1. Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

Our goal is to prove theorem 4.14 of [Voev98], which we restate in terms of $\mathbb{P}^1$-spectra.

Theorem 1.1. Let (X,x) be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let $\mathcal{Y}$ be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

We will prove this theorem by following [Mor03] and [Mor05] by Fabien Morel.

Remark 1. To prove this theorem, Morel carefully analyzes how to pass from spaces in the projective model structure to the $\mathbb{A}^1$ stable homotopy category of $\mathbb{P}^1$ spectra. From the projective model structure on spaces, we construct a model of the left Bousfield localization of spaces at the class of maps $\{U_+ \wedge \mathbb{A}^1 \to U \,|\, U \,\mathrm{im}\, \mathrm{Sm}/k\}$. To get to $\mathbb{P}^1$ spectra, we first invert $S^1 \wedge -$ to get a category of S^1 spectra, and then we invert $\mathbb{G}_m \wedge -$ to get a category of $(\mathbb{G}_m, S^1)$ bispectra.

$$\mathcal{H}_{s,\bullet}(k) \to \mathcal{H}_\bullet(k) \to \mathcal{SH}^{S^1}(k) \to \mathcal{SH}(k)$$

The machinery that we set up to prove this theorem will also allow us to establish a t-structure on $\mathcal{SH}(k)$, and identify its heart.

Remark 2. A construction of Ayoub [Ayo08] shows that theorem 1.1 statement is false over general Noetherian base schemes S. The argument below works for infinite fields, however.

2. Assumptions from previous lectures

We briefly recall some of the basic constructions which appear in [Mor03] and [Mor05].

2.1. Facts about Nisnevich topology.

The proof of Voevodsky's connectivity theorem will follow from the following property of Nisnevich sheaf cohomology by a sequence of reductions.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k, and let $X \in \mathrm{Sm}/k$ with Krull dimension d. Then whenever $n > d$, $H^n_{Nis}(X; M) = 0$.

2.2. Unstable model category $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$.

Definition 2.2. Let k be a field, and let Sm/k denote the category of smooth schemes of finite type over k. The category of Morel-Voevodsky spaces over k is the category of simplicial Nisnevich sheaves on Sm/k. We write $\mathrm{Spc}(k) = \Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$ for this category.

The category $\mathrm{Spc}(k)$ may be equipped with several different model category structures. We will work with the injective local model category structure on $\mathrm{Spc}(k)$, which we now define.

Definition 2.3. A map $\mathcal{X} \to \mathcal{Y}$ is an injective weak equivalence if and only if for any $U \in \mathrm{Sm}/k$, the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a weak equivalence of simplicial sets.

A map $\mathcal{X} \to \mathcal{Y}$ is an injective cofibration if and only if for any $U \in \mathrm{Sm}/k$, the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a cofibration of simplicial sets, i.e., a monomorphism.

A map $\mathcal{X} \to \mathcal{Y}$ is an injective fibration if and only if it satisfies the left lifting property with respect to any trivial injective cofibration. That is, for any commutative square below with $\mathcal{A} \to \mathcal{B}$ a trivial cofibration, a lift $\mathcal{B} \to \mathcal{X}$ exists.

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, tail, "\sim"'] & \mathcal{X} \arrow[d] \\ \mathcal{B} \arrow[r] \arrow[ur, dotted] & \mathcal{Y} \end{tikzcd}$$

Denote the homotopy category associated to the injective model category structure on $\mathrm{Spc}(k)$ by $\mathcal{H}_s(k)$. The "s" stands for simplicial.

Definition 2.4. The category of pointed space $\mathrm{Spc}_\bullet(k)$ inherits a model category structure from $\mathrm{Spc}(k)$. The functor $-_+ : \mathrm{Spc}(k) \to \mathrm{Spc}_\bullet(k)$ defined by adding a disjoint basepoint to a given space is a left Quillen functor. The right adjoint is the forgetful functor.

Proposition 2.5. Every object of $\mathrm{Spc}(k)$ and $\mathrm{Spc}_\bullet(k)$ is cofibrant in the injective model category structure.

Definition 2.6. For $X \in \mathrm{Sm}/k$, let rX denote the sheaf associated to the presheaf $U \mapsto \mathrm{Sm}/k(U, X)$. This defines a functor $r : \mathrm{Sm}/k \to \mathrm{Spc}(k)$.

For K a simplicial set, the constant space $cK \in \mathrm{Spc}(k)$ is the sheaf associated to the constant presheaf with value K. The functor $c : \underline{\mathrm{sSet}} \to \mathrm{Spc}(k)$ is a left Quillen functor with right adjoint given by taking sections at $\mathrm{Spec}\, k$.

Proposition 2.7. $\mathrm{Spc}(k)$ is a simplicial model category.

See [Pel08, Chapter 2] for a detailed treatment of the products and internal hom constructions in $\mathrm{Spc}(k)$. We recount those definitions which are essential to our argument.

Definition 2.8. For spaces $\mathcal{X}$ and $\mathcal{Y}$, the product $\mathcal{X} \times \mathcal{Y}$ in $\mathrm{Spc}(k)$ is given by $U \mapsto \mathcal{X}(U) \times \mathcal{Y}(U)$. For spaces $\mathcal{X}$ and $\mathcal{Y}$, the internal hom $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ in $\mathrm{Spc}(k)$ is given by the formula

$$(U, m) \in \mathrm{Sm}/k \times \Delta \mapsto \mathrm{Hom}_{\Delta^{op}\mathrm{Shv}}(X \times rU \times c\Delta^n, Y).$$

Proposition 2.9. The product and internal hom defined above give $\mathrm{Spc}(k)$ the structure of a closed monoidal model category. See [H-Mod, Chapter 4] or [Pel08, §1.7] for the definition.

Proof. The adjunction between $\mathcal{X} \times -$ and $\underline{Hom}(\mathcal{X}, -)$ is given by the following map.

$$\eta : \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z})) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$$

For $g \in \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$, we define $\eta(g)$ by

$$\begin{aligned} \mathcal{X}_n(U) \times \mathcal{Y}_n(U) &\xrightarrow{\eta(g)(U,n)} \mathcal{Z}_n(U) \\ (a,b) &\longmapsto g(U,n)(b)(U,n)(a, \mathrm{id}_U, \mathrm{id}_{\Delta^n}). \end{aligned}$$

For $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, the map $(\eta^{-1} f)(U,n) : \mathcal{Y}_n(U) \to \underline{Hom}(\mathcal{X}, \mathcal{Z})_n(U)$ is given by sending $y \in \mathcal{Y}_n(U)$ to the map

$$\begin{aligned} \mathcal{X}_m(V) \times \mathrm{Sm}/k(V,U) \times \Delta^n_m &\xrightarrow{(\eta^{-1} f)(U,n)(y)(V,m)} \mathcal{Z}_m(V) \\ (x, \phi, \alpha) &\longmapsto f(V,m)(x, \mathcal{Y}(\phi)(y \circ \alpha)) \end{aligned}$$

where we identify y with a map $y : \Delta^n \to \mathcal{Y}(V)$, and $\alpha : \Delta^m \to \Delta^n$. $\square$

Definition 2.10. Let $\mathcal{X}$ and $\mathcal{Y}$ be spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{Hom}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y}$, where at $(U,n) \in (\mathrm{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times rU \times c\Delta^n \to \mathcal{Y}$ to $g(U,n)(x, \mathrm{id}, \mathrm{id}) \in \mathcal{Y}_n(U)$.

For pointed spaces $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$, the pointed internal hom $\underline{Hom}_\bullet(\mathcal{X}, \mathcal{Y})$ is the fiber of ev_x over y, i.e., $ev_x^{-1}(y)$.

Definition 2.11. Let $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$ be pointed spaces. The wedge of $\mathcal{X}$ and $\mathcal{Y}$, denoted by $\mathcal{X} \vee \mathcal{Y}$, is the pushout of the following diagram.

$$\begin{tikzcd} pt \arrow[r, "x"] \arrow[d, "y"] & \mathcal{X} \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \vee \mathcal{Y} \end{tikzcd}$$

The smash product $\mathcal{X} \wedge \mathcal{Y}$ is the space given by the pushout of the following diagram, with basepoint $\mathcal{X} \vee \mathcal{Y}$.

$$\begin{tikzcd} \mathcal{X} \vee \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{X} \times \mathcal{Y} \arrow[d] \\ pt \arrow[r] & \mathcal{X} \wedge \mathcal{Y} \end{tikzcd}$$

Proposition 2.12. The category of pointed spaces $\mathrm{Spc}_\bullet(k)$ is also a closed monoidal category with product $\wedge$ and internal hom $\underline{\mathrm{Hom}}_\bullet$.

2.3. $\mathbb{A}^1$ **localization.**

Definition 2.13. A space $\mathcal{X}$ is called $\mathbb{A}^1$ local if for any smooth scheme U, the canonical map

$$\mathrm{Hom}(rU, \mathcal{X}) \to \mathrm{Hom}(rU \times \mathbb{A}^1, \mathcal{X})$$

is a bijection.

Definition 2.14. A map $f : \mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$ weak equivalence if

$$\mathrm{Hom}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}(\mathcal{X}, \mathcal{Z})$$

is a bijection for every $\mathbb{A}^1$ local space $\mathcal{Z}$.

The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \to U \,|\, U \in \mathrm{Sm}/k\}$. We deonte the category of spaces with the model structure obtained by left Bousfield localization by $L_W\mathrm{Spc}(k)$ and its homotopy category by $\mathcal{H}(k)$. See [Hir, Chapter 3] for the general theory of Bousfield localization. One thing we obtain is a localization functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \to \mathcal{H}(k)$ which is a left Quillen functor. In particular, $L_{\mathbb{A}^1}$ sends sends $\mathbb{A}^1$ weak equivalences to isomorphisms.

The model category $L_W\mathrm{Spc}(k)$ is constructed as follows. The underlying category of $L_W\mathrm{Spc}(k)$ is $\mathrm{Spc}(k)$, but the weak equivalences are the $\mathbb{A}^1$-local weak equivalences. The cofibrations are the cofibrations in the injective model structure on $\mathrm{Spc}(k)$. The fibrations are what they need to be, i.e., those maps which satisfy the left lifting property with respect to trivial cofibrations.

In order to effectively work with the $\mathcal{H}(k)$, we require a means of constructing fibrant replacements in $L_W\mathrm{Spc}(k)$. Morel accomplishes this by constructing another model category with homotopy category $\mathcal{H}(k)$.

Definition 2.15. Let $\mathrm{Spc}^{\mathbb{A}^1}(k)$ denote the full subcategory of $\mathrm{Spc}(k)$ of $\mathbb{A}^1$ local spaces. The inclusion $\mathrm{Spc}^{\mathbb{A}^1}(k) \to \mathrm{Spc}(k)$ is a left Quillen functor with left adjoint L^∞. See [Mor05, Proposition 3.2.3] for the construction.

Proposition 2.16. The homotopy category of $\mathrm{Spc}^{\mathbb{A}^1}(k)$ is equivalent to $\mathcal{H}(k)$.

Definition 2.17. Let $\mathcal{X}$ be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on Sm/k associated to $U \to \pi_0(\mathcal{X}(U))$. A space $\mathcal{X}$ is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let $(\mathcal{X}, x)$ be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on Sm/k given by

$$U \to \pi_n(\mathcal{X}(U)).$$

A pointed space $\mathcal{X}$ is called n-connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

Proposition 2.18. Let $\mathcal{X}$ be a 0-connected simplicial sheaf. Then $L^\infty\mathcal{X}$ is also 0-connected.

For a sheaf of abelian groups M on Sm/k and a natural number n, a Dold-Kan construction gives a simplifical presheaf $K(M, n)$. It is called the Eilenberg-MacLane spectrum of type (M, n) and has homotopy sheaves as expected.

$$\pi_m(K(M,n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}$$

Proposition 2.19. For $X \in \mathrm{Sm}/k$ and M a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M,n)) \cong H^n_{Nis}(X; M).$$

It therefore follows that

$$\mathcal{H}_\bullet(k)(rX_+, K(M,n)) \cong H^n_{Nis}(X; M).$$

Notation 1. For a pointed space $\mathcal{X}$, Let $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ denote the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^\infty \mathcal{X})$. The sheaf $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is also the sheafification of the presheaf given by

$$U \in \mathrm{Sm}/k \mapsto \mathcal{H}_\bullet(k)(S^n \wedge U_+, \mathcal{X}).$$

2.4. S^1 **spectra.** The functor $\Sigma_s : \mathrm{Spc}_\bullet(k) \to \mathrm{Spc}_\bullet(k)$ given by $\Sigma_s \mathcal{X} = S^1 \wedge \mathcal{X}$ is a left Quillen functor on $\mathrm{Spc}_\bullet(k)$, with right adjoint Ω_s, where $\Omega_s \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(S^1, \mathcal{X})$. This follows since S^1 is a cofibrant object of $\mathrm{Spc}_\bullet(k)$. Note, however, that the derived functor Σ_s is not an equivalence of homotopy categories. We may invert this functor, i.e., make a new category where Σ_s is an equivalence of homotopy categories, by creating a category of S^1 spectra by using the general machinery developed in [H-Spt]. Here the "s" in Σ_s and Ω_s stands for "simplicial circle".

Definition 2.20. Let $\mathrm{Spt}^{S^1}(k)$ denote the category of S^1 spectra of spaces over k. An object $E \in \mathrm{Spt}^{S^1}(k)$ is a sequence of pointed spaces $E_i \in \mathrm{Spc}_\bullet(k)$ equipped with bonding maps $\sigma_i : S^1 \wedge E_i \to E_{i+1}$. A map of spectra $f : E \to F$ consists of a sequence of maps of spaces $f_i : E_i \to F_i$ which are compatible with the bonding maps.

We first endow this category with the projective model structure (or level-wise model structure), i.e., a map $f : E \to F$ is a weak equivalence if for any n the map $f_n : E_n \to F_n$ is a w.e.; a map $f : E \to F$ is a fibration if for all n the map $f_n : E_n \to F_n$ is a fibration. The cofibrations are those maps satisfying the right lifting property with respect to trivial fibrations.

The projective cofibrations have the following characterization [H-Spt, Proposition 1.15]. A map $f : E \to F$ is a projective cofibration if and only if $f_0 : E_0 \to F_0$ is a cofibration and for any $n \geq 1$, the dotted arrow in the diagram below is a cofibration. Here $P.O.$ denotes the push-out of the diagram.

$$\begin{tikzcd}
S^1 \wedge E_{n-1} \arrow[r, "\sigma_{n-1}"] \arrow[d, "S^1 \wedge f_{n-1}"'] & E_n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\
S^1 \wedge F_{n-1} \arrow[r] \arrow[drr, bend right, "\sigma_{n-1}"] & P.O. \arrow[dr, dotted] & \\
 & & F_n
\end{tikzcd}$$

This model structure does not actually invert Σ_s. To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.21. A map $f : E \to F$ of S^1 spectra is a stable equivalence if for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \to \pi_n(F)$ is an isomorphism.

The stable model category structure on $\mathrm{Spt}^{S^1}(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such. Consult [H-Spt] for more details.

Denote the homotopy category of $\mathrm{Spt}^{S^1}(k)$ by $\mathcal{SH}_s^{S^1}(k)$.

Definition 2.22. Consider the class of maps $W = \{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \to \Sigma^\infty U_+ \,|\, U \in \mathrm{Sm}/k\}$ in $\mathrm{Spt}^{S^1}(k)$. The left Bousfield localization of $\mathrm{Spt}^{S^1}(k)$ with respect to W exists, and we write $L_W\mathrm{Spt}^{S^1}(k)$ for the resulting model category. Denote the homotopy category associated to $L_W\mathrm{Spt}^{S^1}(k)$ by $\mathcal{SH}^{S^1}(k)$.

Remark 3. Let $\mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$ denote the full subcategory of $\mathrm{Spt}^{S^1}(k)$ consisting of $\mathbb{A}^1$-local spectra. The inclusion $\mathrm{Spt}^{S^1,\mathbb{A}^1}(k) \to \mathrm{Spt}^{S^1}(k)$ has a left adjoint L^∞ which has a similar construction as was considered for spaces. This adjunction is indeed a Quillen functor, and so just as in the unstable setting, we may identify the stable $\mathbb{A}^1$ homotopy category with the homotopy category of the $\mathbb{A}^1$-local S^1 spectra. See [Mor05, §4.2] for details.

For S^1 spectra E and F, we calculate the stable $\mathbb{A}^1$ homotopy group$\mathcal{SH}^{S^1}(k)(E,F)$ by

$$\begin{aligned}\mathcal{SH}^{S^1}(k)(E,F) &= \mathcal{SH}^{S^1}(k)(L^\infty E, L^\infty F)\\ &= \mathcal{SH}_s^{S^1}(k)(E, L^\infty F)\end{aligned}$$

Here we consider the model for $\mathcal{SH}^{S^1}(k)$ given by Bousfield localization, then translate to the category of $\mathbb{A}^1$ local spectra using L^∞. The second equality follows from the Quillen adjunction $\mathrm{Spt}^{S^1,\mathbb{A}^1}(k) \hookrightarrow \mathrm{Spt}^{S^1}(k)$.

If we assume E is cofibrant and F is fibrant, we get the formula

$$\mathcal{SH}^{S^1}(k)(E,F) = \mathrm{Spt}^{S^1}(k)(E, L^\infty F).$$

Definition 2.23. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$. That is,

$$\pi_n(E) = \mathrm{colim}_r\, \pi_{n+r}(E_r).$$

In particular, for a $U \in \mathrm{Sm}/k$, we have

$$\pi_n(E)(U) = \mathrm{colim}_r\, \pi_{n+r}(E_r)(U).$$

Definition 2.24. An S^1 spectrum E is said to be n-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.25. There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \to \mathrm{Spt}^{S^1}(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ where the bonding maps come from associativity of smash product. The right adjoint to this functor is given by "evaluation at 0", i.e., $\Omega^\infty(E) = E_0$.

Remark 4. The right derived functor $R\Omega^\infty : \mathcal{SH}_s^{S^1}(k) \to \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i\, \Omega_s^i E_i.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

Remark 5. The left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k) \to \mathrm{Spt}^{S^1}(k)$ factors through the category of $\mathbb{A}^1$-local spaces. This follows by [Mor05, Remark 4.1.3]. Furthermore, since a map $f \in \mathrm{Spt}^{S^1}(k)(\Sigma^\infty \mathcal{X}, E)$ is determined by $f_0 : \mathcal{X} \to E_0$ by the adjunction $\Sigma_s^\infty \dashv \Omega_s^\infty$, one can show that $\Sigma_s^\infty \mathcal{X} \to \Sigma^\infty L^\infty \mathcal{X}$ is an $\mathbb{A}^1$-weak equivalence. So in the case of suspension spectra, we may use $\Sigma^\infty L^\infty \mathcal{X}$ as an $\mathbb{A}^1$ localization. In particular, for any $n \in \mathbb{Z}$ the sheaf $\pi_n^{\mathbb{A}^1} \Sigma_s^\infty \mathcal{X}$ is isomorphic to $\pi_n \Sigma^\infty L^\infty \mathcal{X}$.

Remark 6. The stable homotopy category is symmetric monoidal, with smash product $\wedge$ and internal hom $\underline{\mathrm{Hom}}$. Using symmetric spectra, one can give these constructions on the category of spectra [HSS]. The S^1 stable homotopy category is a triangulated category. The shift is given by S^1 suspension, and distinguished triangles are those triangles isomorphic to the cone of a map

$$X \xrightarrow{f} Y \to C(f) \to X[1].$$

Proposition 2.26. Let $U \in \mathrm{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then there is a canonical isomoprhism

$$H^n_{Nis}(U; M) \to \mathcal{SH}^{S^1}(\Sigma^\infty U_+, HM[n]).$$

This is [Mor05, Lemma 3.2.3].

2.5. **Weak connectedness.**

Proposition 2.27. Let k be an infinite field, and consider $\mathcal{X}$ be a pointed space. If for any finitely generated field F over k, $\pi_0(\mathcal{X})(F) = 0$, then the sheaf $\pi_0(\mathcal{X})$ is trivial.

Proof. The proof follows along the lines of [Mor05, Lemma 6.1.3]. □

Remark 7. The analgous statement for S^1-spectra also holds.

2.6. t-**structures.**

Definition 2.28. Let $\mathfrak{C}$ be a triangulated category. A t-structure on $\mathfrak{C}$ is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

(1) For any $X \in \mathfrak{C}_{\geq 0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
(2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
(3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \to X \to Z \to Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$..

The heart of a t-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.29. Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n > 0$.

Theorem 2.30. The pair $(\mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t-structure on $\mathcal{SH}_s^{S^1}(k)$.

Remark 8. For a space $\mathcal{X}$, there is a Postnikov tower associated to it

$$\cdots P^n(\mathcal{X}) \to P^{n-1}(\mathcal{X}) \to \cdots \to P^0(\mathcal{X}) \to P^{-1}(\mathcal{X})$$

constructed in [MV99, p. 57]. The main construction needed is the Moore-Postnikov tower of a simplicial set [GJ91, VI.3.4]. For a simplicial set K and $n \in \mathbb{N}$, define $K^{(n)} = \mathrm{im}(K \to \mathrm{cosk}_n K)$. This is a convenient way to define the Moore construction.

For a space $\mathcal{X}$, we then define $P^n\mathcal{X}$ to be the space given by sheafification of $U \mapsto \mathcal{X}(U)^{(n)}$.

Now for E an S^1 spectrum, let $E_{\leq 0}$ be the spectrum with $(E_{\leq 0})_n = P^n(E_n)$. The bonding maps come from the canonical map

$$S^1 \wedge P^n(E_n) \to P^{n+1}(S^1 \wedge E_n).$$

See [Mor05, Lemma 3.2.1] for more on this construction.

2.7. **Connectivity results.**

Proposition 2.31. [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}^{S^1}(k) \to \mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$ identifies the $\mathbb{A}^1$-localized S^1 stable homotopy category with the homotopy category of $\mathbb{A}^1$-local S^1 spectra.

Theorem 2.32. Let k be an infinite field. Consider $E \in \mathcal{SH}^{S^1}(k)$ and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L^\infty E = 0$.

Theorem 2.33. The pair $(\mathcal{SH}^{S^1}_{\geq 0}(k), \mathcal{SH}^{S^1}_{\leq 0}(k))$ is a t-structure on the category $\mathcal{SH}^{S^1}(k)$.

Proof. This is just the restriction of the t-structure to the $\mathbb{A}^1$-local objects. $\square$

Definition 2.34. Let M be a sheaf of Abelian groups on Sm/k with respect to the Nisnevich topology. We say M is strictly $\mathbb{A}^1$ invariant if for all $n \geq 0$ and all $X \in \mathrm{Sm}/k$, the map $H^n_{Nis}(X; M) \to H^m_{Nis}(X \times \mathbb{A}^1; M)$ is an isomorphism. Let $\underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$ denote the full subcategory of sheaves of Abelian groups on Sm/k in the Nisnevich topology consisting of the strictly $\mathbb{A}^1$ invariant sheaves.

Definition 2.35. If $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$ is a sheaf of Abelian groups, the Eilenberg-MacLane spectrum associated to M is the S^1 spectrum HM given by $HM_n = K(M, n)$. The bonding maps come from the usual identification of $\Omega_s K(M, n) \cong K(M, n-1)$.

Proposition 2.36. HM is $\mathbb{A}^1$ local iff M is strictly $\mathbb{A}^1$ invariant.

Proposition 2.37. The heart of the homotopy t structure is equivalent to the category of strictly $\mathbb{A}^1$ invariant sheaves.

3. Inverting $\mathbb{G}_m \wedge -$; $\mathbb{P}^1$ spectra

3.1. $\mathbb{G}_m$ **suspension and loops.** We always consider $\mathbb{G}_m$ to be pointed at 1 unless otherwise specified.

Definition 3.1. On the category $\mathrm{Spt}^{S^1}(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps. Smashing with $\mathbb{G}_m$ is also a functor on the unstable category of pointed spaces, and we give it the same name Σ_t.

Definition 3.2. The functor Σ_t on $\mathrm{Spc}_\bullet(k)$ has a right adjoint denoted Ω_t. It is given by the formula $\Omega_t \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \mathcal{X})$.

The functor Σ_t on $\mathrm{Spt}^{S^1}(k)$ also has a right adjoint Ω_t given by the internal hom functor, i.e., $\Omega_t E = \underline{\mathrm{Hom}}(\Sigma^\infty \mathbb{G}_m, E)$.

Proposition 3.3. The functor Σ_t is a left Quillen functor on $\mathrm{Spt}^{S^1}(k)$ and on $\mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$. Furthermore, Σ_t is a triangulated functor on $\mathcal{SH}^{S^1}(k)$.

Lemma 3.4. Let $E \in \mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$ be a -1 connected spectrum. Then $\Sigma_t E$ is again -1 connected.

Proof. The claim is clear when $E = \Sigma_s^\infty \mathcal{X}$ a pointed space, since $\Sigma_t E = \Sigma_s^\infty \mathbb{G}_m \wedge \mathcal{X}$ is still a suspension spectrum, and so -1 connected.

Now consider a general -1 connected spectrum E. By [Mor05, Lemma 3.3.4], E is weak equivalent to $\mathrm{hocolim}\, E^i$ where $E^0 = *$, and for each n, there is a family $X_\alpha \in \mathrm{Sm}/k$ and natural numbers $n_\alpha \geq 0$ for which

$$\vee_\alpha \Sigma_s^\infty X_{\alpha,+}[n_\alpha - 1] \to E^{n-1} \to E^n$$

is an exact triangle. An induction argument establishes that $\Sigma_t E^n$ is still -1 connected for all n; hence $\Sigma_t E = \mathrm{hocolim}\, \Sigma_t E^n$ is also -1 connected. Should $\Sigma_t E$ fail to be $\mathbb{A}^1$-local, we may simply apply L^∞ to get an $\mathbb{A}^1$-local representative of $\Sigma_t E$. By the connectivity theorem, $L^\infty \Sigma_t E$ will again be -1 connected. □

3.2. **Contraction in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$, category of pointed sheaves of sets.**

Definition 3.5. Let G be a sheaf of pointed sets on Sm/k. The contraction of G is the sheaf G_{-1} given by the formula

$$U \in \mathrm{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map ev_1 is the map induced by $ev_1 : \mathrm{Spec}(k) \to \mathbb{G}_m$, i.e., $k[x, x^{-1}] \to k$ given by $x \mapsto 1$.

Note that indeed G_{-1} is a sheaf since it is the kernel of the morphism of sheaves $G(-) \to G(- \times \mathbb{G}_m)$. The sheaf $G(- \times \mathbb{G}_m)$ may also be written as $\underline{\mathrm{Hom}}(\mathbb{G}_m, G)$ when we think of G as a space.

Proposition 3.6. If G is the trivial sheaf of abelian groups, then so is its contraction G_{-1}.

Proposition 3.7. Contraction is an exact functor on the category $\underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k, Nis)$.

For any sheaf $G \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$ and any $X \in \mathrm{Sm}/k$, $G(\mathbb{G}_m \times X) = G_{-1}(X) \oplus G(X)$.

3.3. **Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$.**

Proposition 3.8. If G is a sheaf of Abelian groups, then $G_{-1} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $-\wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

Proof. For this category, $\underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)$ and G_{-1} both have sections at X given by $ker(ev_1 : G(X \times \mathbb{G}_m) \to G(X))$. See definition 2.10. □

Remark 9. If G is a sheaf of Abelian groups, we may consider G as a space by declaring $G_n = G$ for all n and giving identity maps for the structure maps. In particular, G is a pointed space at 0.

We can then realize the contraction as a $\mathbb{G}_m$ loop space $G_{-1}(X) = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)(X)$.

Remark 10. We now describe the construction of the canonical map $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ for an S^1 spectrum E.

First observe that for any $U \in \mathrm{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\mathrm{Spt}_s(k)(S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma^\infty \mathbb{G}_m, E) \times \mathrm{Spc}(k)(U, \mathbb{G}_m) \to \pi_0(E)(U)$$

given by sending (f, α) to the composition

$$\Sigma_s^\infty U_+ \xrightarrow{\mathrm{id} \wedge \Sigma_s^\infty \alpha} S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \to \pi_n(E).$$

This map descends to the smash product, so we have

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \to \pi_n(E).$$

But by the adjunction $-\wedge \mathbb{G}_m \dashv \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, -)$ on $\mathrm{Spc}_\bullet(k)$ we have a morphism

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Remark 11. If $E = HM$ is an Eilenberg-MacLane spectrum associated to a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups M, we show

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

Proposition 3.9. For $M \in \underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$, the spectrum $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

Proof. We evaluate $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ at fields F which are finitely generated over k. We consider the special case $F = k$, but the argument works in general.

Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(k)$, we have $\Sigma^\infty \mathbb{P}^1[-1] = \Sigma^\infty \mathbb{G}_m$. Therefore

$$\begin{aligned}
\pi_{-n}\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(\mathrm{Spec}\, k) &= \mathcal{SH}^{S^1}(k)(\Sigma^\infty S^0[-n], \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)) \\
&= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM[n]) \\
&= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1[-1], HM[n]) \\
&= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1, HM[n+1]) \\
&= \tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M).
\end{aligned}$$

As the cohomological dimension of $\mathbb{P}^1$ is less than or equal to 1, we then have $\tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$. Here $\tilde{H}^n_{Nis}(X; M)$ denotes the kernel of

$$\mathcal{SH}^{S^1}_s(k)(\Sigma^\infty X_+, HM[n]) \to \mathcal{SH}^{S^1}_s(k)(\Sigma^\infty S^0, HM[n])$$

induced by $S^0 \to X_+$, where this is obtained by choosing a point in $X(k)$. It follows that

$$\tilde{H}^n(X; M) \oplus H^n(\mathrm{Spec}(k); M) \cong H^n(X; M).$$

Since M is strictly $\mathbb{A}^1$ invariant, it follows that $M(\mathrm{Spec}\, k) \cong M(\mathbb{P}^1)$. Hence $\tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M)$ can be non-zero only for $n = 0$.

For $n \neq 0$, since $\mathcal{SH}^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM[n])$ vanishes at fields, a base change argument shows that indeed the sheaf $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weakly trivial when $n \neq 0$. So then it follows that the sheaf is indeed trivial by 2.27.

We now calculate at $Spec(k)$ for $n = 0$

$$\begin{aligned}
\pi_0 \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(Spec(k)) &= \mathcal{SH}^{S^1}_s(k)(\Sigma^\infty \mathbb{G}_m, HM) \\
&= \tilde{H}^0(\mathbb{G}_m; M) \\
&= \ker(\mathcal{SH}^{S^1}_s(k)(\Sigma^\infty \mathbb{G}_{m,+}, HM) \to \mathcal{SH}^{S^1}_s(k)(\Sigma^\infty S^0, HM)) \\
&= M_{-1}(\mathrm{Spec}\, k)
\end{aligned}$$

We now know that the associated homotopy sheaves $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all n. So they are weak equivalent by [Mor05, Lemma 3.2.5]. $\square$

Proposition 3.10. For any spectrum $E \in \mathcal{SH}^{S^1}(k)$, the homotopy sheaves of $\underline{\mathrm{Hom}}(\mathbb{G}_m, E)$ are calculated by $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \cong \pi_n(E)_{-1}$

Proof. Consider the Postnikov tower for E.

$$\begin{tikzcd}
 & E = \mathrm{hocolim}\, E_{\leq n} \arrow[d] \\
 & \vdots \arrow[d] \\
H\pi_{n+1}E \arrow[r] & E_{\leq n+1} \arrow[d] \\
H\pi_n E \arrow[r] & E_{\leq n} \arrow[d] \\
 & \vdots
\end{tikzcd}$$

Since $\underline{\mathrm{Hom}}(\mathbb{G}_m, -)$ is a triangulated functor, we get triangles

$$H(\pi_{n+1}E)_{-1} \to \underline{\mathrm{Hom}}(\mathbb{G}_m, E_{\leq n+1}) \to \underline{\mathrm{Hom}}(\mathbb{G}_m, E_{\leq n})$$

If there is some i for which $E = E_{\geq i}$, an easy induction argument establishes that $(\pi_n E)_{-1} \cong \pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, E)$. To pass to the general case, use $E = \mathrm{holim}\, E_{\geq i}$.

Hence the following tower is indeed the Postnikov tower for $\underline{\mathrm{Hom}}(\mathbb{G}_m, E)$.

$$
\begin{tikzcd}
 & \underline{\mathrm{Hom}}(\mathbb{G}_m, E) \arrow[d] \\
 & \vdots \arrow[d] \\
H(\pi_{n+1}E)_{-1} \arrow[r] & \underline{\mathrm{Hom}}(\mathbb{G}_m, E_{\leq n+1}) \arrow[d] \\
H(\pi_n E)_{-1} \arrow[r] & \underline{\mathrm{Hom}}(\mathbb{G}_m, E_{\leq n}) \arrow[d] \\
 & \vdots
\end{tikzcd}
$$

□

3.4. **Inverting $\mathbb{G}_m \wedge -$; $(\mathbb{G}_m, S^1)$ bi-spectra.** The functor Σ_t on $\mathrm{Spt}^{S^1}(k)$ is a left Quillen functor. We may therefore apply the general machinery of [H-Spt] to create a model category in which Σ_t is invertible. The construction of Hovey may be described as $(\mathbb{G}_m, S^1)$ bispectra.

Definition 3.11. A $(\mathbb{G}_m, S^1)$ bi-spectrum of spaces over k consists of a bigraded family of spaces $E_{i,j}$, $i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \to E_{i,j+1}$ and $\mu_{i,j} : \mathbb{G}_m \wedge E_{i,j} \to E_{i+1,j}$ for which the following diagram commutes .

$$
\begin{tikzcd}
S^1 \wedge \mathbb{G}_m \wedge E_{i,j} \arrow[r, "S^1 \wedge \tau_{i,j}"] \arrow[d] & S^1 \wedge E_{i+1,j} \arrow[dd, "\sigma_{i+1,j}"] \\
\mathbb{G}_m \wedge S^1 \wedge E_{i,j} \arrow[d, "\mathbb{G}_m \wedge \sigma_{i,j}"'] & \\
\mathbb{G}_m \wedge E_{i,j+1} \arrow[r, "\mu_{i,j+1}"] & E_{i+1,j+1}
\end{tikzcd}
$$

Let $\mathrm{Spt}^{(\mathbb{G}_m, S^1)}(k)$ denote the category of bispectra.

Remark 12. Note that a $(\mathbb{G}_m, S^1)$ bispectrum is just a $\mathbb{G}_m$-spectrum of S^1 spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a $(\mathbb{G}_m, S^1)$ bi-spectrum $E_{i,j}$ as a sequence of S^1 spectra $E_{i,*}$.

Definition 3.12. Let E be a $(\mathbb{G}_m, S^1)$ bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \mathrm{Sm}/k \mapsto \mathrm{colim}_r\, \mathcal{H}_\bullet(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$. We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

Proposition 3.13. If E is a $(\mathbb{G}_m, S^1)$ bispectrum, the presheaf of homotopy groups may also be calculated as

$$\tilde{\pi}_{n+m\alpha}E(U) = \mathrm{colim}_{s,r}\, \mathcal{H}_{\bullet}(k)(\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[DLØRV, p 217]

Definition 3.14. A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable weak equivalence if the following induced map is an isomorphism for all $U \in \mathrm{Sm}/k$.

$$f_* : \tilde{\pi}_{n+m\alpha}(E)(U) \to \tilde{\pi}_{n+m\alpha}(F)(U)$$

Definition 3.15. A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable cofibration if $f_0 : E_{0,*} \to F_{0,*}$ is a cofibration of S^1 spectra and the map $P.O. \to F_{n+1}$ is a cofibration in the following diagram.

$$\begin{tikzcd}
\mathbb{G}_m \wedge E_{n,*} \arrow[r] \arrow[d] & E_{n+1,*} \arrow[d] \arrow[ddr, bend left, "f_{n+1}"] & \\
\mathbb{G}_m \wedge F_{n,*} \arrow[r] \arrow[drr, bend right] & P.O. \arrow[dr, dotted] & \\
& & F_{n+1,*}
\end{tikzcd}$$

Proposition 3.16. The category $\mathrm{Spt}^{(\mathbb{G}_m,S^1),\mathbb{A}^1}(k)$ of $(\mathbb{G}_m, S^1)$ bispectra with $\mathbb{A}^1$ stable weak equivalences and $\mathbb{A}^1$ stable cofibrations is a model category. Denote the associated homotopy category of $\mathrm{Spt}^{(\mathbb{G}_m,S^1),\mathbb{A}^1}(k)$ by $\mathcal{SH}(k)$.

Proposition 3.17. The fibrant bi-spectra are the Ω_t-spectra. [H-Spt, Theorem 3.4]

Proposition 3.18. There is a left Quillen functor $\Sigma_t^\infty : \mathrm{Spt}_s^{\mathbb{A}^1}(k) \to \mathrm{Spt}_{s,t}^{\mathbb{A}^1}(k)$ given by $(\Sigma_t^\infty E)_{i,j} = \mathbb{G}_m^i \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}_m^i \wedge E_j \longrightarrow \mathbb{G}_m^i \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}^m \wedge \mathbb{G}_m^i \wedge E_j \to \mathbb{G}_m^{i+1} E_j.$$

The right adjoint to Σ_t^∞ is denoted by Ω_t^∞ and is given by $\Omega_t^\infty(E) = E_{0,*}$.
The right derived functor $R\Omega_t^\infty(E)$ is given by the formula

$$R\Omega_t^\infty(E) = \mathrm{colim}_i\, \Omega_t^i E_{i,*}.$$

3.5. **Connectivity of $(\mathbb{G}_m, S^1)$ bispectra.**

Definition 3.19. A $(\mathbb{G}_m, S^1)$ bispectrum E is said to be n-connected if for all $k \leq n$ and all $m \in \mathbb{Z}$, the homotopy sheaves $\pi_{k+m\alpha}E$ vanish.

Proposition 3.20. Let $E \in \mathrm{Spt}^{S^1,\mathbb{A}^1}(k)$. If E is -1 connected, then so too is the $(\mathbb{G}_m, S^1)$ bi-spectrum $\Sigma_t^\infty E$.

Proof. We calculate

$$\begin{aligned}\pi_{n+m\alpha}(\Sigma_t^\infty E) &= \pi_n(R\Omega_t^\infty \Omega_t^m \Sigma_t^\infty E)\\ &= \pi_n(\operatorname{colim}_i \Omega_t^{m+i}\Sigma_t^i E)\\ &= \operatorname{colim}_i \pi_n(\Sigma_t^i E)_{-(m+i)}\\ &= 0.\end{aligned}$$

This follows since $\Sigma_t E$ is -1 connected whenever E is -1 connected, and the effect of Ω_t^{m+i} on homotopy sheaves is contraction. □

3.6. **t-structure on $\mathcal{SH}(k)$.**

Definition 3.21. Let $\mathcal{SH}(k)_{\geq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha}E = 0$ whenever $n < 0$.

Let $\mathcal{SH}(k)_{\leq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha}E = 0$ whenever $n > 0$.

Definition 3.22. For a $(\mathbb{G}_m, S^1)$ bispectrum E, let $E_{\leq 0}$ denote the spectrum with $(E_{\leq 0})_n = (E_n)_{\leq 0}$. The bonding maps are given by

$$\mathbb{G}_m \wedge P^j(E_{i,j}) \cong P^j(\mathbb{G}_m \wedge E_{i,j}) \to P^j(E_{i+1,j}).$$

The equivalence $\mathbb{G}_m \wedge P^j(\mathcal{X}) \cong P^j(\mathbb{G}_m \wedge \mathcal{X})$ follows by checking on stalks, and the fact that any stalk of $\mathbb{G}_m$ is just a disjoint union of points.

Theorem 3.23. The pair $(\mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ defines a t-structure on $\mathcal{SH}(k)$.

Proof. Property (2) of a t-structure is clear.

We now establish property (1) of a t-structure. Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. We must show $\mathcal{SH}(k)(E, F[-1]) = 0$. When E is in the image of Σ_t^∞, the result follows by using the adjuction $\Sigma_t^\infty \dashv R\Omega_t^\infty$ and using the t-structure on S^1 spectra. In particular, for $U \in \mathrm{Sm}/k$ we have $\mathcal{SH}(k)(S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty U_+, F[-1]) = 0$ for $n \geq 0$ and $m \in \mathbb{Z}$.

For a general $E \in \mathcal{SH}(k)_{\geq 0}$, we may write $E = \operatorname{hocolim} E^i$ where the E^i are built up as in [Mor05, 3.3.4], but we allow smashing with $\mathbb{G}_m$. Precisely, we take $E^0 = pt$, and each E^i is obtained from E^{i-1} as the cone of a map

$$\bigvee_\alpha S^{n_\alpha} \wedge \mathbb{G}_m^{m_\alpha} \wedge \Sigma^\infty X_{\alpha,+} \to E^{i-1}$$

for some family of $X_\alpha \in \mathrm{Sm}/k$ and indices $n_\alpha \geq 0$, $m_\alpha \in \mathbb{Z}$.

A standard 5-lemma argument using the long exact sequence obtained by applying $\mathcal{SH}(k)(-, F[-1])$ to the triangle

$$\vee S^{n_\alpha} \wedge \mathbb{G}_m^{m_\alpha} \wedge \Sigma^\infty X_{\alpha,+} \to E^{i-1} \to E^i$$

shows that for all $i \in \mathbb{N}$, $\mathcal{SH}(k)(E^i, F[-1]) = 0$. Furthermore, these long exact sequences show that for all $i \geq 1$, $\mathcal{SH}(k)(E^i, F[-2]) \to \mathcal{SH}(k)(E^{i-1}, F[-2])$ is surjective. Hence $\varprojlim^1 \mathcal{SH}(k)(E^i, F[-2]) = 0$, and so

$$\begin{aligned}\mathcal{SH}(k)(E, F[-1]) &= \mathcal{SH}(k)(\operatorname{colim} E^i, F[-1])\\ &= \varprojlim \mathcal{SH}(k)(E^i, F[-1])\\ &= 0.\end{aligned}$$

We now establish property (3) of a t-structure. The functor $(-)_{\leq 0}$ has already been defined. For $k \in \mathbb{Z}$, let $(-)_{\leq k}$ is a functor on $\mathrm{Spt}_s(k)$ and we may extend it to a functor on $\mathcal{SH}(k)$ in the same way as for the case $k = 0$. Define $E_{\geq 0}$ to be the homotopy fiber of the canonical map $E \to E_{\leq -1}$. The long exact sequence of homotopy groups shows that $(-)_{\geq 0}$ has the correct homotopy groups. The uniqueness of the triangle follows by properties of triangulated categories. $\square$

3.7. The heart of the t-structure on $\mathcal{SH}(k)$.

Definition 3.24. A homotopy module over k is a pair (M_*, μ_*) consisting of a $\mathbb{Z}$ graded strictly $\mathbb{A}^1$ invariant sheaf M_* and an isomorphism $\mu_n : M_n \cong (M_{n+1})_{-1}$.

Lemma 3.25. If E is a bi-spectrum, then

$$R\Omega_t^\infty E \to \underline{\mathrm{Hom}}(\mathbb{G}_m, R\Omega_t^\infty(E \wedge \mathbb{G}_m))$$

is an isomorphism.

Lemma 3.26. Let $E \in \mathcal{SH}(k)$. For a fixed $n \in \mathbb{Z}$, the collection $\pi_n(E)_m$ forms a homotopy module.

Lemma 3.27. If (M_*, μ_*) is a homotopy module over k, then there is a $(\mathbb{G}_m, S^1)$ bispectrum HM_* with $(HM_*)_{n,n} = K(M_n, n)$ with evident structure maps.

Theorem 3.28. The heart of the t-structure $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is denoted $\pi_*^{\mathbb{A}^1}(k)$ and is equivalent to the category of homotopy modules. The equivalence is given explicitly by the functors $\pi_0(-)_*$ and $H(-)$.

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