# Sucsan and lack of identification

TT
Hanken

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**Abstract**

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## 1 Introduction

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## 2 The problem

Lee, Sandberg and Sucarrat (in press) explain that their theory applies, among other things, to the GARCH(1,1) model with time-varying unconditional variance (the MTV-GARCH(1,1) model), where $\sigma_t^2 = h_t(\boldsymbol{\theta}, \boldsymbol{\xi})g(t/T, \boldsymbol{\theta})\eta_t^2$, with $\eta_t \sim \text{iid}(0, 1)$, and the conditional variance of $\phi_t = \varepsilon_t / g^{1/2}(t/T, \boldsymbol{\theta})$ equals

$$h_t(\boldsymbol{\theta}, \boldsymbol{\xi}) = \omega + \alpha\phi_{t-1}^2 + \beta h_{t-1}(\boldsymbol{\theta}, \boldsymbol{\xi}) \tag{1}$$

where $\boldsymbol{\xi} = (\omega, \alpha, \beta)'$ with $\omega > 0$, $\alpha > 0$ and $\beta \geq 0$. Furthermore,

$$g(t/T, \boldsymbol{\theta}) = \delta_0 + \delta_1(1 + \exp\{-\gamma(\frac{t}{T} - c)\})^{-1} \tag{2}$$

where $g(t/T, \boldsymbol{\theta})$ is a positive-valued function of $\boldsymbol{\theta} = (\delta_0, \delta_1, \gamma, c)'$ such that $\gamma \neq 0$ and $\delta_i \neq 0$, $i = 0, 1$. Because both (1) and (2) contain a nonzero intercept and $\gamma$ is not restricted, the model for $\varepsilon_t$ with the time-varying unconditional variance is not identified. This means that its parameters cannot be estimated consistently.

Amado and Teräsvirta (2013), and later Silvennoinen and Teräsvirta (2024), solve this problem by assuming $\gamma > 0$ and the intercept $\delta_0$ known. Amado and Teräsvirta (2013) fix $\delta_0$ in advance, whereas Silvennoinen and

Teräsvirta (2024) estimate it assuming $h_t(\boldsymbol{\theta}, \boldsymbol{\xi}) \equiv 1$. They do that for purely numerical reasons and do not treat the resulting value as an estimate of an unknown parameter $\delta_0$ but as a known value.

Lee et al. (in press) show that their QML estimates of $g(t/T, \boldsymbol{\theta})$ are consistent and asymptotically normal under rather general conditions. A look at their two-step method of estimating the parameters of the MTV-GARCH model shows that it is somewhat similar to the first step in Silvennoinen and Teräsvirta (2024). The difference is that while Silvennoinen and Teräsvirta (2024) assume $\delta_0$ to be a known constant, Lee et al. (in press) assume the whole function $g(t/T, \boldsymbol{\theta})$ to be completely known. This solves the identification problem and allows consistent estimation of the GARCH parameters under regularity conditions given in Francq and Zakoïan (2004). The standard deviation estimates of the estimates of $\boldsymbol{\xi}$ are, however, valid only under this assumption.

# 3 Simulations

Consider the unidentified MTV-GARCH model simulated by Lee et al. (in press). It is defined by $\varepsilon_t = h_t^{1/2}(\boldsymbol{\theta}, \boldsymbol{\xi})g^{1/2}(t/T, \boldsymbol{\theta})\eta_t$ with $\eta_t \sim \mathrm{iid}\mathcal{N}(0, 1)$, where

$$h_t(\boldsymbol{\theta}, \boldsymbol{\xi}) = 0.1 + 0.1\frac{\varepsilon_{t-1}^2}{g((t-1)/T, \boldsymbol{\theta})} + 0.8h_{t-1}(\boldsymbol{\theta}, \boldsymbol{\xi})$$

and

$$g(t/T, \boldsymbol{\theta}) = 10 + 1.5(1 + \exp\{-10(\frac{t}{T} - 0.5)\})^{-1} \quad (3)$$

with $x > 0$. The logistic function in (3) is quite smooth. Its value approaches 0.1 at around $t/T = 0.3$ and rises to about 0.9 when $t/T \approx 0.7$.

The simulation results of Lee et al. (in press) can be found in their Table 4.1. They use both the two-step QMLE described above and what they call the 'iterative method' in which they estimate the unidentified model with two unknown intercepts. In the second step they apply variance targeting, see, for example, Francq, Horváth and Zakoïan (2011), which, together with $\gamma > 0$, identifies the model. This would obviously allow efficient estimation of all parameters as in Amado and Teräsvirta (2013) and Silvennoinen and Teräsvirta (2024), but Lee et al. (in press) do not pursue this alternative.

Comparing the two-step and the 'iterative' method amounts to comparing estimates of an identified model with ones from an unidentified one. To make the comparison fair, we identify the unidentified model by assuming $\gamma > 0$ and $\delta_0 =$ constant; see Silvennoinen and Teräsvirta (2024).

We simulate

- 'estimating', i.e., determining $\delta_0$ as in Silvennoinen and Teräsvirta (2024)
- setting $\delta_0 = 10$ (the true value)
- estimating $\gamma$ as it stands
- estimating $\eta$ in $\gamma = e^{\eta}$ and transforming back to $\gamma$, see Goodwin, Holt and Prestemon (2011). Big differences not to be expected as transition fairly smooth. (Maybe this need not be done.)
- $T = 2000, 4000$ should be enough.

We report

- the results after the first iteration
- the final results, including the number of iterations (on average)

Note: The standard deviations for the 'iterative' estimation in Table 4.1 are from the simulations. The 'true' standard deviations do not exist because the Hessian is singular.

# 4 Conclusion

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# References

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