1. Voevodsky's connectivity theorem for \mathbb{P}^1 -spectra

The following is from an email from Aravind to me concerning the proof of the connectivity theorem for \mathbb{P}^1 spectra.

Most of the proof appears in Morel's Trieste notes (in the section on the homotopy t-structure for P^1 -spectra). First you prove a connectivity result for P^1 -stable homotopy sheaves by using Morel's S^1 -stable connectivity theorem and studying what happens under G_m loops and G_m -suspension: suspension preserves connectivity, and Morel shows that taking G_m -loops has the effect of making a "contraction". Then, you globalize this.

Our goal is to prove theorem 4.14 of [Voev98], which should be restated in terms of \mathbb{P}^1 -spectra.

Theorem 1.1. Let X be a pointed smooth scheme over $\operatorname{Spec}(k)$ where k is an infinite field. Let \mathcal{Y} be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^{\infty}X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^{\infty}\mathcal{Y}) = 0.$$

Remark 1. We can formulate the theorem by using homotopy sheaves. Indeed, if we can show that for any pointed space \mathcal{Y} is -1 connected, i.e., the homotopy sheaves $\pi_{n+\alpha m}^{\mathbb{A}^1}(\mathcal{Y})$ vanish for $n \leq -1$, the result will follow, as

$$\mathcal{SH}(k)(\Sigma^{\infty}X, S^{n} \wedge \mathbb{G}_{m}^{m} \wedge \Sigma^{\infty}\mathcal{Y}) = \mathcal{SH}(k)(S^{-n} \wedge \mathbb{G}_{m}^{-m} \wedge \Sigma^{\infty}X, \Sigma^{\infty}\mathcal{Y})$$
$$= \pi_{-n-m\alpha}^{\mathbb{A}^{1}}(X).$$

and by the connectivity theorem, this will vanish whenever -n < 0, i.e., whenever n > 0. This is a stronger statement than [Voev98, theorem 4.14].

The line of attack is then to show that for a pointed space \mathcal{Y} , the S^1 suspsension spectrum $\Sigma_s^{\infty} \mathcal{Y}$ in the stable model category is -1 connected. Then we establish the first connectivity theorem which ensures the \mathbb{A}^1 localization of $\Sigma_s^{\infty} \mathcal{Y}$ is -1 connected. Finally, we show that inverting \mathbb{G}_m does not affect the connectivity, i.e., $\Sigma_m^{\infty} \Sigma_s^{\infty} \mathcal{Y}$ is again -1 connected.

2. Assumptions from previous lectures

2.1. Facts about Nisnevich topology. Points and neighborhoods. Distinguished squares determine the topology. Relation to etale, zariski, and fpqc topology.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k, and let $X \in \text{Sm}/k$ with Krull dimension d. Then whenever n > d, $H_{Nis}^n(X; M) = 0$.

Proposition 2.2. [Mor04, 2.4.1] For any $X \in \text{Sm}/k$, and for any $x \in X(k)$, there is an isomorphism of pointed sheaves of sets in the Nisnevich topology

$$X/(X - \{x\}) \cong \mathbb{A}^n/(\mathbb{A}^n - \{0\}).$$

- 2.2. Unstable model category $\Delta^{op} \text{Shv}(\text{Sm}/k, Nis)$.
 - (1) Don't forget the adjunction for sheafification $a_{Nis} \dashv U$. i.e., to give a map $a_{Nis}\mathcal{X} \to \mathcal{Y}$, of sheaves, it is equivalent to just give a map $\mathcal{X} \to U\mathcal{Y}$.
 - (2) Give the unstable model category the injective model structure to start. (This is Morel's choice, so we stick with it)
 - (3) Weak equivalences: $\mathcal{X} \to \mathcal{Y}$ iff for all U the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ a simplicial w.e.

- (4) Cofibs: $\mathcal{X} \to \mathcal{Y}$ is a cofib. iff for any $U \in \text{Sm}/k$ the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a monomorphism.
- (5) Fibrations: what they need to be.
- (6) The ass. htpy. cat. to the injective model category structure is denoted $\mathcal{H}_s(k)$. The pointed htpy. cat. is denoted $\mathcal{H}_{s,\bullet}(k)$.
- (7) Important properties/constructions in these categories: Cartesian closed, i.e., have smash/tensor \times and \wedge , with right adjoints \underline{Hom} and $\underline{Hom}_{\bullet}$. Representable sheaf functor and constant simplicial set functor.

Definition 2.3. Internal hom. (Ref: P. Pelaez)

Let $U \in \text{Sm}/k$. Let Δ_U^n denote the simplicial sheaf given by

$$(V,m) \in \operatorname{Sm}/k \times \Delta^{op} \to \operatorname{Sm}/k(V,U) \times \Delta^n_m.$$

In other words, $\Delta_U^n = (rU) \times c\Delta^n$.

Let $\mathcal X$ and $\mathcal Y$ be spaces. The internal hom in the unpointed category is given by the formula

$$(U, m) \in \operatorname{Sm}/k \times \Delta \to \operatorname{Hom}_{\Delta^{op}\operatorname{Shv}}(X \times \Delta_U^m, Y).$$

How to describe the adjunction?

$$\operatorname{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \operatorname{Hom}(\mathcal{Y}, Hom(\mathcal{X}, \mathcal{Z}))$$

Certainly we may send a map $g \in \text{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$ to the map $\eta(g)$ given by

$$\eta(g)(U,n) : \mathcal{X}_n(U) \times \mathcal{Y}_n(U) \to \mathcal{Z}_n(U)$$

$$(a,b) \to g(U,n)(U,n)(a, \mathrm{id}, \mathrm{id}).$$

Why is this a bijection?

Definition 2.4. Internal hom in the pointed category. Consider \mathcal{X} and \mathcal{Y} pointed spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{Hom}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y}$, where at $(U, n) \in (\mathrm{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times \Delta_U^n \to \mathcal{Y}$ to $g(U, n)(x, \mathrm{id}, \mathrm{id}) \in \mathcal{Y}_n(U)$. This makes sense as we have the map

$$g(U,n): \mathcal{X}_n(U) \times \operatorname{Sm}/k(U,U) \times \Delta_n^n \to \mathcal{Y}_n(U).$$

The pointed internal hom $\underline{Hom}_{\bullet}(\mathcal{X},\mathcal{Y})$ is the fiber $ev_x^{-1}(y)$.

2.3. \mathbb{A}^1 localization.

- (1) See [?, Prop 2.3.3.] for details on the various properties of fibrant objects in the unstable motivic category.
- (2)

Definition 2.5. A space \mathcal{X} is called \mathbb{A}^1 local if for any smooth scheme U, the canonical map

$$\operatorname{Hom}(rU,\mathcal{X}) \to \operatorname{Hom}(rU \times \mathbb{A}^1,\mathcal{X})$$

is a bijection.

(3)

Definition 2.6. A map $f: \mathcal{X} \to \mathcal{Y}$ is an \mathbb{A}^1 weak equivalence if

$$\operatorname{Hom}(\mathcal{Y}, \mathcal{Z}) \to \operatorname{Hom}(\mathcal{X}, \mathcal{Z})$$

is a bijection for every \mathbb{A}^1 local space \mathbb{Z} .

- (4) The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \to U \mid U \in \operatorname{Sm}/k\}.$
- (5) The general theory of Bousfield localization then gives a localization functor $L_{\mathbb{A}^1}$: $\mathcal{H}_s(k) \to L_W \mathcal{H}_s(k)$; denote the category $L_W \mathcal{H}_s(k)$ by $\mathcal{H}(k)$. The localization functor sends \mathbb{A}^1 weak equivalences to isomorphisms.

See [?, Definition 3.3.1]. Keep the same underlying category, the weak equivalences are the \mathbb{A}^1 -local weak equivalences, do not change the cofibrations from the ones in the injective model structure on $\Delta^{op} \text{Shv}(\text{Sm}/k, Nis)$, and let the fibrations be what they need to be.

- (6) Morel writes $L_{\mathbb{A}^1}$ for the (derived) functor which sends a space (without base point) \mathcal{X} to an \mathbb{A}^1 -localizion.
- (7) Morel gives an explicit construction which takes a pointed space \mathcal{Y} and produces a space $L^{\infty}\mathcal{Y}$ which is \mathbb{A}^1 local and a map $\mathcal{Y} \to L^{\infty}\mathcal{Y}$ which is an \mathbb{A}^1 weak equivalence.

Let \mathcal{Y}_f be the functorial fibrant replacement of \mathcal{Y} with respect to the injective model structure. Consider \mathbb{A}^1 to be pointed at 0. Morel defines $L^{(1)}(\mathcal{Y})$ to be the cone of the map $ev_1: \underline{Hom}_{\bullet}(\mathbb{A}^1, \mathcal{Y}_f) \to \mathcal{Y}_f$. So there is a map $\mathcal{Y} \to L^{(1)}(\mathcal{Y})$ obtained from the trivial cofibration $\mathcal{Y} \to \mathcal{Y}_f$ and the defining map $\mathcal{Y}_f \to L^{(1)}(\mathcal{Y})$.

So $L^{(1)}(-)$ is a functor with a natural transformation $\eta: \mathrm{id} \to L^{(1)}(-)$. Define by induction $L^{(n)}(\mathcal{Y}) = L^{(1)}(L^{(n-1)}(\mathcal{Y}))$. There is thus a directed system $L^{(n-1)}(\mathcal{Y}) \to L^{(n)}(\mathcal{Y})$. Denote the direct limit/hocolim of this directed system by $L^{\infty}(\mathcal{Y})$.

Proposition 2.7. The natural morphism $\mathcal{Y} \to L^{\infty}(\mathcal{Y})$ is an \mathbb{A}^1 weak equivalence, and $L^{\infty}(\mathcal{Y})$ is \mathbb{A}^1 local.

(8)

Definition 2.8. Let \mathcal{X} be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on Sm/k associated to $U \to \pi_0(\mathcal{X}(U))$. A space \mathcal{X} is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let (\mathcal{X}, x) be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on Sm/k given by

$$U \to \pi_n(\mathcal{X}(U)).$$

A pointed space \mathcal{X} is called *n*-connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

(9)

Proposition 2.9. Let \mathcal{X} be a 0-connected simplicial sheaf. Then $L^{\infty}\mathcal{X}$ is also 0-connected.

Proof. M.V. IHES paper? Sketch of argument? It shouldn't be too hard by chasing components around. \Box

(10) For a sheaf of abelian groups M on Sm/k and a natural number n, a Dold-Kan construction gives a simplifical presheaf K(M,n). It is called the Eilenberg-MacLane spectrum of type (M,n) and has homotopy sheaves as expected:

$$\pi_m(K(M,n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}.$$

Important equation for this construction. For $X \in \text{Sm}/k$, M a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M, n)) \cong H^n_{Nis}(X; M).$$

(11) Use square brackets to denote the maps in the unstable pointed (motivic) homotopy category, i.e., $[\mathcal{X}, \mathcal{Y}] = \mathcal{H}_{\bullet}(k)(\mathcal{X}, \mathcal{Y})$.

Use $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ for the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^{\infty}\mathcal{X})$. This is also obtained by sheafifying the presheaf given by

$$U \in \mathrm{Sm}/k \mapsto [S^n \wedge U_+, \mathcal{X}].$$

2.4. Homotopy purity, connectedness calculations.

Proposition 2.10. Weak n-connectedness is equivalent to n-connectedness

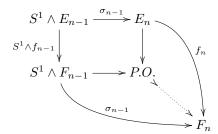
Proof. Details?
$$\Box$$

Proposition 2.11. If V is an irreducible, smooth k-scheme, and $U \subseteq V$ is a dense open subset, the space $L_{\mathbb{A}^1}(V/U)$ is 0-connected.

Proof. Seems to use that local structure of spaces is given by $\mathbb{A}^n/(\mathbb{A}^n - \{0\}) \wedge L_+$. How do all the reductions work? Also uses some results specific to working over perfect fields. \square

2.5. S^1 -spectra.

Definition 2.12. Let $\operatorname{Spt}_s(k)$ denote the category of S^1 -spectra of spaces $\Delta^{op}\operatorname{Shv}(\operatorname{Sm}/k)$. We first endow this category with the projective model structure, i.e., a map $f: E \to F$ is a weak equivalence iff for any n the map $f_n: E_n \to F_n$ is a w.e.; a map $f: E \to F$ is a fibration iff for all n the map $f_n: E_n \to F_n$ is a fibration. The cofibrations are characterized by the property that $f: E \to F$ is a cofib iff $f_0: E_0 \to F_0$ is a cofib and for any $n \ge 1$



This model structure does not actually invert $S^1 \wedge -$. To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.13. A map $f: E \to F$ of S^1 -spectra is a stable equivalence iff for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f): \pi_n(E) \to \pi_n(F)$ is an isomorphism.

The stable model category structure on $\operatorname{Spt}_s(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such.

Definition 2.14. Let $\operatorname{Spt}_s^{\mathbb{A}^1}(k)$ denote the category of S^1 spectra endowed with the stable model category structure localized at the collection of maps $\{\Sigma^{\infty}U_+ \wedge \mathbb{A}^1 \to \Sigma^{\infty}U_+ | U \in \operatorname{Sm}/k\}$.

Let $\mathcal{SH}^{S^1}(k)$ denote the homotopy category associated to $\operatorname{Spt}_s^{\mathbb{A}^1}(k)$. We will use $\mathcal{SH}_s^{S^1}(k)$ to denote the homotopy category of $\operatorname{Spt}_s(k)$.

Remark 2. There is a functor L^{∞} on the category of S^1 spectra which is similar to the unstable construction.

So we can use the functor L^{∞} as an \mathbb{A}^1 localization functor. To be precise, we let $\operatorname{Spt}_s^{\mathbb{A}^1loc}(k)$ be the subcategory of $\operatorname{Spt}_s(k)$ consisting of the \mathbb{A}^1 local spectra. We may equip $\operatorname{Spt}_s^{\mathbb{A}^1loc}(k)$ with a model structure with weak equivalences the \mathbb{A}^1 weak equivalences, the cofibrations are the stable cofibrations, and the fibrations are what they have to be.

The functor $L^{\infty}: \operatorname{Spt}_{s}(k) \to \operatorname{Spt}_{s}^{\mathbb{A}^{1}loc}(k)$ is left adjoint to the inclusion functor, and is a left Quillen functor. The homotopy category of $\operatorname{Spt}_{s}^{\mathbb{A}^{1}loc}(k)$ is categorically equivalent to $\mathcal{SH}^{S^{1}}(k)$. [Mor05, Corollary 4.2.3].

For spectra E and F, we may compute $[E,F]^{\mathbb{A}^1} := \mathcal{SH}^{S^1}(k)(E,F)$ by calculating $[L^{\infty}E,L^{\infty}F]^{\mathbb{A}^1}$. Note that this is $\mathcal{SH}_s^{S^1}(k)(E,L^{\infty}F)$ by using the adjunction. If we assume E is cofibrant and $L^{\infty}F$ is fibrant, we get the formula

$$[E, F]^{\mathbb{A}^1} = \operatorname{Spt}_s(k)(E, L^{\infty}F).$$

Definition 2.15. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{Ab}(Sm/k, Nis)$. That is,

$$\pi_n(E) = \operatorname{colim}_r \pi_{n+r}(E_r).$$

In particular, for a $U \in \text{Sm}/k$, we have

$$\pi_n(E)(U) = \operatorname{colim}_r \pi_{n+r}(E_r)(U).$$

Definition 2.16. An S^1 -spectrum E is said to be n-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.17. There is a left Quillen functor $\Sigma_s^{\infty} : \operatorname{Spc}_{\bullet} \to \operatorname{Spt}_s(k)$ given by $(\Sigma^{\infty} \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ with the evident bonding maps. The right adjoint to this functor is given by "evaluation at 0", i.e., $\Omega^{\infty}(E) = E_0$.

Remark 3. The right derived functor $R\Omega^{\infty}: \mathcal{SH}^{S^1}_s(k) \to \mathcal{H}_{\bullet}(k)$ is given by the formula

$$R\Omega^{\infty}(E) = \operatorname{colim}_{i} \Omega_{s}^{i} E_{i}.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

Remark 4. We also get a left Quillen functor $\Sigma_s^{\infty} : \operatorname{Spc}_{\bullet}^{\mathbb{A}^1}(k) \to \operatorname{Spt}_s^{\mathbb{A}^1}(k)$ given by the same formula as above.

What is the fibrant replacement functor in $\operatorname{Spt}_s^{\mathbb{A}^1}(k)$?

Remark 5. The stable homotopy category is symmetric monoidal, with smash product \land and internal hom <u>Hom</u>. Using symmetric spectra, one can give these constructions on the category of spectra.

The stable homotopy category is a triangulated category.

Proposition 2.18. Let $U \in \text{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\text{Ab}}(\text{Sm}/k)$. Then there is a canonical isomorphism

$$H^n_{Nis}(U;M) \to \mathcal{SH}^{S^1}(\Sigma^{\infty}U_+, HM[n]).$$

Proof. This is [Mor05, Lemma 3.2.3].

2.6. t structure.

Definition 2.19. Let \mathfrak{C} be a triangulated category. A t-structure on \mathfrak{C} is a pair of full subcategories $(\mathfrak{C}_{>0},\mathfrak{C}_{<0})$ which satisfies

- (1) For any $X \in \mathfrak{C}_{>0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\operatorname{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
- (2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
- (3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \to X \to Z \to Y[1]$$

for which $Y \in \mathfrak{C}_{>0}$, $Z \in \mathfrak{C}_{<0}[-1]$..

The heart of a t-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.20 (t-structure on $\mathcal{SH}_s^{S^1}(k)$). Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver n < 0.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver n > 0.

Theorem 2.21. The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t-structure on $\mathcal{SH}_s^{S^1}(k)$.

2.7. Connectivity results.

Proposition 2.22. [Mor03, Lemma4.2.4] The functor $L^{\infty} : \operatorname{Spt}_{s}^{S^{1}}(k) \to \operatorname{Spt}_{s,\mathbb{A}^{1}}^{S^{1}}(k)$ identifies the \mathbb{A}^{1} -localized S^{1} stable homotopy category with the homotopy category of \mathbb{A}^{1} -local S^{1} spectra.

Theorem 2.23 (S^1 stable connectivity theorem). Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever n < 0 the sheaf $\pi_n E = 0$. Then for all n < 0, $\pi_n L_{\mathbb{A}^1} E = 0$.

Theorem 2.24. The pair $(\mathcal{SH}^{S^1}_{\geq 0}(k), \mathcal{SH}^{S^1}_{\leq 0}(k))$ is a *t*-structure on the category $\mathcal{SH}^{S^1}(k)$.

Definition 2.25. Strictly \mathbb{A}^1 invariant sheaf of Abelian groups.

If M is strictly \mathbb{A}^1 invariant sheaf of groups, define the Eilenberg-MacLane spectrum HM associated to it.

Proposition 2.26. HM is \mathbb{A}^1 local iff M is strictly \mathbb{A}^1 invariant.

Proposition 2.27. The heart of the homotopy t structure is equivalent to the category of strictly \mathbb{A}^1 invariant sheaves.

3. Inverting
$$\mathbb{G}_m$$
: \mathbb{P}^1 spectra

3.1. \mathbb{G}_m suspension and loops. We always consider \mathbb{G}_m to be pointed at 1 unless otherwise specified.

Definition 3.1. On the category $\operatorname{Spt}_s(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps.

Smashing with \mathbb{G}_m is also a functor on the unstable category of pointed spaces, and we give it the same name Σ_m .

Definition 3.2. The functor Σ_m on $\operatorname{Spc}_{\bullet}(k)$ has a right adjoint denoted Ω_t . It is given by the formula $\Omega_t \mathcal{X} = \operatorname{\underline{Hom}}_{\bullet}(\mathbb{G}_m, \mathcal{X})$.

The functor Σ_t on $\operatorname{Spt}_s^{S^1}(k)$ also has a right adjoint Ω_t given by the internal hom functor, i.e., $\Omega_t E = \operatorname{Hom}(\Sigma^\infty \mathbb{G}_m, E)$.

Proposition 3.3. The functor Σ_t is a left Quillen functor on $\operatorname{Spt}_s(k)$ and on $\operatorname{Spt}_s^{\mathbb{A}^1}(k)$. Furthermore, Σ_t is an exact functor on $\mathcal{SH}^{S^1}(k)$.

3.2. Contraction in Ab(Sm/k, Nis).

Definition 3.4. Let G be sheaf of pointed sets on Sm/k. The contraction of G is the sheaf $G_{-1} = G_{con}$ given by the formula

$$U \in \operatorname{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map ev_1 is the map induced by $ev_1 : \operatorname{Spec}(k) \to \mathbb{G}_m$, i.e., $k[x, x^{-1}] \to k$ given by $x \mapsto 1$.

Note that indeed G_{-1} is a sheaf since it is the kernel of the morphism of sheaves $G(-) \to G(-\times \mathbb{G}_m)$.

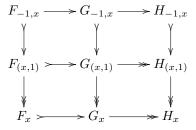
Proposition 3.5. If G is the trivial sheaf of abelian groups, then so is its contraction G_{-1} .

Proposition 3.6. Contraction is an exact functor on the category $\underline{Ab}(Sm/k, Nis)$.

Proof. Functoriality can be established by using the universal property of kernels.

To show exactness, let $F \to G \to H$ be a short exact sequence of sheaves. We must show $F_{-1} \to G_{-1} \to H_{-1}$ is still exact. It suffices to check exactness at the level of stalks.

Since $id = ev_1 \circ \pi : G(X) \to G(X \times \mathbb{G}_m) \to G(X)$, we have $ev_1 : G(-) \to G(-\times \mathbb{G}_m)$ is a surjection on all smooth schemes. Hence ev_1 is an epimorphism. For a Nisnevich point x of X, The following commutative diagram then establishes the exactness of the contraction by the 3x3 lemma.



3.3. Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$.

Proposition 3.7. If G is a sheaf of Abelian groups, then $G_{-1} \cong \underline{\text{Hom}}_{\bullet}(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $- \wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

Proof. For this category, $\underline{\operatorname{Hom}}_{\bullet}(\mathbb{G}_m, G)$ and G_{-1} both have sections at X given by $\ker(\operatorname{ev}_1 : G(X \times \mathbb{G}_m) \to G(X))$. See description of pointed internal hom for this.

Remark 6. If G is a sheaf of Abelian groups, we may consider G as a space by declaring $G_n = G$ for all n and giving identity maps for the structure maps. In particular, G is a pointed space at 0.

We can then realize the contraction as a \mathbb{G}_m loop space $G_{-1}(X) = \underline{\mathrm{Hom}}_{\bullet}(\mathbb{G}_m, G)(X)$.

Remark 7. Construction of canonical map $\pi_n(\underline{\operatorname{Hom}}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ for an S^1 spectrum E.

First observe that for any $U \in \text{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\operatorname{Spt}_s(k)(S^n \wedge \Sigma_s^{\infty} U_+ \wedge \Sigma^{\infty} \mathbb{G}_m, E) \times \operatorname{Spc}(k)(U, \mathbb{G}_m) \to \pi_0(E)(U)$$

given by sending (f, α) to the composition

$$\Sigma_s^{\infty} U_+ \xrightarrow{\operatorname{id} \wedge \Sigma_s^{\infty} \alpha} S^{n} \wedge \Sigma_s^{\infty} U_+ \wedge \Sigma_s^{\infty} \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\operatorname{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \to \pi_n(E).$$

Does this map descend to the smash? Yes, since if either map is a constant map, then so is the composition.

We thus get a map of sheaves of pointed sets

$$\pi_n(\underline{\operatorname{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \to \pi_n(E).$$

But by the adjunction $-\wedge \mathbb{G}_m \dashv \underline{\mathrm{Hom}}_{\bullet}(\mathbb{G}_m, -)$ on $\mathrm{Spc}_{\bullet}(k)$ we have a morphism

$$\pi_n(\underline{\operatorname{Hom}}(\mathbb{G}_m, E)) \to \underline{\operatorname{Hom}}_{\bullet}(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Why is it a map of sheaves of abelian groups?

Remark 8. If E = HM is an Eilenberg-MacLane spectrum associated to a strictly \mathbb{A}^1 invariant sheaf of abelian groups M, we show

$$\pi_n(\underline{\operatorname{Hom}}(\mathbb{G}_m, HM)) \to \underline{\operatorname{Hom}}_{\bullet}(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\operatorname{Hom}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

Lemma 3.8. Let $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then $K(M,r)_{-1} \simeq K(M_{-1},r)$.

Proof. Let $X \in \text{Sm}/k$. The simplicial set $K(M,r)_{-1}(X)$ by the fibration

$$K(M,r)_{-1}(X) \to K(M,r)(\mathbb{G}_m \times X) \to K(M,r)(X).$$

There is thus a long exact sequence of homotopy groups, which ammounts to

$$\pi_r K(M,r)_{-1}(X) \longrightarrow \pi_r K(M,r)(\mathbb{G}_m \times X) \longrightarrow \pi_r K(M,r)(X) \longrightarrow \pi_{r-1} K(M,r)_{-1}(X).$$

The remaining homotopy groups are evidently trivial. As $\pi_r K(M,r)(\mathbb{G}_m \times X) \to \pi_r K(M,r)(X)$ is surjective, we see $\pi_{r-1}K(M,r)_{-1}(X) = 0$. Therefore,

$$\pi_n K(M, r)_{-1}(X) = \begin{cases} 0 & \text{if } n \neq r \\ M_{-1}(X) & \text{if } n = r \end{cases}.$$

We conclude that $K(M,r)_{-1} \simeq K(M_{-1},r)$.

Proposition 3.9. For $M \in \underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$, the spectrum $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

Proof. Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_{\bullet}(k)$, we have $\Sigma^{\infty} \mathbb{P}^1[-1] = \Sigma^{\infty} \mathbb{G}_m$. Therefore

$$\begin{split} \mathcal{SH}_{s}^{S^{1}}(k)(\Sigma^{\infty}\mathbb{G}_{m}, HM[n]) &= \mathcal{SH}_{s}^{S^{1}}(k)(\Sigma^{\infty}\mathbb{P}^{1}[-1], HM[n]) \\ &= \mathcal{SH}_{s}^{S^{1}}(k)(\Sigma^{\infty}\mathbb{P}^{1}, HM[n+1]) \\ &= \tilde{H}_{Nis}^{n+1}(\mathbb{P}^{1}; M). \end{split}$$

As the cohomological dimension of \mathbb{P}^1 is less than or equal to 1, we then have $\tilde{H}_{Nis}^{n+1}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$.

Now¹ for $U \in \text{Sm}/k$, we calculate for $n \neq 0$

$$\pi_{n}\underline{\operatorname{Hom}}(\mathbb{G}_{m}, HM)(U) = \mathcal{SH}_{s}^{S^{1}}(k)(S^{n} \wedge \Sigma^{\infty}\mathbb{G}_{m} \wedge \Sigma^{\infty}U_{+}, HM)$$

$$= \operatorname{colim}_{r} \mathcal{H}_{\bullet}^{s}(k)(S^{n+r} \wedge U_{+}, \underline{\operatorname{Hom}}_{\bullet}(\mathbb{G}_{m}, K(M, r)))$$

$$= \operatorname{colim}_{r} \pi_{n+r}(K(M, r)_{-1})(U)$$

$$= \operatorname{colim}_{r} \pi_{n+r}(K(M_{-1}, r))(U)$$

$$= \operatorname{colim}_{r} 0$$

$$= 0.$$

To establish the second point, we calculate for $U \in \text{Sm}/k$

$$\pi_0 \underline{\operatorname{Hom}}(\mathbb{G}_m, HM)(U) = \mathcal{SH}_s^{S^1}(k)(\Sigma^{\infty}\mathbb{G}_m \wedge \Sigma^{\infty}U_+, HM)$$

$$= \operatorname{colim}_r \mathcal{H}_{\bullet}^s(k)(S^r \wedge U_+, \underline{\operatorname{Hom}}_{\bullet}(\mathbb{G}_m, K(M, r)))$$

$$= \operatorname{colim}_r \pi_r(K(M, r)_{-1})(U)$$

$$= \operatorname{colim}_r \pi_r(K(M_{-1}, r))(U)$$

$$= \operatorname{colim}_r M_{-1}(U)$$

$$= M_{-1}(U)$$

$$= \pi_0 H(M_{-1})(U).$$

We now know that the associated homotopy sheaves $\pi_n \underline{\text{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all n. So they are weak equivalent by [Mor05, Lemma 3.2.5].

3.4. Inverting $\mathbb{G}_m \wedge -$; (\mathbb{G}_m, S^1) bi-spectra. The functor Σ_t on $\operatorname{Spt}_s(k)$ is a left Quillen functor. We may therefore apply the general machinery of [?] to create a model category in which Σ_t is invertible. The construction of Hovey may be described as (\mathbb{G}_m, S^1) bispectra.

Definition 3.10. A (\mathbb{G}_m, S^1) bi-spectrum of spaces over k consists of a bigraded family of spaces $E_{i,j}, i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \to E_{i,j+1}$ and $\mu_{i,j} : S^1 \to E_{i,j+1}$

¹Given the following argument, is the part before this necessary? Does this argument actually work?

 $\mathbb{G}_m \wedge E_{i,j} \to E_{i+1,j}$ for which the following diagram commutes.

$$S^{1} \wedge \mathbb{G}_{m} \wedge E_{i,j} \xrightarrow{S^{1} \wedge \tau_{i,j}} S^{1} \wedge E_{i+1,j}$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

Remark 9. Note that a (\mathbb{G}_m, S^1) bispectrum is just a \mathbb{G}_m -spectrum of S^1 – spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a (\mathbb{G}_m, S^1) bi-spectrum $E_{i,j}$ as a sequence of S^1 spectra $E_{i,*}$.

Definition 3.11. Let E be a (\mathbb{G}_m, S^1) bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \operatorname{Sm}/k \mapsto \operatorname{colim}_r \mathcal{H}_{\bullet}(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$. We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

Proposition 3.12. If is E a (\mathbb{G}_m, S^1) bispectrum, the presheaf of homotopy groups may also be calculated as

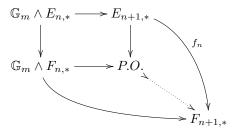
$$\tilde{\pi}_{n+m\alpha}(U) = \operatorname{colim}_{s,r} \mathcal{H}_{\bullet}(k) (\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[DLØRV, p 217]

Definition 3.13. A morphism $f: E \to F$ of (\mathbb{G}_m, S^1) bispectra is an \mathbb{A}^1 stable weak equivalence if the following induced map is an isomorphism for all $U \in \text{Sm}/k$.

$$f_*: \tilde{\pi}_{n+m\alpha}(E)(U) \to \tilde{\pi}_{n+m\alpha}(F)(U)$$

Definition 3.14. A morphism $f: E \to F$ of (\mathbb{G}_m, S^1) bispectra is an \mathbb{A}^1 stable cofibration if $f_0: E_{0,*} \to F_{0,*}$ is a cofibration of S^1 spectra and the map $P.O. \to F_{n+1}$ is a cofibration in the following diagram.



Proposition 3.15. The category $\operatorname{Spt}_{s,t}^{\mathbb{A}^1}(k)$ of (\mathbb{G}_m, S^1) bispectra with \mathbb{A}^1 stable weak equivalences and \mathbb{A}^1 stable cofibrations is a model category. Denote the associated homotopy category of $\operatorname{Spt}_{s,t}(k)$ by $\mathcal{SH}(k)$.

Proposition 3.16. The fibrant bi-spectra are the Ω_t -spectra. [H-Spt, Theorem 3.4]

Proposition 3.17. There is a left Quillen functor $\Sigma_t^{\infty} : \operatorname{Spt}_s^{\mathbb{A}^1}(k) \to \operatorname{Spt}_{s,t}^{\mathbb{A}^1}(k)$ given by $(\Sigma_t^{\infty} E)_{i,j} = \mathbb{G}_m^i \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}_m^i \wedge E_j \longrightarrow \mathbb{G}_m^i \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}^m \wedge \mathbb{G}_m^i \wedge E_j \to \mathbb{G}_m^{i+1} E_j.$$

The right adjoint to Σ_t^{∞} is denoted by Ω_t^{∞} and is given by $\Omega_t^{\infty}(E) = E_{0,*}$. The right derived functor $R\Omega_t^{\infty}(E)$ is given by the formula

$$R\Omega_t^{\infty}(E) = \operatorname{colim}_i \Omega_t^i E_{i,*}.$$

Show connectivity of S^1 spectra is preserved by taking Σ_t^{∞} .

Prove that [Mor03, Definition 5.2.1] gives a t-structure on $\mathcal{SH}(k)$.

Now show/realize that Voevodsky's connectivity theorem holds.

References

- [A-1974] Adams, J.F., Stable Homotopy and Generalized Homology. Chicago Lectures in Mathematics, (1974).
- [B] Blander, Benjamin, Local Projective Model Structures on Simplicial Presheaves. K-Theory, 24 (2001) 283–301.
- [DHI] Dugger, Dan; Hollander, Sharon; Isaksen, Dan, Hypercovers and simplicial presheaves.
- [DLØRV] Dundas, B.; Levine, M.; Østvær, P.; Röndigs, O.; Voevodsky, V., *Motivic Homotopy Theory*. Springer (2000).
- [Hir] Phillip, Hirschhorn, Model Categories and Their Localization. AMS (2003).
- [H-Mod] Hovey, Mark, Model Categories. online preprint (1991).
- [H-Spt] Hovey, Mark, Spectra and symmetric spectra in general model categories. journal? (2001).
- [J] Jardine, J.F., Simplicial presheaves. Journal of Pure and Applied Algebra, 47 (1987) 35–87.
- [Mor03] Morel, Fabien, An introduction to \mathbb{A}^1 homotopy theory.
- [Mor04] Morel, Fabien, On the motivic π_0 of the sphere spectrum. NATO science series.
- [Mor05] Morel, Fabien, The stable \mathbb{A}^1 connectivity theorems. preprint (2004).
- [Voev98] Voevodsky, Vladimir. A¹-Homotopy Theory. Doc. Math. J., (1998) pp. 579–604.