

1. VOEVODSKY'S CONNECTIVITY THEOREM FOR \mathbb{P}^1 -SPECTRA

Our goal is to prove theorem 4.14 of [Voev98], which we restate in terms of \mathbb{P}^1 -spectra.

Theorem 1.1. Let (X, x) be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let \mathcal{Y} be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

Remark 1. We will also show that for any pointed space \mathcal{Y} , the \mathbb{P}^1 suspension spectrum $\Sigma^\infty \mathcal{Y}$ is -1 connected, i.e., the homotopy sheaves $\pi_{n+\alpha m}^{\mathbb{A}^1}(\mathcal{Y})$ vanish for $n \leq -1$. This will allow us to establish a t -structure on $\mathcal{SH}(k)$.

The line of attack is then to show that for a pointed space \mathcal{Y} , the S^1 suspension spectrum $\Sigma_s^\infty \mathcal{Y}$ in the stable model category is -1 connected. Then we establish the first connectivity theorem which ensures the \mathbb{A}^1 localization of $\Sigma_s^\infty \mathcal{Y}$ is -1 connected. Finally, we show that inverting \mathbb{G}_m does not affect the connectivity, i.e., $\Sigma_t^\infty \Sigma_s^\infty \mathcal{Y}$ is again -1 connected.

Remark 2. A construction of Ayoub [?] shows that theorem 1.1 statement is false over general Noetherian base schemes S . The argument below works for infinite fields, however.

2. ASSUMPTIONS FROM PREVIOUS LECTURES

2.1. Facts about Nisnevich topology.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k , and let $X \in \mathrm{Sm}/k$ with Krull dimension d . Then whenever $n > d$, $H_{Nis}^n(X; M) = 0$.

2.2. Unstable model category $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$. Give the unstable model category the injective model structure to start. (This is Morel's choice, so we stick with it)

Weak equivalences: $\mathcal{X} \rightarrow \mathcal{Y}$ iff for all U the map $\mathcal{X}(U) \rightarrow \mathcal{Y}(U)$ a simplicial w.e.

Cofibs: $\mathcal{X} \rightarrow \mathcal{Y}$ is a cofib. iff for any $U \in \mathrm{Sm}/k$ the map $\mathcal{X}(U) \rightarrow \mathcal{Y}(U)$ is a monomorphism.

Fibrations: what they need to be.

The ass. htpy. cat. to the injective model category structure is denoted $\mathcal{H}_s(k)$. The pointed htpy. cat. is denoted $\mathcal{H}_{s,\bullet}(k)$.

Important properties/constructions in these categories: Cartesian closed, i.e., have smash/tensor \times and \wedge , with right adjoints \underline{Hom} and \underline{Hom}_\bullet . Representable sheaf functor and constant simplicial set functor.

Definition 2.2. Internal hom. (Ref: P. Pelaez)

Let $U \in \mathrm{Sm}/k$. Let Δ_U^n denote the simplicial sheaf given by

$$(V, m) \in \mathrm{Sm}/k \times \Delta^{op} \rightarrow \mathrm{Sm}/k(V, U) \times \Delta_m^n.$$

In other words, $\Delta_U^n = (rU) \times c\Delta^n$.

Let \mathcal{X} and \mathcal{Y} be spaces. The internal hom in the unpointed category is given by the formula

$$(U, m) \in \mathrm{Sm}/k \times \Delta \rightarrow \mathrm{Hom}_{\Delta^{op}\mathrm{Shv}}(X \times \Delta_U^m, Y).$$

How to describe the adjunction?

$$\mathrm{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$$

Certainly we may send a map $g \in \text{Hom}(\mathcal{Y}, \underline{\text{Hom}}(\mathcal{X}, \mathcal{Z}))$ to the map $\eta(g)$ given by

$$\begin{aligned} \eta(g)(U, n) : \mathcal{X}_n(U) \times \mathcal{Y}_n(U) &\rightarrow \mathcal{Z}_n(U) \\ (a, b) &\rightarrow g(U, n)(U, n)(a, \text{id}, \text{id}). \end{aligned}$$

Why is this a bijection?

Definition 2.3. Internal hom in the pointed category. Consider \mathcal{X} and \mathcal{Y} pointed spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{\text{Hom}}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y}$, where at $(U, n) \in (\text{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times \Delta_U^n \rightarrow \mathcal{Y}$ to $g(U, n)(x, \text{id}, \text{id}) \in \mathcal{Y}_n(U)$. This makes sense as we have the map

$$g(U, n) : \mathcal{X}_n(U) \times \text{Sm}/k(U, U) \times \Delta_U^n \rightarrow \mathcal{Y}_n(U).$$

The pointed internal hom $\underline{\text{Hom}}_\bullet(\mathcal{X}, \mathcal{Y})$ is the fiber $ev_x^{-1}(y)$.

2.3. \mathbb{A}^1 localization.

Definition 2.4. A space \mathcal{X} is called \mathbb{A}^1 local if for any smooth scheme U , the canonical map

$$\text{Hom}(rU, \mathcal{X}) \rightarrow \text{Hom}(rU \times \mathbb{A}^1, \mathcal{X})$$

is a bijection.

Definition 2.5. A map $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an \mathbb{A}^1 weak equivalence if

$$\text{Hom}(\mathcal{Y}, \mathcal{Z}) \rightarrow \text{Hom}(\mathcal{X}, \mathcal{Z})$$

is a bijection for every \mathbb{A}^1 local space \mathcal{Z} .

The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \rightarrow U \mid U \in \text{Sm}/k\}$.

The general theory of Bousfield localization then gives a localization functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \rightarrow L_W \mathcal{H}_s(k)$; denote the category $L_W \mathcal{H}_s(k)$ by $\mathcal{H}(k)$. The localization functor sends \mathbb{A}^1 weak equivalences to isomorphisms.

See [?, Definition 3.3.1]. Keep the same underlying category, the weak equivalences are the \mathbb{A}^1 -local weak equivalences, do not change the cofibrations from the ones in the injective model structure on $\Delta^{op} \text{Shv}(\text{Sm}/k, \text{Nis})$, and let the fibrations be what they need to be.

Morel writes $L_{\mathbb{A}^1}$ for the (derived) functor which sends a space (without base point) \mathcal{X} to an \mathbb{A}^1 -localization.

Morel gives an explicit construction which takes a pointed space \mathcal{Y} and produces a space $L^\infty \mathcal{Y}$ which is \mathbb{A}^1 local and a map $\mathcal{Y} \rightarrow L^\infty \mathcal{Y}$ which is an \mathbb{A}^1 weak equivalence.

Let \mathcal{Y}_f be the functorial fibrant replacement of \mathcal{Y} with respect to the injective model structure. Consider \mathbb{A}^1 to be pointed at 0. Morel defines $L^{(1)}(\mathcal{Y})$ to be the cone of the map $ev_1 : \underline{\text{Hom}}_\bullet(\mathbb{A}^1, \mathcal{Y}_f) \rightarrow \mathcal{Y}_f$. So there is a map $\mathcal{Y} \rightarrow L^{(1)}(\mathcal{Y})$ obtained from the trivial cofibration $\mathcal{Y} \rightarrow \mathcal{Y}_f$ and the defining map $\mathcal{Y}_f \rightarrow L^{(1)}(\mathcal{Y})$.

So $L^{(1)}(-)$ is a functor with a natural transformation $\eta : \text{id} \rightarrow L^{(1)}(-)$. Define by induction $L^{(n)}(\mathcal{Y}) = L^{(1)}(L^{(n-1)}(\mathcal{Y}))$. There is thus a directed system $L^{(n-1)}(\mathcal{Y}) \rightarrow L^{(n)}(\mathcal{Y})$. Denote the direct limit/hocolim of this directed system by $L^\infty(\mathcal{Y})$.

Proposition 2.6. The natural morphism $\mathcal{Y} \rightarrow L^\infty(\mathcal{Y})$ is an \mathbb{A}^1 weak equivalence, and $L^\infty(\mathcal{Y})$ is \mathbb{A}^1 local.

Definition 2.7. Let \mathcal{X} be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on \mathbf{Sm}/k associated to $U \rightarrow \pi_0(\mathcal{X}(U))$. A space \mathcal{X} is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let (\mathcal{X}, x) be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on \mathbf{Sm}/k given by

$$U \rightarrow \pi_n(\mathcal{X}(U)).$$

A pointed space \mathcal{X} is called n -connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

Proposition 2.8. Let \mathcal{X} be a 0-connected simplicial sheaf. Then $L^\infty \mathcal{X}$ is also 0-connected.

Proof. M.V. IHES paper? Sketch of argument? It shouldn't be too hard by chasing components around. \square

For a sheaf of abelian groups M on \mathbf{Sm}/k and a natural number n , a Dold-Kan construction gives a simplicial presheaf $K(M, n)$. It is called the Eilenberg-MacLane spectrum of type (M, n) and has homotopy sheaves as expected:

$$\pi_m(K(M, n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}.$$

Important equation for this construction. For $X \in \mathbf{Sm}/k$, M a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M, n)) \cong H_{Nis}^n(X; M).$$

It therefore follows that

$$\mathcal{H}_\bullet(k)(rX_+, K(M, n)) \cong H_{Nis}^n(X; M).$$

Use square brackets to denote the maps in the unstable pointed (motivic) homotopy category, i.e., $[\mathcal{X}, \mathcal{Y}] = \mathcal{H}_\bullet(k)(\mathcal{X}, \mathcal{Y})$.

Use $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ for the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^\infty \mathcal{X})$. This is also obtained by sheafifying the presheaf given by

$$U \in \mathbf{Sm}/k \mapsto [S^n \wedge U_+, \mathcal{X}].$$

2.4. Homotopy purity, connectedness calculations.

Proposition 2.9. Let \mathcal{X} be a pointed space. Then if for any finitely generated field F over k , $\pi_0(\mathcal{X})(F) = 0$, then the sheaf $\pi_0(\mathcal{X})$ is trivial.

Proof. Check at points. Use Gabber presentation lemma. \square

2.5. S^1 -spectra.

Definition 2.10. Let $\mathbf{Spt}_s(k)$ denote the category of S^1 -spectra of spaces $\Delta^{op}\mathbf{Shv}(\mathbf{Sm}/k)$. We first endow this category with the projective model structure, i.e., a map $f : E \rightarrow F$ is a weak equivalence iff for any n the map $f_n : E_n \rightarrow F_n$ is a w.e.; a map $f : E \rightarrow F$ is a fibration iff for all n the map $f_n : E_n \rightarrow F_n$ is a fibration. The cofibrations are characterized

by the property that $f : E \rightarrow F$ is a cofib iff $f_0 : E_0 \rightarrow F_0$ is a cofib and for any $n \geq 1$

$$\begin{array}{ccc}
 S^1 \wedge E_{n-1} & \xrightarrow{\sigma_{n-1}} & E_n \\
 S^1 \wedge f_{n-1} \downarrow & & \downarrow \\
 S^1 \wedge F_{n-1} & \longrightarrow & P.O. \\
 & \searrow \sigma_{n-1} & \downarrow f_n \\
 & & F_n
 \end{array}$$

This model structure does not actually invert $S^1 \wedge -$. To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.11. A map $f : E \rightarrow F$ of S^1 -spectra is a stable equivalence if for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \rightarrow \pi_n(F)$ is an isomorphism.

The stable model category structure on $\text{Spt}_s(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such.

Definition 2.12. Let $\text{Spt}_s^{\mathbb{A}^1}(k)$ denote the category of S^1 spectra endowed with the stable model category structure localized at the collection of maps $\{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \rightarrow \Sigma^\infty U_+ \mid U \in \text{Sm}/k\}$.

Let $\mathcal{SH}^{S^1}(k)$ denote the homotopy category associated to $\text{Spt}_s^{\mathbb{A}^1}(k)$. We will use $\mathcal{SH}_s^{S^1}(k)$ to denote the homotopy category of $\text{Spt}_s(k)$.

Remark 3. There is a functor L^∞ on the category of S^1 spectra which is similar to the unstable construction.

So we can use the functor L^∞ as an \mathbb{A}^1 localization functor. To be precise, we let $\text{Spt}_s^{\mathbb{A}^1 \text{loc}}(k)$ be the subcategory of $\text{Spt}_s(k)$ consisting of the \mathbb{A}^1 local spectra. We may equip $\text{Spt}_s^{\mathbb{A}^1 \text{loc}}(k)$ with a model structure with weak equivalences the \mathbb{A}^1 weak equivalences, the cofibrations are the stable cofibrations, and the fibrations are what they have to be.

The functor $L^\infty : \text{Spt}_s(k) \rightarrow \text{Spt}_s^{\mathbb{A}^1 \text{loc}}(k)$ is left adjoint to the inclusion functor, and is a left Quillen functor. The homotopy category of $\text{Spt}_s^{\mathbb{A}^1 \text{loc}}(k)$ is categorically equivalent to $\mathcal{SH}^{S^1}(k)$. [Mor05, Corollary 4.2.3].

For spectra E and F , we may compute $[E, F]^{\mathbb{A}^1} := \mathcal{SH}^{S^1}(k)(E, F)$ by calculating $[L^\infty E, L^\infty F]^{\mathbb{A}^1}$. Note that this is $\mathcal{SH}_s^{S^1}(k)(E, L^\infty F)$ by using the adjunction. If we assume E is cofibrant and $L^\infty F$ is fibrant, we get the formula

$$[E, F]^{\mathbb{A}^1} = \text{Spt}_s(k)(E, L^\infty F).$$

Definition 2.13. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\text{Ab}}(\text{Sm}/k, \text{Nis})$. That is,

$$\pi_n(E) = \text{colim}_r \pi_{n+r}(E_r).$$

In particular, for a $U \in \text{Sm}/k$, we have

$$\pi_n(E)(U) = \text{colim}_r \pi_{n+r}(E_r)(U).$$

Definition 2.14. An S^1 -spectrum E is said to be n -connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.15. There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \rightarrow \mathrm{Spt}_s(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ with the evident bonding maps. The right adjoint to this functor is given by “evaluation at 0”, i.e., $\Omega^\infty(E) = E_0$.

Remark 4. The right derived functor $R\Omega^\infty : \mathcal{SH}_s^{S^1}(k) \rightarrow \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i \Omega_s^i E_i.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

Remark 5. We also get a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k) \rightarrow \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ given by the same formula as above.

What is the fibrant replacement functor in $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$?

Remark 6. The stable homotopy category is symmetric monoidal, with smash product \wedge and internal hom $\underline{\mathrm{Hom}}$. Using symmetric spectra, one can give these constructions on the category of spectra.

The stable homotopy category is a triangulated category.

Proposition 2.16. Let $U \in \mathrm{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then there is a canonical isomorphism

$$H_{Nis}^n(U; M) \rightarrow \mathcal{SH}^{S^1}(\Sigma^\infty U_+, HM[n]).$$

Proof. This is [Mor05, Lemma 3.2.3]. □

2.6. t -structures.

Definition 2.17. Let \mathfrak{C} be a triangulated category. A t -structure on \mathfrak{C} is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

- (1) For any $X \in \mathfrak{C}_{\geq 0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
- (2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
- (3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \rightarrow X \rightarrow Z \rightarrow Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$.

The heart of a t -structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.18 (t -structure on $\mathcal{SH}_s^{S^1}(k)$). Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenever $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenever $n > 0$.

Theorem 2.19. The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t -structure on $\mathcal{SH}_s^{S^1}(k)$.

Proof. Fill in details. □

2.7. Connectivity results.

Proposition 2.20. [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}_s^{S^1}(k) \rightarrow \mathrm{Spt}_{s, \mathbb{A}^1}^{S^1}(k)$ identifies the \mathbb{A}^1 -localized S^1 stable homotopy category with the homotopy category of \mathbb{A}^1 -local S^1 spectra.

Theorem 2.21 (S^1 stable connectivity theorem). Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L_{\mathbb{A}^1} E = 0$.

Theorem 2.22. The pair $(\mathcal{SH}_{\geq 0}^{S^1}(k), \mathcal{SH}_{\leq 0}^{S^1}(k))$ is a t -structure on the category $\mathcal{SH}^{S^1}(k)$.

Proof. This is just the restriction of the t -structure to the \mathbb{A}^1 -local objects. \square

Definition 2.23. Strictly \mathbb{A}^1 invariant sheaf of Abelian groups.

If M is strictly \mathbb{A}^1 invariant sheaf of groups, define the Eilenberg-MacLane spectrum HM associated to it.

Proposition 2.24. HM is \mathbb{A}^1 local iff M is strictly \mathbb{A}^1 invariant.

Proposition 2.25. The heart of the homotopy t structure is equivalent to the category of strictly \mathbb{A}^1 invariant sheaves.

3. INVERTING $\mathbb{G}_m \wedge -$; \mathbb{P}^1 SPECTRA

3.1. \mathbb{G}_m suspension and loops. We always consider \mathbb{G}_m to be pointed at 1 unless otherwise specified.

Definition 3.1. On the category $\mathrm{Spt}_s(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps. In order for Σ_t to be a left Quillen functor, we must replace \mathbb{G}_m with a cofibrant object. We do this, and work with this object instead of \mathbb{G}_m .

Smashing with \mathbb{G}_m is also a functor on the unstable category of pointed spaces, and we give it the same name Σ_t .

Definition 3.2. The functor Σ_t on $\mathrm{Spc}_\bullet(k)$ has a right adjoint denoted Ω_t . It is given by the formula $\Omega_t \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \mathcal{X})$.

The functor Σ_t on $\mathrm{Spt}_s^{S^1}(k)$ also has a right adjoint Ω_t given by the internal hom functor, i.e., $\Omega_t E = \underline{\mathrm{Hom}}(\Sigma^\infty \mathbb{G}_m, E)$.

Proposition 3.3. The functor Σ_t is a left Quillen functor on $\mathrm{Spt}_s(k)$ and on $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. Furthermore, Σ_t is an exact (or homological) functor on $\mathcal{SH}^{S^1}(k)$, meaning it sends exact triangles to exact triangles.

Lemma 3.4. Let $E \in \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ be a -1 connected spectrum. Then $\Sigma_t E$ is again -1 connected.

Proof. The claim is clear when $E = \Sigma_s^\infty \mathcal{X}$ a pointed space, since $\Sigma_t E = \Sigma_s^\infty \mathbb{G}_m \wedge \mathcal{X}$ is still a suspension spectrum, and so -1 connected.

Now consider a general -1 connected spectrum E . By [Mor05, Lemma 3.3.4], E is weak equivalent to $\mathrm{hocolim} E^i$ where $E^0 = *$, and for each n , there is a family $X_\alpha \in \mathrm{Sm}/k$ and natural numbers $n_\alpha \geq 0$ for which

$$\vee_\alpha \Sigma_s^\infty X_{\alpha,+}[n_\alpha - 1] \rightarrow E^{n-1} \rightarrow E^n$$

is an exact triangle. An induction argument establishes that $\Sigma_t E^n$ is still -1 connected for all n ; hence $\Sigma_t E = \text{hocolim } \Sigma_t E^n$ is also -1 connected. Should $\Sigma_t E$ fail to be \mathbb{A}^1 -local, we may simply apply L^∞ to get an \mathbb{A}^1 -local representative of $\Sigma_t E$. By the connectivity theorem, $L^\infty \Sigma_t E$ will again be -1 connected. \square

3.2. Contraction in $\underline{\text{Ab}}(\text{Sm}/k, Nis)$, category of pointed sheaves of sets.

Definition 3.5. Let G be sheaf of pointed sets on Sm/k . The contraction of G is the sheaf $G_{-1} = G_{con}$ given by the formula

$$U \in \text{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map ev_1 is the map induced by $ev_1 : \text{Spec}(k) \rightarrow \mathbb{G}_m$, i.e., $k[x, x^{-1}] \rightarrow k$ given by $x \mapsto 1$.

Note that indeed G_{-1} is a sheaf since it is the kernel of the morphism of sheaves $G(-) \rightarrow G(- \times \mathbb{G}_m)$. The sheaf $G(- \times \mathbb{G}_m)$ may also be written as $\underline{\text{Hom}}(\mathbb{G}_m, G)$ when we think of G as a space.

Proposition 3.6. If G is the trivial sheaf of abelian groups, then so is its contraction G_{-1} .

Proposition 3.7. Contraction is an exact functor on the category $\underline{\text{Ab}}_{st\mathbb{A}^1}(\text{Sm}/k, Nis)$.¹

For any sheaf $G \in \underline{\text{Ab}}(\text{Sm}/k, Nis)$ and any $X \in \text{Sm}/k$, $G(\mathbb{G}_m \times X) = G_{-1}(X) \oplus G(X)$.

3.3. Homotopy sheaves of $\underline{\text{Hom}}(\mathbb{G}_m, E)$.

Proposition 3.8. If G is a sheaf of Abelian groups, then $G_{-1} = \underline{\text{Hom}}_\bullet(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $- \wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

Proof. For this category, $\underline{\text{Hom}}_\bullet(\mathbb{G}_m, G)$ and G_{-1} both have sections at X given by $\ker(ev_1 : G(X \times \mathbb{G}_m) \rightarrow G(X))$. See description of pointed internal hom for this. \square

Remark 7. If G is a sheaf of Abelian groups, we may consider G as a space by declaring $G_n = G$ for all n and giving identity maps for the structure maps. In particular, G is a pointed space at 0.

We can then realize the contraction as a \mathbb{G}_m loop space $G_{-1}(X) = \underline{\text{Hom}}_\bullet(\mathbb{G}_m, G)(X)$.

Remark 8. Construction of canonical map $\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \rightarrow \pi_n(E)_{-1}$ for an S^1 spectrum E .

First observe that for any $U \in \text{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\text{Spt}_s(k)(S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m, E) \times \text{Spc}(k)(U, \mathbb{G}_m) \rightarrow \pi_0(E)(U)$$

given by sending (f, α) to the composition

$$\Sigma_s^\infty U_+ \xrightarrow{\text{id} \wedge \Sigma_s^\infty \alpha} S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \rightarrow \pi_n(E).$$

Does this map descend to the smash? Yes, since if either map is a constant map, then so is the composition.

We thus get a map of sheaves of pointed sets

$$\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \rightarrow \pi_n(E).$$

¹I may be able to prove this after I establish the homotopy sheaves of $\Omega_t HM$.

But by the adjunction $- \wedge \mathbb{G}_m \dashv \underline{\text{Hom}}_{\bullet}(\mathbb{G}_m, -)$ on $\text{Spc}_{\bullet}(k)$ we have a morphism

$$\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \rightarrow \underline{\text{Hom}}_{\bullet}(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Why is it a map of sheaves of abelian groups?

Remark 9. If $E = HM$ is an Eilenberg-MacLane spectrum associated to a strictly \mathbb{A}^1 invariant sheaf of abelian groups M , we show

$$\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, HM)) \rightarrow \underline{\text{Hom}}_{\bullet}(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\underline{\text{Hom}}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

Proposition 3.9. For $M \in \underline{\text{Ab}}_{st\mathbb{A}^1}(\text{Sm}/k)$, the spectrum $\underline{\text{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

Proof. Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_{\bullet}(k)$, we have $\Sigma^{\infty}\mathbb{P}^1[-1] = \Sigma^{\infty}\mathbb{G}_m$. Therefore

$$\begin{aligned} \mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}\mathbb{G}_m, HM[n]) &= \mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}\mathbb{P}^1[-1], HM[n]) \\ &= \mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}\mathbb{P}^1, HM[n+1]) \\ &= \tilde{H}_{Nis}^{n+1}(\mathbb{P}^1; M). \end{aligned}$$

As the cohomological dimension of \mathbb{P}^1 is less than or equal to 1, we then have $\tilde{H}_{Nis}^{n+1}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$. By $\tilde{H}_{Nis}^n(X; M)$ I mean the kernel of $\mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}X_+, HM[n]) \rightarrow \mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}S^0, HM[n])$ induced by $S^0 \rightarrow X_+$, where this is obtained by choosing a point in $X(k)$. But what if $X(k)$ is empty?

$$\tilde{H}^n(X; M) \oplus H^n(\text{Spec}(k); M) \cong H^n(X; M).$$

So as $M(\text{Spec } k) \cong M(\mathbb{P}^1)$ (follows since M is strictly \mathbb{A}^1 invariant) $M(\mathbb{P}^1) = \text{pullback}(M(\mathbb{A}^1) \rightarrow M(\mathbb{G}_m) \rightarrow M(\mathbb{A}^1) = M(\text{Spec } k))$, we see that the only possible value of n this will not vanish at is 1.

Since this vanishes at fields, a base change argument shows that indeed the sheaf $\pi_n \underline{\text{Hom}}(\mathbb{G}_m, HM)$ is weakly trivial when $n \neq 0$. So then it follows that the sheaf is indeed trivial by the argument giving weakly n-connected is equivalent to being n-connected.²

We now calculate at $\text{Spec}(k)$

$$\begin{aligned} \pi_0 \underline{\text{Hom}}(\mathbb{G}_m, HM)(\text{Spec}(k)) &= \mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}\mathbb{G}_m, HM) \\ &= \tilde{H}^0(\mathbb{G}_m; M) \\ &= \ker(\mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}\mathbb{G}_{m,+}, HM) \rightarrow \mathcal{H}_s^{S^1}(k)(\Sigma^{\infty}S^0, HM)) \\ &= M_{-1}(\text{Spec } k) \end{aligned}$$

We now know that the associated homotopy sheaves $\pi_n \underline{\text{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all n . So they are weak equivalent by [Mor05, Lemma 3.2.5]. \square

Proposition 3.10. For any spectrum $E \in \mathcal{H}^{S^1}(k)$, the homotopy sheaves of $\underline{\text{Hom}}(\mathbb{G}_m, E)$ are calculated by $\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \cong \pi_n(E)_{-1}$

Proof. Reduce to the case of Eilenberg-MacLane spectra by using the Postnikov tower. \square

²Reword this to use the modified lemma with gabber presentation.

3.4. Inverting $\mathbb{G}_m \wedge -$; (\mathbb{G}_m, S^1) bi-spectra. The functor Σ_t on $\text{Spt}_s(k)$ is a left Quillen functor. We may therefore apply the general machinery of [H-Spt] to create a model category in which Σ_t is invertible. The construction of Hovey may be described as (\mathbb{G}_m, S^1) bispectra.

Definition 3.11. A (\mathbb{G}_m, S^1) bi-spectrum of spaces over k consists of a bigraded family of spaces $E_{i,j}$, $i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \rightarrow E_{i,j+1}$ and $\mu_{i,j} : \mathbb{G}_m \wedge E_{i,j} \rightarrow E_{i+1,j}$ for which the following diagram commutes .

$$\begin{array}{ccc}
 S^1 \wedge \mathbb{G}_m \wedge E_{i,j} & \xrightarrow{S^1 \wedge \tau_{i,j}} & S^1 \wedge E_{i+1,j} \\
 \downarrow & & \downarrow \sigma_{i+1,j} \\
 \mathbb{G}_m \wedge S^1 \wedge E_{i,j} & & \\
 \downarrow \mathbb{G}_m \wedge \sigma_{i,j} & & \\
 \mathbb{G}_m \wedge E_{i,j+1} & \xrightarrow{\mu_{i,j+1}} & E_{i+1,j+1}
 \end{array}$$

Remark 10. Note that a (\mathbb{G}_m, S^1) bispectrum is just a \mathbb{G}_m -spectrum of S^1 -spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a (\mathbb{G}_m, S^1) bi-spectrum $E_{i,j}$ as a sequence of S^1 spectra $E_{i,*}$.

Definition 3.12. Let E be a (\mathbb{G}_m, S^1) bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \text{Sm}/k \mapsto \text{colim}_r \mathcal{H}_\bullet(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$.

We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

Proposition 3.13. If E is a (\mathbb{G}_m, S^1) bispectrum, the presheaf of homotopy groups may also be calculated as

$$\tilde{\pi}_{n+m\alpha}E(U) = \text{colim}_{s,r} \mathcal{H}_\bullet(k)(\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[DLØRV, p 217]

Definition 3.14. A morphism $f : E \rightarrow F$ of (\mathbb{G}_m, S^1) bispectra is an \mathbb{A}^1 stable weak equivalence if the following induced map is an isomorphism for all $U \in \text{Sm}/k$.

$$f_* : \tilde{\pi}_{n+m\alpha}(E)(U) \rightarrow \tilde{\pi}_{n+m\alpha}(F)(U)$$

Definition 3.15. A morphism $f : E \rightarrow F$ of (\mathbb{G}_m, S^1) bispectra is an \mathbb{A}^1 stable cofibration if $f_0 : E_{0,*} \rightarrow F_{0,*}$ is a cofibration of S^1 spectra and the map $P.O. \rightarrow F_{n+1,*}$ is a cofibration in the following diagram.

$$\begin{array}{ccc}
 \mathbb{G}_m \wedge E_{n,*} & \longrightarrow & E_{n+1,*} \\
 \downarrow & & \downarrow \\
 \mathbb{G}_m \wedge F_{n,*} & \longrightarrow & P.O. \\
 & \searrow & \nearrow \\
 & & F_{n+1,*}
 \end{array}$$

f_{n+1}

Proposition 3.16. The category $\text{Spt}_{s,t}^{\mathbb{A}^1}(k)$ of (\mathbb{G}_m, S^1) bispectra with \mathbb{A}^1 stable weak equivalences and \mathbb{A}^1 stable cofibrations is a model category. Denote the associated homotopy category of $\text{Spt}_{s,t}^{\mathbb{A}^1}(k)$ by $\mathcal{SH}(k)$.

Proposition 3.17. The fibrant bi-spectra are the Ω_t -spectra. [H-Spt, Theorem 3.4]

Proposition 3.18. There is a left Quillen functor $\Sigma_t^\infty : \text{Spt}_s^{\mathbb{A}^1}(k) \rightarrow \text{Spt}_{s,t}^{\mathbb{A}^1}(k)$ given by $(\Sigma_t^\infty E)_{i,j} = \mathbb{G}_m^i \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}_m^i \wedge E_j \longrightarrow \mathbb{G}_m^i \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}_m^m \wedge \mathbb{G}_m^i \wedge E_j \rightarrow \mathbb{G}_m^{i+1} E_j.$$

The right adjoint to Σ_t^∞ is denoted by Ω_t^∞ and is given by $\Omega_t^\infty(E) = E_{0,*}$.

The right derived functor $R\Omega_t^\infty(E)$ is given by the formula

$$R\Omega_t^\infty(E) = \text{colim}_i \Omega_t^i E_{i,*}.$$

Proof.

□

3.5. Connectivity of (\mathbb{G}_m, S^1) bispectra.

Definition 3.19. A (\mathbb{G}_m, S^1) bispectrum E is said to be n -connected if for all $k \leq n$ and all $m \in \mathbb{Z}$, the homotopy sheaves $\pi_{k+m\alpha} E$ vanish.

Proposition 3.20. Let $E \in \text{Spt}_s^{\mathbb{A}^1}(k)$. Consider $\Sigma_t^\infty E \in \text{Spt}_{s,t}^{\mathbb{A}^1}(k)$. If E is -1 connected, then so too is $\Sigma_t^\infty E$.

Proof. We calculate

$$\begin{aligned} \pi_{n+m\alpha}(\Sigma_t^\infty E) &= \pi_n(R\Omega_t^\infty \Omega_t^m \Sigma_t^\infty E) \\ &= \pi_n(\text{colim}_i \Omega_t^{m+i} \Sigma_t^i E) \\ &= \text{colim}_i \pi_n(\Sigma_t^i E)_{-(m+i)} \\ &= 0. \end{aligned}$$

This follows since $\Sigma_t E$ is -1 connected whenever E is -1 connected, and the effect of Ω_t^{m+i} on homotopy sheaves is contraction. □

3.6. t -structure on $\mathcal{SH}(k)$.

Definition 3.21. Let $\mathcal{SH}(k)_{\geq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha} E = 0$ whenever $n < 0$.

Let $\mathcal{SH}(k)_{\leq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha} E = 0$ whenever $n > 0$.

Theorem 3.22. The triple $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ defines a t -structure.

Proof. What needs to be done:

- (1) Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. Show $\mathcal{SH}(k)(E, F[-1]) = 0$.
- (2) Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. Show $E[1] \in \mathcal{SH}(k)_{\geq 0}$ and $F[-1] \in \mathcal{SH}(k)_{\leq 0}$.
This is clear by invertibility of $[1]$ in $\mathcal{SH}(k)$.

- (3) Given $E \in \mathcal{SH}(k)$, construct $E_{\geq 0}$ and $E_{< 0}$ which fit into an exact triangle

$$E_{\geq 0} \rightarrow E \rightarrow E_{< 0} \rightarrow E_{\geq 0}[1].$$

We can construct the functor $(-)_{\geq 0}$. For a bi-spectrum E , we define $(E_{\geq 0})_n = (E_n)_{\geq 0}$, where we are applying the truncation functor for S^1 -spectra. The bonding maps are obtained by using the isomorphism $\mathbb{G}_m \wedge (E_n)_{\geq 0} \rightarrow (\mathbb{G}_m \wedge E_n)_{\geq 0}$. It then follows that the cone of $E_{\geq 0} \rightarrow E$ satisfies the properties needed, and we make take this as the definition of $E_{< 0}$, or $E_{\leq -1}$. The uniqueness follows by general properties of triangulated categories.

□

Definition 3.23. A homotopy module over k is a pair (M_*, μ_*) consisting of a \mathbb{Z} graded strictly \mathbb{A}^1 invariant sheaf M_* and an isomorphism $\mu_n : M_n \cong (M_{n+1})_{-1}$.

Lemma 3.24. If E is a bi-spectrum, then

$$R\Omega_t^\infty E \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}_m, R\Omega_t^\infty(E \wedge \mathbb{G}_m))$$

is an isomorphism.

Proof.

□

Lemma 3.25. Let $E \in \mathcal{SH}(k)$. For a fixed $n \in \mathbb{Z}$, the collection $\pi_n(E)_m$ forms a homotopy module.

Lemma 3.26. If (M_*, μ_*) is a homotopy module over k , then there is a (\mathbb{G}_m, S^1) bispectrum HM_* with $(HM_*)_{n,n} = K(M_n, n)$ with evident structure maps.

Proof.

□

Theorem 3.27. The heart of the t -structure $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is denoted $\pi_*^{\mathbb{A}^1}(k)$ and is equivalent to the category of homotopy modules. The equivalence is given explicitly by the functors $\pi_0(-)_*$ and $H(-)$.

Proof.

□

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