Outline of Algebraic Cobordism Course

1 Spectra, presheaves, formal group laws

1.1 Stable homotopy category

Want: a nice stable homotopy category, whatever that is! The idea, I believe, is that in the stable homotopy category, call it SHot, one should have a way of realizing spaces $X, Y \in$ Top in SHot, call the realizations $\Sigma^\infty X$, so that $f, g : X \to Y$ which are homotopic then satisfy $\Sigma^\infty f = \Sigma^\infty g$ and there should be an operation Σ extending suspension on Top so that $\underline{\text{SHot}}(\Sigma^\infty X, \Sigma^\infty Y) \cong \underline{\text{SHot}}(\Sigma^\infty \Sigma X, \Sigma^\infty \Sigma Y)$. Furthermore, the extended suspension operation should have an inverse Ω, so that we see Σ is an equivalence of categories. We would like the "universal" category with these properties I suppose.

Does such a category exist? What kind of description does it have? How do we work with it? For general $\mathbb{X}, \mathbb{Y} \in$ SHot, what does $\underline{\text{SHot}}(\mathbb{X}, \mathbb{Y})$ mean? Can we relate this back to the regular homotopy category? Such a category exists for any category with a "suspension" functor. See Heller below.

Adams gives a treatment of it using spectra below.

1. Adams, *Stable Homotopy and Generalised Homology*

    i. Works with Whitehead spectra and Ω-spectra.

    ii. Defines homotopy groups of spectra, long exact sequence of homotopy groups of a spectrum pair.

    iii. A cofinal subspectrum $E' \subset E$ is a subspectrum (structure maps required to be homeomorphisms) for which given any $K \subseteq E_n$ a subcomplex, there exists an m such that the image of $\Sigma^m K \to \cdots \to E_{n+m}$ winds up in E'_{n+m}.

    iv. Approaches smash product of spectra by defining "functions" of spectra, and then defines "maps" of spectra as equivalences classes of function on cofinal subspectra which agree on their intersection or some shared cofinal cofinal subspectrum. "cells now–maps later".

    v. Then he defines "morphism" of spectra by modding out by a homotopy relation.

    vi. Maps of degree other than 0 are allowed.

    vii. Prop 2.8 on p. 145 gives a description for $[E, F]_r$ in the homotopy category of spectra in terms of the expected notion when $E = \Sigma^\infty X$.

    viii. pp 152–153 detail that suspension is an equivalence and how to see $[X, Y]$ is an abelian group.

    ix. The homotopy category of spectra is additive, there are cofibers, and a cofibration sequence. Prop 3.9 p. 155. There is a fibration sequence.

    x. What is meant by Thm 3.12?

    xi. The smash construction is done in his stable category. It is commutative, associative, has unit, up to coherent isom. in the stable category.

    xii. Smash product $X \wedge Y$ is messy. Essentially choosing a "diagonal" in the 1st quadrant lattice $X_n \wedge Y_m$. He calls a particular choice a "handicrafted smash product", and would like to show they are equivalent.

xiii. 30 pages of technical details getting handicrafted smash products to work out.

2. Bousfield; Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets.*

   Also discuss spectra and the stable homotopy category, more with a view of closed model categories.

3. Heller, *Stable Homotopy Categories.*

   (a) "Categories with suspension", $(\mathcal{C}, \Sigma)$, $\Sigma : \mathcal{C} \to \mathcal{C}$ functor.

   (b) stable and weakly stable functors of categories with suspension. Let $(\mathcal{C}, \Sigma)$ and $(\mathcal{C}', \Sigma)$ be CwS, $F : \mathcal{C} \to \mathcal{C}'$ functor. It is said to be stable if $F\Sigma = \Sigma F : \mathcal{C} \to \mathcal{C}'$ and is said to be weakly stable if there is a natural equivalence $\theta : F\Sigma \to \Sigma F$.

   (c) A CwS is said to be stable if $\Sigma : \mathcal{C} \to \mathcal{C}$ is an automorphism, or equivalence of categories.

   (d) To any CwS $(\mathcal{C}, \Sigma)$ there is a universal stable CwS associated to $\mathcal{C}$, which we call $s\mathcal{C}$. There is a weakly stable morphism $S : \mathcal{C} \to s\mathcal{C}$ with $\sigma : \Sigma S \to S\Sigma$. It enjoys the following universal property:

   Proposition 1. *If $\mathcal{A}$ is a stable category, $F : \mathcal{C} \to \mathcal{A}$ is weakly stable with $\theta : F\Sigma \to \Sigma F$, then there is a unique functor $G : s\mathcal{C} \to \mathcal{A}$ with $G \circ S = F$, and $G\sigma = \theta$*

   In our stable homotopy category, we would like to be able to talk about fibrations, cofibrations, long exact sequences, etc. We also would like to somehow realize that $\underline{\text{SHot}}(X, Y)$ is not just a set, but an abelian group. So we would like our category to be additive (in the hom. alg. sense) and triangulated.

   We would also like to be able to use the objects of our stable homotopy category to define homology and cohomology theories.

   Relation between $s\,\underline{\text{hTop}}$ and the category of symmetric spectra? Relation with symmetric sequences of Hovey?

   Heller on homology theories A homology theory is a functor $h : \mathfrak{C} \to \mathcal{A}$ with a natural transformation ∂ where $\mathcal{A}$ is a stable abelian category, and $\mathfrak{C}$ has the following properties:

   (a) it is a category with suspension (suggests stability will be involved somehow)

   (b) it has a homotopy relation (suggests model category)

   (c) it has a collection of triangles (have cofibration sequence)

4. Hovey; Shipley; Smith, *Symmetric Spectra.*

5. Hovey; Palmieri; Strickland, *Axiomatic Stable Homotopy Theory.*

6. Puppe, *Stabile Homotopietheorie I.*

7. Stong, *Notes on Cobordism Theory.*

8. Weibel, *An Introduction to Homological Algebra.*

   How does the perspective here relate?

9. Whitehead, *Generalized Homology Theories.*

   Introduces spectra and Ω-spectra. Works only with CW complexes and topological categories. Works out many properties and theorems specific to this case.

10. Brown, *Cohomology Theories.*

    Contains the celebrated Brown representability theorem.

1.2 Formal Group Laws

Adams, *Stable Homotopy and Generalised Homology*

Definition 1. A commutative formal group law is a ring R with a "multiplication" $\mu(x, y) \in R[[x, y]]$ which satisfies:

1. $\mu(x, 0) = x$;
2. $\mu(0, y) = y$;
3. $\mu(x, y) = \mu(y, x)$;
4. $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$.

A morphism of formal group laws $f : (R, \mu) \to (S, \nu)$ is a ring homomorphism $f : R \to S$ such that $f_*\mu = \nu$.

Theorem 1. There is a universal commutative formal group law. That is, there is a commutative FGL $(\mathbb{L}, \mu)$ such that if (R, ν) is any commutative FGL, there exists a unique morphism $f : (\mathbb{L}, \mu) \to (R, \nu)$.

Proof. Let $\mathbb{L} = \mathbb{Z}[a_{ij} \,|\, i, j \in \mathbb{N}]/I$ where I is a particular ideal constructed below, and $\mu(x, y) = x + y + \sum a_{ij}x^iy^j$. The ideal I is taken to be the one generated by:

1. $a_{ij} - a_{ji}$ to ensure commutativity;
2. b_{ijk} where $\mu(\mu(x, y), z) - \mu(x, \mu(y, z)) = \sum_{i,j,k\geq 1} b_{ijk}x^iy^jz^k$.

By construction, $\mathbb{L}$ with $\mu(x, y) = x+y+\sum a_{ij}x^iy^j$ is a commutative formal group law. We abuse notation and write a_{ij} for the class $[a_{ij}] \in \mathbb{L}$.

It is also clearly universal. If (R, ν), $\nu = x + y + \sum b_{ij}x^iy^j$ is a commutative FGL, there is a unique ring morphism $f : \mathbb{Z}[a_{ij} \,|\, i, j \in \mathbb{N}] \to R$ where $f(a_{ij}) = b_{ij}$. Since ν is a commuatative FGL, all the relations of $f(I) = 0$, and so f factors uniquely through $\tilde{f} : \mathbb{L} \to R$. This map is a morphism of commutative FGLs, and the proof is complete. □

Theorem 2. There is an isomorphism $\mathbb{L} \cong \mathbb{Z}[x_2, x_4, x_6, ...]$ where $x_2 = a_{11}$, $x_4 = a_{12}$, $x_6 = a_{22} - a_{13}$, $\ldots$.

Proof. Given in section 7, p. 64. It is a bit messy. □

1.3 FGLs from oriented cohomology theories

Definition 2. A ring spectrum is a spectrum $\mathbb{E}$ equipped with maps $m : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$, $u : \mathbb{S} \to \mathbb{E}$ so that the following diagrams commute. If we are working with symmetric spectra, the diagrams are to commute on the nose, if we are not working with symmetric spectra, we need the diagrams to commute in the (stable) homotopy category of spectra.

$$\begin{tikzcd}
\mathbb{E} \wedge \mathbb{E} \wedge \mathbb{E} \arrow[r, "m\wedge \mathrm{id}"] \arrow[d, "\mathrm{id}\wedge m"'] & \mathbb{E} \wedge \mathbb{E} \arrow[d, "m"] \\
\mathbb{E} \wedge \mathbb{E} \arrow[r, "m"] & \mathbb{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \wedge \mathbb{E} \arrow[r, "u\wedge \mathrm{id}"] \arrow[dr, "\pi"'] & \mathbb{E} \wedge \mathbb{E} \arrow[d, "m"] & \mathbb{E} \wedge \mathbb{S} \arrow[l, "\mathrm{id}\wedge u"'] \arrow[dl, "\pi"] \\
& \mathbb{E} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E} \wedge \mathbb{E} \arrow[r, "\tau"] \arrow[dr, "m"'] & \mathbb{E} \wedge \mathbb{E} \arrow[d, "m"] \\
& \mathbb{E}
\end{tikzcd}$$

That is, a ring spectrum is a commutative monoid in the category of symmetric spectra.

Definition 3. Let $\mathbb{E}$ be a commutative ring spectrum (perhaps also an Ω-spectrum). Observe that a commutative ring spectrum has canonically defined cohomology classes $[u_n] \in \mathbb{E}^n(\mathbb{S}^n)$ given by the map $u_n : \mathbb{S}^n \to E_n$.

An orientation of $\mathbb{E}$ is the choice of a generator $x \in \mathbb{E}^2(\mathbb{CP}^\infty)$ so that for $\iota : \mathbb{CP}^1 \to \mathbb{CP}^\infty$ the standard inclusion, we have $\iota^* x = [u_2]$.

Why does this definition give an "orientation" to a cohomology theory? See my comments on this in my Study Guide.

Theorem 3. If $\mathbb{E}$ is an oriented cohomology theory, then we can make some specific cohomology and homology calculations. They are proven in Adams. The notation $\mathbb{E}^*(pt) = \pi_*(\mathbb{E})$ for the coefficients of the cohomology theory $\mathbb{E}$ is used in Adams. For a ring spectrum $\mathbb{E}$, the coefficients $\mathbb{E}^*(pt)$ is a ring.

1. $\mathbb{E}^*(\mathbb{CP}^n) \cong \mathbb{E}^*(pt)[x]/(x^{n+1})$
2. $\mathbb{E}^*(\mathbb{CP}^\infty \cong \mathbb{E}^*(pt)[[x]]$
3. $\mathbb{E}^*(\mathbb{CP}^n \times \mathbb{CP}^m) \cong \mathbb{E}^*(pt)[x_1, x_2]/(x_1^{n+1}, x_2^{n+2})$
4. $\mathbb{E}^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{E}^*(pt)[[x_1, x_2]]$

Assuming this theorem, we now construct a FGL for $\mathbb{E}$. We are to think of x as the "first Chern class" for our generalized cohomology theory. With this, we are able to construct a theory of characteristic classes using the splitting principle.

There is a map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ which classifies the tensor product of the universal line bundles $E_1 = \pi_1^* E$, $E_2 = \pi_2^* E$ over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ where E is the universal line bundle over $\mathbb{CP}^\infty$ and π_1 and π_2 are the canonical projections. With m chosen in this fashion—it will be unique up to homotopy by basic classifying space theory—we then investigate what properties $m^*(x) \in \mathbb{E}^*(pt)[[x_1, x_2]]$ has. We may write $m^*(x) = \mu(x_1, x_2) = x_1 + x_2 + \sum b_{ij} x_1^i x_2^j$ where $b_{ij} \in \mathbb{E}^*(pt)$, and it is indeed the case that μ is a FGL. How does one see this? Adams says it is easy. First one has to make sense of what is being asked, then it is quite easy after all.

We need to check a few things. First off, let us make sense of $\mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))$. Consider the diagram

$$\begin{tikzcd} E_1 \otimes E_2 \otimes E_3 \arrow[r] \arrow[d, "\xi_1\xi_2\xi_3"'] & E \arrow[d, "\xi"] \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r, "M"] & \mathbb{CP}^\infty \end{tikzcd}.$$

It is written ambiguosly without parentheses because they don't matter in this diagram; they are all equivalent up to homeomorphism. Therefore $M^*(x) = \mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))$. Likewise, we are to interpret $\mu(x, 0) = x$ as

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r, "\pi_1"] & \mathbb{CP}^\infty \end{tikzcd}$$

so pulling back x under pi_1^* in $\mathbb{E}^*(pt)[[x_1, x_2]]$ is just x_1. Likewise the commutativity can be seen.

This method of getting a formal product is just a different way of stating the definition in Levine and Morel. Adams shows the usual topological examples where $\mathbb{E} = H\mathbb{Z}$ the Eilenberg-MacLane spectrum where $m^*(x) = x_1 + x_2$. One can also take $\mathbb{E} = K = BGL$, the spectrum for K-theory, and see that $m^*x = x_1 + x_2 + x_1x_2$.

Perhaps the other way of writing down the FGL for $\mathbb{E}^*$ makes it more clear how to work with it and verify the properties. The FGL $\mu(x, y)$ is the element of $\mathbb{E}^*(pt)[[x, y]]$ such that for any line bundles L_1, L_2 over $\mathbb{CP}^\infty$, with classifying maps $\xi_1, \xi_2 : \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, define $c_1(L_i) = \xi_i^*(x)$. Then we have

$$\mu(c_1(L_1), c_1(L_2)) = c_1(L_1 \otimes L_2).$$

In the first definition, it is clear that it exists. In the second definition, one has to explain why it exists, but it is immediate how one verifies the FGL properties.

Theorem 4. The FGL associated with the oriented cohomology theory MU^* is the universal FGL. That is, there is a map $\phi : \mathbb{L} \to MU^*(pt)$ which is an isomorphism so that the push-forward of the universal FGL agrees with the FGL on MU^*.

This is discussed in section 8 of Adams's blue book.

Levine; Morel. *Algebraic Cobordism.*

2 Model Categories

2.1 Generalities

1. J. P. May, *More Concise Algebraic Topology*
2. Davis & Kirk, *Lecture Notes in Algebraic Topology.*
3. Daniel Quillen, *Homotopical Algebra.*
4. Kathryn Hess, *Model Categories in Algebraic Topology.*

2.2 Model structures on particular categories

2.3 Cohomology theories

3 Algebraic Cobordism

4 Topological Cobordism

5 Motivic Cohomology

6 Complete reference list

1. Adams, *Stable Homotopy and Generalised Homology*

2. Artin, *Grothendieck Topologies.*

3. Blander, *Local projective model structures on simplicial presheaves.*

4. Boardman, *Stable homotopy theory.*

5. Bott; Tu, *Differential Forms in Algebraic Topology.*

6. Bousfield; Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets.*

7. Brown, *Cohomology Theories.*

8. Davis & Kirk, *Lecture Notes in Algebraic Topology.*

9. Hatcher, *Algebraic Topology.*

10. Heller, *Stable Homotopy Categories.*

11. Hess, *Model Categories in Algebraic Topology.*

12. Hilton & Stammbach, *A Course in Homological Algebra.*

13. Hopkins; Quick, *Hodge filtered complex bordism.*

14. Hovey, *Model categories.*

15. Hovey, *Spectra and symmetric spectra in general model categories.*

16. Hovey; Shipley; Smith, *Symmetric Spectra.*

17. Hovey; Palmieri; Strickland, *Axiomatic Stable Homotopy Theory.*

18. Jardine, *Fields Lectures: Presheaves of spectra* (2007).

19. Jardine, *Fields Lectures: Simplicial presheaves.*

20. Jardine, *Presheaves of symmetric spectra.*

21. Jardine, *Stable homotopy theory of simplicial presheaves.*

22. Levine; Morel. *Algebraic Cobordism.*

23. Mac Lane, *Categories for the Working Mathematician.*

24. May, *A Concise Course in Algebraic Topology.*

25. May, *More Concise Algebraic Topology*

26. May, *Simplicial Objects in Algebraic Topology.*

27. Mazza; Voevodsky; Weibel, *Lecture Notes on Motivic Cohomology.*

28. Milnor & Stasheff *Characteristic Classes.*

29. Panin; Pimenov; Röndigs, *A universality theorem for Voevodsky's algebraic cobordism spectrum.*

30. Panin; Pimenov; Röndigs, *On the relation of Voevodsky's algebraic cobordism to Quillen's K-theory.*

31. Panin; Pimenov; Röndigs, *On Voevodsky's algebraic K-theory spectrum BGL.*

32. Puppe, *Stabile Homotopietheorie I.*

33. Quillen, *Homotopical Algebra.*

34. Quillen, *On the formal group laws of unoriented and complex cobordism theory.*

35. Stong, *Notes on Cobordism Theory.*

36. Vick, *Homology Theory.*

37. Voevodsky, *Seattle lectures: K-theory and motivic cohomology.*

38. Weibel, *An Introduction to Homological Algebra.*

39. Weston, *An Introduction to Cobordism Theory.*

40. Whitehead, *Generalized Homology Theories.*