1. Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

The following is from an email from Aravind to me concerning the proof of the connectivity theorem for $\mathbb{P}^1$ spectra.

> Most of the proof appears in Morel's Trieste notes (in the section on the homotopy t-structure for P^1-spectra). First you prove a connectivity result for P^1-stable homotopy sheaves by using Morel's S^1-stable connectivity theorem and studying what happens under G_m loops and G_m-suspension: suspension preserves connectivity, and Morel shows that taking G_m-loops has the effect of making a "contraction". Then, you globalize this.

Our goal is to prove theorem 4.14 of [Voev98], which should be restated in terms of $\mathbb{P}^1$-spectra.

Theorem 1.1. Let X be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let $\mathcal{Y}$ be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

Remark 1. We can formulate the theorem by using homotopy sheaves. Indeed, if we can show that for any pointed space $\mathcal{Y}$ is -1 connected, i.e., the homotopy sheaves $\pi_{n+\alpha m}^{\mathbb{A}^1}(\mathcal{Y})$ vanish for $n \leq -1$, the result will follow, as

$$\begin{aligned}\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) &= \mathcal{SH}(k)(S^{-n} \wedge \mathbb{G}_m^{-m} \wedge \Sigma^\infty X, \Sigma^\infty \mathcal{Y}) \\ &= \pi_{-n-m\alpha}^{\mathbb{A}^1}(X).\end{aligned}$$

and by the connectivity theorem, this will vanish whenever $-n < 0$, i.e., whenever $n > 0$. This is a stronger statement than [Voev98, theorem 4.14].

The line of attack is then to show that for a pointed space $\mathcal{Y}$, the S^1 suspsension spectrum $\Sigma_s^\infty \mathcal{Y}$ in the stable model category is -1 connected. Then we establish the first connectivity theorem which ensures the $\mathbb{A}^1$ localization of $\Sigma_s^\infty \mathcal{Y}$ is -1 connected. Finally, we show that inverting $\mathbb{G}_m$ does not affect the connectivity, i.e., $\Sigma_t^\infty \Sigma_s^\infty \mathcal{Y}$ is again -1 connected.

Remark 2. A construction of Ayoub [?] shows that this statement is false over general Noetherian base schemes S. But is indeed true over (infinite?) fields.

2. Assumptions from previous lectures

2.1. **Facts about Nisnevich topology.** Points and neighborhoods. Distinguished squares determine the topology. Relation to etale, zariski, and fpqc topology.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k, and let $X \in \mathrm{Sm}/k$ with Krull dimension d. Then whenever $n > d$, $H^n_{Nis}(X; M) = 0$.

Proposition 2.2. [Mor04, 2.4.1] For any $X \in \mathrm{Sm}/k$, and for any $x \in X(k)$, there is an isomorphism of pointed sheaves of sets in the Nisnevich topology

$$X/(X - \{x\}) \cong \mathbb{A}^n/(\mathbb{A}^n - \{0\}).$$

2.2. **Homotopy purity, connectedness calculations.**

Proposition 2.3. Weak n-connectedness is equivalent to n-connectedness

Proof. Details? □

Proposition 2.4. If V is an irreducible, smooth k-scheme, and $U \subseteq V$ is a dense open subset, the space $L_{\mathbb{A}^1}(V/U)$ is 0-connected.

Proof. Seems to use that local structure of spaces is given by $\mathbb{A}^n/(\mathbb{A}^n - \{0\}) \wedge L_+$. How do all the reductions work? Also uses some results specific to working over perfect fields. □

2.3. S^1**-spectra.**

Definition 2.5. Let $\mathrm{Spt}_s(k)$ denote the category of S^1-spectra of spaces $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k)$. We first endow this category with the projective model structure, i.e., a map $f : E \to F$ is a weak equivalence iff for any n the map $f_n : E_n \to F_n$ is a w.e.; a map $f : E \to F$ is a fibration iff for all n the map $f_n : E_n \to F_n$ is a fibration. The cofibrations are characterized by the property that $f : E \to F$ is a cofib iff $f_0 : E_0 \to F_0$ is a cofib and for any $n \geq 1$

$$\begin{tikzcd}
S^1 \wedge E_{n-1} \arrow[r, "\sigma_{n-1}"] \arrow[d, "S^1 \wedge f_{n-1}"'] & E_n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\
S^1 \wedge F_{n-1} \arrow[r] \arrow[drr, bend right, "\sigma_{n-1}"] & P.O. \arrow[dr, dotted] & \\
 & & F_n
\end{tikzcd}$$

This model structure does not actually invert $S^1 \wedge -$. To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.6. A map $f : E \to F$ of S^1-spectra is a stable equivalence iff for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \to \pi_n(F)$ is an isomorphism.

The stable model category structure on $\mathrm{Spt}_s(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such.

Definition 2.7. Let $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$ denote the category of S^1 spectra endowed with the stable model category structure localized at the collection of maps $\{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \to \Sigma^\infty U_+ \mid U \in \mathrm{Sm}/k\}$.

Let $\mathcal{SH}^{S^1}(k)$ denote the homotopy category associated to $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. We will use $\mathcal{SH}_s^{S^1}(k)$ to denote the homotopy category of $\mathrm{Spt}_s(k)$.

Remark 3. There is a functor L^∞ on the category of S^1 spectra which is similar to the unstable construction.

So we can use the functor L^∞ as an $\mathbb{A}^1$ localization functor. To be precise, we let $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ be the subcategory of $\mathrm{Spt}_s(k)$ consisting of the $\mathbb{A}^1$ local spectra. We may equip $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ with a model structure with weak equivalences the $\mathbb{A}^1$ weak equivalences, the cofibrations are the stable cofibrations, and the fibrations are what they have to be.

The functor $L^\infty : \mathrm{Spt}_s(k) \to \mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ is left adjoint to the inclusion functor, and is a left Quillen functor. The homotopy category of $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ is categorically equivalent to $\mathcal{SH}^{S^1}(k)$. [Mor05, Corollary 4.2.3].

For spectra E and F, we may compute $[E,F]^{\mathbb{A}^1} := \mathcal{SH}^{S^1}(k)(E,F)$ by calculating $[L^\infty E, L^\infty F]^{\mathbb{A}^1}$. Note that this is $\mathcal{SH}_s^{S^1}(k)(E, L^\infty F)$ by using the adjunction. If we assume E is cofibrant and $L^\infty F$ is fibrant, we get the formula

$$[E,F]^{\mathbb{A}^1} = \mathrm{Spt}_s(k)(E, L^\infty F).$$

Definition 2.8. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$. That is,

$$\pi_n(E) = \mathrm{colim}_r\, \pi_{n+r}(E_r).$$

In particular, for a $U \in \mathrm{Sm}/k$, we have

$$\pi_n(E)(U) = \mathrm{colim}_r\, \pi_{n+r}(E_r)(U).$$

Definition 2.9. An S^1-spectrum E is said to be n-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.10. There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \to \mathrm{Spt}_s(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ with the evident bonding maps. The right adjoint to this functor is given by "evaluation at 0", i.e., $\Omega^\infty(E) = E_0$.

Remark 4. The right derived functor $R\Omega^\infty : \mathcal{SH}_s^{S^1}(k) \to \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i\, \Omega_s^i E_i.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

Remark 5. We also get a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k) \to \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ given by the same formula as above.

What is the fibrant replacement functor in $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$?

Remark 6. The stable homotopy category is symmetric monoidal, with smash product $\wedge$ and internal hom $\underline{\mathrm{Hom}}$. Using symmetric spectra, one can give these constructions on the category of spectra.

The stable homotopy category is a triangulated category.

Proposition 2.11. Let $U \in \mathrm{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then there is a canonical isomoprhism

$$H^n_{Nis}(U; M) \to \mathcal{SH}^{S^1}(\Sigma^\infty U_+, HM[n]).$$

Proof. This is [Mor05, Lemma 3.2.3]. $\square$

2.4. *t*-structures.

Definition 2.12. Let $\mathfrak{C}$ be a triangulated category. A t-structure on $\mathfrak{C}$ is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

1. For any $X \in \mathfrak{C}_{\geq 0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
2. $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$

(3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \to X \to Z \to Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$..

The heart of a t-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.13 (t-structure on $\mathcal{SH}_s^{S^1}(k)$)**.** Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenever $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n > 0$.

Theorem 2.14. The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t-structure on $\mathcal{SH}_s^{S^1}(k)$.

2.5. **Connectivity results.**

Proposition 2.15. [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}_s^{S^1}(k) \to \mathrm{Spt}_{s,\mathbb{A}^1}^{S^1}(k)$ identifies the $\mathbb{A}^1$-localized S^1 stable homotopy category with the homotopy category of $\mathbb{A}^1$-local S^1 spectra.

Theorem 2.16 (S^1 stable connectivity theorem)**.** Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L_{\mathbb{A}^1} E = 0$.

Theorem 2.17. The pair $(\mathcal{SH}_{\geq 0}^{S^1}(k), \mathcal{SH}_{\leq 0}^{S^1}(k))$ is a t-structure on the category $\mathcal{SH}^{S^1}(k)$.

Definition 2.18. Strictly $\mathbb{A}^1$ invariant sheaf of Abelian groups.

If M is strictly $\mathbb{A}^1$ invariant sheaf of groups, define the Eilenberg-MacLane spectrum HM associated to it.

Proposition 2.19. HM is $\mathbb{A}^1$ local iff M is strictly $\mathbb{A}^1$ invariant.

Proposition 2.20. The heart of the homotopy t structure is equivalent to the category of strictly $\mathbb{A}^1$ invariant sheaves.

3. Inverting $\mathbb{G}_m \wedge -$; $\mathbb{P}^1$ spectra

3.1. $\mathbb{G}_m$ **suspension and loops.** We always consider $\mathbb{G}_m$ to be pointed at 1 unless otherwise specified.

Definition 3.1. On the category $\mathrm{Spt}_s(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps.

Smashing with $\mathbb{G}_m$ is also a functor on the unstable category of pointed spaces, and we give it the same name Σ_m.

Definition 3.2. The functor Σ_m on $\mathrm{Spc}_\bullet(k)$ has a right adjoint denoted Ω_t. It is given by the formula $\Omega_t \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \mathcal{X})$.

The functor Σ_t on $\mathrm{Spt}_s^{S^1}(k)$ also has a right adjoint Ω_t given by the internal hom functor, i.e., $\Omega_t E = \underline{\mathrm{Hom}}(\Sigma^\infty \mathbb{G}_m, E)$.

Proposition 3.3. The functor Σ_t is a left Quillen functor on $\mathrm{Spt}_s(k)$ and on $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. Furthermore, Σ_t is an exact functor on $\mathcal{SH}^{S^1}(k)$.

Lemma 3.4. Let $E \in \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ be a -1 connected spectrum. Then $\Sigma_t E$ is again -1 connected.

Proof. The claim is clear when $E = \Sigma_s^\infty \mathcal{X}$ a pointed space, since $\Sigma_t E = \Sigma_s^\infty \mathbb{G}_m \wedge \mathcal{X}$ is still a suspension spectrum, and so -1 connected.

Now consider a general -1 connected spectrum E. By [Mor05, Lemma 3.3.4], E is weak equivalent to $\mathrm{hocolim}\, E^i$ where $E^0 = *$, and for each n, there is a family $X_\alpha \in \mathrm{Sm}/k$ and natural numbers $n_\alpha \geq 0$ for which

$$\vee_\alpha \Sigma_s^\infty X_{\alpha,+}[n_\alpha - 1] \to E^{n-1} \to E^n$$

is an exact triangle. An induction argument establishes that $\Sigma_t E^n$ is still -1 connected for all n; hence $\Sigma_t E = \mathrm{hocolim}\, \Sigma_t E^n$ is also -1 connected. Should $\Sigma_t E$ fail to be $\mathbb{A}^1$-local, we may simply apply L^∞ to get an $\mathbb{A}^1$-local representative of $\Sigma_t E$. By the connectivity theorem, $L^\infty \Sigma_t E$ will again be -1 connected. □

3.2. **Contraction in** $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$**, category of pointed sheaves of sets.**

Definition 3.5. Let G be sheaf of pointed sets on Sm/k. The contraction of G is the sheaf $G_{-1} = G_{con}$ given by the formula

$$U \in \mathrm{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map ev_1 is the map induced by $ev_1 : \mathrm{Spec}(k) \to \mathbb{G}_m$, i.e., $k[x, x^{-1}] \to k$ given by $x \mapsto 1$.

Note that indeed G_{-1} is a sheaf since it is the kernel of the morphism of sheaves $G(-) \to G(- \times \mathbb{G}_m)$. The sheaf $G(- \times \mathbb{G}_m)$ may also be written as $\underline{\mathrm{Hom}}(\mathbb{G}_m, G)$ when we think of G as a space.

Proposition 3.6. If G is the trivial sheaf of abelian groups, then so is its contraction G_{-1}.

Proposition 3.7. Contraction is an exact functor on the category $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$. Furthermore, for any sheaf $G \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$ and any $X \in \mathrm{Sm}/k$, $G(\mathbb{G}_m \times X) = G_{-1}(X) \oplus G(X)$.

Proof. Functoriality can be established by using the universal property of kernels.

To show exactness, let $F \to G \to H$ be a short exact sequence of sheaves. We must show $F_{-1} \to G_{-1} \to H_{-1}$ is still exact. It suffices to check exactness at the level of stalks.

Since $\mathrm{id} = ev_1 \circ \pi : G(X) \to G(X \times \mathbb{G}_m) \to G(X)$, we have $ev_1 : G(-) \to G(- \times \mathbb{G}_m)$ is a surjection on all smooth schemes. Hence ev_1 is an epimorphism. For a Nisnevich point x of X, The following commutative diagram then establishes the exactness of the contraction by the 3x3 lemma.

$$\begin{tikzcd}
F_{-1,x} \arrow[r] \arrow[d, tail] & G_{-1,x} \arrow[r] \arrow[d, tail] & H_{-1,x} \arrow[d, tail] \\
F_{(x,1)} \arrow[r, tail] \arrow[d, two heads] & G_{(x,1)} \arrow[r, two heads] \arrow[d, two heads] & H_{(x,1)} \arrow[d, two heads] \\
F_x \arrow[r, tail] & G_x \arrow[r, two heads] & H_x
\end{tikzcd}$$

The second point follows since $\pi : X \times \mathbb{G}_m \to X$ induces a section of the evaluation at 1 map $ev_1 : G(X \times \mathbb{G}_m) \to G(X)$. □

3.3. **Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$.**

Proposition 3.8. If G is a sheaf of Abelian groups, then $G_{-1} \cong \underline{\mathrm{Hom}_\bullet}(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $- \wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

Proof. For this category, $\underline{\mathrm{Hom}_\bullet}(\mathbb{G}_m, G)$ and G_{-1} both have sections at X given by $ker(ev_1 : G(X \times \mathbb{G}_m) \to G(X))$. See description of pointed internal hom for this. □

Remark 7. If G is a sheaf of Abelian groups, we may consider G as a space by declaring $G_n = G$ for all n and giving identity maps for the structure maps. In particular, G is a pointed space at 0.

We can then realize the contraction as a $\mathbb{G}_m$ loop space $G_{-1}(X) = \underline{\mathrm{Hom}_\bullet}(\mathbb{G}_m, G)(X)$.

Remark 8. Construction of canonical map $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ for an S^1 spectrum E.

First observe that for any $U \in \mathrm{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\mathrm{Spt}_s(k)(S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma^\infty \mathbb{G}_m, E) \times \mathrm{Spc}(k)(U, \mathbb{G}_m) \to \pi_0(E)(U)$$

given by sending (f, α) to the composition

$$\Sigma_s^\infty U_+ \xrightarrow{\mathrm{id} \wedge \Sigma_s^\infty \alpha} S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \to \pi_n(E).$$

Does this map descend to the smash? Yes, since if either map is a constant map, then so is the composition.

We thus get a map of sheaves of pointed sets

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \to \pi_n(E).$$

But by the adjunction $- \wedge \mathbb{G}_m \dashv \underline{\mathrm{Hom}_\bullet}(\mathbb{G}_m, -)$ on $\mathrm{Spc}_\bullet(k)$ we have a morphism

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \underline{\mathrm{Hom}_\bullet}(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Why is it a map of sheaves of abelian groups?

Remark 9. If $E = HM$ is an Eilenberg-MacLane spectrum associated to a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups M, we show

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)) \to \underline{\mathrm{Hom}_\bullet}(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

Lemma 3.9. Let $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then $K(M, r)_{-1} \simeq K(M_{-1}, r)$.

Proof. Let $X \in \mathrm{Sm}/k$. The simplicial set $K(M, r)_{-1}(X)$ by the fibration[1]

$$K(M, r)_{-1}(X) \to K(M, r)(\mathbb{G}_m \times X) \to K(M, r)(X).$$

There is thus a long exact sequence of homotopy groups, which ammounts to

$$\pi_r K(M, r)_{-1}(X) \rightarrowtail \pi_r K(M, r)(\mathbb{G}_m \times X) \longrightarrow \pi_r K(M, r)(X) \twoheadrightarrow \pi_{r-1} K(M, r)_{-1}(X).$$

[1]Is this actually a fibration? We may assume $K(M, r)$ is always fibrant, but I don't see why ev_1 is a fibration. Can I work with the "derived contraction"? This argument works for that.

The remaining homotopy groups are evidently trivial. As $\pi_r K(M,r)(\mathbb{G}_m \times X) \to \pi_r K(M,r)(X)$ is surjective, we see $\pi_{r-1} K(M,r)_{-1}(X) = 0$. Therefore,

$$\pi_n K(M,r)_{-1}(X) = \begin{cases} 0 & \text{if } n \neq r \\ M_{-1}(X) & \text{if } n = r \end{cases}.$$

We conclude that $K(M,r)_{-1} \simeq K(M_{-1}, r)$. ☐

Proposition 3.10. For $M \in \underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$, the spectrum $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

Proof. Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(k)$, we have $\Sigma^\infty \mathbb{P}^1[-1] = \Sigma^\infty \mathbb{G}_m$. Therefore

$$\begin{aligned} \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM[n]) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1[-1], HM[n]) \\ &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1, HM[n+1]) \\ &= \tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M). \end{aligned}$$

As the cohomological dimension of $\mathbb{P}^1$ is less than or equal to 1, we then have $\tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$.

Now[2] for $U \in \mathrm{Sm}/k$, we calculate for $n \neq 0$

$$\begin{aligned} \pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(U) &= \mathcal{SH}_s^{S^1}(k)(S^n \wedge \Sigma^\infty \mathbb{G}_m \wedge \Sigma^\infty U_+, HM) \\ &= \mathrm{colim}_r\, \mathcal{H}^s_\bullet(k)(S^{n+r} \wedge U_+, \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, K(M,r))) \\ &= \mathrm{colim}_r\, \pi_{n+r}(K(M,r)_{-1})(U) \\ &= \mathrm{colim}_r\, \pi_{n+r}(K(M_{-1},r))(U) \\ &= \mathrm{colim}_r\, 0 \\ &= 0. \end{aligned}$$

To establish the second point, we calculate for $U \in \mathrm{Sm}/k$

$$\begin{aligned} \pi_0 \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(U) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m \wedge \Sigma^\infty U_+, HM) \\ &= \mathrm{colim}_r\, \mathcal{H}^s_\bullet(k)(S^{r} \wedge U_+, \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, K(M,r))) \\ &= \mathrm{colim}_r\, \pi_{r}(K(M,r)_{-1})(U) \\ &= \mathrm{colim}_r\, \pi_{r}(K(M_{-1},r))(U) \\ &= \mathrm{colim}_r\, M_{-1}(U) \\ &= M_{-1}(U) \\ &= \pi_0 H(M_{-1})(U). \end{aligned}$$

We now know that the associated homotopy sheaves $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all n. So they are weak equivalent by [Mor05, Lemma 3.2.5]. ☐

An alternate argument to see $H(M_{-1}) \cong \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ would be to use the identification of the heart of the homotopy t-structure.

[2] Given the following argument, is the part before this necessary? Does this argument actually work?

OK, so since $(-)_{-1}$ is defined as a kernel/equalizer in both $\underline{\mathrm{Ab}}(\mathrm{Sm}/k)$ and in $\mathcal{SH}_s^{S^1}(k)$ $\underline{\mathrm{Hom}}(\mathbb{G}_m, -)$ is also defined by the analogous equalizer, since $\pi_0 \dashv H(-)$ on the heart $\pi(k)$ we get that $H(M_{-1}) \cong \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$.

Proposition 3.11. For any spectrum $E \in \mathcal{SH}^{S^1}(k)$, the homotopy sheaves of $\underline{\mathrm{Hom}}(\mathbb{G}_m, E)$ are calculated by $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \cong \pi_n(E)_{-1}$

Proof. Reduce to the case of Eilenberg-MacLane spectra by using the Postnikov tower. Need $\underline{\mathrm{Hom}}(\mathbb{G}_m, -)$ to be exact functor on $\mathcal{SH}^{S^1}(k)$. Is it? Details? $\square$

3.4. **Inverting $\mathbb{G}_m \wedge -$; $(\mathbb{G}_m, S^1)$ bi-spectra.** The functor Σ_t on $\mathrm{Spt}_s(k)$ is a left Quillen functor. We may therefore apply the general machinery of [H-Spt] to create a model category in which Σ_t is invertible. The construction of Hovey may be described as $(\mathbb{G}_m, S^1)$ bispectra.

Definition 3.12. A $(\mathbb{G}_m, S^1)$ bi-spectrum of spaces over k consists of a bigraded family of spaces $E_{i,j}$, $i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \to E_{i,j+1}$ and $\mu_{i,j} : \mathbb{G}_m \wedge E_{i,j} \to E_{i+1,j}$ for which the following diagram commutes .

$$\begin{tikzcd}
S^1 \wedge \mathbb{G}_m \wedge E_{i,j} \arrow[r, "S^1 \wedge \tau_{i,j}"] \arrow[d] & S^1 \wedge E_{i+1,j} \arrow[dd, "\sigma_{i+1,j}"] \\
\mathbb{G}_m \wedge S^1 \wedge E_{i,j} \arrow[d, "\mathbb{G}_m \wedge \sigma_{i,j}"'] & \\
\mathbb{G}_m \wedge E_{i,j+1} \arrow[r, "\mu_{i,j+1}"] & E_{i+1,j+1}
\end{tikzcd}$$

Remark 10. Note that a $(\mathbb{G}_m, S^1)$ bispectrum is just a $\mathbb{G}_m$-spectrum of S^1-spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a $(\mathbb{G}_m, S^1)$ bi-spectrum $E_{i,j}$ as a sequence of S^1 spectra $E_{i,*}$.

Definition 3.13. Let E be a $(\mathbb{G}_m, S^1)$ bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \mathrm{Sm}/k \mapsto \mathrm{colim}_r\, \mathcal{H}_\bullet(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$.
We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

Proposition 3.14. If is E a $(\mathbb{G}_m, S^1)$ bispectrum, the presheaf of homotopy groups may also be calculated as

$$\tilde{\pi}_{n+m\alpha}E(U) = \mathrm{colim}_{s,r}\, \mathcal{H}_\bullet(k)(\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[DLØRV, p 217]

Definition 3.15. A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable weak equivalence if the following induced map is an isomorphism for all $U \in \mathrm{Sm}/k$.

$$f_* : \tilde{\pi}_{n+m\alpha}(E)(U) \to \tilde{\pi}_{n+m\alpha}(F)(U)$$

Definition 3.16. A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable cofibration if $f_0 : E_{0,*} \to F_{0,*}$ is a cofibration of S^1 spectra and the map $P.O. \to F_{n+1}$ is a cofibration in the following diagram.

$$\begin{tikzcd}
\mathbb{G}_m \wedge E_{n,*} \arrow[r] \arrow[d] & E_{n+1,*} \arrow[d] \arrow[ddr, bend left, "f_{n+1}"] & \\
\mathbb{G}_m \wedge F_{n,*} \arrow[r] \arrow[drr, bend right] & P.O. \arrow[dr, dotted] & \\
& & F_{n+1,*}
\end{tikzcd}$$

Proposition 3.17. The category $\mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$ of $(\mathbb{G}_m, S^1)$ bispectra with $\mathbb{A}^1$ stable weak equivalences and $\mathbb{A}^1$ stable cofibrations is a model category. Denote the associated homotopy category of $\mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$ by $\mathcal{SH}(k)$.

Proposition 3.18. The fibrant bi-spectra are the Ω_t-spectra. [H-Spt, Theorem 3.4]

Proposition 3.19. There is a left Quillen functor $\Sigma^\infty_t : \mathrm{Spt}^{\mathbb{A}^1}_s(k) \to \mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$ given by $(\Sigma^\infty_t E)_{i,j} = \mathbb{G}^i_m \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}^i_m \wedge E_j \longrightarrow \mathbb{G}^i_m \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}^m \wedge \mathbb{G}^i_m \wedge E_j \to \mathbb{G}^{i+1}_m E_j.$$

The right adjoint to Σ^∞_t is denoted by Ω^∞_t and is given by $\Omega^\infty_t(E) = E_{0,*}$.

The right derived functor $R\Omega^\infty_t(E)$ is given by the formula

$$R\Omega^\infty_t(E) = \mathrm{colim}_i\, \Omega^i_t E_{i,*}.$$

Proof. □

3.5. **Connectivity of $(\mathbb{G}_m, S^1)$ bispectra.**

Definition 3.20. A $(\mathbb{G}_m, S^1)$ bispectrum E is said to be n-connected if for all $k \leq n$ and all $m \in \mathbb{Z}$, the homotopy sheaves $\pi_{k+m\alpha}E$ vanish.

Lemma 3.21. Consider $E = \Sigma^\infty_t \Sigma^\infty_s \mathcal{X}$ for some pointed space $\mathcal{X}$. Then E is -1 connected.

Proof. We calculate for $n < 0$

$$\begin{aligned}
\mathcal{SH}(k)(S^n \wedge \mathbb{G}^m_m \wedge \Sigma^\infty U_+, E) &= \mathcal{SH}(k)(S^n \wedge \Sigma^\infty U_+, \Omega^m_t E) \\
&= \mathcal{SH}^{S^1}(k)(S^n \wedge \Sigma^\infty U_+, R\Omega^\infty_t \Omega^m_t E) \\
&= \mathcal{SH}^{S^1}(k)(S^n \wedge \Sigma^\infty U_+, \mathrm{colim}_i\, \Omega^{i+m}_t \Sigma^i_t \Sigma^\infty_s X) \\
&= \mathrm{colim}\, \mathcal{SH}^{S^1}(k)(S^n \wedge \Sigma^\infty U_+, \Omega^{m+i}_t \Sigma^i_t \Sigma^\infty_s X) \\
&= 0
\end{aligned}$$

as Ω_t preserves connectivity, as does Σ_t for suspension spectra. □

Proposition 3.22. Let $E \in \mathrm{Spt}^{\mathbb{A}^1}_s(k)$. Consider $\Sigma^\infty_t E \in \mathrm{Spt}^{\mathbb{A}^1}_{s,t}(k)$. If E is -1 connected, then so too is $\Sigma^\infty_t E$.

Proof. We calculate

$$\begin{aligned}
\pi_{n+m\alpha}(\Sigma_t^\infty E) &= \pi_n(R\Omega_t^\infty \Omega_t^m \Sigma_t^\infty E) \\
&= \pi_n(\operatorname{colim}_i \Omega_t^{m+i}\Sigma_t^i E) \\
&= \operatorname{colim}_i \pi_n(\Sigma_t^i E)_{-(m+i)} \\
&= 0.
\end{aligned}$$

This follows since $\Sigma_t E$ is -1 connected whenever E is -1 connected, and the effect of Ω_t^{m+i} on homotopy sheaves is contraction. □

3.6. **t-structure on $\mathcal{SH}(k)$.**

Definition 3.23. Let $\mathcal{SH}(k)_{\geq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha}E = 0$ whenever $n < 0$.

Let $\mathcal{SH}(k)_{\leq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $pi_{n+m\alpha}E = 0$ whenever $n > 0$.

Theorem 3.24. The triple $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ defines a t-structure.

Proof. What needs to be done:

(1) Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. Show $\mathcal{SH}(k)(E, F[-1]) = 0$.
(2) Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. Show $E[1] \in \mathcal{SH}(k)_{\geq 0}$ and $F[-1] \in \mathcal{SH}(k)_{\leq 0}$. This is clear by invertibility of $[1]$ in $\mathcal{SH}(k)$.
(3) Given $E \in \mathcal{SH}(k)$, construct $E_{\geq 0}$ and $E_{<0}$ which fit into an exact triangle

$$E_{\geq 0} \to E \to E_{<0} \to E_{\geq 0}[1].$$

□

Definition 3.25. A homotopy module over k is a pair (M_*, μ_*) consisting of a $\mathbb{Z}$ graded strictly $\mathbb{A}^1$ invariant sheaf M_* and an isomorphism $\mu_n : M_n \cong (M_{n+1})_{-1}$.

Lemma 3.26. If E is a bi-spectrum, then

$$R\Omega_t^\infty E \to \underline{\mathrm{Hom}}(\mathbb{G}_m, R\Omega_t^\infty(E \wedge \mathbb{G}_m))$$

is an isomorphism.

Proof. □

Lemma 3.27. Let $E \in \mathcal{SH}(k)$. For a fixed $n \in \mathbb{Z}$, the collection $\pi_n(E)_m$ forms a homotopy module.

Lemma 3.28. If (M_*, μ_*) is a homotopy module over k, then there is a $(\mathbb{G}_m, S^1)$ bispectrum HM_* with $(HM_*)_{n,n} = K(M_n, n)$ with evident structure maps.

Proof. □

Theorem 3.29. The heart of the t-structure $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is denoted $\pi_*^{\mathbb{A}^1}(k)$ and is equivalent to the category of homotopy modules. The equivalence is given explicitly by the functors $\pi_0(-)_*$ and $H(-)$.

Proof. □

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