10, 15],\ [0, 1, 5, 3, 12, 4, 9, 8, 10, 11],\\
&[0, 1, 4, 2, 5, 3, 11, 10, 13, 14],\ [0, 1, 8, 3, 9, 2, 7, 5, 10, 11],\ [0, 1, 10, 3, 8, 2, 13, 4, 6, 15],\\
&[3, 0, 7, 2, 9, 5, 12, 6, 13, 14]\big\},\\
B_3 = \big\{&[1, 3, 0, 5, 10, 15, 12, 7, 11, 4],\ [1, 5, 0, 4, 11, 13, 14, 6, 12, 15],\ [3, 6, 0, 4, 9, 12, 13, 5, 10, 11],\\
&[2, 7, 6, 11, 3, 0, 12, 1, 8, 10],\ [0, 3, 11, 5, 12, 9, 14, 6, 7, 10],\ [0, 3, 9, 1, 8, 10, 15, 4, 5, 12],\\
&[9, 14, 0, 1, 2, 3, 8, 4, 5, 11]\big\},\\
B_4 = \big\{&[0, 5, 10, 13, 15, 2, 11, 9, 1, 4],\ [0, 4, 11, 6, 14, 1, 2, 5, 7, 12],\ [0, 4, 9, 12, 15, 5, 14, 13, 6, 7],\\
&[0, 3, 12, 6, 15, 2, 5, 11, 4, 7],\ [0, 2, 11, 8, 10, 4, 5, 6, 7, 9],\ [0, 2, 5, 1, 4, 8, 13, 14, 10, 15],\\
&[0, 1, 5, 6, 14, 2, 12, 3, 4, 9]\big\},\\
B_5 = \big\{&[1, 3, 0, 5, 10, 6, 9, 13, 2, 4, 7], [1, 5, 0, 4, 11, 7, 9, 10, 6, 12, 15], [3, 6, 0, 4, 9, 7, 10, 11, 8, 13, 14],\\
&[0, 12, 3, 6, 11, 1, 2, 13, 5, 7, 14], [0, 3, 11, 1, 2, 10, 13, 15, 4, 5, 8], [0, 3, 9, 1, 8, 5, 13, 14, 4, 6, 11],\\
&[9, 14, 0, 1, 2, 6, 7, 12, 4, 5, 11]\big\}.
\end{aligned}$$

Then for $k \in [1, 5]$ an H_k-decomposition of $L^{(3)}_{8,8}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $j \mapsto j + 1 \pmod{16}$ on the vertices.

Example 4 Let $V\left(K^{(3)}_{1,\overline{8},\overline{8}} \cup L^{(3)}_{\overline{8,8}}\right) = \{f_1\} \cup \mathbb{Z}_{16}$ with vertex partition $\big\{\{f_1\}, \{0, 2, \dots, 14\}, \{1, 3, \dots, 15\}\big\}$. Now, let

$$\begin{aligned}
B_1 = \big\{&[1, 0, f_1, 5, 2, 3, 8, 9, 6],\ [10, 11, 0, 3, 15, 1, 2, 4, 7],\ [5, 4, 0, 15, 9, 1, 8, 10, 3],\\
&[10, 9, 0, 15, 4, 1, 3, 12, 14],\ [4, 3, 0, 15, 10, 1, 9, 11, 6],\ [7, 12, 0, 13, 4, 3, 8, 14, 11],\\
&[12, 3, 0, 13, 8, 6, 9, 14, 5],\ [11, 6, 0, 9, 3, f_1, 7, 14, 5]\big\},\\
B_2 = \big\{&[2, 0, 1, 5, 8, 7, 12, 10, 14, 15],\ [3, 0, 1, 6, 13, 7, 14, 9, 10, 15],\ [0, 1, 5, 3, 12, 4, 9, 8, 10, 11],\\
&[0, 1, 4, 2, 5, 3, 11, 10, 13, 14],\ [0, 1, 8, 3, 9, 2, 7, 5, 10, 11],\ [0, 1, 10, 3, 8, 2, 13, 4, 6, 15],\\
&[3, 0, 7, 2, 9, 5, 12, 6, 13, f_1],\ [f_1, 0, 1, 4, 7, 10, 15, 6, 13, 14]\big\},\\
B_3 = \big\{&[1, 3, 0, 5, 10, 15, 12, 7, 11, 4],\ [1, 5, 0, 4, 11, 13, 14, 6, 12, 15],\ [3, 6, 0, 4, 9, 12, 13, 5, 10, 11],\\
&[2, 7, 6, 11, 3, 0, 12, 1, 8, 10],\ [0, 3, 11, 5, 12, 9, 14, 6, 7, 10],\ [0, 3, 9, 1, 8, 10, 15, 4, 5, 12],\\
&[9, 14, 0, 1, f_1, 10, 15, 4, 5, 11],\ [0, 2, 1, 4, f_1, 6, 13, 8, 9, 14]\big\},\\
B_4 = \big\{&[0, 5, 10, 13, 15, 2, 3, f_1, 4, 11],\ [1, 4, 9, 11, 2, 3, 14, f_1, 10, 13],\ [0, 4, 11, 6, 14, 1, 2, 5, 7, 12],\\
&[0, 4, 9, 12, 15, 5, 14, 13, 6, 7],\ [0, 3, 12, 6, 15, 2, 5, 11, 4, 7],\ [0, 2, 11, 8, 10, 4, 5, 6, 7, 9],\\
&[0, 2, 5, 1, 4, 8, 13, 14, 10, 15],\ [0, 1, 5, 6, 14, 2, 12, 3, 4, 9]\big\},
\end{aligned}$$

$$B_5=\{[1,3,0,5,10,6,9,13,2,4,7],[1,5,0,4,11,7,9,10,6,12,15],[3,6,0,4,9,7,10,11,8,13,14],$$
$$[0,12,3,6,11,1,2,13,5,7,14],[0,3,11,1,2,10,13,15,4,5,8],[0,3,9,1,8,5,13,14,4,6,11],$$
$$[4,5,f_1,10,15,0,9,14,6,7,8],[0,3,f_1,8,15,6,7,12,4,5,11]\}.$$

Then for $k \in [1,5]$ an H_k-decomposition of $K^{(3)}_{1,\overline{8},\overline{8}} \cup L^{(3)}_{\overline{8},\overline{8}}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_1 \mapsto f_1$ and $j \mapsto j+1 \pmod{16}$ on the vertices.

Example 5 Let $V\left(K^{(3)}_{12} \setminus K^{(3)}_4\right) = \mathbb{Z}_8 \cup \{f_1, f_2, f_3, f_4\}$ with $\{f_1, f_2, f_3, f_4\}$ as the set of vertices in the hole. Now, let

$$B_1 = \{[7,5,0,1,4,2,f_1,3,6],\ [0,2,f_2,1,4,f_3,3,5,7],\ [1,4,f_3,0,2,5,f_4,f_2,6],$$
$$[5,3,f_4,1,2,4,7,6,0],\ [f_3,0,f_1,f_2,1,f_4,2,4,7]\},$$
$$B_1' = \{[0,1,f_3,2,3,4,5,f_2,6],\ [0,1,f_2,2,3,4,5,f_1,6],\ [0,1,f_1,2,3,4,5,f_3,6],$$
$$[1,2,f_3,3,4,6,7,f_2,0],\ [1,2,f_2,3,4,6,7,f_1,0],\ [1,2,f_1,3,4,6,7,f_3,0],$$
$$[6,2,5,1,f_2,7,4,0,f_1],\ [7,3,6,2,f_2,0,5,1,f_1],\ [0,4,7,3,f_2,1,6,2,f_1],$$
$$[1,5,0,4,f_2,2,7,3,f_1],\ [2,6,1,5,f_4,3,0,4,f_3],\ [3,7,2,6,f_4,4,1,5,f_3],$$
$$[4,0,3,7,f_4,5,2,6,f_3],\ [5,1,4,0,f_4,6,3,7,f_3]\},$$

$$B_2 = \{[f_1,0,f_4,1,3,4,7,2,5,6],\ [f_2,0,2,4,f_4,7,f_3,3,5,6],\ [0,1,2,3,7,4,6,f_1,5,f_3],$$
$$[f_3,2,4,3,6,1,f_4,f_2,5,0],\ [f_4,0,1,2,4,3,6,f_1,f_2,5]\},$$
$$B_2' = \{[f_1,0,1,2,3,4,5,f_2,6,7],\ [f_2,0,1,2,3,4,5,f_3,6,7],\ [f_3,0,1,2,3,4,5,f_1,6,7],$$
$$[f_1,1,2,3,4,5,6,f_2,0,7],\ [f_2,1,2,3,4,5,6,f_3,0,7],\ [f_3,1,2,3,4,5,6,f_1,0,7],$$
$$[7,f_4,3,1,6,2,5,0,4,f_3],\ [0,f_4,4,2,7,3,6,1,5,f_3],\ [1,f_4,5,3,0,4,7,2,6,f_3],$$
$$[2,f_4,6,4,1,5,0,3,7,f_3],\ [3,f_2,7,5,2,6,1,4,0,f_1],\ [4,f_2,0,6,3,7,2,5,1,f_1],$$
$$[5,f_2,1,7,4,0,3,6,2,f_1],\ [6,f_2,2,0,5,1,4,7,3,f_1]\},$$

$$B_3 = \{[0,2,f_1,1,4,7,f_2,3,5,f_3],\ [0,2,f_4,1,4,7,f_3,3,5,f_2],\ [3,5,7,f_4,f_3,2,f_1,0,1,4],$$
$$[f_3,0,f_2,1,f_4,3,f_1,2,4,7],\ [0,f_4,1,2,7,f_1,f_2,3,4,6]\},$$
$$B_3' = \{[0,1,f_1,2,3,4,f_2,5,6,f_3],\ [1,2,f_1,3,4,5,f_2,6,7,f_3],\ [0,1,f_2,2,3,4,f_3,5,6,f_1],$$
$$[1,2,f_3,0,7,6,f_2,4,5,f_1],\ [2,3,f_3,1,0,7,f_1,5,6,f_2],\ [1,2,f_2,0,7,6,f_1,4,5,f_3],$$
$$[2,3,7,6,5,1,f_3,0,4,f_2],\ [3,4,0,7,6,2,f_3,1,5,f_2],\ [4,5,1,0,7,3,f_3,2,6,f_2],$$
$$[5,6,2,1,0,4,f_3,3,7,f_2],\ [6,7,3,2,1,5,f_4,4,0,f_1],\ [7,0,4,3,2,6,f_4,5,1,f_1],$$
$$[0,1,5,4,3,7,f_4,6,2,f_1],\ [1,2,6,5,4,0,f_4,7,3,f_1]\},$$
$$B_4 = \{[0,f_3,f_1,1,3,7,f_2,f_4,4,6],\ [3,0,1,2,5,6,f_3,4,7,f_2],\ [f_1,7,f_4,2,5,0,f_3,f_2,1,3],$$
$$[1,5,0,3,f_1,4,6,7,f_4,f_3]\},$$
$$B_4' = \{[0,1,f_1,2,3,4,5,f_2,6,7],\ [0,1,f_2,2,3,4,5,f_3,6,7],\ [0,1,f_3,2,3,4,5,f_1,6,7],$$
$$[1,2,f_1,3,4,5,6,f_2,0,7],\ [1,2,f_2,3,4,5,6,f_3,0,7],\ [1,2,f_3,3,4,5,6,f_1,0,7],$$
$$[f_1,0,4,5,6,f_3,7,2,f_4,1],\ [f_1,1,5,6,7,f_3,0,3,f_4,2],\ [f_1,2,6,7,0,f_3,1,4,f_4,3],$$
$$[f_1,3,7,0,1,f_3,2,5,f_4,4],\ [f_2,4,0,1,2,f_3,3,6,f_4,5],\ [f_2,5,1,2,3,f_3,4,7,f_4,6],$$
$$[f_2,6,2,3,4,f_3,5,0,f_4,7],\ [f_2,7,3,4,5,f_3,6,1,f_4,0],\ [f_3,0,4,7,2,f_1,f_2,1,3,5],$$
$$[f_3,1,5,0,3,f_1,f_2,2,4,6],\ [f_3,2,6,1,4,f_1,f_2,3,5,7],\ [f_3,3,7,2,5,f_1,f_2,4,6,0],$$
$$[f_4,4,0,3,6,f_1,f_2,5,7,1],\ [f_4,5,1,4,7,f_1,f_2,6,0,2],\ [f_4,6,2,5,0,f_1,f_2,7,1,3],$$
$$[f_4,7,3,6,1,f_1,f_2,0,2,4]\},$$

$$B_5 = \{[0, f_4, 1, f_2, f_3, 3, 5, f_1, 4, 6, 7], [1, f_3, f_1, 5, f_4, 3, 6, f_2, 0, 2, 4], [4, 7, f_4, 3, 5, 0, 2, f_2, 1, 6, f_3], [2, 5, 0, 3, f_1, 4, 6, f_3, f_2, f_4, 7]\},$$
$$B_5' = \{[0, 1, f_1, 2, 3, f_2, 4, 5, f_3, 6, 7], [0, 1, f_2, 2, 3, f_3, 4, 5, f_1, 6, 7], [0, 1, f_3, 2, 3, f_1, 4, 5, f_2, 6, 7], [1, 2, f_1, 3, 4, f_2, 5, 6, f_3, 7, 0], [1, 2, f_2, 3, 4, f_3, 5, 6, f_1, 7, 0], [1, 2, f_3, 3, 4, f_1, 5, 6, f_2, 7, 0], [2, 1, 0, f_1, 4, 3, 6, 7, 5, f_3, f_4], [3, 2, 1, f_1, 5, 4, 7, 0, 6, f_3, f_4], [4, 3, 2, f_1, 6, 5, 0, 1, 7, f_3, f_4], [5, 4, 3, f_1, 7, 6, 1, 2, 0, f_3, f_4], [6, 5, 4, f_2, 0, 7, 2, 3, 1, f_3, f_4], [7, 6, 5, f_2, 1, 0, 3, 4, 2, f_3, f_4], [0, 7, 6, f_2, 2, 1, 4, 5, 3, f_3, f_4], [1, 0, 7, f_2, 3, 2, 5, 6, 4, f_3, f_4], [3, 6, 4, 0, f_3, f_2, f_1, 7, 1, 2, 5], [4, 7, 5, 1, f_3, f_2, f_1, 0, 2, 3, 6], [5, 0, 6, 2, f_3, f_2, f_1, 1, 3, 4, 7], [6, 1, 7, 3, f_3, f_2, f_1, 2, 4, 5, 0], [7, 2, 0, 4, f_4, f_2, f_1, 3, 5, 6, 1], [0, 3, 1, 5, f_4, f_2, f_1, 4, 6, 7, 2], [1, 4, 2, 6, f_4, f_2, f_1, 5, 7, 0, 3], [2, 5, 3, 7, f_4, f_2, f_1, 6, 0, 1, 4]\}.$$

Then for $k \in [1, 5]$ an H_k-decomposition of $K_{12}^{(3)} \setminus K_4^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in [1, 4]$, and $j \mapsto j + 1 \pmod 8$ on the vertices along with the H_k-blocks in B_k'.

3 Main Results

We now turn our attention to settling the spectrum problem for each of the hypergraphs of interest. First, we establish our fundamental construction which is based on the approach in [6].

Lemma 3 *Let n, x, and r be nonnegative integers such that $nx + r \geq 3$. There exists a decomposition of $K_{nx+r}^{(3)}$ that is comprised of isomorphic copies of each of the following under the given conditions:*

- $K_r^{(3)}$ *if* $x = 0$,
- $K_{n+r}^{(3)}$ *if* $x \geq 1$,
- $K_{n+r}^{(3)} \setminus K_r^{(3)}$ *if* $x \geq 2$,
- $K_{r,\overline{n},\overline{n}}^{(3)} \cup L_{\overline{n},\overline{n}}^{(3)}$ *if* $x \geq 2$,
- $K_{n,n,n}^{(3)}$ *if* $x \geq 3$.

Furthermore, if $x \geq 1$ and $r \geq 3$, the decomposition contains exactly one copy of $K_{n+r}^{(3)}$.

Proof If $x \in \{0, 1\}$, the decomposition is trivial. Similarly, if $n = 0$, the result is trivial because $K_r^{(3)} = K_{n+r}^{(3)} = K_{nx+r}^{(3)}$ while $K_{n+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$, and $K_{n,n,n}^{(3)}$ are all empty (i.e., contain no edges). For the remainder of the proof, we assume that $x \geq 2$ and $n \geq 1$.

Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = n$. Then, the result follows from the fact that the complete 3-uniform hypergraph on the vertex set $V_0 \cup V_1 \cup \cdots \cup V_x$, which is $nx + r$ vertices, can be viewed as the (edge-disjoint) union

$$K^{(3)}_{V_1\cup V_0} \cup \bigcup_{2\le i\le x}\left(K^{(3)}_{V_i\cup V_0}\setminus K^{(3)}_{V_0}\right) \cup \bigcup_{1\le i<j\le x}\left(K^{(3)}_{V_0,V_i,V_j}\cup L^{(3)}_{V_i,V_j}\right) \cup \bigcup_{1\le i<j<k\le x}\left(K^{(3)}_{V_i,V_j,V_k}\right). \qquad \square$$

We also establish some basic decompositions.

Lemma 4 *Let $k\in[1,5]$ and let r, s be positive integers such that if $k\in\{3,5\}$ then $r\neq 3$ and $s\ge 2$. There exists an H_k-decomposition of $K^{(3)}_{r,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$. Furthermore, there exists an H_k-decomposition of $K^{(3)}_{2s,8,8}$.*

Proof If $r=1$, then an H_k-decomposition of $K^{(3)}_{1,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$ follows from Example 4. If $r=2$, then an H_k-decomposition of $K^{(3)}_{2,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$ either follows directly from Example 11 or, if $k\in\{1,2,4\}$, from Examples 12 and 3. Similarly, if $s=1$ (and thus $k\in\{1,2,4\}$), then an H_k-decomposition of $K^{(3)}_{2,8,8}$ follows from Example 12. Hence, for the remainder of the proof, we assume that $r\ge 3$ and $s\ge 2$.

First, we use Lemma 2 to prove the existence of an H_k-decomposition of $K^{(3)}_{2s,8,8}$. If $k\in\{1,2,4\}$, then the result follows from s copies of an H_k-decomposition of $K^{(3)}_{2,8,8}$. Now if $k\in\{3,5\}$ and $s\in\{2,3\}$, then the result follows from Examples 13 and 14, and we henceforth assume that $s\ge 4$. If s is even, then an H_k-decomposition of $K^{(3)}_{2s,8,8}$ follows from $s/2$ copies of an H_k-decomposition of $K^{(3)}_{4,8,8}$; whereas, if s is odd, then the result follows from one copy of an H_k-decomposition of $K^{(3)}_{6,8,8}$ along with $(s-3)/2$ copies of an H_k-decomposition of $K^{(3)}_{4,8,8}$.

Next, we prove the existence of an H_k-decomposition of $K^{(3)}_{r,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$. If $r=3$ (and thus $k\in\{1,2,4\}$), then the result follows from an H_k-decomposition of $K^{(3)}_{1,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$ along with an H_k-decomposition of $K^{(3)}_{2,8,8}$. Similarly, for any $k\in[1,5]$ and any odd $r\ge 5$, the result follows from an H_k-decomposition of $K^{(3)}_{1,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$ along with an H_k-decomposition of $K^{(3)}_{(r-1),8,8}$. Finally, for any $k\in[1,5]$ and any even $r\ge 4$, the result follows from an H_k-decomposition of $K^{(3)}_{r,8,8}$ along with an H_k-decomposition of $L^{(3)}_{8,8}$ (see Example 3). $\square$

Lemma 5 *For $k\in[1,5]$ there exists an H_k-decomposition of M where $M\in\left\{K^{(3)}_{17}, K^{(3)}_{18}, K^{(3)}_{17}\setminus K^{(3)}_{9}, K^{(3)}_{18}\setminus K^{(3)}_{10}\right\}$.*

Proof If $k=5$ or if $k\neq 1$ and $M\in\left\{K^{(3)}_{17},\ K^{(3)}_{17}\setminus K^{(3)}_{9}\right\}$, then the result follows from Examples 9, 10, 17, and 18. For an H_1-decomposition of $K^{(3)}_{17}$, we apply Lemma 3 with $n=8$, $x=2$, and $r=1$ and note that $K^{(3)}_{r}=K^{(3)}_{1}$ is empty while $K^{(3)}_{n+r}=K^{(3)}_{n+r}\setminus K^{(3)}_{r}=K^{(3)}_{9}$. Hence, an H_1-decomposition of $K^{(3)}_{17}$ follows from two copies of an H_1-decomposition of $K^{(3)}_{9}$ along with an H_1-decomposition of $K^{(3)}_{1,\overline{8},\overline{8}}\cup L^{(3)}_{\overline{8},\overline{8}}$, which follow from Examples 1 and 4, respectively. Furthermore, an

H_1-decomposition of $K_{17}^{(3)} \setminus K_9^{(3)}$ follows from the removal of one of the copies of the H_1-decomposition of $K_9^{(3)}$. Next, we consider the remaining results with $k \in [1, 4]$. For an H_k-decomposition of $K_{18}^{(3)}$, we apply Lemma 3 with $n = 8$, $x = 2$, and $r = 2$ and note that $K_r^{(3)} = K_2^{(3)}$ is again empty while $K_{n+r}^{(3)} = K_{n+r}^{(3)} \setminus K_r^{(3)} = K_{10}^{(3)}$. Hence, an H_k-decomposition of $K_{18}^{(3)}$ follows from two copies of an H_k-decomposition of $K_{10}^{(3)}$ (see Example 2) along with an H_k-decomposition of $K_{2,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, which follows from Lemma 4. Furthermore, an H_k-decomposition of $K_{18}^{(3)} \setminus K_{10}^{(3)}$ follows from the removal of one of the copies of the H_k-decomposition of $K_{10}^{(3)}$. □

Finally, we turn to settling the spectrum problem for these hypergraphs.

Main Theorem *Let $k \in [1, 5]$ and let $v \geq 3$. There exists an H_k-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4,$ or $6 \pmod 8$ and $v \geq |V(H_k)|$.*

Proof In the case where $v = 9$, an H_1-decomposition of $K_9^{(3)}$ follows from Example 1. Similarly, in the case where $v = 10$ and $k \in [1, 4]$, an H_k-decomposition of $K_{10}^{(3)}$ follows from Example 2. For the remainder of the proof, we assume that $v \geq 12$. Hence, we can write $v = 8x + r$ for some integers $x \geq 1$ and $r \in \{4, 6, 8, 9, 10\}$. By Lemma 3, $K_v^{(3)}$ can be decomposed into copies of $K_{8+r}^{(3)}$, $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, and $K_{8,8,8}^{(3)}$. The result then follows from the existence of an H_k-decomposition of each of the following:

- $K_{12}^{(3)}$, $K_{14}^{(3)}$, and $K_{16}^{(3)}$ (see Examples 6, 7, and 8, respectively);
- $K_{17}^{(3)}$ and $K_{18}^{(3)}$ (see Lemma 5);
- $K_{12}^{(3)} \setminus K_4^{(3)}$, $K_{14}^{(3)} \setminus K_6^{(3)}$, and $K_{16}^{(3)} \setminus K_8^{(3)}$ (see Examples 5, 15, and 16, respectively);
- $K_{17}^{(3)} \setminus K_9^{(3)}$ and $K_{18}^{(3)} \setminus K_{10}^{(3)}$ (see Lemma 5);
- $K_{4,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, $K_{6,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, $K_{8,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, $K_{9,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, $K_{10,\overline{8},\overline{8}}^{(3)} \cup L_{\overline{8},\overline{8}}^{(3)}$, and $K_{8,8,8}^{(3)}$ (see Lemma 4). □

Given the known results on the other 3-uniform linear forests with 4 edges, we arrive at the following corollary to our Main Theorem.

Corollary 1 *Let H be a 3-uniform linear forest of size 4. There exists a nontrivial H-design of order v if and only if $v \equiv 0, 1, 2, 4,$ or $6 \pmod 8$ and $v \geq |V(H)|$ with the exception that there is no decomposition of $K_9^{(3)}$ into copies of the 4-star with a single vertex center.*

Acknowledgements This research is supported by grant number A1659815 from the Division of Mathematical Sciences at the National Science Foundation. This work was done while all but the first author were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

Appendix A

We now give the remaining H_k-decompositions not explicitly shown in Sect. 2. As with Examples 3 through 5, if H_k has r_k vertices, the H_k-blocks listed in sets B_k and B_k' are displayed as $[v_1, v_2, \ldots, v_{r_k}]$ rather than as $H_k[v_1, v_2, \ldots, v_{r_k}]$.

Example 6 Let $V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{f_1\}$ and let

$$\begin{aligned}
B_1 = \big\{&[10, 3, 0, 1, 2, 6, 7, 8, f_1],\ [10, 7, 0, 1, 6, 2, 3, 5, f_1],\ [9, 3, 0, 2, 4, 5, 7, 10, f_1],\\
&[10, 6, 0, 1, 3, 2, 7, 9, 4],\ [8, 4, 0, 3, 6, 7, f_1, 5, 10]\big\},\\
B_2 = \big\{&[f_1, 0, 10, 1, 9, 2, 8, 3, 5, 7],\ [0, 1, 4, 3, 6, 2, f_1, 7, 8, 9],\ [0, 1, 3, 2, 5, 4, f_1, 6, 8, 9],\\
&[0, 1, 5, 2, 6, 4, 7, 3, 8, 9],\ [1, 3, 9, 4, 5, 7, 0, 2, 6, 8]\big\},\\
B_3 = \big\{&[1, 2, 0, 3, 6, 7, 10, 4, 8, 9],\ [1, 8, 0, 2, 7, 5, 3, 9, 10, f_1],\ [2, 5, 0, 1, 6, 7, 9, 3, 4, 8],\\
&[3, 7, 0, 4, f_1, 5, 10, 6, 8, 9],\ [2, 6, 0, 3, f_1, 7, 9, 5, 8, 10]\big\},\\
B_4 = \big\{&[0, 1, 2, 3, 5, 6, 8, 10, f_1, 4],\ [0, 1, 4, f_1, 8, 5, 7, 10, 2, 6],\ [0, 1, 6, 8, 3, 10, 2, 5, f_1, 7],\\
&[0, 1, 5, 6, 3, 2, 7, 9, f_1, 10],\ [0, 1, 7, 8, 4, 5, 10, 3, f_1, 6]\big\},\\
B_5 = \big\{&[0, 1, f_1, 4, 10, 3, 7, 9, 5, 6, 8],\ [0, 2, f_1, 1, 5, 3, 4, 9, 6, 8, 10],\ [1, 5, 0, 2, 6, 3, 4, 10, 7, 8, 9],\\
&[2, 8, 0, 3, 7, 1, 4, 10, 5, 6, 9],\ [3, 10, 2, 4, 5, 1, 6, 9, 0, 8, f_1]\big\}.
\end{aligned}$$

Then for $k \in [1, 5]$ an H_k-decomposition of $K_{12}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_1 \mapsto f_1$ and $j \mapsto j + 1 \pmod{11}$ on the vertices.

Example 7 Let $V\left(K_{14}^{(3)}\right) = \mathbb{Z}_{13} \cup \{f_1\}$ and let

$$\begin{aligned}
B_1 = \big\{&[12, 3, 0, 1, 2, 8, 9, 10, f_1],\ [12, 7, 0, 1, 6, 4, 5, 3, f_1],\ [4, 2, 0, 1, 10, 3, 11, 8, f_1],\\
&[11, 5, 0, 2, 6, 3, 10, 7, 8],\ [12, 10, 0, 1, 7, 2, 8, 4, f_1],\ [11, 8, 0, 2, 9, 3, 7, 12, f_1],\\
&[10, 6, 0, 3, 8, 4, 9, 2, f_1]\big\},\\
B_2 = \big\{&[f_1, 0, 1, 8, 11, 10, 12, 2, 3, 7],\ [f_1, 0, 4, 1, 6, 2, 8, 3, 5, 9],\ [0, 1, 2, 3, 12, 4, 5, 6, 8, 10],\\
&[0, 1, 11, 2, 10, 3, 6, 4, 7, 12],\ [0, 1, 6, 2, 8, 5, 11, 3, 4, 10],\ [0, 1, 8, 3, 4, 6, 10, 5, 9, 11],\\
&[0, 1, 3, 4, 8, 6, 9, 5, 7, 10]\big\},\\
B_3 = \big\{&[3, 7, 0, 4, f_1, 5, 10, 12, 1, 8],\ [2, 6, 0, 3, f_1, 7, 9, 12, 5, 10],\ [4, 8, 0, 3, 9, 11, 12, f_1, 1, 7],\\
&[0, 3, 8, 10, 5, 6, 2, 11, 12, 7],\ [1, 2, 0, 3, 6, 7, 10, 11, 12, 5],\ [1, 8, 0, 2, 7, 5, 3, 9, 10, f_1],\\
&[2, 5, 0, 1, 6, 7, 9, 3, 4, 8]\big\},\\
B_4 = \big\{&[0, 1, 2, 3, 5, 6, 7, 10, f_1, 4],\ [0, 1, 5, 6, 11, 7, 9, 12, f_1, 4],\ [0, 1, 7, 10, 2, 6, 9, 12, f_1, 3],\\
&[0, 1, 8, 9, 4, 12, 2, 3, f_1, 6],\ [0, 1, 11, 7, 9, 6, 10, 2, f_1, 4],\ [0, 2, 6, 8, 1, 12, 5, 9, f_1, 10],\\
&[0, 2, 7, 9, 3, 4, 6, 1, 5, 11]\big\},\\
B_5 = \big\{&[0, 1, 2, 4, 7, 11, f_1, 3, 5, 6, 9], [0, 2, 6, 1, 10, 9, 12, 3, 11, f_1, 7], [3, 8, 2, f_1, 0, 1, 4, 9, 6, 5, 12],\\
&[8, 4, 3, 0, 6, 1, 2, 11, f_1, 5, 12], [11, 4, 3, 0, 9, 5, 6, 8, 12, 2, f_1], [1, f_1, 2, 4, 11, 3, 6, 8, 0, 10, 12],\\
&[5, 3, 1, 0, 9, 2, 8, 10, 4, 6, 12]\big\}.
\end{aligned}$$

Then for $k \in [1,5]$ an H_k-decomposition of $K_{14}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_1 \mapsto f_1$ and $j \mapsto j+1 \pmod{13}$ on the vertices.

Example 8 Let $V\left(K_{16}^{(3)}\right) = \mathbb{Z}_{14} \cup \{f_1, f_2\}$ and let

$$\begin{aligned}
B_1 = \big\{&[5, 3, f_1, 1, 2, 4, 7, 11, f_2],\ [5, 3, f_2, 1, 2, 4, 7, 11, f_1],\ [7, 1, f_1, 0, 5, 13, f_2, 2, 8],\\
&[13, 5, 0, 1, 4, 7, 8, 9, 2],\ [13, 10, 0, 1, 9, 2, 3, 5, 7],\ [12, 4, 0, 2, 5, 6, 8, 13, 1],\\
&[12, 8, 0, 2, 7, 3, 5, 10, 13],\ [7, 3, 0, 11, 8, 6, 9, 13, 5],\ [13, 9, 0, 1, 5, 7, 11, 6, f_2]\big\},\\
B_1' = \big\{&H[1, 2, 3, 4, 6, 7, 12, 5, f_1],\ [2, 3, 4, 5, 7, 8, 13, 6, f_1],\ [3, 4, 5, 6, 8, 9, 0, 7, f_1],\\
&[4, 5, 6, 7, 9, 10, 1, 8, f_1],\ [5, 6, 7, 8, 10, 11, 2, 9, f_1],\ [6, 7, 8, 9, 11, 12, 3, 10, f_1],\\
&[7, 8, 9, 10, 12, 13, 4, 11, f_1],\ [8, 9, 10, 11, 13, 0, 5, 12, f_2],\ [9, 10, 11, 12, 0, 1, 6, 13, f_2],\\
&[10, 11, 12, 13, 1, 2, 7, 0, f_2],\ [11, 12, 13, 0, 2, 3, 8, 1, f_2],\ [12, 13, 0, 1, 3, 4, 9, 2, f_2],\\
&[13, 0, 1, 2, 4, 5, 10, 3, f_2],\ [0, 1, 2, 3, 5, 6, 11, 4, f_2]\big\},
\end{aligned}$$

$$\begin{aligned}
B_2 = \big\{&[f_1, 0, 1, 2, 4, 5, 8, 9, 10, 12],\ [f_2, 0, 1, 2, 4, 5, 8, 9, 10, 13],\ [f_1, 4, 10, 5, 9, 6, 11, 13, 7, 8],\\
&[f_2, 4, 10, 5, 9, 6, 11, 1, 7, 8],\ [0, 1, 9, 2, 8, 3, 7, 4, 10, 13],\ [0, 1, 5, 2, 6, 3, 9, 4, 8, 12],\\
&[0, 1, 7, 2, 9, 10, 12, 4, 8, 13],\ [0, 1, 11, 2, 7, 3, 10, 6, 8, 9],\ [0, 1, 10, 2, 11, 4, 6, 7, f_1, f_2]\big\},\\
B_2' = \big\{&[0, f_1, 7, 1, 2, 3, 6, 8, 10, 13],\ [1, f_1, 8, 2, 3, 4, 7, 9, 11, 0],\ [2, f_1, 9, 3, 4, 5, 8, 10, 12, 1],\\
&[3, f_1, 10, 4, 5, 6, 9, 11, 13, 2],\ [4, f_1, 11, 5, 6, 7, 10, 12, 0, 3],\ [5, f_1, 12, 6, 7, 8, 11, 13, 1, 4],\\
&[6, f_1, 13, 7, 8, 9, 12, 0, 2, 5],\ [7, f_2, 0, 8, 9, 10, 13, 1, 3, 6],\ [8, f_2, 1, 9, 10, 11, 0, 2, 4, 7],\\
&[9, f_2, 2, 10, 11, 12, 1, 3, 5, 8],\ [10, f_2, 3, 11, 12, 13, 2, 4, 6, 9],\ [11, f_2, 4, 12, 13, 0, 3, 5, 7, 10],\\
&[12, f_2, 5, 13, 0, 1, 4, 6, 8, 11],\ [13, f_2, 6, 0, 1, 2, 5, 7, 9, 12]\big\},
\end{aligned}$$

$$\begin{aligned}
B_3 = \big\{&[3, 9, 0, 10, 1, 5, 8, f_2, 6, 12],\ [f_1, 13, 1, 0, 6, 7, 11, 2, 3, 5],\ [10, 13, f_1, 1, f_2, 4, 6, 3, 8, 12],\\
&[1, 12, 9, 2, 0, 4, 8, f_2, 3, 6],\ [13, 2, 1, 0, 8, 5, 9, 3, f_1, 7],\ [10, 2, 0, f_1, 1, 5, f_2, 4, 7, 9],\\
&[0, 5, f_1, 7, 1, 4, 8, 11, 12, f_2],\ [9, 5, 3, 0, 8, 7, 2, 11, 12, 4],\ [2, f_2, 11, 6, 4, 0, 1, 5, 7, 13]\big\},
\end{aligned}$$

$$\begin{aligned}
B_3' = \big\{&[7, f_1, 0, 1, 2, 4, 6, 3, 5, 8],\ [8, f_1, 1, 2, 3, 5, 7, 4, 6, 9],\ [9, f_1, 2, 3, 4, 6, 8, 5, 7, 10],\\
&[10, f_1, 3, 4, 5, 7, 9, 6, 8, 11],\ [11, f_1, 4, 5, 6, 8, 10, 7, 9, 12],\ [12, f_1, 5, 6, 7, 9, 11, 8, 10, 13],\\
&[13, f_1, 6, 7, 8, 10, 12, 9, 11, 0],\ [0, f_2, 7, 8, 9, 11, 13, 10, 12, 1],\ [1, f_2, 8, 9, 10, 12, 0, 11, 13, 2],\\
&[2, f_2, 9, 10, 11, 13, 1, 12, 0, 3],\ [3, f_2, 10, 11, 12, 0, 2, 13, 1, 4],\ [4, f_2, 11, 12, 13, 1, 3, 0, 2, 5],\\
&[5, f_2, 12, 13, 0, 2, 4, 1, 3, 6],\ [6, f_2, 13, 0, 1, 3, 5, 2, 4, 7]\big\},
\end{aligned}$$

$$\begin{aligned}
B_4 = \big\{&[0, 1, 2, 3, 5, 6, 7, 10, f_1, 13],\ [0, 1, 5, 6, f_2, 7, 8, 13, 2, 9],\ [0, 1, 7, 9, 11, 13, 2, 4, f_2, 6],\\
&[0, 1, 8, f_1, 12, 2, 4, 7, f_2, 13],\ [0, f_1, f_2, 1, 6, 2, 3, 12, 4, 10],\ [0, 1, 11, 12, f_1, 2, 4, 8, 10, 3],\\
&[0, 3, 6, 8, f_1, 2, 9, 12, f_2, 1],\ [0, 1, 12, f_1, 3, 2, 4, 9, f_2, 13],\ [0, 2, 10, 13, 4, 7, 12, 3, f_1, 9]\big\},\\
B_4' = \big\{&[7, f_1, 0, 1, 9, 6, 10, 2, 5, 11],\ [8, f_1, 1, 2, 10, 7, 11, 3, 6, 12],\ [9, f_1, 2, 3, 11, 8, 12, 4, 7, 13],\\
&[10, f_1, 3, 4, 12, 9, 13, 5, 8, 0],\ [11, f_1, 4, 5, 13, 10, 0, 6, 9, 1],\ [12, f_1, 5, 6, 0, 11, 1, 7, 10, 2],\\
&[13, f_1, 6, 7, 1, 12, 2, 8, 11, 3],\ [0, f_2, 7, 8, 2, 13, 3, 9, 12, 4],\ [1, f_2, 8, 9, 3, 0, 4, 10, 13, 5],\\
&[2, f_2, 9, 10, 4, 1, 5, 11, 0, 6],\ [3, f_2, 10, 11, 5, 2, 6, 12, 1, 7],\ [4, f_2, 11, 12, 6, 3, 7, 13, 2, 8],\\
&[5, f_2, 12, 13, 7, 4, 8, 0, 3, 9],\ [6, f_2, 13, 0, 8, 5, 9, 1, 4, 10]\big\},
\end{aligned}$$

$$\begin{aligned}
B_5 = \big\{&[1, 5, 13, 4, 9, 0, f_1, f_2, 2, 3, 6],\ [0, f_2, 6, f_1, 12, 13, 8, 7, 1, 5, 9],\\
&[0, f_2, 5, f_1, 10, 6, 8, 9, 1, 4, 11],\ [0, f_2, 4, f_1, 8, 2, 5, 7, 9, 11, 13],\\
&[0, f_2, 3, f_1, 6, 4, 17, 13, 8, 11, 12],\ [0, f_2, 2, f_1, 4, 5, 8, 13, 6, 10, 11],\\
&[0, f_2, 1, f_1, 2, 3, 6, 10, 5, 7, 12],\ [0, 3, 6, 7, 13, 2, 4, 10, 9, 11, 5],\\
&[5, 0, 1, 2, 9, 3, 4, 12, 6, 11, 13]\big\},
\end{aligned}$$

$$B_5' = \{[2, 1, 0, f_1, 7, 3, 4, 6, 8, 10, 13],\ [3, 2, 1, f_1, 8, 4, 5, 7, 9, 11, 0],$$
$$[4, 3, 2, f_1, 9, 5, 6, 8, 10, 12, 1],\ [5, 4, 3, f_1, 10, 6, 7, 9, 11, 13, 2],$$
$$[6, 5, 4, f_1, 11, 7, 8, 10, 12, 0, 3],\ [7, 6, 5, f_1, 12, 8, 9, 11, 13, 1, 4],$$
$$[8, 7, 6, f_1, 13, 9, 10, 12, 0, 2, 5],\ [9, 8, 7, f_2, 0, 10, 11, 13, 1, 3, 6],$$
$$[10, 9, 8, f_2, 1, 11, 12, 0, 2, 4, 7],\ [11, 10, 9, f_2, 2, 12, 13, 1, 3, 5, 8],$$
$$[12, 11, 10, f_2, 3, 13, 0, 2, 4, 6, 9],\ [13, 12, 11, f_2, 4, 0, 1, 3, 5, 7, 10],$$
$$[0, 13, 12, f_2, 5, 1, 2, 4, 6, 8, 11],\ [1, 0, 13, f_2, 6, 2, 3, 5, 7, 9, 12]\}.$$

Then for $k \in [1, 5]$ an H_k-decomposition of $K_{16}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{14}$ on the vertices along with the H_k-blocks in B_k'.

Example 9 Let $V\left(K_{17}^{(3)}\right) = \mathbb{Z}_{17}$ and let

$$B_2 = \{[0, 3, 13, 4, 12, 5, 11, 7, 8, 9],\ [0, 3, 11, 4, 8, 5, 10, 6, 7, 9],\ [0, 1, 11, 2, 4, 3, 10, 5, 7, 15],$$
$$[0, 2, 13, 3, 12, 4, 10, 5, 6, 9],\ [0, 1, 12, 2, 11, 3, 9, 5, 6, 10],\ [0, 1, 7, 2, 5, 3, 6, 8, 11, 16],$$
$$[0, 1, 9, 2, 8, 4, 11, 5, 7, 14],\ [0, 1, 6, 2, 12, 4, 9, 8, 10, 14],\ [0, 14, 16, 4, 5, 1, 10, 6, 9, 11],$$
$$[0, 1, 8, 2, 7, 3, 4, 9, 12, 16]\},$$

$$B_3 = \{[1, 2, 0, 3, 6, 7, 10, 8, 11, 16],\ [1, 8, 0, 2, 7, 5, 3, 4, 6, 14],\ [2, 5, 0, 1, 6, 7, 9, 3, 4, 8],$$
$$[4, 8, 0, 9, 3, 5, 14, 1, 6, 12],\ [0, 12, 2, 13, 6, 7, 16, 8, 11, 15],\ [0, 1, 15, 3, 4, 8, 13, 2, 5, 12],$$
$$[0, 4, 10, 15, 5, 7, 11, 8, 12, 3],\ [0, 1, 14, 15, 4, 2, 11, 3, 6, 16],\ [0, 3, 12, 1, 4, 2, 15, 8, 10, 5],$$
$$[0, 1, 13, 14, 7, 9, 15, 3, 4, 12]\},$$

$$B_4 = \{[0, 1, 2, 3, 5, 6, 7, 10, 11, 15],\ [0, 1, 6, 7, 13, 16, 3, 12, 15, 5],\ [0, 1, 8, 10, 5, 2, 4, 11, 15, 6],$$
$$[0, 1, 9, 11, 4, 13, 3, 8, 12, 2],\ [0, 1, 10, 12, 14, 9, 2, 5, 8, 11],\ [0, 1, 11, 14, 6, 2, 7, 13, 15, 3],$$
$$[0, 1, 12, 16, 5, 14, 3, 6, 8, 2],\ [0, 1, 13, 15, 2, 16, 3, 6, 8, 14],\ [0, 1, 14, 2, 11, 8, 3, 5, 7, 15],$$
$$[0, 1, 15, 2, 7, 3, 6, 12, 4, 10]\},$$

$$B_5 = \{[1, 2, 0, 3, 16, 4, 6, 12, 7, 10, 13],\ [1, 3, 0, 4, 16, 5, 7, 14, 6, 9, 13],$$
$$[1, 9, 0, 5, 16, 6, 8, 13, 7, 10, 15],\ [1, 11, 0, 6, 16, 2, 4, 12, 3, 7, 10],$$
$$[1, 10, 0, 7, 16, 2, 4, 13, 5, 11, 14],\ [1, 12, 0, 4, 5, 7, 9, 13, 3, 11, 14],$$
$$[1, 14, 0, 2, 4, 3, 8, 10, 5, 9, 13],\ [1, 15, 0, 2, 13, 3, 8, 16, 4, 9, 14],$$
$$[2, 5, 0, 3, 10, 4, 8, 14, 1, 6, 12],\ [2, 14, 0, 4, 11, 1, 5, 13, 3, 6, 15]\}.$$

Then for $k \in [2, 5]$ an H_k-decomposition of $K_{17}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $j \mapsto j + 1 \pmod{17}$ on the vertices.

Example 10 Let $V\left(K_{18}^{(3)}\right) = \mathbb{Z}_{17} \cup \{f_1\}$ and let

$$B_5 = \{[0, 1, f_1, 2, 4, 6, 8, 14, 7, 10, 13],\ [0, 6, f_1, 2, 7, 3, 5, 12, 8, 11, 15],$$
$$[1, 9, 0, 5, 16, 6, 8, 13, 7, 10, 15],\ [1, 11, 0, 6, 16, 2, 4, 12, 3, 7, 10],$$
$$[1, 10, 0, 7, 16, 2, 4, 13, 5, 11, 14],\ [1, 12, 0, 4, 5, 7, 9, 13, 3, 11, 14],$$
$$[1, 14, 0, 2, 4, 3, 8, 10, 5, 9, 13],\ [1, 15, 0, 2, 13, 3, 8, 16, 4, 9, 14],$$
$$[2, 5, 0, 3, 10, 4, 8, 14, 1, 6, 12],\ [2, 14, 0, 4, 11, 1, 5, 13, 3, 6, 15],$$
$$[0, 3, f_1, 1, 5, 6, 7, 10, 12, 13, 14],\ [0, 8, f_1, 6, 16, 9, 10, 14, 12, 13, 15]\}.$$

Then an H_5-decomposition of $K_{18}^{(3)}$ consists of the orbits of the H_5-blocks in B_5 under the action of the map $f_1 \mapsto f_1$ and $j \mapsto j + 1 \pmod{17}$ on the vertices.

Example 11 Let $V\left(K^{(3)}_{2,\overline{8},\overline{8}} \cup L_{\overline{8},\overline{8}}\right) = \{f_1, f_2\} \cup \mathbb{Z}_{16}$ with the vertex partition $\big\{\{f_1, f_2\}, \{0, 2, \ldots, 14\}, \{1, 3, \ldots, 15\}\big\}$. Now, let

$$
\begin{aligned}
B_3 = \big\{&[1, 3, 0, 5, 10, 15, 12, 7, 11, 4],\ [1, 5, 0, 4, 11, 13, 14, 6, 12, 15],\ [3, 6, 0, 4, 9, 12, 13, 5, 10, 11],\\
&[2, 7, 6, 11, 3, 0, 12, 1, 8, 10],\ [0, 3, 11, 5, 12, 9, 14, 6, 7, 10],\ [0, 3, 9, 1, 8, 10, 15, 4, 5, 12],\\
&[4, f_1, 11, 12, f_2, 5, 10, 0, 1, 2], [14, 15, f_1, 10, 5, f_2, 12, 0, 1, 6], [f_1, 9, 6, f_2, 3, 1, 12, 4, 5, 11]\big\},\\
B_5 = \big\{&[1, 3, 0, 5, 10, 6, 9, 13, 2, 4, 7],\ [1, 5, 0, 4, 11, 7, 9, 10, 6, 12, 15],\\
&[3, 6, 0, 4, 9, 7, 10, 11, 8, 13, 14],\ [0, 12, 3, 6, 11, 1, 2, 13, 5, 7, 14],\\
&[0, 3, 11, 1, 2, 10, 13, 15, 4, 5, 8],\ [0, 3, 9, 1, 8, 5, 13, 14, 4, 6, 11],\\
&[4, 5, f_1, 10, 15, 0, 9, 14, 6, 7, f_2],\ [0, 3, f_1, 8, 15, f_2, 7, 12, 4, 5, 11],\\
&[4, 11, f_2, 10, 13, 0, 1, 2, 6, 7, 12]\big\}.
\end{aligned}
$$

Then for $k \in \{3, 5\}$ an H_k-decomposition of $K^{(3)}_{2,\overline{8},\overline{8}} \cup L^{(3)}_{\overline{8},\overline{8}}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{16}$ on the vertices.

Example 12 Let $V\left(K^{(3)}_{2,8,8}\right) = \{f_1, f_2\} \cup \mathbb{Z}_{16}$ with vertex partition $\big\{\{f_1, f_2\}, \{0, 2, \ldots, 14\}, \{1, 3, \ldots, 15\}\big\}$. Now, let

$$
\begin{aligned}
B_1 &= \big\{[0, 1, f_1, 2, 5, 3, 8, 15, f_2],\ [0, 1, f_2, 2, 5, 3, 8, 15, f_1]\big\},\\
B_2 &= \big\{[f_1, 0, 1, 2, 5, 4, 9, f_2, 6, 13],\ [f_2, 0, 1, 2, 5, 4, 9, f_1, 6, 13]\big\},\\
B_4 &= \big\{[0, 1, f_1, 2, 5, 4, 9, f_2, 6, 13],\ [0, 1, f_2, 2, 5, 4, 9, f_1, 6, 13]\big\}.
\end{aligned}
$$

Then for $k \in \{1, 2, 4\}$ an H_k-decomposition of $K^{(3)}_{2,8,8}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{16}$ on the vertices.

Example 13 Let $V\left(K^{(3)}_{4,8,8}\right) = \{f_1, f_2, f_3, f_4\} \cup \mathbb{Z}_{16}$ with the vertex partition $\big\{\{f_1, f_2, f_3, f_4\}, \{0, 2, \ldots, 14\}, \{1, 3, \ldots, 15\}\big\}$. Now, let

$$
\begin{aligned}
B_3 = \big\{&[0, f_1, 4, f_2, 1, f_3, 2, 3, f_4, 5],\ [0, f_2, 4, f_3, 1, f_4, 2, 3, f_1, 5],\\
&[0, f_3, 4, f_4, 1, f_1, 2, 3, f_2, 5],\ [0, f_4, 4, f_1, 1, f_2, 2, 3, f_3, 5]\big\},\\
B_5 = \big\{&[0, f_1, 4, f_2, 1, 2, f_3, 3, 5, f_4, 7],\ [0, f_2, 4, f_3, 1, 2, f_4, 3, 5, f_1, 7],\\
&[0, f_3, 4, f_4, 1, 2, f_1, 3, 5, f_2, 7],\ [0, f_4, 4, f_1, 1, 2, f_2, 3, 5, f_3, 7]\big\}.
\end{aligned}
$$

Then for $k \in \{3, 5\}$ an H_k-decomposition of $K^{(3)}_{4,8,8}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in [1, 4]$, and $j \mapsto j + 1 \pmod{16}$ on the vertices.

Example 14 Let $V\left(K^{(3)}_{6,8,8}\right) = \{f_1, f_2, f_3, f_4, f_5, f_6\} \cup \mathbb{Z}_{16}$ with vertex partition $\big\{\{f_1, f_2, \ldots, f_6\}, \{0, 2, \ldots, 14\}, \{1, 3, \ldots, 15\}\big\}$. Now, let

$$B_3 = \{[0, f_1, 4, f_2, 1, f_3, 2, 3, f_4, 5],\ [0, f_2, 4, f_3, 1, f_4, 2, 3, f_5, 5],\ [0, f_3, 4, f_4, 1, f_5, 2, 3, f_6, 5],$$
$$[0, f_4, 4, f_5, 1, f_6, 2, 3, f_1, 5],\ [0, f_5, 4, f_6, 1, f_1, 2, 3, f_2, 5],\ [0, f_6, 4, f_1, 1, f_2, 2, 3, f_3, 5]\},$$
$$B_5 = \{[0, f_1, 4, f_2, 1, 2, f_3, 3, 5, f_4, 7],\ [0, f_2, 4, f_3, 1, 2, f_4, 3, 5, f_5, 7],$$
$$[0, f_3, 4, f_4, 1, 2, f_5, 3, 5, f_6, 7],\ [0, f_4, 4, f_5, 1, 2, f_6, 3, 5, f_1, 7],$$
$$[0, f_5, 4, f_6, 1, 2, f_1, 3, 5, f_2, 7],\ [0, f_6, 4, f_1, 1, 2, f_2, 3, 5, f_3, 7]\}.$$

Then for $k \in \{3, 5\}$ an H_k-decomposition of $K^{(3)}_{6,8,8}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in [1, 6]$, and $j \mapsto j + 1 \pmod{16}$ on the vertices.

Example 15 Let $V\left(K^{(3)}_{14} \setminus K^{(3)}_{6}\right) = \mathbb{Z}_8 \cup \{f_1, f_2, f_3, f_4, f_5, f_6\}$ with $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ as the set of vertices in the hole. Now, let

$$B_1 = \{[7, 5, 0, 1, 4, 2, f_1, 3, 6], [1, 6, f_2, 3, 5, f_3, 4, f_5, f_6], [1, 4, f_3, 0, 2, 5, f_4, f_2, 6],$$
$$[0, 1, f_5, 2, 4, 3, f_1, f_2, 5], [0, 1, f_6, 2, 4, 3, f_1, f_3, 5], [0, 3, f_5, f_3, 1, f_2, 4, 5, f_4],$$
$$[0, 3, f_6, f_3, 1, f_2, 2, f_1, f_4], [3, 6, 0, 2, 4, f_6, f_4, f_5, 1]\},$$
$$B_1' = \{[0, 1, f_3, 2, 3, 4, 5, f_2, 6],\ [0, 1, f_2, 2, 3, 4, 5, f_1, 6],\ [0, 1, f_1, 2, 3, 4, 5, f_3, 6],$$
$$[1, 2, f_3, 3, 4, 6, 7, f_2, 0],\ [1, 2, f_2, 3, 4, 6, 7, f_1, 0],\ [1, 2, f_1, 3, 4, 6, 7, f_3, 0],$$
$$[6, 2, 5, 1, f_2, 7, 4, 0, f_1],\ [7, 3, 6, 2, f_2, 0, 5, 1, f_1], [0, 4, 7, 3, f_2, 1, 6, 2, f_1],$$
$$[1, 5, 0, 4, f_2, 2, 7, 3, f_1],\ [2, 6, 1, 5, f_4, 3, 0, 4, f_3],\ [3, 7, 2, 6, f_4, 4, 1, 5, f_3],$$
$$[4, 0, 3, 7, f_4, 5, 2, 6, f_3],\ [5, 1, 4, 0, f_4, 6, 3, 7, f_3],\ [6, 5, 4, 0, f_5, 2, f_4, 0, 3],$$
$$[7, 6, 5, 1, f_5, 3, f_4, 1, 4],\ [0, 7, 6, 2, f_5, 4, f_4, 2, 5],\ [1, 0, 7, 3, f_5, 5, f_4, 3, 6],$$
$$[2, 1, 0, 4, f_6, 6, f_4, 4, 7],\ [3, 2, 1, 5, f_6, 7, f_4, 5, 0],\ [4, 3, 2, 6, f_6, 0, f_4, 6, 1],$$
$$[5, 4, 3, 7, f_6, 1, f_4, 7, 2]\},$$

$$B_2 = \{[f_5, 0, 1, 2, 4, 3, 6, 7, f_6, f_1],\ [f_6, 6, 7, 3, 5, 1, 4, 2, f_5, f_2],\ [f_3, 2, 4, 3, 6, 1, f_4, f_2, 5, 0],$$
$$[f_4, 0, 1, 2, 4, 3, 6, f_3, f_5, 5],\ [f_2, 0, 2, 4, f_4, 7, f_3, 3, 5, 6],\ [f_1, 0, 3, 1, f_4, 2, f_5, 4, 5, 6],$$
$$[2, 4, 6, f_6, f_2, f_1, 0, 7, f_5, f_4],\ [f_6, f_5, 3, f_4, 4, f_3, 6, 2, f_2, f_1]\},$$
$$B_2' = \{[7, f_4, 3, 1, 6, 2, 5, 0, 4, f_3],\ [0, f_4, 4, 2, 7, 3, 6, 1, 5, f_3],\ [1, f_4, 5, 3, 0, 4, 7, 2, 6, f_3],$$
$$[2, f_4, 6, 4, 1, 5, 0, 3, 7, f_3],\ [3, f_2, 7, 5, 2, 6, 1, 4, 0, f_1],\ [4, f_2, 0, 6, 3, 7, 2, 5, 1, f_1],$$
$$[5, f_2, 1, 7, 4, 0, 3, 6, 2, f_1],\ [6, f_2, 2, 0, 5, 1, 4, 7, 3, f_1],\ [4, 5, 1, 7, 3, f_5, 0, f_1, 6, f_3],$$
$$[5, 6, 2, 0, 4, f_5, 1, f_1, 7, f_3],\ [6, 7, 3, 1, 5, f_5, 2, f_1, 0, f_3],\ [7, 0, 4, 2, 6, f_5, 3, f_1, 1, f_3],$$
$$[0, 1, 5, 3, 7, f_6, 4, f_1, 2, f_3],\ [1, 2, 6, 4, 0, f_6, 5, f_1, 3, f_3],\ [2, 3, 7, 5, 1, f_6, 6, f_1, 4, f_3],$$
$$[3, 4, 0, 6, 2, f_6, 7, f_1, 5, f_3],\ [f_1, 0, 1, 2, 3, 4, 5, f_2, 6, 7],\ [f_2, 0, 1, 2, 3, 4, 5, f_3, 6, 7],$$
$$[f_3, 0, 1, 2, 3, 4, 5, f_1, 6, 7],\ [f_1, 1, 2, 3, 4, 5, 6, f_2, 0, 7],\ [f_2, 1, 2, 3, 4, 5, 6, f_3, 0, 7],$$
$$[f_3, 1, 2, 3, 4, 5, 6, f_1, 0, 7]\},$$

$$B_3 = \{[0, 2, f_1, 1, 4, 7, f_2, 3, 5, f_3],\ [0, 2, f_4, 1, 4, 7, f_3, 3, 5, f_2],\ [3, 5, 7, f_4, f_3, 2, f_2, 0, 1, 4],$$
$$[0, 1, f_5, 4, 2, 5, f_6, f_1, f_4, 7],\ [0, 1, f_6, 4, 2, 5, f_5, f_2, f_1, 7],\ [f_1, 2, f_6, f_2, 3, f_3, f_5, 0, 1, 6],$$
$$[0, f_5, f_4, 5, 6, f_3, f_6, 2, 4, 7],\ [f_1, 2, f_5, f_2, 4, f_4, f_6, 0, 1, 3\},$$

$$\begin{aligned}
B_3' = \{&[0,1,f_1,2,3,4,f_2,5,6,f_3],\ [1,2,f_1,3,4,5,f_2,6,7,f_3],\ [0,1,f_2,2,3,4,f_3,5,6,f_1],\\
&[1,2,f_3,0,7,6,f_2,4,5,f_1],\ [2,3,f_3,1,0,7,f_1,5,6,f_2],\ [1,2,f_2,0,7,6,f_1,4,5,f_3],\\
&[2,3,7,6,5,1,f_3,0,4,f_2],\ [3,4,0,7,6,2,f_3,1,5,f_2],\ [4,5,1,0,7,3,f_3,2,6,f_2],\\
&[5,6,2,1,0,4,f_3,3,7,f_2],\ [6,7,3,2,1,5,f_4,4,0,f_1],\ [7,0,4,3,2,6,f_4,5,1,f_1],\\
&[0,1,5,4,3,7,f_4,6,2,f_1],\ [1,2,6,5,4,0,f_4,7,3,f_1],\ [0,4,f_5,f_6,5,f_1,f_3,f_2,f_4,1],\\
&[1,5,f_5,f_6,6,f_1,f_3,f_2,f_4,2],\ [2,6,f_5,f_6,7,f_1,f_3,f_2,f_4,3],\\
&[3,7,f_5,f_6,0,f_1,f_3,f_2,f_4,4],\ [0,4,f_6,f_5,1,f_1,f_3,f_2,f_4,5],\\
&[1,5,f_6,f_5,2,f_1,f_3,f_2,f_4,6],\ [2,6,f_6,f_5,3,f_1,f_3,f_2,f_4,7],\\
&[3,7,f_6,f_5,4,f_1,f_3,f_2,f_4,0]\},
\end{aligned}$$

$$\begin{aligned}
B_4 = \{&[0,f_3,f_1,1,3,7,f_2,f_4,4,6],\ [3,0,1,2,5,6,f_3,4,7,f_2],\ [0,1,f_5,2,4,6,7,f_6,3,5],\\
&[f_1,0,f_5,f_4,1,f_6,f_3,2,4,f_2],\ [f_1,0,f_6,f_4,2,f_2,f_5,1,3,4],\\
&[0,f_4,f_3,f_2,1,2,f_5,f_6,3,6],\ [0,f_4,3,f_2,f_6,7,f_3,f_5,1,4]\},
\end{aligned}$$

$$\begin{aligned}
B_4' = \{&[0,1,f_1,2,3,4,5,f_2,6,7],\ [0,1,f_2,2,3,4,5,f_3,6,7],\ [0,1,f_3,2,3,4,5,f_1,6,7],\\
&[1,2,f_1,3,4,5,6,f_2,0,7],\ [1,2,f_2,3,4,5,6,f_3,0,7],\ [1,2,f_3,3,4,5,6,f_1,0,7],\\
&[f_1,0,4,5,6,f_3,7,2,f_4,1],\ [f_1,1,5,6,7,f_3,0,3,f_4,2],\ [f_1,2,6,7,0,f_3,1,4,f_4,3],\\
&[f_1,3,7,0,1,f_3,2,5,f_4,4],\ [f_2,4,0,1,2,f_3,3,6,f_4,5],\ [f_2,5,1,2,3,f_3,4,7,f_4,6],\\
&[f_2,6,2,3,4,f_3,5,0,f_4,7],\ [f_2,7,3,4,5,f_3,6,1,f_4,0],\ [f_3,0,4,7,2,f_1,f_2,1,3,5],\\
&[f_3,1,5,0,3,f_1,f_2,2,4,6],\ [f_3,2,6,1,4,f_1,f_2,3,5,7],\ [f_3,3,7,2,5,f_1,f_2,4,6,0],\\
&[f_4,4,0,3,6,f_1,f_2,5,7,1],\ [f_4,5,1,4,7,f_1,f_2,6,0,2],\ [f_4,6,2,5,0,f_1,f_2,7,1,3],\\
&[f_4,7,3,6,1,f_1,f_2,0,2,4],\ [f_5,4,0,1,5,7,f_4,f_1,3,6],\ [f_5,5,1,2,6,0,f_4,f_1,4,7],\\
&[f_5,6,2,3,7,1,f_4,f_1,5,0],\ [f_5,7,3,4,0,2,f_4,f_1,6,1],\ [f_6,0,4,5,1,3,f_4,f_1,7,2],\\
&[f_6,1,5,6,2,4,f_4,f_1,0,3],\ [f_6,2,6,7,3,5,f_4,f_1,1,4],\ [f_6,3,7,0,4,6,f_4,f_1,2,5]\},
\end{aligned}$$

$$\begin{aligned}
B_5 = \{&[6,f_4,f_6,f_5,7,1,3,f_3,0,2,5],\ [5,f_3,f_4,f_5,7,f_2,3,6,0,2,4],\\
&[f_5,1,f_3,5,f_6,3,4,6,f_1,0,2],\ [f_5,2,f_2,f_6,5,3,6,f_4,0,1,4],\\
&[f_3,1,f_2,f_4,4,2,5,6,f_1,0,3],\ [f_5,6,f_1,f_6,7,f_4,4,5,f_3,0,3],\\
&[3,f_2,f_1,f_4,5,2,4,f_6,f_5,0,1],\ [0,2,f_5,3,6,f_4,5,7,f_6,1,4]\},
\end{aligned}$$

$$\begin{aligned}
B_5' = \{&[0,1,f_1,2,3,f_2,4,5,f_3,6,7],\ [0,1,f_2,2,3,f_3,4,5,f_1,6,7],\ [0,1,f_3,2,3,f_1,4,5,f_2,6,7],\\
&[1,2,f_1,3,4,f_2,5,6,f_3,7,0],\ [1,2,f_2,3,4,f_3,5,6,f_1,7,0],\ [1,2,f_3,3,4,f_1,5,6,f_2,7,0],\\
&[1,2,0,f_1,4,f_2,3,7,f_6,5,6],\ [2,3,1,f_1,5,f_2,4,0,f_6,6,7],\ [3,4,2,f_1,6,f_2,5,1,f_6,7,0],\\
&[4,5,3,f_1,7,f_2,6,2,f_6,0,1],\ [5,6,4,f_3,0,f_4,7,3,f_6,1,2],\ [6,7,5,f_3,1,f_4,0,4,f_6,2,3],\\
&[7,0,6,f_3,2,f_4,1,5,f_6,3,4],\ [0,1,7,f_3,3,f_4,2,6,f_6,4,5],\\
&[1,3,4,0,f_5,f_2,5,7,f_1,f_3,2],\ [2,4,5,1,f_5,f_2,6,0,f_1,f_3,3],\\
&[3,5,6,2,f_5,f_2,7,1,f_1,f_3,4],\ [4,6,7,3,f_5,f_2,0,2,f_1,f_3,5],\\
&[5,7,0,4,f_6,f_2,1,3,f_1,f_3,6],\ [6,0,1,5,f_6,f_2,2,4,f_1,f_3,7],\\
&[7,1,2,6,f_6,f_2,3,5,f_1,f_3,0],\ [0,2,3,7,f_6,f_2,4,6,f_1,f_3,1]\}.
\end{aligned}$$

Then for $k \in [1,5]$ an H_k-decomposition of $K_{14}^{(3)} \setminus K_6^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in [1,6]$, and $j \mapsto j+1 \pmod 8$ on the vertices along with the H_k-blocks in B_k'.

Example 16 Let $V\left(K_{16}^{(3)} \setminus K_8^{(3)}\right) = \mathbb{Z}_8 \cup \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ with the set $\{f_1, \ldots, f_8\}$ as the set of vertices in the hole. Now, let

$$\begin{aligned}
B_1 = \{&[7,5,0,1,4,2,f_1,3,6],\ [0,6,f_2,2,5,f_3,4,f_5,f_6],\ [1,4,f_3,0,2,5,f_4,f_2,6],\\
&[0,1,f_5,2,4,3,f_1,f_2,5],\ [0,1,f_6,2,4,3,f_1,f_3,5],\ [0,3,f_5,f_3,1,f_2,4,5,f_4],\\
&[5,6,f_7,1,3,f_1,0,2,4],\ [0,1,f_8,2,4,3,f_1,f_4,5],\ [f_3,0,f_8,f_2,1,2,f_4,f_6,3],\\
&[0,3,f_7,f_3,1,2,f_2,f_6,4],\ [0,3,f_8,f_5,1,2,f_6,4,7],\ [f_5,0,f_7,f_4,1,2,f_6,f_3,3]\},
\end{aligned}$$

$$\begin{aligned}
B_1' = \{&[0,1,f_3,2,3,4,5,f_2,6],\ [0,1,f_2,2,3,4,5,f_1,6],\ [0,1,f_1,2,3,4,5,f_3,6],\\
&[1,2,f_3,3,4,6,7,f_2,0],\ [1,2,f_2,3,4,6,7,f_1,0],\ [1,2,f_1,3,4,6,7,f_3,0],\\
&[6,2,5,1,f_2,7,4,0,f_1],\ [7,3,6,2,f_2,0,5,1,f_1],\ [0,4,7,3,f_2,1,6,2,f_1],\\
&[1,5,0,4,f_2,2,7,3,f_1],\ [2,6,1,5,f_4,3,0,4,f_3],\ [3,7,2,6,f_4,4,1,5,f_3],\\
&[4,0,3,7,f_4,5,2,6,f_3],\ [5,1,4,0,f_4,6,3,7,f_3],\ [6,5,4,0,f_5,2,f_4,0,3],\\
&[7,6,5,1,f_5,3,f_4,1,4],\ [0,7,6,2,f_5,4,f_4,2,5],\ [1,0,7,3,f_5,5,f_4,3,6],\\
&[2,1,0,4,f_6,6,f_4,4,7],\ [3,2,1,5,f_6,7,f_4,5,0],\ [4,3,2,6,f_6,0,f_4,6,1],\\
&[5,4,3,7,f_6,1,f_4,7,2],\ [1,7,4,f_5,f_4,0,f_7,f_8,2],\ [2,0,5,f_5,f_4,1,f_7,f_8,3],\\
&[3,1,6,f_5,f_4,2,f_7,f_8,4],\ [4,2,7,f_5,f_4,3,f_7,f_8,5],\ [5,3,0,f_5,f_4,4,f_8,f_7,6],\\
&[6,4,1,f_5,f_4,5,f_8,f_7,7],\ [7,5,2,f_5,f_4,6,f_8,f_7,0],\ [0,6,3,f_5,f_4,7,f_8,f_7,1]\},
\end{aligned}$$

$$\begin{aligned}
B_2 = \{&[f_8,f_3,2,f_2,1,f_1,0,3,5,f_5],\ [f_6,0,1,2,4,3,6,5,f_5,f_1],\ [f_7,f_4,2,f_3,1,f_2,0,3,4,f_5],\\
&[f_7,0,1,2,4,3,6,5,f_6,f_1],\ [f_8,f_7,0,f_6,1,f_4,2,3,f_5,f_3],\ [f_6,f_5,0,f_4,1,f_3,2,f_1,7,f_7],\\
&[f_5,f_4,2,0,3,4,f_8,5,f_7,f_6],\ [f_8,0,1,2,4,3,6,5,f_2,f_6],\ [f_1,0,f_4,1,4,3,5,6,f_5,f_2],\\
&[0,1,2,4,6,f_7,f_5,5,f_3,f_1],\ [f_2,0,2,4,f_4,7,f_3,3,5,6],\ [f_4,0,1,2,4,3,6,f_1,f_2,5],\\
&[f_3,2,4,3,6,1,f_4,f_2,5,0]\},
\end{aligned}$$

$$\begin{aligned}
B_2' = \{&[7,f_4,3,1,6,2,5,0,4,f_3],\ [0,f_4,4,2,7,3,6,1,5,f_3],\ [1,f_4,5,3,0,4,7,2,6,f_3],\\
&[2,f_4,6,4,1,5,0,3,7,f_3],\ [3,f_2,7,5,2,6,1,4,0,f_1],\ [4,f_2,0,6,3,7,2,5,1,f_1],\\
&[5,f_2,1,7,4,0,3,6,2,f_1],\ [6,f_2,2,0,5,1,4,7,3,f_1],\ [0,1,5,4,f_5,3,7,2,6,f_6],\\
&[1,2,6,5,f_5,4,0,3,7,f_6],\ [2,3,7,6,f_5,5,1,4,0,f_6],\ [3,4,0,7,f_5,6,2,5,1,f_6],\\
&[4,5,1,0,f_7,7,3,6,2,f_8],\ [5,6,2,1,f_7,0,4,7,3,f_8],\ [6,7,3,2,f_7,1,5,0,4,f_8],\\
&[7,0,4,3,f_7,2,6,1,5,f_8],\ [f_1,0,1,2,3,4,5,f_2,6,7],\ [f_2,0,1,2,3,4,5,f_3,6,7],\\
&[f_3,0,1,2,3,4,5,f_1,6,7],\ [f_1,1,2,3,4,5,6,f_2,0,7],\ [f_2,1,2,3,4,5,6,f_3,0,7],\\
&[f_3,1,2,3,4,5,6,f_1,0,7]\},
\end{aligned}$$

$$\begin{aligned}
B_3 = \{&[0,2,f_1,1,4,7,f_2,3,5,f_3],\ [0,2,f_4,1,4,7,f_3,3,5,f_2],\ [3,5,7,f_4,f_3,2,f_2,0,1,4],\\
&[0,1,f_5,4,2,5,f_6,f_1,f_4,7],\ [0,1,f_6,4,2,5,f_5,f_2,f_1,7],\ [0,f_5,f_4,5,6,f_3,f_6,2,4,7],\\
&[f_8,2,f_1,4,f_7,5,6,0,1,3],\ [f_7,0,f_2,1,f_8,2,3,f_4,f_6,4],\ [f_8,0,f_3,1,f_7,2,4,f_2,f_5,3],\\
&[f_7,1,f_4,3,f_8,2,4,0,5,7],\ [f_8,0,f_5,1,f_7,2,5,f_1,f_6,3],\ [f_7,0,f_6,1,f_8,2,5,f_3,f_5,3]\},
\end{aligned}$$

$$\begin{aligned}
B_3' = \{&[0,1,f_1,2,3,4,f_2,5,6,f_3],\ [1,2,f_1,3,4,5,f_2,6,7,f_3],\ [0,1,f_2,2,3,4,f_3,5,6,f_1],\\
&[1,2,f_3,0,7,6,f_2,4,5,f_1],\ [2,3,f_3,1,0,7,f_1,5,6,f_2],\ [1,2,f_2,0,7,6,f_1,4,5,f_3],\\
&[2,3,7,6,5,1,f_3,0,4,f_2],\ [3,4,0,7,6,2,f_3,1,5,f_2],\ [4,5,1,0,7,3,f_3,2,6,f_2],\\
&[5,6,2,1,0,4,f_3,3,7,f_2],\ [6,7,3,2,1,5,f_4,4,0,f_1],\ [7,0,4,3,2,6,f_4,5,1,f_1],\\
&[0,1,5,4,3,7,f_4,6,2,f_1],\ [1,2,6,5,4,0,f_4,7,3,f_1],\ [0,4,f_5,f_6,5,f_1,f_3,f_2,f_4,1],\\
&[1,5,f_5,f_6,6,f_1,f_3,f_2,f_4,2],\ [2,6,f_5,f_6,7,f_1,f_3,f_2,f_4,3],\\
&[3,7,f_5,f_6,0,f_1,f_3,f_2,f_4,4],\ [0,4,f_6,f_5,1,f_1,f_3,f_2,f_4,5],\\
&[1,5,f_6,f_5,2,f_1,f_3,f_2,f_4,6],\ [2,6,f_6,f_5,3,f_1,f_3,f_2,f_4,7],\\
&[3,7,f_6,f_5,4,f_1,f_3,f_2,f_4,0],\ [0,4,f_7,f_8,1,f_1,f_5,f_2,f_6,2],\\
&[1,5,f_7,f_8,2,f_1,f_5,f_2,f_6,3],\ [2,6,f_7,f_8,3,f_1,f_5,f_2,f_6,4],\\
&[3,7,f_7,f_8,4,f_1,f_5,f_2,f_6,5],\ [0,4,f_8,f_7,5,f_1,f_5,f_2,f_6,6],
\end{aligned}$$

$[1, 5, f_8, f_7, 6, f_1, f_5, f_2, f_6, 7]$, $[2, 6, f_8, f_7, 7, f_1, f_5, f_2, f_6, 0]$,
$[3, 7, f_8, f_7, 0, f_1, f_5, f_2, f_6, 1]\big\}$,

$B_4 = \big\{[0, f_3, f_1, 1, 3, 7, f_2, f_4, 4, 6], [f_1, 0, f_6, f_4, 2, f_2, f_5, 1, 3, 4], [0, f_4, f_7, 1, f_1, 2, f_2, f_8, 3, 6],$
$[0, f_3, f_8, 1, f_5, 2, f_2, f_7, 3, f_6], [0, f_6, f_8, 1, f_1, 2, f_3, f_7, 3, f_5], [0, 1, f_7, 2, 4, 6, 7, f_8, 3, 5],$
$[0, f_7, f_8, 1, f_4, 2, f_2, f_3, 3, f_6], [0, 3, f_5, 1, 2, 4, 7, f_6, 5, 6], [0, 2, f_2, 4, 7, 3, f_3, f_5, 1, f_1],$
$[0, f_4, f_5, 1, f_6, 2, f_7, 5, 7, 4], [4, 0, 1, 3, f_3, f_4, 2, 5, f_2, f_6]\big\}$,

$B'_4 = \big\{[0, 1, f_1, 2, 3, 4, 5, f_2, 6, 7], [0, 1, f_2, 2, 3, 4, 5, f_3, 6, 7], [0, 1, f_3, 2, 3, 4, 5, f_1, 6, 7],$
$[1, 2, f_1, 3, 4, 5, 6, f_2, 0, 7], [1, 2, f_2, 3, 4, 5, 6, f_3, 0, 7], [1, 2, f_3, 3, 4, 5, 6, f_1, 0, 7],$
$[f_1, 0, 4, 5, 6, f_3, 7, 2, f_4, 1], [f_1, 1, 5, 6, 7, f_3, 0, 3, f_4, 2], [f_1, 2, 6, 7, 0, f_3, 1, 4, f_4, 3],$
$[f_1, 3, 7, 0, 1, f_3, 2, 5, f_4, 4], [f_2, 4, 0, 1, 2, f_3, 3, 6, f_4, 5], [f_2, 5, 1, 2, 3, f_3, 4, 7, f_4, 6],$
$[f_2, 6, 2, 3, 4, f_3, 5, 0, f_4, 7], [f_2, 7, 3, 4, 5, f_3, 6, 1, f_4, 0], [f_3, 0, 4, 7, 2, f_1, f_2, 1, 3, 5],$
$[f_3, 1, 5, 0, 3, f_1, f_2, 2, 4, 6], [f_3, 2, 6, 1, 4, f_1, f_2, 3, 5, 7], [f_3, 3, 7, 2, 5, f_1, f_2, 4, 6, 0],$
$[f_4, 4, 0, 3, 6, f_1, f_2, 5, 7, 1], [f_4, 5, 1, 4, 7, f_1, f_2, 6, 0, 2], [f_4, 6, 2, 5, 0, f_1, f_2, 7, 1, 3],$
$[f_4, 7, 3, 6, 1, f_1, f_2, 0, 2, 4], [f_5, 4, 0, 1, 5, 7, f_4, f_1, 3, 6], [f_5, 5, 1, 2, 6, 0, f_4, f_1, 4, 7],$
$[f_5, 6, 2, 3, 7, 1, f_4, f_1, 5, 0], [f_5, 7, 3, 4, 0, 2, f_4, f_1, 6, 1], [f_6, 0, 4, 5, 1, 3, f_4, f_1, 7, 2],$
$[f_6, 1, 5, 6, 2, 4, f_4, f_1, 0, 3], [f_6, 2, 6, 7, 3, 5, f_4, f_1, 1, 4], [f_6, 3, 7, 0, 4, 6, f_4, f_1, 2, 5],$
$[f_7, 4, 0, 2, f_6, f_3, f_4, 5, 7, f_5], [f_7, 5, 1, 3, f_6, f_3, f_4, 6, 0, f_5], [f_7, 6, 2, 4, f_6, f_3, f_4, 7, 1, f_5],$
$[f_7, 7, 3, 5, f_6, f_3, f_4, 0, 2, f_5], [f_8, 0, 4, 6, f_6, f_3, f_4, 1, 3, f_5], [f_8, 1, 5, 7, f_6, f_3, f_4, 2, 4, f_5],$
$[f_8, 2, 6, 0, f_6, f_3, f_4, 3, 5, f_5], [f_8, 3, 7, 1, f_6, f_3, f_4, 4, 6, f_5]\big\}$,

$B_5 = \big\{[0, 4, 1, 2, 6, f_2, f_3, 3, 5, 7, f_4], [6, f_6, f_8, 7, f_7, 3, f_3, f_4, 0, f_1, f_2],$
$[7, f_6, f_7, 3, 6, 2, 5, f_2, 4, f_5, f_8], [4, f_7, f_5, 3, f_6, 0, 2, f_2, 1, 6, f_3],$
$[1, f_7, f_4, 2, f_8, 0, 3, 5, 4, 6, 7], [0, f_5, f_4, 1, f_6, 2, 4, 6, 3, 5, f_8],$
$[7, f_8, f_3, 6, f_7, 2, f_1, f_5, 0, 3, f_6], [4, f_5, f_3, 6, f_6, 0, 3, f_1, 1, 2, f_4],$
$[5, f_8, f_2, 3, f_7, 2, 4, f_6, 0, 1, f_5], [0, f_5, f_2, f_6, 4, 2, 5, f_4, 1, 3, f_3],$
$[0, f_3, f_1, f_4, 4, 6, 7, f_7, 1, 3, f_5], [f_1, f_6, 1, f_2, f_4, 4, 7, f_8, 3, 5, f_7],$
$[0, f_8, f_1, 1, f_7, 6, 7, f_6, 2, 5, f_5]\big\}$,

$B'_5 = \big\{[0, 1, f_1, 2, 3, f_2, 4, 5, f_3, 6, 7], [0, 1, f_2, 2, 3, f_3, 4, 5, f_1, 6, 7], [0, 1, f_3, 2, 3, f_1, 4, 5, f_2, 6, 7],$
$[1, 2, f_1, 3, 4, f_2, 5, 6, f_3, 7, 0], [1, 2, f_2, 3, 4, f_3, 5, 6, f_1, 7, 0], [1, 2, f_3, 3, 4, f_1, 5, 6, f_2, 7, 0],$
$[5, 6, 7, 3, f_2, 1, 2, f_8, 0, 4, f_1], [6, 7, 0, 4, f_2, 2, 3, f_8, 1, 5, f_1], [7, 0, 1, 5, f_2, 3, 4, f_8, 2, 6, f_1],$
$[0, 1, 2, 6, f_2, 4, 5, f_8, 3, 7, f_1], [1, 2, 3, 7, f_3, 5, 6, f_8, 0, 4, f_4], [2, 3, 4, 0, f_3, 6, 7, f_8, 1, 5, f_4],$
$[3, 4, 5, 1, f_3, 7, 0, f_8, 2, 6, f_4], [4, 5, 6, 2, f_3, 0, 1, f_8, 3, 7, f_4], [1, 3, 0, 4, f_5, 5, 7, f_1, 2, 6, f_6],$
$[2, 4, 1, 5, f_5, 6, 0, f_1, 3, 7, f_6], [3, 5, 2, 6, f_5, 7, 1, f_1, 4, 0, f_6], [4, 6, 3, 7, f_5, 0, 2, f_1, 5, 1, f_6],$
$[5, 7, 4, 0, f_7, 1, 3, f_1, 6, 2, f_8], [6, 0, 5, 1, f_7, 2, 4, f_1, 7, 3, f_8], [7, 1, 6, 2, f_7, 3, 5, f_1, 0, 4, f_8],$
$[0, 2, 7, 3, f_7, 4, 6, f_1, 1, 5, f_8]\big\}$.

Then for $k \in [1, 5]$ an H_k-decomposition of $K_{16}^{(3)} \setminus K_8^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in [1, 8]$, and $j \mapsto j + 1 \pmod 8$ on the vertices along with the H_k-blocks in B'_k.

Example 17 Let $V\left(K_{17}^{(3)} \setminus K_9^{(3)}\right) = \mathbb{Z}_8 \cup \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$ with $\{f_1, \ldots, f_9\}$ as the set of vertices in the hole. Now, let

$$\begin{aligned}
B_2 = \big\{&[f_8, f_3, 2, f_2, 1, f_1, 0, 3, 5, f_5], [f_6, 0, 1, 2, 4, 3, 6, 5, f_5, f_1], [f_7, f_4, 2, f_3, 1, f_2, 0, 3, 4, f_5],\\
&[f_7, 0, 1, 2, 4, 3, 6, 5, f_6, f_1], [f_8, f_7, 0, f_6, 1, f_4, 2, 3, f_5, f_3], [f_6, f_5, 0, f_4, 1, f_3, 2, f_1, 7, f_7],\\
&[f_8, 0, 1, 2, 4, 3, 6, 5, f_2, f_6], [f_5, f_4, 2, 0, 3, 4, f_8, 5, f_7, f_6], [0, 1, 2, 4, 6, f_7, f_5, 5, f_3, f_1],\\
&[f_9, f_4, 0, f_3, 1, f_1, 2, 3, 6, f_2], [f_2, 0, f_5, 1, f_4, 2, f_9, 3, 5, 6], [f_4, 0, 1, 2, 4, 3, 6, 7, f_9, f_5],\\
&[f_1, 0, 3, 1, f_4, 2, f_2, 4, 7, f_3], [f_9, 0, 1, 2, 4, 3, 6, f_2, f_3, 5], [f_9, f_8, 7, f_7, 6, f_6, 5, 1, f_4, f_3]\big\},\\
B_2' = \big\{&[f_1, 0, 1, 2, 3, 4, 5, f_2, 6, 7], [f_2, 0, 1, 2, 3, 4, 5, f_3, 6, 7], [f_3, 0, 1, 2, 3, 4, 5, f_1, 6, 7],\\
&[f_1, 1, 2, 3, 4, 5, 6, f_2, 0, 7], [f_2, 1, 2, 3, 4, 5, 6, f_3, 0, 7], [f_3, 1, 2, 3, 4, 5, 6, f_1, 0, 7],\\
&[f_1, 0, 4, 1, 3, 2, 6, 5, 7, f_2], [f_1, 0, 6, 2, 4, 3, 5, 1, 7, f_2], [f_1, 0, 2, 1, 5, 3, 7, 4, 6, f_3],\\
&[f_2, 0, 2, 1, 3, 4, 6, 5, 7, f_1], [f_3, 0, 2, 1, 3, 5, 7, f_1, 4, 6], [f_2, 0, 6, 2, 4, 3, 5, 1, 7, f_3],\\
&[f_3, 0, 6, 2, 4, 3, 5, 1, 7, f_1], [7, f_2, 3, 1, 6, 2, 5, 0, 4, f_3], [0, f_2, 4, 2, 7, 3, 6, 1, 5, f_3],\\
&[1, f_2, 5, 3, 0, 4, 7, 2, 6, f_3], [2, f_2, 6, 4, 1, 5, 0, 3, 7, f_3], [3, f_4, 7, 5, 2, 6, 1, 4, 0, f_5],\\
&[4, f_4, 0, 6, 3, 7, 2, 5, 1, f_5], [5, f_4, 1, 7, 4, 0, 3, 6, 2, f_5], [6, f_4, 2, 0, 5, 1, 4, 7, 3, f_5],\\
&[0, 1, 5, 4, f_6, 3, 7, 2, 6, f_7], [1, 2, 6, 5, f_6, 4, 0, 3, 7, f_7], [2, 3, 7, 6, f_6, 5, 1, 4, 0, f_7],\\
&[3, 4, 0, 7, f_6, 6, 2, 5, 1, f_7], [4, 5, 1, 0, f_8, 7, 3, 6, 2, f_9], [5, 6, 2, 1, f_8, 0, 4, 7, 3, f_9],\\
&[6, 7, 3, 2, f_8, 1, 5, 0, 4, f_9], [7, 0, 4, 3, f_8, 2, 6, 1, 5, f_9]\big\},\\
B_3 = \big\{&[0, 2, f_1, 1, 4, 7, f_2, 3, 5, f_3], [0, 2, f_4, 1, 4, 7, f_3, 3, 5, f_2], [3, 5, 7, f_4, f_3, 2, f_2, 0, 1, 4],\\
&[0, 1, f_5, 4, 2, 5, f_6, f_1, f_4, 7], [0, 1, f_6, 4, 2, 5, f_5, f_2, f_1, 7], [f_1, 2, f_5, f_2, 4, f_4, f_6, 0, 1, 3],\\
&[f_1, 2, f_6, f_2, 3, f_3, f_5, 0, 1, 6], [0, f_5, f_4, 5, 6, f_3, f_6, 2, 4, 7], [0, 1, f_7, 2, 4, 7, f_8, f_1, f_9, 6],\\
&[0, 1, f_8, 2, 4, 7, f_9, f_1, f_7, 6], [0, 1, f_9, 2, 4, 7, f_7, f_1, f_8, 6], [0, f_7, f_2, 1, f_8, f_3, 2, f_4, f_9, 3],\\
&[f_2, 0, f_9, 1, f_3, f_7, 2, f_4, f_8, 3], [f_9, 0, f_5, 1, f_8, f_6, 2, f_4, f_7, 3],\\
&[0, f_8, f_7, 1, f_9, f_6, 2, f_3, 3, 4], [f_6, 0, f_7, f_5, 1, f_8, f_9, f_2, 2, 3]\big\},\\
B_3' = \big\{&[f_8, 1, 5, 6, f_1, 2, 3, f_9, 0, 4], [f_7, 3, 7, 0, f_1, 4, 5, f_9, 2, 6], [f_7, 5, 1, 0, f_1, 3, 4, f_8, 2, 6],\\
&[f_1, 6, 7, 3, f_9, 1, 5, f_8, 0, 4], [f_1, 1, 2, 6, f_7, 0, 4, f_8, 3, 7], [2, 3, 7, 6, 5, 1, f_3, 0, 4, f_2],\\
&[3, 4, 0, 7, 6, 2, f_3, 1, 5, f_2], [4, 5, 1, 0, 7, 3, f_3, 2, 6, f_2], [5, 6, 2, 1, 0, 4, f_3, 3, 7, f_2],\\
&[6, 7, 3, 2, 1, 5, f_4, 4, 0, f_1], [7, 0, 4, 3, 2, 6, f_4, 5, 1, f_1], [0, 1, 5, 4, 3, 7, f_4, 6, 2, f_1],\\
&[1, 2, 6, 5, 4, 0, f_4, 7, 3, f_1], [0, 4, f_5, f_6, 5, f_1, f_3, f_2, f_4, 1], [1, 5, f_5, f_6, 6, f_1, f_3, f_2, f_4, 2],\\
&[2, 6, f_5, f_6, 7, f_1, f_3, f_2, f_4, 3], [3, 7, f_5, f_6, 0, f_1, f_3, f_2, f_4, 4],\\
&[0, 4, f_6, f_5, 1, f_1, f_3, f_2, f_4, 5], [1, 5, f_6, f_5, 2, f_1, f_3, f_2, f_4, 6],\\
&[2, 6, f_6, f_5, 3, f_1, f_3, f_2, f_4, 7], [3, 7, f_6, f_5, 4, f_1, f_3, f_2, f_4, 0]\big\},\\
B_4 = \big\{&[0, f_3, f_1, 1, 3, 7, f_2, f_4, 4, 6], [f_1, 0, f_6, f_4, 2, f_2, f_5, 1, 3, 4], [0, f_3, f_8, 1, f_5, 2, f_2, f_7, 3, f_6],\\
&[0, f_6, f_8, 1, f_1, 2, f_3, f_7, 3, f_5], [0, 3, f_5, 1, 2, 4, 7, f_6, 5, 6], [0, 1, f_7, 2, 4, 6, 7, f_8, 3, 5],\\
&[0, f_4, f_7, 1, f_1, 2, f_2, f_8, 3, 6], [0, f_8, f_9, 1, f_7, 2, f_4, f_5, 3, f_3], [0, f_6, f_9, 1, f_5, 2, f_2, f_3, 3, 4],\\
&[0, f_4, f_9, 1, 3, 5, f_8, f_7, 4, 7], [0, f_1, f_9, 1, f_3, 2, 5, f_2, 3, f_6], [0, f_5, f_6, 1, f_3, 5, f_9, 2, 4, f_2],\\
&[0, f_2, f_9, 1, 2, 3, 4, 6, f_1, f_5], [4, 0, 1, 3, f_3, 2, 5, f_4, 6, f_8]\big\},\\
B_4' = \big\{&[0, 7, f_1, 3, 4, f_2, 1, 2, 6, f_9], [f_1, 2, 3, 7, f_9, 0, 1, f_2, 5, 6], [1, 2, f_1, 6, 7, f_9, 0, 4, 5, f_2],\\
&[6, f_1, 5, 1, f_9, 3, 4, f_2, 0, 7], [0, 1, f_1, 4, 5, 2, 3, f_2, 6, 7], [f_1, 0, 4, 5, 6, f_3, 7, 2, f_4, 1],\\
&[f_1, 1, 5, 6, 7, f_3, 0, 3, f_4, 2], [f_1, 2, 6, 7, 0, f_3, 1, 4, f_4, 3], [f_1, 3, 7, 0, 1, f_3, 2, 5, f_4, 4],\\
&[f_2, 4, 0, 1, 2, f_3, 3, 6, f_4, 5], [f_2, 5, 1, 2, 3, f_3, 4, 7, f_4, 6], [f_2, 6, 2, 3, 4, f_3, 5, 0, f_4, 7],\\
&[f_2, 7, 3, 4, 5, f_3, 6, 1, f_4, 0], [f_3, 0, 4, 7, 2, f_1, f_2, 1, 3, 5], [f_3, 1, 5, 0, 3, f_1, f_2, 2, 4, 6],
\end{aligned}$$

$[f_3, 2, 6, 1, 4, f_1, f_2, 3, 5, 7]$, $[f_3, 3, 7, 2, 5, f_1, f_2, 4, 6, 0]$, $[f_4, 4, 0, 3, 6, f_1, f_2, 5, 7, 1]$,
$[f_4, 5, 1, 4, 7, f_1, f_2, 6, 0, 2]$, $[f_4, 6, 2, 5, 0, f_1, f_2, 7, 1, 3]$, $[f_4, 7, 3, 6, 1, f_1, f_2, 0, 2, 4]$,
$[f_5, 4, 0, 1, 5, 7, f_4, f_1, 3, 6]$, $[f_5, 5, 1, 2, 6, 0, f_4, f_1, 4, 7]$, $[f_5, 6, 2, 3, 7, 1, f_4, f_1, 5, 0]$,
$[f_5, 7, 3, 4, 0, 2, f_4, f_1, 6, 1]$, $[f_6, 0, 4, 5, 1, 3, f_4, f_1, 7, 2]$, $[f_6, 1, 5, 6, 2, 4, f_4, f_1, 0, 3]$,
$[f_6, 2, 6, 7, 3, 5, f_4, f_1, 1, 4]$, $[f_6, 3, 7, 0, 4, 6, f_4, f_1, 2, 5]$, $[f_7, 4, 0, 2, f_6, f_3, f_4, 5, 7, f_5]$,
$[f_7, 5, 1, 3, f_6, f_3, f_4, 6, 0, f_5]$, $[f_7, 6, 2, 4, f_6, f_3, f_4, 7, 1, f_5]$, $[f_7, 7, 3, 5, f_6, f_3, f_4, 0, 2, f_5]$,
$[f_8, 0, 4, 6, f_6, f_3, f_4, 1, 3, f_5]$, $[f_8, 1, 5, 7, f_6, f_3, f_4, 2, 4, f_5]$, $[f_8, 2, 6, 0, f_6, f_3, f_4, 3, 5, f_5]$,
$[f_8, 3, 7, 1, f_6, f_3, f_4, 4, 6, f_5]\}$,

$B_5 = \{[0, 1, f_7, 2, 4, f_8, 3, 6, f_1, f_9, 7]$, $[0, 1, f_8, 2, 4, f_9, 3, 6, f_1, f_7, 7]$,
$[0, 1, f_9, 2, 4, f_7, 3, 6, f_1, f_8, 7]$, $[0, f_7, f_9, f_8, 1, f_3, 2, 3, f_2, 4, 5]$,
$[f_6, 0, f_8, f_7, 1, f_2, f_9, 2, f_4, 3, 4]$, $[f_2, 0, f_7, f_3, 1, f_5, f_8, 2, f_6, f_9, 3]$,
$[f_2, 0, f_8, f_3, 1, f_4, f_7, 2, f_5, f_9, 3]$, $[f_5, 0, f_7, f_6, 1, f_4, f_8, 2, f_3, f_9, 3]$,
$[6, f_4, f_6, f_5, 7, 1, 3, f_3, 0, 2, 5]$, $[5, f_3, f_4, f_5, 7, f_2, 3, 6, 0, 2, 4]$,
$[f_5, 1, f_3, 5, f_6, 3, 4, 6, f_1, 0, 2]$, $[f_5, 2, f_2, f_6, 5, 3, 6, f_4, 0, 1, 4]$,
$[f_3, 1, f_2, f_4, 4, 2, 5, 6, f_1, 0, 3]$, $[f_5, 6, f_1, f_6, 7, f_4, f_9, 5, f_3, 0, 3]$,
$[3, f_3, f_1, f_4, 5, 2, 4, f_6, f_5, 0, 1]$, $[0, 2, f_5, 3, 6, f_4, 5, 7, f_6, 1, 4]\}$,

$B'_5 = \{[1, 2, f_1, 5, 6, f_8, 0, 4, f_9, 3, 7]$, $[3, 4, f_1, 7, 0, f_8, 1, 5, f_9, 2, 6]$, $[2, 3, f_1, 6, 7, f_7, 1, 5, f_9, 0, 4]$,
$[0, 1, f_1, 4, 5, f_7, 3, 7, f_8, 2, 6]$, $[0, 4, f_7, 2, 6, f_9, 1, 5, f_8, 3, 7]$, $[1, 2, 0, f_1, 4, f_2, 3, 7, f_6, 5, 6]$,
$[2, 3, 1, f_1, 5, f_2, 4, 0, f_6, 6, 7]$, $[3, 4, 2, f_1, 6, f_2, 5, 1, f_6, 7, 0]$, $[4, 5, 3, f_1, 7, f_2, 6, 2, f_6, 0, 1]$,
$[5, 6, 4, f_3, 0, f_4, 7, 3, f_6, 1, 2]$, $[6, 7, 5, f_3, 1, f_4, 0, 4, f_6, 2, 3]$, $[7, 0, 6, f_3, 2, f_4, 1, 5, f_6, 3, 4]$,
$[0, 1, 7, f_3, 3, f_4, 2, 6, f_6, 4, 5]$, $[2, f_1, f_2, 1, 7, 3, 5, 6, f_5, 0, 4]$, $[3, f_1, f_2, 2, 0, 4, 6, 7, f_5, 1, 5]$,
$[4, f_1, f_2, 3, 1, 5, 7, 0, f_5, 2, 6]$, $[5, f_1, f_2, 4, 2, 6, 0, 1, f_5, 3, 7]$, $[6, f_1, f_2, 5, 3, 7, 1, 2, f_6, 0, 4]$,
$[7, f_1, f_2, 6, 4, 0, 2, 3, f_6, 1, 5]$, $[0, f_1, f_2, 7, 5, 1, 3, 4, f_6, 2, 6]$, $[1, f_1, f_2, 0, 6, 2, 4, 5, f_6, 3, 7]\}$.

Then for $k \in [2, 5]$ an H_k-decomposition of $K_{17}^{(3)} \setminus K_9^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $f_i \mapsto f_i$, for $i \in [1, 9]$, and $j \mapsto j + 1 \pmod 8$ on the vertices along with the H_k-blocks in B'_k.

Example 18 Let $V\left(K_{18}^{(3)} \setminus K_{10}^{(3)}\right) = \mathbb{Z}_8 \cup \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}\}$ with $\{f_1, \ldots, f_{10}\}$ as the set of vertices in the hole. Now, let

$B_5 = \{[6, f_9, f_8, 7, f_{10}, 0, f_1, f_3, 1, 2, 4]$, $[4, f_9, f_7, 5, f_{10}, 0, f_1, f_4, 1, 3, 6]$,
$[2, f_6, f_7, f_8, 3, 4, 6, f_1, 0, 1, 5]$, $[4, f_9, f_6, 6, f_{10}, 0, 2, f_3, 3, 5, f_2]$,
$[2, f_5, f_6, f_8, 3, 0, 1, f_4, f_{10}, 4, 7]$, $[2, f_9, f_5, f_{10}, 3, 0, 1, f_6, 4, 7, f_7]$,
$[2, f_7, f_5, f_8, 3, 1, f_2, f_3, 4, 7, f_9]$, $[3, f_9, f_4, f_{10}, 4, f_1, f_5, 2, 1, f_2, f_6]$,
$[3, f_7, f_4, f_8, 6, 2, 4, f_9, 5, 7, f_{10}]$, $[2, f_3, f_6, 4, f_4, f_1, f_{10}, 1, f_5, 0, 3]$,
$[f_9, 2, f_3, f_{10}, 4, f_1, 0, 3, f_2, f_8, 1]$, $[f_7, 2, f_3, f_8, 3, f_2, f_5, 4, f_1, f_9, 1]$,
$[f_4, 1, f_3, f_5, 2, f_2, f_{10}, 3, f_9, 6, 7]$, $[1, f_4, f_2, f_7, 5, f_1, f_8, 3, f_6, 2, 4]$,
$[f_2, f_9, 5, 6, f_8, f_4, 2, 7, f_{10}, 3, 4]$, $[1, f_6, f_1, 4, f_7, f_8, 3, 5, f_5, 0, 2]$,
$[1, 3, f_7, 5, 6, 4, 7, f_8, f_4, 0, 2]$, $[f_2, 2, 5, 6, f_5, f_3, 0, 3, f_6, 4, 7]\}$,

$B'_5 = \{[0, 1, f_1, 2, 3, f_2, 4, 5, f_3, 6, 7]$, $[0, 1, f_2, 2, 3, f_3, 4, 5, f_1, 6, 7]$,
$[0, 1, f_3, 2, 3, f_1, 4, 5, f_2, 6, 7]$, $[1, 2, f_1, 3, 4, f_2, 5, 6, f_3, 7, 0]$,
$[1, 2, f_2, 3, 4, f_3, 5, 6, f_1, 7, 0]$, $[1, 2, f_3, 3, 4, f_1, 5, 6, f_2, 7, 0]$,
$[1, 2, 0, 4, f_1, f_2, 3, 7, 6, f_9, f_{10}]$, $[2, 3, 1, 5, f_1, f_2, 4, 0, 7, f_9, f_{10}]$,

$$[3, 4, 2, 6, f_1, f_2, 5, 1, 0, f_9, f_{10}],\ [4, 5, 3, 7, f_1, f_2, 6, 2, 1, f_9, f_{10}],$$
$$[5, 6, 4, 0, f_3, f_4, 7, 3, 2, f_9, f_{10}],\ [6, 7, 5, 1, f_3, f_4, 0, 4, 3, f_9, f_{10}],$$
$$[7, 0, 6, 2, f_3, f_4, 1, 5, 4, f_9, f_{10}],\ [0, 1, 7, 3, f_3, f_4, 2, 6, 5, f_9, f_{10}],$$
$$[1, 3, 4, 0, f_5, 2, 6, f_6, f_1, f_2, 7],\ [2, 4, 5, 1, f_5, 3, 7, f_6, f_1, f_2, 0],$$
$$[3, 5, 6, 2, f_5, 4, 0, f_6, f_1, f_2, 1],\ [4, 6, 7, 3, f_5, 5, 1, f_6, f_1, f_2, 2],$$
$$[5, 7, 0, 4, f_7, 6, 2, f_8, f_1, f_2, 3],\ [6, 0, 1, 5, f_7, 7, 3, f_8, f_1, f_2, 4],$$
$$[7, 1, 2, 6, f_7, 0, 4, f_8, f_1, f_2, 5],\ [0, 2, 3, 7, f_7, 1, 5, f_8, f_1, f_2, 6],$$
$$[f_9, 0, 4, f_4, f_5, 1, 3, 5, 2, 6, 7],\ [f_9, 1, 5, f_4, f_5, 2, 4, 6, 3, 7, 0],$$
$$[f_9, 2, 6, f_4, f_5, 3, 5, 7, 4, 0, 1],\ [f_9, 3, 7, f_4, f_5, 4, 6, 0, 5, 1, 2],$$
$$[f_{10}, 4, 0, f_4, f_5, 5, 7, 1, 6, 2, 3],\ [f_{10}, 5, 1, f_4, f_5, 6, 0, 2, 7, 3, 4],$$
$$[f_{10}, 6, 2, f_4, f_5, 7, 1, 3, 0, 4, 5],\ [f_{10}, 7, 3, f_4, f_5, 0, 2, 4, 1, 5, 6]\big\}.$$

Then an H_5-decomposition of $K_{18}^{(3)} \setminus K_{10}^{(3)}$ consists of the orbits of the H_5-blocks in B_5 under the action of the map $f_i \mapsto f_i$, for $i \in [1, 10]$, and $j \mapsto j + 1 \pmod 8$ on the vertices along with the H_5-blocks in B_5'.

References

1. Adams, P., Bryant, D., Buchanan, M.: A survey on the existence of G-designs. J. Combin. Des. **16**, 373–410 (2008)
2. Armstrong, A., Bunge, R.C., Duncan, W., El-Zanati, S.I., Koe, K., Stutzman, R.: On decompositions and maximum packings of complete 3-uniform hypergraphs with the symmetric triple-star of size 4. Electron. J. Graph. Theory App. **9**, 451–468 (2021)
3. Bailey, R.F., Stevens, B.: Hamilton decompositions of complete k-uniform hypergraphs. Discret. Math. **310**, 3088–3095 (2010)
4. Baranyai, Z.S.: On the factorization of the complete uniform hypergraph. In: Infinite and Finite Sets, Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam, vol. 10, pp. 91–108 (1975)
5. Bermond, J.-C., Germa, A., Sotteau, D.: Hypergraph-designs. Ars Combin. **3**, 47–66 (1977)
6. Bryant, D., Herke, S., Maenhaut, B., Wannasit, W.: Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs. Australas. J. Combin. **60**, 227–254 (2014)
7. Bryant, D.E., McCourt, T.A.: Existence results for G-designs. http://wiki.smp.uq.edu.au/G-designs/
8. Bunge, R.C., El-Zanati, S.I., Hamanaka, L., Hatzer, C., Mills, T., Nowak, L.: On loose 4-path decompositions of complete 3-uniform hypergraphs, submitted
9. Bunge, R.C., El-Zanati, S.I., Jeffries, J., Vanden Eynden, C.: Edge orbits and cyclic and r-pyramidal decompositions of complete uniform hypergraphs. Discret. Math. **341**, 3348–3354 (2018)
10. Glock, S., Kühn, D., Lo, A., Osthus, D.: The existence of designs via iterative absorption, 63 pages (2017). arXiv:1611.06827v2
11. Glock, S., Kühn, D., Lo, A., Osthus, D.: Hypergraph F-designs for arbitrary F, 72 pages (2017). arXiv:1706.01800
12. Keevash, P.: The existence of designs, 39 pages (2018). arXiv:1401.3665v2
13. Kuhl, J., Schroeder, M.W.: Hamilton cycle decompositions of k-uniform k-partite hypergraphs. Australas. J. Combin. **56**, 23–37 (2013)
14. Lonc, Z.: Solution of a delta-system decomposition problem. J. Combin. Theory, Ser. A **55**, 33–48 (1990)

15. Meszka, M., Rosa, A.: Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles. Australas. J. Combin. **45**, 291–302 (2009)
16. Schroeder, M.W.: On Hamilton cycle decompositions of r-uniform r-partite hypergraphs. Discret. Math. **315**, 1–8 (2014)
17. Wilson, R.M.: Decompositions of complete graphs into subgraphs isomorphic to a given graph. In: Nash-Williams, C.St.J.A., Sheehan, J. eds., Proceedings of the Fifth British Combinatorial Conference, pp. 647–659, Congruent Number, XV (1975)

Exterior Corners on Bargraphs of Motzkin Words

Toufik Mansour and José L. Ramírez

Abstract In this paper we introduce the concept of Motzkin words and Motzkin bargraphs. Over this family of bargraphs we study the exterior corner statistic. Generating functions and exact combinatorial formulas of the exterior corners are established.

Keywords Motzkin word · Generating function · External corners

2010 Mathematics Subject Classification 05A15 · 05A19

1 Introduction

The Motzkin numbers are a well-known sequence, having many combinatorial interpretations. For example, the Motzkin sequence first appears in the problem of counting the number of ways of connecting a subset of n points on a circle by non-intersecting chords [19]. The Motzkin numbers can be defined by the combinatorial formula (cf. [3, 11]):

$$m_n = \frac{1}{n+1}\sum_{i\geq 0}\binom{n+1}{i}\binom{n+1-i}{i+1}, \quad n \geq 0.$$

T. Mansour
Department of Mathematics, University of Haifa, 3498838 Haifa, Israel
e-mail: tmansour@univ.haifa.ac.il

J. L. Ramírez (✉)
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
e-mail: jlramirezr@unal.edu.co

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_30

Moreover, the generating function is given by

$$\sum_{n\geq 0} m_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

The first few values of the Motzkin sequence are

$$1,\ 1,\ 2,\ 4,\ 9,\ 21,\ 51,\ 127,\ 323,\ 835,\ 2188, \ldots$$

which correspond to the sequence A001006 in [22].

There are several families of restricted words enumerated by the Motzkin sequence. For example, the words of length n generated by the grammar $S \to \lambda|(SS)$ are enumerated by m_n (cf. [2]). In this paper, we study a different family of Motzkin words inspired by the ECO method and the generating tree defined by the rewritten rule $(k) \hookrightarrow (1)(2)\cdots(k-1)(k+1)$ (see [8] for more details).

A word $w = w_1 w_2 \cdots w_n$ over the set of positive integers is called a *Motzkin word* if $w_1 = 1$, $1 \leq w_k \leq w_{k-1} + 1$, and $w_{k-1} \neq w_k$ for $k = 2, \ldots, n$. Let $\mathcal{M}_n$ denote the set of Motzkin words of length n. For example,

$$\mathcal{M}_5 = \{\texttt{12121}, \texttt{12123}, \texttt{12312}, \texttt{12321}, \texttt{12323}, \texttt{12341}, \texttt{12342}, \texttt{12343}, \texttt{12345}\}.$$

The cardinality of the set $\mathcal{M}_n$ is given by the Motzkin number m_{n-1} (see Corollary 2.2).

A *bargraph* is a self-avoiding lattice path in the first quadrant with steps up $u = (0, 1)$, horizontal $h = (1, 0)$, and down $d = (0, -1)$ that starts at the origin and ends on the x-axis. Notice that bargraphs are a particular family of polyominoes. The concept of polyomino was introduced by Golomb in 1953 [12]. Since then, several authors have tried to enumerate different families of polyominoes. We remark that bargraphs provide a rich source of combinatorial ideas and have been studied in connection with several discrete structures such as words, set partitions, permutations, graphs, among others (see for example [4–6, 13, 15–17] and references contained therein).

A Motzkin word $w = w_1 \cdots w_n$ can be represented as a bargraph, whose i-th column contains w_i cells for $1 \leq i \leq n$. In Fig. 1 we show the Motzkin bargraphs associated to the elements in $\mathcal{M}_5$.

An *exterior corner* in a bargraph is a point of intersection of a horizontal (h) step with an up (u) step or the intersection of a down (d) step with a horizontal (h) step. Let $\texttt{cor}(w)$ denote the number of *exterior corners* associated to the bargraph of the Motzkin word w. Let $\texttt{cor}_{hu}(w)$ and $\texttt{cor}_{dh}(w)$ denote the number of corners of type hu and dh, respectively. For example, for the Motzkin bargraph in Fig. 2 we denote by red (resp. blue) the corners of type hu (resp. dh). Moreover, $\texttt{cor}(w) = 14$, $\texttt{cor}_{hu}(w) = 9$, and $\texttt{cor}_{dh}(w) = 5$.

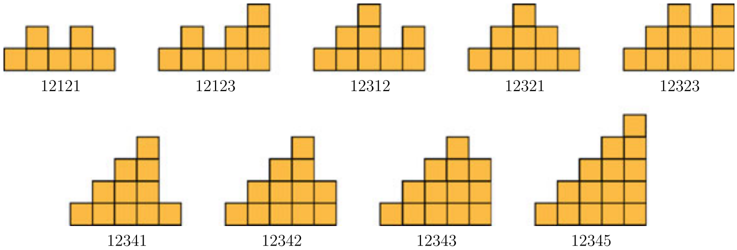
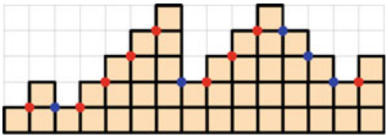


Fig. 1 Motzkin words and their associated Motzkin bargraphs

Fig. 2 Exterior corners of a Motzkin bargraph



The distribution of exterior corners has been studied for example for parallelogram polyominoes [10] and for bargraphs associated to set partitions, compositions and Catalan polyominoes [9, 17].

In this paper, we consider the exterior corner statistic of Motzkin words over their associated bargraphs. The outline of the paper is this: in Sect. 2, we enumerate the Motzkin words/bargraphs by using Riordan arrays. In Sect. 3, we give the generating function for the exterior corners statistic over the set of Motzkin words of length n.

2 A Connection with the Riordan Arrays

We recall that an infinite lower triangular matrix is called a (proper) Riordan array [21] if its k-th column satisfies the generating function $g(x)\,(f(x))^k$ for $k \geq 0$, where $g(x)$ and $f(x)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. The matrix corresponding to the pair $f(x), g(x)$ is denoted by $(g(x), f(x))$. If we multiply (g, f) by a column vector $(c_0, c_1, \dots)^T$ with the generating function $h(x) = \sum_{i\geq 0} c_i x^i$, then the resulting column vector has generating function $gh(f)$. This property is known as the Fundamental Theorem of Riordan arrays or summation property.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x))).$$

We recall that the set of all Riordan matrices is a group under the operator " $*$ " [21]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is given by

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ \overline{f}\right)(x), \overline{f}(x)\right), \tag{1}$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$. Rogers [20] introduced the concept of the A-*sequence*. Specifically, Rogers observed that every element $d_{n+1,k+1}$ of a Riordan matrix (not belonging to row 0 or column 0) can be expressed as a linear combination of the elements in the preceding row. Merlini et al. [18] introduced the Z-*sequence*, which characterizes column 0, except for the element $d_{0,0}$. Summarizing, an infinite lower triangular array $\mathcal{D} = (d_{n,k})_{n,k\geq 0}$ is a Riordan array if and only if there are two sequences $(a_0 \neq 0, a_1, a_2, \dots)$ and $(z_0, z_1, z_2, \dots)$ such that

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, \quad n, k = 0, 1, \dots \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots, \quad n = 0, 1, \dots \end{aligned}$$

Therefore, the A-sequence, Z-sequence, and the element $d_{0,0}$ completely characterize a proper Riordan array. Throughout this article, we suppose $d_{0,0} = 1$. The sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are called A-sequence and Z-sequence, respectively. Additionally, if $\mathcal{D} = (g(x), f(x))$, then the generating functions $A(x)$ and $Z(x)$ of the A-sequence and Z-sequence of $\mathcal{D}$ satisfy the equalities $f(x) = xA(f(x))$ and $g(x) = 1/(1 - xZ(f(x)))$, see [14].

Let $m(n)$ be the number of Motzkin words of length n, that is $m(n) = |\mathcal{M}_n|$. Let $m(n,i)$ denote the number of Motzkin words of length n that end with letter i. It is clear that $m(n) = \sum_{i\geq 1} m(n,i)$. The first few rows of the matrix $[m(n,i)]_{n,i\geq 1}$ are

$$[m(n,i)]_{n,i\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 6 & 7 & 3 & 4 & 0 & 1 & 0 & 0 \\ 15 & 14 & 12 & 4 & 5 & 0 & 1 & 0 \\ 36 & 37 & 24 & 18 & 5 & 6 & 0 & 1 \end{pmatrix}$$

In Theorem 2.1 we show that the infinite matrix $[m(n+1, i+1)]_{n,i\geq 0}$ is a Riordan array. Let $\mathcal{M}_{n,i}$ denote the set of the Motzkin words of length n that end with letter i.

Theorem 2.1 *The infinite triangular matrix* $\mathbb{M} = [m(n+1, k+1)]_{n,k\geq 0}$ *has a Riordan array expression given by*

$$\mathbb{M} = \left(\frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1+x)}, \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2(1+x)} \right). \tag{2}$$

Moreover,

$$m(n,k) = \frac{k}{n} \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \binom{2j - k - 1}{j - 1}. \tag{3}$$

Proof If we delete the last letter (column) in a Motzkin word (polyomino) in $\mathcal{M}_{n+1,1}$ we obtain a Motzkin word in $\cup_{j\geq 2}\mathcal{M}_{n,j}$, then $m(n+1,1) = \sum_{j\geq 2} m(n,j)$. Given any word w in $\mathcal{M}_{n+1,k}$ for $k \geq 2$, it can be decomposed as $w = w'k$, where $w' \in \mathcal{M}_{n,i}$, for $k-1 \leq i$ with $i \neq k$. Then we have the equality

$$m(n+1,k) = m(n,k-1) + m(n,k+1) + \cdots + m(n,n).$$

Therefore, the matrix $\mathbb{M} = [m(n+1,k+1)]_{n,k\geq 0}$ is a Riordan array with the Z-sequence given by $(0,1,1,1,\ldots)$ and A-sequence $(1,0,1,1,\ldots)$. Additionally, the generating functions for these two sequences are

$$Z(x) = \sum_{j\geq 1} x^j = \frac{x}{1-x} \quad \text{and} \quad A(x) = 1 + \sum_{j\geq 2} x^j = \frac{1-x+x^2}{1-x}.$$

If $\mathbb{M} = (g, f)$, then

$$f(x) = xA(f(x)) = x\frac{1-f(x)+f(x)^2}{1-f(x)}.$$

Solving the above equation we have

$$f(x) = \frac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)}.$$

Moreover,

$$g(x) = \frac{1}{1-xZ(f(x))} = \frac{1}{1-xZ(f(x))} = \frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)}.$$

Finally, from the definition of Riordan array we have that

$$m(n+1,k+1) = [x^n]g(x)f(x)^k = [x^{n-1}]f(x)^{k+1}.$$

Notice that $f(x) = x\Phi(f(x))$, where $\Phi(u) = (1-u+u^2)/(1-u)$, then as an application of the Lagrange inversion we obtain (3). □

From the summation property we obtain the following corollary.

Corollary 2.2 *The number of Motzkin words $m(n)$ is given by the $(n-1)$-th Motzkin number.*

Proof Multiplying the right-hand side of the equality (2) by the vector $(1,1,1,\ldots)^T$, which has generating function $1/(1-x)$, and using the summation property, the resulting vector has generating function

$$\left(\frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)}, \frac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)}\right)\frac{1}{1-x} = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

By comparing the n-th coefficient we obtain the desired result. □

3 The Exterior Corner Statistic

The goal of the current section is to enumerate the exterior corner statistic over the set of Motzkin bargraphs. Generating function established here is employed to give combinatorial formulas for the total number of exterior corners.

Define

$$C_i(x; p, q) := \sum_{n\geq 1} x^n \sum_{w\in\mathcal{M}_{n,i}} p^{\mathtt{cor}_{hu}(w)} q^{\mathtt{cor}_{dh}(w)}, \quad i \geq 1.$$

By definition, we have for $i = 1$ the following functional equation, see Fig. 3 for a graphical representation of this decomposition.

$$C_1(x; p, q) = x + qx \sum_{j\geq 2} C_j(x; p, q). \tag{4}$$

For $i \geq 2$ we have the following relation, see Fig. 4.

$$C_i(x; p, q) = pxC_{i-1}(x; p, q) + xq \sum_{j\geq i+1} C_j(x; p, q), \qquad i \geq 2. \tag{5}$$

Define $C(x; p, q; v) = \sum_{i\geq 1} C_i(x; p, q)v^{i-1}$. Then by multiplying (5) by v^{i-1} and summing over $i \geq 2$ with using (4), we obtain

$$C_1(x; p, q) = x + qx\left(C(x; p, q; 1) - C_1(x; p, q)\right),$$

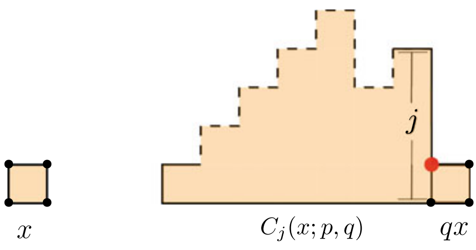


Fig. 3 Decomposition of the Motzkin word in $\mathcal{M}_{n,1}$ and the exterior corners

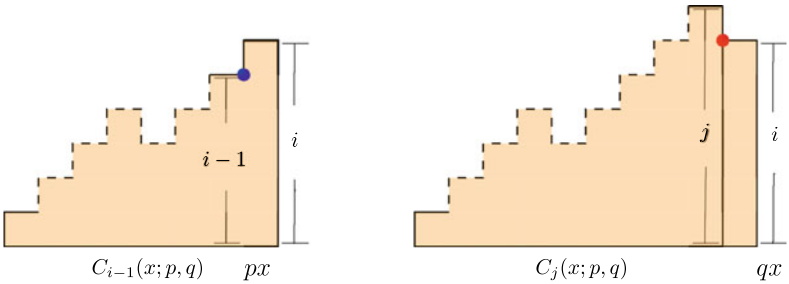


Fig. 4 Decomposition of the Motzkin words in $\mathcal{M}_{n,i}$ and the exterior corners

and

$$\begin{aligned}
C(x;p,q;v) - C_1(x;p,q) &= px\sum_{j\geq 2} C_{i-1}(x;p,q)v^{i-1} + qx\sum_{i\geq 2}\sum_{j\geq i+1} C_j(x;p,q)v^{i-1} \\
&= pxv\sum_{j\geq 1} C_i(x;p,q)v^{i-1} + qx\sum_{j\geq 2} C_j(x;p,q)\sum_{i=2}^{j-1} v^{i-1} \\
&= pxvC(x;p,q;v) + \frac{qxv}{1-v}\sum_{j\geq 2} C_j(x;p,q) - \frac{qx}{1-v}\sum_{i\geq 2} C_j(x;p,q)v^{i-1} \\
&= pxvC(x;p,q;v) + \frac{qxv}{1-v}\left(C(x;p,q;1) - C_1(x;p,q)\right) \\
&\quad - \frac{qx}{1-v}\left(C(x;p,q;v) - C_1(x;p,q)\right),
\end{aligned}$$

which leads to

$$C(x;p,q;v) = x + \left(pxv - \frac{xq}{1-v}\right)C(x;p,q;v) + \frac{qx}{1-v}C(x;p,q;1). \tag{6}$$

Theorem 3.1 *The generating function for the number of nonempty Motzkin bargraphs according to the length (number of columns) and the exterior corners of type* dh *is given by*

$$C(x;1,q;1) = \frac{1-x-\sqrt{1-2x+(1-4q)x^2}}{2qx}.$$

Proof If $p=1$ in (6), then

$$\left(1 - xv + \frac{xq}{1-v}\right)C(x;1,q;v) = x + \frac{qx}{1-v}C(x;1,q;1). \tag{7}$$

To solve the functional equation (7) we use the kernel method (cf. [1]). Indeed, by taking $v=v_0$ such that $1 - xv_0 + \frac{xq}{1-v_0} = 0$, namely

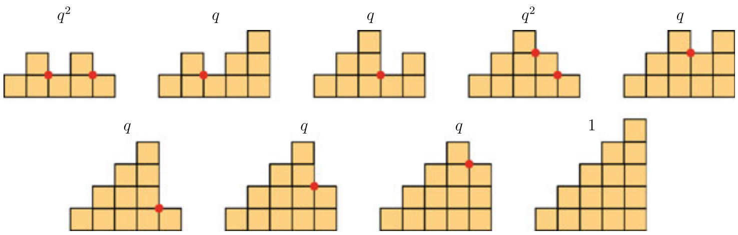


Fig. 5 Motzkin bargraphs and their exterior corners of type dh

$$v_0 = \frac{1 + x - \sqrt{1 - 2x + (1 - 4q)x^2}}{2x},$$

we have

$$C(x; 1, q; 1) = \frac{v_0 - 1}{q} = \frac{1 - x - \sqrt{1 - 2x + (1 - 4q)x^2}}{2qx}. \square$$

As a series expansion, the generating function $C(x; 1, q; 1)$ begins with

$$C(x; 1, q; 1) = x + x^2 + (q + 1)x^3 + (3q + 1)x^4 + \left(\mathbf{2q^2 + 6q + 1}\right)\boldsymbol{x^5} + \left(10q^2 + 10q + 1\right)x^6 + \cdots$$

Notice that Fig. 5 shows the Motzkin bargraphs corresponding to the bold coefficient in the above series.

The coefficients of $C(x; 1, q; 1)$ corresponds to the triangular array of Motzkin polynomials coefficients, array A055151 in the OEIS [22]. Therefore, it is possible to verify that

$$C(x; 1, q; 1) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \frac{n!}{(n - 2k)!k!(k + 1)!} q^k \right) x^{n+1}. \tag{8}$$

Let $w_{dh}(n)$ be the total number of exterior corners of type dh over all Motzkin bargraphs of length n. The first few values are

$$0,\ 0,\ 1,\ 3,\ 10,\ 30,\ 90,\ 266,\ 784,\ 2304,\ 6765,\ 19855,\ 58278, \ldots$$

which correspond to the sequence A014531 in [22].

From the generating functions we obtain the popularity of exterior corners in the Motzkin words of length n. In [7] the notion of popularity was introduced in the context of pattern based statistics.

Corollary 3.2 *The generating function for the total number of exterior corners of type dh over all Motzkin polyominoes of length n is given by*

$$\sum_{n\geq 1} w_{dh}(n)x^n = \frac{1-2x-x^2-(1-x)\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}$$
$$= x^3+3x^4+10x^5+30x^6+90x^7+266x^8+784x^9+2304x^{10}+6765x^{11}+\cdots.$$

Moreover,

$$w_{dh}(n) = \sum_{k=0}^{n} \frac{(n-1)!k}{(n-2k-1)!k!(k+1)!} = \sum_{k=0}^{n} \binom{n-1}{k}\binom{n-1-k}{k+2}.$$

Proof From Theorem 3.1 we have

$$\sum_{n\geq 1} w_{dh}(n)x^n = \frac{\partial}{\partial q} C(x;1,q;1)\bigg|_{q=1} = \frac{1-2x-x^2-(1-x)\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}.$$

From (8), we obtain the closed form expression. □

Theorem 3.3 *The generating function for the number of nonempty Motzkin bargraphs according to the length and the exterior corners of type hu is given by*

$$C(x;p,1;1) = \frac{1-px-\sqrt{1-2px-4px^2+p^2x^2}}{2px}.$$

Proof If $q=1$ in (6) we obtain

$$\left(1-pxv+\frac{x}{1-v}\right)C(x;1,q;v) = x+\frac{x}{1-v}C(x;p,1;1). \tag{9}$$

By taking $v=v_0$ such that $1-pxv_0+\frac{x}{1-v_0}=0$, namely

$$v_0 = \frac{1+px-\sqrt{1-2px-(4-p)px^2}}{2px},$$

we have

$$C(x;p,1;1) = -x(1-v_0)/x = v_0-1 = \frac{1-px-\sqrt{1-2px-4px^2+p^2x^2}}{2px}.\square$$

□

As a series expansion, the generating function $C(x;p,1;1)$ begins with

$$C(x;p,1;1) = x+px^2+\left(p^2+p\right)x^3+\left(p^3+3p^2\right)x^4$$
$$+\boldsymbol{\left(p^4+6p^3+2p^2\right)x^5}+\left(p^5+10p^4+10p^3\right)x^6+\cdots.$$

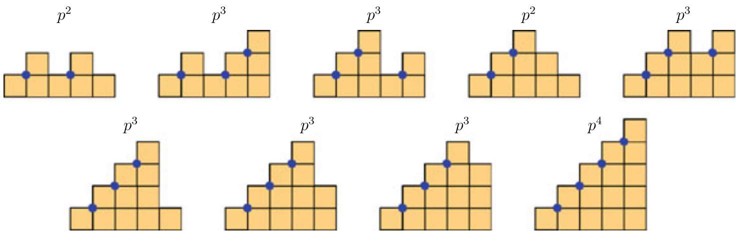


Fig. 6 Motzkin bargraphs and their exterior corners of type *hu*

The Fig. 6 shows the Motzkin bargraphs corresponding to the bold coefficient in the above series.

Analogously, we have the expression:

$$C(x; p, 1; 1) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{n!}{(n-2k)!k!(k+1)!}p^{n-k}\right)x^{n+1}. \tag{10}$$

Let $w_{hu}(n)$ be the total number of exterior corners of type hu over all Motzkin polyominoes of length n. The first few values are

$$0,\ 0,\ 1,\ 3,\ 9,\ 26,\ 75,\ 216,\ 623,\ 1800,\ 5211,\ 15115,\ 43923\ldots$$

which correspond to the sequence A005774 in [22].

Corollary 3.4 *The generating function for the total number of exterior corners of type hu over all Motzkin polyominoes of length n is given by*

$$\begin{aligned}\sum_{n\geq 1} w_{hu}(n)x^n &= \frac{1-2x-x^2-\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}\\ &= x^2+3x^3+9x^4+26x^5+75x^6+216x^7+623x^8+1800x^9+5211x^{10}+\cdots.\end{aligned}$$

Moreover,

$$w_{hu}(n) = \sum_{k=0}^{n}\frac{(n-1)!(n-1-k)}{(n-2k-1)!k!(k+1)!}.$$

Finally, if $p = q$ in (6) we obtain the functional equation

$$C(x; q; v) := C(x; q, q; v) = x + \left(qxv - \frac{xq}{1-v}\right)C(x; q; v) + \frac{qx}{1-v}C(x; q; 1). \tag{11}$$

Then

$$\left(1 - qxv + \frac{xq}{1-v}\right) C(x;q;v) = x + \frac{qx}{1-v} C(x;q;1).$$

By taking $v = v_0$ such that $1 - qxv_0 + \frac{xq}{1-v_0} = 0$, namely

$$v_0 = \frac{1 - qx - \sqrt{1 - 2qx - 3q^2x^2}}{2qx},$$

we obtain $C(x;q;1) = -x(1-v_0)/qx = (v_0 - 1)/q$.

Corollary 3.5 *The generating function for the number of nonempty Motzkin bargraphs according to the length and the exterior corners is given by*

$$\begin{aligned} C(x;q,q;1) &= \frac{1 - qx - \sqrt{1 - 2qx - 3q^2x^2}}{2q^2x} \\ &= x + qx^2 + 2q^2x^3 + 4q^3x^4 + 9q^4x^5 + 21q^5x^6 + 51q^6x^7 + \cdots . \end{aligned}$$

Let $w(n,i)$ denote the number of exterior corners of the Motzkin bargraphs of length n that end with a column of height i. The first few values are

$$[w(n,i)]_{n,i\geq 1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 3 & 0 & 0 & 0 & 0 \\ 12 & 8 & 12 & 0 & 4 & 0 & 0 & 0 \\ 30 & 35 & 15 & 20 & 0 & 5 & 0 & 0 \\ 90 & 84 & 72 & 24 & 30 & 0 & 6 & 0 \\ 252 & 259 & 168 & 126 & 35 & 42 & 0 & 7 \\ & & \vdots & & & & & \end{pmatrix}$$

From decomposition given in Figs. 3 and 4 we obtain the following recurrence relation for the sequence $w(n,i)$. For $n \geq 2$ and $2 \leq i \leq n$

$$w(n,i) = w(n-1,i-1) + m(n-1,i-1) + \sum_{j=i+1}^{n-1} (w(n-1,j) + m(n-1,j)),$$

with $w(n,1) = \sum_{j=2}^{n-1}(w(n-1,j) + m(n-1,j))$ for $n \geq 2$ and $w(1,1) = 0$. Remember that $m(n,k)$ is given in (2).

Let $w(n)$ be the total number of exterior corners over all Motzkin words of length n. The first few values are

$$0,\ 1,\ 4,\ 12,\ 36,\ 105,\ 306,\ 889,\ 2584,\ 7515, \ldots$$

which correspond to the sequence A290380 in [22]. From Corollary 3.5 we have

$$\sum_{n\geq 1} w(n)x^n = \frac{\partial}{\partial q} C(x; q, q; 1)\bigg|_{q=1} = \frac{2-3x-3x^2-(2-x)\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}.$$

Notice that all the sequences obtained in this paper have different combinatorial interpretations in [22].

References

1. Banderier, C., Bousquet-Mélou, M., Denise, A., Flajolet, P., Gardy, D., Gouyou-Beauchamps, D.: Generating functions for generating trees. Discrete Math. **246**, 29–55 (2000)
2. Baril, J.-L., Pallo, J.-M.: Motzkin subposets and Motzkin geodesics in Tamari lattices. Inform. Process. Lett. **114**, 31–37 (2014)
3. Bernhart, F.R.: Catalan, Motzkin, and Riordan numbers. Discrete Math. **204**, 73–112 (1999)
4. Blecher, A., Brennan, C., Knopfmacher, A.: Levels in bargraphs. Ars Math. Contemp. **9**, 287–300 (2015)
5. Blecher, A., Brennan, C., Knopfmacher, A.: Peaks in bargraphs. Trans. R. Soc. S. Afr. **71**, 97–103 (2016)
6. Blecher, A., Brennan, C., Knopfmacher, A.: Combinatorial parameters in bargraphs. Quaest. Math. **39**, 619–635 (2016)
7. Bóna, M.: Surprising symmetries in objects counted by Catalan numbers. Electron. J. Combin. **19**, P62 (2012)
8. Bracucci, E., Del Lungo, A., Perogola, E., Pinzani, R.: ECO: a methodology for the enumeration of combinatorial objects. J. Diff. Equ. Appl. **5**, 435–490 (1999)
9. Callan, D., Mansour, T., Ramírez, J.L.: Statistics on bargraphs of Catalan words. J. Autom. Lang. Comb. **26**, 177–196 (2021)
10. Delest, M., Dubernard, J.P., Dutour, I.: Parallelogram polyominoes and corners. J. Symbol. Comput. **20**, 503–515 (1995)
11. Donaghey, R., Shapiro, L.W.: Motzkin numbers. J. Combin. Theory Ser. A **23**(3), 291–301 (1997)
12. Golomb, S.W.: Checker boards and polyominoes. Am. Math. Monthly **61**, 675–682 (1954)
13. Guttmann, A.J. ed.: Polygons, Polyominoes and Polycubes. Lecture Notes Physics, vol. 775. Springer, Heidelberg, Germany (2009)
14. He, T.-X., Sprugnoli, R.: Sequence characterization of Riordan arrays. Discrete Math. **309**, 3962–3974 (2009)
15. Mansour, T., Ramírez, J.L., Toquica, D.A.: Counting lattice points on bargraphs of Catalan words. Math. Comput. Sci. **15**, 701–713 (2021)
16. Mansour, T., Shabani, ASh.: Enumerations on bargraphs. Discrete Math. Lett. **2**, 65–94 (2019)
17. Mansour, T., Shabani, ASh., Shattuck, M.: Counting corners in compositions and set partitions presented as bargraphs. J. Diff. Equ. Appl. **24**, 992–1015 (2018)
18. Merlini, D., Rogers, D.G., Sprugnoli, R., Cecilia Verri, M.: On some alternative characterizations of Riordan arrays. Canadian J. Math. **49**(2), 301–320 (1997)

19. Motzkin, T.: Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products. Bull. Am. Math. Sot. **54**, 352–360 (1948)
20. Rogers, D.G.: Pascal triangles, Catalan numbers and renewal arrays. Discrete Math. **22**, 301–310 (1978)
21. Shapiro, L.W., Getu, S., Woan, W., Woodson, L.: The Riordan group. Discrete Appl. Math. **34**, 229–239 (1991)
22. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences. http://oeis.org/

Some Connections Between Restricted Dyck Paths, Polyominoes, and Non-Crossing Partitions

Rigoberto Flórez, José L. Ramírez, Fabio A. Velandia, and Diego Villamizar

Abstract A *Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and consists of the same number of North-East steps U and South-East steps D. A *valley* is a subpath of the form DU. A Dyck path is called *restricted d-Dyck* if the difference between any two consecutive valleys is at least d (right-hand side minus left-hand side) or if it has at most one valley. In this paper we give some connections between restricted d-Dyck paths and both, the non-crossing partitions of $[n]$ and some subfamilies of polyominoes. We also give generating functions to count several aspects of these combinatorial objects.

Keywords Dyck path · Polyomino · Partition · Bijection · Generating function

2010 Mathematics Subject Classification 05A15 · Secondary 05B50

R. Flórez (✉)
Department of Mathematical Sciences, The Citadel, Charleston, SC, USA
e-mail: rigo.florez@citadel.edu

J. L. Ramírez · F. A. Velandia
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
e-mail: jlramirezr@unal.edu.co

F. A. Velandia
e-mail: fvelandias@unal.edu.co

D. Villamizar
Escuela de Ciencias Exactas e Ingeniería, Universidad Sergio Arboleda, Bogotá, Colombia
e-mail: diego.villamizarr@correo.usa.edu.co

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_31

1 Introduction

A *Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and consists of (the same number of) North-East steps $U = (1, 1)$ and South-East steps $D = (1, -1)$. A *peak* is a subpath of the form UD, and a *valley* is a subpath of the form DU. We define the *valley vertex* of DU to be the lowest point (a local minimum) of DU. Following [9], we define the vector $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$, called *valley vertices*, formed by all y-coordinates (listed from left to right) of all valley vertices of a Dyck path. For a fixed $d \in \mathbb{Z}$, we introduce a new family of lattice paths called *restricted d-Dyck* or *d-Dyck* (for simplicity). Namely, a Dyck path P is a *d-Dyck*, if either P has at most one valley, or if its valley vertex vector ν satisfies that $\nu_{i+1} - \nu_i \geq d$, where $1 \leq i < k$. For example, in Fig. 1 we see that $\nu = (0, 2, 5, 7)$, and that $\nu_{i+1} - \nu_i \geq 2$, for $1 \leq i < 4$. So, the figure depicts a 2-Dyck path of length 28 (or semi-length 14). Another well-known example, is the set of 0-Dyck paths, known in literature as non-decreasing Dyck paths (see for example, [1, 3, 4, 6, 8]). A second classic example occurs when $d \to -\infty$, giving rise to Dyck paths. We say that a polyomino P is *directed* if for a given cell $c \in P$ there is a path totally contained in P joining c with the bottom left-hand corner cell and using only north and east steps. We say that P is *column-convex* if every vertical path joining the bottom cell with the top cell in the same column is fully contained in P. A polyomino that is both directed and column-convex is denoted by dccp [2].

Deutsch and Prodinger [5] give a bijection between the set of non-decreasing Dyck paths of length $2n$ and the set of directed column-convex polyominoes (dccp). For $d \geq 0$, we say that a dccp is *d-restricted* if either it is formed by exactly one or two columns or if the difference between any two of its consecutive initial altitudes is at least d (right-hand side minus left-hand side, but not including the first initial altitude). The left-hand side of Fig. 2 depicts a 2-polyomino, where the initial altitudes are $(0, 0, 2, 4)$. In this paper we give a combinatorial expression and a generating function to count the number of d-restricted polyominoes of area n. When $d = 0$, we obtain the result given by Deutsch and Prodinger.

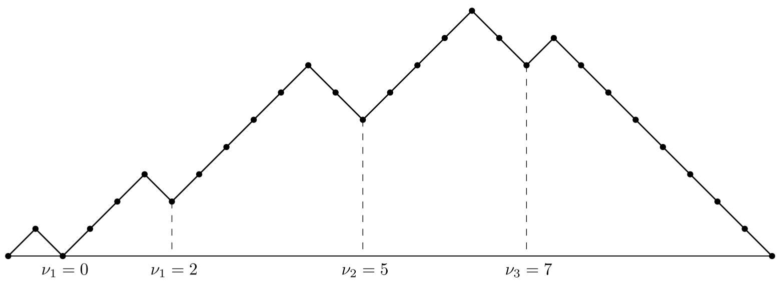
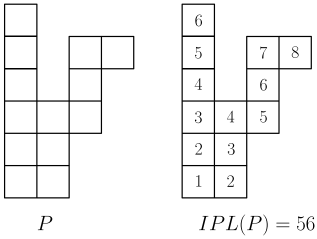


Fig. 1 A 2-Dyck path of length 28

Fig. 2 A 2-polyomino P with TIPL equal to 56



The *internal path length* of a cell in a dccp is the minimum number of steps needed to reach the cell, starting at the origin of the dccp and moving from one cell to any one of the two adjacent cells. The *total internal path length* of dccp is the sum of the internal path length over the set of its cells. In this paper we give a generating function to count the total internal path length (TIPL) when the polyominoes are in the family of the d-restricted. The left-hand side of Fig. 2 shows the polyomino, while the right-hand side shows the internal path lengths of each cell, from with the total internal path length is seen to be 56.

A fixed partition P of $[n]$ is called *non-crossing* if every edge of the form $\{n_1, n_2\} \subset [n]$ of its associated graph (defined in Sect. 3) connecting two distinct elements of P do not cross each other. In this paper we extend this concept (see restricted d-partitions of $[n]$ on Page 380) and give a connection between the d-Dyck paths and the non-crossing partitions of $[n]$.

2 A Connection with the Polyominoes

The *area* of a polyomino is the number of its cells. The right-hand side of Fig. 3 shows a dccp of area 12. The entries of the vector $A = (0, 0, 2, 5)$ represent the initial altitude (height) of each column of the polyomino and the entries of the vector $B = (2, 4, 7, 6)$ represent the final altitude (height) of each column of the polyomino.

Deutsch and Prodinger [5] give a bijection between the set of non-decreasing Dyck paths of length $2n$ and the set of dccp of area n. The bijection from [5] can be described as follows. Let $A = (0, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$ be vectors formed by the initial altitudes and final altitudes, from left to right, of the columns of a dccp. If D is a non-decreasing Dyck path with valley vertices vector $V = (a_2, \ldots, a_k)$ and peak vertices vector $P' = (b_1, b_2, \ldots, b_k)$ from left to right, then A and B are bijectively related with V and P', respectively. For example, the dccp in the right-hand side of Fig. 3 maps bijectively to the Dyck path (on the left-hand side). Note that the valley vertices vector and the peak vertices vector of the path in Fig. 3 are $V = (0, 2, 5)$ and $P' = (2, 4, 7, 6)$, respectively. Clearly these two vectors, V and P', are bijectively related with $A = (0, 0, 2, 5)$ and $B = (2, 4, 7, 6)$.

We say that a dccp is *d-restricted* for $d \geq 0$, if either it is formed by exactly one or two columns or if its initial altitudes vector $A = (0, a_2, \ldots, a_k)$ satisfies that

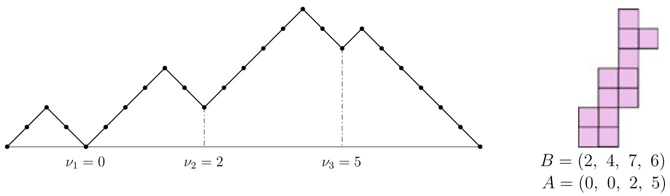


Fig. 3 A bijection between dccp polyomino and non-decreasing Dyck path

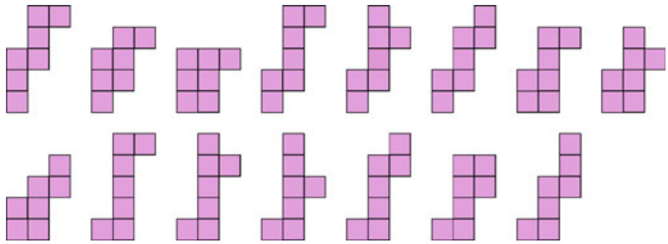


Fig. 4 All 2-restricted polyominoes of area 7 with 3 columns

$a_{i+1} - a_i \geq d$ for all $2 \leq i \leq k - 1$. In Fig. 4 we show all 2-restricted polyominoes of area 7, with exactly 3 columns.

From [9] we know that the number of d-Dyck paths of semi-length n for $d \geq 1$ is given by

$$\sum_{k=0}^{\lfloor \frac{n+d-2}{d} \rfloor} \binom{n - (d-1)(k-1)}{2k}. \tag{1}$$

From the bijection described in the second paragraph of this section we conclude that the set of d-Dyck paths of semi-length n are bijectively related with the set of the d-polyominoes of area n. Therefore, we have this result.

Proposition 2.1 *The expression on* (1) *also counts the total number of d-restricted polyominoes of area n.*

We use $\mathcal{P}_d(n)$ to denote the family of d-restricted polyominoes of length n (n columns). Associated to this concept we define these three sets.

$$\mathcal{P}_d^*(n) = \{P \in \mathcal{P}_d(n) : a_{i+1} - a_i \geq d, \text{ for all } i \geq 1\},$$

$$\mathcal{P}_d = \bigcup_{n \geq 0} \mathcal{P}_d(n), \quad \text{and} \quad \mathcal{P}_d^* = \bigcup_{n \geq 0} \mathcal{P}_d^*(n).$$

Notice that the elements in $\mathcal{P}_d^*(n)$ satisfy that the difference between the initial altitude of the second column and the first column is at least d.

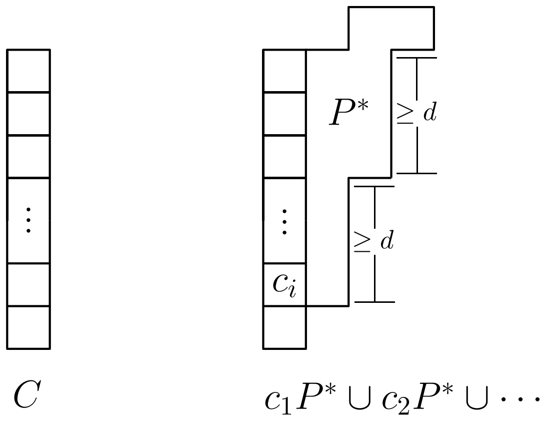


Fig. 5 Decomposition of a d-restricted polyomino

Theorem 2.2 *Let $v_d(n)$ be the number of d-restricted polyominoes of area n. Then the generating function of the sequence $v_d(n)$ is given by*

$$V_d(x) = \sum_{n\geq 0} v_d(n)x^n = \frac{1-2x+2x^2-x^{d+1}}{(1-x)(1-2x+x^2-x^{d+1})}. \tag{2}$$

Proof For any d-restricted polyomino $P \in \mathcal{P}_d$, then P is either a (possibly empty) column or can be obtained by "gluing" a polyomino P^* in $\mathcal{P}_d^*$ to the right-hand side of column C. That is, the lowest level of P^* must be at the same level of a chosen cell c_i in C as shown on the right-hand side of Fig. 5.

Let $V_d^*(x)$ be the generating function of the area of the polyominoes in $\mathcal{P}_d^*$. Let $C(x)$ be the generating function of the area of the nonempty columns (polyominoes with only one column). Notice that if the column has height n, then this case contributes to the generating function the term x^n. Thus,

$$1 + C(x) = 1 + \sum_{n\geq 1} x^n = 1 + \frac{x}{1-x} = \frac{1}{1-x}.$$

So, $\partial(C(x))/\partial x = \frac{x}{(1-x)^2}$ is the area generating function of the nonempty single column polyominoes with a labeled cell. From the symbolic method (see [7]), we obtain the functional equation

$$V_d(x) = \underbrace{(1 + C(x))}_{(1)} + \underbrace{\frac{\partial(C(x))}{\partial x} V_d^*(x)}_{(2)} = \frac{1}{1-x} + \frac{x}{(1-x)^2} V_d^*(x).$$

The expression (1) in the above equality corresponds to the generating function for the area of the columns (possible empty). The expression (2) is the generating function for the d-restricted polyominoes with at least two columns such that the

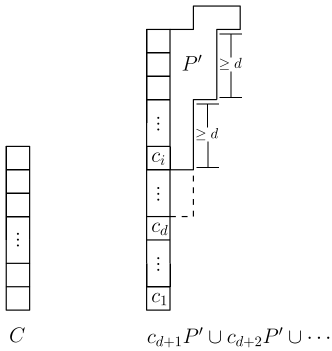


Fig. 6 Decomposition of the polyominoes in $\mathcal{P}^*$

polyomino $\mathcal{P}^*$ that starts in column 2 is in $\mathcal{P}_d^*$. In order to find $V_d^*(x)$ we can apply a similar decomposition to the family $\mathcal{P}_d^*$ as shown in Fig. 6. Notice that any $P^* \in \mathcal{P}_d^*$ with at least one column can be obtained by attaching a polyomino $P' \in \mathcal{P}_d^*$ to any but the first d cells of a column of area greater than d (this is necessary to preserve the inequalities on the initial altitudes vector). Thus

$$V_d^*(x) = \frac{1}{1-x} + x^d \cdot \frac{x}{(1-x)^2} V_d^*(x),$$

this implies that

$$V_d^*(x) = \frac{1-x}{1-2x+x^2-x^{d+1}}.$$

Therefore, we obtain the desired result. □

2.1 Total Path Length of the Polyominoes

Let P be a dccp. The *internal path length* (IPL) of a cell c in P is the minimum number of steps needed to reach c starting at the origin (the bottom leftmost cell) of P and moving from one cell to any one of the two adjacent cells. The *total internal path length* of P (TIPL) is defined to be the sum of the IPL over the set of its cells. For example, Fig. 2 shows a 2-restricted polyomino such that each cell is labeled with the minimum number of steps required to walk from the origin. So, the TIPL of this polyomino is 56 (it can be obtained adding by up all of these labels). We give a generating function to count the TIPL. This result generalizes the result given by Barcucci et al. [2] for the case $d = 0$.

We use $t_d(n)$ to denote the total internal path length of all the d-polyominoes of area n. The following theorem gives a generating function for the sequence $t_d(n)$.

Theorem 2.3 *For $d \geq 0$, we have the rational generating function*

$$T_d(x) := \sum_{n\geq 0} t_d(n)x^n = \frac{f_d(x)}{g_d(x)},$$

where $g_d(x) = 2((x-1)^3 - (x-1)x^{d+1})^3$ and

$$\begin{aligned} f_d(x) = &-\left(d^2+7d+18\right)x^{d+3} + 2\left(2d^2+10d+13\right)x^{d+4} - 2\left(3d^2+9d+11\right)x^{d+5} \\ &+ 4\left(d^2+d+2\right)x^{d+6} - \left(d^2-3d-10\right)x^{2d+4} + \left(2d^2-4d-6\right)x^{2d+5} + 6x^{d+2} \\ -(d-1)dx^{d+7} &- 6x^{2d+3} - (d-1)dx^{2d+6} + 2x^{3d+4} - 6x^7 + 22x^6 - 30x^5 + 20x^4 - 10x^3 + 6x^2 - 2x. \end{aligned}$$

Proof Let $T_d^*(x)$ be the generating function of the TIPL of all d-restricted polyominoes of area n in $\mathcal{P}_d^*$. We use again the decomposition given in Fig. 5. Since the TIPL of a single column with n cells is $n(n+1)/2$, we have

$$T_d(x) = \sum_{n\geq 0} \frac{n(n+1)}{2} x^n + Q_d(x) = \frac{x}{(1-x)^3} + Q_d(x),$$

where $Q_d(x)$ is the generating function of the TIPL of d-restricted polyominoes with at least two columns. According to the decomposition given in Fig. 5, the TIPL contribution in $Q_d(t)$ can be divided into three parts:

Part (1). *The TIPL contribution of the family $\mathcal{P}_d^*$ to the right of the first column.* Given that the whole family of d-restricted polyominoes in $\mathcal{P}_d^*$ can be attached to a particular cell, the corresponding generating function is given by

$$\left(\sum_{n\geq 1} nx^n\right) T_d^*(x) = \frac{x}{(1-x)^2} T_d^*(x). \tag{3}$$

That is, the expression (3) is the product of the generating function of the cells in a column (without contributions to the TIPL) and the generating function of the TIPL in the family $\mathcal{P}_d^*$.

Part (2). *The TIPL contribution of the first column.* In this case, the TIPL of a column equals the contributions of the smaller columns ending at a cell having a d-restricted polyomino in $\mathcal{P}_d^*$ is attached. In order to distinguish cells, the generating function $\mathcal{S} := x\frac{d}{dx}\sum_{m\geq 0}\binom{m+d+1}{2}x^n$ must be considered. This TIPL contribution must be considered for each non-empty polyomino in $\mathcal{P}_d^*$ that is attached. Thus, the generating function of the TIPL contributed by the first column is given by

$$\mathcal{S} \cdot (V_d^*(x) - 1) = \mathcal{S} \cdot V_d^*(x) - \mathcal{S}.$$

Part (3). *The TIPL contribution of the cells of $\mathcal{P}_d^*$, relative to the origin on the first column.* Similarly, $x\frac{d}{dx}V_d(x)$ is the generating function of the cells in $\mathcal{P}_d^*$ (the altitude of this cell is greater than or equal to d). However, the generating function associated to the TIPL contribution of every cell must be modified, not only to exclude attachments to the first d cells but also to add the increase on d units to the TIPL. The TIPL of the upper cells, with the origin placed on the $(d+1)$-th cell, is represented by

$$x^d \sum_{m \geq 0} \binom{m+1}{2} x^m = x^d \cdot \frac{x}{(1-x)^3}.$$

For each one of the upper cells, their IPL relative to the column of the origin, equals the previously described IPL increased by d. In terms of generating functions, the increase in d units for every upper cell is represented by

$$dx^d \sum_{m \geq 0} m x^m = dx^d \cdot \frac{x}{(1-x)^2}.$$

Thus, the TIPL contribution of every upper cell is represented by the sum

$$x^d \cdot \frac{x}{(1-x)^3} + dx^d \cdot \frac{x}{(1-x)^2} = x^d \left(\frac{x}{(1-x)^3} + d \frac{x}{(1-x)^2} \right).$$

Therefore, the generating function representing the TIPL contribution of the first column is

$$x^d \left(\frac{x}{(1-x)^3} + d \frac{x}{(1-x)^2} \right) x \frac{d}{dx} V_d^*(x).$$

From the previous analysis we have

$$\begin{aligned}
T_d^*(x) &= \frac{x}{(1-x)^3} + Q_d^*(x) \\
&= \frac{x}{(1-x)^3} + \frac{x^{d+1}}{(1-x)^2} T_d^*(x) \\
&+ x^d \left(\sum_{m \geq 0} m \binom{m+d+1}{2} x^m \right) V_d^*(x) - x^d \sum_{m \geq 0} m \binom{m+d+1}{2} x^m \\
&+ x^d \left(\frac{x}{(1-x)^3} + d \frac{x}{(1-x)^2} \right) x \frac{d}{dx} V_d^*(x) \\
&= \frac{x}{(1-x)^3} + \frac{x^{d+1}}{(1-x)^2} T_d^*(x) \\
&+ x^d (V_d^*(x) - 1) \cdot x \frac{d}{dx} \left(\sum_{m \geq 0} m \binom{m+d+1}{2} x^m \right) \\
&+ x^d \left(\frac{x}{(1-x)^3} + d \frac{x}{(1-x)^2} \right) x \frac{d}{dx} V_d^*(x).
\end{aligned}$$

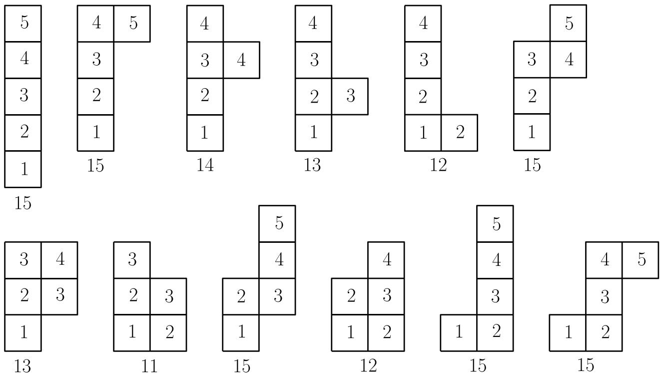


Fig. 7 The TIPL and IPL of each 2-restricted polyomino of area 5

Solving for $T_d^*(x)$ and afterwards for $T_d(x)$, the result follows. □

In particular for $d = 0$ we recover the following result for the directed column-convex polyominoes.

Corollary 2.4 *([2], Theorem 4.1) The generating functions for the TIPL of the directed column-convex polyominoes is given by*

$$T_0(x) = \frac{x\left(3x^4 - 9x^3 + 8x^2 - 4x + 1\right)}{(1-x)\left(x^2 - 3x + 1\right)^3}.$$

For example, the series expansion of $T_2(x)$ is

$$\begin{aligned} T_2(x) &= \frac{x\left(4x^6 - 14x^5 + 15x^4 - 8x^3 + 2x^2 - x + 1\right)}{(x-1)\left(x^3 - x^2 + 2x - 1\right)^3} \\ &= x + 6x^2 + 23x^3 + 65x^4 + \mathbf{165}x^5 + 401x^6 + 932x^7 + 2081x^8 + 4516x^9 + \cdots . \end{aligned}$$

Thus, the TIPL of all 2-restricted polyominoes of area 5 is equal to 165. Figure 7 shows both the IPL and the TIPL of each 2-restricted polyomino of area 5.

3 Restricted Non-Crossing Partitions

In this section, we describe a connection between the d-Dyck paths and the non-crossing partitions. Before doing so, let us recall some terminology and make a few definitions. A *partition* of $[n] := \{1, 2, \ldots, n\}$ is a collection of mutually disjoint non-empty sets whose union is $[n]$. An element of the partition is called a *block*. The cardinality of the set of partitions of $[n]$ having exactly k blocks is given by the Stirling number of the second kind $\left\{ {n \atop k} \right\}$. The set of all partitions of $[n]$ is enumerated

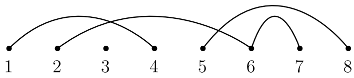


Fig. 8 Graph representation of $\pi = \{\{1, 4\}, \{2, 6, 7\}, \{3\}, \{5, 8\}\}$

by $B(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$, the n-th Bell number. For $n, k \geq 0$, we use $\Pi(n, k)$ to denote the set of all partitions of $[n]$ having k blocks, and use $\Pi(n)$ to denote $\cup_{k=0}^{n} \Pi(n, k)$. For example, $B(3) = 5$, with the corresponding partitions being

$$\{\{1, 2, 3\}\}, \quad \{\{1, 2\}, \{3\}\}, \quad \{\{1, 3\}, \{2\}\}, \quad \{\{1\}, \{2, 3\}\}, \quad \{\{1\}, \{2\}, \{3\}\}.$$

Suppose that π in $\Pi(n, k)$ is represented as $\pi = B_1/B_2/\cdots/B_k$, where B_i is a block of π, for $1 \leq i \leq k$. (Note that different blocks are separated by the symbol $/$.) The graph on the vertex set $[n]$ whose edge set consists of arcs connecting the elements of each block in numerical order is called the *graph representation* of π. For example, in Fig. 8 we depict the graph representation of the set partition $\pi = \{\{1, 4\}, \{2, 6, 7\}, \{3\}, \{5, 8\}\} \in \Pi(8, 4)$.

A set partition is called *non-crossing* if none of the edges on the graph representation cross —in the graph representation. Let $\mathtt{NC}(n)$ denote the set of non-crossing set partitions of $[n]$. It is well-known that $|\mathtt{NC}(n)| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. Here we sketch the bijection between $\mathtt{NC}(n)$ and the Dyck paths of semi-length n.

Let P be a Dyck path of semi-length n. This path can be represented as a word over the alphabet U and D. We use U to denote a North-East step $(1, 1)$ and use D to denote the South-East step $(1, -1)$. Therefore, any Dyck path can be written as $U^{a_1} D^{b_1} \cdots U^{a_n} D^{b_n}$, where $a_i, b_i \geq 0$ (factor it in such a way that if $a_i = 0$, then $b_j = a_j = 0$ for $j > i$), $\sum_{j\geq 1}^{i} b_j \leq \sum_{j\geq 1}^{i} a_j$ for every $1 \leq i \leq n$, and if $i = n$ the equality holds. Enumerate, starting with 1, all U steps. Notice that if we write the Dyck path P as $P = P_1 P_2 \cdots P_{2n-1} P_{2n}$, with $P_i \in \{U, D\}$, then every D step, say $P_j = D$, on P has a corresponding U step, say $P_{j'} = U$, such that $j' < j$ and either $j' = j - 1$ or $\widetilde{P} = P_{j'+1} P_{j'+2} \cdots P_{j-1}$ is a Dyck path. Now, for every $1 \leq j \leq n$ take $b_j \neq 0$ consecutive D steps, match them with their corresponding U steps, and then form a block with the subscripts of the (corresponding) U labels. At the end of this procedure we obtain a partition of $[n]$. It is known that this partition is non-crossing (see, for example, [11]). This process can be inverted and it is a bijection. We denote this bijection by Φ. For example, for the Dyck path P in Fig. 9, we have that $P = U^4 D^2 U^5 D U D^7$. That is, $b_1 = 2$, $b_2 = 1$, and $b_3 = 7$. The first two South-East steps, D^2, on the left-hand side correspond to the labels 3 and 4. The next South-East step corresponds, D, to the label 9, and finally the last seven South-East steps correspond, D^7, to the labels $1, 2, 5, 6, 7, 8$, and 10. Therefore, $\Phi(P) = \{\{1, 2, 5, 6, 7, 8, 10\}, \{3, 4\}, \{9\}\}$.

Following the bijection Φ, we can consider the following characterization for the family of d-Dyck paths in terms of partitions. We denote by $\mathtt{NC}_d(n)$ the set

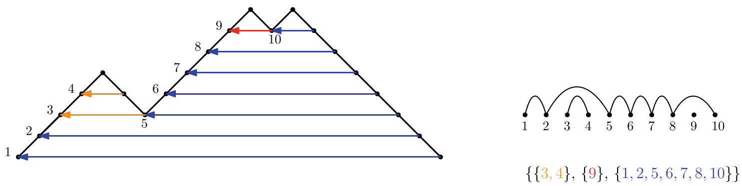


Fig. 9 A bijection between a non-crossing partition and a Dyck path

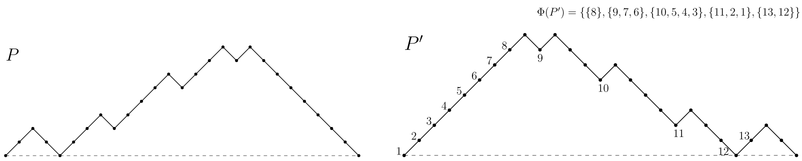
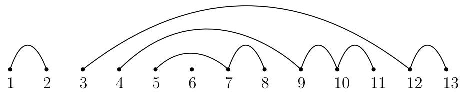


Fig. 10 A Bijection between a non-crossing partitions and a Dyck path

Fig. 11 Graphical representation of $\Phi(P')^R$



$$\{B_1/B_2/\cdots/B_k \in \mathrm{NC}(n) : |([n] \setminus [\mathsf{a}_{i+1}]) \cap B_i| \geq d,\ \mathsf{a}_{i+1} = \max(B_{i+1}) \text{ for } 1 < i < k\}. \quad (4)$$

That is, a partition $\pi = B_1/B_2/\cdots/B_k$ belongs to $\mathrm{NC}_d(n)$ if and only if for all $1 < i < k$, there are at least d elements in the block B_i bigger than the maximum element in B_{i+1}.

The *reverse* of a partition $\pi = B_1/B_2/\cdots/B_k$ of $[n]$, is the partition $\pi^R = B'_1/B'_2/\cdots/B'_k$, where $B'_i = n + 1 - B_i := \{n + 1 - \ell : \ell \in B_i\}$. It is clear that this operation is a bijection.

We now construct a bijection to show that $|\mathrm{NC}_2(13)| = r_2(13)$. For this goal, we consider the 2-Dyck path of semi-length 13, given on left-hand side of Fig. 10, $P = U^2D^2U^3DU^4DU^3DUD^8$; and the path P' obtained from P —interchanging the roles of U and D—, (see the right-hand side of Fig. 10). That is, $P' = D^2U^2D^3UD^4UD^3UDU^8$. The path P' can be seen as a reflection of P with respect to the $(x = 2n)$-axis. The valleys vector of P is $\nu = (0, 2, 5, 7)$ and the valleys vector of P' is $\nu' = (7, 5, 2, 0)$. Observe that the valleys vector of P' satisfy that $\nu'_{i+1} - \nu'_i \leq -2$. Applying the bijection Φ to P', we obtain that

$$\Phi(P') = \{\{1, 2, 11\}, \{3, 4, 5, 10\}, \{6, 7, 9\}, \{8\}, \{12, 13\}\},$$

and

$$\Phi(P')^R = \{\{1, 2\}, \{3, 12, 13\}, \{4, 9, 10, 11\}, \{5, 7, 8\}, \{6\}\} \in \mathrm{NC}_2(13).$$

A graph representation of $\Phi(P')^R$ is depicted in Fig. 11.

We now give a general statement of the previous example. Thus, this theorem gives a bijection between the set $\mathrm{NC}_d(n)$ and the set of d-Dyck paths of length $2n$.

Theorem 3.1 *If $d > 0$ and $n \geq 0$, then the family of d-Dyck paths of length $2n$ and $NC_d(n)$ are bijectively related. Furthermore,*

$$|NC_d(n)| = \sum_{k=0}^{\lfloor \frac{n+d-2}{d} \rfloor} \binom{n-(d-1)(k-1)}{2k}.$$

Proof Let P be a d-Dyck path represented by a word over the alphabet $\{U, D\}$. Let P' be the path traversed backwards, that means, exchange the U's for $D's$ and vice versa and reverse the string. So the path P' is the reflection with respect to the $(x = 2n)$-axis. From this transformation, we can see that P' has the property that the valleys heights satisfy that $\nu_{i+1} - \nu_i \leq -d$.

We now apply the bijection Φ to P'. Every valley in P' gives rise to a new block (containing its label) and each block has at least d labels less than the label of the valley present in that block. Note that the first block of the partition does not follow this rule. This gives that the number of blocks in the partition is equal to the number of valleys plus one (the first block). Now applying the reverse to $\Phi(P')$ we have that the condition of being *smaller* becomes being *larger*, that is $\Phi(P')^R \in \mathrm{NC}_d(n)$. This gives the characterization. □

The previous theorem gives rise to the question: what kind of interesting results do we obtain dropping the "non-crossing condition"? With this question in mind we introduce the *restricted d-Bell numbers*: let

$$\Pi_d(n) = \{\pi = B_1/B_2/\cdots/B_k \in \Pi(n) : \text{for } 1 < i < k, |([n] \setminus [a]) \cap B_i| \geq d\},$$

where $a = \max(B_{i+1})$ with $d \geq 0$. Notice how $\mathrm{NC}_d(n) \subseteq \Pi_d(n)$, so the extension with respect to $\mathrm{NC}_d(n)$, defined in (4), is just considering all partitions in $\Pi(n)$ instead of just the non-crossing ones in $\mathrm{NC}(n)$. We use $B_d(n)$ to denote the cardinality of $\Pi_d(n)$; an element $\pi \in \Pi_d(n)$ is called *a restricted d-partition of* $[n]$. For example, the partition $\pi = \{\{\{1\}, \{2, 10, 11\}, \{3, 8, 9\}, \{4, 6, 7\}, \{5\}\}\}$ is a restricted 2-partition of the set $[11]$. Notice that for $d = 0$ we recover the Bell numbers $B(n)$.

Theorem 3.2 gives an answer to our question. Thus, this theorem gives a recurrence relation to calculate the number of restricted d-Bell numbers.

Theorem 3.2 *The restricted d-Bell number $B_d(n)$ satisfies the recurrence relation*

$$B_d(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B_d(k-d),$$

with $B_d(n) = 1$ for $n \leq 1$. Furthermore, if $d \longrightarrow \infty$, then $B_d(n) = 2^{n-1}$.

Table 1 Values of $B_d(n)$ for $d = 0, 1, 2, 3, 4$

$d\backslash n$	0	1	2	3	4	5	6	7	8	9	10
$d=0$	1	1	2	5	15	52	203	877	4140	21147	115975
$d=1$	1	1	2	4	9	23	65	199	654	2296	8569
$d=2$	1	1	2	4	8	17	40	104	291	857	2634
$d=3$	1	1	2	4	8	16	33	73	177	467	1309
$d=4$	1	1	2	4	8	16	32	65	138	315	782

Proof Let $\pi = B_1/\cdots/B_k$ be a partition in $\Pi_d(n)$ and let $\pi^R = B'_k/B'_{k-1}/\cdots/B'_2/B'_1$ be the reverse of π. It is easy to see that $n \in B'_1$. From the condition on the partition π^R, we have that for $i > 1$, the block B'_i has at least d elements smaller than the minimum element in B'_{i+1}. We select a $(n-1-k)$-set $X \subseteq [n-1]$ satisfying the condition that $\{n\} \cup X$ is equal to B'_1. Let M_d be the set of d minimal elements of $[n-1] \setminus X$. We now create a restricted d-partition of $[n-1] \setminus (X \cup M_d)$ and attach M_d to the smallest block. This procedure gives the the desired recursion.

Finally, if $d > n$, then we cannot have three or more blocks in our partition. If there are more than two blocks, then we need an infinite number of elements to be placed in the middle partition. Since $\left\{ {n \atop 1} \right\} = 1$, $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$, adding them we get $B_d(n) = 2^{n-1}$. $\square$

The numbers $B_d(n)$ agree with the sequence A210545 in [10] shifting d units in the row. In Table 1 we show the first few values of the sequence $B_d(n)$.

Acknowledgements The first author was partially supported by The Citadel Foundation. The second author was partially supported by the Facultad de Ciencias at Universidad Nacional de Colombia, Project No. 53490.

References

1. Barcucci, E., Del Lungo, A., Fezzi, A., Pinzani, R.: Nondecreasing Dyck paths and q-Fibonacci numbers. Discrete Math. **170**, 211–217 (1997)
2. Barcucci, E., Pinzani, R., Sprugnoli, R.: Directed column-convex polyominoes by recurrence relations. Lecture Notes Comput. Sci. **668**, 282–298 (1993)
3. Czabarka, E., Flórez, R., Junes, L.: Some enumerations on non-decreasing Dyck paths. Electron. J. Combin. **22**, # P1.3, 1–22 (2015)
4. Czabarka, E., Flórez, R., Junes, L., Ramírez, J.L.: Enumerations of peaks and valleys on non-decreasing Dyck paths. Discrete Math. **341**, 2789–2807 (2018)
5. Deutsch, E., Prodinger, H.: A bijection between directed column-convex polyominoes and ordered trees of height at most three. Theoret. Comput. Sci. **307**, 319–325 (2003)
6. Elizalde, S., Flórez, R., Ramírez, J.L.: Enumerating symmetric peaks in nondecreasing Dyck paths. Ars Math. Contemp. **21**, #P2.04 (2021)
7. Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009)

8. Flórez, R., Junes, L., Ramírez, J.L.: Enumerating several aspects of non-decreasing Dyck paths. Discrete Math. **342**, 3079–3097 (2019)
9. Flórez, R., Ramírez, J.L., Velandia, F.A., Villamizar, D.: A refinement of Dyck paths: a combinatorial approach. Discrete Math. Algorithms Appl. In press
10. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences. http://oeis.org/
11. Zhao, H., Zhong, Z.: Two statistics linking Dyck paths and non-crossing partitions. Electron. J. Combin. **18**, Paper 83, 12 pp (2011)

Powers of Two as Sums of Two Balancing Numbers

Jeremiah Bartz, Bruce Dearden, Joel Iiams, and Julia Peterson

Abstract The sequence of balancing numbers $(B_n)_{n\geq 1}$ is given by $B_1 = 1$, $B_2 = 6$, and $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 3$. We consider the exponential Diophantine equations $B_n = 2^a$ and $B_n + B_m = 2^a$. Using Baker's theory of logarithmic forms, Matveev's Theorem, and an additional reduction theorem, we establish bounds on the space of possible solutions. This remaining space is sufficiently small that the problem of identifying solutions is reduced to a computational search which is carried out by a simple computer program.

Keywords Integer sequences · Recursion relations · Balancing numbers

1 Introduction

Recently, there has been much interest in identifying terms or sums of terms of well known integer sequences which are perfect powers. For instance, Bugeaud, Mignotte, and Siksek singled out all perfect powers in the Fibonacci sequence, namely 0, 1, 8, and 144 [9]. Bravo and Luca identified all sums of two Fibonacci numbers and sums of two Lucas numbers that are powers of two [7, 8]. The sums of two Padovan numbers which are powers of two were ascertained by Lomelí and Hernández [18]. Aboudja, Hernane, Rihane, and Togbé found the sums of two Pell numbers that give perfect powers under certain index conditions [1].

J. Bartz (✉) · B. Dearden · J. Iiams · J. Peterson
University of North Dakota, Grand Forks, ND 58202, USA
e-mail: jeremiah.bartz@und.edu

B. Dearden
e-mail: bruce.dearden@und.edu

J. Iiams
e-mail: joel.iiams@und.edu

J. Peterson
e-mail: julia.peterson.1@und.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_32

Initially introduced by Behera and Panda [5], the sequence of balancing numbers $(B_n)_{n\geq 1}$ is given by $B_1 = 1$, $B_2 = 6$, and $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 3$. Interest in balancing numbers arose from observing that

$$T(B_n - 1) + T(B_n) = T(m_n) \tag{1}$$

for some positive integer m_n where $T(i) = i(i+1)/2$ denotes the ith triangular number. Equation (1) can be viewed as a generalization of almost isosceles Pythagorean triples [11] where square numbers are replaced by triangular numbers. The sequence of balancing numbers appears in The Online Encyclopedia of Integer Sequences [19] as A001109. Generalizations and special properties of balancing numbers have been studied by Panda [20], Panda and Panda [21, 22], Panda and Rout [24, 28], Panda and Ray [23], the authors [3, 4], and many others [10, 12, 13, 15–17, 25–27, 29]. In this paper, we find all solutions to the exponential Diophantine equations $B_n = 2^a$ and $B_n + B_m = 2^a$. More precisely, we prove the following.

Theorem 1 *The only solutions to $B_n + B_m = 2^a$ in positive integers (n, m, a) with $n \geq m$ are given by $B_1 + B_1 = 2$. In particular, the only balancing number which is a power of two is $B_1 = 1$.*

The theorem is proved using Baker's theory of logarithmic forms, Matveev's Theorem, and an additional reduction theorem to establish bounds on the space of possible solutions. This remaining space is sufficiently small that the problem of identifying solutions is reduced to a computational search which is carried out by a simple computer program. These methods can be used to solve the analogous powers-of-two problem for any sequence given by a second-order linear homogenous recurrence relation with constant coefficients. This paper resolves the problem for balancing numbers. After establishing Theorem 1, we conjecture which sums of two balancing numbers are perfect powers based on our computational searches.

2 Results from the Theory of Logarithmic Forms and Balancing Numbers

In order to obtain bounds on the space of possible solutions to $B_n = 2^a$ and $B_n + B_m = 2^a$, we utilize two results from the theory of logarithmic forms. Both use the notion of logarithmic height of an algebraic number and their well-known elementary properties. Note that $\log(x)$ denotes the natural logarithm for the remainder of the paper.

Definition 1 Suppose γ be an algebraic number in a real algebraic number field $\mathbb{K}$ of degree d. Let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$. Denote the conjugates of γ as $\gamma = \gamma^{(1)}, \ldots, \gamma^{(d)}$. Then the *logarithmic height* of γ is

$$h(\gamma) = \frac{1}{d}\left(\log\gamma + \sum_{i=1}^{d} \log\max\left\{|\gamma^{(i)}|, 1\right\}\right).$$

Proposition 1 *Let γ, γ_1, and γ_2 be algebraic numbers and $n \in \mathbb{Z}$. Then*

(a) $h(\gamma_1 + \gamma_2) \le h(\gamma_1) + h(\gamma_2) + \log(2)$*;*
(b) $h(\gamma_1\gamma_2) \le h(\gamma_1) + h(\gamma_2)$*;*
(c) $h(\gamma^n) = |n|h(\gamma)$*;*
(d) if $\gamma = \frac{p}{q} \in \mathbb{Q}$ *with* $\gcd(p, q) = 1$ *and* $q > 0$*, then* $h(\gamma) = \log\max\{|p|, q\}$.

Next we present two useful results for establishing the bounds on the space of possible solutions to the exponential Diophantine equations. The second result is due to Bravo, Gómez, and Luca [6] and is a version of the reduction method due to Baker and Davenport [2].

Theorem 2 (Metveev's Theorem) *Assume that $\gamma_1, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree d. Let $b_1, \ldots, b_t \in \mathbb{Z} \setminus \{0\}$, $B \ge \max\{|b_1|, \ldots, |b_t|\}$, and*

$$\Lambda = \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers with

$$A_i \ge \max\{d \cdot h(\gamma_i), \log(\gamma_i), 0.16\} \text{ for } i = 1, 2, \ldots, t.$$

If $\Lambda \neq 0$, then

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d^2(1 + \log d)(1 + \log B)A_1 \cdots A_t.$$

Theorem 3 (Bravo-Gómez-Luca Theorem) *Suppose M is a positive integer. Let τ, μ, $A > 0$, $B > 1$ be real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := ||\mu q|| - M||\tau q|| > 0$ where $||x||$ denotes the distance from x to the nearest integer. Then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < \frac{A}{B^w}$$

in positive integers u, v, and w with

$$u \le M \quad \text{and} \quad w \ge \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

Lastly we identify two inequalities which will be handy. The first inequality is specific to balancing numbers; the second is a general result due to Guzmán Sánchez and Luca [14].

The Binet formula for balancing numbers is

$$B_n = c_1(3+\sqrt{8})^n + c_2(3-\sqrt{8})^n \tag{2}$$

where $c_1 = \frac{1}{2\sqrt{8}}$ and $c_2 = -\frac{1}{2\sqrt{8}}$. Put $\alpha = 3+\sqrt{8}$ and $\beta = 3-\sqrt{8}$.

Lemma 1 *For $n \geq 1$, we have $\alpha^{n-1} \leq B_n < \alpha^n$.*

Proof The inequality clearly holds for $n = 1$. Next assume that $\alpha^{k-1} \leq B_k < \alpha^k$ holds for some $k \geq 1$. Multiplying by α yields $\alpha^k \leq \alpha B_k < \alpha^{k+1}$. Then observe

$$\alpha B_k = c_1\alpha^{k+1} + c_2\alpha\beta^k < c_1\alpha^{k+1} + c_2\beta^{k+1} = B_{k+1}$$

which establishes $\alpha^k \leq B_{k+1}$. Lastly we have

$$B_{k+1} = c_1\alpha^{k+1} + c_2\beta^{k+1} < c_1\alpha^{k+1} < \alpha^{k+1}$$

which completes the inductive step. □

Lemma 2 *Suppose $s \geq 1$, $T > (4s^2)^s$, and $T > x/(\log x)^s$. Then $x < 2^s T(\log T)^s$.*

3 Solutions to $B_n = 2^a$

Assume $n \geq 1$ and $B_n = 2^a$. By Lemma 1, we have $\alpha^{n-1} \leq B_n = 2^a < \alpha^n$. Taking logarithms and rearranging gives

$$(n-1)\frac{\log\alpha}{\log 2} \leq a < n\frac{\log\alpha}{\log 2}.$$

Observing $2 < \frac{\log\alpha}{\log 2} < 3$, it follows that

$$2(n-1) < a < 3n. \tag{3}$$

If $n \leq 100$, then $a < 300$. Implementing a *Mathematica* program to check for solutions to $B_n = 2^a$ with $1 \leq n \leq 100$ and $0 \leq a < 300$ only returns one solution, namely $B_1 = 1$.

Now assume $n > 100$. Then (3) implies $a > 198$. Using (2), we write $B_n = 2^a$ as $c_1\alpha^n + c_2\beta^n = 2^a$. This implies

$$|2^a - c_1\alpha^n| = |c_2\beta^n| = |c_2|\,|\beta|^n < 1$$

because $|c_2| < 1$ and $|\beta| < 1$. Dividing by $c_1\alpha^n$ and noting $c_1\alpha > 1$ yields

$$|2^a\alpha^{-n}c_1^{-1} - 1| < \frac{1}{c_1\alpha^n} < \frac{1}{\alpha^{n-1}}. \tag{4}$$

Let $\Lambda = 2^a\alpha^{-n}c_1^{-1} - 1$. Then $\Lambda \neq 0$. To see this, consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ defined by $\sigma(\alpha) = \beta$. If $\Lambda = 0$, then $\sigma(\Lambda) = 0$ and $2^a = c_1\alpha^n$. Thus $2^a = \sigma(c_1\alpha^n) = c_2\beta^n$, a contradiction because $c_2\beta^n < 0$.

Next we apply Matveev's Theorem to Λ with $\gamma_1 = 2$, $\gamma_2 = \alpha$, $\gamma_3 = c_1$, $b_1 = a$, $b_2 = -n$, $b_3 = -1$, and $d = 2$. Noting (3) we take $B = 3n$. From the definition and properties of logarithmic heights, it follows that $h(\gamma_1) = \log 2$, $h(\gamma_2) \leq h(3) + \log(\sqrt{8}) = \log(12\sqrt{2})$, and $h(\gamma_3) = \log(4\sqrt{2})$. Choosing $A_1 = 1.4$, $A_2 = 5.7$, and $A_3 = 3.5$, Matveev's Theorem yields

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)(1 + \log 3n)(1.4)(5.7)(3.5).$$

Thus

$$\log|\Lambda| > -2.70849 \times 10^{13} \cdot (1 + \log 3n). \tag{5}$$

Combining (4) and (5), we have

$$(n-1)\log\alpha < 2.70849 \times 10^{13} \cdot (1 + \log 3n) < 2.70849 \times 10^{13} \cdot 3\log n$$

where the last inequality holds under our assumption that $n > 100$. Hence we conservatively deduce that $n < 4.609 \times 10^{13} \cdot \log n$ and $n < 1.61396 \times 10^{15}$.

To lower the bound, we will apply the Bravo-Gómez-Luca Theorem. Consider

$$\Gamma = a\log 2 - n\log\alpha + \log\left(\frac{1}{c_1}\right)$$

so $e^\Gamma - 1 = \Lambda$. Since $\Lambda \neq 0$, it follows that $\Gamma \neq 0$. If $\Gamma > 0$, we see using (4) that

$$0 < \Gamma \leq e^\Gamma - 1 = |e^\Gamma - 1| = |\Lambda| < \frac{1}{\alpha^{n-1}}. \tag{6}$$

On the other hand, if $\Gamma < 0$ we have $1 - e^\Gamma = |e^\Gamma - 1| = |\Lambda| < 1/2$ using (4) again and our assumption that $n > 100$. Then $e^{-\Gamma} = e^{|\Gamma|} < 2$ and

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < \frac{2}{\alpha^{n-1}}. \tag{7}$$

Combining (6) and (7) yields

$$0 < |\Gamma| < \frac{2}{\alpha^{n-1}}.$$

Using the definition of Γ and dividing by $\log\alpha$ gives

$$0 < |a\tau - n + \mu| < \frac{2}{\alpha^{n-1}\log(\alpha)} = \frac{2\alpha}{\alpha^n\log(\alpha)} < \frac{7}{\alpha^n}$$

where

$$\tau = \frac{\log 2}{\log \alpha} \quad \text{and} \quad \mu = \frac{\log(1/c_1)}{\log \alpha}.$$

Since $a < 3n$ by (3), the bound on n obtained from Matveev's Theorem implies $M = 4.84188 \times 10^{15}$ is an upper bound on a. Observe the 32nd convergent of τ is

$$\frac{p_{32}}{q_{32}} = \frac{26\,195\,788\,993\,692\,987}{66\,618\,684\,047\,814\,617}.$$

Now $q_{32} > 6M$ and $\varepsilon = ||q_{32}\mu|| - M||q_{32}\tau|| = 0.45600573296\ldots > 0$. By the Bravo-Gómez-Luca Theorem with $A = 7$ and $B = \alpha$, we conclude

$$n < \frac{\log(Aq_{32}/\varepsilon)}{\log B} < 24.$$

This contradicts the assumption of $n > 100$ and completes the proof for $B_n = 2^a$.

4 Proof of Theorem 1

Consider $B_n + B_m = 2^a$. If $n = m$, the equation reduces to $B_n = 2^{a-1}$ whose only solution is $B_1 = 1$ as shown in Sect. 3. Assume $n > m \geq 1$. By Lemma 1, we see $\alpha^{n-1} < B_n < B_n + B_m = 2^a$ and $2^a = B_n + B_m < 2B_n < 2\alpha^n < \alpha^{n+1}$. Thus

$$\alpha^{n-1} < 2^a < \alpha^{n+1}.$$

Taking logarithms gives $(n-1)\log\alpha < a\log 2 < (n+1)\log\alpha$. Since $2 < \frac{\log\alpha}{\log 2} < 3$, it follows that

$$2(n-1) < a < 3(n+1). \tag{8}$$

If $n \leq 100$, then $a < 303$. Implementing a *Mathematica* program to check for solutions to $B_n + B_m = 2^a$ with $1 \leq m < n \leq 100$ and $0 \leq a < 303$ returns one solution, namely $B_1 + B_1 = 2$.

Now assume $n > 100$. Then (8) shows $a > 198$. Using (2), we can express $B_n + B_m = 2^a$ as $c_1\alpha^n + c_2\beta^n + B_m = 2^a$. Using Lemma 1, this implies

$$|2^a - c_1\alpha^n| = |c_2\beta^n + B_m| \leq |c_2|\,|\beta|^n + \alpha^m < 1 + \alpha^m < \alpha^{m+1}$$

since $|c_2| < 1$ and $|\beta| < 1$. Dividing by $c_1\alpha^n$ and noting $c_1\alpha > 1$ yields

$$|2^a\alpha^{-n}c_1^{-1} - 1| < \frac{\alpha^{m+1}}{c_1\alpha^n} < \frac{1}{\alpha^{n-m-2}}.$$

Let $\Lambda_1 = 2^a\alpha^{-n}c_1^{-1} - 1$. Since $B_n + B_m = 2^a$, Lemma 1 gives

$$c_1\alpha^n = |c_1\alpha^n| = |B_n - c_2\beta^n| \leq |B_n| + |c_2||\beta|^n < B_n + 1 \leq B_n + B_m = 2^a$$

which is equivalent to $\Lambda_1 > 0$. Applying Matveev's Theorem to Λ_1 using the same values of γ_1, γ_2, γ_3, b_1, b_2, b_3, B, A_1, A_2, and A_3 as before implies

$$(n-m)\log\alpha < 2.70849 \times 10^{13}(1 + \log 3n). \tag{9}$$

Now we obtain a bound for n. From $B_n + B_m = 2^a$ and (2) we have

$$|2^a - c_1\alpha^m(\alpha^{n-m} + 1)| = |c_2\beta^n + c_2\beta^m| < 2|c_2||\beta| < 1.$$

Dividing by $c_1\alpha^m(\alpha^{n-m} + 1)$ and noting $c_1\alpha > 1$ yields

$$\left|2^a\alpha^{-m}(c_1(\alpha^{n-m} + 1))^{-1} - 1\right| < \frac{1}{c_1(\alpha^n + \alpha^m)} < \frac{1}{c_1\alpha^n} < \frac{1}{\alpha^{n-1}}. \tag{10}$$

Set $\Lambda_2 = 2^a\alpha^{-m}(c_1(\alpha^{n-m} + 1))^{-1} - 1$. Then $\Lambda_2 \neq 0$. To see this, again consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ defined by $\sigma(\alpha) = \beta$. If $\Lambda_2 = 0$, then $\sigma(\Lambda_2) = 0$ and $2^a = \sigma(c_1\alpha^n + c_1\alpha^m) = c_2\beta^n + c_2\beta^m$, a contradiction since $c_2\beta^n + c_2\beta^m < 0$.

We apply Matveev's Theorem to Λ_2 with $\gamma_1 = 2$, $\gamma_2 = \alpha$, $\gamma_3 = c_1(\alpha^{n-m} + 1)$, $b_1 = a, b_2 = -m, b_3 = -1$, and $d = 2$. Again noting (3) and $m < n$ we take $B = 3n$. As observed earlier, $h(\gamma_1) = \log 2$, $h(\gamma_2) \leq \log(12\sqrt{2})$, and $h(c_1) = 4\sqrt{2}$. Using Lemma 1 and (9), we have

$$h(\gamma_3) \leq h(c_1) + |n - m|\, h(\alpha) + \log 2 < 2.70850 \times 10^{13}(1 + \log 3n)$$

since $\frac{h(\alpha)}{\log(\alpha)} < 1$ and using the assumption that $n > 100$. Choosing $A_1 = 1.4$, $A_2 = 5.7$, and $A_3 = 5.417 \times 10^{13}(1 + \log 3n)$, Matveev's Theorem gives

$$\log|\Lambda_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)(1 + \log 3n)(1.4)(5.7)(5.417 \times 10^{13}(1 + \log 3n)).$$

Thus

$$\log|\Lambda_2| > -4.19197 \times 10^{26}(1 + \log 3n)^2. \tag{11}$$

A comparison of (10) and (11), we deduce

$$n \log\alpha < 4.19197 \times 10^{26}(1 + \log 3n)^2 < 4.19197 \times 10^{26}(3\log n)^2$$

under our assumptions. Hence $n < 5.1057 \times 10^{27}(\log n)^2$. By Lemma 2 with $T = 5.1057 \times 10^{27}$ and $s = 2$, we conservatively deduce that $n < 8.31302 \times 10^{31}$.

To lower the bounds, we invoke the Bravo-Gómez-Luca Theorem twice. First consider

$$\Gamma_1 = a \log 2 - n \log \alpha + \log \left(\frac{1}{c_1} \right)$$

so $e^{\Gamma_1} - 1 = \Lambda_1$. Since $\Lambda_1 > 0$, we have $\Gamma_1 > 0$. Thus

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = |\Lambda_1| < \frac{1}{\alpha^{n-m-2}}.$$

Using the definition of Γ_1 and dividing by $\log \alpha$ gives

$$0 < |a\tau - n + \mu| < \frac{1}{\alpha^{n-m-2} \log(\alpha)} < \frac{20}{\alpha^{n-m}}$$

where

$$\tau = \frac{\log 2}{\log \alpha} \quad \text{and} \quad \mu = \frac{\log(1/c_1)}{\log \alpha}.$$

Since $a < 3(n+1)$ by (8), the bound on n obtained from Matveev's Theorem implies $M = 2.49391 \times 10^{32}$ is an upper bound on a. Observe the 70th convergent of τ is

$$\frac{p_{70}}{q_{70}} = \frac{6\,530\,249\,862\,426\,579\,539\,040\,549\,586\,100\,324}{16\,607\,121\,566\,104\,479\,739\,413\,038\,500\,833\,989}.$$

Now $q_{70} > 6M$ and $\varepsilon = ||q_{70}\,\mu|| - M||q_{70}\,\tau|| = 0.49703\ldots > 0$. By the Bravo-Gómez-Luca Theorem with $A = 20$ and $B = \alpha$, we conclude

$$n - m < \frac{\log(Aq_{70}/\varepsilon)}{\log B} < 47.$$

To lower the bound on n, consider

$$\Gamma_2 = a \log 2 - m \log \alpha - \log(c_1(\alpha^{n-m} + 1))$$

so $e^{\Gamma_2} - 1 = \Lambda_2 \neq 0$. Hence $\Gamma_2 \neq 0$. If $\Gamma_2 > 0$, we see that

$$0 < \Gamma_2 \leq e^{\Gamma_2} - 1 = |\Lambda_2| < \frac{1}{\alpha^{n-1}}$$

by (10). However, if $\Gamma_2 < 0$, then $1 - e^{\Gamma_2} = |e^{\Gamma_2} - 1| = |\Lambda_2| < 1/2$ since $n > 100$ by assumption. Thus $e^{-\Gamma_2} = e^{|\Gamma_2|} < 2$. It follows that

$$0 < |\Gamma_2| < e^{|\Gamma_2|} - 1 = e^{|\Gamma_2||\Lambda_2|} < \frac{2}{\alpha^{n-1}}.$$

Combining both cases, we have

$$0 < |\Gamma_2| < \frac{2}{\alpha^{n-1}}.$$

Using the definition of Γ_2 and dividing by $\log \alpha$ gives

$$0 < |a\tau - m + \mu| < \frac{2}{\alpha^{n-1}\log(\alpha)} < \frac{7}{\alpha^n}$$

where

$$\tau = \frac{\log 2}{\log \alpha} \quad \text{and} \quad \mu = \frac{-\log(c_1(\alpha^{n-m}+1))}{\log \alpha}.$$

Put $\ell = n - m$ and consider

$$\mu_\ell = \frac{-\log(c_1(\alpha^\ell + 1))}{\log \alpha} \qquad \text{for } \ell = 1, 2, \ldots, 46.$$

Using a Python program, we find that the 70th convergent of τ shows that $\varepsilon_\ell = ||q_{70}\,\mu_\ell|| - M||q_{70}\,\tau|| > 0.00247838 > 0$ for $\ell = 1, \ldots, 46$. Applying the Bravo-Gómez-Luca Theorem with $A = 7$, $B = \alpha$, and $M = 2.49391 \times 10^{32}$, the maximum value of $\log(7 \cdot q_{70}/\varepsilon_\ell)/\log \alpha$ for $\ell = 1, \ldots, 46$ is less than 50, hence $n < 50$. This contradicts the assumption that $n > 100$ and ends the proof.

5 Perfect Powers Conjectures

Based on the computational evidence gathered in our investigations, we present two conjectures on the general perfect power problem for balancing numbers and sums of two balancing numbers.

Conjecture 1 The unique solution to the Diophantine equation $B_n = a^k$ with $k \geq 2$ is $B_1 = 1$.

Conjecture 2 The unique solution to the Diophantine equation $B_n + B_m = a^k$ in positive integers n, m, a, and k with $n \geq m$ and $k \geq 2$ is given by $B_1 + B_3 = 6^2$.

References

1. Aboudja, H., Hernane, M., Rihane, S.E., Togbé, A.: On perfect powers that are sums of two Pell numbers. Period. Math. Hungar. **82**, 11–15 (2021)
2. Baker, A., Davenport, H.: The equations $3X^2 - 2 = Y^2$ and $8X^2 - 7 = Z^2$. Q. J. Math. **20**(2), 129–137 (1969)

3. Bartz, J., Dearden, B., Iiams, J.: Almost gap balancing numbers. Integers **18**(A79), 1–12 (2018)
4. Bartz, J., Dearden, B., Iiams, J.: Classes of gap balancing numbers. Rocky Mountain J. Math **51**(2), 399–411 (2021)
5. Behera, A., Panda, G.K.: On the square roots of triangular numbers. Fibonacci Quart. **37**, 98–105 (1999)
6. Bravo, J., Gómez, A., Luca, F.: Powers of two as sums of two k-Fibonacci numbers. Miskoc Math. Notes. **17**(1), 85–100 (2016)
7. Bravo, J., Luca, F.: On the Diophantine equation $F_n + F_m = 2^a$. Quaest. Math. **39**(3), 391–400 (2016)
8. Bravo, J., Luca, F.: Powers of two as sums of two Lucas numbers. J. Integer Seq. **17**(A14.8.3), 1–12 (2014)
9. Bugeaud, Y., Mignotte, M., Siksek, S.: Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. Ann. Math. **163**, 969–1018 (2006)
10. Castillo, R.: A survey of triangular numbers, factorial and some advanced numbers. Indian J. Sci. Tech. **9**(14), 1–7 (2016)
11. Choi, E.: Almost and nearly isosceles Pythagorean triples. Int. J. Math. Math. Sci. **2016**(A5189057), 1–6 (2016)
12. Dash, K.K., Ota, R.S., Dash, S.: Sequence t-balancing numbers. Int. J. Contemp. Math. Sci. **7**(48), 2305–2310 (2012)
13. Dash, K.K., Ota, R.S., Dash, S.: t-balancing numbers. Int. J. Contemp. Math. Sci. **7**(41), 1999–2012 (2012)
14. Guzmán Sánchez, S., Luca, F.: Linear combinations of factorials and S-units in a binary recurrence sequence. Ann. Math. Qué. **38**, 169–188 (2014)
15. Kovács, T., Liptai, K., Olajos, P.: On (a, b)-balancing numbers. Publ. Math. Debrecen. **77**(3–4), 485–498 (2010)
16. Liptai, K.: Lucas balancing numbers. Acta Math. Univ. Ostrav. **14**(1), 43–47 (2006)
17. Liptai, K., Luca, F., Pinter, A., Szalay, L.: Generalized balancing numbers. Indag. Math. (N.S.). **20**, 87–100 (2009)
18. Lomelí, A., Hernández, S.: Powers of two as sums of two Padovan numbers. Integers **18**(A84), 1–11 (2018)
19. OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences. http://oeis.org (2021)
20. Panda, G.K.: Sequence balancing and cobalancing numbers. Fibonacci Quart. **45**, 265–271 (2007)
21. Panda, G.K., Panda, A.K.: Almost balancing numbers. J. Indian Math. Soc. (N. S.) **82**, 147–156 (2015)
22. Panda, G.K., Panda, A.K.: Circular balancing numbers. Fibonacci Quart. **55**, 309–314 (2017)
23. Panda, G.K., Ray, P.K.: Cobalancing numbers and cobalancers. Int. J. Math. Math. Sci. **8**, 1189–1200 (2005)
24. Panda, G.K., Rout, S.S.: Gap balancing numbers. Fibonacci Quart. **51**, 239–249 (2013)
25. Patel, B.K., Irmak, N., Ray, P.K.: Incomplete balancing and Lucas balancing numbers. Math. Rep. (Bucur.) **20**(70.1), 59–72 (2018)
26. Ray, P.K.: On the properties of k-balancing numbers. Ains Shams Engr. J. **9**(3), 395–402 (2018)
27. Ray, P.K.: Balancing polynomials and their derivatives. Ukrainian Math. J. **69**(4), 646–663 (2017)
28. Rout, S.S., Panda, G.K.: k-gap balancing numbers. Period. Math. Hungar. **70**, 109–121 (2015)
29. Tekcan, A., Tayat, M., Özbek, M.: The Diophantine equation $8x^2 - y^2 + 8x(1 + t) + (2t + 1)^2 = 0$ and t-balancing numbers. ISRN Comb. **2014**, 1–5 (2014)

Sprague-Grundy Functions for Certain Infinite Acyclic Graphs

Aaron Meyerowitz

Abstract We study the Sprague-Grundy functions of certain 2-player combinatorial games with the positive integers as the positions. One such game, called SAD (Subtract A Divisor), allows an integer to be reduced by a proper divisor. The game graph has the positive integers as vertices with a directed edge from q to $q - d$ for each proper divisor $d \mid q$. This is an acyclic directed graph with only finitely many vertices reachable from each given one. It thus has, as is usual, a unique Sprague-Grundy function. It turns out that this function is the 2-adic valuation ν. If we reverse the direction of the edges the vertices are still the positive integers with a directed edge from q to $q + d$ for any divisor $d \mid q$ including $d = q$. We consider the various Sprague-Grundy functions of this unbounded graph, one of which is ν. If we restrict the graph to a finite interval $[1, T]$, there is again a unique Sprague-Grundy function, g_T. We investigate $\lim_{T\to\infty} g_T$ and provide strong empirical evidence that the limit is ν.

Keywords Combinatorial game · Number theory · Sprague-Grundy functions

1 Introduction

We consider two-player normal-play games with the positive integers as positions. The two players alternate turns and a player with no possible move loses. At each position the move options are the same independent of whose turn it is. In combinatorial game theory, this is referred to as an *impartial* game. The game graph of such a game is the graph with vertices the positive integers and a directed edge from q to r if one option at q is to move to r. The analysis of combinatorial games relies heavily on certain colorings of their game graphs.

A. Meyerowitz (✉)
Florida Atlantic University, Boca Raton, USA
e-mail: meyerowi@fau.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_33

A Grundy coloring of a directed graph is a coloring of the points with non-negative integers so that each point has a directed edge to some point of each smaller color but none of the same color. For a finite acyclic directed graph, there is a unique such coloring. One can define it by induction on the length of the longest path starting at a vertex v. Each vertex gets the least color not assigned to the other end of an edge leaving it. In particular, a vertex with no edges leaving it gets color 0. By the same reasoning, there is also a unique Grundy coloring for an infinite directed graph such that from each vertex only finitely many vertices can be reached via directed paths.

It is usual to assume that there a finite number of moves from each position (an assumption we will keep) and that, from any given position, the game terminates in a finite number of moves (an assumption we will discard at times.) If both hold, then the game graph, even if infinite, has exactly the two condition above which implies a unique Grundy coloring.

Consider the directed graph $\mathbb{N}^{\downarrow}$ whose nodes are the positive integers with a directed edge from q to $q-d$ for each proper divisor $d|q$. This is the game-tree of a two player game: Each position is a positive integer and, when it is q, the player whose turn it is can reduce it by subtracting a proper divisor. A player faced with the position 1 has no move and loses. When q is even, the next player can win by always leaving an odd position. This forces the other player to again leave an even position.

If this is just one heap in a NIM type game, a more refined version of this even/odd dichotomy is needed. Consider the greedy coloring of $\mathbb{N}^{\downarrow}$ which assigns $g(1)=0$ and $g(q)=\textbf{mex}\{g(q-d) : d|q \text{ and } d<q\}$, the smallest integer not assigned to any of the $q-d$. This turns out to be the 2-adic valuation $g(q)=\nu(q)=k$ where $q=2^k m$ with m odd.

The famous theory discovered by both Sprague [1] and Grundy [2] uses these values $g(q)$ and the operation of nim-addition to allow the analysis of such a multi-heap game or other disjunctive sum of a finite number of normal play impartial games. One comprehensive account is Winning Ways [3].

This important concept of a Grundy coloring was generalized to undirected graphs and directed graphs with possibly unbounded paths in [4]. In this case it may not be unique.

Reverse the direction of the edges in $\mathbb{N}^{\downarrow}$ so we have another graph $\mathbb{N}^{\uparrow}$ with an edge $q\to q+d$ for each divisor $d|q$ including $d=q$. Now we have unbounded paths. $g(q)=\nu(q)$ also satisfies $g(q)=\textbf{mex}\{q+d : d|q\}$, it is a Grundy coloring. However it is not the only one.

This paper studies that situation as well as the restriction of that graph to $\{1, 2, \cdots, T\}$ when there is a unique Grundy coloring.

2 Definitions

$\mathbb{N}$ denotes the positive integers and $\mathbb{N}_0$ the non-negative integers. Recall that for finite $S \subset \mathbb{N}_0$, **mex**S is the smallest non-negative integer not in S. Hence, mex$S \leq |S|$.

We consider directed acyclic graphs $G = (V, E)$. The only graphs used here are

- $\mathbb{N}^{\downarrow}$ which has vertices the positive integers and an edge $q \to q - d$ for every *proper* divisor $d|q$.
  - We can consider this a combinatorial game, called the SAD game. (Subtract A Divisor). A player without a move (i.e. faced with position 1) is the loser.
- $\mathbb{N}^{\uparrow}$ which has the same vertices and edges, but with direction reversed. So an edge $p \to p + d$ for every divisor $d|p$.
- Either of the two above restricted to some finite interval.
  - $\mathbb{N}^{\downarrow}$ restricted to $[1, T]$ is just a terminal segment of the SAD game, so nothing new.
  - $\mathbb{N}^{\uparrow}$ restricted to $[1, T]$ is called the AAD game (with target T.)
  - One might consider either of the two restricted to some interval $[S, T]$, but that is not studied here.

However, we could ask some of the questions here for other directed acyclic graph so some generality in definitions is aimed at, as long as it doesn't complicate things. For example this game SAD is essentially the game SALIQUOT discussed in [5]. They allow the move $q \to 0$ but this just shifts the coloring $g(q)$ up by 1. In that version it doesn't make sense to reverse the edges, the vertex 0 would be on an infinite number of edges. This is only one of a number of games they consider determined by decreasing number theoretic functions. They too might be given a similar analysis for increasing versions.

A *proper coloring* of G is a partition of vertices into classes so that no edge has both ends in the same class. For convenience we may specify a coloring using a map $c : V \longrightarrow \mathbb{N}_0$.

A *Sprague-Grundy function*, or SGf for G is a vertex labelling $g : V \longrightarrow \mathbb{N}_0$ which enjoys the property that $g(u) = \mathbf{mex}\{g(v) : u \to v \in E\}$. So this is always a proper coloring.

If G has a unique Sprague-Grundy function then we call it **the** SGf of G. As is well known, this is the case when there are no infinite directed paths. In any case, $g(u) = 0$ for terminal vertices, if there are any.

After the next section (which you may wish to skip on a first reading) we describe **the** SGf of $\mathbb{N}^{\downarrow}$. Later we show that it is **a** SGf for $\mathbb{N}^{\uparrow}$, but not the only one.

3 Main Results

Recall that the 2-adic valuation of a positive integer n is the non-negative integer defined by

$$\nu(n) = \nu_2(n) = \max\{e : 2^e | n\}.$$

In other words, the unique e so that $\frac{n}{2^e}$ is an odd integer.

- For $\mathbb{N}^{\downarrow}$ the SGf is $g(n) = \nu(n)$. This is not a new result, see [5], for example. However we spell out a proof in Sect. 4 since we need it elsewhere.
- For $\mathbb{N}^{\uparrow}$,
  - $g(n) = \nu(n)$ is one SGf but not the only one.
  - Another is obtained by choosing a j and setting

$$g(m) = \begin{cases} j & \nu(m) = j + 1 \\ j + 1 & \nu(m) = j \\ \nu(m) & \text{otherwise} \end{cases}$$

    In other words, $g(m) = \sigma\nu(m)$ for the *adjacent transposition* $\sigma = (j\ j + 1)$.
  - In fact any σ whose cycle decomposition consists of (disjoint) adjacent transpositions works, but no others do. Furthermore, any of these is a SGf for $\mathbb{N}^{\updownarrow}$ the graph obtained from $\mathbb{N}^{\downarrow}$ by allowing edges to go both ways. This is shown in Sect. 5.

- Let g_T be the SGf of the AAD game with target T. We discuss this in Sect. 6.
  - $g_T = \nu$ (restricted to $[1, T]$) exactly for T of the form $2^j - 1$
  - The game is always a second player win for T odd with the strategy "always add 1."
  - There is strong evidence that the game is a second player win except for $T = 2$ and $T = 6$
  - Computations suggest that a relatively compact strategy for even T (depending on T) can given by "always add 1 except for this table of exceptions."
  - Let $T = N_n$ be the largest T with $g_T(n) \neq \nu(n)$, or ∞ if there is none. It is not established that N_n is ever finite, but it certainly seems to be the case that it always is. In other words, $\lim_{T\to\infty} g_T = \nu$. For $n = 2, 4, 8, 16$, it appears that $N_n \approx 16n^2$ in that there is an exceptional T that large, but no further exceptions are found up to $3 \cdot 2^{14}$. It also appears that $N_m \ll N_{2^k}$ when $m < 2^k$.

4 $\mathbb{N}^{\downarrow}$, the SAD Game and Its SGf

The SAD game: The starting position is a single heap of $T > 0$ objects, or simply a positive integer T. Two players alternately remove objects from the heap. When the heap has n objects, the next player must remove a proper divisor d of n thus leaving the other player with the position $n - d$. The player who receives position 1 has no moves and is the loser.

It is not hard to convince oneself that the P-positions for SAD are the odd integers. The strategy $2k \to 2k - 1$ i.e. "faced with an even integer, subtract 1" wins for a player faced with an even pile. So $g(2k + 1) = 0$. At any rate, it is clear that $g(1) = 0$.

Recall that the 2-adic valuation of a positive integer n is the non-negative integer defined by

$$\nu(n) = \nu_2(n) = \max\{e : 2^e|n\}.$$

In other words, the unique e so that $\frac{n}{2^e}$ is an odd integer.

Claim 1 $\nu(a \pm b) \begin{cases} = \min(\nu(a), \nu(b)) \text{ when } \nu(a) \neq \nu(b) \\ > \nu(a) \text{ when } \nu(a) = \nu(b) \end{cases}$.

Here we assume $b < a$ when considering $a - b$. The easy proof is left to the reader.
An example: when $a = 200 = 8 \cdot 25$ and $b = 56 = 8 \cdot 7$ we have $\nu(a) = \nu(b) = 3$, $\nu(a + b) = \nu(256) = 8$ and $\nu(a - b) = \nu(144) = 4$.

Hence

Claim 2 Suppose $\nu(n) = e$ and $D = \{d : d|n\}$

- $\nu(d) \leq e$ when $d \in D$.
- for $d \in D$ with $\nu(d) < \nu(n)$ we have $\nu(n - d) = \nu(n + d) = \nu(d)$.
- for $d \in D$ with $\nu(d) = \nu(n) = e$ we have $\nu(n - d) > \nu(n)$ and $\nu(n + d) > \nu(n)$
- for $e' < e$ we have $2^{e'} \in D$ with $\nu(n \pm 2^{e'}) = \nu(2^{e'}) = e'$. Furthermore, these are the two closest integers to n with $\nu(\cdot) = e'$.
- $\nu(n + n) = \nu(n) + 1$.

Note: For the pair $(p, D) = (2, \mathbb{Z})$ of a (prime) element p in a Euclidean Domain D, the two claims above stem from the fact that every $n \in D$ has an expression $n = pq + r$ with $r \in \{0, 1\}$. This , in turn, means that there is a unique integer e with $n = p^e(pQ + 1)$ for some Q and allows the definition $\nu_p(n) = e$. Other pairs (p, D) where this holds include $(p, \mathbb{Z}[d])$ for $(p, d) = (1 + i, i), (\sqrt{2}, \sqrt{2}), (2, \frac{1+\sqrt{5}}{2})$ except that in the first two examples $\nu_p(n + n) = \nu_p(n) + 2$ as $p^2|2$. That alone might might transfer over most of the results below.

Claim 3 For $\mathbb{N}^{\downarrow}$, the SGf is $g(n) = \nu(n)$

Proof The claim is true for $n = 1$. Suppose $n = 2^e(2k + 1)$ and that the claim holds for all smaller positive integers. From claim 2 we have that $\{g(n - d) : d|n\} = \{\nu(n - d) : d|n\}$ includes $0, 1, \cdots, e - 1$, and possibly some integers greater than e, but not e itself.

For any particular SAD game, $g(n)$ is just $\nu(n)$ restricted to $[1, T]$. After all $\{n - d : d|n\}$ is always the same. □

For the strategy of (one heap) SAD we only need to know that the positions with $g(n) = 0$, the P-positions, are exactly the odd integers. Now that we know $g(n)$ we also know winning strategies for games with multiple SAD heaps. Once a player is in a N-position they can win by only subtracting divisors of the form 2^j, even if the opponent is allowed to subtract any divisor. A move in a pile may increase the value of that pile. Of course, as with any combinatorial game, the next player then has a move to restore the previous value of that heap, or change it to anything less than the current value.

5 SGfs for $\mathbb{N}^{\uparrow}$ and the AAD Games

Theorem 1 *The 2-adic valuation ν is a SGf for $\mathbb{N}^{\uparrow}$.*

The proof above works. But we spell it out, with a small observation about $\nu(n + n)$ to aid in the proof of the next theorem.

Proof Let $\nu(n) = e$ so $n = 2^e(2k + 1)$. Then $\nu(n + 2^{e'}) = e'$ for each $0 \le e' < e$. Also, $\nu(n + n) = e + 1$. There may be other divisors d with $\nu(n + d) > e$, but none with $\nu(n + d) = e$. Hence $B = \{\nu(n + d) : d|n\}$ contains each of $0, 1, \cdots e - 1$, contains $e + 1$, and perhaps contains other values greater than e, but not e itself. Accordingly, $\mathbf{mex} B = e$. □

Another SGf arises from ν by swapping two adjacent values

$$g(m) = \begin{cases} j & \nu(m) = j + 1 \\ j + 1 & \nu(m) = j \\ \nu(m) & \text{otherwise} \end{cases}$$

In other words, $g(m) = \sigma\nu(m)$ for the adjacent transposition $\sigma = (j\ \ j + 1)$. In fact any σ whose cycle decomposition consists of (disjoint) adjacent transpositions works, but no others do.

Theorem 2 *The following condition on a permutation σ is both sufficient and necessary for $g(m) = \sigma\nu(m)$ to be a SGf for $\mathbb{N}^{\uparrow}$: There is a set J (possibly infinite) of non-negative integers,no two adjacent, and $\sigma = \prod_{j \in J}(j\ \ j + 1)$.*

Proof First, let σ be a permutation of this form. Again, let $\nu(n) = e$. Then for $B = \{\nu(n+d) : d|n\}$,
$B \supset \{0, \cdots, e-1, e+1\}$ but $e \notin B$.
However we are concerned with $B' = \sigma B$. where
$B' \supset \{\sigma 0, \cdots, \sigma(e-1), \sigma(e+1)\}$ but $\sigma e \notin B'$. There are three cases and in all three, $\mathbf{mex}B' = \sigma e$.

- When $\sigma e = e$, $B' \supset \{0, \cdots, e-2, e-1, \sigma(e+1)\}$ but $e \notin B'$.
- When $\sigma e = e-1, \sigma(e-1) = e$ so $B' \supset \{0, \cdots, e-2, e, \sigma(e+1)\}$ but $e-1 \notin B'$.
- When $\sigma e = e+1, \sigma(e+1) = e$ so $B' \supset \{0, \cdots, e-2, e-1, e\}$ but $e+1 \notin B'$.

Now suppose that σ is some permutation of $\mathbb{N}_0$ with $g : n \mapsto \sigma\nu(n)$ a SGf for $\mathbb{N}^{\uparrow}$. For $n = 2^e$ one has $g(n) = \sigma e = \mathbf{mex}S$ for a certain set of $e+1$ distinct non-negative integers: $S = \{\sigma 0, \sigma 1, \cdots, \sigma(e-1), \sigma(e+1)\}$. Hence $\sigma e \leq |S| = e+1$. Let $J = \{e : \sigma e = e+1\}$ and let $P(m)$ be the statement that $\{\sigma 0, \sigma 1, \cdots, \sigma(m-1)\} = \{0, 1, \cdots, m-1\}$. Then $P(m+1)$ is false when $m \in J$.
We will now show that

- no two elements of J are adjacent
- $\sigma = \prod_{j \in J}(j\ j+1)$.

$P(0)$ is vacuously true. Suppose $P(e)$ is true. Then

$$\sigma e = \mathbf{mex}\{0, \cdots, e-1, \sigma(e+1)\}.$$

Either $\sigma e = e$, in which case $P(e+1)$ is true, or else $\sigma e = e+1$ in which case $\sigma(e+1) = e$ and $e \in J$. Then $P(e+1)$ is not true but $P(e+2)$ is. □

The AAD game: There is a top position (or target) $T > 3$ known to both players. The two players alternately increase the position but cannot exceed T. So the player who moves to T is the winner. The starting position is 1. The first player is forced to move to 2. Then the second player may respond 3 or 4. When the position is n the next player must add a divisor d of n thus leaving the other player with the position $n+d$ subject to the restriction $n+d \leq T$. Here $d = n$ is permitted, provided that $2n \leq T$. The player who receives position T has no moves and hence loses.

Let g_T be the SGf for this game with target T.

It is not hard to show that for odd T the strategy $2k \to 2k+1$ is a second player win. In these cases the P-positions are exactly the odd integers. So, $g_T(n) = 0$ iff n is odd iff $\nu(n) = 0$. However $g_T(n) = \nu(n)$ is only true some of the time. Here are a few cases with T even. The columns are exactly those $\begin{pmatrix} n \\ g_T(n) \\ \nu(n) \end{pmatrix}$ where $g_T(n) \neq \nu(n)$. The final column is for $n = T$.

The numbers in boldface are those n where either n is odd but $g_T(n) \neq 0$ or n is even but $g_T(n) = 0$. In other words, the odd P-positions and the even N-positions. This is all the information one need for a strategy.

$$\begin{pmatrix} 4 & 8 & 12 & \mathbf{13} & 20 & 24 & \mathbf{25} & \mathbf{26} \\ 3 & 2 & 3 & 2 & 3 & 2 & 1 & 0 \\ 2 & 3 & 2 & 0 & 2 & 3 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 16 & 18 & 20 & \mathbf{21} & 26 & \mathbf{27} & \mathbf{28} \\ 3 & 1 & 5 & 4 & 2 & 2 & 1 & 0 \\ 1 & 4 & 1 & 2 & 0 & 1 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{9} & \mathbf{11} & \mathbf{12} & \mathbf{13} & 14 & \mathbf{15} & 16 & 18 & \mathbf{25} & \mathbf{26} & \mathbf{27} & \mathbf{29} & \mathbf{30} \\ 2 & 2 & 0 & 1 & 4 & 3 & 1 & 4 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 4 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 14 & 16 & 20 & 26 & 28 & 30 & \mathbf{31} & \mathbf{32} \\ 4 & 3 & 2 & 4 & 2 & 1 & 2 & 1 & 0 \\ 2 & 1 & 4 & 2 & 1 & 2 & 1 & 0 & 5 \end{pmatrix}$$

In service of the next theorem we note:

Lemma 1 *Suppose that $T > 1$ and that a given $n < T$ enjoys the property that $g_T(m) = \nu(m)$ for every element of $\{m : n < m \leq T\}$. Let $\nu(n) = f$. So n is an odd multiple of 2^f.*

1. *$g_T(n) \leq f$. In particular, for n odd, $g_T(n) = f = 0$.*
2. *If n is odd or $n + 2^{f-1} \leq T$ then also $g_T(n) = f = \nu(n)$.*
3. *If n is even and $n + 2^{f-1} > T$ then $g_T(n) < f = \nu(n.)$*

Proof Because of the property n enjoys, we know that $g_T(n) = \mathbf{mex}A$ for

$$A = \{\nu(n+d) : d|n \text{ and } n + d \leq T\} \subseteq A' = \{\nu(n+d) : d|n\}.$$

Also, T is odd because $g_T(T) = 0$.

1. $\mathbf{mex}(A) \leq \mathbf{mex}(A') = f$.
2. We already know this for n odd. Since $n + 2^{f'} \in A$ for each $0 \leq f' < f$ and $\nu(n + 2^{f'}) = f'$, we have $g_T(n) \geq f$.
3. Since the least integer greater than n with $\nu(n') = f - 1$, namely $n' = n + 2^{f-1}$, is not in A, it follows that $g_T(n) \leq f - 1$.

□

Theorem 3 *The T with $g_T = \nu$ are precisely those one less than a power of 2.*

Proof We first show that, for $T = 2^e - 1$, we have $g_T(n) = \nu(n)$ for all $1 \leq n \leq T$. We prove this by induction on n descending from T to 1. We may thus assume $g_T(m) = \nu(m)$ for all $n < m \leq T$. Let $\nu(n) = f$. For $f = 0$, n is odd and we already know $g_T(n) = 0$. For n even we have $n \leq n' = 2^e - 2^f$ as this is the greatest integer under 2^e with $\nu(n') = f$. Hence $n + 2^{f-1} \leq T$ and, from the lemma, $g_T(n) = f$.

It remains to show that, for T not of the form $2^e - 1$, there is an $n \leq T$ with $g_T(n) \neq \nu(n)$. We may assume that T is odd. Otherwise $g_T(T) = 0 < \nu(T)$. So there are unique integers $0 < f < e$ with $2^e - 2^f < T < 2^e - 2^{f-1}$. Let $n = 2^e - 2^f$ and $n' = 2^e - 2^{f-1}$. Then $\nu(n) = f$ and, since $T < n + 2^{f-1} = n'$, the lemma gives $g_T(n) < f$. □

Conjecture 1 $\lim_{T\to\infty} g_T(n) = \nu(n)$. That is, For each n there is an $N = N_n$ so that $g_T(n) = \nu(n)$ for all $T > N$.

Here is a table with some computational results supporting this conjecture.

n	N_n	$\nu(n)$
2	72	1
4	282	2
8	910	3
9	30	0
12	84	2
14	38	1
16	4698	4
18	61	1
24	276	3
28	180	2
30	62	1
32	12110	5
36	164	2
48	1176	4
56	361	3
60	342	2
64	24221	6
72	334	3
80	590	4
96	5460	5
104	756	3
112	1260	4
116	408	2
120	714	3
128	48443	7
452	2050	2
904	4101	3
7624	45858	3
8190	16382	1
16380	32765	2

The function g_T were computed for all T up to $3\ 2^{14} + 1 = 49153$ and a record was kept of the minimal exception, least n with $g_T(n) \neq \nu(n)$.

Only certain numbers show up as a minimal exception. The table records the first few that do with their presumptive N_n The only odd one is 9 for $T = 30$.

The row n N_n $\nu(n) = 16$ 4968 4 records that:

- $\nu(16) = 4$
- $g_{4968}(16) \neq \nu(16)$ but
- $g_{4968}(n) = \nu(n)$ for all $n < 16$.
- $g_T(16) = \nu(16)$ for all $4968 < T < 49153$.

The fact that no row begins with 20 indicates that all T (in the range checked) with $g_T(20) \neq \nu(20)$ also have $g_T(n) \neq \nu(n)$ for some $n < 20$.

Again the values N_n are only relative to a the calculations stopping at $3\ 2^{14} + 1 = 49153$. The ones which are much less than this might be strongly suspected of being correct. For $n = 2, 4, 8, 16, 32$ it seems that $N_n \approx 16n^2$.

As n increases, most of the minimal exceptions which show up seem to have somewhat large $\nu(n)$.

References

1. Sprague, R.: Über Mathematische Kampsfspiele. Tôhoku J. Math. **41**, 438–444 (1936)
2. Grundy, P.: Mathematics and games. Eureka **2**, 6–8 (1939)
3. Berlekamp, E.R., Conway, J.H. , Guy, R.K.: Winning ways for your mathematical plays. Academic, Inc. London-New York (1982)
4. Christen, C., Selkow, S.: Some perfect colorings of graphs. J. Combinat. Theory Ser. B **27**, 49–49 (1979)
5. Iannucci, D., Larson, U.: Game values of arithmetic functions INTEGERS s **21B**, A14 (2021)

New Absolute Irreducibility Testing Criteria and Factorization of Multivariate Polynomials

Carlos Agrinsoni, Heeralal Janwa, and Moises Delgado

Abstract In this chapter, we introduce a new concept of *degree-gap of a multivariate polynomial*. We effectively utilize this concept to present a bound on the number of absolutely irreducible factors of an infinite family of multivariate polynomials. We also show how this result can be used to guarantee that a polynomial is absolutely irreducible. We also present an algorithm for testing the absolute irreducibility of multivariate polynomials over finite fields. We discuss the ramifications and applications of our results to algebraic geometry, coding theory, cryptography, finite geometry, and other research domains.

Keywords Multivariate polynomial · Factorization · Number of factors · Absolutely irreducible polynomials

1 Introduction

The problem of testing whether a polynomial is irreducible (respectively, absolutely irreducible) is a long-standing problem that is very important in mathematics. Finding practical criteria to test irreducibility is fundamental for applications in pure and applied mathematics. Eisenstein's criterion is a classical method to test irreducibility of a polynomial of one variable [10]. A multivariate polynomial defined over a field $\mathbb{F}$ is absolutely irreducible if it is irreducible over the algebraic closure of $\mathbb{F}$. Eisenstein's criterion can be generalized to multivariate versions using Newton

C. Agrinsoni (✉)
Department of Mathematics, Purdue University, 150 N University St, West Lafayette, IN 47907-2067, USA
e-mail: cagrinso@purdue.edu

H. Janwa
Department of Mathematics, College of Natural Sciences, University of Puerto Rico, Rio Piedras, 17 University Ave. Ste 1701, San Juan, PR 00925-2537, USA

M. Delgado
University of Puerto Rico, 205 Calle Antonio R. Barceló, Cayey, PR 00736, USA

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_34

polygons, valuations, and prime ideals to test irreducibility and in some cases absolute irreducibility [6, 26, 31].

Indeed, one of the key problems in algebraic geometry and its applications in coding theory, cryptography, and other disciplines is to determine whether the variety defined by a set of polynomials is absolutely irreducible; i.e., it remains irreducible in the algebraic closure of the defining field. One important place one needs this is when one wants to apply the Riemann-Roch theorem. Another important case arises in the Weil conjectures and their applications to find bounds on the number of rational points and also bounds on exponential sums. In our case, we consider the hypersurfaces defined by a multivariate polynomial. The Eisenstein criterion for irreducibility works only over the defining fields.

Similar to the case of irreducibility, the absolute irreducibility property is important in many practical applications. Some areas of mathematics in which the absolute irreducibility property has applications are algebraic geometry [13], combinatorics [36], coding theory [18, 19, 35], cryptography [15], finite geometry [16, 17], function field sieve [1], permutation polynomials [30], and scattered polynomials [3]. Some problems that have been solved or partially solved by proving the absolute irreducibility include the exceptional APN conjecture [15], and the Segre–Bartocci conjecture [16]. Important results in algebraic geometry, the Weil conjectures and Deligne's theorem [8, 13], require that the underlying variety is absolutely irreducible. The generalization of Weil conjectures to singular curves also requires that the underlying curve be absolutely irreducible [2]. In all the applications, these theorems are critically used; therefore, proving the absolute irreducibility of multivariate polynomials is fundamental.

There are only a handful of criteria for absolute irreducibility known so far. Two generalizations of Eisenstein's Criterion using Newton polygons are the Eisenstein–Dumas criterion [9, 38] and the Stepanov's criterion (sometimes called Stepanov–Schmidt criterion) [32–34]. These criteria can be generalized even further using Newton polytopes [11]. Other strategies to test absolute irreducibility use algebraic geometry techniques. It is a well-known result that if a curve (respectively hypersurface) defined by a polynomial is non-singular, then the polynomial is absolutely irreducible. Janwa, Wilson, and McGuire prove an algorithm to test absolute irreducibility by using Bezout's Theorem and intersection multiplicities [18, 19]. This method is a powerful tool due to its universality property. Several authors have used this technique to prove the exceptional almost perfect nonlinear conjecture for monomials [15, 18–20]. Hernando and McGuire use this technique to prove the Segre–Bartocci's conjecture [16]. Other methods of testing absolute irreducibility are the Noether's irreducibility forms [25] and the degree difference of the first two forms [7]. Lemma 1 observed in [7] will be generalized in Sect. 2. This generalization will be a direct consequence of Theorem 1. In this chapter, polynomials will be written as graded degree homogeneous forms. If $F(\mathbf{X})$ is non-homogeneous of degree m, then we will write $F(\mathbf{X}) = F_m(\mathbf{X}) + H(\mathbf{X})$, and we will set $d = \deg(H)$.

Lemma 1 (Delgado and Janwa [7]) *Let K be a field. Let $G(X, Y, Z) \in K[X, Y, Z]$ be a polynomial whose graded homogeneous representation is:* $G = G_b + G_a +$

$\cdots + G_0$, *where* G_i *is* 0 *or homogeneous of degree* $i \in \{0, \ldots, b\}$. *We also assume that* $b > 2a$ *and that* G_b *factors into distinct irreducible factors over* $\overline{K}$ *and* $(G_a, G_b) = 1$. *Then,* G *is absolutely irreducible.*

There are some algorithms to test absolute irreducibility tests in the literature. For more information see Gao [12], Kaltofen [23], von zur Gathen [37], and Heintz and Sieveking [14]. One can also devise algorithms to factor the polynomials. Berlekamp gave the first efficient algorithm to factor polynomials in one variable defined over finite fields [4, 5]. Later, Lenstra, Lenstra and Lovasz [29] gave a polynomial-time algorithm for univariate polynomials over the rational numbers, while Kaltofen gave one for multivariate polynomials [21, 22, 24]. The first polynomial-time algorithm for factorization of multivariate polynomials over finite fields is due to Lenstra [28]. If a polynomial factors, one can ask how many absolutely irreducible factors it has. Weinberger proved that for univariate polynomials with rational coefficients there exist polynomial-time algorithms to determine the number of factors of the polynomial, assuming the Riemann hypothesis [39]. For an irreducible multivariate polynomial defined over a finite field, one can show that the factors are conjugates of a fixed absolutely irreducible polynomial of a certain degree (Kopparty and Yekhanin [27]) as stated in Lemma 2. We are going to use this lemma later in Sect. 3 to obtain an algorithm to test absolute irreducibility.

Lemma 2 (Kopparty, and Yekhanin [27]) *Suppose* $p(\boldsymbol{X}) \in \mathbb{F}_q[X_1, \ldots, X_n]$ *is of degree* d *and is irreducible in* $\mathbb{F}_q[X_1, \ldots, X_n]$. *Then there exists* r *with* $r \mid d$ *and an absolute irreducible polynomial* $h(\boldsymbol{X}) \in \mathbb{F}_{q^r}[X_1, \ldots, X_n]$ *of degree* d/r *such that*

$$p(\boldsymbol{X}) = c \prod_{\sigma \in G} \sigma(h(\boldsymbol{X}))$$

where $G = Gal(\mathbb{F}_{q^r}/\mathbb{F}_q)$ *and* $c \in \mathbb{F}_q$. *Furthermore, if* $p(\boldsymbol{X})$ *is homogeneous, then so is* $h(\boldsymbol{X})$.

Remark: Using this lemma, we can make the following useful observation. We don't know if it is previously known.

Proposition 1 *Let* $f(\boldsymbol{X}) \in \mathbb{F}_q[X_1, \ldots, X_n]$ *be a polynomial of degree* m. *If* $f(\boldsymbol{X})$ *is irreducible over* $\mathbb{F}_q, \mathbb{F}_{q^2}, \ldots, \mathbb{F}_{q^m}$, *then* $f(\boldsymbol{X})$ *is absolutely irreducible.*

This chapter is organized as follows. In section two, we will give a new definition called the degree-gap of a polynomial. Then we are going to prove that for an infinite family of multivariate polynomials, the degree-gap of the factors is greater or equal to the degree-gap of the original polynomial. Using this theorem, we are going to classify the factorization of all the binomials (two homogeneous term polynomials) in which the highest degree form is square free. We will also give an upper bound on the number of factors a polynomial of this family could have based on the degree-gap and the degree of the polynomial. We will also prove many consequences of this theorem. In the third section we will provide two algorithms to test absolute irreducibility based on testing irreducibility in multiple subfields, and give applications to some concrete classes of polynomials.

2 Our Concept of a Degree-Gap and a Bound on the Number of Factors of an Infinite Family of Multivariate Polynomials

For the rest of the chapter, we will assume that the polynomial $F(\mathbf{X}) \in \mathbb{F}_q[X_1, \ldots, X_n]$ is a polynomial in which the highest degree form is square free. In the following definition, we introduce a new concept called the degree-gap of a polynomial. Using this conceptual framework, one of our results is a bound on the number of factors a family of polynomials can have; see Theorem 1. We will also give sufficient conditions to guarantee that every factor of a polynomial has the same degree-gap.

Definition 1 Let $F(\mathbf{X}) \in \mathbb{F}_q[X_1, \ldots, X_n]$ be a polynomial of degree m with at least two forms. We define the *degree-gap* $DG(F)$ as the difference between the degrees of the two highest degree forms of the polynomial. If $F(\mathbf{X})$ is a homogeneous polynomial, then $DG(F)$ is defined to be infinity.

Notice that if $F(\mathbf{X}) \in \mathbb{F}_q[X_1, \ldots, X_n]$ satisfies that $\deg(F(\mathbf{X})) < DG(F(\mathbf{X}))$, then $F(\mathbf{X})$ is a homogeneous polynomial.

Example 1 The polynomial $F(X_1, X_2, X_3, X_4, X_5) = X_1^{16} + X_1X_4^{15} + X_1^{10} + X_2^{10} + X_3^{10} + X_1^5X_4^5 + X_4^4X_5^6 + X_2^3X_3^7 + X_1^2X_5^5 + X_1^8 + X_2^7 + X_1X_2X_3X_4X_5 + 1$ defined over $\mathbb{F}_2$ has degree-gap $DG(F) = 6$.

Theorem 1 *Let $F(X) \in \mathbb{F}_q[X_1, \ldots, X_n]$ and let $F(X) = F_m(X) + H(X)$ (where $\deg(H(X)) < m$). If $F_m(X)$ is square free, then every factor of $F(X)$ has the degree-gap at least that of $F(X)$.*

Proof If $F(\mathbf{X})$ is a homogeneous polynomial, then this result is immediate since a homogeneous polynomial factors as the product of homogeneous polynomials. Without loss of generality we can assume that $F(\mathbf{X})$ is not a homogeneous polynomial. It is clear that every homogeneous factor of $F(\mathbf{X})$ satisfies the stated property.

We may assume that without loss of generality that $(F_m, H) = 1$ that is there are no homogeneous factors. If $DG(F) = 1$, then this is a trivial result. Suppose that $DG(F) = k > 1$. Assume that $F(\mathbf{X}) = P(\mathbf{X})Q(\mathbf{X})$ with graded homogeneous decomposition as follows:

$$F(\mathbf{X}) = (P_s(\mathbf{X}) + \cdots + P_0(\mathbf{X}))(Q_t(\mathbf{X}) + \cdots + Q_0(\mathbf{X})),$$

where P_i, (respectively Q_j) are homogeneous polynomials of degree i (respectively degree j) or zero, and $DG(Q) \geq DG(P)$. Assume that $DG(F) > DG(P)$. Let $j = DG(P)$, then we have the following equation

$$0 = F_{m-j} = \sum_{i=1}^{j} P_{s-i}Q_{t-j+i}. \tag{1}$$

By the degree-gap of $P(\mathbf{X})$ we have that $P_{s-1} = \cdots = P_{s-j+1} = 0$ (respectively by the degree-gap of $Q(\mathbf{X})$ $Q_{t-1} = \cdots = Q_{t-j+1} = 0$). Substituting these in Eq. 1 we obtain

$$0 = \sum_{i=1}^{j} P_{s-i} Q_{t-j+i} = P_s Q_{t-j} + P_{s-j} Q_t,$$

implying that $P_s Q_{t-j} = Q_t P_{s-j}$. Since $(P_s, Q_t) = 1$ as $F_m(\mathbf{X})$ is square free, we obtain that $P_s \mid P_{s-j}$ that is $P_{s-j} = 0$. This is a contradiction with $DG(P) = j$. Therefore, $DG(F) \leq DG(P) \leq DG(Q)$. Since $P(\mathbf{X})$ and $Q(\mathbf{X})$ are arbitrary factors, we can conclude that every factor of $F(\mathbf{X})$ has the degree-gap at least that of $F(\mathbf{X})$.

Corollary 1 *Let* $F(X) = F_m(X) + H(X) \in \mathbb{F}_q[X_1, \ldots, X_n]$, $F_m(X)$ *square free, and* $(F_m, H) = 1$. *If* $P(X)$ *is a factor of* $F(X)$, *then* $\deg(P) \geq DG(F)$.

This corollary follows directly from the proof of Theorem 1.

The following corollary gives a bound on the number of factors a polynomial satisfying certain conditions can have.

Corollary 2 *Let* $F(X) = F_m(X) + H(X) \in \mathbb{F}_q[X_1, \ldots, X_n]$. *If* $F_m(X)$ *is square free and* $(F_m, H) = 1$, *then* $F(X)$ *can have at most* $\left\lfloor \frac{\deg(F)}{DG(F)} \right\rfloor$ *factors.*

Proof Let $G(\mathbf{X})$ be a factor of $F(\mathbf{X})$. By Theorem 1 and Corollary 1 we have $\deg(G) \geq DG(F)$. Then $F(\mathbf{X})$ can have at most $\left\lfloor \frac{\deg(F)}{DG(F)} \right\rfloor$ factors.

Using this corollary, we can prove a generalization of Lemma 1 for finite fields in two different ways. First, we generalize to any multivariate polynomial and second, we weaken the greatest common divisor condition.

Corollary 3 *Let* $F(X) = F_m(X) + H(X) \in \mathbb{F}_q[X_1, \ldots, X_n]$, $F_m(X)$ *is square free, and* $(F_m, H) = 1$. *If* $P(X)$ *is a factor of* $F(X)$, *then* $\deg(P) \geq DG(F)$.

Proof By Corollary 2, $F(X)$ can have at most one factor.

The following corollary gives sufficient conditions to guarantee that the degree-gap is preserved through factorization.

Corollary 4 *Let* $F(X) = F_m(X) + H(X) \in \mathbb{F}_q[X_1, \ldots, X_n]$. *If* $F_m(X)$ *is square free,* $(F_m, H_d) = 1$, *and* $P(X)$ *is a factor of* $F(X)$, *then* $DG(P) = DG(F)$.

Proof Assume that $F(\mathbf{X}) = (P_s(\mathbf{X}) + \cdots + P_0(\mathbf{X}))(Q_t(\mathbf{X}) + \cdots + Q_0(\mathbf{X})$. Then by the proof of Theorem 1 we have the following system of equations.

$$F_m(\mathbf{X}) = P_s(\mathbf{X}) Q_t(\mathbf{X})$$

$$H_d(\mathbf{X}) = P_s(\mathbf{X}) Q_{t-e}(\mathbf{X}) + P_{s-e}(\mathbf{X}) Q_t(\mathbf{X}). \tag{2}$$

Since $(F_m, H_d) = 1$, we have that $P_{s-e}(\mathbf{X}) \neq 0$, and $Q_{t-e}(\mathbf{X}) \neq 0$. Therefore, $DG(P) = DG(F)$.

Using this theorem, we can show that many polynomials are absolutely irreducible. The following corollary proves that a class of polynomials is absolutely irreducible. Let $T_F(\mathbf{X})$ denote the tangent cone of a polynomial; i.e., the lowest degree form of the polynomial.

Corollary 5 *Let* $F(X) = F_m(X) + H(X) \in \mathbb{F}_q[X_1, \dots, X_n]$*, where* $\deg(F) = m$ *and* $\deg(H) = d < m$*. If* $F_m(X)$ *is square free,* $(F_m, H) = 1$ *and* $\deg(T_F) > \deg(F) - 2DG(F)$*, then* $F(X)$ *is absolutely irreducible.*

Proof Assume that $F(\mathbf{X}) = P(\mathbf{X})Q(\mathbf{X})$, then $DG(P) \geq DG(F)$ and $DG(Q) \geq DG(Q)$. Then $\deg(T_P) \leq \deg(P) - DG(P) \leq \deg(P) - DG(F)$ and $\deg(T_Q) \leq \deg(Q) - DG(Q) \leq \deg(Q) - DG(F)$. Therefore, $\deg(T_F) = \deg(T_P) + \deg(T_Q) \leq \deg(Q) - DG(F) + \deg(P) - DG(F) = \deg(F) - 2DG(F)$.

One can characterize the factorization of all Generalized binomials in which the highest degree form is square free.

Corollary 6 *Let* $F(X) = F_m(X) + F_d(X) \in \mathbb{F}_q[X_1, \dots, X_n]$*, where* $\deg(F) = m$*. If* $F_m(X)$ *is square free and* $(F_m, F_d) = 1$*, then* $F(X)$ *is absolutely irreducible.*

Proof Assume that $F(\mathbf{X}) = P(\mathbf{X})Q(\mathbf{X})$. Then by Corollary 4 $DG(P) = DG(F)$ and $DG(Q) = DG(F)$. Therefore, $T_F(\mathbf{X}) = T_P(\mathbf{X})T_Q(\mathbf{X})$, and $\deg(T_F) = \deg(T_P) + \deg(T_Q) \leq m - 2DG(F) < m - DG(F) = d$.

Remark: Every polynomial $F(\mathbf{X}) = F_m(\mathbf{X}) + F_d(\mathbf{X}) \in \mathbb{F}_q[X_1, \dots, X_n]$, where $F_m(\mathbf{X})$ is square free, can be written as follows

$$F(\mathbf{X}) = L(\mathbf{X})Q(\mathbf{X}),$$

where $L(\mathbf{X}) = (F_m, F_d)$ and $Q(\mathbf{X}) \in \mathbb{F}_q[X_1, \dots, X_n]$ is absolutely irreducible.

Corollary 7 *Let* $F(X) = F_m(X) + H(X) \in \mathbb{F}_q[X_1, \dots, X_n]$*, where* $\deg(F) = m$*,* $\deg(H) = d$*,* $F_m(X)$ *is square free and* $(F_m, H) = 1$*. If* $F(X)$ *is irreducible in* $\mathbb{F}_{q^i}$ *for* $i = 1, \dots, \left\lfloor \frac{\deg(F)}{DG(F)} \right\rfloor$*, then* F *is absolute irreducible.*

Proof Assume $F(\mathbf{X})$ factors over $\mathbb{F}_{q^r}$, where $r > \left\lfloor \frac{\deg(F)}{DG(F)} \right\rfloor$. Then, by Lemma 2 $F(\mathbf{X})$ has r factors, but this contradicts Corollary 2.

3 Absolute Irreducibility Testing Algorithms

From Corollary 7 we can derive the following algorithm to test absolute irreducibility.

Example 2 $F(X_1, X_2, X_3, X_4, X_5) = \prod_{\alpha \in \mathbb{F}_{2^4}} (X_1 + X_2 + X_3 + \alpha X_4 + (\alpha + 1) X_5) + X_1^{10} + X_2^{10} + X_3^{10} + X_1^5 X_4^5 + X_4^4 X_5^6 + X_2^3 X_3^7 + X_1^2 X_5^5 + X_1^8 + X_2^7 + X_1 X_2 X_3 X_4 X_5 + 1$ defined over $\mathbb{F}_2$. Then,

Algorithm 1: Absolute irreducibility testing

```
[H] Result: The polynomial is absolutely irreducible or not
F(X) ← polynomial in F_q[X] satisfying conditions in Theorem 1;
AbsoluteIrreducible(F);
t ← 1;
while t * DG(F) ≤ deg(F) do
    if if F(X) is irreducible in F_{q^t}(X) then
        t ← t + 1;
    else
        return(F(X) is not absolutely irreducible);
        exit;
    end
end
return(F(X) is absolutely irreducible)
```

1. $F(X_1, X_2, X_3, X_4, X_5)$ is irreducible over $\mathbb{F}_2$.
2. $F(X_1, X_2, X_3, X_4, X_5)$ is irreducible over $\mathbb{F}_{2^2}$.
3. $3 * DG(F) > \deg(F)$. Therefore, $F(X_1, X_2, X_3, X_4, X_5)$ is absolutely irreducible.

We can improve the algorithm as follows. Using Lemma 2 we can reduce substantially the number of extension fields needed to be tested in order to determine if the polynomial is absolutely irreducible.

Algorithm 2: Absolute irreducibility testing

```
Result: The polynomial is absolutely irreducible or not
F(X) ← polynomial in F_q[X] satisfying conditions in Theorem 1;
AbsoluteIrreducible2(F);
s ← list(divisors of deg(F)) in increasing order;
i ← 0;
while s[i] * DG(F) ≤ deg(F) do
    if if F(X) is irreducible in F_{q^{s[i]}}(X) then
        i ← i + 1;
    else
        return(F(X) is not absolutely irreducible);
        exit;
    end
end
return(F(X) is absolutely irreducible)
```

Example 3 $F(X_1, X_2, X_3) = (X_1 + X_2)(\prod_{\alpha \in \mathbb{F}_{2^3} - \mathbb{F}_2}(X_1 + \alpha X_2 + (\alpha + 1)X_3)) + X_1^5 + X_2^5 + X_3^5 + X_3^2 + X_1 + X_2$ defined over $\mathbb{F}_2$. Then

1. $F(X_1, X_2, X_3)$ is irreducible over $\mathbb{F}_2$.
2. $7 * DG(F) > \deg(F)$. Therefore, $F(X_1, X_2, X_3)$ is absolutely irreducible.

Complexity of our algorithm: If there exists an irreducibility testing algorithm over an extension field $\mathbb{F}_{q^m}$ (for a polynomial $F(\mathbf{X}) \in \mathbb{F}[\mathbf{X}]$), then by Proposition 1, we need to apply such an algorithm for extension degrees m, $1 \leq m \leq \deg(F)$ (i.e., a maximum of $\deg(F)$ times). However, we show that we can cut down such complexity considerably. We use Lenstra's factorization of an algorithm to test multivariate polynomials for irreducibility over $\mathbb{F}_q$ in the first step of the while loop. That step is polynomial-time in the degrees of the polynomial to be factored. Lenstra's algorithm makes use of a new basis reduction algorithm for lattices over $\mathbb{F}_q[Y]$. We need to run the while loop a maximum of $\left\lfloor \frac{\deg(F)}{DG(F)} \right\rfloor$ times. In practice, for many applications, we have found that the algorithm is very fast for most polynomials we tried.

Acknowledgements Carlos A. Agrinsoni's work is supported by the NASA Training Grant No. NNX15AI11H, and 80NSSC20M0052. The opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NASA. The first author thanks the Department of Mathematics, University of Puerto Rico, Rio Piedras, where most of the work on this chapter was accomplished.

References

1. Adleman, L.M.: The function field sieve. In: Algorithmic Number Theory (Ithaca, NY, 1994). Lecture Notes in Computer Science, vol. 877, pp. 108–121. Springer, Berlin (1994)
2. Aubry, Y., Perret, M.: A Weil theorem for singular curves. In: Arithmetic, Geometry and Coding Theory (Luminy, 1993), pp. 1–7. de Gruyter, Berlin (1996)
3. Bartoli, D., Montanucci, M.: Towards the full classification of exceptional scattered polynomials (2019). arXiv:1905.11390
4. Berlekamp, E.R.: Factoring polynomials over finite fields. Bell Syst. Tech. J. **46**, 1853–1859 (1967)
5. Berlekamp, E.R.: Factoring polynomials over large finite fields. Math. Comp. **24**, 713–735 (1970)
6. Cassels, J.W.S.: Local fields. London Mathematical Society Student Texts, vol. 3. Cambridge University Press, Cambridge (1986)
7. Delgado, M., Janwa, H.: On the completion of the exceptional APN conjecture in the Gold degree case and absolutely irreducible polynomials. Congr. Numer. **229**, 135–142 (2017)
8. Deligne, P.: La conjecture de Weil. i. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* **43**(1), 273–307 (1974)
9. Dumas, G.: Sur quelques cas d'irréductibilité des polynômes à coefficients rationnels. Journal de Mathématiques Pures et Appliquées **2**, 191–258 (1906)
10. Eisenstein, G.: Über die Irreductibilität und einige andere Eigenschaften der Gleichung, von welcher die Theilung der ganzen Lemniscate abhängt. J. Reine Angew. Math. **39**, 160–179 (1850)
11. Gao, S.: Absolute irreducibility of polynomials via Newton polytopes. J. Algebra **237**(2), 501–520 (2001)
12. Gao, S.: Factoring multivariate polynomials via partial differential equations. Math. Comp. **72**(242), 801–822 (2003)
13. Hartshorne, R.: Algebraic Geometry. Springer, New York-Heidelberg (1977). Graduate Texts in Mathematics, No. 52
14. Heintz, J., Sieveking, M.: Absolute primality of polynomials is decidable in random polynomial time in the number of variables. In: Automata, Languages and Programming (Akko, 1981). Lecture Notes in Computer Science, vol. 115, pp. 16–28. Springer, Berlin-New York (1981)

15. Hernando, F., McGuire, G.: Proof of a conjecture on the sequence of exceptional numbers, classifying cyclic codes and APN functions. J. Algebra **343**, 78–92 (2011)
16. Hernando, F., McGuire, G.: Proof of a conjecture of Segre and Bartocci on monomial hyperovals in projective planes. Des. Codes Cryptogr. **65**(3), 275–289 (2012)
17. Hirschfeld, J.W.P.: Projective Geometries over Finite Fields. The Clarendon Press, Oxford University Press, New York (1979). Oxford Mathematical Monographs
18. Janwa, H., McGuire, G.M., Wilson, R.M.: Double-error-correcting cyclic codes and absolutely irreducible polynomials over GF(2). J. Algebra **178**(2), 665–676 (1995)
19. Janwa, H., Wilson, R.M.: Hyperplane sections of Fermat varieties in $\mathbf{P}^3$ in char. 2 and some applications to cyclic codes. In: Applied Algebra, Algebraic Algorithms and Error-correcting Codes (San Juan, PR, 1993), Lecture Notes in Computer Science, vol. 673, pp. 180–194. Springer, Berlin (1993)
20. Jedlicka, D.: APN monomials over GF(2^n) for infinitely many n. Finite Fields Appl. **13**(4), 1006–1028 (2007)
21. Kaltofen, E.: A polynomial reduction from multivariate to bivariate integral polynomial factorization. In: Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing, pp. 261–266 (1982)
22. Kaltofen, E.: A polynomial-time reduction from bivariate to univariate integral polynomial factorization. In: 23rd Annual Symposium on Foundations of Computer Science (Chicago, Ill., 1982), pp. 57–64. IEEE, New York (1982)
23. Kaltofen, E.: Fast parallel absolute irreducibility testing. J. Symb. Comput. **1**(1), 57–67 (1985)
24. Kaltofen, E.: Polynomial-time reductions from multivariate to bi- and univariate integral polynomial factorization. SIAM J. Comput. **14**(2), 469–489 (1985)
25. Kaltofen, E.: Effective Noether irreducibility forms and applications. J. Comput. Syst. Sci. **50**(2), 274–295 (1995). 23rd Symposium on the Theory of Computing (New Orleans, LA, 1991)
26. Khanduja, S.K., Saha, J.: On a generalization of Eisenstein's irreducibility criterion. Mathematika **44**(1), 37–41 (1997)
27. Kopparty, S., Yekhanin, S.: Detecting rational points on hypersurfaces over finite fields. In: Twenty-Third Annual IEEE Conference on Computational Complexity, pp. 311–320. IEEE Computer Society, Los Alamitos, CA (2008)
28. Lenstra, A.K.: Factoring multivariate polynomials over finite fields. J. Comput. Syst. Sci. **30**(2), 235–248 (1985)
29. Lenstra, A.K., Lenstra, H.W., Lovász, L.: Factoring polynomials with rational coefficients. Math. Ann. **261**(4), 515–534 (1982)
30. Lidl, R., Niederreiter, H.: Finite fields. Encyclopedia of Mathematics and its Applications, vol. 20. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA (1983). With a foreword by P. M. Cohn
31. MacLane, S.: The Schönemann-Eisenstein irreducibility criteria in terms of prime ideals. Trans. Amer. Math. Soc. **43**(2), 226–239 (1938)
32. Schmidt, W.: Equations over Finite Fields: An Elementary Approach, 2nd edn. Kendrick Press, Heber City, UT (2004)
33. Stepanov, S.A.: Congruences with two unknowns. Izv. Akad. Nauk SSSR Ser. Mat. **36**, 683–711 (1972)
34. Stepanov, S.A.: Rational points of algebraic curves over finite fields. In: Current Problems of Analytic Number Theory (Proc. Summer School, Minsk, 1972) (Russian), pp. 223–243, vol. 272 (1974)
35. Stichtenoth, H.: Algebraic Function Fields and Codes. Universitext. Springer, Berlin (1993)
36. Szőnyi, T.: Some applications of algebraic curves in finite geometry and combinatorics. In: Surveys in combinatorics, 1997 (London). London Mathematical Society Lecture Note Series, vol. 241, pp. 197–236. Cambridge University Press, Cambridge (1997)
37. von zur Gathen, J.: Irreducibility of multivariate polynomials. J. Comput. Syst. Sci. **31**(2), 225–264 (1985). Special Issue: Twenty-Fourth Annual Symposium on the Foundations of Computer Science (Tucson, Ariz., 1983)

38. Wan, D.Q.: Minimal polynomials and distinctness of Kloosterman sums. Finite Fields and Their Applications **1**(2), 189–203 (1995). Special issue dedicated to Leonard Carlitz
39. Weinberger, P.J.: Finding the number of factors of a polynomial. J. Algorithms **5**(2), 180–186 (1984)

Resolution of a Conjecture on the Covering Radius of Linear Codes

Juan Carlos Orozco and Heeralal Janwa

Abstract The covering radius of a q-ary block code C of length n is defined as the smallest integer $R = R(C)$ such that all vectors in $\mathbb{F}_q^n$ are within Hamming distance R of some codeword of C. By $[n, k, d]R$ code, we mean an $[n, k, d]$ code having covering radius R. The covering radius of a code is one of the fundamental parameters of a code and gives its suitability for data compression, list decoding radius, and has many other applications. The upper bound of Janwa (1986) relates all the fundamental parameters as $R(C) \leq \mathscr{H}(C) := n - \sum_{i=1}^{k} \lceil \frac{d}{2^i} \rceil$. Which can be expressed as $n - g_q + d - \lceil d/q^k \rceil$. If $n_q(k, d)$ denotes the minimum length of any code of dimension k and distance over $\mathbb{F}_q$ it was conjectured by Janwa that under certain conditions $g_q(k, d)$ (the Griesmer length) can be replaced by $n_q(k, d)$. Janwa (1989) and Janwa and Mattson (1999), proved three of the four cases and conjectured that the final case is true. In this article, we give a resolution of this conjecture. These bounds have helped us in determing the exact covering radius of codes from Hermitian curves in most cases, and yielding close bounds in the rest of them.

Keywords Bounds on the covering radius of codes · Griesmer bound · Optimal codes · Maximal codes · Algebraic coding theory

---

J. C. Orozco (✉)
Statistical Institute and Computerised Information Systems, Faculty of Business Administration
University of Puerto Rico Río Piedras Campus, 15 Ave Universidad Suite 1501, San Juan, PR 00925-2535, USA
e-mail: juan.orozco@upr.edu

H. Janwa
Department of Mathematics, University of Puerto Rico, Rio Piedras, College of Natural Sciences, 17 University Ave. Ste 1701, San Juan, PR 00925-2537, USA

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_35

1 Introduction

The covering radius is a fundamental parameter of a code [3, 13]. It has important applications not only in computer and communications science and mathematic but also in combinatorics (see comprehensive discussion in [11, 14]). The minimum distance gives the code measure of error-correcting capability. The covering radius gives the measure of rate distortion for data compression. In mathematics, the covering radius of a code corresponds to the minimal covering of the space while the minimum distance gives the maximum packing. The covering radius also gives the ultimate capability of list-decoding, a subject that has become very important and very active during the past decades, important applications to complexity theory and theoretical computer science [23]. Several Rolf-Nevanlinna prizes have been awarded for results that are intimately connected with list-decoding. In general, the radius of coverage has been thoroughly investigated during the last decades. In the comprehensive book [2], and the subsequent bibliography maintained by the authors of the book, one can find more than a thousand important articles devoted to covering radius.

The problem of computing covering radii is known to be both NP-hard and co-NP-hard (in fact, Π_2^p-comple, [23]), and hence strictly harder than any NP-complete problem unless NP = co-NP. Even for binary cyclic codes, supercomputers (the 1024 nodes nCube and 32 nodes connection-machines) were used in computing the exact covering radius for for lengths $n \leq 63$ (for both even and odd lengths) and redundancy up to 28 [8]. Estimating the radius of coverage is a very difficult task. Likewise, the idea of deriving good lower and upper bounds on this parameter is hard. Only for a class of binary Reed-Muller codes, duals of binary BCH codes, and Reed-Solomon codes, one has exact values or upper and lower bounds on the covering radius (using deep results from algebraic geometry) [2].

Very few bounds are known about the covering radius (see Sect. 3 below for a brief survey).

Only one upper bound connected all the four fundamentals of a code and it is due to Janwa [10, 17]. Janwa and Mattson [15] in 1999, gave an improvement to that bound for three of the subcases and made a conjecture about the remaining case. In this article, we prove several results and, as a final consequence, we give a resolution of that conjecture.

The article is organized as follows. In Sect. 1, we present an introduction the article with a brief historical perspective. In Sect. 2, we provide a background from coding theory and give preliminary results on the covering radius. In Sect. 3, we give a brief survey of known results on the covering radius. In the main Sect. 4, we prove new results to give the resolution of the said conjecture, in Sect. 5, we give some results on maximal codes that are needed to apply some of the bounds. In the final section, we give some applications of our results.

2 Preliminaries

In this section, we introduce the fundamental definitions of coding theory and present some preliminary results which will be used in the following sections. References to these results can be found in Huffman and Pless [26], Van Lint [28], or Ling and Xing [20].

Definition 1 A code is any non-empty subset of $\mathbb{F}_q^n$, the code is linear if it is an $\mathbb{F}_q$–linear subspace of $\mathbb{F}_q^n$. The number n is the length of the code. The dimension k of a linear code is its dimension as a vector space.

Definition 2 The Hamming distance d is given by $d(x, y) = |\{i \in \{1, 2, \ldots, n\} : x_i \neq y_i\}|$ where $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n)$, where x and y are elements of F_q^n.

Definition 3 The minimum distance of code $C \subset \mathbb{F}_q^n$ is given by $d(C) = \min \{d(x, y) : x, y \in C, x \neq y\}$.

Definition 4 For each $x \in \mathbb{F}_q^n$, define the Hamming weight of x by $wt(x) = |\{i \in \{1, 2, \ldots, n\} : x_i \neq 0\}|$, $wt(x)$ is the number of nonzero position of x.

Definition 5 Let $C \subset \mathbb{F}_q^n$ be a code. The minimum weight of C, $wt(C)$, is defined as the smallest of the weights of the nonzero codewords of C.

Lemma 1 *If $x, y \in \mathbb{F}_q^n$, then $d(x, y) = wt(x - y)$.*

Proposition 1 *Let C be a linear code over $\mathbb{F}_q$. Then $d(C) = wt(C)$.*

Let $C \subset \mathbb{F}_q^n$ be a linear code of length n, dimension k and minimum distance d. We say that C is an $[n, k, d]$ code over F_q.

Proposition 2 *A code C is t-error-correcting if and only if $d(C) \geq 2t + 1$; i.e., a code with distance d is an exactly $\lfloor \frac{d-1}{2} \rfloor$-error correcting code.*

Definition 6 Let C be an $[n, k, d]$ linear code over F_q. A matrix G is called a generator matrix if its rows span C.

We say the matrix G is in standard form if $G = (I_k|A)$, where I_k is the identity matrix and A is some $k \times (n - k)$ matrix. A generator matrix G gives a way of encoding information, by taking a message vector, m, of length k, and assigning it to mG, a codeword in C.

Definition 7 Let $C \subset \mathbb{F}_q^n$ be a code. The dual code $C^\perp$ is the code $C^\perp = \{x \in \mathbb{F}_q^n : \langle x, y\rangle = 0, \forall y \in C\}$, where $\langle \cdot, \cdot\rangle$ is the dot product.

Definition 8 A check matrix H for C is any matrix whose rows generate $C^\perp$.

Proposition 3 ([12]) *If C is an $[n, k, d]$ code and $G = (I_k|A)$ is a generator matrix for C, then $H = (-A^T|I_{n-k})$ is a check matrix for C.*

Definition 9 We say a square matrix M is a monomial matrix if every row and column of M contains exactly one non zero element of $\mathbb{F}_q$.

Every monomial matrix is the product of a permutation matrix and a diagonal matrix. We say that two codes C and C' are equivalent (and we write $C \sim C'$) if there exists a monomial matrix M such that the matrix product CM equals C'. Equivalent codes have equal invariants, for example the dimension, minimum distance, and covering radius.

Definition 10 Let C be a linear code over $\mathbb{F}_q$, and let $u \in \mathbb{F}_q^n$ be any vector of length n; we define the coset of C determined by u to be the set

$$C + u = \{v + u : v \in C\}.$$

Definition 11 Let $C + u$ be a coset of C. We say that a vector $l \in C + u$ is called a coset leader if $wt(x) \geqslant wt(l),\ \forall x \in C + u$.

Definition 12 Let C be an $[n, k, d]$ linear code over $\mathbb{F}_q$ and H a check matrix for C. Then for any $w \in \mathbb{F}_q^n$, the syndrome of w, is defined to be the vector $S(w) = Hw^T$.

Proposition 4 (The Singleton bound) *All linear $[n, k, d]$ codes of length n, dimension k, minimum distance d, satisfy $k + d \leqslant n + 1$.*

Note: the Singleton bound is independent of q. The linear $[n, k, d]$ code over $\mathbb{F}_q$ with $d = n - k + 1$ is called an MDS code, here MDS stands for "maximum distance separable". For example, the classical Reed-Solomon codes are MDS codes.

Definition 13 For any vector $x \in \mathbb{F}_q^n$ and any integer $r \geqslant 0$, the sphere of radius r and center x, denoted $B_r(x)$, is the set $B_r(x) := \{y \in \mathbb{F}_q^n : d(x, y) \leqslant r\}$.

Proposition 5 *Let d be the minimum distance of the code C and $t = \lfloor \frac{d-1}{2} \rfloor$, then the spheres of radius t over the different codewords are disjoint.*

Definition 14 For a given $q > 1$, a positive integer n and an integer $r \geqslant 0$, we define $V_q^n(r)$ to be $V_q^n(r) = \begin{cases} \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{r}(q-1)^r & 0 \leqslant r \leqslant n \\ q^n & r \geqslant n \end{cases}$

Lemma 2 *For all integers $r \geqslant 0$, a sphere of radius r in $\mathbb{F}_q^n$ contains exactly $V_q^n(r)$ vectors.*

Proposition 6 (Hamming bound) *Let C be an $[n, k, d]$ code over $\mathbb{F}_q$ with $d \geqslant 2t + 1$ then, $\sum_{i=0}^{t} \binom{n}{i}(q-1)^i \leqslant q^{n-k}$. In particular, $\sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{i}(q-1)^i \leq q^{n-k}$.*

Definition 15 A q-ary code that attains the Hamming (or sphere-packing) bound, i.e., one which has $\dfrac{q^n}{\sum_{i=0}^{t} \binom{n}{i}(q-1)^i}$ codewords, is called a perfect code.

Definition 16 The support of a vector $c \in C$, denoted by Supp(c), is defined as supp$(c) := \{i | c_i \neq 0, 1 \leq i \leq n\}$.

Definition 17 Let C be an $[n, k, d]$ code over $\mathbb{F}_q$, we can puncture C by deleting the same coordinate i in each codeword.

Let C be an $[n, k, d]$ linear code; and let $i \in \{1, \ldots, n\}$. If we delete the ith coordinate of every codeword $c \in C$, then, we obtain a new code C^*.

We call C^* the punctured code of C (on the ith coordinate). Also, we call the process of deleting the ith coordinate, puncturing.

Definition 18 Let $x \in \mathbb{F}_q^n$. The residual code of $C \subset F_q^n$ with respect to x, denoted by $Res(C, x)$ is the code of the length $n - wt(x)$ obtained from C by puncturing on all the coordinates of Supp(x).

Lemma 3 ([20]) *If C is an $[n, k, d]$ linear code over $\mathbb{F}_q$ and $c \in C$ is a codeword of weight d, then $Res(C; c)$ is an $[n - d, k - 1, d' \geqslant \lceil \frac{d}{q} \rceil]$.*

Proposition 7 ([20] (The Griesmer Bound))
Let C be q-ary code with parameters $[n, k, d]$, where $k \geqslant 1$. Then $n \geqslant g_q(k, d) := \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$.

We will denote $g_2(k, d)$ by $g(k, d)$. Note that $g_q(k, d)$ is strictly increasing in d (and in k).
An $[n, k, d]$ code C with $n = g_q(k, d)$ is called a Griesmer code.

Definition 19 Let $n_q(k, d) := \min\{n : \exists\ [n, k, d]$ q-ary code$\}$.

The function $n_q(k, d)$ is strictly increasing in k and d (see [4], p. 196).
An $[n, k, d]$ code C with $n = n_q(k, d)$ is called an optimal code. That is, an optimal code is a code having the smallest possible length over all codes with parameters k and d.

Corollary 1 $n_q(k, d) \geq g_q(k, d) := \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$.

3 Known-Results

In this section we give some fundamental results about the covering radius with background that we will need in the rest of the paper. In most places we give references. The rest can be found in [13], and some in [26].

There are other upper and lower bounds known on the covering radius. Some of these bounds are in the context of saturated configurations for covering radius 2 or 3 (see [25, 27, 27, 28]). A good survey of results in large fields is [6]. Some are asymptotic results. Other results are in terms of the length function (see the recent article [7].

Here, we discuss some known bounds on the covering radius that are explicit functions of the four fundamental parameters of the code.

Definition 20 The covering radius of a q-ary code C of length n is defined as the smallest integer $R = R(C)$ such that all vectors in $\mathbb{F}_q^n$ are within Hamming distance R of some codeword of C. By an $[n, k, d]R$ code we mean an $[n, k, d]$ code having covering radius R.

The covering radius is the smallest integer R such that the union of the Hamming spheres of radius R centered at the codewords is the whole space $\mathbb{F}_q^n$, i.e $R(C) := \min\{R \in \mathbb{N} : \bigcup_{c\in C} B_R(c) = \mathbb{F}_q^n\}$. Or equivalently, $R(C) := \max_{x\in\mathbb{F}_q^n} \{\min_{c\in C} d(x, c)\}$. In other words, the covering radius measures the distance between the code and the farthest-off vector in the space.

Two equivalent codes C and $C' = CM$ have the same covering radius (here M is a monomial matrix). Because, $R(C) = \max_{x\in\mathbb{F}_q^n}\{\min_{c\in C} d(x, c)\} = \max_{x\in\mathbb{F}_q^n} \{\min_{c\in C} d(xM, cM)\} = \max_{yM^{-1}\in\mathbb{F}_q^n}\{\min_{c\in C} d(y, cM)\} = \max_{y\in\mathbb{F}_q^n}\{\min_{c'\in C'} d(y, c')\} = R(C')$.

Definition 21 [2] Let $C \subset F_q^n$ be a linear code with covering radius R. If $y \in \mathbb{F}_q^n$ satisfies $d(y, c) \geq R$ for all $c \in C$, then we call y a deep hole.

Recall that a vector that has minimum weight in a coset is called a *coset leader*.

Proposition 8 *Let C be a linear code with check matrix H. Then*

(a) $R(C)$ is the weight of any coset leader of C of largest weight;
(b) $R(C)$ is the smallest among the numbers s such that every syndrome is a combination of s or fewer columns of the check matrix H.

Proof See Theorem 1.1 [26], p. 757.

Proposition 9 (Redundancy bound) *For every $[n, k, d]$ code C, $R(C) \leqslant n - k$.*

Proof See Corollary 8.1.4 [2], p. 217.

Proposition 10 (Delsarte bound) *Let C be a $[n, k, d]$ code. Also, let $A_i := \{x \in C^\perp : wt(x) = i\}$, and $s^\perp := |\{i \in \{1, \dots, n\} : A_i \neq \emptyset\}|$. Then $R(C) \leq s^\perp$.*

Proof See Theorem 8.3.7 [2], p. 232.

Lemma 4 (Supercode Lemma) *Let $C_0 \subset C$ and $D(C_0, C)$ be the maximum distance of a vector in C to C_0, i.e., $D(C_0, C) := \max_{x\in C\setminus C_0} d(x, C_0)$, where $d(x, C_0) := \min\{d(x, y)|x \in C_0\}$. Then $R(C_0) \geqslant D(C_0, C)$. In particular, $R(C_0) \geqslant d(C)$. If C and C_0 are linear codes, then $R(C_0) \geqslant \max\{wt(x + C_0) : x \in C \setminus C_0\}$.*

Proof See Lemma 8.2.1 [2], p. 222.

Proposition 11 (Sphere Covering bound) *Let C be a $[n, k, d]R$ code over $\mathbb{F}_q$, then $\sum_{i=0}^{R} \binom{n}{i}(q - 1)^i \geqslant q^{n-k}$.*

Proof See [26], p. 757.

Corollary 2 *Let C be an $[n, k, d]$ code. Then $R(C) \geqslant \lfloor \frac{d-1}{2} \rfloor$.*

Proof The result follows from the fact that the covering radius is at least as large as the packing radius, which the error-correcting capability of the code. See [26], p. 757.

Definition 22 An $[n, k, d]$ code is called maximal if the addition of any new codeword to C reduces its minimum distance.

Proposition 12 ([2]) *Let C be an $[n, k, d]R$ linear code. Then, C is maximal if and only if $R < d$.*

Recall that, if C is a Griesmer code, then C is a maximal code (see [4], Theorem 4.1, p. 196).

Proposition 13 ([10]) *An $[n, 1, n]$ q-ary code has covering radius $R = \lfloor \frac{(q-1)n}{q} \rfloor$.*

Theorem 1 ([22]) *Let C be an $[n, k, d]$ code. Let J be any subset of the coordinate-places of C. Consider a generator matrix $G(C)$ of C of the form,*

$$G(C) = \begin{pmatrix} G(C_J) & A \\ 0 & G(C_0) \end{pmatrix}$$

where we have permuted the columns so that J is at the left. The code C_J is the projection of C onto the coordinates-places J; C_0 is shortening the subcode of C of codewords which have $0's$ on J; shortened by removal of the coordinates J. Then $R(C) \leqslant R(C_J) + R(C_0)$.

The following bound is due to Janwa [10].

Theorem 2 (Janwa [10]) *Let C be an $[n, k, d]$ code over $\mathbb{F}_q$ with covering radius $R(C)$. Then*

$$R(C) \leqslant \mathscr{H}(C) := n - \sum_{i=1}^{k} \lceil \frac{d}{q^i} \rceil$$

We will use several times the following Proposition given in [2].

Proposition 14 *Let C be a binary $[n, k, d]$ code with covering radius R and, let $x \in \mathbb{F}_2^n$ such that $d(x, C) = wt(x) = R$.*

Then $Res(C; x)$ is an $[n - R, k, d' \geqslant \lceil \frac{d}{2} \rceil]$ code.

Proof See Lemma 8.1.8 [2], p. 218.

Proposition 15 ([15]) *Let C be an $[n, k, d]$ code over $\mathbb{F}_q$. If x is a coset leader of C, then $Res(C; x)$ is an $[n - wt(x), k, d' \geqslant \lceil \frac{d}{q} \rceil]$ code.*

Proposition 16 ([15]) *Let C be an $[n, k, d]$ code over $\mathbb{F}_q$ with covering radius R. Then, $R \leqslant n - n_q(k, \lceil \frac{d}{q} \rceil) \leqslant \mathcal{H}(C)$.*

Proposition 17 $n_q(k, d) \geqslant n_q(k-1, \lceil \dfrac{d}{q} \rceil) + d$.

Proof Recall that $n_q(k, d)$ is the smallest number n such that there exists an $[n, k, d]$ code, C, over $\mathbb{F}_q$. So, there exists an $[n_q(k, d), k, d]$ code, C_0.

Since $d(C_0) = d$, we have a codeword $c \in C_0$ with $wt(c) = d$.

By Lemma 3, $Res(C_0; c)$ is an $[n_q(k, d) - d, k-1, d' \geqslant \lceil \dfrac{d}{q} \rceil]$ code.

Therefore,

$$n_q(k-1, \lceil \frac{d}{q} \rceil) \leqslant n_q(k-1, d') \leqslant n_q(k, d) - d, \text{ or } n_q(k, d) \geqslant n_q(k-1, \lceil \frac{d}{q} \rceil) + d$$

Proposition 18 ([15]) *Let C be an $[n, k, d]$ code over $\mathbb{F}_q$.*

In the Janwa bound, $R(C) \leqslant \mathcal{H}(C) = n - g_q(k, \lceil \dfrac{d}{q} \rceil)$, equality holds only if there exists an $[g_q(k, \lceil \dfrac{d}{q} \rceil), k, \lceil \dfrac{d}{q} \rceil]$ code.

In this section, we present the first new results of this work. First, we derive a new upper bound on the covering radius of an $[n, k, d]_q$ code. This new result improves the Janwa bound. Moreover, we show that an improvement on the lower bound of the minimum distance of residual codes implies an improvement on the upper bound of the covering radius. Also, we give improvements on the Janwa bound that rely on the quantities $n_q(k, d)$ and $g_q(k, d)$ or on the code being maximal. Finally, we conclude the section with some new results about maximal codes.

4 Another Improvement to the Janwa Bound and Resolution of a Conjecture

We first give some new bounds on the covering radius of $[n, k, d]_q$ codes. In the next section we will also give an improvement of the bounds given in [15].

Now we have one more possibility to rewrite the bound given in Theorem 2, given in the following proposition. Here we give another Griesmer decomposition:

$$R(C) \leqslant \mathcal{H}(C) = \mathcal{H}(n, k, d) = n - g_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil$$

and prove that such a replacement is possible.

Proposition 19 *Let C be an $[n, k, d]_q R$ code. Then there are four different decompositions of $\mathcal{H}(C)$: a) $n - g_q(k, d) + d - \lceil \frac{d}{q^k} \rceil$; b) $n - g_q(k+1, d) + d$; c) $n - g_q(k, \lceil \frac{d}{q} \rceil)$; d) $n - g_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil$.*

Proof For proofs of (a), (b) and (c) see [15]. The proof of (e) is given by Theorem 2. Part (d) follows from the Griesmer decomposition.

The Griesmer bound states that $n_q(k, d) \geq g_q(k, d)$. An open problem of improving the bound $H(C)$ was whether the function $g_q()$ can be replaced by the function $n_q()$, thus providing a stronger bond. Using several known and new results, Janwa and Mattson [15] proved that in parts (a)–(c) such improvements are possible; Part (a) and (b) are true under the condition of maximally. The following conjecture is about from Janwa and Mattson [15] (1999).

Conjecture 1 The bound, $R(C) \leq n - n_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil$, is true under similar conditions.

We give a resolution of this conjecture after 25 years, in Theorem 3 below. For this, we first prove an important Lemma.

Lemma 5 *Let C be an $[n, k, d]$ code over $\mathbb{F}_q$. Also, let x be a coset leader such that $wt(x) = R(C) =: R$. Take, $C_1 = Res(C; x)$. By Proposition 15, we know that C_1 is an $[n - R, k, d^{(1)} \geqslant \lceil \frac{d}{q} \rceil]$ code. Let $z_1 \in C_1$ such that $wt(z_1) = d^{(1)}$.*

Then $C_2 := Res(C_1; z_1)$ is an $[n - R - d^{(1)}, k - 1, d^{(2)} \geqslant \lceil \frac{d}{q^2} \rceil]$ code.

Proof Since C_1 is an $[n - R, k, d^{(1)} \geqslant \lceil \frac{d}{q} \rceil]$ code, we have by Lemma 3, $C_2 := Res(C_1; z_1)$ is an $[n - R - d^{(1)}, k - 1, d^{(2)} \geqslant \lceil \frac{d^{(1)}}{q} \rceil]$ code. But $d^{(1)} \geqslant \lceil \frac{d}{q} \rceil$. So $d^{(2)} \geqslant \lceil \frac{d}{q^2} \rceil$. Therefore C_2 is an $[n - R - d^{(1)}, k - 1, d^{(2)} \geqslant \lceil \frac{d}{q^2} \rceil]$ code.

Now we prove the main theorem that resolves the conjecture that has stood for 25 years.

Theorem 3 *Let C be an $[n, k, d]_q R$ code.*
Then, $R \leqslant n - n_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil \leqslant \mathscr{H}(C)$.

Proof Lemma 5 implies that there is an $[n - R - d^{(1)}, k - 1, d^{(2)} \geq \lceil \frac{d}{q^2} \rceil]$ code C_2. Then $n(C_2) = n - R - d^{(1)} \geq n_q(k - 1, d^{(2)}) \geq n_q(k-1, \lceil \frac{d}{q^2} \rceil)$, the latter inequality holding because $n_q(k, d)$ is monotone in d. Since $d^{(1)} \geqslant \lceil \frac{d}{q} \rceil$, therefore $R \leqslant n - n_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil$.

On the other hand, by Griesmer bound we have

$$n - n_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil \leqslant n - g_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil$$

$$= n - \sum_{i=0}^{k-2} \lceil \frac{d}{q^{i+2}} \rceil - \lceil \frac{d}{q} \rceil = n - \sum_{i=2}^{k} \lceil \frac{d}{q^i} \rceil - \lceil \frac{d}{q} \rceil = n - \sum_{i=1}^{k} \lceil \frac{d}{q^i} \rceil = \mathscr{H}(C).$$

Therefore, $R \leqslant\; n - n_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil \leqslant \mathscr{H}(C)$.

After our result, in the following theorem, we collect in one place all the improvements that were contemplated.

Theorem 4 *Let C an $[n, k, d]R$ code. Then :*

(a) If C is maximal, then $R \leqslant n - n_q(k, d) + d - \lceil \frac{d}{q^k} \rceil \leqslant \mathscr{H}(n, k, d)$;
(b) If C is maximal, then $R \leqslant n - n_q(k+1, d) + d \leqslant \mathscr{H}(n, k, d)$;
(c) $R \leqslant n - n_q(k, \lceil \frac{d}{q} \rceil) \leqslant \mathscr{H}(n, k, d)$;
(d) $R \leqslant n - n_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil \leqslant \mathscr{H}(n, k, d)$.

Proof For proofs (a), (b) and (c) see ([15], Theorem 2, p. 164, Theorem 3, p. 165 and Theorem 4, p. 166 respectively). The proof of (d) follows from Proposition 3; (also, see Bhandari [1], for $q = 2$).

We have used Theorem 4 in determining exactly (or coming close to it) the covering radius of algebraic geometric codes from Hermitian curves. For other practical applications of Theorem 4, we need the knowledge of the function $n_q(k, d)$. Griesmer bound has been improved in dozens of articles over the years [9, 18, 19, 21]. These lead to improvements of the original bound.

We make the following observation: suppose that we could improve the Griesmer bound by a quantity $l_q(k, d)$. In other words, suppose that we know that $n_q(k, d) \geq g_q(k, d) + l_q(k, d)$. Then, we obtain the following result.

Corollary 3 *Let C be $[n, k, d]R$ code and suppose that $n_q(k, d) \geq g_q(k, d) + l_q(k, d)$. Then:*

(a) If C is maximal, then $R \leqslant\; n - g_q(k, d) + d - \lceil \frac{d}{q^k} \rceil - l_q(k, d)$;
(b) If C is maximal, then $R \leqslant\; n - g_q(k+1, d) + d - l_q(k+1, d)$;
(c) $R \leqslant\; n - g_q(k, \lceil \frac{d}{q} \rceil) - l_q(k, \lceil \frac{d}{q} \rceil)$;
(d) $R \leqslant\; n - g_q(k-1, \lceil \frac{d}{q^2} \rceil) - \lceil \frac{d}{q} \rceil - l_q(k-1, \lceil \frac{d}{q^2} \rceil)$.

Thus, we see that any further improvement of the Griesmer bound allows us to compute better upper bounds for the covering radius.

Theorem 5 *Let C be an $[n, k, d]R$ MDS code over $\mathbb{F}_q$. Then $R \leqslant d - \lceil \dfrac{d}{q} \rceil$.*

Proof Let x be a coset leader of C such that $wt(x) = R$. Then by Proposition 15, there is an $Res(C; x) = [n - R, k, d' \geqslant \lceil \frac{d}{q} \rceil]$ code. By the Singleton bound $d' \leqslant n - R - k + 1$. Since C is an MDS code, $d = n - k + 1$. So we have $R \leqslant n - k + 1 - d' = d - d' \leqslant d - \lceil \frac{d}{q} \rceil$.

5 Some Results on Maximal Codes

To apply Theorem 4(a), (b), we need to first prove that the given code is maximal. In this section we give some results on maximality.

Theorem 6 *If C is an $[n, k, d]_q R$ code such that $R < d$, then:*

(a) There exists an $[n + (d - R), k + 1, d]$ code.
(b) $n_q(k + 1, d) \leq n - (R - d)$.
(c) $R \leqslant \min\{d - (n_q(k + 1, d) - n), d - 1\}$.

Proof (a) Let x be a coset leader of weight R. Then, $\langle C, (x, \mathbf{1}_{d-R}) \rangle$ is an $[n + (d - R), k + 1, d]$ code, where $\mathbf{1}_{d-R}$ is the all $1's$ vector of length $d - R$.
(b) By a) there is an $[n + (d - R), k + 1, d]$ code. Therefore by definition of $n_q(k + 1, d)$, we have $n_q(k + 1, d) \leq n + (d - R) = n - (R - d)$.
(c) The proof follows immediately from b).

Theorem 7 *Let C be an $[n, k, d]_q$ code. Then if C is not a maximal code, $n \geq n_q(k + 1, d)$.*

Proof (i) Since C is not a maximal code, there exists an $[n, k + 1, d]$ code. By the definition of $n_q(k + 1, d)$ we have $n \geqslant n_q(k + 1, d)$.

Theorem 8 *If $C_1 \subset C_2$ and C_2 is a maximal code, then $R(C_1) > R(C_2)$.*

Proof If $C_1 \subset C_2$, then by Lemma 4, $R(C_1) \geqslant d_2$. Now, given that C_2 is maximal, by Proposition 6 $R(C_2) < d_2$. Therefore $R(C_1) > R(C_2)$.

Remark 1 If C is an MDS code, then C is a maximal code.

Intuitively, this is obvious. We provide the following chain of reasoning. Let $C := [n, k, d]$ be an MDS code. Recall that $d = n - k + 1$.

We claim $n = g_q[k, n - k + 1]$. Note that, $n \geqslant n_q(k, d) \geqslant g_q[k, n - k + 1]$ and $g_q[k, n - k + 1] = \sum_{i=0}^{k-1} \lceil \frac{n - k + 1}{q^i} \rceil = (n - k + 1) + \sum_{i=1}^{k-1} \lceil \frac{n - k + 1}{q^i} \rceil \geqslant (n - k + 1) + k - 1 = n$. Thus, $n = g_q(k, n - k + 1)$, so C is a Griesmer code.

But, $R(C) \leqslant \mathscr{H}(n, k, d) = n - g_q(k, d) + d - \lceil \frac{d}{q^k} \rceil \leqslant d - \lceil \frac{d}{q^k} \leqslant d - 1$.

Therefore, C is maximal.

6 Comparison of the Bounds and Some Applications

In principle, if we know information about the $n_q(k, d)$ function for sufficiently large parameters, we can compare bounds we have given. What we have discovered is that our bounds are quite effective in applications for nested sequences codes. In Orozco [16, 24], we have obtained exact values or close upper and lower bounds on the covering radius of codes from Hermtian curves. Ours are the first results on the covering radius of families of algebraic geometric codes obtained by the Janwa bound, or the improvements discussed in this article. We have not been able to use any other bonds that are applicable to these algebraic geometric codes. In that sense, our bounds are better.

Acknowledgements We would like to thank Carlos Agrinsoni and Andres Arroyo for helpful discussions. We also thank the reviewers for a careful reading of the manuscript and for helpful suggestions and comments.

References

1. Bhandari, M.C., Garg, M.S.: Optimum codes of dimension 3 and 4 over GF(4). IEEE Trans. Inform. Theory **38**(5), 1564–1567 (1992)
2. Cohen, G., Honkala, I., Litsyn, S., Lobstein, A.: Covering Codes, pp. I–537 (1997)
3. Cohen, G., Karpovsky, M., Mattson, H., Schatz, J.: Covering radius–survey and recent results. IEEE Trans. Inf. Theory **31**(3), 328–343 (1985)
4. Cohen, G., Litsyn, S., Lobstein, A., Mattson, H.: Covering radius 1985–1994. AAECC **8** (1997)
5. Davydov, A.: Constructions and families of covering codes and saturated sets of points in projective geometry. IEEE Trans. Inf. Theory **41**(6), 2071–2080 (1995)
6. Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: Linear nonbinary covering codes and saturating sets in projective spaces (2009). arXiv:0904.3835
7. Davydov, A.A., Marcugini, S., Pambianco, F.: Upper bounds on the length function for covering codes (2021). arxiv:2108.13609
8. Dougherty, R., Janwa, H.: Covering radius computations for binary cyclic codes. Math. Comput. **57**(195), 415–434 (1991)
9. Hamada, N., Helleseth, T., Ytrehus, Ø.: On the construction of [q 4+ q 2- q, 5, q 4- q 3+ q 2–2 q; q]-codes meeting the griesmer bound. Des. Codes Crypt. Des. Codes Crypt. **2**, 225–229 (1992)
10. Janwa, H.: Relations among parameters of codes. Ph.D. dissertation (1986)
11. Janwa, H.: On the optimality and covering radii of some algebraic geometric codes. In: Presented at the Workshop on Coding Theory (1988)
12. Huffman, W.C., Pless, V.: Fundamentals of Error-Correcting Codes, 1st edn., p. 633 (2003)
13. Janwa, H.: On the parameters of algebraic geometric codes. In: Applied Algebra, Algebraic Algorithms and Error-correcting Codes. New Orleans, LA, 1991, volume 539 of Lecture Notes in Computer Science, pp. 19–28. Springer, Berlin (1991)
14. Janwa, H., Jr, H.F.M.: The covering radius and normality of t-dense codes. IEEE Trans. Inform. Theory (1986)
15. Janwa, H., Mattson, J.H.F.: Some upper bounds on the covering radii of linear codes over $\mathbf{F}_q$ and their applications. Des. Codes Cryptogr. **18**(1–3), 163–181 (1999). Designs and codes—a memorial tribute to Ed Assmus

16. Janwa, H., Orozco, J., Arroyo, A.: Covering radius of codes from hermitian curves (2022)
17. Justesen, J., Hoeholdt, T.: A cours in error correcting codes. Eur. Math. Soc. **42**, 99–114 (2008)
18. Kageyama, Y., Maruta, T.: On the geometric constructions of optimal linear codes. Des. Codes Crypt. **81**, 469–480 (2016)
19. Klein, A.: On codes meeting the griesmer bound. Discret. Math. **274**(1–3), 289–297 (2004)
20. Ling, S., Xing, C.: Coding Theory, A First Course, pp. I–223 (2004)
21. Ma, W., Luo, J.: Nonexistence of linear codes meeting the griesmer bound. Discret. Math. **345**(4), 112744 (2022)
22. Mattson, J.H.F.: An improved upper bound on covering radius. In: Applied Algebra, Algorithmics and Error-Correcting Codes (Toulouse, 1984), volume 228 of Lecture Notes in Computer Science, pp. 90–106. Springer, Berlin (1986)
23. McLoughlin, A.: The complexity of computing the covering radius of a code. IEEE Trans. Inf. Theory **30**(6), 800–804 (1984)
24. Orozco, J.C.: On the covering radius of hermitian codes. Graduate thesis, University of Puerto Rico (2017)
25. Ostergard, P., Blass, U.: On the size of optimal binary codes of length 9 and covering radius 1. IEEE Trans. Inf. Theory **47**(6), 2556–2557 (2001)
26. Pless, V.S., Huffman, W.C.: Handbook of Coding Theory, pp. I–537 (1998)
27. Östergård, P.R., Weakley, W.D.: Switching of covering codes. Discret. Math. **341**(6), 1778–1788 (2018)
28. van Lint, J.H.: Algebraic geometric codes. In: Coding Theory and Design Theory, Part I, volume 20 of *The IMA Volumes in Mathematics and its Applications*, pp. 137–162. Springer, New York (1990)

Quantum Error-Correcting Codes Over Small Fields From AG Codes

Heeralal Janwa and Fernando L. Piñero–González

Abstract To overcome decades of obstacles and constraints implied by the no-cloning theorem, Calderbank and Shor constructed codes to control errors in quantum computers. Their construction uses a pair of binary or quaternary error-control codes for classical channels. Classic codes such as Reed-Muller codes provide such pairs. However, their performance is not so good. In this article we use codes from algebraic curves over high degree extensions of $\mathbb{F}_2$ to construct the self–orthogonal binary code or quaternary code pairs. We also present some results on the parameters of the resulting subfield codes over $\mathbb{F}_2$ or $\mathbb{F}_4$ from Hermitian curves, Norm–Trace curves, quasi–Hermitian curves, Castle curves and others. Several of these results are novel and provide a pathway to make progress towards making quantum computers feasible and practical during the next decade.

Keywords Quantum error-correcting codes · AG codes · Subfield subcodes · Hermitian · Quasi-Hermitian · Giuletti–Korchmaros curve · Suzuki curves

1 Background and Motivation

One of the seminal breakthrough of the twentieth century was the proposal by Richard Feynman to build computers based on quantum mechanical principles that would solve problems that were infeasible on foreseeable classical computers. Among the obstacles that were to be overcome were quantum decoherence and quantum entanglement. It was soon realized that the random outputs of such computers would be feasible only if one were to be able to correct a massive number of errors that would result as a process of a sequence of unitary operators as part of any algorithm. Earlier

H. Janwa
University of Puerto Rico at Rio Piedras, San Juan, PR 00925, USA
e-mail: heeralal.janwa@upr.edu

F. L. Piñero–González (✉)
University of Puerto Rico – Ponce Campus, Ponce, PR 00716, USA
e-mail: fernando.pinero1@upr.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_36

attempts were not successful because of the no-cloning theorem in quantum mechanics (meaning that even a simple repetition code would not be proposed). Calderbank et al. [5] obtained a fundamental breakthrough when they showed that one could obtain qubit quantum error-correcting codes from classical binary codes under certain self-orthogonality conditions. This meant that obvious candidates such as the BCH codes, the Reed-Muller codes and such yield quantum error-correcting codes. However, their parameters are still far from what can theoretically be obtained. In this article, we carry forward the research by obtaining binary and quaternary codes from algebraic geometry (AG) codes. We do acknowledge that scores and scores of articles exist that have constructed non-binary quantum from AG codes. Indeed, soon after the results of Calderbank et al., the concept of generalized qubit in an abstract Hilbert space was defined where one can use non-binary alphabets. However, for the physical layer type practical applications, to a functioning quantum computer, one needs improved binary (or quaternary) classical codes that would result in improved error-correcting codes. In this article, we construct binary and quaternary codes. We show that all the algebraic and algebraic geometric methods that were thought obsolete after the advent of low density parity (LDPC) check codes, have now become very relevant for very practical application in quantum computing. We will see in this article that we employ all the methods developed since the advent of algebraic coding theory in the construction of our codes.

In a remarkable development, Peter Shor showed that quantum computers would be able to break the RSA cryptosystem. Among other developments, to quote the CEO of IBM, Arvind Krishna: “We have put a roadmap out that by 2023 we expect to be at a 1000-qbit computer. So, by 2023 or 2024, we’re going to be able to start solving problems that could have a large impact. There’s some hard engineering challenges between now and then, reduce the errors in these machines, make sure they can stay up for long periods of time, make sure that they are fully programmable, but I have confidence we’re going to get there. And as we get there, problems in materials, problems in risk, problems in financial modeling such as pricing, maybe EV battery technology, then going down the road a little bit, problems around supply chains, how to minimize fuel consumption, maybe weather predictions and modeling-those that are probably a little bit harder-are all problems that will be in the realm of quantum computers and that means you bring so much value when you think about the climate change crisis, we think about lightweight materials, we think about EVs, there’s so much promise in what these technologies can deliver us a few years down the road, and now it’s a few years, no longer a few decades.” (Washington Post Live, 2021) [26].

QECC over binary, ternary, and quaternary fields are needed for practical applications. There have been dozens of articles constructing QECC from algebraic geometric codes over large finite fields (see, for example, [3, 6–8, 10, 16, 18–23, 27]). In particular one can consider QECC from hyperbolic codes over large finite fields obtained by Christensen and Geil [6]. All QECC codes from curves or varieties over finite fields extend the lengths of the codes in the same way AG Goppa codes extend the length of Reed–Solomon codes. However, one must work over large finite fields. Often such large fields are imposed by the defining curves. For practical applications, this is their inherent limitations. We overcome this constraint and address the main problem of obtaining practical and applicable (for implemention at the physical layer or in software algorithms) QECC from AG codes over smaller fields (especially over binary, ternary, and quaternary ones.) This distinction has to be clear from the outset to see the worth of our efforts.

Indeed, our article is a supplement to all the cited articles (and others), and often we even use results obtained by them in large finite fields (for example, those from hyperbolic AG codes). Some of our main results (such as Corollary 1 and Lemma 1 follow as consequences of the seminal work of Delsarte. Other results are perhaps not deep, but we are the first to observe them for important applications.

We succeed in this worthy endeavor (obtaining binary and quaternary codes) by appealing to and adapting the results and algorithms we had developed in our prior article [13, 13–15, 24]. To emphasize, we obtain binary, ternary, and quaternary QECC from AG codes that no other authors have obtained so far. One of our key ideas is to use a Self–Orthogonal code pair over a large field $\mathbb{F}_q$ to derive a binary self–orthogonal code pair, which yields the binary quantum error–correcting code. This is an outcome of a string of articles by us on subfield subcodes of algebraic geometric codes. Another key idea is that by constructions of AG codes via L-spaces or via Gröbner bases, several authors, including us, have constructed quantum AG codes albeit over large finite fields. Our approach in such cases is to use some known and some new orthogonal codes over large alphabet size fields to exploit such explicit L-spaces and Froebenius action on them to construct binary and quaternary codes.

We say that a quantum code of length n, dimension k and minimum distance d by is an

$$[[n, k, d]]$$

quantum code. We recall the following from Calderbank and Shor (1994), Steane (1996), and Calderbank et al. [5]:

There are versions of the following theorems for q-ary quantum codes. However that is not our concern here.

Proposition 1 *Let C_1 and C_2 be two binary codes of length n and dimensions k_1 and k_2, respectively, such that $C_2 \subseteq C_1$. Then there exists a binary quantum code of length n, dimension $k_1 - k_2$, and minimum distance* $\min\{d(C_1 \setminus C_2), d(C_2^\perp \setminus C_1^\perp)\}$.

Corollary 1 *Let C_1 and C_2 be two binary codes of length n and dimensions k_1 and k_2, respectively, such that $C_2^\perp \subseteq C_1$. Then there exists a binary quantum code of length n and dimension $k_1 + k_2 - n$, and minimum distance* $\min\{d(C_1 \setminus C_2^\perp), d(C_2 \setminus C_1^\perp)\} \geq \min\{d(C_1), d(C_2)\}$

Corollary 2 *Let C_1 be a binary code of length n and dimension k_1 such that $C_1^\perp \subseteq C_1$. Then there exists a binary quantum code of length n and dimension $2k_1 - n$, and minimum distance $d(C_1 \setminus C_1^\perp) \geq d(C_1)$.*

2 Quantum Error-Correcting Codes Over Small Fields from Self-orthogonal Codes Over Large Fields

In this section $q_0 = p^r$ and $q = q_0^m$ represent two powers of prime p such that $\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$. We denote the trace function of $\mathbb{F}_q$ over $\mathbb{F}_{q_0}$ by $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}$. For this section, we refer to [30].

Definition 1 Let C be a code over $\mathbb{F}_q$ of length n. The subfield subcode of C is defined as

$$C|\mathbb{F}_{q_0} := C \cap \mathbb{F}_{q_0}^n.$$

Definition 2 Let C be a code over $\mathbb{F}_q$ of length n. We define the trace code of C as

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) := \{(\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(c_1), \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(c_2), \ldots, \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(c_n))|c \in C\}.$$

Both $C|\mathbb{F}_{q_0}$ and $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)$ are linear codes over $\mathbb{F}_{q_0}$ of length n. In fact:

Proposition 2 (Delsarte's Theorem)

$$C|\mathbb{F}_{q_0}^{\perp} = \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C^{\perp}).$$

As $\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$, the map $x \mapsto x^{q_0}$ is an automorphism (called the Froebenius automorphism) of $\mathbb{F}_q$ which fixes $\mathbb{F}_{q_0}$ pointwise. In fact, the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_{q_0}$ is the cyclic group of order m generated by this automorphism. We use the automorphism to define a code over $\mathbb{F}_q$ as follows:

Definition 3 Let C be a code over $\mathbb{F}_q$ of length n. We define

$$C^{q_0} := \{((c_1^{q_0}, c_2^{q_0}, \ldots, c_n^{q_0}) \mid (c_1, c_2, \ldots, c_n) \in C)\}.$$

Stichtenoth [30] showed that $C|\mathbb{F}_{q_0}$ and $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)$ may be considered as codes over the larger field $\mathbb{F}_q$:

Proposition 3 *Suppose $q = {q_0}^m$. Let C be a code over $\mathbb{F}_q$. Then:*

$$\bigcap_{i=0}^{m-1} C^{q_0^i} \textit{ is the code generated by } C|\mathbb{F}_{q_0} \textit{ over } \mathbb{F}_q.$$
$$\sum_{i=0}^{m-1} C^{q_0^i} \textit{ is the code generated by } \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \textit{ over } \mathbb{F}_q.$$

We denote the code $\bigcap_{i=0}^{m-1} C^{q_0^i}$ by C^0 and denote $\sum_{i=0}^{m-1} C^{q_0^i}$ by $C^\wedge$. From properties of sums of spaces and dual codes, we restate Dersarte's theorem as simply: $(C^0)^\perp = (C^\perp)^\wedge$.

We end this section with the following corollary to Delsarte's Theorem.

Lemma 1 *Let $C \subseteq C^\perp$ be a self–orthogonal code. If for every $c, c' \in C$ and i the equation*

$$c^{q_0^i} \cdot c' = 0$$

holds, then $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)^{\perp}$.

Proof Let C be a code over $\mathbb{F}_q$. Let $c, c' \in C$. If $c^{q_0^i} \cdot c' = 0$, then $c' \cdot d = 0$ for any $d \in (C^{\wedge})^{\perp}$. This implies that $C \subseteq (C^{\wedge})^{\perp}$. As the hypothesis $c^{q_0^i} \cdot c' = 0$ holds for any i, it also holds for $c^{q_0^{i-j}} \cdot c' = 0$, which also holds after raising both sides to q_o^j. Therefore $c^{q_0^i} \cdot (c')^{q_0^j} = 0$. As in the previous argument: $C^{q_0^j} \subseteq (C^{\wedge})^{\perp}$.

Therefore, $C^{\wedge} \subseteq (C^{\wedge})^{\perp}$. Proposition 3 states that $C^{\wedge} \subseteq (C^{\perp})^0$. Applying Delsarte's theorem, we obtain

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq (C^{\perp})|\mathbb{F}_{q_0} = \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)^{\perp}$$

thus completing the proof of the Lemma.

Lemma 2 *Let* $\mathbb{F}_{q_0}$ *be a subfield of* $\mathbb{F}_q$. *Suppose* C *is a* q*–ary linear code such that* $C \subseteq C^{\perp}$. *If* $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq C^{\perp}$ *then*

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)$$

is a quantum error-correcting code.

Proof Suppose that C is a code such that $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq C^{\perp}$. As $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)$ is fixed under the automorphism $x \mapsto x^{q_0}$, the code $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)$ is a q_0–invariant subspace of $C^{\perp}$. Therefore, it is contained in the largest q_0–invariant subspace of $C^{\perp}$, the subfield subcode $(C^{\perp})|\mathbb{F}_{q_0}$. Thus,

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq (C^{\perp})|\mathbb{F}_{q_0}.$$

Delsarte's Theorem implies

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)^{\perp}.$$

This property of self–orthogonal trace codes also holds for subcodes.

Corollary 3 *Let* $\mathbb{F}_{q_0}$ *be a subfield of* $\mathbb{F}_q$. *Suppose* C *is a* q*–ary linear code such that*

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq C^{\perp}$$

then for any subcode $D \leq C$ *we have that*

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(D)$$

is a quantum error-correcting code.

Proof We need to prove that if C satisfies $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)^{\perp}$ and $D \subseteq C$, then

$$\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(D) \subseteq D^{\perp}|\mathbb{F}_{q_0}.$$

From the definition of the trace code, it is clear that $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(D) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)$. By hypothesis, $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)^{\perp}$. The dual codes satisfy the following: $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C)^{\perp} \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(D)^{\perp}$. Putting everything together, we obtain $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(D) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(D)^{\perp}$

We may also consider codes which are not self–orthogonal, but which are contained in a code isometric to its dual.

Definition 4 Let C be a linear code. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a vector of nonzero elements of $\mathbb{F}_q$. We define the code $\alpha C := \{(\alpha_1 c_1, \alpha_2 c_2, \dots, \alpha_n c_n) \,|c \in C\}$.

Definition 5 Let C be a linear code. We say C is isometric self–dual if $C \subseteq (\alpha C)^{\perp}$

Lemma 3 *Let C be a linear code. Suppose that C is isometric self–orthogonal. That is $C \subseteq (\alpha C)^{\perp}$. If there exists $\beta \in \mathbb{F}_q^n$ such that $\beta^2 = \alpha$ then the code βC is self-orthogonal.*

Proof We aim to prove that βC is self-orthogonal. As C is isometric self–dual, we have that

$$C \subseteq (\alpha C)^{\perp}.$$

Note that this implies that

$$C \subseteq \alpha^{-1} C^{\perp}.$$

Scaling both codes by β we obtain

$$\beta C \subseteq \beta\alpha^{-1} C^{\perp}.$$

As $\alpha = \beta^2$ it follows that $\beta\alpha^{-1} = \beta^{-1}$. Therefore

$$\beta C \subseteq \beta^{-1} C^{\perp}$$

which implies

$$\beta C \subseteq (\beta C)^{\perp}.$$

3 Quantum AG Codes from Self–orthogonal AG Codes

A key observation is that via L-spaces or via Gröbner bases, one can often get classes of self-orthogonal AG codes over large fields. In fact, the latter construction has generated a substantial amount of literature to construct quantum AG codes albeit over large finite fields. However, to get practical quantum codes for applications to modern quantum computers, one needs binary or quaternary AG quantum codes. This is precisely what we do, by combining AG codes over large fields with subfield codes over small ones using a Self–Orthogonal code pair over a large field $\mathbb{F}_q$ to make a binary self–orthogonal code pair. Lemma 3 implies we can apply our results to isometric dual AG code classes (such as several AG codes from Castle curves) with little difficulty.

Definition 6 Let $\mathcal{F}$ be an algebraic function field over $\mathbb{F}_q$ where $q = p^m$. Let $D = P_1 + P_2 + \cdots + P_n$ be a divisor of places of degree 1 and let Q be a place of degree 1 not appearing in the support of D. Let s be a nonnegative integer.

The *Algebraic Geometry Code* $C_L(D, sQ)$ is defined as

$$C_L(D, sQ)\{(f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in \mathcal{L}(sQ)\}.$$

We say that the class of codes $C_L(D, sQ)$ is *self–orthogonal* if and only if for every s there exists $s^\perp$ such that $C_L(D, sQ)^\perp = C_L(D, s^\perp Q)$.

Proposition 4 ([31]) *Suppose that $\mathcal{F}$ is an algebraic function field of genus g. The code $C_L(D, sQ)$ is an $[n, \geq s - g + 1, \geq n - g]$ code.*

Proposition 5 ([31]) *Suppose that $\mathcal{F}$ is an algebraic function field of genus g. If $C_L(D, sQ)$ is a class of self–orthogonal codes,then $s^\perp = n + 2g - 2 - s$*

Lemma 4 *Suppose that $\mathcal{F}$ is an algebraic function field of genus g. If $f \in \mathcal{L}((n + 2g - 2)Q)$, then*

$$\sum_{i=1}^{n} f(P_i) = 0$$

It follows from the fact that $C_L(D, (n + 2g - 2)Q)^\perp = C_L(D, 0Q)$.

Munuera, Tenorio and Torres worked on self–orthogonal trace codes from AG codes in [22, Proposition 5.6]. We state the bound on s which yield a self–orthogonal trace code from $m \leq s \leq \frac{n+2g-2}{2p^{\lfloor \frac{m}{2} \rfloor}}$ to $s < \frac{n+2g-1}{p^{\lfloor \frac{m}{2} \rfloor}+1}$. For example, with Hermitian codes, we ensure that we have self–orthogonal trace codes up to $s = q^2 - 2$, which is an improvement on $s = q + 1$ as in [22]. We obtain an such improvement as we work over the prime subfield F_p as opposed to [22] where they work over an extension field tied to their curve (more specifically the Castle curve).

Proposition 6 *Let $C_L(D, sQ)$ be an AG code from a self–orthogonal class defined over $\mathbb{F}_q$ where $q = p^m$. Let $s < \frac{n+2g-1}{p^{\lfloor \frac{m}{2} \rfloor}+1}$. Then*

$$Tr(C_L(D, sQ)) \subseteq Tr(C_L(D, sQ))^\perp = C_L(D, s^\perp Q)|\mathbb{F}_p.$$

Proof Lemma 4 implies that if $v_Q(f) + v_Q(g) < n + 2g - 1$ then

$$\sum_{i=1}^{n} f(P_i)g(P_i) = 0.$$

Let f, g be such that $v_Q(f), v_Q(g) < \frac{n+2g-1}{p^{\lfloor \frac{m}{2} \rfloor}+1}$. If $p^r \leq p^{\lfloor \frac{m}{2} \rfloor}$ then $v_Q(f) + v_Q(g^{p^r}) < \frac{n+2g-1}{p^{\lfloor \frac{m}{2} \rfloor}+1}(p^{\lfloor \frac{m}{2} \rfloor} + 1) = n + 2g - 1$. This implies that

$$\sum f(P_i)g(P_i)^{p^r} = 0.$$

If $p^{\lfloor \frac{m}{2} \rfloor} < p^r < q$ then note that $p^{m-r} < p^{\lfloor \frac{m}{2} \rfloor}$. In this case $(f^{p^{m-r}}) + (g) < \frac{n+2g-1}{p^{\lfloor \frac{m}{2} \rfloor}+1}(p^{\lfloor \frac{m}{2} \rfloor} + 1) = n + 2g - 1$. This implies that

$$\sum_{i=1}^{n} f(P_i)^{p^{m-r}} g(P_i) = 0.$$

Raising the sum to p^r implies that

$$\sum_{i=1}^{n} (f(P_i)g(P_i)^{p^r}) = 0.$$

As all p–powers of all codewords of $C_L(D, sQ)$ are orthogonal to each other, it follows from Lemma 1 that the code $\mathcal{T}r(C_L(D, sQ))$ is self–orthogonal.

That is

$$\mathcal{T}r(C_L(D, sQ)) \subseteq \mathcal{T}r(C_L(D, sQ))^{\perp} = C_L(D, s^{\perp}Q)|_{\mathbb{F}_p}$$

The proof of Proposition 6 follows from taking s to be small enough such that $v_Q(f) + p^{\lfloor \frac{m}{2} \rfloor} v_Q(g) < n + 2g - 1$. This bound was developed with codes over the prime subfield $\mathbb{F}_p$ in mind. In the case for Quaternary subfields (or $\mathbb{F}_{p^2}$ in general) we can improve the bound depending on the parity of m and Euclidean or Hermitian orthogonality.

Lemma 5 *Let $q = 4^m$ where $m = 2r + 1$ is odd. If $s \leq \frac{n+2g-1}{1+4^r}$ then the code $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ)$ is Euclidean self–orthogonal.*

If $s \leq \frac{n+2g-1}{1+2(4^r)}$ then the code $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ)$ is Hermitian self–orthgonal.

Proof For Euclidean self–orthgonality, this is precisely the statement of Proposition 6. For Hermitian self–orthogonality, we prove that

$$\sum_{i=1}^{n} f(P_i)^{4^a} g(P_i)^{2(4^j)} = 0.$$

Without loss of generality, we may assume $a = 0$. By hypothesis for $j \leq r$, we have $v_Q(fg^{2(4^j)}) < n + 2g - 1$. If $r + 1 \leq j \leq 2r$, then we consider $(fg^{2(4^j)})^{2(4^{2r-j})} = f^{2(4^{2r-j})g}$. Note that $0 \leq 2r - j \leq r - 1$, which means $f^{2(4^{2r-j})g}$ satisfies the conditions of the lemma.

If m is even, the bounds for Hermitian self–orthogonality are improved.

Lemma 6 *Let* $q = 4^m$ *where* $m = 2r$ *is even. If* $s \leq \frac{n+2g-1}{1+4^r}$ *then the code* $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ)$ *is Euclidean self–orthogonal.*

If

$$s \leq \frac{n+2g-1}{1+2(4^{r-1})}$$

then the code $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ)$ *is Hermitian self–orthgonal.*

Proof For Euclidean self–orthgonality, this is precisely the statement of Proposition 6. For Hermitian self–orthogonality, shall prove that

$$\sum_{i=1}^{n} f(P_i)^{4^a} g(P_i)^{2(4^j)} = 0.$$

Without loss of generality, we may assume $a = 0$. By hypothesis for $j \leq r-1$, we have $v_Q(fg^{2(4^j)}) < n + 2g - 1$. If $r \leq j \leq 2r-1$, then we consider $(fg^{2(4^j)})^{2(4^{2r-1-j})} = f^{2(4^{2r-1-j})g}$. Note that $0 \leq 2r-1-j \leq r-1$, which means that $f^{2(4^{2r-1-j})g}$ satisfies the conditions of the theorem

The bound on s is better for Euclidean orthogonality when working over odd extensions of $\mathbb{F}_4$ and the bound on s for Hermitian orthogonality is better when working over even extensions of $\mathbb{F}_4$.

Until now most constructions of Quantum Error Correcting Codes consider small subfields. Most AG codes are defined over $\mathbb{F}_{q^2}$ and the orthogonal pairs constructed are usually from the fields $\mathbb{F}_{q^2}$ or $\mathbb{F}_q$. We prove that it is possible to find a self–orthogonal code pair over $\mathbb{F}_2$ or $\mathbb{F}_4$ using AG codes. We shall construct binary and quaternary quantum codes from the following curves, which give self–orthogonal AG codes.

4 AG Curves for Binary Quantum Codes

Our methods work with the following curves:

- The Hermitian Curve: $x^{q+1} = y^q + y$ over $\mathbb{F}_{q^2}$
- The Norm–Trace curve $x^{\frac{q^r-1}{q-1}} - y^{q^{r-1}} - y^{q^{r-2}} - \cdots - y^q - y$ over $\mathbb{F}_{q^r}$
- The Quasi–Hermitian Curve $x^{q^r+1} - y^q - y$ over $\mathbb{F}_{q^{2r}}$
- The Generalized–Hermitian Curve $\sum\limits_{0 \leq i < j \leq r-1} x^{q^i+q^j} - \sum_{i=0}^{r-1} y^{q^i}$ over $\mathbb{F}_{q^{2r}}$
- The Suzuki curve $y^q - y = x^{q_0}(x^q - x)$ over $\mathbb{F}_q$ where $q0 = 2^n$ and $q = 2q_0^2$.
- The Giuletti–Korchmaros curve $y^{q+1} = x^q + x, z^{q^2-q+1} = y^{q^2} - y$ over $\mathbb{F}_{q^6}$.
- The Gunüeri–Garcia–Stichtenoth curve $y^{q+1} = x^q + x,\ z^{\frac{q^N+1}{q+1}} = y^{q^2} - y$ over $\mathbb{F}_{q^{2N}}$, N odd.

For some background work, we refer to [1, 2, 4, 5, 9, 17, 23, 25, 28, 29].

We compute the parameters of self–orthogonal trace codes of One–Point AG codes. The dimensions are computed directly and the minimum distances are given by the Feng–Rao bound.

Definition 7 (*(Hyperbolic Codes)* [12]) Let $V \subseteq \mathbb{F}_q^2$ be a set of finite points. Let $\delta \geq 2$ be an integer. Define $L_\delta = \{X^i Y^j \mid (i+1)(j+1) \leq \delta - 1\}$. The *Hyperbolic Code of designed distance* δ is the affine variety code

$$C(V, L_\delta) := \{ev_V(f) \mid f \in \langle L_\delta \rangle\}.$$

The hyperbolic code $C(V, L_\delta)$ is a linear code of distance δ. We shall take $L_\delta \subseteq \mathcal{L}(sQ)$ where $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ)$ is a self–orthogonal code. This implies that the subcode $C(V, L_\delta)$ is also self–orthogonal.

We use Propositon 6 and Lemmas 5 and 6 to establish that an AG code $C_L(D, sQ)$ satisfies $\mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ)) \subseteq \mathcal{T}r_{\mathbb{F}_q/\mathbb{F}_{q_0}}(C_L(D, sQ))^\perp$ thereby giving a self–orthogonal code pair. Corollary 3 implies self–orthogonality is preserved for subcodes $D \subseteq C_L(D, sQ)$. In particular, hyperbolic codes have a designed minimum distance while using fewer parity check equations than $C_L(D, sQ)$, Now we present several code parameters found by explicit computations of Trace codes of Hyperbolic codes. Our improvement is twofold. First, we increase the threshold for the pole order of two self–orthogonal functions. Second, we use hyperbolic codes instead of AG codes. In several cases, a Hyperbolic codes achieves a distance bound with fewer parity check equations (and thus better dimension!) as the corresponding AG code. Now we present selected quaternary self–orthogonal codes from AG curves and their resulting binary quantum codes. The dimension is found by direct computation and the distance is the designed distance of the Hyperbolic codes.

5 Quantum Codes from Hyperbolic Sets

Here we present the parameters of several Binary self–orthogonal codes, Euclidean Quaternary self–orthogonal codes and Hermitian self–orthogonal codes obtained from Hyperbolic codes evaluated from several AG curves. The following curves showcase the improvements obtained by taking quaternary codes and changing the type of orthogonality, depending on the field extension degree of the alphabet of the original code. Hyperbolic codes over large fields have been well studied (e.g., [6, 11]). Quantum codes over large fields from hyperbolic codes have also been obtained (see [6]) (Tables 1, 2, 3, 4, 5, 6, 7 and 8).

Table 1 Self–orthogonal hyperbolic codes from $x^3 = y^2 + y$

Hermitian curve $x^3 = y^2 + y$ over $\mathbb{F}_4$		
Binary code	Euclidean code	Hermitian code
[8, 8, 1]	[8, 8, 1]	[8, 8, 1]
[8, 7, 2]	[8, 7, 2]	[8, 7, 2]
N/A	[8, 5, 3]	N/A
N/A	[8, 4, 4]	N/A

Table 2 Binary quantum error correcting codes from $x^3 = y^2 + y$

Hermitian curve $x^3 = y^2 + y$ over $\mathbb{F}_4$		
Binary code	Euclidean code	Hermitian code
[[8, 8, 1]]	[[8, 8, 1]]	[[8, 8, 1]]
[[8, 6, 2]]	[[8, 6, 2]]	[[8, 6, 2]]
N/A	[[8, 2, 3]]	N/A
N/A	[[8, 0, 4]]	N/A

Table 3 Self–orthogonal Hyperbolic codes from $x^5 = y^4 + y$

Hermitian curve $x^5 = y^4 + y$ over $\mathbb{F}_{16}$		
Binary code	Euclidean code	Hermitian code
$[64, 64, 1]_2$	$[64, 64, 1]_4$	$[64, 64, 1]_4$
$[64, 63, 2]_2$	$[64, 63, 2]_4$	$[64, 63, 2]_4$
$[64, 59, 3]_2$	$[64, 59, 3]_4$	$[64, 59, 3]_4$
$[64, 55, 4]_2$	$[64, 55, 4]_4$	$[64, 55, 4]_4$
N/A	N/A	$[64, 49, 6]_4$
N/A	N/A	$[64, 43, 8]_4$

Table 4 Binary quantum error correcting codes from $x^5 = y^4 + y$

Hermitian curve $x^5 = y^4 + y$ over $\mathbb{F}_{16}$		
Binary code	Euclidean code	Hermitian code
$[[64, 64, 1]]_2$	$[[64, 64, 1]]_2$	$[[64, 64, 1]]_2$
$[[64, 62, 2]]_2$	$[[64, 62, 2]]_2$	$[[64, 62, 2]]_2$
$[[64, 54, 3]]_2$	$[[64, 54, 3]]_2$	$[[64, 54, 3]]_2$
$[[64, 46, 4]]_2$	$[[64, 46, 4]]_2$	$[[64, 46, 4]]_2$
N/A	N/A	$[[64, 34, 6]]_4$
N/A	N/A	$[[64, 22, 8]]_4$

Table 5 Self–orthogonal Hyperbolic codes from $x^{21} = y^{16} + y^4 + y$

Norm–trace curve $x^{21} = y^{16} + y^4 + y$ over $\mathbb{F}_{64}$		
Binary code	Euclidean code	Hermitian code
N/A	$[1024, 1017, 3]_4$	$[1024, 1017, 3]_4$
$[1024, 1011, 4]_2$	$[1024, 1011, 4]_4$	$[1024, 1011, 4]_4$
$[1024, 993, 6]_2$	$[1024, 1002, 6]_4$	$[1024, 1002, 6]_4$
N/A	$[1024, 990, 7]_4$	$[1024, 990, 7]_4$
$[1024, 969, 8]_2$	$[1024, 984, 8]_4$	$[1024, 984, 8]_4$
N/A	$[1024, 972, 9]_4$	N/A
N/A	$[1024, 969, 10]_4$	N/A
N/A	$[1024, 957, 11]_4$	N/A
N/A	$[1024, 951, 12]_4$	N/A
N/A	$[1024, 933, 14]_4$	N/A

Table 6 Binary quantum error correcting codes from $x^{21} = y^{16} + y^4 + y$

Norm–trace curve $x^{21} = y^{16} + y^4 + y$ over $\mathbb{F}_{64}$		
Binary code	Euclidean code	Hermitian code
N/A	$[[1024, 1010, 3]]_2$	$[[1024, 1010, 3]]_2$
$[[1024, 998, 4]]_2$	$[[1024, 998, 4]]_2$	$[[1024, 998, 4]]_2$
$[[1024, 963, 6]]_2$	$[[1024, 980, 6]]_2$	$[[1024, 980, 6]]_2$
N/A	$[[1024, 956, 7]]_2$	$[[1024, 956, 7]]_2$
$[[1024, 915, 8]]_2$	$[[1024, 944, 8]]_2$	$[[1024, 944, 8]]_2$
N/A	$[[1024, 920, 9]]_2$	N/A
N/A	$[[1024, 914, 10]]_2$	N/A
N/A	$[[1024, 890, 11]]_2$	N/A
N/A	$[[1024, 878, 12]]_2$	N/A
N/A	$[[1024, 842, 14]]_2$	N/A

Table 7 Self–orthogonal hyperbolic codes from quasi–hermitian curve $x^{17} = y^4 + y$

Quasi–hermitian curve $x^{17} = y^4 + y$ over $\mathbb{F}_{256}$		
Binary code	Euclidean code	Hermitian code
N/A	$[1024, 1016, 3]_4$	$[1024, 1016, 3]_4$
$[1024, 1009, 4]_2$	$[1024, 1009, 4]_4$	$[1024, 1009, 4]_4$
$[1024, 985, 6]_2$	$[1024, 997, 6]_4$	$[1024, 997, 6]_4$
N/A	$[1024, 985, 7]_4$	$[1024, 985, 7]_4$
$[1024, 961, 8]_2$	$[1024, 981, 8]_4$	$[1024, 981, 8]_4$
N/A	$[1024, 969, 9]_4$	$[1024, 969, 9]_4$
$[1024, 937, 10]_2$	$[1024, 965, 10]_4$	$[1024, 965, 10]_4$
N/A	$[1024, 977, 11]_4$	$[1024, 977, 11]_4$
$[1024, 921, 12]_2$	$[1024, 953, 12]_4$	$[1024, 953, 12]_4$
$[1024, 889, 14]_2$	$[1024, 937, 14]_4$	$[1024, 937, 14]_4$
N/A	$[1024, 929, 15]_4$	$[1024, 929, 15]_4$
$[1024, 873, 16]_2$	$[1024, 921, 16]_4$	$[1024, 921, 16]_4$
N/A	N/A	$[1024, 909, 18]_4$
N/A	N/A	$[1024, 902, 19]_4$
N/A	N/A	$[1024, 897, 20]_4$
N/A	N/A	$[1024, 885, 21]_4$
N/A	N/A	$[1024, 881, 22]_4$
N/A	N/A	$[1024, 877, 23]_4$
N/A	N/A	$[1024, 873, 24]_4$
N/A	N/A	$[1024, 857, 26]_4$
N/A	N/A	$[1024, 853, 27]_4$
N/A	N/A	$[1024, 845, 28]_4$
N/A	N/A	$[1024, 833, 30]_4$
N/A	N/A	$[1024, 825, 31]_4$
N/A	N/A	$[1024, 821, 32]_4$

Table 8 Binary quantum error correcting codes from quasi–hermitian curve $x^{17} = y^4 + y$

Quasi–hermitian curve $x^{17} = y^4 + y$ over $\mathbb{F}_{256}$		
Binary code	Euclidean code	Hermitian code
N/A	$[[1024, 1008, 3]]_2$	$[[1024, 1008, 3]]_2$
$[[1024, 994, 4]]_2$	$[[1024, 994, 4]]_2$	$[[1024, 994, 4]]_2$
$[[1024, 946, 6]]_2$	$[[1024, 970, 6]]_2$	$[[1024, 970, 6]]_2$
N/A	$[[1024, 946, 7]]_2$	$[[1024, 946, 7]]_2$
$[[1024, 898, 8]]_2$	$[[1024, 938, 8]]_2$	$[[1024, 938, 8]]_2$
N/A	$[[1024, 914, 9]]_2$	$[[1024, 914, 9]]_2$
$[[1024, 850, 10]]_2$	$[[1024, 906, 10]]_2$	$[[1024, 906, 10]]_2$
N/A	$[[1024, 890, 11]]_2$	$[[1024, 890, 11]]_2$
$[[1024, 818, 12]]_2$	$[[1024, 882, 12]]_2$	$[[1024, 882, 12]]_2$
$[[1024, 754, 14]]_2$	$[[1024, 850, 14]]_2$	$[[1024, 850, 14]]_2$
N/A	$[[1024, 825, 15]]_2$	$[[1024, 825, 15]]_2$
$[[1024, 722, 16]]_2$	$[[1024, 821, 16]]_2$	$[[1024, 821, 16]]_2$
N/A	N/A	$[[1024, 794, 18]]_2$
N/A	N/A	$[[1024, 778, 19]]_2$
N/A	N/A	$[[1024, 770, 20]]_2$
N/A	N/A	$[[1024, 746, 21]]_2$
N/A	N/A	$[[1024, 738, 22]]_2$
N/A	N/A	$[[1024, 730, 23]]_2$
N/A	N/A	$[[1024, 722, 24]]_4$
N/A	N/A	$[[1024, 690, 26]]_2$
N/A	N/A	$[[1024, 682, 27]]_2$
N/A	N/A	$[[1024, 666, 28]]_2$
N/A	N/A	$[[1024, 642, 30]]_2$
N/A	N/A	$[[1024, 626, 31]]_2$
N/A	N/A	$[[1024, 618, 32]]_2$

Acknowledgements The authors would like to thank Eddie Arrieta for his insightful comments and fruitful conversations. The authors would also like to express their gratitude to the anonymous referees, whose comments have improved our article.

References

1. Ashikhmin, A., Knill, E.: Nonbinary quantum stabilizer codes. IEEE Trans. Inf. Theory **47**(7), 3065–3072 (2001)
2. Ashikhmin, A.E., Barg, A.M., Knill, E., Litsyn, S.N.: Quantum error detection. I. Statement of the problem. IEEE Trans. Inf. Theory **46**(3), 778–788 (2000)

3. Bartoli, D., Montanucci, M., Zini, G.: Ag codes and ag quantum codes from the ggs curve. Des. Codes Crypt. **86**(10), 2315–2344 (2018)
4. Calderbank, A.R., Rains, E.M., Shor, P.W., Sloane, N.J.A.: Quantum error correction and orthogonal geometry. Phys. Rev. Lett. **78**(3), 405–408 (1997)
5. Calderbank, A.R., Rains, E.M., Shor, P.W., Sloane, N.J.A.: Quantum error correction via codes over GF(4). IEEE Trans. Inf. Theory **44**(4), 1369–1387 (1998)
6. Christensen, R.B., Geil, O.: On steane-enlargement of quantum codes from cartesian product point sets. Quantum Inf. Process. **19**(7), 192 (2020)
7. Elezi, A.: Error correcting quantum codes and algebraic curves. In: Advances on Superelliptic Curves and Their Applications, volume 41 of NATO Science for Peace and Security Series D: Information and Communication Security, pp. 286–304. IOS, Amsterdam (2015)
8. Elezi, A., Shaska, T.: Quantum codes from superelliptic curves. Albanian J. Math. **5**(4), 175–191 (2011)
9. Fletcher, A.S., Shor, P.W., Win, M.Z.: Channel-adapted quantum error correction for the amplitude damping channel. IEEE Trans. Inf. Theory **54**(12), 5705–5718 (2008)
10. Galindo, C., Hernando, F., Matsumoto, R., Ruano, D.: Entanglement-assisted quantum error-correcting codes over arbitrary finite fields. Quantum Inf. Process. **18**(4), 116 (2019)
11. Geil, O.: On codes from norm-trace curves. Finite Fields Their Appl. **9**(3), 351–371 (2003)
12. Høholdt, T., van Lint, J.H., Pellikaan, R.: Order functions and evaluation codes. In: Mora, T., Mattson, H.F. (eds.) Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, 12th International Symposium, AAECC-12, Toulouse, France, June 23-27, 1997, Proceedings, volume 1255 of Lecture Notes in Computer Science, pp. 138–150. Springer, Berlin (1997)
13. Janwa, H., Piñero, F.: On parameters of subfield subcodes of extended norm-trace codes (2016). arXiv:1604.05777
14. Janwa, H., Piñero, F.: On the parameters of subfield subcodes of norm trace codes. In: Advances in Mathematics of Communications (AMC), pp. 379–388 (2017)
15. Janwa, H., Piñero, F., Arrieta, E.: Quantum error-correcting codes over small fields from ag curves. In: Springer Proceedings in Mathematics and Statistics (submitted), pp. 1–32 (2022)
16. Jin, L.: Quantum stabilizer codes from maximal curves. IEEE Trans. Inf. Theory **60**(1), 313–316 (2014)
17. Jordan, S.P., Farhi, E., Shor, P.W.: Error-correcting codes for adiabatic quantum computation. Phys. Rev. A (3) **74**(5), 052322 (2006)
18. Kim, J.-L., Matthews, G.L.: Quantum error-correcting codes from algebraic curves. In: Advances in Algebraic Geometry Codes, Volume 5 of Series Coding Theory; Cryptology, pp. 419–444. World Sci. Publ., Hackensack, NJ (2008)
19. Kim, J.-L., Walker, J.: Nonbinary quantum error-correcting codes from algebraic curves. Discret. Math. **308**(14), 3115–3124 (2008)
20. La Guardia, G.G., Pereira, F.R.F.: Good and asymptotically good quantum codes derived from algebraic geometry. Quantum Inf. Process. **16**(6), 165 (2017)
21. Montanucci, M., Timpanella, M., Zini, G.: Ag codes and ag quantum codes from cyclic extensions of the suzuki and ree curves. J. Geom. **109**(1), 23 (2018)
22. Munuera, C., Tenório, W., Torres, F.: Quantum error-correcting codes from algebraic geometry codes of castle type. Quantum Inf. Process. **15**(10), 4071–4088 (2016)
23. Munuera, C., Tenório, W., Torres, F.: Quantum error-correcting codes from algebraic geometry codes of Castle type. Quantum Inf. Process. **15**(10), 4071–4088 (2016)
24. Piñero, F., Janwa, H.: On the subfield subcodes of hermitian codes. Des. Codes Crypt. **70**(1–2), 157–173 (2014)
25. Pollatsek, H.: Quantum error correction: classic group theory meets a quantum challenge. Am. Math. Monthly **108**(10), 932–962 (2001)
26. Post, W.: Washington post live. quantum computers are a few years, no longer a few decades (2021). https://www.youtube.com/watch?v=74anf6Lmms
27. Sarvepalli, P.K., Klappenecker, A.: Nonbinary quantum codes from hermitian curves. In: Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, Volume 3857 of Lecture Notes in Computer Science, pp. 136–143. Springer, Berlin (2006)

28. Shor, P.W.: Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM Rev. **41**(2), 303–332 (1999)
29. Shor, P.W., Smith, G., Smolin, J.A., Zeng, B.: High performance single-error-correcting quantum codes for amplitude damping. IEEE Trans. Inf. Theory **57**(10), 7180–7188 (2011)
30. Stichtenoth, H.: On the dimension of subfield subcodes. IEEE Trans. Inf. Theory **36**(1), 90–93 (1990)
31. Stichtenoth, H.: Algebraic Function Fields and Codes, 2nd edn. Springer Publishing Company, Incorporated (2008)

A Go-Up Code Construction from Linear Codes Yielding Additive Codes for Quantum Stabilizer Codes

Arrieta A. Eddie and Heeralal Janwa

Abstract Given a code C over the finite field $\mathbb{F}_q$, where q is a power of a prime number, some constructions exist that permit us to obtain a new code from C over $\mathbb{F}_q$ or over a subfield of $\mathbb{F}_q$, such as subfield subcodes. However in some important applications, one needs codes over an extension field, for example in quantum error-correcting codes (QECC). We propose a technique that we call **Go-Up** construction, which allows us to obtain an additive or a linear code over $\mathbb{F}_{q^m}$ from any set of m linear codes over $\mathbb{F}_q$. We show under what condition this code is a self-orthogonal or self-dual code. Thus we are able to give new constructions of quantum stabilizer codes from our codes that are additive. We present several such classes of QECC. Our GU codes also have applications to algebraic coding theory, finite geometries, finite group theory, and also to combinatorial objects such as strongly regular graphs, and few-weight codes (see [3]).

Keywords Go-Up construction · Self-orthogonality · Quantum stabilizer codes · Quantum error correction

1 Introduction

We recall here some background from classical coding theory, see [18, 28], and for finite fields the encyclopedia [17]. A code C over the finite field $\mathbb{F}_q$ of length n, with M codewords and minimum Hamming distance d is a collection of M elements in $\mathbb{F}_q^n$ such that any two differ in at least d positions. We call the code additive if it is a subgroup of $\mathbb{F}_q^n$, i.e., when the coordinatewise sum of two codewords again is a codeword, and we call the code linear when it is a subspace of $\mathbb{F}_q^n$.

A. A. Eddie (✉)
Inter American University of Puerto Rico, Bayamón, PR 00957, USA
e-mail: earrieta@bayamon.inter.edu

H. Janwa
University of Puerto Rico at Rio Piedras, San Juan, PR 00925, USA
e-mail: heeralal.janwa@upr.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_37

Given a code C over $\mathbb{F}_q$, there are constructions that permit us to obtain a new code from C, for example, extension, shortening, lengthening, puncturing, and concatenation on the same field (see [6, 18]). Subfield subcodes and trace codes yield codes over a subfield of $\mathbb{F}_q$ (see [10, 26]). In some cases, one needs to construct codes from C to an extension field by extending scalars [11] or by lifting [21]. Relationship between parameters of subfield subcodes and trace codes is given in Delsarte [10, 26, 27].

However, for applications to quantum stabilizer codes (QSC), we need to construct a new code over an extension field $\mathbb{F}_{q^m}$ from a class of codes over $\mathbb{F}_q$. Such a code need only be an additive code (see Theorem 5). With this aim, in this article, we take a family of m codes C_i, $0 \leq i \leq m-1$, over $\mathbb{F}_q$ and construct additive codes in $\mathbb{F}_{q^m}$; we call this a **Go-Up** construction (see Definition 2). Then we determine the parameters of $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$. From the resulting additive codes in $\mathbb{F}_{q^m}$, we construct quantum stabilizer codes over F_q. We note that in the very special case, when all the codes are the same, then the code we obtain is lifting of C_0.

We recall that

$$\mathbb{F}_{q^m} = \mathbb{F}_q[x]/(p(x))$$

where $p(x) \in F_q[x]$ is a monic irreducible polynomial over F_q of degree m. We recall from Galois theory, a finite field with q^m elements is unique up to isomorphism. However, for combinatorial structures, and for complexity of computations the multiplicative and additive structure depend upon the specific irreducible or primitive polynomials $p(x)$, to impart a concrete representation to the quotient ring structure.

To illustrate this, for one of the important case in this article, when $m = 2$, we write it $p(x) = x^2 + \beta x + \alpha$. We take $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $p(\delta) = 0$, that is,

$$\delta^2 = -\alpha - \delta\beta$$

For $\lambda = \lambda_1 + \delta\lambda_2$ and $\theta = \theta_1 + \delta\theta_2$ in $\mathbb{F}_{q^2}$,

$$\begin{aligned}\lambda\theta &= (\lambda_1 + \delta\lambda_2)(\theta_1 + \delta\theta_2) = (\lambda_1 + \lambda_2 x)(\theta_1 + \theta_2 x) mod(p(x))\\ &= \lambda_1\theta_1 - \lambda_2\alpha\theta_2 + \delta(\lambda_2\theta_1 + (\lambda_1 - \lambda_2\beta)\theta_2)\end{aligned}$$

Unless otherwise specified, we write $\mathbb{F}_{q^2} = \mathbb{F}(\delta)$.

In our case the scalar multiplication depends on the polynomial $p(x)$ and the prime number p. For example, if $p(x) = x^2 + x + 1$, that is, $\alpha = 1$ and $\beta = 1$, we get

$$\lambda\theta = \lambda_1\theta_1 - \lambda_2\theta_2 + \delta(\lambda_2\theta_1 + (\lambda_1 - \lambda_2)\theta_2)$$

For the binary case, $x^2 + x + 1$ is irreducible over $\mathbb{F}_2$ and we take $\delta \in \mathbb{F}_4$ such that $\delta^2 + \delta + 1 = 0$, then

$$\lambda\theta = \lambda_1\theta_1 + \lambda_2\theta_2 + \delta(\lambda_2\theta_1 + (\lambda_1 + \lambda_2)\theta_2)$$

For $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{F}_q^n$, we define

$$\mathbf{a} + \delta\mathbf{b} = (a_1 + \delta b_1, \ldots, a_n + \delta b_n) \in \mathbb{F}_{q^2}^n.$$

A Goppa code $\mathbf{\Gamma}(L, g(x))$ is defined by the Goppa polynomial $g(x)$ of degree t over the extension field $\mathbb{F}_{q^m}$ and an accessory subset L of $\mathbb{F}_{q^m}$ (see [18, 19].) Let

$$g(x) = g_0 + g_1 x + \cdots + g_t x^t \in \mathbb{F}_{q^m}[x] \;\; and \;\; L = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{F}_{q^m}$$

be such that $g(\alpha_i) \neq 0$ for all $\alpha_i \in L$. With a vector $\mathbf{c} = (c_1, \ldots, c_n)$ in $\mathbb{F}_q^n$ we associate the function

$$R_c(x) = \sum_{i=1}^{n} \frac{c_i}{x - \alpha_i}$$

where $\frac{1}{x-\alpha_i}$ is the unique polynomial such that $(x - \alpha_i)\frac{1}{x-\alpha_i} \equiv 1 \pmod{g(x)}$. We observe that

$$\frac{1}{x - \alpha_i} = \frac{g(\alpha_i) - g(x)}{x - \alpha_i} g(\alpha_i)^{-1} \pmod{g(x)}$$

Definition 1 The Goppa code $\mathbf{\Gamma}(L, g(x)) \subset \mathbb{F}_q^n$ consists of all vectors $\mathbf{c} = (c_1, \ldots, c_n)$ such that

$$R_c(x) \equiv 0 \pmod{g(x)}.$$

Theorem 1 *The Goppa code $\mathbf{\Gamma}(L, g(x))$ of size n and polynomial of degree t is an $[n, k, d]_q$ linear code over $\mathbb{F}_q$ such that $k \geq n - mt$ and $d \geq t + 1$.*

The article is organized as follows. In Sect. 1, we define our Go-Up construction (Definition 2). In Section 3, we give applications of our method to construct quantum stabilizer error correcting codes (Theorem 6). Section 4 shows some explicit examples of stabilizer codes.

2 Go-Up Construction ($\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$)

For the properties and applications of GU codes introduced in this article, we need an explicit representation and reformulation of the Euclidean, Hermitian, and trace Hermitian inner products.

2.1 A Reformulation of the Euclidean, Hermitian, and Trace Hermitian Inner Products

We illustrate again with $m = 2$, where $p(x) = x^2 + \beta x + \alpha$ is irreducible over $\mathbb{F}_q$.

Let $\mathbf{x} = \mathbf{a} + \delta\mathbf{b}$, $\mathbf{y} = \mathbf{c} + \delta\mathbf{d}$ be elements of $\mathbb{F}_{q^2}^n$ and

$$tr : \mathbb{F}_{q^2} \to \mathbb{F}_q, \quad x \to x + \overline{x}$$

the trace of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, where $\overline{x} = x^q$. If δ is a primitive element of $\mathbb{F}_{q^2}$, i.e., $p(x)$ is a primitive polynomial, the set $\{1, \ \delta\}$ is a polynomial basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ because the matrix

$$A = \begin{bmatrix} 1 & \delta \\ 1 & \delta^q \end{bmatrix}$$

is such that $det(A) = \delta^q - \delta \neq 0$. If $\delta^q - \delta = 0$, then $\delta^{q-1} = 1$ which is not possible, because δ is primitive and $q - 1 < q^2 - 1$ (see [17]).

Observe that the linear operator

$$L_\delta : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}, \quad x \to \delta x$$

has the matrix

$$[L_\delta] = \begin{bmatrix} 0 & -\alpha \\ 1 & -\beta \end{bmatrix}$$

as a representation over the polynomial basis. Then

$$tr(\delta) = \delta + \delta^q = tr[L_\delta] = -\beta$$

and

$$N(\delta) = \delta\delta^q = \delta^{q+1} = det[L_\delta] = \alpha$$

Therefore we get:

a. *Euclidean inner product:*

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} := \sum_i x_i y_i &= (\mathbf{a} + \delta\mathbf{b}) \cdot (\mathbf{c} + \delta\mathbf{d}) \\ &= (\mathbf{a} \cdot \mathbf{c} - \alpha\mathbf{b} \cdot \mathbf{d}) + \delta(\mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} - \beta\mathbf{b} \cdot \mathbf{d}) \end{aligned} \tag{1}$$

It is bilinear over $\mathbb{F}_{q^2}$ and symmetric. By linearity, for any subset C in $\mathbb{F}_{q^2}^n$,

$$C^{\perp_E} = \{\boldsymbol{y} = \boldsymbol{c} + \delta\boldsymbol{d} \in \mathbb{F}_{q^2}^n \mid \boldsymbol{x} \cdot \boldsymbol{y} = 0 \ \ \forall \boldsymbol{x} \in C\}$$

is linear over $\mathbb{F}_{q^2}$.

b. *Trace of the Euclidean inner product:*

$$\begin{aligned}\mathbf{x} \cdot_T \mathbf{y} &:= tr[(\mathbf{a} + \delta\mathbf{b}) \cdot (\mathbf{c} + \delta\mathbf{d})] \\ &= 2(\mathbf{a} \cdot \mathbf{c} - \alpha\mathbf{b} \cdot \mathbf{d}) + (\delta + \delta^q)(\mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} - \beta\mathbf{b} \cdot \mathbf{d})\end{aligned}$$

That is,

$$\mathbf{x} \cdot_T \mathbf{y} = 2(\mathbf{a} \cdot \mathbf{c} - \alpha\mathbf{b} \cdot \mathbf{d}) - \beta(\mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} - \beta\mathbf{b} \cdot \mathbf{d}) \tag{2}$$

In general, $C^{\perp_T} = \{\boldsymbol{y} = \boldsymbol{c} + \delta\boldsymbol{d} \in \mathbb{F}_{q^2}^n \mid \boldsymbol{x} \cdot_T \boldsymbol{y} = 0 \ \ \forall\boldsymbol{x} \in C\}$ is not linear over $\mathbb{F}_{q^2}$ because the trace is not. Since

$$\boldsymbol{x} \cdot_T (\delta\boldsymbol{y}) = tr(\boldsymbol{x} \cdot (\delta\boldsymbol{y})) = tr((\delta\boldsymbol{x}) \cdot \boldsymbol{y}) = \delta\boldsymbol{x} \cdot_T \boldsymbol{y},$$

if $C \subset \mathbb{F}_{q^2}^n$ is linear over $\mathbb{F}_{q^2}$, then $C^{\perp_T}$ is linear over $\mathbb{F}_{q^2}$.

c. *Hermitian inner product:*

$$\begin{aligned}\mathbf{x} \cdot_H \mathbf{y} &:= \sum_i x_i\overline{y_i} = (\mathbf{a} + \delta\mathbf{b}) \cdot (\mathbf{c} + \delta^q\mathbf{d}) \\ &= \mathbf{a} \cdot \mathbf{c} + \delta^q\mathbf{a} \cdot \mathbf{d} + \delta\mathbf{b} \cdot \mathbf{c} + \delta^{q+1}\mathbf{b} \cdot \mathbf{d} = a \cdot \mathbf{c} \\ &\quad + (\delta^q + \delta - \delta)\mathbf{a} \cdot \mathbf{d} + \delta\mathbf{b} \cdot \mathbf{c} + \alpha\mathbf{b} \cdot \mathbf{d}\end{aligned}$$

That is,

$$\mathbf{x} \cdot_H \mathbf{y} = (\mathbf{a} + \delta\mathbf{b}) \cdot (\mathbf{c} + \delta^q\mathbf{d}) = (\mathbf{a} \cdot \mathbf{c} + \alpha\mathbf{b} \cdot \mathbf{d} - \beta\mathbf{a} \cdot \mathbf{d}) + \delta(\mathbf{b} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{d}) \tag{3}$$

Since $(\delta\boldsymbol{x}) \cdot_H \boldsymbol{y} = \delta(\boldsymbol{x} \cdot_H \boldsymbol{y})$ and $\boldsymbol{x} \cdot_H (\delta\boldsymbol{y}) = \overline{\delta}(\boldsymbol{x} \cdot_H \boldsymbol{y}) = \delta^q(\boldsymbol{x} \cdot_H \boldsymbol{y})$, the Hermitian inner product is linear in the first coordinate and conjugate linearity in the second. From the conjugate linearity we get that for any subset C of $\mathbb{F}_{q^2}^n$

$$C^{\perp_H} = \{\boldsymbol{y} = \boldsymbol{c} + \delta\boldsymbol{d} \in \mathbb{F}_{q^2}^n \mid \boldsymbol{x} \cdot_H \boldsymbol{y} = 0 \ \ \forall\boldsymbol{x} \in C\}$$

is linear over $\mathbb{F}_{q^2}$.

d. *Trace of the Hermitian inner product:*

$$\mathbf{x} \cdot_{TH} \mathbf{y} := tr(\mathbf{x} \cdot_H \mathbf{y}) = 2(\mathbf{a} \cdot \mathbf{c} + \delta^{q+1}\mathbf{b} \cdot \mathbf{d}) + (\delta + \delta^q)(\mathbf{b} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{d})$$

That is,

$$\mathbf{x} \cdot_{TH} \mathbf{y} = 2(\mathbf{a} \cdot \mathbf{c} + \alpha\mathbf{b} \cdot \mathbf{d}) + \beta(\mathbf{a} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{c}) \tag{4}$$

Then for any subset C in $\mathbb{F}_{q^2}$, $C^{\perp_{TH}} = \{\boldsymbol{y} = \boldsymbol{c} + \delta \boldsymbol{d} \in \mathbb{F}_{q^2}^n \mid \boldsymbol{x} \cdot_{TH} \boldsymbol{y} = 0 \ \forall \boldsymbol{x} \in C\}$ is not linear over $\mathbb{F}_{q^2}$. Since

$$\boldsymbol{x} \cdot_{TH} (\delta \boldsymbol{y}) = tr(\boldsymbol{x} \cdot_H (\delta \boldsymbol{y})) = tr((\overline{\delta} \boldsymbol{x}) \cdot_H \boldsymbol{y}) = (\overline{\delta} \boldsymbol{x}) \cdot_{TH} \boldsymbol{y},$$

if $C \subset \mathbb{F}_{q^2}^n$ is linear over $\mathbb{F}_{q^2}$, then $C^{\perp_{TH}}$ is linear over $\mathbb{F}_{q^2}$.

2.2 $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$ *Codes*

Definition 2 Let q be a power of a prime number p, take linear codes C_i in $\mathbb{F}_q^n$ with $0 \le i \le m-1$ and $\{1, \ \delta, \ \ldots, \ \delta^{m-1}\}$ being a polynomial basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. We define a **Go-Up** code over $\mathbb{F}_{q^m}$, denoted $\mathbf{GU}(C_0, C_1, \cdots, C_{m-1})$, by setting it equal to

$$\{\boldsymbol{a}_0 + \delta \boldsymbol{a}_1 + \cdots + \delta^{m-1} \boldsymbol{a}_{m-1} \mid \boldsymbol{a}_i \in C_i, \quad 0 \le i \le m-1\}$$

This definition is in a sense gluing or *an amalgamation of the m codes* $C_i's$. When all the m codes are the same code C_0 we denote the Go-Up by $\mathbf{GU}(m, \ C_0)$.

Example 1 To get an insight into our construction, we take elementary but important codes to begin with. We take the dual of the binary Hamming code, that is, C_0 is the binary Simplex code $S_3(2)$ with parameters $[7, \ 3, \ 4]_2$ and $C_1 = C_0^{\perp_E}$ the binary Hamming code $H_3(2)$ (see [6]). We obtain $\mathbf{GU}(\mathbf{C_0}, \mathbf{C_1})$. We first observe the following. The code $\mathbf{GU}(2, \ C_0) = \mathbf{GU}(2, \ S_3(2))$ is given by Table 1.

$C_0 = S_3(2)$			
(0, 0, 0, 0, 0, 0, 0)	(0, 1, 1, 1, 1, 0, 0)	(1, 0, 1, 1, 0, 1, 0)	(1, 1, 0, 1, 0, 0, 1)
(1, 1, 0, 0, 1, 1, 0)	(1, 0, 1, 0, 1, 0, 1)	(0, 1, 1, 0, 0, 1, 1)	(0, 0, 0, 1, 1, 1, 1)

Now, $C_1 = H_3(2)$ is the $[7, \ 4, \ 3]_2$ binary Hamming code, where we use the following matrix as a parity check matrix for the Hamming code.

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Since $H_3(2) = S_3(2) \cup (\boldsymbol{a} + S_3(2))$, we have

$$\mathbf{GU}(C_0, \ C_1) = S_3(2) + \delta H_3(2) = (S_3(2) + \delta S_3(2)) \cup (S_3(2) + \delta(\boldsymbol{a} + S_3(2)))$$

Taking $\boldsymbol{a} = (1, \ 1, \ 1, \ 0, \ 0, \ 0, \ 0) \in H_3(2)$, Table 2 gives us the elements of the coset $\boldsymbol{a} + S_3(2)$

The elements of $S_3(2) + \delta S_3(2)$ are given in Table 1 and the elements of $S_3(2) + \delta(\boldsymbol{a} + S_3(2))$ are given in Table 3

Table 1 $\mathbf{GU}(2,\ S_3(2))$

$C_0+\delta C_0$			
$(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)$	$(0,\ \delta,\ \delta,\ \delta,\ \delta,\ 0,\ 0)$	$(\delta,\ 0,\ \delta,\ \delta,\ 0,\ \delta,\ 0)$	$(\delta,\ \delta,\ 0,\ \delta,\ 0,\ 0,\ \delta)$
$(\delta,\ \delta,\ 0,\ 0,\ \delta,\ \delta,\ 0)$	$(\delta,\ 0,\ \delta,\ 0,\ \delta,\ 0,\ \delta)$	$(0,\ \delta,\ \delta,\ 0,\ 0,\ \delta,\ \delta)$	$(0,\ 0,\ 0,\ \delta,\ \delta,\ \delta,\ \delta)$
$(0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0)$	$(0,\ \delta^2,\ \delta^2,\ \delta^2,\ \delta^2,\ 0,\ 0)$	$(\delta,\ 1,\ \delta^2,\ \delta^2,\ 1,\ \delta,\ 0)$	$(\delta,\ \delta^2,\ 1,\ \delta^2,\ 1,\ 0,\ \delta)$
$(\delta,\ \delta^2,\ 1,\ 1,\ \delta^2,\ \delta,\ 0)$	$(\delta,\ 1,\ \delta^2,\ 1,\ \delta^2,\ 0,\ \delta)$	$(0,\ \delta^2,\ \delta^2,\ 1,\ 1,\ \delta,\ \delta)$	$(0,\ 1,\ 1,\ \delta^2,\ \delta^2,\ \delta,\ \delta)$
$(1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 0)$	$(1,\ \delta,\ \delta^2,\ \delta^2,\ \delta,\ 1,\ 0)$	$(\delta^2,\ 0,\ \delta^2,\ \delta^2,\ 0,\ \delta^2,\ 0)$	$(\delta^2,\ \delta,\ 1,\ \delta^2,\ 0,\ 1,\ \delta)$
$(\delta^2,\ \delta,\ 1,\ 1,\ \delta,\ \delta^2,\ 0)$	$(\delta^2,\ 0,\ \delta^2,\ 1,\ \delta,\ 1,\ \delta)$	$(1,\ \delta,\ \delta^2,\ 1,\ 0,\ \delta^2,\ \delta)$	$(1,\ 0,\ 1,\ \delta^2,\ \delta,\ \delta^2,\ \delta)$
$(1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1)$	$(1,\ \delta^2,\ \delta,\ \delta^2,\ \delta,\ 0,\ 1)$	$(\delta^2,\ 1,\ \delta,\ \delta^2,\ 0,\ \delta,\ 1)$	$(\delta^2,\ \delta^2,\ 0,\ \delta^2,\ 0,\ 0,\ \delta^2)$
$(\delta^2,\ \delta^2,\ 0,\ 1,\ \delta,\ \delta,\ 1)$	$(\delta^2,\ 1,\ \delta,\ 1,\ \delta,\ 0,\ \delta^2)$	$(1,\ \delta^2,\ \delta,\ 1,\ 0,\ \delta,\ \delta^2)$	$(1,\ 1,\ 0,\ \delta^2,\ \delta,\ \delta,\ \delta^2)$
$(1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 0)$	$(1,\ \delta^2,\ \delta,\ \delta,\ \delta^2,\ 1,\ 0)$	$(\delta^2,\ 1,\ \delta,\ \delta,\ 1,\ \delta^2,\ 0)$	$(\delta^2,\ \delta^2,\ 0,\ \delta,\ 1,\ 1,\ \delta)$
$(\delta^2,\ \delta^2,\ 0,\ 0,\ \delta^2,\ \delta^2,\ 0)$	$(\delta^2,\ 1,\ \delta,\ 0,\ \delta^2,\ 1,\ \delta)$	$(1,\ \delta^2,\ \delta,\ 0,\ 1,\ \delta^2,\ \delta)$	$(1,\ 1,\ 0,\ \delta,\ \delta^2,\ \delta^2,\ \delta)$
$(1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1)$	$(1,\ \delta,\ \delta^2,\ \delta,\ \delta^2,\ 0,\ 1)$	$(\delta^2,\ 0,\ \delta^2,\ \delta,\ 1,\ \delta,\ 1)$	$(\delta^2,\ \delta,\ 1,\ \delta,\ 1,\ 0,\ \delta^2)$
$(\delta^2,\ \delta,\ 1,\ 0,\ \delta^2,\ \delta,\ 1)$	$(\delta^2,\ 0,\ \delta^2,\ 0,\ \delta^2,\ 0,\ \delta^2)$	$(1,\ \delta,\ \delta^2,\ 0,\ 1,\ \delta,\ \delta^2)$	$(1,\ 0,\ 1,\ \delta,\ \delta^2,\ \delta,\ \delta^2)$
$(0,\ 1,\ 1,\ 0,\ 0,\ 1,\ 1)$	$(0,\ \delta^2,\ \delta^2,\ \delta,\ \delta,\ 1,\ 1)$	$(\delta,\ 1,\ \delta^2,\ \delta,\ 0,\ \delta^2,\ 1)$	$(\delta,\ \delta^2,\ 1,\ \delta,\ 0,\ 1,\ \delta^2)$
$(\delta,\ \delta^2,\ 1,\ 0,\ \delta,\ \delta^2,\ 1)$	$(\delta,\ 1,\ \delta^2,\ 0,\ \delta,\ 1,\ \delta^2)$	$(0,\ \delta^2,\ \delta^2,\ 0,\ 0,\ \delta^2,\ \delta^2)$	$(0,\ 1,\ 1,\ \delta,\ \delta,\ \delta^2,\ \delta^2)$
$(0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1)$	$(0,\ \delta,\ \delta,\ \delta^2,\ \delta^2,\ 1,\ 1)$	$(\delta,\ 0,\ \delta,\ \delta^2,\ 1,\ \delta^2,\ 1)$	$(\delta,\ \delta,\ 0,\ \delta^2,\ 1,\ 1,\ \delta^2)$
$(\delta,\ \delta,\ 0,\ 1,\ \delta^2,\ \delta^2,\ 1)$	$(\delta,\ 0,\ \delta,\ 1,\ \delta^2,\ 1,\ \delta^2)$	$(0,\ \delta,\ \delta,\ 1,\ 1,\ \delta^2,\ \delta^2)$	$(0,\ 0,\ 0,\ \delta^2,\ \delta^2,\ \delta^2,\ \delta^2)$

Table 2 $\boldsymbol{a}+S_3(2)$

$\boldsymbol{a}+S_3(2)$			
$(1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0)$	$(1,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0)$	$(0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0)$	$(0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 1)$
$(0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 0)$	$(0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1)$	$(1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1)$	$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$

Table 3 $S_3(2)+\delta(\boldsymbol{a}+S_3(2))$

$S_3(2)+\delta(\boldsymbol{a}+S_3(2))$			
$(\delta,\ \delta,\ \delta,\ 0,\ 0,\ 0,\ 0)$	$(\delta,\ 0,\ 0,\ \delta,\ \delta,\ 0,\ 0)$	$(0,\ \delta,\ 0,\ \delta,\ 0,\ \delta,\ 0)$	$(0,\ 0,\ \delta,\ \delta,\ 0,\ 0,\ \delta)$
$(0,\ 0,\ \delta,\ 0,\ \delta,\ \delta,\ 0)$	$(0,\ \delta,\ 0,\ 0,\ \delta,\ 0,\ \delta)$	$(\delta,\ 0,\ 0,\ 0,\ 0,\ \delta,\ \delta)$	$(\delta,\ \delta,\ \delta,\ \delta,\ \delta,\ \delta,\ \delta)$
$(\delta,\ \delta^2,\ \delta^2,\ 1,\ 1,\ 0,\ 0)$	$(\delta,\ 1,\ 1,\ \delta^2,\ \delta^2,\ 0,\ 0)$	$(0,\ \delta^2,\ 1,\ \delta^2,\ 1,\ \delta,\ 0)$	$(0,\ 1,\ \delta^2,\ \delta^2,\ 1,\ 0,\ \delta)$
$(0,\ 1,\ \delta^2,\ 1,\ \delta^2,\ \delta,\ 0)$	$(0,\ \delta^2,\ 1,\ 1,\ \delta^2,\ 0,\ \delta)$	$(\delta,\ 1,\ 1,\ 1,\ 1,\ \delta,\ \delta)$	$(\delta,\ \delta^2,\ \delta^2,\ \delta^2,\ \delta^2,\ \delta,\ \delta)$
$(\delta^2,\ \delta,\ \delta^2,\ 1,\ 0,\ 1,\ 0)$	$(\delta^2,\ 0,\ 1,\ \delta^2,\ \delta,\ 1,\ 0)$	$(1,\ \delta,\ 1,\ \delta^2,\ 0,\ \delta^2,\ 0)$	$(1,\ 0,\ \delta^2,\ \delta^2,\ 0,\ 1,\ \delta)$
$(1,\ 0,\ \delta^2,\ 1,\ \delta,\ \delta^2,\ 0)$	$(1,\ \delta,\ 1,\ 1,\ \delta,\ 1,\ \delta)$	$(\delta^2,\ 0,\ 1,\ 1,\ 0,\ \delta^2,\ \delta)$	$(\delta^2,\ \delta,\ \delta^2,\ \delta^2,\ \delta,\ \delta^2,\ \delta)$
$(\delta^2,\ \delta^2,\ \delta,\ 1,\ 0,\ 0,\ 1)$	$(\delta^2,\ 1,\ 0,\ \delta^2,\ \delta,\ 0,\ 1)$	$(1,\ \delta^2,\ 0,\ \delta^2,\ 0,\ \delta,\ 1)$	$(1,\ 1,\ \delta,\ \delta^2,\ 0,\ 0,\ \delta^2)$
$(1,\ 1,\ \delta,\ 1,\ \delta,\ \delta,\ 1)$	$(1,\ \delta^2,\ 0,\ 1,\ \delta,\ 0,\ \delta^2)$	$(\delta^2,\ 1,\ 0,\ 1,\ 0,\ \delta,\ \delta^2)$	$(\delta^2,\ \delta^2,\ \delta,\ \delta^2,\ \delta,\ \delta,\ \delta^2)$
$(\delta^2,\ \delta^2,\ \delta,\ 0,\ 1,\ 1,\ 0)$	$(\delta^2,\ 1,\ 0,\ \delta,\ \delta^2,\ 1,\ 0)$	$(1,\ \delta^2,\ 0,\ \delta,\ 1,\ \delta^2,\ 0)$	$(1,\ 1,\ \delta,\ \delta,\ 1,\ 1,\ \delta)$
$(1,\ 1,\ \delta,\ 0,\ \delta^2,\ \delta^2,\ 0)$	$(1,\ \delta^2,\ 0,\ 0,\ \delta^2,\ 1,\ \delta)$	$(\delta^2,\ 1,\ 0,\ 0,\ 1,\ \delta^2,\ \delta)$	$(\delta^2,\ \delta^2,\ \delta,\ \delta,\ \delta^2,\ \delta^2,\ \delta)$
$(\delta^2,\ \delta,\ \delta^2,\ 0,\ 1,\ 0,\ 1)$	$(\delta^2,\ 0,\ 1,\ \delta,\ \delta^2,\ 0,\ 1)$	$(1,\ \delta,\ 1,\ \delta,\ 1,\ \delta,\ 1)$	$(1,\ 0,\ \delta^2,\ \delta,\ 1,\ 0,\ \delta^2)$
$(1,\ 0,\ \delta^2,\ 0,\ \delta^2,\ \delta,\ 1)$	$(1,\ \delta,\ 1,\ 0,\ \delta^2,\ 0,\ \delta^2)$	$(\delta^2,\ 0,\ 1,\ 0,\ 1,\ \delta,\ \delta^2)$	$(\delta^2,\ \delta,\ \delta^2,\ \delta,\ \delta^2,\ \delta,\ \delta^2)$
$(\delta,\ \delta^2,\ \delta^2,\ 0,\ 0,\ 1,\ 1)$	$(\delta,\ 1,\ 1,\ \delta,\ \delta,\ 1,\ 1)$	$(0\,\delta^2,\ 1,\ \delta,\ 0,\ \delta^2,\ 1)$	$(0,\ 1,\ \delta^2,\ \delta,\ 0,\ 1,\ \delta^2)$
$(0,\ 1,\ \delta^2,\ 0,\ \delta,\ \delta^2,\ 1)$	$(0,\ \delta^2,\ 1,\ 0,\ \delta,\ 1,\ \delta^2)$	$(\delta,\ 1,\ 1,\ 0,\ 0,\ \delta^2,\ \delta^2)$	$(\delta,\ \delta^2,\ \delta^2,\ \delta,\ \delta,\ \delta^2,\ \delta^2)$
$(\delta,\ \delta,\ \delta,\ 1,\ 1,\ 1,\ 1)$	$(\delta,\ 0,\ 0,\ \delta^2,\ \delta^2,\ 1,\ 1)$	$(0,\ \delta,\ 0,\ \delta^2,\ 1,\ \delta^2,\ 1)$	$(0,\ 0,\ \delta,\ \delta^2,\ 1,\ 1,\ \delta^2)$
$(0,\ 0,\ \delta,\ 1,\ \delta^2,\ \delta^2,\ 1)$	$(0,\ \delta,\ 0,\ 1,\ \delta^2,\ 1,\ \delta^2)$	$(\delta,\ 0,\ 0,\ 1,\ 1,\ \delta^2,\ \delta^2)$	$(\delta,\ \delta,\ \delta,\ \delta^2,\ \delta^2,\ \delta^2,\ \delta^2)$

Finally, the $2^7 = 128$ codewords of the additive code $S_3(2) + \delta H_3(2)$ are given in Tables 1 and 3. Observe that for any nonzero $\boldsymbol{x} = \boldsymbol{a} + \delta \boldsymbol{b}$ in $S_3(2) + \delta H_3(2)$, we get $\omega(\boldsymbol{x}) \in \{3,\ 4,\ 5,\ 6,\ 7\}$.

Theorem 2 *Let C_i be an $[n,\ k_i,\ d_i]_q$ linear code over $\mathbb{F}_q$, $0 \le i \le m-1$. $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$ is additive over $\mathbb{F}_{q^m}$ with minimum distance*

$$d = min\{d_0,\ \cdots,\ d_{m-1}\}$$

and $q^{(k_0+\cdots+k_{m-1})}$ codewords over $\mathbb{F}_q$. In addition $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$ is linear over $\mathbb{F}_{q^m}$ if and only if $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1}) = \mathbf{GU}(m,\ C_0)$ and $\mathbf{GU}(m,\ C_0)$ is an $[n,\ k_0,\ d_0]_{q^m}$ linear code.

Proof Given $\boldsymbol{x} = \boldsymbol{a}_0 + \delta \boldsymbol{a}_1 + \cdots + \delta^{m-1}\boldsymbol{a}_{m-1} \in \mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$, where $\boldsymbol{a}_i \in C_i,\ \ 0 \le i \le m-1$

$$\boldsymbol{x} = \sum_{i=0}^{m-1} \delta^i \boldsymbol{a}_i = \left(\sum_{i=0}^{m-1} \delta^i \boldsymbol{a}_{i1},\ \ldots,\ \sum_{i=0}^{m-1} \delta^i \boldsymbol{a}_{in}\right) \tag{5}$$

That is

$$\boldsymbol{x} = \begin{pmatrix} 1, & \delta, & \cdots, & \delta^{m-1} \end{pmatrix} \begin{pmatrix} a_{01} & \cdots & a_{0n} \\ a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{(m-1)1} & \cdots & a_{(m-1)n} \end{pmatrix} \tag{6}$$

We can see that the weight of $\boldsymbol{x}$ is the number of nonzero columns of the $m \times n$ matrix in Eq. (6). Let $\boldsymbol{b}_i$ be a minimum weight codeword in C_i. Then $\boldsymbol{x} = \mathbf{0} + \delta\mathbf{0} + \cdots + \delta^i \boldsymbol{b}_i + \cdots + \delta^{m-1}\mathbf{0}$, is such that $\omega(\boldsymbol{x}) = d_i$, that is, the minimum distance of $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$ is

$$d = min\{d_0,\ \ldots,\ d_{m-1}\}$$

where d_i is the minimum distance of C_i.

Now, for $\boldsymbol{x} = \sum_{i=0}^{m-1} \delta^i \boldsymbol{a}_i$ and $\boldsymbol{y} = \sum_{i=0}^{m-1} \delta^i \boldsymbol{b}_i$ in $\mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$, from Eq. (5), we get that $\boldsymbol{x} + \boldsymbol{y} = \sum_{i=0}^{m-1} \delta^i(\boldsymbol{a}_i + \boldsymbol{b}_i) \in \mathbf{GU}(C_0, C_1, \ldots, C_{m-1})$. Then $\mathbf{GU}(C_0, C_1, \cdots, C_{m-1})$ is an additive code over $\mathbb{F}_{q^m}$.

On the other hand, let $p(x) = x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + r_0$ be an irreducible polynomial of degree m over $\mathbb{F}_q$ such that $p(\delta) = 0$, that is, $\delta^m = -r_0 - r_1\delta - \cdots - r_{m-1}\delta^{m-1}$. Then

$$\delta \boldsymbol{x} = \sum_{i=0}^{m-1} \delta^{i+1}\boldsymbol{a}_i = \sum_{i=0}^{m-2} \delta^{i+1}\boldsymbol{a}_i - (\sum_{i=0}^{m-1} \delta^i r_i)\boldsymbol{a}_{m-1} \tag{7}$$

$$= (-r_0\boldsymbol{a}_{m-1}) + \delta(\boldsymbol{a}_0 - r_1\boldsymbol{a}_{m-1}) + \cdots + \delta^{m-1}(\boldsymbol{a}_{m-2} - r_{m-1}\boldsymbol{a}_{m-1}) \tag{8}$$

If $\delta \boldsymbol{x} \in \mathbf{GU}(C_0, C_1, \cdots, C_{m-1})$, we get that $\boldsymbol{a}_{i-1} - r_i \boldsymbol{a}_{m-1} \in C_i$, for $1 \le i \le m-1$ and $r_0 \boldsymbol{a}_{m-1} \in C_0$. Thus $C_{m-1} \subset C_0$. Since $\boldsymbol{a}_0 - r_1 \boldsymbol{a}_{m-1} \in C_1$ and $\boldsymbol{a}_0 \in C_0$, then $C_0 \subset C_1$. We have $\boldsymbol{a}_1 - r_2 \boldsymbol{a}_{m-1} \in C_1$, and for $i = 2$, $\boldsymbol{a}_1 - r_2 \boldsymbol{a}_{m-1} \in C_2$, i.e., $C_1 \subset C_2$. Continuing, for $i = m-1$, we obtain that $\boldsymbol{a}_{m-2} - r_{m-1} \boldsymbol{a}_{m-1} \in C_{m-1}$, i.e., $\boldsymbol{a}_{m-2} \in C_{m-1}$, then $C_{m-2} \subset C_{m-1}$. That is, $C_{m-1} \subset C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_{m-2} \subset C_{m-1}$. Thus, $C_0 = C_1 = C_2 = \cdots = C_{m-2} = C_{m-1}$ and $\mathbf{GU}(C_0, C_1, \cdots, C_{m-1}) = \mathbf{GU}(m,\ C_0)$.

If $\boldsymbol{x} \in \mathbf{GU}(m,\ C_0)$, from Eq. (7), $\delta \boldsymbol{x} \in \mathbf{GU}(m,\ C_0)$. That is, $\mathbf{GU}(m,\ C_0)$ is linear over $\mathbb{F}_{q^m}$ and if C_0 is an $[n,\ k_0,\ d_0]_q$ linear code, $|\mathbf{GU}(m,\ C_0)| = q^{k_0} q^{k_0} \ldots q^{k_0} = (q^m)^{k_0}$, i.e., $dim_{\mathbb{F}_{q^m}}(\mathbf{GU}(m,\ C_0)) = k_0$.

We observe that if all the codes C_i are equal to a given code C_0, then the code we obtain is lifting of C_0 (see [21]).

2.3 Orthogonality of GU Codes with Respect to Euclidean, Hermitian, and Trace Hermitian Inner Products

Theorem 3 *Let $C_0 \subset \mathbb{F}_q^n$ be a linear code and m an even number. Then*

$$\mathbf{GU}(m,\ C_0)^{\perp_E} = \mathbf{GU}(m,\ C_0^{\perp_E}) = \mathbf{GU}(m,\ C_0)^{\perp_{TH}}$$

and

$$C_0 \subset C_0^{\perp_E} \rightarrow \ \mathbf{GU}(m,\ C_0) \subset \mathbf{GU}(m,\ C_0)^{\perp_E}$$

$$C_0 = C_0^{\perp_E} \rightarrow \ \mathbf{GU}(m,\ C_0) = \mathbf{GU}(m,\ C_0)^{\perp_E}$$

Proof Suppose that C_0 is an $[n,\ k_0,\ d_0]_q$ linear code over $\mathbb{F}_q$. We know by Theorem 2 that $C = \mathbf{GU}(m,\ C_0)$ is an $[n,\ k_0,\ d_0]_{q^m}$ linear code over $\mathbb{F}_{q^m}$. Given $\boldsymbol{y} \in \mathbf{GU}(m,\ C_0^{\perp_E})$, we write $\boldsymbol{y} = \sum_{i=0}^{m-1} \delta^i \boldsymbol{b}_i$ where $\boldsymbol{b}_i \in C_0^{\perp_E}$. Writing $\boldsymbol{x} = \sum_{i=0}^{m-1} \delta^i \boldsymbol{a}_i \in \mathbf{GU}(m,\ C_0)$ and since $x_i = \sum_{j=0}^{m-1} \delta^j a_{ji}$ and $y_i = \sum_{k=0}^{m-1} \delta^k b_{ki}$. From Eq. (6) we get

$$\begin{aligned}
\boldsymbol{x} \cdot \boldsymbol{y} &= \sum_{i=1}^{n} x_i y_i = (\boldsymbol{a}_0 + \delta \boldsymbol{a}_1 + \cdots + \delta^{m-1} \boldsymbol{a}_{m-1}) \cdot (\boldsymbol{b}_0 + \delta \boldsymbol{b}_1 + \cdots + \delta^{m-1} \boldsymbol{b}_{m-1}) \\
&= \left(\sum_{k=0}^{m-1} \delta^k \boldsymbol{a}_0 \cdot \boldsymbol{b}_k \right) + \delta \left(\sum_{k=0}^{m-1} \delta^k \boldsymbol{a}_1 \cdot \boldsymbol{b}_k \right) + \cdots + \delta^{m-1} \left(\sum_{k=0}^{m-1} \delta^k \boldsymbol{a}_{m-1} \cdot \boldsymbol{b}_k \right) \\
&= 0
\end{aligned}$$

That is, $\mathbf{GU}(m,\ C_0^{\perp_E}) \subset \mathbf{GU}(m,\ C_0)^{\perp_E}$. But

$$|\mathbf{GU}(m,\ C_0^{\perp_E})| = q^{n-k_0} q^{n-k_0} \cdots q^{n-k_0} = (q^m)^{n-k_0} = |\mathbf{GU}(m,\ C_0)^{\perp_E}|$$

i.e., $\mathbf{GU}(m,\ C_0)^{\perp_E} = \mathbf{GU}(m,\ C_0^{\perp_E})$. Observe that if $C_0 \subset C_0^{\perp_E}$, $\mathbf{GU}(m,\ C_0) \subset \mathbf{GU}(m,\ C_0)^{\perp_E}$.

On the other hand, we want to find the Hermitian dual of the code $C \subset \mathbb{F}_{q^m}^n$, in this case $\overline{\boldsymbol{y}} = \boldsymbol{y}^{\sqrt{q^m}}$, that is, we take m an even number and the Frobenius automorphism with fixed field $\mathbb{F}_{q^{m/2}}$. Then $\overline{\boldsymbol{y}} = (\boldsymbol{b}_0 + \overline{\delta}\boldsymbol{b}_1 + \cdots + \overline{\delta^{(m-1)}}\boldsymbol{b}_{m-1})$, because $\boldsymbol{b}_j \in \mathbb{F}_q^n$ and we get that

$$\boldsymbol{x} \cdot_H \boldsymbol{y} := \boldsymbol{x} \cdot \overline{\boldsymbol{y}} = (\boldsymbol{a}_0 + \delta\boldsymbol{a}_1 + \cdots + \delta^{m-1}\boldsymbol{a}_{m-1}) \cdot (\boldsymbol{b}_0 + \overline{\delta}\boldsymbol{b}_1 + \cdots + \overline{\delta^{(m-1)}}\boldsymbol{b}_{m-1}) = 0$$

because each $\boldsymbol{a}_i \cdot \boldsymbol{b}_j = 0$ for $0 \leq i,\ j \leq m-1$ and therefore

$$\boldsymbol{x} \cdot_{TH} \boldsymbol{y} = tr(\boldsymbol{x} \cdot_H \boldsymbol{y}) = 0$$

where

$$tr : \mathbb{F}_{q^m} \to \mathbb{F}_{q^{m/2}}, \quad x \to x + x^{q^{m/2}}$$

the trace of $\mathbb{F}_{q^m}$ over $\mathbb{F}_{q^{m/2}}$. That is, $\mathbf{GU}(m,\ C_0)^{\perp_E} \subset \mathbf{GU}(m,\ C_0)^{\perp_{TH}}$ and they have the same size. Therefore equality holds.

In the language of [11], $\mathbf{GU}(m,\ C_0)$ may be thought of as the Galois extension of C_0 from $\mathbb{F}_q$ to $\mathbb{F}_{q^m}$. Therefore, parts of the proof of the following result are also in the cited paper. We give the proof for the completeness using our notation.

Theorem 4 *Given $C \subset \mathbb{F}_{q^m}^n$ and $C^q = \{\boldsymbol{x}^q \mid \boldsymbol{x} \in C\} \subset C$, C is linear over $\mathbb{F}_{q^m}$ if and only if $C = \mathbf{GU}(m,\ C_0)$, where C_0 is the subfield subcode of C over $\mathbb{F}_q$. Since C is Frobenius invariant, $tr(C) = \{tr(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{x}^q + \cdots + \boldsymbol{x}^{q^{m-1}} \mid \boldsymbol{x} \in C\} = C_0$.*

Proof Let $C_0 \subset \mathbb{F}_q^n$ be a linear code and $C = \mathbf{GU}(m,\ C_0)$. Given $\boldsymbol{x} \in C$, $\boldsymbol{x} = \boldsymbol{a}_0 + \delta\boldsymbol{a}_1 + \cdots + \delta^{m-1}\boldsymbol{a}_{m-1}$, where $\{1,\ \delta,\ \ldots,\ \delta^{m-1}\}$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^m}$ and $\boldsymbol{a}_i \in C_0$, $0 \leq i \leq m-1$. Then

$$\boldsymbol{x}^q = \boldsymbol{a}_0 + \delta^q\boldsymbol{a}_1 + \cdots + \delta^{q(m-1)}\boldsymbol{a}_{m-1}$$

Since $\delta^{qj} \in \mathbb{F}_{q^m}$, with $1 \leq j \leq m-1$, $\delta^{qj} = \alpha_{0j} + \alpha_{1j}\delta + \cdots + \alpha_{(m-1j)}\delta^{m-1}$ for some $\alpha_{ij} \in \mathbb{F}_q$, we get that $\boldsymbol{x}^q \in C$. Thus, $C^q \subset C$, that is, C is Frobenius invariant.

On the other hand, let $C \subset \mathbb{F}_{q^m}^n$ be a linear code and C_0 its subfield subcode over $\mathbb{F}_q$. From Theorem 2, $\mathbf{GU}(m,\ C_0)$ is a linear code over $\mathbb{F}_{q^m}$, then $\boldsymbol{a}_0 + \delta\boldsymbol{a}_1 + \cdots + \delta^{m-1}\boldsymbol{a}_{m-1}$ is an element of C. Therefore $\mathbf{GU}(m,\ C_0) \subset C$. If in addition $C^q \subset C$, we obtain that for $\boldsymbol{x} \in C$, $\boldsymbol{x} + \boldsymbol{x}^q + \cdots + \boldsymbol{x}^{q^{m-1}} = tr(\boldsymbol{x}) \in C$. Thus $dim_{\mathbb{F}_q}(tr(C)) \leq dim_{\mathbb{F}_{q^m}}(C)$ and from Delsarte's theorem

$$\begin{aligned} dim(tr(C)) &= dim((C^{\perp_E})_0)^{\perp_E} = n - dim(C^{\perp_E})_0 \geq n - dim(C^{\perp_E}) \\ &= dim(C) \geq dim(C_0) \end{aligned}$$

hence

$$dim_{\mathbb{F}_q}(tr(C)) = dim_{\mathbb{F}_{q^m}}(C) \;\; and \;\; dim_{\mathbb{F}_q}(C_0) = dim_{\mathbb{F}_{q^m}}(C)$$

Also,

$$dim_{\mathbb{F}_{q^m}}(\mathbf{GU}(m,\ C_0)) = dim_{\mathbb{F}_q}(C_0) = k_0$$

therefore we obtain that

$$C = \mathbf{GU}(m,\ C_0).$$

3 Applications to Quantum Stabilizer Codes: Theoretical Results

In this section, we formulate the quantum mechanical framework for quantum stabilizer codes that are used in correcting errors in quantum computing. We use this framework and our Go-Up construction to derive quantum stabilizer codes from Euclidean self-orthogonal linear codes. The principal references are [1, 4, 5, 8, 12, 15, 16, 20], the books [14, 22, 23, 29], and the seminal works [7, 9]. Quantum error-correcting codes are a key ingredient in carrying out information processing based on quantum mechanics [7, 8, 15]. In the late 1990's Gottesman [13] and independently Calderbank et al. [7] proposed a method to construct quantum codes from classical codes. These codes are the most studied class of quantum codes together with the CSS construction introduced by Shor et al. [9] and independently by Steane [25].

3.1 A $\star$ Product and An Alternating and Bilinear form in an Arbitrary Basis

Here we show that a $\star$ product can be defined that yields an alternating and bilinear form in any basis. Given $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}$ and $\boldsymbol{d}$ in $\mathbb{F}_q^n$, let $\boldsymbol{x} = \boldsymbol{a}\alpha_1 + \boldsymbol{b}\alpha_2$ and $\boldsymbol{y} = \boldsymbol{c}\alpha_1 + \boldsymbol{d}\alpha_2$ in $\mathbb{F}_{q^2}^n$ where $\{\alpha_1,\ \alpha_2\}$ is a basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, it is known the number of such ordered bases is $(q^2-1)(q-1)$ and the matrix

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1^q & \alpha_2^q \end{bmatrix}$$

is such that $det(A) = \alpha_1\alpha_2^q - \alpha_1^q\alpha_2 \neq 0$.

Then

$$\mathbf{x}\cdot\mathbf{y}^q = (\boldsymbol{a}\alpha_1 + \boldsymbol{b}\alpha_2)\cdot(\alpha_1^q\boldsymbol{c} + \alpha_2^q\boldsymbol{d}) = \alpha_1\alpha_1^q\boldsymbol{a}\cdot\boldsymbol{c} + \alpha_1\alpha_2^q\boldsymbol{a}\cdot\boldsymbol{d} + \alpha_2\alpha_1^q\boldsymbol{b}\cdot\boldsymbol{c} + \alpha_2\alpha_2^q\boldsymbol{b}\cdot\boldsymbol{d}$$

and

$$\mathbf{x}^q\cdot\mathbf{y} = (\boldsymbol{a}\alpha_1^q + \boldsymbol{b}\alpha_2^q)\cdot(\boldsymbol{c}\alpha_1 + \boldsymbol{d}\alpha_2) = \alpha_1\alpha_1^q\boldsymbol{a}\cdot\boldsymbol{c} + \alpha_1^q\alpha_2\boldsymbol{a}\cdot\boldsymbol{d} + \alpha_1\alpha_2^q\boldsymbol{b}\cdot\boldsymbol{c} + \alpha_2\alpha_2^q\boldsymbol{b}\cdot\boldsymbol{d}$$

That is,

$$\frac{\mathbf{x}\cdot\mathbf{y^q}-\mathbf{x}^q\cdot\mathbf{y}}{\alpha_1\alpha_2^q-\alpha_1^q\alpha_2}=\frac{(\boldsymbol{a}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{c})(\alpha_1\alpha_2^q-\alpha_1^q\alpha_2)}{\alpha_1\alpha_2^q-\alpha_1^q\alpha_2}=\boldsymbol{a}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{c}$$

We define the following alternating and bilinear form over $\mathbb{F}_p$, denoted by $\star$,

$$\boldsymbol{x}\star\boldsymbol{y}=tr(\frac{\boldsymbol{x}\cdot\boldsymbol{y}^q-\boldsymbol{x}^q\cdot\boldsymbol{y}}{\alpha_1\alpha_2^q-\alpha_1^q\alpha_2})=tr(\boldsymbol{a}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{c}) \tag{9}$$

Then $\mathbf{x}\star\mathbf{x}=0$. In this definition "$tr$" is the trace function from $\mathbb{F}_q$ to its prime subfield $\mathbb{F}_p$.

Since tr is linear over $\mathbb{F}_p$ and $-1\in\mathbb{F}_p$ we get that $\mathbf{x}\star\mathbf{y}=-(\mathbf{y}\star\mathbf{x})$. For $r\in\mathbb{F}_p$, $(r\mathbf{x})\star\mathbf{y}=r(\mathbf{x}\star\mathbf{y})$ by linearity of tr and for $\mathbf{z}=\boldsymbol{v}\alpha_1+\boldsymbol{w}\alpha_2\in\mathbb{F}_{q^2}^n$ we have

$$(\mathbf{x}+\mathbf{z})\star\mathbf{y}=\mathbf{x}\star\mathbf{y}+\mathbf{z}\star\mathbf{x}\ \ and\ \ \mathbf{x}\star(\mathbf{y}+\mathbf{z})=\mathbf{x}\star\mathbf{y}+\mathbf{x}\star\mathbf{z}$$

Therefore "$\star$" is a skew-symmetric and bilinear form over $\mathbb{F}_p$.

Observe that for $q=p$, p a prime number,

$$\mathbf{x}\star\mathbf{y}=\boldsymbol{a}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{c}$$

which corresponds, for $p=2$, to Eq. (4), the symplectic inner product obtained in [7].

Also, observe that $\mathbf{x}\star\mathbf{y}=-(\mathbf{x}*_a\mathbf{y})$, where "$*_a$" is the alternating form given in [15].

Therefore, given any basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, $\mathbf{x}\perp_\star\mathbf{y}$ if and only if $\mathbf{x}\perp_{*_a}\mathbf{y}$.

3.2 Quantum Mechanical Formulation of Quantum Stabilizer Codes

It is known that any channel comes with an underlying alphabet, letters of the alphabet are the smallest unit of information that can be sent across the channel. In classical error correcting codes the alphabet can be the finite field $\mathbb{F}_q$. In the quantum scenario the analogous to $\mathbb{F}_q$ is a finite dimensional Hilbert space $\mathbb{H}$. We take $\mathbb{H}=\mathbb{C}^q$ or in particular $\mathbb{H}=\mathbb{C}^2$ in the binary case (see [15, 22, 24]).

Definition 3 The basic unit of quantum information is the **quantum bit** coined as **qubit** by Schumacher [24] and its state space is the two state space $\mathbb{C}^2$. The basis states are denoted in the Dirac notation by $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ where $|\mathbf{0}\rangle$ is the column vector $\binom{1}{0}\in\mathbb{C}^2$ and $|\mathbf{1}\rangle$ is the column $\binom{0}{1}\in\mathbb{C}^2$. With this notation a qubit is any system whose state vector $|\boldsymbol{u}\rangle$ can be written as $|\boldsymbol{u}\rangle=u_1|\mathbf{0}\rangle+u_2|\mathbf{1}\rangle$ where $u_1,u_2\in\mathbb{C}$ and $|u_1|^2+|u_2|^2=1$ (see [7, 9]).

Definition 4 A $q-$ary quantum code Q of length n is a q^k dimensional subspace of $\mathbb{C}^{q^n}$.

The Hilbert space $\mathbb{C}^{q^n}$ is identified with the $n-$fold tensor product of the Hilbert space $\mathbb{C}^q$ and it is thought as the state space of a $q-$ary system in the same way as the values 0 and 1 can be thought of as the possible states of a bit in a bit string, (see [4, 15]).

Definition 5 Given two complex or real $n \times m$ matrices A and B the Frobenius inner product is given by

$$(A, B)_F = \sum_{i,j} b_{ij}\overline{a_{i,j}} = Tr(\overline{A}^t B) \tag{10}$$

In order to select an appropriate error model in $q-$ary quantum stabilizer codes, we take $\mathbb{H}$ being a complex finite Hilbert space of dimension m. In view of linearity of quantum mechanics, if we can correct errors E and E_1, we can correct any linear combination of them, $rE + sE_1$ where $r, r_1 \in \mathbb{C}$, so we only need to consider whether the code can correct a **basis of errors**. We want to find a basis of the complex vector space $Hom(\mathbb{H}, \mathbb{H})$ representing a discrete set of errors, such a basis contains m^2 complex linear independent operators, $\mathbb{B} = \{E_1, \ldots, E_{m^2}\}$, with the following properties (see [16]):

a. It is a set of unitary operators and $I \in \mathbb{B}$.
b. Taking the matrix representation and the Frobenius inner product, $\mathbb{B}$ is a set of orthonormal unitary operators, that is, $(E_i, E_j)_F = m\delta_{i,j}$. If $E_i = I$, we get that $(I, E_j)_F = Tr(E_j) = 0$ for all $E_j \neq I \in \mathbb{B}$.
c. Since the set of unitary operators is a group under composition, we take $\mathbb{B}$ such that $E_iE_j = w_{ij}E_{i\circ j}$ for some operation $\circ$ on the set of indices. Observe that

$$I = \overline{(E_iE_j)}^t(E_iE_j) = \overline{w_{ij}}w_{ij}\overline{E_{i\circ j}}^t E_{i\circ j} = |w_{ij}|^2 I$$

then $|w_{ij}|^2 = 1$

If the conditions **a**, **b**, and **c** hold, then $\mathbb{B}$ is called a **nice error basis**. Now, since each E_i is a unitary operator, $|det(E_i)| = 1$. If in addition, we take each E_i such that $det(E_i) = 1$, the basis $\mathbb{B}$ is called a **very nice error basis**. For example, taking $\mathbb{H} = \mathbb{C}^2$, the set of Pauli matrices

$$\mathbb{B} = \{I_2,\ \sigma_x,\ \sigma_y,\ \sigma_z\} \tag{11}$$

is a nice error basis for $Hom(\mathbb{C}^2, \mathbb{C}^2)$ and

$$\mathbb{B} = \{I_2,\ i\sigma_x,\ i\sigma_y,\ i\sigma_z\} \tag{12}$$

is a very nice error basis.

In the classical scenario the only errors are the so called position errors, they are elements of $\mathbb{F}_q^n$ that act additively on words. In the quantum setting there are two type of error, **position and phase errors**. These errors are best explained using Weyl Operators (see [15, 30]). Taking $\mathbb{H} = \mathbb{C}^q$, for $a, b \in \mathbb{F}_q$, we define the unitary operators U_a (position error), V_b (phase error) and x in $\mathbb{C}^q$ by

$$U_a|x\rangle = |x + a\rangle \tag{13}$$

$$V_b|x\rangle = w^{tr(bx)}|x\rangle \tag{14}$$

$$E_{ab}|x\rangle = U_a V_b|x\rangle = w^{tr(bx)}|x + a\rangle \tag{15}$$

where $w = exp(2\pi i/p)$ is a primitive pth root of unity, tr denotes the trace operation from the extension $\mathbb{F}_q$ to its prime field $\mathbb{F}_p$, and for $r \in \mathbb{F}_p$, $w^r = exp(2r\pi i/p)$. Then

$$\mathbb{B}_1(q) = \{E_{ab} : a,\ b \in \mathbb{F}_q\}$$

is a nice error basis on $\mathbb{C}^q$ (see [15], Lemma 1).

For $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbb{F}_q^n$, we write $U_{\boldsymbol{a}} = U_{a_1} \otimes \cdots \otimes U_{a_n}$ and $V_{\boldsymbol{a}} = V_{a_1} \otimes \cdots \otimes V_{a_n}$ for the tensor product of n error operators. The set

$$\mathbb{B}_n(q) = \{E_{\boldsymbol{ab}} = U_{\boldsymbol{a}} V_{\boldsymbol{b}} : \boldsymbol{a},\ \boldsymbol{b} \in \mathbb{F}_q^n\}$$

where $E_{\boldsymbol{ab}}|\boldsymbol{v}\rangle = U_{\boldsymbol{a}} V_{\boldsymbol{b}}|\boldsymbol{v}\rangle = w^{tr(\boldsymbol{b}\cdot\boldsymbol{v})}|\boldsymbol{v} + \boldsymbol{a}\rangle$, is a nice error basis on the Hilbert space $\mathbb{C}^{q^n}$ (see [15, 16]).

Since

$$E_{\boldsymbol{ab}} E_{\boldsymbol{cd}} = w^{tr(\boldsymbol{b}\cdot\boldsymbol{c})} U_{\boldsymbol{a}+\boldsymbol{c}} V_{\boldsymbol{b}+\boldsymbol{d}} = w^{tr(\boldsymbol{b}\cdot\boldsymbol{c})} E_{\boldsymbol{a}+\boldsymbol{c}\ \boldsymbol{b}+\boldsymbol{d}} \tag{16}$$

the set

$$\xi_n = \{w^r E_{\boldsymbol{ab}} : \boldsymbol{a},\ \boldsymbol{b} \in \mathbb{F}_q^n,\ r \in \mathbb{F}_p\}$$

is a finite group of order pq^{2n}, **called the error group** associated with the nice error basis $\mathbb{B}_n(q)$ (see [15].)

From [4, 15] we have.

Definition 6 The stabilizer **S** is some Abelian subgroup of ξ_n and the coding space $Q \subset \mathbb{C}^{q^n}$ is the space of vectors fixed by **S**. That is

$$Q = \bigcap_{E \in \mathbf{S}} \{|\boldsymbol{v}\rangle \in \mathbb{C}^{q^n} \mid E|\boldsymbol{v}\rangle = |\boldsymbol{v}\rangle\}$$

Q is called a stabilizer code and it is the space with all eigenvalues $+1$.

That is, a Quantum stabilizer code is defined as the joint eigenspace of the operators of a commutative subgroup **S** of ξ_n.

Now, we take the centralizer of $\mathbf{S}$ in ξ_n,

$$\{E \in \xi_n : EE_1 = E_1E \ \ \forall E_1 \in \mathbf{S}\}$$

and $\mathbf{S}Z(\xi_n)$ denotes the group generated by the subgroup $\mathbf{S}$ and $Z(\xi_n)$, the center of ξ_n. From [15], Lemma 11, we have

Lemma 1 ([15]) *Suppose that* $\mathbf{S}$ *is the stabilizer group of a stabilizer code* Q *of dimension* $dim\ Q > 1$. *An error* E *in* ξ_n *is detectable by the quantum code* Q *if and only if either* E *is an element of* $\mathbf{S}Z(\xi_n)$ *or* E *does not belong to the centralizer of* $\mathbf{S}$ *in* ξ_n.

From Eq. (16) we get that: Two elements $E = w^r E_{\boldsymbol{ab}}$ and $E_1 = w^{r_1} E_{\boldsymbol{cd}}$ of the error group ξ_n satisfy the relation

$$E_1E = w^{tr(\boldsymbol{a}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{c})}EE_1$$

In particular, the elements E and E_1 commute if and only if $tr(\boldsymbol{a}\cdot\boldsymbol{d} - \boldsymbol{b}\cdot\boldsymbol{c}) = 0$. When $q = p$, we get $tr(\boldsymbol{a}\cdot\boldsymbol{d} - \boldsymbol{b}\cdot\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{d} - \boldsymbol{b}\cdot\boldsymbol{c}$ and if $q = p = 2$ it corresponds to Eq. (4).

The weight of an element $E_{\boldsymbol{ab}} \in \xi_n$ is the number of nonidentity tensor components and the weight of a scalar multiple of the identity matrix is zero. A quantum code Q has minimum distance d if and only if it can detect all errors in ξ_n of weight less than d, but can not detect some error of weight d. **It is pure to** t if and only if its stabilizer subgroup $\mathbf{S}$ does not contain nonscalar matrices of weight less than t. When $t = d$, we say that Q is pure. A $q-$ary stabilizer code of length n, dimension q^k and minimum distance d is denoted $[[n,\ k,\ d]]_q$.

In order to get a relation between additive or linear codes in $\mathbb{F}_{q^2}^n$ with stabilizer codes, we define a bijective map φ that maps each element $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}_q^{2n}$ to a vector $\boldsymbol{a}\alpha_1 + \boldsymbol{b}\alpha_2 \in \mathbb{F}_{q^2}^n$, where $\{\alpha_1, \alpha_2\}$ is a basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. From Definition 6, $\mathbf{S}$ is the set of errors E, that have no effect on the encoded state, $\mathbf{S}$ is the analog of the zero subgroup in the classical coding case. When $\varphi(\mathbf{S}) = \overline{\mathbf{S}}$ is $\star-$self-orthogonal, we get that for all $E \in \mathbf{S}$ and all $E_1 \in \varphi^{-1}(\overline{\mathbf{S}}^{\perp_\star}) = \mathbf{S}^{\perp_\star}$, $EE_1 = E_1E$. **Then $\mathbf{S}^{\perp_\star} \setminus \mathbf{S}$ is the set of the undetectable errors.** Observe that Q is invariant under the elements of $\mathbf{S}^{\perp_\star}$. On the other hand, the errors which fail to commute with some element of $\mathbf{S}$ move codewords into an orthogonal subspace to the code, so can be detected by Q (see [5]).

From Theorem 15 in [15], we have

Theorem 5 ([15]) *An* $[[n,\ k,\ d]]_q$ *stabilizer code exists if and only if there exists an additive subcode* D *of* $\mathbb{F}_{q^2}^n$ *of cardinality* $q^n/q^k = q^{n-k}$ *such that* D *is* $*_a-$*self-orthogonal and weight of* $D^{\perp_a} \setminus D$ *is* d *if* $k > 0$ *and weight of* $D^{\perp_a}$ *is* d *if* $k = 0$. *Where the weight of the additive code* D *is equal to* $min\{\omega(\boldsymbol{c}) \mid \boldsymbol{c} \in D,\ \boldsymbol{c} \neq \mathbf{0}\}$.

Proposition 1 ([15]) (Quantum Singleton Bound) *An* $[[n,\ k,\ d]]_q$ *stabilizer code with* $k > 0$ *satisfies*

$$q^k \leq q^{n-2d+2}.$$

3.3 Quantum Stabilizer Codes from GU Construction

Now we can prove the following result.

Lemma 2 *Let $C_0 \in \mathbb{F}_q^n$ be an Euclidean self-orthogonal code. Then the trace-alternating duality for $C = \mathbf{GU}(2,\ C_0)$ is the same as Euclidean duality.*

Proof We take C_0 and C_1 linear codes over $\mathbb{F}_q$ such that $C_0 \subset C_1^{\perp_E}$, where $C_1^{\perp_E}$ is the Euclidean dual of C_1. Then

$$C_0 + \delta C_1 \subset (C_0 + \delta C_1)^{\perp_\star} = (C_0 + \delta C_1)^{\perp_{*a}},$$

because for any $\boldsymbol{a} + \delta \boldsymbol{b}$ and $\boldsymbol{c} + \delta \boldsymbol{d}$ in $C_0 + \delta C_1$, and since $C_0 \subset C_1^{\perp_E}$, we get $\boldsymbol{a} \cdot \boldsymbol{d} = 0 = \boldsymbol{b} \cdot \boldsymbol{c}$. That is $C = C_0 + \delta C_1$ is self-orthogonal with respect to the alternating form $\star$. In particular if $C_0 = C_1$, i.e., C_0 is Euclidean-self-orthogonal, the linear code $\mathbf{GU}(2,\ C_0)$ is $\star-$self-orthogonal. Given $\mathbf{y} = \boldsymbol{c} + \delta \boldsymbol{d} \in \mathbf{GU}(2,\ C_0^{\perp_E})$, for any $\mathbf{x} = \boldsymbol{a} + \delta \boldsymbol{b} \in \mathbf{GU}(2,\ C_0)$,

$$\mathbf{x} \star \mathbf{y} = tr(\boldsymbol{a} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{c}) = tr(0 - 0) = 0$$

Thus,

$$\mathbf{GU}(2,\ C_0^{\perp_E}) \subset \mathbf{GU}(2,\ C_0)^{\perp_\star},$$

and since $|\mathbf{GU}(2,\ C_0^{\perp_E})| = q^{2n-2k_0} = |\mathbf{GU}(2,\ C_0)^{\perp_\star}|$, we get

$$\mathbf{GU}(2,\ C_0)^{\perp_{*a}} = \mathbf{GU}(2,\ C_0)^{\perp_\star} = \mathbf{GU}(2,\ C_0)^{\perp_E}.$$

Therefore, we just need to establish classical Euclidean duality for our construction and to work with classical self-orthogonal codes.

We observe that

$$\mathbf{GU}(2,\ C_0)^{\perp_\star} \setminus \mathbf{GU}(2,\ C_0) = \{\boldsymbol{c} + \delta \boldsymbol{d} : \boldsymbol{c},\ \boldsymbol{d} \in C_0^{\perp_E} \setminus C_0\}$$

Let $d = \omega_H(\mathbf{GU}(2,\ C_0)^{\perp_\star} \setminus \mathbf{GU}(2,\ C_0)) = \omega_H(C_0^{\perp_E} \setminus C_0) \geq d_0^{\perp_E} = \omega_H(C_0^{\perp_E})$, because $C_0^{\perp_E} \setminus C_0 \subset C_0^{\perp_E}$. Then we get

Theorem 6 *Let $C \in \mathbb{F}_{q^2}^n$ be an $[n,\ k,\ d]_{q^2}$ Euclidean, i.e., classical, self-orthogonal code that is Frobenius invariant. Then C yields an $[[n,\ n-2k,\ \geq d^{\perp_E}]]_q$ quantum stabilizer code that is pure to d.*

Proof Let $C \subset \mathbb{F}_{q^2}^n$ be a linear code and C_0 its subfield subcode over $\mathbb{F}_q$. From Theorem 2

$$\mathbf{GU}(2,\ C_0) = \{\boldsymbol{a} + \delta \boldsymbol{b} \mid \boldsymbol{a},\ \boldsymbol{b} \in C_0\}$$

is a linear code over $\mathbb{F}_{q^2}$. From Theorem 4 and taking $m = 2$, we get

$$C = \mathbf{GU}(2,\ C_0)$$

Using Theorem 3 and Lemma 2, $C^{\perp_E} = \mathbf{GU}(2,\ C_0)^{\perp_E} = \mathbf{GU}(2,\ C_0^{\perp_E}) = C^{\perp_\star}$ and applying Theorem 5 we obtain an $[[n,\ n-2k,\ \geq d^{\perp_E}]]_q$ quantum stabilizer code. It is pure to $d = \omega_H(C) = \omega_H(\mathbf{GU}(2,\ C_0)) = \omega_H(C_0)$

4 Explicit Classes of Quantum Stabilizer Codes

Example 2 If $C_0 = \boldsymbol{R}(1, m)$ is the first order Reed-Muller code of parameters $[2^m,\ m+1,\ 2^{m-1}]_2$, we have C_0 is self-orthogonal because $C_0^{\perp_E} = \boldsymbol{R}(m-2, m)$, $C_0^{\perp_E}$ has parameters $[2^m,\ 2^m - m - 1,\ 4]_2$, and $\frac{m+1}{2^m} \leq \frac{1}{2}$ for all $m \geq 3$ (see [18]). That is

$$R_{\boldsymbol{R}(1,m)} \leq \frac{1}{2}\ \ \forall\, m \geq 3$$

Thus, from the first order Reed-Muller code, $C_0 = \boldsymbol{R}(1, m)$, we obtain the binary quantum stabilizer code with parameters $[[2^m,\ 2^m - 2m - 2,\ 4]]_2$. We get that $d = 4$, because the minimum weight of $C_0^{\perp_E} \setminus C_0$ is equal to the minimum weight of $C_0^{\perp_E} = 4$, for all $m \geq 3$. From Proposition 1, the parameters verify the quantum Singleton bound

$$2(d-1) \leq n - k$$

In this case, $n - k = 2m + 2$, that is,

$$4 \leq m + 2$$

For $m = 3$, we obtain $[[8,\ 0,\ 4]]_2$ from the Euclidean self-dual binary code $C_0 = \boldsymbol{R}(1, 3)$, which permits us to get the **Euclidean self-dual** linear code $C = \mathbf{GU}(2,\ \boldsymbol{R}(1, 3))$ over $\mathbb{F}_4$.

We get $d = 4$ when $m = 3$, because $d \leq 5$ and it is known that except for trivial codes, codes with $d \leq 2$, there are only two binary quantum MDS codes, $[[5, 1, 3]]_2$ and $[[6, 0, 4]]_2$ (see [7], Theorem 24).

Example 3 We take $H_k(2)$, the binary Hamming code of parameters $[2^k - 1,\ 2^k - 1 - k,\ 3]_2$ and C_0 as its dual with parameters $[2^k - 1,\ k,\ 2^{k-1}]_2$, i.e., $C_0 = S_k(2)$ the binary Simplex code of dimension k (see [6]). Since the columns of a parity check matrix for a binary Hamming code consist of all possible nonzero binary words of length k, that is, all the elements of $\mathbb{F}_2^k \setminus \{0\}$. We can think that if α is a primitive element of $\mathbb{F}_{2^k}$ a parity check matrix is equivalent to

$$H = [\alpha^{2^k-2}, \ldots, \alpha,\ 1]$$

Then

$$HH^t = \left(\sum_{i=0}^{2^k-2} \alpha^i\right)^2$$

because the characteristic is two. But $\sum_{i=0}^{2^k-2} \alpha^i = 0$ since $0 = \alpha^{p^r} - \alpha = \alpha(\alpha - 1)(\alpha^{p^r-2} + \cdots + \alpha + 1)$ with $\alpha \neq 0$ and $\alpha - 1 \neq 0$, in our case $p = 2$. Thus

$$HH^t = 0 \tag{17}$$

and since H is a generator matrix of $S_k(2)$, the binary Simplex code of dimension k is Euclidean self-orthogonal with rate $\frac{k}{2^k-1} \leq \frac{1}{2}$ for all $k \geq 3$. That is

$$R_{C_0} \leq \frac{1}{2} \;\; \forall\, k \geq 3$$

and since $dis(H_k(2) \setminus S_k(2)) = 3$, we obtain a binary stabilizer code of parameters $[[2^k - 1,\; 2^k - 1 - 2k,\; 3]]_2$

Example 4 We take $h(x) = x^t$, $L = \{1, \alpha, \ldots, \alpha^{n-1}\} \subset \mathbb{F}_{2^m}$ where $\alpha \in \mathbb{F}_{2^m}$ and $ord(\alpha) = n$. If $t < \frac{n}{2}$, then we will obtain a linear quantum code of parameters $[[n,\; n - 2k_{\mathbf{\Gamma}(L,x^t)^{\perp_E}},\; d]]_2$ where d is the minimum distance of $\mathbf{\Gamma}(L, x^t) \setminus \mathbf{\Gamma}(L, x^t)^{\perp_E}$.

Proof A parity check matrix of $\mathbf{\Gamma}(L,\; h(x))$ is given by

$$H = \begin{pmatrix} 1 & \alpha^{t-1} & \alpha^{2(t-1)} & \cdots & \alpha^{(n-1)(t-1)} \\ 1 & \alpha^{t-2} & \alpha^{2(t-2)} & \cdots & \alpha^{(n-1)(t-2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} h^{-1}(1) & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & h^{-1}(\alpha^{n-1}) \end{pmatrix}$$

Then

$$H = \begin{pmatrix} 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \\ 1 & \alpha^{-2} & \alpha^{-4} & \cdots & \alpha^{-2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha^{-t} & \alpha^{-2t} & \cdots & \alpha^{-(n-1)t} \end{pmatrix}$$

and

$$(HH^t)_{ij} = h^{-1}(1)h^{-1}(1) + h^{-1}(\alpha)\alpha^i h^{-1}(\alpha)\alpha^j + \cdots + h^{-1}(\alpha^{n-1})\alpha^{(n-1)i} h^{-1}(\alpha^{n-1})\alpha^{(n-1)j}$$

i.e., $(HH^t)_{ij} = 1 + \alpha^{-(i+j)} + (\alpha^{-(i+j)})^2 + \cdots + (\alpha^{-(i+j)})^{n-1}$, where $2 \leq i + j \leq 2t$. Since $t < \frac{n}{2}$, $\alpha^{-(i+j)} \neq 1$ and

$$(HH^t)_{ij} = (\alpha^{-(i+j)} - 1)^{-1}((\alpha^{-(i+j)})^n - 1) = 0$$

Thus, $HH^t \equiv 0$ and

$$\mathbf{\Gamma}(L, x^t)^{\perp_E} \subset \mathbf{\Gamma}(L, x^t)$$

Taking d as the minimum weight of $(\mathbf{\Gamma}(L, x^t) \setminus \mathbf{\Gamma}(L, x^t)^{\perp_E})$ and $C_0 = \mathbf{\Gamma}(L, x^t)^{\perp_E}$, we get the stabilizer group $C = \mathbf{GU}(2,\ C_0)$ which permits us to construct a linear quantum code with parameters $[[n,\ n - 2k_{\mathbf{\Gamma}(L,x^t)^{\perp_E}},\ d]]_2$.

Example 5 Let $C_0 \subset \mathbb{F}_2^n$ be an $[n,\ k,\ d]_2$ Euclidean self-orthogonal linear code. **The additive code** $C = C_0 + \delta C_0^{\perp_E} \subset \mathbb{F}_4^n$ is self-dual with respect to the trace of the Hermitian inner product and we get a pure $[[n,\ 0,\ d]]_2$ quantum stabilizer code (see [7]).

Proof We define

$$\mathbf{S} = \{E = i^r U_{\boldsymbol{a}} V_{\boldsymbol{b}} \in \xi_n \mid \mathbf{a} + \delta \mathbf{b} \in C_0 + \delta C_0^{\perp_E},\ r \in \mathbb{Z}_4,\ i^2 = -1\}$$

which has $|C_0 + \delta C_0^{\perp_E}| = 2^k 2^{n-k} = 2^n$ elements. For $E = i^r U_{\mathbf{a}} V_{\mathbf{b}}$ and $E_1 = i^{r_1} U_{\mathbf{c}} V_{\mathbf{d}}$ in $\mathbf{S}$, we have

$$EE_1|\boldsymbol{v}\rangle = i^{r+r_1}(-1)^{(\mathbf{b}+\mathbf{d}).\mathbf{v}+\mathbf{b}.\mathbf{c}}|\mathbf{v} + \mathbf{a} + \mathbf{c}\rangle = i^s U_{\mathbf{a}+\mathbf{c}} V_{\mathbf{b}+\mathbf{d}}|\mathbf{v}\rangle$$

and

$$E_1 E|\boldsymbol{v}\rangle = i^{r+r_1}(-1)^{(\mathbf{b}+\mathbf{d}).\mathbf{v}+\mathbf{a}.\mathbf{d}}|\mathbf{v} + \mathbf{a} + \mathbf{c}\rangle = i^s U_{\mathbf{a}+\mathbf{c}} V_{\mathbf{b}+\mathbf{d}}|\mathbf{v}\rangle = EE_1|\boldsymbol{v}\rangle$$

Since $EE_1 \in \mathbf{S}$ and $E^{-1} = i^{-r} U_{\mathbf{a}} V_{\mathbf{b}} \in \mathbf{S}$, we get that $\mathbf{S}$ is an Abelian subgroup of ξ_n and $\overline{\mathbf{S}} = C_0 + \delta C_0^{\perp_E} = C$.

Since $C_0 \subset C_0^{\perp_E}, C^{\perp_{TH}} = C_0 + \delta C_0^{\perp_E} = C$. We take $q = 2$ and applying the Theorem 5, we get a single quantum state pure code with parameters $[[n,\ 0,\ d]]_2$.

Such a code might be useful in testing whether certain storage locations for qubits are decohering faster than they should (see [7]). In the particular case when we have the additive code $C = S_k(2) + \delta H_k(2)$, we get a single quantum state code with parameters $[[2^k - 1,\ 0,\ 2^{k-1}]]_2$ see Example 1.

Example 6 Given C_0 an $[n,\ k_0,\ d_0]_2$ and C_1 an $[n,\ k_1,\ d_1]_2$ codes such that $C_0 \subset C_1^{\perp_E}$. Then we get an $[[n,\ n - (k_0 + k_1),\ d]]_2$ quantum stabilizer code, where $d = dis\{C_1^{\perp_E} + \delta C_0^{\perp_E} \setminus C_0 + \delta C_1\}$.

Proof We take

$$\mathbf{S} = \{E = i^r U_{\boldsymbol{a}} V_{\boldsymbol{b}} \in \xi_n \mid \mathbf{a} + \delta\mathbf{b} \in C_0 + \delta C_1,\ r \in \mathbb{Z}_4,\ i^2 = -1\}$$

That is, $\overline{\mathbf{S}} = C_0 + \delta C_1$. For any two elements of $\mathbf{S}$, $E = i^r U_{\mathbf{a}} V_{\mathbf{b}}$ and $E_1 = i^{r_1} U_{\mathbf{c}} V_{\mathbf{d}}$ we have

$$EE_1|\mathbf{v}\rangle = Ei^{r_1}(-1)^{\mathbf{d}\cdot\mathbf{v}}|\mathbf{v}+\mathbf{c}\rangle = i^{r+r_1}(-1)^{\mathbf{d}\cdot\mathbf{v}+\mathbf{b}\cdot\mathbf{v}+\mathbf{b}\cdot\mathbf{c}}|\mathbf{v}+\mathbf{c}+\mathbf{a}\rangle,$$

and

$$E_1E|\mathbf{v}\rangle = i^{r_1+r}(-1)^{\mathbf{b}\cdot\mathbf{v}+\mathbf{d}\cdot\mathbf{v}+\mathbf{d}\cdot\mathbf{a}}|\mathbf{v}+\mathbf{a}+\mathbf{c}\rangle,$$

then

$$EE_1 = E_1E$$

when

$$\mathbf{d}\cdot\mathbf{v}+\mathbf{b}\cdot\mathbf{v}+\mathbf{b}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{v}+\mathbf{d}\cdot\mathbf{v}+\mathbf{a}\cdot\mathbf{d}$$

for any $|\mathbf{v}\rangle$, that is, when

$$\mathbf{a}\cdot\mathbf{d} = \mathbf{b}\cdot\mathbf{c}.$$

Since $C_0 \subset C_1^{\perp}$,

$$\mathbf{a}\cdot\mathbf{d} = 0 = \mathbf{b}\cdot\mathbf{c}$$

and we obtain that the operators E and E_1 commute and $\overline{\mathbf{S}} \subset \overline{\mathbf{S}}^{\perp_{TH}}$.

We observe that $EE_1|\mathbf{v}\rangle = i^{r+r_1}(-1)^{(\mathbf{b}+\mathbf{d})\cdot\mathbf{v}+\mathbf{b}\cdot\mathbf{c}}|\mathbf{v}+\mathbf{a}+\mathbf{c}\rangle = i^s U_{\mathbf{a}+\mathbf{c}} V_{\mathbf{b}+\mathbf{d}}|\mathbf{v}\rangle$, because $\mathbf{b}\cdot\mathbf{c} = 0$, where $s = r + r_1 \in \mathbb{Z}_4$, then $EE_1 \in \mathbf{S}$. Since $E^{-1} = i^{-r}U_{\mathbf{a}}V_{\mathbf{b}}$, $E^{-1} \in \mathbf{S}$ and $E = U_{\mathbf{0}}V_{\mathbf{0}} \in \mathbf{S}$ is the identity element, we get that $\mathbf{S}$ is an Abelian subgroup of ξ_n.

In this case we take

$$\mathbf{S}^{\perp_{TH}} = \{g = i^r U_{\boldsymbol{a}} V_{\boldsymbol{b}} \in \xi_n \mid \mathbf{a} + \delta\mathbf{b} \in C_1^{\perp} + \delta C_0^{\perp}\}$$

Since $k_0 = dim(C_0)$ and $k_1 = dim(C_1)$, we get that $|\overline{\mathbf{S}}| = 2^{k_0+k_1} = 2^{n-(n-k_0-k_1)}$.

But

$$C_1^{\perp} + \delta C_0^{\perp} \setminus C_0 + \delta C_1 = \{\mathbf{a} + \delta\mathbf{b} \in C_1^{\perp} + \delta C_0^{\perp} \mid a \in C_1^{\perp} \setminus C_0,\ \boldsymbol{b} \in C_0^{\perp} \setminus C_1\},$$

then $d = min\{dis(C_1^{\perp} \setminus C_0),\ dis(C_0^{\perp} \setminus C_1)\} = dis(C_1^{\perp} + \delta C_0^{\perp} \setminus C_0 + \delta C_1)$. Thus, by Theorem 5 (taking q = 2) and *Theorem* 9 from [7], we get an $[[n,\ n-(k_0+k_1),\ d]]_2$ additive quantum code.

In the particular case when we have the additive code $C = C_0 + \delta C_1$, where C_0 is the repetition code of parameters $[2^m,\ 1,\ 2^m]_2$ and C_1 the first-order Reed-

Muller code with parameters $[2^m,\ m+1,\ 2^{m-1}]_2$. Since $C_0 \subset C_1 \subset C_1^{\perp}$, we get $[[2^m,\ 2^m - m - 2,\ d]]_2$ an additive quantum code.

5 Some Other Applications of the GO-UP Construction

Our Go-Up construction also has applications to algebraic coding theory, finite geometries, finite group theory, and also to combinatorial objects such as strongly regular graphs, and few-weight codes. We give several important applications in [2, 3].

Acknowledgements We would like to thank the reviewers for their helpful comments. Preliminary research of Eddie Arrieta Arrieta was supported by a Fellowship from UPRRP-DEGI during 2019–2021. The work of Heeralal Janwa is funded in parts by the NASA grants 80NSSC20M0052, 80NSSC19M0167, 80NSSC20M0132 and 80NSSC21M0156.

References

1. Aly, S.A., Klappenecker, A., Sarvepalli, P.K.: On quantum and classical BCH codes. IEEE Trans. Inf. Theory **53**(3), 1183–1188 (2007)
2. Arrieta, A.E.: A GO-UP Construction and Applications. Ph.D. Dissertation,: Department of Mathematics. University of Puerto Rico, Rio Piedras (2021)
3. Arrieta, A.E., and Janwa, H.: A new construction of two-, three- and few-weight codes via our gu codes and their applications. Appl. Algebra Eng. Commun. Comput. (AAECC). final version accepted (2022)
4. Ashikhmin, A., Knill, E.: Nonbinary quantum stabilizer codes. IEEE Trans. Inf. Theory **47**(7), 3065–3072 (2001)
5. Barnum, H., Crepeau, C., Gottesman, D., Smith, A., Tapp, A.: Authentication of Quantum Messages, pp. 449–458. IEEE Press (2002)
6. Bierbrauer, J.: Introduction to Coding Theory, 2nd edn. CRC Press Taylor Francis Group (2017)
7. Calderbank, A.R., Rains, E.M., Shor, P.M., Sloane, N.J.A.: Quantum error correction via codes over **GF**(4). IEEE Trans. Inf. Theory **44**(4), 1369–1387 (1998)
8. Calderbank, A.R., Shor, P., Sloane, N., Rains, E.: Quantum error correction and orthogonal geometry. Phys. Rev. Lett. **78**(3), 405–408 (1997)
9. Calderbank, A.R., Shor, P.W.: Good quantum error-correcting codes exist. Phys. Rev. A **54**(2), 1098–1106 (1996)
10. Delsarte, P.: On subfield subcodes of modified Reed-Solomon codes. IEEE Trans. Inf. Theory **21**(5), 575–576 (1975)
11. Giorgetti, M., Previtali, A.: Galois invariance, trace codes and subfield subcodes. Finite Fields Their Appl. **16**(2), 96–99 (2010)
12. Gottesman, D.: Theory of fault-tolerant quantum computation. Phys. Rev. A **57**(1), 127–137 (1998)
13. Gottesman, D.E.: Stabilizer Codes and Quantum Error Correction. Ph.D. Dissertation, California Institute of Technology (1997)
14. Hoffman, K., Kunze, R.: Linear Algebra, 2nd edn. Prentice-Hall, Inc. (1971)
15. Ketkar, A., Klappenecker, A., Kumar, S., Sarvepalli, P.K.: Nonbinary stabilizer codes over finite fields. IEEE Trans. Inf. Theory **52**(11), 4892–4914 (2006)

16. Knill, E.: Non-Binary Unitary Error Bases and Quantum Codes. Los Alamos National Laboratory (1996)
17. Lidl, R., Niederreiter, H.: Introduction to Finite Fields and their Applications. Cambridge University Press (1994)
18. McWilliams, F.J., Sloane, N.J.A.: The Theory of Error-Correcting Codes. North-Holland Amsterdam (1977)
19. McElliece, R. J.: The Theory of Information and Coding. Addison-Wesley Publishing Company Inc. (1977)
20. Rains, E.M.: Nonbinary quantum codes. IEEE Trans. Inf. Theory **45**(6), 1827–1832 (1999)
21. Rifa, J., Zinoviev, V.A.: On lifting perfect codes. IEEE Trans. Inf. Theory **57**(9), 5918–5925 (2011)
22. Roman, S.: Advanced Linear Algebra, 2nd edn. Springer, Berlin (2005)
23. Renes, J.M.:Quantum information theory. In: Lecture Notes (2015)
24. Schumacher, B.: Quantum coding. Phys. Rev. A **51**(4) (1995)
25. Steane, A.M.: Simple quantum error-correcting codes. Phys. Rev. A **54**, 6 (1996)
26. Stichtenoth, H.: Algebraic Function Fields and Codes, vol. 254. Springer Science and Business Media (2009)
27. Stichtenoth, H.: On the dimension of subfield subcodes. IEEE Trans. Inf. Theory **36**(1), 90–93 (1990)
28. Trappe, W., Washinton, L.C.: Introduction to Cryptography with Coding Theory, 2nd edn. Pearson Prentice Hall (2006)
29. Wilde Mark, M.: From Classical to Quantum Shannon Theory. Cambridge University Press (2019)
30. Wilde Mark, M.: Quantum Information Theory. Cambridge University Press (2013)

Bent and Near-Bent Function Construction and 2-Error-Correcting Codes

Jose W. Velazquez and Heeralal Janwa

Abstract A function $f : F_{2^m} \to F_{2^t}$ is called a vectorial Boolean function in m variables. Whenever $t = 1$ we call these functions Boolean functions. In this work, we study the construction of Gold and Kasami-Welch functions of the form $Tr(\lambda x^d)$ (for $d = 2^l + 1, 2^{2l} - 2^l + 1$ and $\lambda \in F^*_{2^m}$). These functions' nonlinearity property is a measure of their distance to the set of affine functions (the first-order Reed-Muller codes). We generalize a result of Dillon and Dobbertin for conditions under which these functions are bent. We give algorithms that generate and determine the bentness of the functions. We construct 2-error-correcting cyclic codes utilizing Almost-Perfect-Nonlinear (APN) and near-bent exponents. We present theorems that enumerate the Gold and Kasami-Welch functions. We improve previous algorithms used to determine the minimum distance of these codes.

Keywords APN exponents · Bent functions · Boolean functions · Cyclic codes

1 Some New Results on Boolean Bent Functions: The Gold and Kasami-Welch Case

In this chapter, we construct algorithms that generate a Boolean function $g : F_{2^m} \to F_2$ in m variables of the Gold and Kasami-Welch type as constructed by Dillon and Dobbertin in [6]. These exponents are known for their corresponding APN power functions ($f : F_{2^m} \to F_{2^m}$) of the form $f(x) = x^d$. The exponents are defined as $d = 2^l + 1, 2^{2l} - 2^l + 1$ respectively, with $(l, m) = 1$ a necessary condition for these functions to be APN and the construction of 2-error-correcting codes as per Janwa and Wilson [10]. To generate these functions, we construct algorithms (which will be showed in the following subsections) that define the Boolean functions and iterate

J. W. Velazquez (✉) · H. Janwa
University of Puerto Rico at Rio Piedras, San Juan, PR 00925, USA
e-mail: jose.velazquez16@upr.edu

H. Janwa
e-mail: heeralal.janwa@upr.edu

F. Hoffman et al. (eds.), *Combinatorics, Graph Theory and Computing*,
Springer Proceedings in Mathematics & Statistics 448,
https://doi.org/10.1007/978-3-031-52969-6_38

over distinct Gold and Kasami-Welch exponents over the field F_{2^m}. These functions are given by the trace definition which follows from [16]:

Definition 1 *(Trace Function)* For m and t integers such that $t|m$, the trace of an element $x \in F_{2^m}$ is denoted by the vectorial Boolean function

$$Tr_k^m(x) : F_{2^m} \to F_{2^t}$$

such that:

$$Tr_t^m(x) = x + x^q + \cdots + x^{q^{l-1}}, q = 2^t, l = \frac{m}{t} \tag{1}$$

When $t = 1$, we write $Tr(x)$.

We consider the bent property of these functions via the Walsh-Hadamard transform definition as given in [4].

Definition 2 (*Walsh-Hadamard Transform*) The Walsh-Hadamard transform of a Boolean function f in m variables is given by:

$$\hat{F}(\omega) = \sum_{x \in F_{2^m}} (-1)^{\omega * x + f(x)}, \forall \omega \in F_{2^m}. \tag{2}$$

Definition 3 (*Bent Boolean Function*) Let f be a Boolean function in m (even) variables. Then, f is bent if and only if:

$$|\hat{F}(\omega)| = 2^{\frac{m}{2}} \forall \omega \in F_{2^m}.$$

Dillon and Dobbertin in [6] analyzed the Fourier coefficients of $(-1)^{Tr(\lambda x^d)}$ and obtained conditions under which the Kasami-Welch and Gold Boolean functions are bent:

Proposition 1 (Dillon and Dobbertin Gold Bent Functions) *Let $L = F_{2^m}$ and let $(l, m) = 1$. For:*

$$\lambda \in L^*, s_{2^l+1}^{\lambda}(x) = Tr(\lambda x^{2^l+1}) \textit{ and } \rho_{2^l+1}^{\lambda}(x) = (-1)^{s_{2^l+1}^{\lambda}(x)}$$

if m is even and $\lambda \in L^*$ is not a cube, then:

$$\hat{\rho}_{2^l+1}(\alpha) = \pm 1, \ \forall \alpha \in L,$$

i.e., $s_{2^l+1}^{\lambda}(x)$ is bent.

Theorem 1 (Dillon and Dobbertin Kasami-Welch Bent Functions) *Let $L = F_{2^m}$, where m is even but not divisible by 6, and let $K = F_{2^2}$. Let $d = 2^{2l} - 2^l + 1$, $(l, m) = 1$. For:*

$$\lambda \in K^*, s_d^{\lambda} = Tr(\lambda x^d) \text{ and let } \rho_d^{\lambda} = (-1)^{s_d^{\lambda}} = \phi(\lambda x^d)$$

then:

$$\text{(1) If } \lambda \neq 1, \text{ then } s_d^\lambda \text{ is bent; i.e., } \hat{\rho}_d^\lambda(\beta) = \pm 1, \forall \beta \in L$$

$$\text{(2) If } \lambda = 1, \text{ then } \hat{\rho}_d \text{ takes just three values } \{-2, 0, 2\}, \textit{ and}$$

$$\begin{aligned}&\text{supp } \hat{\rho}_d = L \setminus K^*\{\Gamma(\theta) : Tr_{L/K}(\theta^{-d}) \neq 0\}\\ &\text{where } \Gamma(z) = T(z^{2^{3k}+1})/z^{2^{2k}(2^k+1)}\\ &\text{and } \ T(z) = z + z^{2^k} + z^{2^{2k}}.\end{aligned}$$

Later, Carlet generalized this theorem in [4] by reducing the divisibility criteria to m not divisible by 3 and $\lambda \in F_{2^m}$ a non-cube. To study these functions, we take α as a primitive element in F_{2^m} and consider $Tr(\alpha^i x^d)$ where we iterate over i for each Gold and Kasami-Welch exponent. We state the following lemma as an observation from this notation:

Lemma 1 (Primitive Element Representation of a kth Power Element in F_{2^m}) *Let $\alpha \in F_{2^m}$ be a primitive in the field and β a k-th power of another element in the field. Then, $\beta \equiv \alpha^{ik} \pmod{2^m - 1}$ where $i, k \in \mathbb{Z}$.*

Proof Let $\beta \equiv \gamma^k \pmod{2^m - 1} \in F_{2^m}$. Since α is a primitive element in the field, then one of its powers will be equivalent to γ. Let's say $\alpha^i \equiv \gamma \pmod{2^m - 1}$. Then we have: $\beta \equiv \gamma^k \equiv (\alpha^i)^k \equiv \alpha^{ik} \pmod{2^m - 1}$

Thus, if $\beta \in F_{2^m}$ is a k-th power of another element, then you can represent it as α^j where j is a multiple of k. □

We utilize this in the following section.

1.1 A Generalization of a Theorem of Dillon and Dobbertin in the Gold Case

In this section, we consider the Boolean functions in m variables of the form $Tr(\alpha^i x^d)$. We define a list containing the Gold exponents of the form $2^l + 1$ for $0 < l < m$ and iterate over all the exponents in the list. For $\alpha \in F_{2^m}^*$, a primitive element, we iterate over $0 \leq i < 2^m - 1$ (covering all non-zero elements in the field). We utilize SAGE online programming software to construct algorithms that generate these functions. The three main algorithms utilized: select an irreducible polynomial of degree m to construct the finite field and polynomial ring, verify if a Boolean function is bent, and construct these functions and verifies bentness. The irreducible polynomial is found via Conway polynomials as described in [14]. This step is carried out to ensure that every time we construct the field, we use the same modulus.

Algorithms We refer to the SAGE Boolean function page for Boolean function commands [13]. Three algorithms are constructed:

```
def FindModNonCube(m):
  L =[]
  P.<a> = PolynomialRing(GF(2))
  k.<a> = GF(2**m, modulus = 'conway')
  r = k.modulus()
  L.append(r)
  return L
```

This algorithm constructs a polynomial ring over F_2, uses the Conway polynomial method to construct a finite field F_{2^m}, and stores the modulus on a list L. The algorithm returns the modulus which is used to construct polynomial rings and finite fields in other algorithms.

```
def Bentness(f):
  dim = f.nvariables()
  w = f.walsh_hadamard_transform()
  for i in range(2^dim):
    if abs(w[i]) != 2^(dim/2)
      return "Not Bent"
    else:
      i = pass
  return "Bent"
```

This algorithm takes as input a Boolean function "f," finds in how many variables this function is defined and determines its Walsh-Hadamard transform spectrum. There are many ways to input a Boolean function; we use the BooleanFunction(f(x)) command, where $f(x)$ is a polynomial over F_{2^m} and the result is the Boolean function given by $Tr(f(x))$ [13]. Then it verifies if the function is bent by iterating over the spectrum to check if it meets definition 3. The performance of this algorithm is comparable to the built-in "is_bent()" command in SAGE. We compare these in Tables 1 and 2.

We compared these commands for up to 10 variables. We utilized Python memory allocation and time tracking packages. Boolean functions of the form $Tr(\alpha^i x^3)$ iterating for $0 \leq i \leq 2^m - 1$ were computed. We observed that the computational time was similar for both methods, while the maximum memory allocation was slightly lower for the Bentness algorithm.

Table 1 Computation time and memory allocation of is_bent() command

Variables	Bent functions detected	Time (s)	Max memory allocation
4	10	0.108	27703
6	42	1.864	68556
8	170	33.754	125285
10	682	706.255	175985

Table 2 Computation time and memory allocation of Bentness algorithm

Variables	Bent functions detected	Time (s)	Max memory allocation
4	10	0.106	25090
6	42	1.843	44221
8	170	35.895	124533
10	682	710.432	158697

```
def BentTraceIterationGold(m):
  f = FindModNonCube(m)
  R.<x> = GF (2^m, 'a', modulus = f)[]
  k.<a> = GF(2**m, modulus = f)
  print("Variables, Exponent, Power of the element,
Is it Bent?")
  for j in range(1,m):
    for i in range(0,2^m - 1):
      GB = BooleanFunction((a^i)*x ^(G[j]))
      print(m,",", G[j], ", a^",i, " ,",Bentness(GB))
```

The algorithm takes as input the number of variables "m" and then constructs a Boolean polynomial Ring over F_{2^m} such that it assigns "a" as the congruence class $[x]$ modulo f (the irreducible polynomial from the previous algorithm). Then it constructs a finite field as powers of "a" where "a" is a primitive element in the field. Next, it iterates over the possible Gold exponents (which are stored in a list "G") and all the power of "a" to go over all the possible Gold functions. Finally, it verifies if the function is bent and prints the results.

Explicit Families of New Gold Bent Functions Not Obtained by Dillon and Dobbertin: We observed a pattern in the generation of the bent and non-bent functions. If $Tr(\alpha^i x^d)$ was non-bent for i some multiple of an integer k, then it was not bent for all the multiples of k. Furthermore, we noted that k took on values as divisors of the Gold exponent under consideration. We denoted these exceptions by "Mk" meaning "multiples of k." Full tables were computed for up to 10 variables. We established some observations based on the resulting tables and then obtained results for 12–18 variables only considering powers of i that correspond to multiples of divisors of d. We had the following observations for up to 14 variables:

(1) If $(l, m) = 1$, then the exceptions are the multiples of 3.
(2) Let $i, i_2, t, t_2, j \in \mathbb{Z}, m \neq 2^j$, and say $m = 2^i * t$, then if $l = 2^{i_2} * t_2$ for $i \le i_2$, then all cases are exceptions.
(3) For all other cases, the exceptions are multiples of some divisor of d. Special case when $m = 2 \times 1$. Then the exceptions are multiples of d.

We list the obtained tables in Appendix 3.1 (see Tables 4 and 5) noting for what multiples of k the function was not bent. We computed partial results for up to

24 variables (see Table 6) and verify that the deduced conditions apply to these cases. Computational limitations resulted in Table 6 not being completed for 20-24 variables. However, the pattern is still present in the observed results. Note that for 24 variables, when $m = 2 * l, 4 * l, 6 * l, 8 * l$, and $12 * l$, the exceptions are Md. Interestingly, $m = 3 * l$ does not lead to this result as it meets the second case (perhaps suggesting priority for that condition). For 22 variables, when $m = 2 * l$ the exceptions are for Md but for $m = 11 * l$ we see that case 2 applies and we get $M1$ as the exceptions. For 20 variables when $m = 2 * l, 4 * l, 10 * l$ the exceptions are for Md but for $m = 5 * l$ we see that case 2 applies and we get $M1$ as the exceptions. This pattern is observed for 16 and 18 variables. For the remaining cases we note that the exceptions coincide with the smallest Gold exponent in the set of divisors of d. The exception to this is the $M9$ cases that fall under case 4 as if 9 divides the Gold exponent, so does 3. It seems that $M3$ cases only occur under condition 1, so perhaps this could be an if and only if condition, although we do not have strong enough evidence to state this yet. These observations lead to the final set of conditions deduced from the computational results:

(1) If $(l, m) = 1$, then the functions are bent whenever $i \notin M3$.
(2) Let $i, i_2, t, t_2, j \in \mathbb{Z}^+, m \neq 2^j$, and say $m = 2^i * t$, then if $l = 2^{i_2} * t_2$ for $i \leq i_2$, then all cases are exceptions.
(3) Given that case 2 is not met, then if $m = t * l$ for an integer t, then the functions are bent whenever $i \notin Md$.
(4) If the previous cases are not met, then the functions are bent for $i \notin Mk$, where k is the smallest Gold exponent such that $k|d$, $k > 3$.

Based on these conditions, we state the following conjecture on the Gold Boolean bent functions:

A Conjecture in the Gold Case

Conjecture 1 (*New Families of Gold Bent Boolean Functions*) Consider $m, l, i, k \in \mathbb{Z}$ and the finite field F_{2^m}, where m is an even integer. Let $d = 2^l + 1$ be a Gold exponent, and $\alpha \in F_{2^m}$ a primitive element in the field. Let $Tr(\alpha^i x^d)$ be a Gold Boolean function and Mk the set of all the multiples of k. Then:

(1) If $(l, m) = 1$, $Tr(\alpha^i x^d)$ is bent if $i \notin M3$.
(2) Let $i, i_2, t, t_2, j \in \mathbb{Z}^+, m \neq 2^j$, and say $m = 2^i * t$, then if $l = 2^{i_2} * t_2$ for $i \leq i_2$ then $Tr(\alpha^i x^d)$ is not bent.
(3) Given that case 2 is not met, then if $m = t * l$ for $t \in \mathbb{Z}^+$, then $Tr(\alpha^i x^d)$ is bent if $i \notin Md$.
(4) If the previous cases are not met, then $Tr(\alpha^i x^d)$ is bent if $i \notin Mk$ for k the smallest Gold exponent such that $k|d$, $k > 3$.

The first case follows from Dillon and Dobbertin's results.

1.2 A Generalization of a Theorem of Dillon and Dobbertin in the Kasami-Welch Case

In this section, we consider the Boolean functions in m variables of the form $Tr(\alpha^i x^d)$. We define a list of Kasami-Welch exponent: $2^{2l} - 2^l + 1$ for $0 < l < m$ and repeat a similar methodology as the Gold case. For the following section, we develop an additional algorithm to construct the Kasami-Welch bent functions.

Algorithms We construct an algorithm to determine if the Kasami-Welch Boolean function is bent or not.

```
  def BentTraceIterationKas(m):
    f = FindModNonCube(m)
    R.<x> = GF(2ˆm, 'a', modulus = f)[]
    k.<a> = GF(2**m, modulus = f)
    print("Variables, Exponent, Power of the element,
Is it Bent?")
    forjinrange(1,m):
      for i in range(0,2ˆm - 1):
        KB = BooleanFunction((aˆi)*x ˆ(K[j]))
        print(m,",", K[j], ", aˆ",i, " ,",Bentness(KB))
```

The algorithm repeats the same process as the Gold case algorithm. The key difference is that we pre-generated and use a list of Kasami-Welch exponents.

Explicit Families of New Kasami-Welch Bent Functions Not Obtained by Dillon and Dobbertin: We generated the functions of the form $Tr(\alpha^i x^d)$, $d = 2^{2l} - 2^l + 1$ for up to 12 variables and for all values of i such that $0 \leq i \leq 2^m - 2$. We note that for $(l, m) = 1$ our computational results show that Theorem 1 is met as the exception found are when i is a multiple of 3. However, for 6 and 12 variables, we find bent functions. These exist for $(l, m) = 1$. We show the results for the 6 and 12 variable cases in Table 7 in Appendix 3.2.

The results show that the non-divisibility by 6 criteria given in [6] can be removed and still generate some bent functions. The conditions on λ and $(l, m) = 1$, however, are still satisfied for these cases. We propose a generalization of the bent function result of their theorem as follows:

A Conjecture in the Kasami-Welch Case

Conjecture 2 (*Generalized Kasami-Welch Bent Boolean Functions*) Let $L = F_{2^m}$, where m is an even integer, let α be a primitive element in F_{2^m} and the Kasami-Welch exponent $d = 2^{2l} - 2^l + 1$, $(l, m) = 1$. Let $Tr(\alpha^i x^d)$ be the Kasami-Welch Boolean functions and Mk the set of multiples of the integer k. We have:

$$s_d^{\alpha^i}(x) = Tr(\alpha^i x^d) \text{ and let } \rho_d^{\lambda} = (-1)^{s_d^{\lambda}}$$

then

$$\text{If } i \notin M3, \text{ then } s_d^{\alpha^i}(x) \text{ is bent; i.e., } \hat{\rho}_d^{\alpha^i}(\beta) = \pm 1, \forall \beta \in L \tag{3}$$

2 Some New Results on Near-Bent Functions and on the Corresponding Cyclic Codes

In this chapter, we construct cyclic codes in 2 roots utilizing known APN and near-bent exponents. Cyclic codes are those such that if $x = (a_0, a_1, \ldots, a_{n-1})$ is a codeword, then so is $y = (a_{n-1}, a_0, \ldots, a_{n-2})$, $a \in F_2^m$. Define ω as a primitive element in the field F_{2^m} and define $m_i(x)$ as the minimal polynomial of ω^i over F_2. We can define a cyclic code (with 2 roots) of length $2^m - 1$, dimension at least $2^m - 1 - 2m$ and minimum distance d(C) by constructing the generator polynomial $g(x) = m_1(x)m_d(x)$ [10]. We denote these 2 root codes as C_m^d. The weight of a codeword $x = (a_0, a_1, \ldots, a_{n-1})$ is given by $\{\#i|a_i \neq 0\}$.

We call $\{d_0, d_1, d_2, \ldots, d_y\}$ the defining set of the cyclic code with y roots whose generator polynomial is $g(x) = m_{d_0}(x)m_{d_1}(x)m_{d_2}(x)\ldots m_{d_y}(x)$. Janwa and Wilson have shown in [10] that for a cyclic code in 2 roots (defining set $\{1, d_1\}$), if $f(x) = x^{d_1}$ satisfies $(l, m) = 1$ (and thus is APN), then the code is 2-error-correcting. This same condition is present for the construction of near-bent functions of the form $Tr(x^{d_1})$ for d_1 corresponding to the Gold or Kasami-Welch exponents [4]. It is known that if $f(x) = x^d$, then $f(x)$ is almost-bent (AB) if and only if $Tr(x^d)$ is near-bent [4]. Furthermore, if $f(x)$ is AB, then it is APN [2]. Thus, when constructing the codes, we select and analyze the following list of APN and AB exponents (Table 3).

Table 3 Exponents for the power function $f(x) = x^d$ over F_{2^m} such that f is APN. When m is odd, $m = 2r + 1$ [7, 9, 12]

Power functions	Exponent	Conditions	Exponent type
Gold	$2^l + 1$	$(l, m) = 1$	APN/Near-Bent
Kasami-Welch	$2^{2l} - 2^l + 1$	$(l, m) = 1$	APN/Near-Bent
Niho (Even)	$2^r + 2^{\frac{r}{2}} - 1$	m odd, r even	APN/Near-Bent
Niho (odd)	$2^r + 2^{\frac{3r+1}{2}} - 1$	m odd, r odd	APN/Near-Bent
Inverse	$2^n - 2$	m odd	APN
Dobbertin	$2^{\frac{4m}{5}} + 2^{\frac{3m}{5}} + 2^{\frac{2m}{5}} + 2^{\frac{m}{5}} - 1$	m divisible by 5	APN
Welch	$2^r + 3$	m odd	APN/Near-Bent
Nyberg	$2^{2l} + 2^l - 1$	$4l \equiv 3 \pmod{m}$	APN

2.1 Some New Results on the Enumeration of Near-Bent Functions via Cyclotomic Coset Analysis

We note the following, based on Definition 1, the Gold and Kasami-Welch near-bent functions in m variables have the form:

$$Tr(x^d) = x^d + x^{2d} + x^{4d} + \cdots + x^{(2^{m-1})d} \tag{4}$$

Consider the following definition for the 2-cyclotomic cosets $\pmod{2^m - 1}$ from [15]:

Definition 4 (*2-Cyclotomic Coset of* a) Consider $a \in Z_{2^m-1}$, the 2-cyclotomic coset $\pmod{2^m - 1}$ that contains a $(C(a))$ is of the form:

$$\{a, 2a, 4a, \ldots, 2^{j-1}a\} \tag{5}$$

where j is some divisor of m and $2^j * a \equiv a \pmod{2^m - 1}$.

We refer to this as the cyclotomic coset of a $(C(a))$ for the rest of this paper. Observe that the exponents in Definition 1 represent the cyclotomic coset containing d. Thus:

$$Tr(x^d) = Tr(x^{2d}) = \cdots = Tr(x^{(2^{m-1})d}). \tag{6}$$

Now consider the following for the minimal polynomial of α^{d_i} of degree j_2 over F_2:

$$m_{d_i}(x) = a_0 + a_1x^1 + a_2x^2 + \cdots x^{j_2}. \tag{7}$$

Since the coefficients of the polynomial are 0s or 1s, if α^{d_i} is a root, then so is any α^y such that $y \in C(d_i)$ (say $\alpha^{2^k d_i}$, k an integer such that $0 < k < m$):

$$\begin{aligned} m_{d_i}(\alpha^{2^k d_i}) &= a_0 + a_1(\alpha^{2^k d_i})^1 + a_2(\alpha^{2^k d_i})^2 + \cdots (\alpha^{2^k d_0})^{j_2} \\ &= a_0 + a_1(\alpha^{d_i})^{2^k} + a_2(\alpha^{d_i})^{2*2^k} + \cdots (\alpha^{d_i})^{j_2*2^k} \\ = (a_0 + a_1(\alpha^{d_i}) &+ a_2(\alpha^{d_i})^2 + \cdots (\alpha^{d_i})^{j_2})^{2^k} = (0)^{2^k} = 0 \end{aligned}$$

From [8] we can establish a clear relationship between the cyclotomic cosets and the roots of minimal polynomials. It is known that $X^{2^m} - X$ over F_{2^m} is the product of all minimal polynomials over F_2 whose degree divides m. From corollary 3 in [8] we have that $X^{2^m-1} - 1 = \sum_s (m_s(X))$ which iterates over a set of cyclotomic coset representatives. Thus, the size of the cyclotomic cosets must be a divisor of m.

With these results, we reduce the number of exponents needed to construct the cyclic codes and the associated functions. We choose the smallest exponent of the cyclotomic coset (called the cyclotomic coset representative) for these purposes. We do this for the Gold and Kasami-Welch exponents by studying the cyclotomic cosets that contain these exponents for $0 < l < m$. Given this, we determine the number of non-equal Gold near-bent functions and determine an upper bound for the number of non-equal Kasami-Welch near-bent function of the form $Tr(x^d)$:

Theorem 2 (The Number of Gold Exponents in The Cyclotomic Cosets $\pmod{2^m-1}$) *Let 2^l+1 be a Gold exponent. Gold exponents occur in pairs of the form $2^i+1, 2^j+1$ in a cyclotomic coset of size $a \pmod{2^m-1}$ where $j+i=a$.*

First, we establish a lemma that we will use in the proof of this theorem:

Lemma 2 *The sum $\sum_{i=0}^{n-1} a_i 2^i \le 2^n-1$, with coefficients from $\{0,1\}$. Equality holds if and only if all the coefficients are 1.*

Proof Follows from the 2-adic representation of integers. Or by mathematical induction.

The proof of the theorem is as follows:

Proof Let i, j, w, k, a, t and $m \in \mathbb{Z}$

Case 1: m is an odd prime.

We know from the preceding discussion that the size of the cyclotomic cosets $\pmod{2^m-1}$ is a non-trivial divisor of m, and since m is prime, all the cyclotomic cosets (aside from the one composed of only the 0 element) are of size m. Say we have the cyclotomic coset containing 2^i+1: $C(2^i+1)=\{2^i+1, 2*(2^i+1), \dots 2^{m-1}*(2^i+1)\}$. Consider j such that $j+i=m$,$0<j,i<m$ (if $j=i$ then the pair of 2^i+1 is itself) then we have $2^j(2^i+1)\in C(2^i+1)$ and

$$\begin{aligned} 2^j(2^i+1) &\equiv 2^{i+j}+2^j \pmod{2^m-1} \equiv 2^m+2^j \pmod{2^m-1} \\ \equiv 1+2^j \pmod{2^m-1}. \end{aligned} \tag{8}$$

Thus, we found a "pair" of Gold exponents in the same cyclotomic coset. Now we show that no other Gold exponent can be found.

Consider $k \ne j, i,\ 0<k, w<m,\ w\ne j$ such that: $2^w(2^i+1)\equiv 2^k+1 \pmod{2^m-1}$. Note that $w\ne k$ or else we have $(2^{i+k}+2^k)\equiv 2^k+1 \pmod{2^m-1} \to 2^{i+k}\equiv 1 \pmod{2^m-1} \to k=j$ (since both i and k are less than m). Furthermore, $w+i\ne m$ (forces w to equal j) nor $w+i=k$ as this would mean: $2^w(2^i+1)\equiv 2^{w+i}+2^w \pmod{2^m-1}\equiv 2^k+2^w \pmod{2^m-1}\equiv 2^k+1 \pmod{2^m+1}$ $\to 2^w\equiv 1 \pmod{2^m-1}\to w=m$ or $w=0$ which is false.

With the conditions given above, we have:

$$2^w(2^i+1)\equiv 2^{w+i}+2^w \pmod{2^m-1}\equiv 2^k+1 \pmod{2^m-1}. \tag{9}$$

Note that if $w+i>m$, say $w+i=e+m$, then $2^{w+i}\equiv 2^e \pmod{2^m-1}$ with 2^e being a power of 2 less than 2^m. And if $w+i=e<m$ we have the same situation. Thus, on both sides we have the sum of two distinct powers of 2 that are less than 2^m and, since m is a prime odd integer with a minimum value of 3, both sums are less than 2^m-1 (from Lemma 2). The left and right expressions are representatives of

distinct congruence classes $\pmod{2^m - 1}$ and, as such must be distinct. Thus, no such k and w exist, and we get our result.

Case 2: m is an even integer.

As discussed previously, the sizes of the cyclotomic cosets divide m, so in this case, we could have Gold exponents in a coset of length a, where $a|m$. For the case where the size of the coset is m, we can apply the same argument as the previous case to find precisely two Gold exponents in the coset whenever $m > 2$.

Consider the size of the coset to be, $0 < a < m$ that is:

$$C(2^i + 1) = \{2^i + 1, 2 * (2^i + 1) + \cdots 2^{a-1}(2^i + 1)\} \tag{10}$$

such that $a|m$. Consider $0 < j, i < a$, $j + i = a$ we have that $2^j(2^i + 1) \equiv 2^a + 2^j \pmod{2^m - 1} \equiv 2^a(1 + 2^{j-a}) \pmod{2^m - 1} \equiv 1 + 2^{j-a} \pmod{2^m - 1}$.

Furthermore, $j - a = -i$ (since $j + i = a$). And the inverse of a power of 2 $\pmod{2^m - 1}$ is another power of 2 (easy to see, since the cyclotomic coset containing 1 contains all the powers of 2 less than 2^m), let us call $-i = t$. Thus, we have $2^j(2^i + 1) \equiv 1 + 2^t \pmod{2^m - 1}$. This gives us the pair $2^i + 1, 2^t + 1$.

If $t \neq j$ and $t = i$, then $(2^j)(2^i + 1) \equiv 2^i + 1 \pmod{2^m - 1} \to a|\text{j}$ (since a is the size of the cyclotomic coset), but this is absurd as $j < a$. If $t = j, t \neq i$, we have $a = i + j = i + t = 0$ which is a contradiction. If $t = j$ and $t = i$, we get the same contradiction as before. If $t \neq j$ and $t \neq i$, we have: $2^t + 1 \in C(2^i + 1) = C(2^t + 1) = \{2^t + 1, 2^{t+1} + 2^1 + \cdots + 2^{t+a-1} + 2^{a-1}\}$. However, $a = i + j, t + i = 0 \to 2^{t+a-1} + 2^{a-1} \equiv 2^{t+i+j-1} + 2^{a-1} \pmod{2^m - 1} \equiv 2^{j-1} + 2^{a-1} \pmod{2^m - 1} \to 2^j + 2^a \equiv 2^t + 1 \pmod{2^m - 1}$ (due to the size of the coset being a). This is the sum of two distinct powers of 2 less than 2^m which means the equivalence does not hold (due to Lemma 2), thus, contradicting the size of the cyclotomic coset. Thus, such j, i, t cannot exist.

Consider the case $i = t = j = \frac{m}{2}, 0 < i, j \leq a$. Then:

$$2^{\frac{m}{2}}(2^{\frac{m}{2}} + 1) \equiv 2^m + 2^{\frac{m}{2}} \pmod{2^m - 1} \equiv 1 + 2^i \pmod{2^m - 1} \to a|j, \tag{11}$$

that is, $a|\frac{m}{2}$. Since we chose $j \leq a$ then $a = \frac{m}{2}$. Thus, a pair can only exist if $i = j = t = a = \frac{m}{2}$ and the pair is $2^{\frac{m}{2}} + 1$ and itself.

Could there exist some “pair" (other than itself) for $2^i + 1$, (i $= \frac{m}{2}$)? Consider:

$$2^w(2^i + 1) \equiv 2^k + 1 \pmod{2^m - 1}, \tag{12}$$

then $w \neq i, 0 < w, k < a$ and $k \neq w$ or else we have $2^{k+i} + 2^k \equiv 2^k + 1 \pmod{2^m - 1} \to 2^{k+i} \equiv 1 \pmod{2^m - 1} \to k + i \equiv 0 \pmod{m} \to k = \frac{m}{2} \pmod{m} \equiv i \pmod{m}$ which is a contradiction. Thus, with these conditions, we have $2^{w+i} + 2^w \equiv 2^k + 1 \pmod{2^m - 1}$ these are two distinct sums of powers of 2 less than $2^m - 1$, thus, the equivalence cannot hold and thus, such pair does not exist.

Note: Since the condition that $(l, m) = 1$ for the function to be near-bent, this case will not apply as it requires $l = \frac{m}{2}$

Case 3: m a non-prime odd integer.

Once more, we appeal to the argument about the size of the cyclotomic cosets. In this case the size will either be m or some odd integer a that divides m. In case 1 we have shown that if the coset is of size m, then there are precisely two Gold exponents if one is found in the coset; in case 2, we showed that if the size is $a < m$, then if we find a Gold exponent in the cyclotomic coset, the size must be $\frac{m}{2}$, but since m is odd, no such case can happen; thus, if we find a Gold exponent in a coset, there are precisely two of them. $\square$

Corollary 1 (Number of Gold Near-Bent Functions Over F_{2^m}) *Let the Gold near-bent Boolean function in m variables be of the form* $Tr(x^{2^l+1})$ *such that* $1 \leq l \leq m-1$ *and* $(l, m) = 1$. *Then, the number of non-equal Gold near-bent functions is precisely* $\frac{\Phi(m)}{2}$.

Proof The result follows from Theorem 2. The $(l, m) = 1$ condition follows from [4]. For m an odd integer we have precisely two Gold exponents in each cyclotomic coset that contains them. The possible values of l are $1, 2, 3 \ldots m-1$ and only those such that $(l, m) = 1$ lead to near-bent functions. The number of such values of l is given by Euler's Φ function and we divide this total by 2. Thus, $\frac{\Phi(m)}{2}$ will be the total number of Gold near-bent functions generated. $\square$

Theorem 3 (Number of Kasami-Welch Exponents in The Cyclotomic Cosets (mod $2^m - 1$) for m odd) *Let* $2^{2l} - 2^l + 1$ *be a Kasami-Welch exponent and m an odd integer. Then, the Kasami-Welch exponents occur in pairs of the form* $2^{2i} - 2^i + 1, 2^{2j} - 2^j + 1$, $j + i = m$ *in a cyclotomic coset that contains an exponent. Furthermore, there cannot be any such exponents in a cyclotomic coset of size less than m.*

Proof Let i, j, w, k, a, t and $m \in \mathbb{Z}$

Consider: $C(2^{2i} - 2^i + 1) = \{2^{2i} - 2^i + 1, 2(2^{2i} - 2^i + 1) + \cdots + 2^{m-1}(2^{2i} - 2^i + 1)\}$ as the cyclotomic coset of size m containing a Kasami-Welch exponent. Set $0 < i, j < m, i + j = m$. Now, consider:

$$\begin{aligned} 2^{2j}(2^{2i} - 2^i + 1) &= 2^{2(i+j)} - 2^{j+(j+i)} + 2^{2j} \\ &= 2^{2m} - 2^{j+m} + 2^{2j} \equiv 1 - 2^j + 2^{2j} \pmod{2^m - 1}, \end{aligned} \tag{13}$$

thus, we have found a "pair" of Kasami-Welch exponents that belong to the same cyclotomic coset. Note that this j and i meet the same criteria as the ones for the Gold case.

Consider the case where the size of the coset is $a < m$, where $a|m$. Let us say $i + j = a + e \leq m$ with $0 \leq e \leq m - 1$ and $0 < i, j \neq a$, thus, we have:

$$2^{2j}(2^{2i} - 2^i + 1) = 2^{2(a+e)} - 2^{j+a+e} + 2^{2j} = 2^{2a}(2^{2e} - 2^{j+e-a} + 2^{2(j-a)}) \tag{14}$$

and since this is in $C(2^{2i} - 2^i + 1)$ of size a, then we have

$$2^{2a}(2^{2e} - 2^{j+e-a} + 2^{2(j-a)}) \equiv (2^{2e} - 2^{j+e-a} + 2^{2(j-a)}) \pmod{2^m - 1}. \quad (15)$$

We note that $i \neq a$ because otherwise $j = e$ which means Eq. 15 transforms into: $(2^{2j} - 2^{2j-a} + 2^{2(j-a)}) \pmod{2^m - 1}$. For this to be a Kasami-Welch exponent, either 2^{2j} or $2^{2j-2a} \equiv 2^m \pmod{2^m - 1}$. In the first case, it would mean $m|2j$ which is a contradiction, since m is odd and $j < m$. The other case, $2^{2j-2a} \equiv 2^m \pmod{2^m - 1}$, means that $m|2(j-a)$ (given that $a, j < m$ and $j - a \equiv y \pmod{m}$ with $y < m$) which is a contradiction as m is an odd integer.

For the last expression in Eq. 15 to be a Kasami-Welch exponent, we must have that one of these powers of 2 must be equivalent to $2^m \pmod{2^m - 1}$. Assume $e \neq 0$; clearly 2^{2e} cannot be $\equiv 1 \pmod{2^m - 1}$ as $e \neq 0$, $e < m$. We also cannot have, $e = \frac{m}{2}$ as m is odd. Furthermore, since $a \neq j$, $2^{2(j-a)} \equiv 2^y$ for some $0 < y < m$ (based on the cyclotomic coset of powers of 2). Thus, $2^{2(j-a)} \not\equiv 1 \pmod{2^m - 1}$. The remaining term is negative, thus, the only way for this term to be Kasami-Welch is if $e = 0$. Thus, whatever pair lives in the coset that has this form must have $j + i = a$.

Finally consider the case where $j + i = a$, $0 < j, i, a < m$. Considering that $j = a - i$, Eq. 14 transforms into:

$$2^{2j}(2^{2i} - 2^i + 1) = 2^{2a} - 2^{a+j} + 2^{2j} = 2^{2a} - 2^{2a-i} + 2^{2a-2i}. \quad (16)$$

$$= 2^{2a}(1 - 2^{-i} + 2^{-2i}) \equiv 1 - 2^{-i} + 2^{-2i} \pmod{2^m - 1} \quad (17)$$

Say $-i \equiv t \pmod{m}$. Thus, we have:

$$2^{2j}(2^{2i} - 2^i + 1) = 1 - 2^t + 2^{2t}. \quad (18)$$

Since both Kasami-Welch exponents belong to the same cyclotomic coset, we must have $2^a(2^{2t} - 2^t + 1) = 2^{2t} - 2^t + 1$, however, $C(2^{2t} - 2^t + 1) = \{2^{2t} - 2^t + 1, 2(2^{2t} - 2^t + 1) + \cdots + 2^{a-1}(2^{2t} - 2^t + 1)\}$ with $2^{a-1}(2^{2t} - 2^t + 1) = (2^{2t+(j+i)-1} - 2^{t+j+i-1} + 2^{a-1}) = 2^{j+t-1} - 2^{j-1} + 2^{a-1}$.

Multiply by 2:

$$2^{j+t} - 2^j + 2^a \equiv 2^{2t} - 2^t + 1 \pmod{2^m - 1} \quad (19)$$

which is not possible unless $j + t = m$, $j = t$ or $j = i$. For $j + t = m$, we have that $j + i = m - t + i = m = a$ which is a contradiction. For $j = t$, this implies $j + i = -i + i = 0 = a$ which is absurd since $a|m$, and for $j = i$ this implies $a = 2j$ which means a is divisible by 2. This is because m is odd, and $a|m$; thus, a cannot be an even integer. Therefore, no Kasami-Welch exponent pairs can exist in a cyclotomic coset of size less than m. □

From this theorem, we state the following result:

Corollary 2 (Upper Bound for The Number of Kasami-Welch Near-Bent Functions Over F_{2^m}) *Let the Kasami-Welch near-bent Boolean function in m variables be of the form* $Tr(x^{2^{2l}-2^l+1})$ *such that* $1 \le l \le m-1$ *and* $(l, m) = 1$. *Then, the number of non-equal Kasami-Welch near-bent functions is bounded from above by* $\frac{\Phi(m)}{2}$.

Proof The result follows from Theorem 3. The $(l, m) = 1$ condition follows from [4]. For m an odd integer, we know that there are at least two Kasami-Welch exponents in each cyclotomic coset that contains them. The possible values of l are $1, 2, 3 \dots m-1$ and only those such that $(l, m) = 1$ lead to near-bent functions. The number of such values of l is given by Euler's Φ function, and we divide this total by 2 as there are at least two Kasami-Welch exponents per cyclotomic coset that contains them. Thus, $\frac{\Phi(m)}{2}$ will be the maximum number of Kasami-Welch near-bent functions generated.

We conjecture that, as in the Gold case, the number of non-equal near-bent functions of the Kasami-Welch type is precisely $\frac{\Phi(m)}{2}$.

Conjecture 3 (*Exact number of Kasami-Welch Near-Bent Functions Over* F_{2^m})
Let the Kasami-Welch near-bent Boolean function in m variables be of the form $Tr(x^{2^{2l}-2^l+1})$ such that $1 \le l \le m-1$ and $(l, m) = 1$. Then, the number of non-equal near-bent functions generated is precisely $\frac{\Phi(m)}{2}$.

In Appendix 3.3 (see Tables 8 and 9) we note the non-equal near-bent functions corresponding to the Gold and Kasami-Welch cases for up to 13 variables.

A concept we utilize in the following analysis is the that of the weight distribution of cyclic codes and how to represent it. For cyclic codes of length $2^m - 1$, the weight distribution of a code can be represented as a vector:

$$(A_0, A_1, \dots, A_n) \tag{20}$$

where A_w is the number of codewords of weight w. The enumerator polynomial, as defined in [11], is given by:

$$A(x) = \sum_{w=0}^{2^m-1} (A_w x^w) \tag{21}$$

The MacWilliams identity is given by:

Theorem 4 (MacWilliams Identity) *If* $A(x)$ *is the enumerator polynomial of a binary* ($2^m - 1, k$) *code* C, *then the enumerator polynomial* $B(x)$ *of the dual code* $C^\perp$ *is given by:*

$$B(x) = 2^{-k}(1+x)^{2^m-1} A\left(\frac{1-x}{1+x}\right). \tag{22}$$

2.2 An Improved Algorithm to Compute the Minimum Distance of Linear Codes and Classification of APN and Near-Bent Functions

The purpose of this section is to construct cyclic codes in 2 roots utilizing known APN and AB exponents. We test their minimum distance properties and compare them to the results by Janwa and Wilson in [10]. We utilize the GUAVA package of the GAP software system for faster computation of the minimum distance in our algorithm in SAGE. We develop algorithms to study these codes.

Since many of the known APN exponents are defined for an odd number of variables, and we focus mainly on the AB/near-bent exponents, we first define an algorithm that verifies if a Boolean function in m (odd) variables of the form $Tr(x^d)$ is near-bent. The near-bent property is verified via the Walsh-Hadamard transform definition given in [4]:

Definition 5 (*Near-Bent Boolean Functions*)
Let f be a Boolean function in m (odd) variables. Then, f is near-bent if:

$$|\hat{F}(\omega)| \in \{2^{\frac{m+1}{2}}, 0\} \text{for every} \omega \in F_{2^m}.$$

```
def NearBent(f):
  dim = f.nvariables()
 w = f.walsh_hadamard_transform(), k = (dim + 1)/2
  for i in range(2^dim):
    if abs(w[i]) != 0 and abs(w[i]) != 2^k:
      return "Not Near-Bent"
    else:
      i = i
  return "Near-Bent"
```

The input is a Boolean function “f” in m variables and the algorithm determines its Walsh-Hadamard transform spectrum. The Boolean function input is as in the “Bentness” algorithm and thus given by $Tr(f(x))$ [13]. Then it verifies if the function is near-bent by iterating over the spectrum to check if it meets the definition. We note that since $Tr(x^d)$ is near-bent if and only if $f(x) = x^d$, $f : F_{2^m} \to F_{2^m}$ is AB, and if $f(x)$ is AB, then it is APN, this algorithm also verifies if $f(x)$ is APN.

The following algorithm is a general algorithm used to construct cyclic codes of length $n = 2^m - 1$ with defining set $D = \{1, d\}$ where d is some cyclotomic coset representative $\pmod{2^m - 1}$. Properties such as weight distribution of the dual code, number of non-zero weights of the dual code, and minimum distance are computed through SAGE built-in functions.

```
def Ccode2r(m):
  N = [], I = [], D = [], W = [], n = 2^m -1
```

```
    print("Code - Minimum Distance - Defining Set - APN
Exponents - Enumerator Polynomial - Weight Coeffic-
ients - Nonzero Weights - Weight Distribution ")
    for i in range(1,len(CosetsRep[m])):
      cr = CosetsRep[m][i]
      C = codes.CyclicCode(field=GF(2), length=n,
D=[1, cr])
      h = C.check_polynomial()
      DC = codes.CyclicCode(generator_pol = h.reverse(),
length = n)
      sd = DC.spectrum(algorithm = "binary")
      indexlist = [i for i, e in enumerate(sd) if
e != 0]
      indexlist = [y for y in indexlist if y != 0]
      a = [y for y in sd if y != 0], g = 0
      for w in range(0,n+1):
        g = g + sd[w]*(x)^(w)
      cy = Cyclo(cr,n)
      print ("     n", C, " - ", C.minimum_distance(algorithm =
'guava'), " - ", 1, cr, " - ", end = "")
      if set(cy) & set(G) != set():
        print (" Gold",end = "")
      if set(cy) & set(K) != set():
        print (" Kasami Welch",end = "")
      if m%2 == 1:
        I.append(2^m - 2), W.append(2^((m-1)/2) + 3)
        if set(cy) & set(I) != set():
          print (" Inverse",end = "")
        if set(cy) & set(W) != set():
          print (" Welch",end = "")
        if m%4 == 1:
          N.append(2^((m - 1)/2) + 2^((m -1)/4) - 1)
          if set(cy) & set(N) != set():
            print (" Niho even case",end = "")
        if m%4 == 3:
          N.append(2^((m-1)/2) + 2^((3*m-1)/4) - 1)
          if set(cy)& set(N) != set():
            print (" Niho odd case",end = "")
        if m%5 == 0:
          D.append(2^(4*m/5) + 2^(3*m/5) + 2^(2*m/5) +
2^(m/5) - 1)
          if set(cy) & set(D) != set():
            print (" Dobbertin",end = "")
        if set(cy) & set(Ny) != set():
          print (" Nyberg",end = "")
```

```
  nonzl = len(a) - 1
 print(" - $", g, "$ - ", a, " - ", nonzl, " - ", indexlist)
```

The algorithm takes as an input the number of variables "*m*", assigns the length of the code to the variable *n*, and defines empty lists in which its stores the APN exponents considered. The Nyberg, Gold, and Kasami-Welch exponents were pre-generated. The headings of the tables are printed and then it iterates over a list of all the cyclotomic coset representatives $\pmod{2^m - 1}$. The algorithm assigns the defining set, constructs the cyclic code, finds its check polynomial and reverses it to construct the dual code, finds its spectrum (weight distribution), and puts the non-zero weights on a list. Then, it constructs the enumerator polynomial of the dual code. It then finds the cyclotomic coset $\pmod{2^m - 1}$ of the current exponent under consideration, identifies if an element of this coset coincides with one in the cyclotomic coset containing an APN exponent. We obtain the results in Table 10 from this algorithm.

```
def cycloequiv(m):
  f = FindModNoncube(m)
  R.<x> = GF(2^m,'a', modulus = f)[]
 k. < a >= GF(2 ∗ ∗m, modulus = f)
  CRL = RepCyclo(m), n = 2^m -1, Cl = Cyclo(1,n)
  for i in range(len(CRL)):
    if w in range(len(CRL)):
      if (CRL[i]*CRL[w])%(n) in Cl and gcd(CRL[i],n)
== 1:
        print(",f(x) = $x^{", CRL[i],"}$ is cyclotomic
equivalent to g(x) = $x^{", CRL[w],"}$")
```

This algorithm verifies the second condition for cyclotomic-equivalence between power functions over F_{2^m} as in Definition 6 and in [1]. The algorithm takes as an input the number of variables, then finds an irreducible polynomial of degree *m* (over F_{2^m}) and then constructs a Boolean polynomial Ring over F_{2^m} such that it assigns "a" as the congruence class $[x]$ modulo f (the irreducible polynomial from the previous algorithm). Then, it constructs a finite field as powers of "a" where "a" is a primitive element in the field. It then assigns a list of all the cyclotomic coset representatives $\pmod{2^m - 1}$ to the variable CRL, assigns $n = 2^m - 1$ and the cyclotomic coset of $1 \pmod{n}$ to the list CL. Then, it iterates over the list CRL and verifies for every pair of cyclotomic coset representatives if the second condition of the cyclotomic-equivalence definition is met. The results from this algorithm were organized and the exponents corresponding to the list of functions that lead to 2-error-correcting codes (obtained from the second algorithm in this section) were compared and identified as equivalent to an APN function or not.

```
def MinimumDistance2r(m):
  n = 2^m -1
  print("Code - Minimum Distance - Defining Set -
Weight Coefficients - Nonzero Weights")
```

```
    for i in range(0,len(CrepNB[m])):
      cr = CrepNB[m][i],D = [1, cr]
      C = codes.CyclicCode(field=GF(2), length=n,
D=[1, cr])
     h = C.check_polynomial()
     DC = codes.CyclicCode(generator_pol = h.reverse(),
length = n)
      sd = DC.spectrum(algorithm = "binary")
      a = [y for y in sd if y != 0]
      g2 = 2^(-C.dimension())*(1+x)^(n),az = 0
      for w2 in range(0, n+1):
        az = az + sd[w2]*((1-x)/(1+x))^(w2)
      g2 = g2*az
      for i2 in range(1, degree(g2)):
        g2 = sym.diff(g2)
        if g2.subs(x:0) != 0:
          d = i2
          break
      print(C, "-",d, "-", D, "-", len(a) - 1)
```

The algorithm takes as input the number of variables considered, assigns the length of the corresponding codes to n, and prints the table headings. Then, it begins iterating over the list of cyclotomic coset representatives of the near-bent and APN exponents and constructs the cyclic code C with defining set $D = \{1, cr\}$. The generator polynomial for the dual code $C^{\perp}$ is obtained by taking the check polynomial of C and reversing it. The spectrum (weight distribution) of this code is computed, we take the non-zero weights (which correspond to the indices in the distribution) and begin construction of the enumerator polynomial of C by applying the MacWilliams identity. Once this polynomial is obtained, we iterate over the derivatives of the polynomial and evaluate them at $x = 0$, with the first non-zero result meaning that the lowest (non-zero) degree term has been reduced to a constant, this degree corresponding to the minimum weight (which is the same as the minimum distance of a linear code [11]).

2.3 Results

We construct Table 10 in Appendix 3.4 based on the second algorithm in this section. We note that there are 66 distance 5 codes from this construction. There are several functions related to these codes that are not directly identified with APN or AB exponents. In particular, the exponent of the form $2^{m-3} - 1$ (dubbed as the "pre-Inverse") case is interesting, as for an odd integer m, it seems to be a distinct exponent that leads to distance 5 codes. Equivalence analysis of the functions was performed to verify if some correspond through some criteria to known APN functions.

It is important to note that while we consider the cyclotomic coset representative criteria (as discussed at the beginning of this chapter), there are still other equivalence criteria to consider. CCZ-equivalence (named after Carlet, Charpin, and Zinoviev) is the most general equivalence criterion for functions that preserves the APN property [3]. In [5], it is shown that two APN power functions are CCZ-equivalent if and only if they are cyclotomic-equivalent. We state the definition from [1] of this equivalence for APN power functions over F_{2^m}.

Definition 6 (*Cyclotomic-equivalence*) Consider $l, k, a \in \mathbb{Z}$ with $0 \leq a \leq m$. Define $f(x) = x^k$ and $g(x) = x^l$ power functions over F_{2^m}. Then they are cyclotomic-equivalent if there exist a such that for (k,$2^m - 1$) = 1, $l \equiv k * 2^a \pmod{2^m - 1}$, $k * l \equiv 2^a \pmod{2^m - 1}$.

The first condition states that l and k are in the same cyclotomic coset. We have already eliminated this possibility by taking the cyclotomic cosets representatives of the APN and AB exponents and verifying that they are distinct. The second condition states that k and the inverse of l (or vice versa) share a cyclotomic coset. We construct the third algorithm in this section to verify this equivalence and apply it to construct a table that verifies if the observed "new" functions that lead to 2-error-correcting codes are APN or not. Note that since the functions are cyclotomic-equivalent, the corresponding cyclic codes are as well. The results are observed in Table 11.

Note that the "pre-Inverse" case that we observed before is APN only when it is equivalent to a Gold function (5 and 7 variables). From [10] we know that if an exponent d leads to a 2-error-correcting codes, then $f(x) = x^d$ has to be APN. Thus, we establish that the minimum-distance commands from SAGE have some errors. We propose a new minimum distance algorithm using MacWilliams identity and derivatives of the enumerator polynomial for codes with a low number of weights on their respective dual code. This is the fourth algorithm in the previous section. The results from this algorithm are seen in Table 12 and are verified and complemented by the APN conjecture and its relation to 2-error-correcting codes.

In Table 12, we observe that the weight distribution of the dual codes contains mostly 3 non-zero weights. Furthermore, the dimension of the dual codes is at most $2 * m$ while the dimension of the original code is at least $2^m - 1 - 2 * m$, which is a vastly greater number as m increases. We apply Theorem 4 to the enumerator polynomial of the dual code but do not expand the resulting polynomial. We then compute the successive derivatives of the resulting polynomial and evaluate at $x = 0$. The first non-zero result obtained will determine the minimum distance of the code. Thus, a smaller number of codewords to consider, fewer computations by not expanding these polynomials, and results that match the theory show that this algorithm improves the current standards by SAGE.

Acknowledgements This research is funded by the PR Space Grant Consortium, NASA Cooperative Agreement No. 80NSSC20M0052 and NNX15AI11H. We would like to thank the reviewers for reading the manuscript carefully and for their helpful comments and suggestions.

3 Appendix

3.1 Gold Case Tables

Every function in m variables of the form $Tr(\alpha^i x^{2^l+1})$ is bent except the ones listed as exceptions on the following tables. The notation used is:

(1) Mk: set of values of i where i is a multiple of k.
(2) $d = 2^l + 1$; i.e., the Gold exponent
(3) We list the cases that coincide in the same row, with the predictions and results in the position corresponding to the position of the exponent in the "l" column.
(4) α is a primitive element in F_{2^m} (Tables 4, 5 and 6).

Table 4 Gold bent functions exceptions test 4–10 variables

Variables	l	Case	Prediction	Computational results
4	1, 3	1	M3, M3	M3, M3
4	2	3	M5	M5
6	1, 5	1	M3, M3	M3, M3
6	2, 4	2	M1, M1	M1, M1
6	3	3	M9	M9
8	1, 3, 5, 7	1	M3, M3, M3, M3	M3, M3, M3, M3
8	2, 4, 6	3	M5, M17, M (divisor of 65)	M5, M17, M5
10	1, 3, 7, 9	1	M3, M3, M3, M3	M3, M3, M3, M3
10	2, 4, 6, 8	2	M1, M1, M1, M1	M1, M1, M1, M1
10	5	3	M33	M33

Table 5 Gold bent functions exceptions test 12–14 variables. We define i as a divisor of a gold exponent

Variables	l	Case	Prediction	Computational results
12	1, 5,7 , 11	1	M3, M3, M3, M3	M3, M3, M3, M3
12	2, 3, 6, 9, 10	3	M5, Mi, M65, Mi, Mi	M5, M9, M65, M9, M5
12	4, 8	2	M1, M1	M1, M1
14	1, 3, 5, 9, 11 13	1	M3, M3, M3, M3, M3 M3	M3, M3, M3, M3, M3 M3
14	2, 4, 6, 8, 10 12	2	M1, M1, M1, M1, M1 M1	M1, M1, M1, M1, M1 M1
14	7	3	M129	M129

Table 6 Gold bent function exceptions partial test 16–24 variables

Variables	l	Case	Prediction	Computational results
16	1, 3, 5, 7, 9 11, 13, 15	1	M3, M3, M3, M3, M3 M3, M3	M3, M3, M3, M3, M3 M3, M3
16	2, 4, 6, 8,1 0 12, 14	3	M5, M17, Mi, Mi, Mi Mi, Mi	M5, M17, M5, M257, M5 M17, M5
18	1, 5, 7, 11, 13 17	1	M3, M3, M3, M3, M3 M3	M3, M3, M3, M3, M3 M3
18	2, 4, 6, 8, 10 12, 14, 16	2	M1, M1, M1, M1, M1 M1, M1, M1	M1, M1, M1, M1, M1 M1, M1, M1
18	3, 9, 15	3	Mi, M513, Mi	M9, M513, M9
20	1, 3, 7, 9, 11 13	1	M3, M3, M3, M3, M3 M3	M3, M3, M3, M3, M3 M3
20	2, 5, 6, 10, 14	3	M5, Mi, Mi, M1025, Mi	M5, M33, M5, M1025, M5
20	4, 8, 12	2	M1, M1, M1	M1, M1, M1
22	1, 3, 5, 7, 9	1	M3, M3, M3, M3, M3	M3, M3, M3, M3, M3
22	2, 4, 6, 8, 10 12	2	M1, M1, M1, M1, M1 M1	M1, M1, M1, M1, M1 M1
22	11	3	M2049	M2049
24	1, 5, 7, 11, 13	1	M3, M3, M3, M3, M3	M3, M3, M3, M3, M3
24	2, 3, 4, 6, 9 10, 12	3	M5, Mi, M17, Mi, Mi Mi, Mi	M5, M9, M17, M65, M9 M5, M4097
24	8	2	M1	M1

3.2 *Kasami-Welch Case Tables*

Functions in m variables of the form $Tr(\alpha^i x^{2^{2l}-2^l+1})$ in 6 and 12 variables that are bent given are in the table that follows. The notation used is below:

(1) Mk: set of values of i where i is a multiple of k.
(2) $d = 2^{2l} - 2^l + 1$; i.e., the Kasami-Welch exponent
(3) α is a primitive element in F_{2^m} (Table 7).

Table 7 Kasami-Welch bent Boolean function computations 6, 12 variables. The functions are of the form $Tr(\alpha^i x^d)$

Variables	Exponent	Power of the element	Condition	Is it Bent?
6	3	a^i	$(l, m) = 1, i \notin M3$	Bent
6	993	a^i	$(l, m) = 1, i \notin M3$	Bent
12	3	a^i	$(l, m) = 1, i \notin M3$	Bent
12	993	a^i	$(l, m) = 1, i \notin M3$	Bent
12	16257	a^i	$(l, m) = 1, i \notin M3$	Bent
12	4192257	a^i	$(l, m) = 1, i \notin M3$	Bent

3.3 Near-Bent Functions Tables

See Tables 8, 9 and 10.

Table 8 Non-equal Gold near-bent Boolean functions of the form $Tr(x^{2^l+1})$ from 3 to 13 variables. Equivalent functions or cases where $(l, m) \neq 1$ are left blank

Exponent	L	Mod $2^3 - 1$	Mod $2^5 - 1$	Mod $2^7 - 1$	Mod $2^9 - 1$	Mod $2^{11} - 1$	Mod $2^{13} - 1$
3	1	Tr(x^3)	Tr(x^3)	Tr(x^3)	Tr(x^3)	Tr(x^3)	Tr(x^3)
5	2		Tr(x^5)	Tr(x^5)	Tr(x^5)	Tr(x^5)	Tr(x^5)
9	3			Tr(x^9)		Tr(x^9)	Tr(x^9)
17	4				Tr(x^{17})	Tr(x^{17})	Tr(x^{17})
33	5					Tr(x^{33})	Tr(x^{33})
65	6						Tr(x^{65})

Table 9 Non-equal Kasami-Welch Boolean near-bent functions of the form $Tr(x^{2^{2l}-2^l+1})$ from 3 to 13 variables. Equivalent functions or cases where $(l, m) \neq 1$ are left blank

Exponent	L	Mod $2^3 - 1$	Mod $2^5 - 1$	Mod $2^7 - 1$	Mod $2^9 - 1$	Mod $2^{11} - 1$	Mod $2^{13} - 1$
3	1	Tr(x^3)	Tr(x^3)	Tr(x^3)	Tr(x^3)	Tr(x^3)	Tr(x^3)
13	2		Tr(x^7)	Tr(x^{13})	Tr(x^{13})	Tr(x^{13})	Tr(x^{13})
57	3			Tr(x^{23})		Tr(x^{57})	Tr(x^{57})
241	4				Tr(x^{47})	Tr(x^{143})	Tr(x^{241})
993	5					Tr(x^{95})	Tr(x^{287})
4033	6						Tr(x^{191})

Table 10 Cyclic-codes with minimum distance 5 from the 'guava' package in SAGE

n	k	d(C)	Defining set	APN exponent		F(x)	Near-Bent?	Bent?
15	7	5	{1, 3}	Gold	Kasami-Welch	X^3		No
31	21	5	{1, 3}	Gold	Kasami-Welch	X^3	Yes	
31	21	5	{1, 5}	Gold	Niho even	X^5	Yes	
31	21	5	{1, 7}	Welch	Nyberg	X^7	Yes	
31	21	5	{1, 11}	Kasami-Welch		X^11	Yes	
31	21	5	{1, 15	Inverse	Dobbertin	X^15	No	
63	51	5	{1, 3}	Gold	Kasami-Welch	X^3		No
127	113	5	{1, 3}	Gold	Kasami-Welch	X^3	Yes	
127	113	5	{1, 5}	Gold		X^5	Yes	
127	113	5	{1, 9}	Gold		X^9	Yes	
127	113	5	{1, 11}	Welch		X^11	Yes	
127	113	5	{1, 13}	Kasami-Welch		X^13	Yes	
127	113	5	{1, 15}			X^15	Yes	
127	113	5	{1, 23}	Kasami-Welch		X^23	Yes	
127	113	5	{1, 27}			X^27	Yes	
127	113	5	{1, 29}	Niho odd		X^29	Yes	
127	113	5	{1, 43}			X^43	Yes	
127	113	5	{1, 63}	Inverse	Nyberg	X^63	No	
255	239	5	{1, 3}	Gold	Kasami-Welch	X^3		No
255	239	5	{1, 9}	Gold		X^9		No
255	239	5	{1, 15}			X^15		Yes
255	239	5	{1, 39}	Kasami-Welch		X^39		No
255	239	5	{1, 45}			X^45		Yes

(continued)

Table 10 (continued)

n	k	d(C)	Defining set	APN exponent		F(x)	Near-Bent?	Bent?
511	493	5	{1, 3}	Gold	Kasami-Welch	X^3	Yes	
511	493	5	{1, 5}	Gold		X^5	Yes	
511	493	5	{1, 13}	Kasami-Welch		X^13	Yes	
511	493	5	{1, 17}	Gold		X^17	Yes	
511	493	5	{1, 19}	Welch	Niho even	X^19	Yes	
511	493	5	{1, 27}			X^27	Yes	
511	493	5	{1, 31}			X^31	Yes	
511	493	5	{1, 47}	Kasami-Welch		X^47	Yes	
511	493	5	{1, 59}			X^59	Yes	
511	493	5	{1, 63}			X^63	No	
511	493	5	{1, 87}			X^87	Yes	
511	493	5	{1, 103}			X^103	Yes	
511	493	5	{1, 171}			X^171	Yes	
511	493	5	{1, 255}	Inverse		X^255	No	
1023	1003	5	{1, 3}	Gold	Kasami-Welch	X^3		No
1023	1003	5	{1, 9}	Gold		X^9		No
1023	1003	5	{1, 57}	Kasami-Welch		X^57		No
1023	1003	5	{1, 213}	Dobbertin		X^213		No
1023	1003	5	{1, 237}			X^237		No
2047	2025	5	{1, 3}	Gold	Kasami-Welch	X^3	Yes	
2047	2025	5	{1, 5}	Gold		X^5	Yes	
2047	2025	5	{1, 9}	Gold		X^9	Yes	
2047	2025	5	{1, 13}	Kasami-Welch		X^13	Yes	
2047	2025	5	{1, 17}	Gold		X^17	Yes	

(continued)

Table 10 (continued)

n	k	d(C)	Defining set	APN exponent		F(x)	Near-Bent?	Bent?
2047	2025	5	{1, 33}	Gold		X^33	Yes	
2047	2025	5	{1, 35}	Welch		X^35	Yes	
2047	2025	5	{1, 43}			X^43	Yes	
2047	2025	5	{1, 57}	Kasami-Welch		X^57	Yes	
2047	2025	5	{1, 63}			X^63	Yes	
2047	2025	5	{1, 95}	Kasami-Welch		X^95	Yes	
2047	2025	5	{1, 107}			X^107	Yes	
2047	2025	5	{1, 117}			X^117	Yes	
2047	2025	5	{1, 143}	Kasami-Welch		X^143	Yes	
2047	2025	5	{1, 151}			X^151	Yes	
2047	2025	5	{1, 231}			X^231	Yes	
2047	2025	5	{1, 249}	Niho odd		X^249	Yes	
2047	2025	5	{1, 255}			X^255	No	
2047	2025	5	{1, 315}			X^315	Yes	
2047	2025	5	{1, 365}			X^365	Yes	
2047	2025	5	{1, 411}			X^411	Yes	
2047	2025	5	{1, 413}			X^413	Yes	
2047	2025	5	{1, 683}			X^683	Yes	
2047	2025	5	{1, 1023}	Inverse		X^1023	No	

3.4 Minimum Distance Computation Tables

For the following computations, we consider cyclic codes of length $n = 2^m - 1$, dimension k, and minimum distance d(C) over F_{2^n} (Tables 11 and 12).

Table 11 Cyclotomic-equivalence analysis of "new" 2-error-correcting codes functions

n	k	Defining set	$f(x)$	APN?	Equivalence
127	113	$\{1, 15\}$	X^15	Yes	f(x) = x^{15} $\equiv$ g(x) = x^9 , Gold
127	113	$\{1, 27\}$	X^27	Yes	f(x) = x^{27} $\equiv$ g(x) = x^5 , Gold
127	113	$\{1, 43\}$	X^43	Yes	f(x) = x^{43} $\equiv$ g(x) = x^3 , Gold
511	493	$\{1, 27\}$	X^27	Yes	f(x) = x^{27} $\equiv$ g(x) = x^{19} , Niho and Welch
511	493	$\{1, 31\}$	X^31	Yes	f(x) = x^{31} $\equiv$ g(x) = x^{17} , Gold
511	493	$\{1, 59\}$	X^59	Yes	f(x) = x^{59} $\equiv$ g(x) = x^{13} , Kasami-Welch
511	493	$\{1, 63]$	X^63	No	No Equivalence
511	493	$\{1, 87\}$	X^87	Yes	f(x) = x^{87} $\equiv$ g(x) = x^{47} , Kasami-Welch
511	493	$\{1, 103\}$	X^103	Yes	f(x) =x^{103} $\equiv$ g(x) = x^5 , Gold
511	493	$\{1, 171\}$	X^171	Yes	f(x) =x^{171} $\equiv$ g(x) = x^3 , Gold
2047	2025	$\{1, 43\}$	X^43	Yes	f(x) = x^{43} $\equiv$ g(x) = x^{143} , Kasami-Welch
2047	2025	$\{1, 63\}$	X^63	Yes	f(x) =x^{63} $\equiv$ g(x) = x^{33} , Gold
2047	2025	$\{1, 107\}$	X^107	Yes	f(x) =x^{10} $\equiv$ g(x) = x^{249} , Niho
2047	2025	$\{1, 117\}$	X^117	Yes	f(x) =x^{117} $\equiv$ g(x) = x^{35} , Welch
2047	2025	$\{1, 151\}$	X^151	Yes	f(x) =x^{151} $\equiv$ g(x) = x^{95}, Kasami-Welch
2047	2025	$\{1, 231\}$	X^231	Yes	f(x) = x^{231} $\equiv$ g(x) = x^9 , Gold
2047	2025	$\{1, 255\}$	X^255	No	f(x) = x^{255} $\equiv$ g(x) = x^{731} , not APN
2047	2025	$\{1, 315\}$	X^315	Yes	f(x) = x^{315} $\equiv$ g(x) = x^{13}, Kasami-Welch
2047	2025	$\{1, 365\}$	X^365	Yes	f(x) = x^{365} $\equiv$ g(x) = x^{17}, Gold
2047	2025	$\{1, 411\}$	X^411	Yes	f(x) = x^{411} $\equiv$ g(x) = x^5, Gold
2047	2025	$\{1, 413\}$	X^413	Yes	f(x) = x^{413} $\equiv$ g(x) = x^{57}, Kasami-Welch
2047	2025	$\{1, 683\}$	X^683	Yes	f(x) = x^{683} $\equiv$ g(x) = x^3, Gold

Table 12 Minimum distance computations with alternative method to SAGE

Code	d(C)	Defining set	Non-zero weights of $C^{\perp}$
{7, 1}	7	{1, 3}	3
{31, 21}	5	{1, 3}, {1, 5}, {1, 7}, {1, 11}	3
{31, 21}	5	{1, 15}	6
{127, 113}	5	{1, 3}, {1, 5}, {1, 9}, {1, 11} {1, 13}, {1, 15}, {1, 23}, {1, 29}	3
{127, 113}	5	{1, 63}	11
{511, 493}	5	{1, 3}, {1, 5},{1, 9},{1, 13} {1, 17},{1, 19},{1, 47},{1, 57}	3
{511, 493}	4	{1, 63}	15
{511, 493}	5	{1, 255}	23
{2047, 2025}	5	{1, 3},{1, 5},{1, 9},{1, 13} {1, 17},{1, 33},{1, 35},{1, 57} {1, 95},{1, 143},{1, 249}	3
{2047, 2025}	4	{1, 255}	16
{2047, 2025}	5	{1, 1023}	45

References

1. Budaghyan, L., Calderini, M., Villa, I.: On relations between CCZ-and EA-equivalences. Crypt. Commun. **12**(1), 85–100 (2020)
2. Canteaut, A., Charpin, P., Dobbertin, H.: A new characterization of almost bent functions. In: Knudsen, L. (ed.) Fast Software Encryption, pp. 186–200. Springer, Berlin (1999)
3. Canteaut, A., Perrin, L.: On CCZ-equivalence, extended-affine equivalence, and function twisting. Finite Fields Their Appl. **56**, 209–246 (2019)
4. Carlet, C., Mesnager, S.: Four decades of research on bent functions. Des. Codes Crypt. **78**(1), 5–50 (2016)
5. Dempwolff, U.: CCZ equivalence of power functions. Des. Codes Crypt. **86**(3), 665–692 (2018)
6. Dillon, J., Dobbertin, H.: New cyclic difference sets with Singer parameters. Finite Fields Their Appl. **10**(3), 342–389 (2004)
7. McGuire, G.: APN Functions, APN Codes and S-Boxes (2008). https://www.cosic.esat.kuleuven.be/nato_arw/slides_participants/McGuire_slides_nato08.pdf
8. Geraci, J., Bussel, F.V.: A note on cyclotomic cosets, an algorithm for finding coset representatives and size, and a theorem on the quantum evaluation of weight enumerators for a certain class of cyclic codes. CoRR (2007). arxiv:abs/cs/0703129
9. Gorodilova, A.: On a remarkable property of APN Gold functions. IACR Cryptol. ePrint Arch. **2016**, 286 (2016)
10. Janwa, H., Wilson, R.M.: Hyperplane sections of fermat varieties in P3 in char. 2 and some applications to cyclic codes. In: Cohen, G., Mora, T., Moreno, O. (eds.) Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, pp. 180–194. Springer, Berlin (1993)
11. Justesen, J., Høholdt, T.: A Course in Error-Correcting Codes, vol. 1. European Mathematical Society (2004)
12. Kyureghyan, G., Suder, V.: On inverses of APN exponents. In: IEEE International Symposium on Information Theory—ISIT 2012, pp. 1207–1211. Cambridge, MA, United States (2012)
13. Makarim, R., Laigle-Chapui, Y.: Boolean functions (2016). https://doc.sagemath.org/html/en/reference/cryptography/sage/crypto/boolean_function.htm. Accessed 7 Dec 2019

14. Roe, D., Flori, J.-P., Bruin, P.: Routines for conway and pseudo-conway polynomials (2020). https://doc.sagemath.org/html/en/reference/finite_rings/sage/rings/finite_rings/conway_polynomials.html?highlight=conway#module-sage.rings.finite_rings.conway-polynomials. Accessed 10 Feb 2020
15. Wong, D.: Cyclotomic cosets, codes and secret sharing. Malays. J. Math. Sci. **11**, 59–73 (2017)
16. Youssef, A.M., Gong, G.: Hyper-bent functions. In: Pfitzmann, B. (ed.) Advances in Cryptology—EUROCRYPT 2001, pp. 406–419. Springer, Berlin (2001)