Distributions / Inference Exercise Solutions

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① (a) Note that $g(X) = X^2$ is not one-to-one, so we can't just use the change-of-variables formula. One option is to find the MGF and recognize it as the MGF of a χ^2_1 random variable. Alternately, start with the CDF of $Y = X^2$.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) \\
&= \Phi(\sqrt{y}) - (1 - \Phi(\sqrt{y})) \\
&= 2\Phi(\sqrt{y}) - 1
\end{aligned}$$

$$\begin{aligned}
\Rightarrow f_Y(y) &= 2\,\phi(\sqrt{y}) \cdot \tfrac{1}{2} y^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \cdot y^{-\frac{1}{2}}
\end{aligned}$$

$$\Rightarrow Y \sim \chi^2_1$$

(In the final expression, note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.)

① (b) By symmetry, $E[X^k] = 0$ for all odd k.

(c) $$\begin{aligned} \mathrm{Cov}(X, X^2) &= E[X^3] - E[X]E[X^2] \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

(But as in the slides, note that X and X^2 are <u>not</u> independent!)

(d) $$\begin{aligned} E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 + tx} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} \, dx \qquad \text{(complete the square!)} \\ &= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2}}_{\text{This is the pdf of a } N(t,1) \text{ rv, so it integrates to } 1.} \, dx \\ &= e^{\frac{1}{2}t^2} \end{aligned}$$

(e) Given $Z \sim N(\mu, \sigma^2)$, we can write $Z \overset{d}{=} \sigma X + \mu$.
So, using properties of the MGF (see slides), we get

$$M_Z(t) = e^{\mu t} \cdot e^{\frac{1}{2}\sigma^2 t^2}$$

② (a) Let $W = \frac{X_1}{X_1+X_2}$, $T = X_1 + X_2$.

$\Rightarrow X_1 = TW$, $X_2 = T(1-W)$

Using the change-of-variables formula, we get:

$$f_{T,W}(t,w) = f_{X_1,X_2}(x_1,x_2) \cdot \left| \begin{matrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_2}{\partial t} \\ \frac{\partial x_1}{\partial w} & \frac{\partial x_2}{\partial w} \end{matrix} \right|$$

$$= \frac{\xi^a}{\Gamma(a)} \cdot x_1^{a-1} e^{-\xi x_1} \cdot \frac{\xi^b}{\Gamma(b)} x_2^{b-1} e^{-\xi x_2} \cdot \left| \begin{matrix} w & 1-w \\ t & -t \end{matrix} \right|$$

$$= \frac{\xi^a}{\Gamma(a)} (tw)^{a-1} e^{-\xi(tw)} \cdot \frac{\xi^b}{\Gamma(b)} t^{b-1} (1-w)^{b-1} \cdot e^{-\xi t(1-w)} \cdot t$$

$$= \frac{\xi^{a+b}}{\Gamma(a)\Gamma(b)} t^{a+b-1} \cdot e^{-\xi t} \cdot w^{a-1} (1-w)^{b-1}$$

Finally, integrate out t to get $f_W(w)$.

$$f_W(w) = \frac{\xi^{a+b}}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1} \int_0^\infty \underbrace{t^{a+b-1} e^{-\xi t}}_{\text{Kernel of Gamma}(a+b,\, \xi) \text{ r.v.}} dt$$

$$= \frac{\xi^{a+b}}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1} \cdot \frac{\Gamma(a+b)}{\xi^{a+b}}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot w^{a-1} (1-w)^{b-1}$$

$\Rightarrow W \sim \text{Beta}(a, b)$.

② (b) Given $a=b=1$, we have $W \sim Unif(0,1)$.

$$\Rightarrow f_W(w) = 1$$

And given $Y = -\frac{1}{\lambda}\log W$, we get $w = e^{-\lambda y}$.

$$\Rightarrow f(y) = f(w) \cdot \left|\frac{dw}{dy}\right|$$
$$= 1 \cdot \lambda e^{-\lambda y}$$
$$= \lambda e^{-\lambda y}$$

$$\Rightarrow Y \sim Exp(\lambda)$$

(c) $Z = Y^{1/\alpha} \Rightarrow Y = Z^{\alpha}$

$$\Rightarrow f_Z(z) = f_Y(y) \left|\frac{dy}{dz}\right|$$
$$= \lambda e^{-\lambda z^{\alpha}} \cdot \alpha z^{\alpha-1}$$
$$= \alpha\lambda z^{\alpha-1} e^{-\lambda z^{\alpha}}$$

$$\Rightarrow Z \sim Weibull(\alpha, \lambda).$$

② (d) Let $F = \dfrac{U_1/\nu_1}{U_2/\nu_2}$ and $V = \dfrac{U_2}{\nu_2}$.

The change-of-variables formula and some ugly calculations should yield:

$$F \sim F(\nu_1, \nu_2)$$

② (e) Given $X \sim F_X$ and $Y = F_X(X)$, we get $X = F_X^{-1}(Y)$.

It's helpful to remember that the derivative of an inverse function is given by

$$\frac{d}{dx} F^{-1}(x) = \frac{1}{F'(F^{-1}(x))}$$

So, using the change-of-variables formula,

$$f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right|$$

$$= f_X(F_X^{-1}(y)) \cdot \left| \frac{d F_X^{-1}(y)}{dy} \right|$$

$$= f_X(F_X^{-1}(y)) \cdot \frac{1}{f_X(F_X^{-1}(y))}$$

$$= 1$$

$\Rightarrow Y \sim \text{Unif}[0,1]$

(Note that Y has support $[0,1]$ by construction.)

③ (a) $F_{X_{(1)}}(x) = P(X_{(1)} \leq x)$

$$= 1 - P(X_{(1)} > x)$$

$$= 1 - P(X_1, \ldots, X_n > x)$$

$$= 1 - \prod_{i=1}^{n} P(X_i > x)$$

$$= 1 - \prod_{i=1}^{n} (1 - P(X_i \leq x))$$

$$= 1 - \prod_{i=1}^{n} e^{-\lambda x}$$

$$= 1 - e^{-n\lambda x}$$

$\Rightarrow X_{(1)} \sim Exp(n\lambda)$

(b) $M_{X_i}(t) = E_{X_i}[e^{tX_i}]$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \int_0^\infty e^{(t-\lambda)x} \cdot \lambda \, dx$$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big]_0^\infty$$

$$= \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

③ (c) One strategy would be to start with the case of $n=2$, then prove the general case by induction.

Alternately, just use the properties of MGFs:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^n$$

This is the MGF of a Gamma(n, λ) distribution.

$\Rightarrow Y \sim \text{Gamma}(n, \lambda)$.

(d) Z is continuous. Note that $P(Z=z)=0 \quad \forall z \in \mathbb{R}$.

③(e) $P(Z < z) = P(X - W < z)$

$$= P(X < W + z)$$

$$= E[\mathbb{1}(X < W + z)]$$

$$= \sum_{w=0}^{\infty} \int_0^{\infty} \mathbb{1}(x < w + z)\, f_X(x)\, f_W(w)\, dx$$

$$= \sum_{w=0}^{\infty} \frac{e^{-\mu}\mu^w}{w!} \cdot \int_0^{\infty} \mathbb{1}(x < w + z) \cdot \lambda e^{-\lambda x}\, dx$$

$$= \sum_{w=0}^{\infty} \frac{e^{-\mu}\mu^w}{w!} \int_0^{w+z} \lambda e^{-\lambda x}\, dx$$

$$= \sum_{w=0}^{\infty} \frac{e^{-\mu}\mu^w}{w!} \left(1 - e^{-\lambda(w+z)}\right)$$

$$= 1 - e^{-(\lambda z + \mu)} \sum_{w=0}^{\infty} \underbrace{\frac{(e^{-\lambda}\mu)^w}{w!}}_{\text{kernel of Poisson}(e^{-\lambda}\mu)}$$

$$= 1 - e^{-(\lambda z + \mu)} \cdot e^{e^{-\lambda}\mu}$$

$$= 1 - \exp\left(-\lambda z + (e^{-\lambda} - 1)\mu\right)$$

④ (a) Using change of variables with $Y = \frac{X_1}{X_2}$, $Z = X_2$ then integrating out z should yield

$$Y \sim \text{Cauchy}(0,1)$$

(b) $T \sim t_\nu$

(5) (a) Let $Z_n = M_n - \log(n)$. Then:

$$
\begin{aligned}
F_{Z_n}(z) &= P(Z_n \le z) \\
&= P(M_n \le z + \log n) \\
&= \prod_{i=1}^{n} P(X_i \le z + \log n) \\
&= \left(1 - e^{-(z+\log n)}\right)^n \\
&= \left(1 - \frac{e^{-z}}{n}\right)^n \\
&\to \exp(-e^{-z})
\end{aligned}
$$

(Note that $\lim_{n\to\infty} (1+\frac{k}{n})^n = e^k$.)

(b) By WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$ and thus, $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{d} \mu$.

Note that

$$E\left[\frac{g(X_i)}{f(X_i)}\right] = \int \frac{g(x)}{f(x)} f(x)\,dx = \int g(x)\,dx = 1 < \infty$$

so again by WLLN,

$$\frac{1}{n}\sum_{i=1}^{n} \frac{g(X_i)}{f(X_i)} \xrightarrow{p} 1$$

Thus, by Slutsky's Theorem,

$$W_n = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} g(X_i)/f(X_i)} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{\frac{1}{n}\sum_{i=1}^{n} g(X_i)/f(X_i)} \xrightarrow{p} \frac{\mu}{1} = \mu$$

6 1) False

2) True

3) False

4) True

5) False

6) True

7) True

8) False