1. Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

The following is from an email from Aravind to me concerning the proof of the connectivity theorem for $\mathbb{P}^1$ spectra.

> Most of the proof appears in Morel's Trieste notes (in the section on the homotopy t-structure for P^1-spectra). First you prove a connectivity result for P^1-stable homotopy sheaves by using Morel's S^1-stable connectivity theorem and studying what happens under G_m loops and G_m-suspension: suspension preserves connectivity, and Morel shows that taking G_m-loops has the effect of making a "contraction". Then, you globalize this.

Our goal is to prove theorem 4.14 of [Voev98], which should be restated in terms of $\mathbb{P}^1$-spectra.

Theorem 1.1. Let (X, x) be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let (Y, y) be a pointed simplicial sheaf. Then for any $n < \dim(X)$

$$\mathcal{SH}(k)(S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty X, \Sigma^\infty Y) = 0.$$

In particular, if we take $X = S^0$, this theorem says

$$\mathcal{SH}(k)(S^n \wedge \mathbb{G}_m^m \wedge \mathbb{1}, \Sigma^\infty Y) = \tilde{\pi}_n(Y)_{-m} = 0$$

whenever $n < 0$.

We can formulate the theorem by using homotopy sheaves. The theorem says (in Morel's notation) that $\tilde{\pi}_n(\Sigma^\infty Y)_m(X)$ vanishes whenever $n < \dim(X)$.

The theorem is equivalent to the vanishing of the homotopy groups $\pi_n(\Sigma^\infty Y)_m(X)$ when $n < \dim(X)$. (Is this right?) So certainly by [Mor03, Example 5.2.2] we have $\pi_n(\Sigma^\infty Y)_m = 0$ as a sheaf whenever $n < 0$. How to show $\pi_n(\Sigma^\infty Y)_m(X) = 0$ when $0 \leq n < \dim X$?

2. Assumptions from previous lectures

2.1. Facts about Nisnevich topology.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k, and let $X \in \mathrm{Sm}/k$ with Krull dimension d. Then whenever $n > d$, $H^n_{Nis}(X; M) = 0$.

Proposition 2.2. [Mor04, 2.4.1] For any $X \in \mathrm{Sm}/k$, and for any $x \in X(k)$, there is an isomorphism of pointed sheaves of sets in the Nisnevich topology

$$X/(X - \{x\}) \cong \mathbb{A}^n/(\mathbb{A}^n - \{0\}).$$

2.2. S^1-spectra.

Definition 2.3. Let $\mathcal{SH}_s^{S^1}(k)$ denote the homotopy category associated to the projective model structure on Nisnevich sheaves of simplicial S^1-spectra on Sm/k. (Is this the same as localizing the collection of S^1 spectra of spaces equipped with the proj. model str?)

Definition 2.4. Let $\mathcal{SH}^{S^1}(k)$ denote the localization of the model category associated to $\mathcal{SH}_s^{S^1}(k)$ at the collection of maps $E \wedge \mathbb{A}^1 \to E$.

Remark 1. Advantages to using alternate model category structures? E.g., use presheaves instead of sheaves? Use Hovey's method of stabilizing wrt Quillen functor $X \wedge -$?

Definition 2.5. An S^1-spectrum E is said to be n-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.6. Let $\mathfrak{C}$ be a triangulated category. A t-structure on $\mathfrak{C}$ is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

(1) $(\forall X \in \mathfrak{C}_{\geq 0})(\forall Y \in \mathfrak{C}_{\leq 0})(\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$
(2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
(3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle
$$Y \to X \to Z \to Y[1]$$
for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$..

The heart of a t-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.7 (t-structure on $\mathcal{SH}_s^{S^1}(k)$)**.** Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n > 0$.

Theorem 2.8. The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t-structure on $\mathcal{SH}_s^{S^1}(k)$.

Proposition 2.9. [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}_s^{S^1}(k) \to \mathrm{Spt}_{s,\mathbb{A}^1}^{S^1}(k)$ identifies the $\mathbb{A}^1$-localized S^1 stable homotopy category with the homotopy category of $\mathbb{A}^1$-local S^1 spectra.

Theorem 2.10 (S^1 stable connectivity theorem)**.** Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L_{\mathbb{A}^1} E = 0$.

Theorem 2.11. The pair $(\mathcal{SH}_{\geq 0}^{S^1}(k), \mathcal{SH}_{\leq 0}^{S^1}(k))$ is a t-structure on the category $\mathcal{SH}^{S^1}(k)$.

Definition 2.12. Strictly $\mathbb{A}^1$ invariant sheaf of Abelian groups.

If M is strictly $\mathbb{A}^1$ invariant sheaf of groups, define the Eilenberg-MacLane spectrum HM associated to it.

Proposition 2.13. HM is $\mathbb{A}^1$ local iff M is strictly $\mathbb{A}^1$ invariant.

Proposition 2.14. The heart of the homotopy t structure is equivalent to the category of strictly $\mathbb{A}^1$ invariant sheaves.

3. Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$

Definition 3.1. Contraction of a sheaf of pointed sets G (or abelian groups G). G_{-1}

Theorem 3.2. [Mor03, Lemma 4.3.11] $\pi_n(\underline{Hom}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ is iso.

Lemma 3.3. If M is a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups, then
$$\underline{Hom}(\mathbb{G}_m, HM) \cong H(M_{-1})$$

Lemma 3.4. When $n \neq 0$,
$$[\Sigma^\infty \mathbb{G}_m, HM[n]]_s^{S^1} = 0. \tag{1}$$

The following map is an iso.
$$[\Sigma^\infty \mathbb{G}_m, HM]_s^{S^1} \to [S^0, H(M_{-1})]_s^{S^1} \tag{2}$$

References

[A-1974] Adams, J.F., *Stable Homotopy and Generalized Homology.* Chicago Lectures in Mathematics, (1974).

[B] Blander, Benjamin, *Local Projective Model Structures on Simplicial Presheaves.* K-Theory, **24** (2001) 283–301.

[DHI] Dugger, Dan; Hollander, Sharon; Isaksen, Dan, *Hypercovers and simplicial presheaves.*

[DLØRV] Dundas, B.; Levine, M.; Østvær, P.; Röndigs, O.; Voevodsky, V., *Motivic Homotopy Theory.* Springer (2000).

[Hir] Phillip, Hirschhorn, *Model Categories and Their Localization.* AMS (2003).

[H-Mod] Hovey, Mark, *Model Categories.* online preprint (1991).

[H-Spt] Hovey, Mark, *Spectra and symmetric spectra in general model categories.* journal? (2001).

[J] Jardine, J.F., *Simplicial presheaves.* Journal of Pure and Applied Algebra, **47** (1987) 35–87.

[Mor03] Morel, Fabien, *An introduction to $\mathbb{A}^1$ homotopy theory.*

[Mor04] Morel, Fabien, *On the motivic π_0 of the sphere spectrum.* NATO science series.

[Mor05] Morel, Fabien, *The stable $\mathbb{A}^1$ connectivity theorems.* preprint (2004).

[Voev98] Voevodsky, Vladimir. $\mathbb{A}^1$-*Homotopy Theory.* Doc. Math. J., (1998) pp. 579–604.