1. Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

Our goal is to prove theorem 4.14 of [Voev98], which we restate in terms of $\mathbb{P}^1$-spectra.

Theorem 1.1. Let (X, x) be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let $\mathcal{Y}$ be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

We will prove this theorem by following [Mor03] and [Mor05] by Fabien Morel.

Remark 1. To prove this theorem, Morel carefully analyzes how to pass from spaces in the projective model structure to the $\mathbb{A}^1$ stable homotopy category of $\mathbb{P}^1$ spectra. From the projective model structure on spaces, we construct a model of the left Bousfield localization of spaces at the class of maps $\{U_+ \wedge \mathbb{A}^1 \to U \,|\, U \operatorname{im} \mathrm{Sm}/k\}$. To get to $\mathbb{P}^1$ spectra, we first invert $S^1 \wedge -$ to get a category of S^1-spectra, and then we invert $\mathbb{G}_m \wedge -$ to get a category of $(\mathbb{G}_m, S^1)$ bispectra.

$$\mathcal{H}_{s,\bullet}(k) \to \mathcal{H}_\bullet(k) \to \mathcal{SH}^{S^1}(k) \to \mathcal{SH}(k)$$

The line of attack is then to show that for a pointed space $\mathcal{Y}$, the S^1 suspsension spectrum $\Sigma_s^\infty \mathcal{Y}$ in the stable model category is -1 connected. Then we establish the first connectivity theorem which ensures the $\mathbb{A}^1$ localization of $\Sigma_s^\infty \mathcal{Y}$ is -1 connected. Finally, we show that inverting $\mathbb{G}_m$ does not affect the connectivity, i.e., $\Sigma_t^\infty \Sigma_s^\infty \mathcal{Y}$ is again -1 connected.

The machinery that we set up to prove this theorem will also allow us to establish a t-structure on $\mathcal{SH}(k)$, and identify its heart.

Remark 2. A construction of Ayoub [**?**] shows that theorem 1.1 statement is false over general Noetherian base schemes S. The argument below works for infinite fields, however.

2. Assumptions from previous lectures

We briefly recall some of the basic constructions which appear in [Mor03] and [Mor05].

2.1. **Facts about Nisnevich topology.**

The proof of Voevodsky's connectivity theorem will follow from the following property of Nisnevich sheaf cohomology by a sequence of reductions.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k, and let $X \in \mathrm{Sm}/k$ with Krull dimension d. Then whenever $n > d$, $H^n_{Nis}(X; M) = 0$.

2.2. **Unstable model category** $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$**.**

Definition 2.2. Let k be a field, and let Sm/k denote the category of smooth schemes of finite type over k. The category of Morel-Voevodsky spaces over k is the category of simplicial Nisnevich sheaves on Sm/k. We write $\mathrm{Spc}(k) = \Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$ for this category.

The category $\mathrm{Spc}(k)$ may be equipped with several different model category structures. We will work with the injective local model category structure on $\mathrm{Spc}(k)$, which we now define.

Definition 2.3. A map $\mathcal{X} \to \mathcal{Y}$ is an injective weak equivalence if and only if for any $U \in \mathrm{Sm}/k$, the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a weak equivalence of simplicial sets.

A map $\mathcal{X} \to \mathcal{Y}$ is an injective cofibration if and only if for any $U \in \mathrm{Sm}/k$, the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a cofibration of simplicial sets, i.e., a monomorphism.

A map $\mathcal{X} \to \mathcal{Y}$ is an injective fibration if and only if it satisfies the left lifting property with respect to any trivial injective cofibration. That is, for any commutative square below with $\mathcal{A} \to \mathcal{B}$ a trivial cofibration, a lift $\mathcal{B} \to \mathcal{X}$ exists.

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, tail, "\sim"'] & \mathcal{X} \arrow[d] \\ \mathcal{B} \arrow[r] \arrow[ur, dotted] & \mathcal{Y} \end{tikzcd}$$

Denote the homotopy category associated to the injective model category structure on $\mathrm{Spc}(k)$ by $\mathcal{H}_s(k)$. The "s" stands for simplicial.

Definition 2.4. The category of pointed space $\mathrm{Spc}_\bullet(k)$ inherits a model category structure from $\mathrm{Spc}(k)$. The functor $-_+ : \mathrm{Spc}(k) \to \mathrm{Spc}_\bullet(k)$ defined by adding a disjoint basepoint to a given space is a left Quillen functor. The right adjoint is the forgetful functor.

Proposition 2.5. Every object of $\mathrm{Spc}(k)$ and $\mathrm{Spc}_\bullet(k)$ is cofibrant in the injective model category structure.

Definition 2.6. For $X \in \mathrm{Sm}/k$, let rX denote the sheaf associated to the presheaf $U \mapsto \mathrm{Sm}/k(U, X)$. This defines a functor $r : \mathrm{Sm}/k \to \mathrm{Spc}(k)$.

For K a simplicial set, the constant space $cK \in \mathrm{Spc}(k)$ is the sheaf associated to the constant presheaf with value K. The functor $c : \underline{\mathrm{sSet}} \to \mathrm{Spc}(k)$ is a left Quillen functor with right adjoint given by taking sections at $\operatorname{Spec} k$.

Proposition 2.7. $\mathrm{Spc}(k)$ is a simplicial model category.

See [Pel08, Chapter 2] for a detailed treatment of the products and internal hom constructions in $\mathrm{Spc}(k)$. We recount those definitions which are essential to our argument.

Definition 2.8. For spaces $\mathcal{X}$ and $\mathcal{Y}$, the product $\mathcal{X} \times \mathcal{Y}$ in $\mathrm{Spc}(k)$ is given by $U \mapsto \mathcal{X}(U) \times \mathcal{Y}(U)$. For spaces $\mathcal{X}$ and $\mathcal{Y}$, the internal hom $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ in $\mathrm{Spc}(k)$ is given by the formula

$$(U, m) \in \mathrm{Sm}/k \times \Delta \mapsto \mathrm{Hom}_{\Delta^{op}\mathrm{Shv}}(X \times rU \times c\Delta^n, Y).$$

Proposition 2.9. The product and internal hom defined above give $\mathrm{Spc}(k)$ the structure of a closed monoidal model category. See [H-Mod, Chapter 4] or [Pel08, §1.7] for the definition.

Proof. The adjunction between $\mathcal{X} \times -$ and $\underline{Hom}(\mathcal{X}, -)$ is given by the following map.

$$\eta : \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z})) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$$

For $g \in \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$, we define $\eta(g)$ by

$$\begin{array}{ccc} \mathcal{X}_n(U) \times \mathcal{Y}_n(U) & \xrightarrow{\eta(g)(U,n)} & \mathcal{Z}_n(U) \\ (a, b) & \longmapsto & g(U, n)(b)(U, n)(a, \mathrm{id}_U, \mathrm{id}_{\Delta^n}). \end{array}$$

For $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, the map $(\eta^{-1}f)(U,n) : \mathcal{Y}_n(U) \to \underline{Hom}(\mathcal{X},\mathcal{Z})_n(U)$ is given by sending $y \in \mathcal{Y}_n(U)$ to the map

$$\begin{array}{ccc} \mathcal{X}_m(V) \times \mathrm{Sm}/k(V,U) \times \Delta^n_m & \xrightarrow{(\eta^{-1}f)(U,n)(y)(V,m)} & \mathcal{Z}_m(V) \\ (x,\phi,\alpha) & \longmapsto & f(V,m)(x,\mathcal{Y}(\phi)(y\circ\alpha)) \end{array}$$

where we identify y with a map $y : \Delta^n \to \mathcal{Y}(V)$, and $\alpha : \Delta^m \to \Delta^n$. □

Definition 2.10. Let $\mathcal{X}$ and $\mathcal{Y}$ be spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{Hom}(\mathcal{X},\mathcal{Y}) \to \mathcal{Y}$, where at $(U,n) \in (\mathrm{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times rU \times c\Delta^n \to \mathcal{Y}$ to $g(U,n)(x,\mathrm{id},\mathrm{id}) \in \mathcal{Y}_n(U)$.

For pointed spaces $(\mathcal{X},x)$ and $\mathcal{Y},y)$, the pointed internal hom $\underline{Hom}_\bullet(\mathcal{X},\mathcal{Y})$ is the fiber of ev_x over y, i.e., $ev_x^{-1}(y)$.

Definition 2.11. Let $(\mathcal{X},x)$ and $(\mathcal{Y},y)$ be pointed spaces. The wedge of $\mathcal{X}$ and $\mathcal{Y}$, denoted by $\mathcal{X} \vee \mathcal{Y}$, is the pushout of the following diagram.

$$\begin{tikzcd} pt \arrow[r, "x"] \arrow[d, "y"] & \mathcal{X} \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \vee \mathcal{Y} \end{tikzcd}$$

The smash product $\mathcal{X} \wedge \mathcal{Y}$ is the space given by the pushout of the following diagram, with basepoint $\mathcal{X} \vee \mathcal{Y}$.

$$\begin{tikzcd} \mathcal{X} \vee \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{X} \times \mathcal{Y} \arrow[d] \\ pt \arrow[r] & \mathcal{X} \wedge \mathcal{Y} \end{tikzcd}$$

Proposition 2.12. The category of pointed spaces $\mathrm{Spc}_\bullet(k)$ is also a closed monoidal category with product $\wedge$ and internal hom $\underline{\mathrm{Hom}}_\bullet$.

2.3. $\mathbb{A}^1$ localization.

Definition 2.13. A space $\mathcal{X}$ is called $\mathbb{A}^1$ local if for any smooth scheme U, the canonical map

$$\mathrm{Hom}(rU,\mathcal{X}) \to \mathrm{Hom}(rU \times \mathbb{A}^1,\mathcal{X})$$

is a bijection.

Definition 2.14. A map $f : \mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$ weak equivalence if

$$\mathrm{Hom}(\mathcal{Y},\mathcal{Z}) \to \mathrm{Hom}(\mathcal{X},\mathcal{Z})$$

is a bijection for every $\mathbb{A}^1$ local space $\mathcal{Z}$.

The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \to U \,|\, U \in \mathrm{Sm}/k\}$. We deonte the category of spaces with the model structure obtained by left Bousfield localization by $\mathrm{Spc}^{\mathbb{A}^1}(k)$ and its homotopy category by $\mathcal{H}(k)$. See [Hir, Chapter 3] for the general theory of Bousfield localization. One thing we obtain is a localization functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \to L_W \mathcal{H}_s(k)$ which is a left Quillen functor. In particular, $L_{\mathbb{A}^1}$ sends sends $\mathbb{A}^1$ weak equivalences to isomorphisms.

The model category $\mathrm{Spc}^{\mathbb{A}^1}(k)$ is constructed as follows. The underlying category of $\mathrm{Spc}^{\mathbb{A}^1}(k)$ is $\mathrm{Spc}(k)$, but the weak equivalences are the $\mathbb{A}^1$-local weak equivalences. The cofibrations are the cofibrations in the injective model structure on $\mathrm{Spc}(k)$. The fibrations are what they need to be, i.e., those maps which satisfy the left lifting property with respect to trivial cofibrations, i.e., $\mathbb{A}^1$-local weak equivalences which are also cofibrations.

In [Mor05, Proposition 3.2.3], Morel gives an explicit construction which takes a pointed space $\mathcal{Y}$ and produces a space $L^\infty \mathcal{Y}$ which is $\mathbb{A}^1$ local and a map $\mathcal{Y} \to L^\infty \mathcal{Y}$ which is an $\mathbb{A}^1$ weak equivalence.

Definition 2.15. Let $\mathcal{X}$ be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on Sm/k associated to $U \to \pi_0(\mathcal{X}(U))$. A space $\mathcal{X}$ is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let $(\mathcal{X}, x)$ be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on Sm/k given by

$$U \to \pi_n(\mathcal{X}(U)).$$

A pointed space $\mathcal{X}$ is called n-connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

Proposition 2.16. Let $\mathcal{X}$ be a 0-connected simplicial sheaf. Then $L^\infty \mathcal{X}$ is also 0-connected.

For a sheaf of abelian groups M on Sm/k and a natural number n, a Dold-Kan construction gives a simplifical presheaf $K(M, n)$. It is called the Eilenberg-MacLane spectrum of type (M, n) and has homotopy sheaves as expected:

$$\pi_m(K(M,n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}.$$

Important equation for this construction. For $X \in \mathrm{Sm}/k$, M a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M,n)) \cong H^n_{Nis}(X; M).$$

It therefore follows that

$$\mathcal{H}_\bullet(k)(rX_+, K(M,n)) \cong H^n_{Nis}(X; M).$$

Use square brackets to denote the maps in the unstable pointed (motivic) homotopy category, i.e., $[\mathcal{X}, \mathcal{Y}] = \mathcal{H}_\bullet(k)(\mathcal{X}, \mathcal{Y})$.

Use $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ for the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^\infty \mathcal{X})$. This is also obtained by sheafifying the presheaf given by

$$U \in \mathrm{Sm}/k \mapsto [S^n \wedge U_+, \mathcal{X}].$$

2.4. **Homotopy purity, connectedness calculations.**

Proposition 2.17. Let k be an infinite field, and consider $\mathcal{X}$ be a pointed space. If for any finitely generated field F over k, $\pi_0(\mathcal{X})(F) = 0$, then the sheaf $\pi_0(\mathcal{X})$ is trivial.

Proof. Check at points. Use Gabber presentation lemma. □

2.5. S^1**-spectra.**

Definition 2.18. Let $\mathrm{Spt}_s(k)$ denote the category of S^1-spectra of spaces $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k)$. We first endow this category with the projective model structure (or level-wise model structure), i.e., a map $f : E \to F$ is a weak equivalence iff for any n the map $f_n : E_n \to F_n$ is a w.e.; a map $f : E \to F$ is a fibration iff for all n the map $f_n : E_n \to F_n$ is a fibration. The cofibrations are characterized by the property that $f : E \to F$ is a cofib iff $f_0 : E_0 \to F_0$ is a cofib and for any $n \geq 1$

$$\begin{tikzcd}
S^1 \wedge E_{n-1} \arrow[r, "\sigma_{n-1}"] \arrow[d, "S^1\wedge f_{n-1}"'] & E_n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\
S^1 \wedge F_{n-1} \arrow[r] \arrow[drr, bend right, "\sigma_{n-1}"] & P.O. \arrow[dr, dotted] & \\
 & & F_n
\end{tikzcd}$$

This model structure does not actually invert $S^1 \wedge -$. To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.19. A map $f : E \to F$ of S^1-spectra is a stable equivalence if for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \to \pi_n(F)$ is an isomorphism.

The stable model category structure on $\mathrm{Spt}_s(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such.

Definition 2.20. Let $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$ denote the category of S^1 spectra endowed with the stable model category structure localized at the collection of maps $\{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \to \Sigma^\infty U_+ \,|\, U \in \mathrm{Sm}/k\}$.

Let $\mathcal{SH}^{S^1}(k)$ denote the homotopy category associated to $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. We will use $\mathcal{SH}_s^{S^1}(k)$ to denote the homotopy category of $\mathrm{Spt}_s(k)$.

Remark 3. There is a functor L^∞ on the category of S^1 spectra which is similar to the unstable construction.

So we can use the functor L^∞ as an $\mathbb{A}^1$ localization functor. To be precise, we let $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ be the subcategory of $\mathrm{Spt}_s(k)$ consisting of the $\mathbb{A}^1$ local spectra. We may equip $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ with a model structure with weak equivalences the $\mathbb{A}^1$ weak equivalences, the cofibrations are the stable cofibrations, and the fibrations are what they have to be.

The functor $L^\infty : \mathrm{Spt}_s(k) \to \mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ is left adjoint to the inclusion functor, and is a left Quillen functor. The homotopy category of $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ is categorically equivalent to $\mathcal{SH}^{S^1}(k)$. [Mor05, Corollary 4.2.3].

For spectra E and F, we may compute $[E,F]^{\mathbb{A}^1} := \mathcal{SH}^{S^1}(k)(E,F)$ by calculating $[L^\infty E, L^\infty F]^{\mathbb{A}^1}$. Note that this is $\mathcal{SH}_s^{S^1}(k)(E, L^\infty F)$ by using the adjunction. If we assume E is cofibrant and $L^\infty F$ is fibrant, we get the formula

$$[E,F]^{\mathbb{A}^1} = \mathrm{Spt}_s(k)(E, L^\infty F).$$

Definition 2.21. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$. That is,

$$\pi_n(E) = \mathrm{colim}_r\, \pi_{n+r}(E_r).$$

In particular, for a $U \in \mathrm{Sm}/k$, we have

$$\pi_n(E)(U) = \mathrm{colim}_r\, \pi_{n+r}(E_r)(U).$$

Definition 2.22. An S^1-spectrum E is said to be n-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.23. There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \to \mathrm{Spt}_s(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ with the evident bonding maps. The right adjoint to this functor is given by "evaluation at 0", i.e., $\Omega^\infty(E) = E_0$.

Remark 4. The right derived functor $R\Omega^\infty : \mathcal{SH}_s^{S^1}(k) \to \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i\, \Omega_s^i E_i.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

Remark 5. We also get a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k) \to \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ given by the same formula as above.

Remark 6. The stable homotopy category is symmetric monoidal, with smash product $\wedge$ and internal hom $\underline{\mathrm{Hom}}$. Using symmetric spectra, one can give these constructions on the category of spectra.

The stable homotopy category is a triangulated category.

Proposition 2.24. Let $U \in \mathrm{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then there is a canonical isomoprhism

$$H^n_{Nis}(U; M) \to \mathcal{SH}^{S^1}(\Sigma^\infty U_+, HM[n]).$$

Proof. This is [Mor05, Lemma 3.2.3]. □

2.6. ***t*-structures.**

Definition 2.25. Let $\mathfrak{C}$ be a triangulated category. A t-structure on $\mathfrak{C}$ is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

(1) For any $X \in \mathfrak{C}_{\geq 0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
(2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
(3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \to X \to Z \to Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$..

The heart of a t-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.26 (t-structure on $\mathcal{SH}_s^{S^1}(k)$)**.** Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n > 0$.

Theorem 2.27. The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t-structure on $\mathcal{SH}_s^{S^1}(k)$.

Remark 7. For a space $\mathcal{X}$, there is a Postnikov tower associated to it

$$\cdots P^n(\mathcal{X}) \to P^{n-1}(\mathcal{X}) \to \cdots \to P^0(\mathcal{X}) \to P^{-1}(\mathcal{X})$$

constructed in [MV99, p. 57]. The main construction needed is the Moore-Postnikov tower of a simplicial set [GJ91, VI.3.4]. For a simplicial set K and $n \in \mathbb{N}$, define $K^{(n)} = \mathrm{im}(K \to \mathrm{cosk}_n K)$. This is a convenient way to define the Moore construction.

For a space $\mathcal{X}$, we then define $P^n\mathcal{X}$ to be the space given by sheafification of $U \mapsto \mathcal{X}(U)^{(n)}$.

Now for E an S^1-spectrum, let $E_{\leq 0}$ be the spectrum with $(E_{\leq 0})_n = P^n(E_n)$. The bonding maps come from the canonical map

$$S^1 \wedge P^n(E_n) \to P^{n+1}(S^1 \wedge E_n).$$

See [Mor05, Lemma 3.2.1] for more on this construction.

2.7. **Connectivity results.**

Proposition 2.28. [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}_s^{S^1}(k) \to \mathrm{Spt}_{s,\mathbb{A}^1}^{S^1}(k)$ identifies the $\mathbb{A}^1$-localized S^1 stable homotopy category with the homotopy category of $\mathbb{A}^1$-local S^1 spectra.

Theorem 2.29 (S^1 stable connectivity theorem)**.** Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L_{\mathbb{A}^1} E = 0$.

Theorem 2.30. The pair $(\mathcal{SH}_{\geq 0}^{S^1}(k), \mathcal{SH}_{\leq 0}^{S^1}(k))$ is a t-structure on the category $\mathcal{SH}^{S^1}(k)$.

Proof. This is just the restriction of the t-structure to the $\mathbb{A}^1$-local objects. ∎

Definition 2.31. Strictly $\mathbb{A}^1$ invariant sheaf of Abelian groups.

If M is strictly $\mathbb{A}^1$ invariant sheaf of groups, define the Eilenberg-MacLane spectrum HM associated to it.

Proposition 2.32. HM is $\mathbb{A}^1$ local iff M is strictly $\mathbb{A}^1$ invariant.

Proposition 2.33. The heart of the homotopy t structure is equivalent to the category of strictly $\mathbb{A}^1$ invariant sheaves.

3. Inverting $\mathbb{G}_m \wedge -$; $\mathbb{P}^1$ spectra

3.1. $\mathbb{G}_m$ **suspension and loops.**

We always consider $\mathbb{G}_m$ to be pointed at 1 unless otherwise specified.

Definition 3.1. On the category $\mathrm{Spt}_s(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps. In order for Σ_t to be a left Quillen functor, we must replace $\mathbb{G}_m$ with a cofibrant object. We do this, and work with this object instead of $\mathbb{G}_m$.

Smashing with $\mathbb{G}_m$ is also a functor on the unstable category of pointed spaces, and we give it the same name Σ_t.

Definition 3.2. The functor Σ_t on $\mathrm{Spc}_\bullet(k)$ has a right adjoint denoted Ω_t. It is given by the formula $\Omega_t\mathcal{X} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \mathcal{X})$.

The functor Σ_t on $\mathrm{Spt}_s^{S^1}(k)$ also has a right adjoint Ω_t given by the internal hom functor, i.e., $\Omega_t E = \underline{\mathrm{Hom}}(\Sigma^\infty \mathbb{G}_m, E)$.

Proposition 3.3. The functor Σ_t is a left Quillen functor on $\mathrm{Spt}_s(k)$ and on $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. Furthermore, Σ_t is an exact (or homological) functor on $\mathcal{SH}^{S^1}(k)$, meaning it sends exact triangles to exact triangles.

Lemma 3.4. Let $E \in \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ be a -1 connected spectrum. Then $\Sigma_t E$ is again -1 connected.

Proof. The claim is clear when $E = \Sigma_s^\infty \mathcal{X}$ a pointed space, since $\Sigma_t E = \Sigma_s^\infty \mathbb{G}_m \wedge \mathcal{X}$ is still a suspension spectrum, and so -1 connected.

Now consider a general -1 connected spectrum E. By [Mor05, Lemma 3.3.4], E is weak equivalent to $\operatorname{hocolim} E^i$ where $E^0 = *$, and for each n, there is a family $X_\alpha \in \mathrm{Sm}/k$ and natural numbers $n_\alpha \geq 0$ for which

$$\vee_\alpha \Sigma_s^\infty X_{\alpha,+}[n_\alpha - 1] \to E^{n-1} \to E^n$$

is an exact triangle. An induction argument establishes that $\Sigma_t E^n$ is still -1 connected for all n; hence $\Sigma_t E = \operatorname{hocolim} \Sigma_t E^n$ is also -1 connected. Should $\Sigma_t E$ fail to be $\mathbb{A}^1$-local, we may simply apply L^∞ to get an $\mathbb{A}^1$-local representative of $\Sigma_t E$. By the connectivity theorem, $L^\infty \Sigma_t E$ will again be -1 connected. $\square$

3.2. Contraction in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$, category of pointed sheaves of sets.

Definition 3.5. Let G be sheaf of pointed sets on Sm/k. The contraction of G is the sheaf $G_{-1} = G_{con}$ given by the formula

$$U \in \mathrm{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map ev_1 is the map induced by $ev_1 : \mathrm{Spec}(k) \to \mathbb{G}_m$, i.e., $k[x, x^{-1}] \to k$ given by $x \mapsto 1$.

Note that indeed G_{-1} is a sheaf since it is the kernel of the morphism of sheaves $G(-) \to G(- \times \mathbb{G}_m)$. The sheaf $G(- \times \mathbb{G}_m)$ may also be written as $\underline{\mathrm{Hom}}(\mathbb{G}_m, G)$ when we think of G as a space.

Proposition 3.6. If G is the trivial sheaf of abelian groups, then so is its contraction G_{-1}.

Proposition 3.7. Contraction is an exact functor on the category $\underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k, Nis)$.[1]

For any sheaf $G \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$ and any $X \in \mathrm{Sm}/k$, $G(\mathbb{G}_m \times X) = G_{-1}(X) \oplus G(X)$.

3.3. Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$.

Proposition 3.8. If G is a sheaf of Abelian groups, then $G_{-1} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $- \wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

Proof. For this category, $\underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)$ and G_{-1} both have sections at X given by $ker(ev_1 : G(X \times \mathbb{G}_m) \to G(X))$. See description of pointed internal hom for this. $\square$

Remark 8. If G is a sheaf of Abelian groups, we may consider G as a space by declaring $G_n = G$ for all n and giving identity maps for the structure maps. In particular, G is a pointed space at 0.

We can then realize the contraction as a $\mathbb{G}_m$ loop space $G_{-1}(X) = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)(X)$.

[1] I may be able to prove this after I establish the homotopy sheaves of $\Omega_t HM$.

Remark 9. Construction of canonical map $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ for an S^1 spectrum E.

First observe that for any $U \in \mathrm{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\mathrm{Spt}_s(k)(S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma^\infty \mathbb{G}_m, E) \times \mathrm{Spc}(k)(U, \mathbb{G}_m) \to \pi_0(E)(U)$$

given by sending (f, α) to the composition

$$\Sigma_s^\infty U_+ \xrightarrow{\mathrm{id} \wedge \Sigma_s^\infty \alpha} S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \to \pi_n(E).$$

Does this map descend to the smash? Yes, since if either map is a constant map, then so is the composition.

We thus get a map of sheaves of pointed sets

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \to \pi_n(E).$$

But by the adjunction $- \wedge \mathbb{G}_m \dashv \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, -)$ on $\mathrm{Spc}_\bullet(k)$ we have a morphism

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Why is it a map of sheaves of abelian groups?

Remark 10. If $E = HM$ is an Eilenberg-MacLane spectrum associated to a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups M, we show

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

Proposition 3.9. For $M \in \underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$, the spectrum $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

Proof. Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(k)$, we have $\Sigma^\infty \mathbb{P}^1[-1] = \Sigma^\infty \mathbb{G}_m$. Therefore

$$\begin{aligned} \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM[n]) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1[-1], HM[n]) \\ &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1, HM[n+1]) \\ &= \tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M). \end{aligned}$$

As the cohomological dimension of $\mathbb{P}^1$ is less than or equal to 1, we then have $\tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$. By $\tilde{H}^n_{Nis}(X; M)$ I mean the kernel of $\mathcal{SH}_s^{S^1}(k)(\Sigma^\infty X_+, HM[n]) \to \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty S^0, HM[n])$ induced by $S^0 \to X_+$, where this is obtained by choosing a point in $X(k)$. But what if $X(k)$ is empty?

$$\tilde{H}^n(X; M) \oplus H^n(\mathrm{Spec}(k); M) \cong H^n(X; M).$$

So as $M(\mathrm{Spec}\, k) \cong M(\mathbb{P}^1)$ (follows since M is strictly $\mathbb{A}^1$ invariant) $M(\mathbb{P}^1) = \mathit{pullback}(M(\mathbb{A}^1) \to M(\mathbb{G}_m) \to M(\mathbb{A}^1) = M(\mathrm{Spec}\, k))$, we see that the only possible value of n this will not vanish at is 1.

Since this vanishes at fields, a base change argument shows that indeed the sheaf $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weakly trivial when $n \neq 0$. So then it follows that the sheaf is indeed trivial by the argument giving weakly n-connected is equivalent to being n-connected.[2]

[2]Reword this to use the modified lemma with gabber presentation.

We now calculate at $Spec(k)$

$$\begin{aligned}\pi_0\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(Spec(k)) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty\mathbb{G}_m, HM)\\ &= \tilde{H}^0(\mathbb{G}_m; M)\\ &= \ker(\mathcal{SH}_s^{S^1}(k)(\Sigma^\infty\mathbb{G}_{m,+}, HM) \to \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty S^0, HM))\\ &= M_{-1}(\mathrm{Spec}\, k)\end{aligned}$$

We now know that the associated homotopy sheaves $\pi_n\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all n. So they are weak equivalent by [Mor05, Lemma 3.2.5]. □

Proposition 3.10. For any spectrum $E \in \mathcal{SH}^{S^1}(k)$, the homotopy sheaves of $\underline{\mathrm{Hom}}(\mathbb{G}_m, E)$ are calculated by $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \cong \pi_n(E)_{-1}$

Proof. Reduce to the case of Eilenberg-MacLane spectra by using the Postnikov tower. □

3.4. **Inverting $\mathbb{G}_m \wedge -$; $(\mathbb{G}_m, S^1)$ bi-spectra.** The functor Σ_t on $\mathrm{Spt}_s(k)$ is a left Quillen functor. We may therefore apply the general machinery of [H-Spt] to create a model category in which Σ_t is invertible. The construction of Hovey may be described as $(\mathbb{G}_m, S^1)$ bispectra.

Definition 3.11. A $(\mathbb{G}_m, S^1)$ bi-spectrum of spaces over k consists of a bigraded family of spaces $E_{i,j}$, $i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \to E_{i,j+1}$ and $\mu_{i,j} : \mathbb{G}_m \wedge E_{i,j} \to E_{i+1,j}$ for which the following diagram commutes .

$$\begin{tikzcd} S^1 \wedge \mathbb{G}_m \wedge E_{i,j} \arrow[r, "S^1\wedge\tau_{i,j}"] \arrow[d] & S^1 \wedge E_{i+1,j} \arrow[dd, "\sigma_{i+1,j}"] \\ \mathbb{G}_m \wedge S^1 \wedge E_{i,j} \arrow[d, "\mathbb{G}_m\wedge\sigma_{i,j}"'] & \\ \mathbb{G}_m \wedge E_{i,j+1} \arrow[r, "\mu_{i,j+1}"] & E_{i+1,j+1} \end{tikzcd}$$

Remark 11. Note that a $(\mathbb{G}_m, S^1)$ bispectrum is just a $\mathbb{G}_m$-spectrum of S^1-spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a $(\mathbb{G}_m, S^1)$ bi-spectrum $E_{i,j}$ as a sequence of S^1 spectra $E_{i,*}$.

Definition 3.12. Let E be a $(\mathbb{G}_m, S^1)$ bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \mathrm{Sm}/k \mapsto \mathrm{colim}_r\, \mathcal{H}_\bullet(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$.
We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

Proposition 3.13. If is E a $(\mathbb{G}_m, S^1)$ bispectrum, the presheaf of homotopy groups may also be calculated as

$$\tilde{\pi}_{n+m\alpha}E(U) = \mathrm{colim}_{s,r}\, \mathcal{H}_\bullet(k)(\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[DLØRV, p 217]

Definition 3.14. A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable weak equivalence if the following induced map is an isomorphism for all $U \in \mathrm{Sm}/k$.

$$f_* : \tilde{\pi}_{n+m\alpha}(E)(U) \to \tilde{\pi}_{n+m\alpha}(F)(U)$$

Definition 3.15. A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable cofibration if $f_0 : E_{0,*} \to F_{0,*}$ is a cofibration of S^1 spectra and the map $P.O. \to F_{n+1}$ is a cofibration in the following diagram.

$$\begin{tikzcd}
\mathbb{G}_m \wedge E_{n,*} \arrow[r] \arrow[d] & E_{n+1,*} \arrow[d] \arrow[ddr, bend left, "f_{n+1}"] & \\
\mathbb{G}_m \wedge F_{n,*} \arrow[r] \arrow[drr, bend right] & P.O. \arrow[dr, dotted] & \\
 & & F_{n+1,*}
\end{tikzcd}$$

Proposition 3.16. The category $\mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$ of $(\mathbb{G}_m, S^1)$ bispectra with $\mathbb{A}^1$ stable weak equivalences and $\mathbb{A}^1$ stable cofibrations is a model category. Denote the associated homotopy category of $\mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$ by $\mathcal{SH}(k)$.

Proposition 3.17. The fibrant bi-spectra are the Ω_t-spectra. [H-Spt, Theorem 3.4]

Proposition 3.18. There is a left Quillen functor $\Sigma^\infty_t : \mathrm{Spt}^{\mathbb{A}^1}_s(k) \to \mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$ given by $(\Sigma^\infty_t E)_{i,j} = \mathbb{G}^i_m \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}^i_m \wedge E_j \longrightarrow \mathbb{G}^i_m \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}^m \wedge \mathbb{G}^i_m \wedge E_j \to \mathbb{G}^{i+1}_m E_j.$$

The right adjoint to Σ^∞_t is denoted by Ω^∞_t and is given by $\Omega^\infty_t(E) = E_{0,*}$.

The right derived functor $R\Omega^\infty_t(E)$ is given by the formula

$$R\Omega^\infty_t(E) = \mathrm{colim}_i\, \Omega^i_t E_{i,*}.$$

Proof. □

3.5. **Connectivity of** $(\mathbb{G}_m, S^1)$ **bispectra.**

Definition 3.19. A $(\mathbb{G}_m, S^1)$ bispectrum E is said to be n-connected if for all $k \leq n$ and all $m \in \mathbb{Z}$, the homotopy sheaves $\pi_{k+m\alpha}E$ vanish.

Proposition 3.20. Let $E \in \mathrm{Spt}^{\mathbb{A}^1}_s(k)$. Consider $\Sigma^\infty_t E \in \mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$. If E is -1 connected, then so too is $\Sigma^\infty_t E$.

Proof. We calculate

$$\begin{aligned}
\pi_{n+m\alpha}(\Sigma^\infty_t E) &= \pi_n(R\Omega^\infty_t \Omega^m_t \Sigma^\infty_t E) \\
&= \pi_n(\mathrm{colim}_i\, \Omega^{m+i}_t \Sigma^i_t E) \\
&= \mathrm{colim}_i\, \pi_n(\Sigma^i_t E)_{-(m+i)} \\
&= 0.
\end{aligned}$$

This follows since $\Sigma_t E$ is -1 connected whenever E is -1 connected, and the effect of Ω_t^{m+i} on homotopy sheaves is contraction. □

3.6. *t*-structure on $\mathcal{SH}(k)$.

Definition 3.21. Let $\mathcal{SH}(k)_{\geq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha}E = 0$ whenever $n < 0$.

Let $\mathcal{SH}(k)_{\leq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha}E = 0$ whenever $n > 0$.

Definition 3.22. For a $(\mathbb{G}_m, S^1)$ bispectrum E, let $E_{\leq 0}$ denote the spectrum with $(E_{\leq 0})_n = (E_n)_{\leq 0}$. The bonding maps are given by

$$\mathbb{G}_m \wedge P^j(E_{i,j}) \cong P^j(\mathbb{G}_m \wedge E_{i,j}) \to P^j(E_{i+1,j}).$$

The equivalence $\mathbb{G}_m \wedge P^j(\mathcal{X}) \cong P^j(\mathbb{G}_m \wedge \mathcal{X})$ follows by checking on stalks, and the fact that any stalk of $\mathbb{G}_m$ is just a disjoint union of points. 0

Theorem 3.23. The triple $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ defines a t-structure.

Proof. What needs to be done:

(1) Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. We must show $\mathcal{SH}(k)(E, F[-1]) = 0$. When E is in the image of Σ_t^∞, the result follows by using the adjuction $\Sigma_t^\infty \dashv R\Omega_t^\infty$ and using the t-structure on S^1 spectra. In particular, for $U \in \mathrm{Sm}/k$ we have $\mathcal{SH}(k)(S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty U_+, F[-1]) = 0$ for $n \geq 0$ and $m \in \mathbb{Z}$.

For a general $E \in \mathcal{SH}(k)_{\geq 0}$, we may write $E = \operatorname{hocolim} E^i$ where the E^i are built up as in [Mor05, 3.3.4], but we allow smashing with $\mathbb{G}_m$. Precisely, we take $E^0 = pt$, and each E^i is obtained from E^{i-1} as the cone of a map

$$\bigvee_\alpha S^{n_\alpha} \wedge \mathbb{G}_m^{m_\alpha} \wedge \Sigma^\infty X_{\alpha,+} \to E^{i-1}$$

for some family of $X_\alpha \in \mathrm{Sm}/k$ and indices $n_\alpha \geq 0$, $m_\alpha \in \mathbb{Z}$.

A standard 5-lemma argument using the long exact sequence obtained by applying $\mathcal{SH}(k)(-, F[-1])$ to the triangle

$$\vee S^{n_\alpha} \wedge \mathbb{G}_m^{m_\alpha} \wedge \Sigma^\infty X_{\alpha,+} \to E^{i-1} \to E^i$$

shows that for all $i \in \mathbb{N}$, $\mathcal{SH}(k)(E^i, F[-1]) = 0$. Furthermore, these long exact sequences show that for all $i \geq 1$, $\mathcal{SH}(k)(E^i, F[-2]) \to \mathcal{SH}(k)(E^{i-1}, F[-2])$ is surjective. Hence $\varprojlim^1 \mathcal{SH}(k)(E^i, F[-2]) = 0$, and so

$$\begin{aligned}\mathcal{SH}(k)(E, F[-1]) &= \mathcal{SH}(k)(\operatorname{colim} E^i, F[-1]) \\ &= \varprojlim \mathcal{SH}(k)(E^i, F[-1]) \\ &= 0.\end{aligned}$$

(2) Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. Show $E[1] \in \mathcal{SH}(k)_{\geq 0}$ and $F[-1] \in \mathcal{SH}(k)_{\leq 0}$. This is clear by invertibility of [1] in $\mathcal{SH}(k)$.

(3) Given $E \in \mathcal{SH}(k)$, construct $E_{\geq 0}$ and $E_{<0}$ which fit into an exact triangle

$$E_{\geq 0} \to E \to E_{\leq -1} \to E_{\geq 0}[1].$$

The functor $(-)_{\leq 0}$ has already been defined. For $k \in \mathbb{Z}$, let $(-)_{\leq k}$ is a functor on $\mathrm{Spt}_s(k)$ and we may extend it to a functor on $\mathcal{SH}(k)$ in the same way as for the case $k = 0$. Define $E_{\geq 0}$ to be the homotopy fiber of the canonical map $E \to E_{\leq -1}$. The long exact sequence of homotopy groups shows that $(-)_{\geq 0}$ has the correct homotopy groups. The uniqueness of the triangle follows by properties of triangulated categories.

$\square$

3.7. The heart of the t-structure on $\mathcal{SH}(k)$.

Definition 3.24. A homotopy module over k is a pair (M_*, μ_*) consisting of a $\mathbb{Z}$ graded strictly $\mathbb{A}^1$ invariant sheaf M_* and an isomorphism $\mu_n : M_n \cong (M_{n+1})_{-1}$.

Lemma 3.25. If E is a bi-spectrum, then

$$R\Omega_t^\infty E \to \underline{\mathrm{Hom}}(\mathbb{G}_m, R\Omega_t^\infty(E \wedge \mathbb{G}_m))$$

is an isomorphism.

Proof. $\square$

Lemma 3.26. Let $E \in \mathcal{SH}(k)$. For a fixed $n \in \mathbb{Z}$, the collection $\pi_n(E)_m$ forms a homotopy module.

Lemma 3.27. If (M_*, μ_*) is a homotopy module over k, then there is a $(\mathbb{G}_m, S^1)$ bispectrum HM_* with $(HM_*)_{n,n} = K(M_n, n)$ with evident structure maps.

Proof. $\square$

Theorem 3.28. The heart of the t-structure $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is denoted $\pi_*^{\mathbb{A}^1}(k)$ and is equivalent to the category of homotopy modules. The equivalence is given explicitly by the functors $\pi_0(-)_*$ and $H(-)$.

Proof. $\square$

References

[A-1974] Adams, J.F., *Stable Homotopy and Generalized Homology.* Chicago Lectures in Mathematics, (1974).

[B] Blander, Benjamin, *Local Projective Model Structures on Simplicial Presheaves.* K-Theory, **24** (2001) 283–301.

[DHI] Dugger, Dan; Hollander, Sharon; Isaksen, Dan, *Hypercovers and simplicial presheaves.*

[DLØRV] Dundas, B.; Levine, M.; Østvær, P.; Röndigs, O.; Voevodsky, V., *Motivic Homotopy Theory.* Springer (2000).

[GJ91] Goerss, Paul; Jardine, John, *Simplicial Homotopy Theory.* (1991).

[Hir] Phillip, Hirschhorn, *Model Categories and Their Localization.* AMS (2003).

[H-Mod] Hovey, Mark, *Model Categories.* online preprint (1991).

[H-Spt] Hovey, Mark, *Spectra and symmetric spectra in general model categories.* journal? (2001).

[J] Jardine, J.F., *Simplicial presheaves.* Journal of Pure and Applied Algebra, **47** (1987) 35–87.

[Mor03] Morel, Fabien, *An introduction to $\mathbb{A}^1$ homotopy theory.*

[Mor04] Morel, Fabien, *On the motivic π_0 of the sphere spectrum.* NATO science series.

[Mor05] Morel, Fabien, *The stable $\mathbb{A}^1$ connectivity theorems.* preprint (2004).

[MV99] Morel, Fabien; Voevodsky, Vladimir, *$\mathbb{A}^1$-homotopy theory of schemes.* IHES, tome 90 (1999), p. 45–143.

[Pel08] Pelaez, Pablo, *Multiplicative Properties of the Slice Filtration.* December 2008.

[Voev98] Voevodsky, Vladimir. *$\mathbb{A}^1$-Homotopy Theory.* Doc. Math. J., (1998) pp. 579–604.