The stable $\mathbb{A}^1$-connectivity theorem: an outline

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Contents

This is a draft outline of lectures for the USC K-theory summer school on Morel's proof of the stable $\mathbb{A}^1$-connectivity theorem. The precise form of the outline may change slightly as we get closer to the school.

PDFs of all the references can be found at http://math.mit.edu/~hoyois/ktss2015/.

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Overview

The goal of this summer school is to give a proof of Morel's stable $\mathbb{A}^1$-connectivity theorem.

To even state the theorem, we will need to introduce and study some basic properties of the category of S^1-spectra in simplicial presheaves on the category of smooth schemes over a base scheme S (equipped with the Nisnevich topology); for convenience, we will refer to such objects as S^1-spectra over S. Such objects form a model category and the first part of the summer school will be devoted to developing enough algebraic geometry and abstract homotopy theory to work with such objects.

Morel's theorem is about the $\mathbb{A}^1$-homotopy theory of such S^1-spectra, which involves studying a special Bousfield localization of the model category of S^1-spectra above. Abstract techniques show that such a localization exists without too much effort, but in order to prove Morel's theorem, we will need an explicit construction of the localization functor $\mathrm{L}_{\mathbb{A}^1}$. The first 5 lectures will be devoted to building up all these preliminaries and constructing the $\mathbb{A}^1$-localization functor.

Theorem 1 ([Mor05, Theorem 6.1.8]). *Suppose k is an infinite field. If E is a -1-connected object S^1-spectrum over $\operatorname{Spec} k$, then $\mathrm{L}_{\mathbb{A}^1} E$ is again -1-connected.*

Besides the material required to even formulate the statement, the main input to the proof of the theorem is the cohomological dimension of sheaves in the Nisnevich topology on smooth schemes (Lecture 6). The proof (which comprises Lecture 9) of the theorem essentially consists of a long series of reductions. Essentially by definition, the -1-connectedness of a spectrum can be tested on points for the Nisnevich topology, i.e., henselian local schemes. The first reduction consists of using "base-change" arguments (developed in Lecture 7) to reduce to treating the case of fields (this also requires Lecture 8). Once one has reduced to the case of fields, one can use a Postnikov tower argument to reduce to the vanishing theorem (this argument is also in Lecture 6).

The remaining lectures will be concerned with consequences and applications of these results. Lecture 10 introduces the homotopy t-structure on the triangulated category of S^1-spetra and the related notion of strict $\mathbb{A}^1$-invariance of sheaves, which has become central in unstable $\mathbb{A}^1$-algebraic topology. Lecture 11 discusses $\mathbb{P}^1$-spectra and the construction of a corresponding t-structure. This is closely related to Voevodsky's connectivity theorem from his ICM note [Voe98].

1 Day 1

1.1 Introductory lecture: Overview

(By the organizers). This lecture will state the connectivity theorem and some applications and try to put the topic of the summer school in context.

1.2 Lecture 1: Algebro-geometric preliminaries

References: [MV99, §3.1], [Dég99], [TT90, Appendix E], [CTHK97]
Dependencies: none

1. Nisnevich topology on smooth S-schemes
2. Examples of Nisnevich covers
3. Nisnevich neighborhoods, henselization and points for the topology
4. Local structure of smooth pairs.
5. Characterization in terms of distinguished squares; Sheaves for the Nisnevich topology

1.3 Lecture 2: Model structures and simplicial presheaves

References: [DS95] for model categories; [Hov99], [GJ09], [Qui67] for model categories and simplicial sets; for model structures on simplicial presheaves: [Jar87, Jar00, Jar15] (injective), [Bla01] (projective), [Hir03, §11.6] (projective), [Lur09, §A.2.8] (both); [Dugb] is great for motivation. For Bousfield localizations: [Hir03], [Bar10]
Dependencies: none

1. Simplicial sets and the usual model structure ([Hov99, §3.1-3.2], [GJ09, Ch. 1])
2. Simplicial presheaves and model category stuctures (without proofs)
3. Definition of left Bousfield localizations, effect on the homotopy category ([Hir03, §3.3], [Bar10, §4])
4. Examples of localization: p-localization, [Dug01, Example 5.6], local model structure on simplicial presheaves on a Grothendieck site with enough points

1.4 Lecture 3: Spectra and stable homotopy theory

References: for the classical theory, Adams' book probably suffices, but perhaps we should be doing everything in the context of simplicial sets.
Dependencies: none

1. Definition of a spectrum: levelwise and stable model structures, fibrant objects (Ω-spectra), Eilenberg-Mac Lane spectra ([GJ09, p. 528], [BF78, §2])
2. Postnikov tower and truncation functors
3. Structure as a triangulated category
4. Action of simplicial sets on spectra ("smash product")

2 Day 2

2.1 Lecture 4: The unstable $\mathbb{A}^1$-homotopy category

References: [MV99], [Mor03] [Dug01], [Dugb]
Dependencies: Lectures 1 and 2

1. The $\mathbb{A}^1$-local model structure as a left Bousfield localization
2. Characterization of local objects: the Brown-Gersten property (no proof) and $\mathbb{A}^1$-homotopy invariance [Duga]
3. The singular construction and basic properties [MV99, p. 87]
4. Explicit construction of the (unstable) $\mathbb{A}^1$-localization functor
5. Proof of [MV99, §2 Corollary 3.22] as a warm-up to the general case.

2.2 Lecture 5: $\mathbb{A}^1$-localization of S^1-spectra

References: [Mor05, §4.1-4.2], [Mor03]
Dependencies: Lectures 2, 3, and 4

1. The levelwise model structure on presheaves of spectra on the Nisnevich site and its Bousfield localizations: stable, local, $\mathbb{A}^1$-local
2. Equivalent characterizations of being $\mathbb{A}^1$-local
3. Construction of the stable $\mathbb{A}^1$-localization functor

2.3 Lecture 6: A vanishing theorem for $\mathbb{A}^1$-local S^1-spectra

References: [MV99, Prop. 3.1.8], [Mor05, §4.3]. This is the first of the two key technical ingredients in the proof.
Dependencies: Lectures 1, 3, and 5

1. Nisnevich cohomological dimension theorem
2. Vanishing theorem for Eilenberg-MacLane spectra (follows from the previous point)
3. [Mor05, Lemma 4.3.1]: reduction to the previous case by means of Postnikov towers

3 Day 3

3.1 Lecture 7: Base-change and other formal properties

References: [Mor05] or [Ayo07].
Dependencies: Lectures 4 and 5

1. Functoriality: for each $f : S' \to S$, the construction of the pullback $f^* : \mathcal{S}p^{S^1}(S) \to \mathcal{S}p^{S^1}(S')$ between the categories of presheaves of spectra, of its right adjoint f_* and, when f is smooth, of its left adjoint $f_\sharp$ ([Mor05, §5.1], [Ayo07, 4.4.50])
2. Derived functoriality: if $f : S' \to S$ is an arbitrary (resp. smooth) morphism, then $f^* : \mathcal{S}p^{S^1}(S) \to \mathcal{S}p^{S^1}(S')$ (resp. its left adjoint $f_\sharp$) is left Quillen with respect to the $\mathbb{A}^1$-local S^1-stable model structures ([Mor05, §5.2], [Ayo07, 4.5.23])
3. Continuity with respect to inverse limits of smooth morphisms ([Mor05, 5.2.7], [Hoy15, Appendix A])
4. Application to fibers of S^1-spectra ([Mor05, 5.2.8])

3.2 Lecture 8: Gabber's presentation lemma

References: the original is [Gab94, §3], but a much more readable account is [CTHK97, §3]. This lecture is essentially pure algebraic geometry/commutative algebra.
Dependencies: none

1. Gabber's presentation lemma is a very refined form of Noether normalization. So one might want to start by reviewing that.
2. One result which historically precedes the Gabber presentation lemma is [Qui73, Lemma 5.12], which is what Quillen used to prove the Gersten conjecture for K-theory.
3. Statement of Gabber's presentation lemma [CTHK97, Theorem 3.11]
4. Interpretation of Gabber's lemma in terms of Nisnevich squares [CTHK97, Corollary 3.1.2]
5. Sketch of proof of Gabber's presentation lemma

3.3 Lecture 9: Putting everything together

References: [Mor05, §6.1], [Mor03, §4.3]
Dependencies: Lecture 4 (specifically [MV99, Corollary 3.22]), 6, 7, and 8

1. Definition of weak connectedness
2. Proof that weakly connected implies connected [Mor05, 6.1.3] (NB: the alternative argument for perfect fields from [Mor03, Lemma 3.3.7] is wrong)
3. Proof of the main theorem [Mor05, Lemma 6.1.7]
4. Consequence: existence of the homotopy t-structure on spectra

4 Day 4

4.1 Lecture 10: Strict $\mathbb{A}^1$-invariance and the Gersten resolution

References: [Mor03], [CTHK97]
Dependencies: Lectures 3 and 9

1. $\mathbb{A}^1$-local Eilenberg-Mac Lane spectra
2. The heart of the homotopy t-structure: strictly $\mathbb{A}^1$-invariant sheaves
3. Homotopy sheaves are strictly $\mathbb{A}^1$-invariant
4. Strictly $\mathbb{A}^1$-invariant sheaves have Gersten resolutions ([CTHK97, §5]); thus we get a Gersten resolution for homotopy sheaves of theories representable on the S^1-stable homotopy category.
5. To construct strictly $\mathbb{A}^1$-invariant sheaves, it basically suffices to have a Gersten resolution (viz. Rost's theory of cycle modules [Ros96]): unramified Milnor K-theory sheaves.
6. Coniveau spectral sequences.
7. Comparison of Zariski and Nisnevich cohomology for strictly $\mathbb{A}^1$-invariant sheaves [CTHK97, §8]?

4.2 Lecture 11: Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

Reference: [Mor03]
Dependencies: Lectures 9 and 10

1. Contractions: exactness on the category of strictly $\mathbb{A}^1$-invariant sheaves
2. Morel's computation of the homotopy sheaves of $\mathbf{G}_m$-loop spaces ([Mor03, 4.3.11])
3. Definition of the category of $\mathbb{P}^1$-spectra
4. The homotopy t-structure on $\mathbb{P}^1$-spectra [Mor03, §5.2]
5. Proof of Voevodsky's $\mathbb{P}^1$-stable connectivity theorem ([Voe98, 4.14]) over spectra of infinite fields (follows from the previous point)

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