Let me set some notational conventions for the discussion to follow.

(1) Let $\mathrm{Spc}(k)$ denote the category of simplicial sheaves on Sm/k with respect to the Nisnevich topology. We may equip $\mathrm{Spc}(k)$ with the injective model structure as Morel does. Let $\mathrm{Spc}_\bullet(k)$ denote the category of pointed spaces.
(2) Let $W_{\mathbb{A}^1}$ denote the class of maps $\{U \times \mathbb{A}^1 \to U \mid U \in \mathrm{Sm}/k\}$. Denote the left Bousfield localization of $\mathrm{Spc}(k)$ with respect to $W_{\mathbb{A}^1}$ by $L_{\mathbb{A}^1}\mathrm{Spc}(k)$. There is a left Quillen functor $L_{\mathbb{A}^1} : \mathrm{Spc}(k) \to L_{\mathbb{A}^1}\mathrm{Spc}(k)$ given by the identity functor. The analogous constructions exist for pointed spaces.
(3) Denote the full subcategory of $\mathrm{Spc}_\bullet(k)$ consisting of $\mathbb{A}^1$-local objects by $\mathrm{Spc}_\bullet^{\mathbb{A}^1}(k)$. Morel proves that there is a left Quillen functor $L^\infty : \mathrm{Spc}_\bullet(k) \to \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k)$ which induces a Quillen equivalence between $L_{\mathbb{A}^1}\mathrm{Spc}_\bullet(k)$ and $\mathrm{Spc}_\bullet^{\mathbb{A}^1}(k)$.
(4) Let $\mathrm{Spt}^{S^1}(k)$ denote the category of S^1-spectra of spaces equipped with the stable (simplicial) model category structure. Let $W_{\mathbb{A}^1} = \{\Sigma^\infty U_+ \wedge \mathbb{A}^1_+ \to \Sigma^\infty U_+ \mid U \in \mathrm{Sm}/k\}$. Denote the left Bousfield localization of $\mathrm{Spt}^{S^1}(k)$ with respect to $W_{\mathbb{A}^1}$ by $L_{\mathbb{A}^1}\mathrm{Spt}^{S^1}(k)$.
(5) Let $\mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$ denote the full subcategory of $\mathrm{Spt}^{S^1}(k)$ consisting of the $\mathbb{A}^1$-local objects. Morel constructs a left Quillen functor $L^\infty : \mathrm{Spt}^{S^1}(k) \to \mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$ which induces a Quillen equivalence between $\mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$ and $L_{\mathbb{A}^1}\mathrm{Spt}^{S^1}(k)$.

From our discussion, the issue with Morel's argument in the case of a perfect field is Lemma 3.3.9 of [Mor03]. Here is the reproduced statement.

> **Lemma 3.3.9** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial preseheaves. Then the $\mathbb{A}^1$-localization of the cone of f is canonically isomorphic to the $\mathbb{A}^1$-localization of the cone of $L_{\mathbb{A}^1} f : L_{\mathbb{A}^1}\mathcal{X} \to L_{\mathbb{A}^1}\mathcal{Y}$.*

There are some issues here. Presumably he means "simplicial sheaves" instead of "simplicial presheaves." It is unclear whether by "$L_{\mathbb{A}^1}$" Morel wants the functor coming from Bousfield localization or the functor L^∞ he constructs.

If we consider the functor $L_{\mathbb{A}^1} : \mathrm{Spt}^{S^1}(k) \to L_{\mathbb{A}^1}\mathrm{Spt}^{S^1}(k)$ obtained by Bousfield localization, the lemma follows by Morel's explicit description of cones, and since $L_{\mathbb{A}^1}$ is the identity functor. For $f : E \to F$ a map of S^1 spectra, Morel defines $C(f)$ to be the spectrum with $C(f)_n$ the cone of $f_n : E_n \to F_n$ as pointed spaces. Here, the cone of a map of pointed spaces is just the push-out of the following diagram.

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & F_n \wedge \Delta^1 \\ pt & \end{tikzcd}$$

In particular, the cone is just a colimit, which depends only on the category, and not on the model structure. The point is that $E_n \to F_n \wedge \Delta^1$ is a cofibration in both model category structures on $\mathrm{Spc}_\bullet(k)$, so that the homotopy pushout is weak equivalent to the categorical push-out by Bousfield-Kan. So Lemma 3.3.9 is correct in this case.

Now let's consider what happens when we apply L^∞ to the cone of $f : E \to F$. Let's suppose that f is a cofibration in the stable simplicial model structure on S^1 spectra. Recall that L^∞ is a left Quillen functor. It thus preserves cofibrations and preserves colimits. Now, the cone of $f : E \to F$ in $\mathrm{Spt}^{S^1}(k)$ is the homotopy push-out of the following diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \\ pt & \end{tikzcd}$$

But as we are assuming $f : E \to F$ is a cofibration, Bousfield-Kan says that the categorical colimit of this diagram is weak equivalent to the homotopy push-out. Now let us apply L^∞ to the diagram. $L^\infty f$ is still a cofibration, and the colimit of

$$\begin{tikzcd} L^\infty E \arrow[r] \arrow[d] & L^\infty F \\ L^\infty pt & \end{tikzcd}$$

is canonically isomorphic to $L^\infty \operatorname{colim}(pt \leftarrow E \to F) \cong L^\infty C(f)$. Now, since $L^\infty pt \cong pt$, we conclude that the cone of $L^\infty E \to L^\infty F$ is weak equivalent to $L^\infty C(f)$.

Does the above suffice to establish lemma 3.3.9 in general for the functor L^∞?

Even without this lemma, I noticed that one can show that suspension spectra of (simplicially) 0 connected pointed spaces are ($\mathbb{A}^1$) 0 connected because $\Sigma^\infty L^\infty \mathcal{X}$ is weak equivalent to $L^\infty \Sigma^\infty \mathcal{X}$. The result follows by Corollary 3.2.5 of [Mor03].

References

[Mor03] Morel, Fabien, *An introduction to $\mathbb{A}^1$ homotopy theory.*