

1. VOEVODSKY'S CONNECTIVITY THEOREM FOR \mathbb{P}^1 -SPECTRA

Our goal is to prove theorem 4.14 of [?], which we restate in terms of \mathbb{P}^1 -spectra.

Theorem 1.1. Let (X, x) be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let \mathcal{Y} be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

We will prove this theorem by following [?] and [?] by Fabien Morel.

Remark 1. To prove this theorem, Morel carefully analyzes how to pass from spaces in the projective model structure to the \mathbb{A}^1 stable homotopy category of \mathbb{P}^1 spectra. From the projective model structure on spaces, we construct a model of the left Bousfield localization of spaces at the class of maps $\{U_+ \wedge \mathbb{A}^1 \rightarrow U \mid U \text{ in } \mathrm{Sm}/k\}$. To get to \mathbb{P}^1 spectra, we first invert $S^1 \wedge -$ to get a category of S^1 spectra, and then we invert $\mathbb{G}_m \wedge -$ to get a category of (\mathbb{G}_m, S^1) bispectra.

$$\mathcal{H}_{s,\bullet}(k) \rightarrow \mathcal{H}_\bullet(k) \rightarrow \mathcal{SH}^{S^1}(k) \rightarrow \mathcal{SH}(k)$$

The machinery that we set up to prove this theorem will also allow us to establish a t -structure on $\mathcal{SH}(k)$, and identify its heart.

Remark 2. A construction of Ayoub [?] shows that theorem 1.1 statement is false over general Noetherian base schemes S . The argument below works for infinite fields, however.

2. ASSUMPTIONS FROM PREVIOUS LECTURES

We briefly recall some of the basic constructions which appear in [?] and [?].

2.1. Facts about Nisnevich topology. The proof of Voevodsky's connectivity theorem will follow from the following property of Nisnevich sheaf cohomology by a sequence of reductions.

Proposition 2.1. [?, 2.4.1] Let M be a sheaf of abelian groups on Sm/k , and let $X \in \mathrm{Sm}/k$ with Krull dimension d . Then whenever $n > d$, $H_{Nis}^n(X; M) = 0$.

2.2. Unstable model category $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$.

Definition 2.2. Let k be a field, and let Sm/k denote the category of smooth schemes of finite type over k . The category of Morel-Voevodsky spaces over k is the category of simplicial Nisnevich sheaves on Sm/k . We write $\mathrm{Spc}(k) = \Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$ for this category.

The category $\mathrm{Spc}(k)$ may be equipped with several different model category structures. We will work with the injective local model category structure on $\mathrm{Spc}(k)$, which we now define.

Definition 2.3. A map $\mathcal{X} \rightarrow \mathcal{Y}$ is an injective weak equivalence if and only if for any $U \in \mathrm{Sm}/k$, the map $\mathcal{X}(U) \rightarrow \mathcal{Y}(U)$ is a weak equivalence of simplicial sets.

A map $\mathcal{X} \rightarrow \mathcal{Y}$ is an injective cofibration if and only if for any $U \in \mathrm{Sm}/k$, the map $\mathcal{X}(U) \rightarrow \mathcal{Y}(U)$ is a cofibration of simplicial sets, i.e., a monomorphism.

A map $\mathcal{X} \rightarrow \mathcal{Y}$ is an injective fibration if and only if it satisfies the left lifting property with respect to any trivial injective cofibration. That is, for any commutative square below with $\mathcal{A} \rightarrow \mathcal{B}$ a trivial cofibration, a lift $\mathcal{B} \rightarrow \mathcal{X}$ exists.

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{X} \\ \downarrow \sim & \nearrow & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{Y} \end{array}$$

Denote the homotopy category associated to the injective model category structure on $\mathrm{Spc}(k)$ by $\mathcal{H}_s(k)$. The “s” stands for simplicial.

Definition 2.4. The category of pointed space $\mathrm{Spc}_\bullet(k)$ inherits a model category structure from $\mathrm{Spc}(k)$. The functor $-_+ : \mathrm{Spc}(k) \rightarrow \mathrm{Spc}_\bullet(k)$ defined by adding a disjoint basepoint to a given space is a left Quillen functor. The right adjoint is the forgetful functor.

Proposition 2.5. Every object of $\mathrm{Spc}(k)$ and $\mathrm{Spc}_\bullet(k)$ is cofibrant in the injective model category structure.

Definition 2.6. For $X \in \mathrm{Sm}/k$, let rX denote the sheaf associated to the presheaf $U \mapsto \mathrm{Sm}/k(U, X)$. This defines a functor $r : \mathrm{Sm}/k \rightarrow \mathrm{Spc}(k)$.

For K a simplicial set, the constant space $cK \in \mathrm{Spc}(k)$ is the sheaf associated to the constant presheaf with value K . The functor $c : \mathbf{sSet} \rightarrow \mathrm{Spc}(k)$ is a left Quillen functor with right adjoint given by taking sections at $\mathrm{Spec} k$.

Proposition 2.7. $\mathrm{Spc}(k)$ is a simplicial model category.

See [?, Chapter 2] for a detailed treatment of the products and internal hom constructions in $\mathrm{Spc}(k)$. We recount those definitions which are essential to our argument.

Definition 2.8. For spaces \mathcal{X} and \mathcal{Y} , the product $\mathcal{X} \times \mathcal{Y}$ in $\mathrm{Spc}(k)$ is given by $U \mapsto \mathcal{X}(U) \times \mathcal{Y}(U)$. For spaces \mathcal{X} and \mathcal{Y} , the internal hom $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ in $\mathrm{Spc}(k)$ is given by the formula

$$(U, m) \in \mathrm{Sm}/k \times \Delta \mapsto \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Shv}}(X \times rU \times c\Delta^n, Y).$$

Proposition 2.9. The product and internal hom defined above give $\mathrm{Spc}(k)$ the structure of a closed monoidal model category. See [?, Chapter 4] or [?, §1.7] for the definition.

Proof. The adjunction between $\mathcal{X} \times -$ and $\underline{\mathrm{Hom}}(\mathcal{X}, -)$ is given by the following map.

$$\eta : \mathrm{Hom}(\mathcal{Y}, \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Z})) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$$

For $g \in \mathrm{Hom}(\mathcal{Y}, \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Z}))$, we define $\eta(g)$ by

$$\begin{aligned} \mathcal{X}_n(U) \times \mathcal{Y}_n(U) &\xrightarrow{\eta(g)(U, n)} \mathcal{Z}_n(U) \\ (a, b) &\longmapsto g(U, n)(b)(U, n)(a, \mathrm{id}_U, \mathrm{id}_{\Delta^n}). \end{aligned}$$

For $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$, the map $(\eta^{-1}f)(U, n) : \mathcal{Y}_n(U) \rightarrow \underline{Hom}(\mathcal{X}, \mathcal{Z})_n(U)$ is given by sending $y \in \mathcal{Y}_n(U)$ to the map

$$\begin{aligned} \mathcal{X}_m(V) \times \mathrm{Sm}/k(V, U) \times \Delta_m^n &\xrightarrow{(\eta^{-1}f)(U, n)(y)(V, m)} \mathcal{Z}_m(V) \\ (x, \phi, \alpha) &\longmapsto f(V, m)(x, \mathcal{Y}(\phi)(y \circ \alpha)) \end{aligned}$$

where we identify y with a map $y : \Delta^n \rightarrow \mathcal{Y}(V)$, and $\alpha : \Delta^m \rightarrow \Delta^n$. \square

Definition 2.10. Let \mathcal{X} and \mathcal{Y} be spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{Hom}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y}$, where at $(U, n) \in (\mathrm{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times rU \times c\Delta^n \rightarrow \mathcal{Y}$ to $g(U, n)(x, \mathrm{id}, \mathrm{id}) \in \mathcal{Y}_n(U)$.

For pointed spaces (\mathcal{X}, x) and (\mathcal{Y}, y) , the pointed internal hom $\underline{Hom}_\bullet(\mathcal{X}, \mathcal{Y})$ is the fiber of ev_x over y , i.e., $ev_x^{-1}(y)$.

Definition 2.11. Let (\mathcal{X}, x) and (\mathcal{Y}, y) be pointed spaces. The wedge of \mathcal{X} and \mathcal{Y} , denoted by $\mathcal{X} \vee \mathcal{Y}$, is the pushout of the following diagram.

$$\begin{array}{ccc} pt & \xrightarrow{x} & \mathcal{X} \\ \downarrow y & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \vee \mathcal{Y} \end{array}$$

The smash product $\mathcal{X} \wedge \mathcal{Y}$ is the space given by the pushout of the following diagram, with basepoint $\mathcal{X} \vee \mathcal{Y}$.

$$\begin{array}{ccc} \mathcal{X} \vee \mathcal{Y} & \longrightarrow & \mathcal{X} \times \mathcal{Y} \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathcal{X} \wedge \mathcal{Y} \end{array}$$

Proposition 2.12. The category of pointed spaces $\mathrm{Spc}_\bullet(k)$ is also a closed monoidal category with product \wedge and internal hom \underline{Hom}_\bullet .

2.3. \mathbb{A}^1 localization.

Definition 2.13. A space \mathcal{X} is called \mathbb{A}^1 local if for any smooth scheme U , the canonical map

$$\mathrm{Hom}(rU, \mathcal{X}) \rightarrow \mathrm{Hom}(rU \times \mathbb{A}^1, \mathcal{X})$$

is a bijection.

Definition 2.14. A map $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an \mathbb{A}^1 weak equivalence if

$$\mathrm{Hom}(\mathcal{Y}, \mathcal{Z}) \rightarrow \mathrm{Hom}(\mathcal{X}, \mathcal{Z})$$

is a bijection for every \mathbb{A}^1 local space \mathcal{Z} .

The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \rightarrow U \mid U \in \mathrm{Sm}/k\}$. We denote the category of spaces with the model structure obtained by left Bousfield localization by $L_W \mathrm{Spc}(k)$ and its homotopy category by $\mathcal{H}(k)$. See [?, Chapter 3] for the general theory of Bousfield localization. One thing we obtain is a localization functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \rightarrow \mathcal{H}(k)$

which is a left Quillen functor. In particular, $L_{\mathbb{A}^1}$ sends \mathbb{A}^1 weak equivalences to isomorphisms.

The model category $L_W\mathrm{Spc}(k)$ is constructed as follows. The underlying category of $L_W\mathrm{Spc}(k)$ is $\mathrm{Spc}(k)$, but the weak equivalences are the \mathbb{A}^1 -local weak equivalences. The cofibrations are the cofibrations in the injective model structure on $\mathrm{Spc}(k)$. The fibrations are what they need to be, i.e., those maps which satisfy the left lifting property with respect to trivial cofibrations.

In order to effectively work with the $\mathcal{H}(k)$, we require a means of constructing fibrant replacements in $L_W\mathrm{Spc}(k)$. Morel accomplishes this by constructing another model category with homotopy category $\mathcal{H}(k)$.

Definition 2.15. Let $\mathrm{Spc}^{\mathbb{A}^1}(k)$ denote the full subcategory of $\mathrm{Spc}(k)$ of \mathbb{A}^1 local spaces. The inclusion $\mathrm{Spc}^{\mathbb{A}^1}(k) \rightarrow \mathrm{Spc}(k)$ is a left Quillen functor with left adjoint L^∞ . See [?, Proposition 3.2.3] for the construction.

Proposition 2.16. The homotopy category of $\mathrm{Spc}^{\mathbb{A}^1}(k)$ is equivalent to $\mathcal{H}(k)$.

Definition 2.17. Let \mathcal{X} be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on Sm/k associated to $U \rightarrow \pi_0(\mathcal{X}(U))$. A space \mathcal{X} is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let (\mathcal{X}, x) be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on Sm/k given by

$$U \rightarrow \pi_n(\mathcal{X}(U)).$$

A pointed space \mathcal{X} is called n -connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

Proposition 2.18. Let \mathcal{X} be a 0-connected simplicial sheaf. Then $L^\infty\mathcal{X}$ is also 0-connected.

For a sheaf of abelian groups M on Sm/k and a natural number n , a Dold-Kan construction gives a simplifical presheaf $K(M, n)$. It is called the Eilenberg-MacLane spectrum of type (M, n) and has homotopy sheaves as expected.

$$\pi_m(K(M, n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}$$

Proposition 2.19. For $X \in \mathrm{Sm}/k$ and M a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M, n)) \cong H_{Nis}^n(X; M).$$

It therefore follows that

$$\mathcal{H}_\bullet(k)(rX_+, K(M, n)) \cong H_{Nis}^n(X; M).$$

Notation 1. For a pointed space \mathcal{X} , Let $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ denote the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^\infty\mathcal{X})$. The sheaf $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is also the sheafification of the presheaf given by

$$U \in \mathrm{Sm}/k \mapsto \mathcal{H}_\bullet(k)(S^n \wedge U_+, \mathcal{X}).$$

2.4. Weak connectedness.

Proposition 2.20. Let k be an infinite field, and consider \mathcal{X} be a pointed space. If for any finitely generated field F over k , $\pi_0(\mathcal{X})(F) = 0$, then the sheaf $\pi_0(\mathcal{X})$ is trivial.

For spaces or spectra??

Proof. The proof follows along the lines of [?, Lemma 6.1.3]. \square

2.5. S^1 spectra. The functor $\Sigma_s : \mathrm{Spc}_\bullet(k) \rightarrow \mathrm{Spc}_\bullet(k)$ given by $\Sigma_s \mathcal{X} = S^1 \wedge \mathcal{X}$ is a left Quillen functor on $\mathrm{Spc}_\bullet(k)$, with right adjoint Ω_s , where $\Omega_s \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(S^1, \mathcal{X})$. This follows since S^1 is a cofibrant object of $\mathrm{Spc}_\bullet(k)$. Note, however, that the derived functor Σ_s is not an equivalence of homotopy categories. We may invert this functor, i.e., make a new category where Σ_s is an equivalence of homotopy categories, by creating a category of S^1 spectra by using the general machinery developed in [?]. Here the “s” in Σ_s and Ω_s stands for “simplicial circle”.

Definition 2.21. Let $\mathrm{Spt}^{S^1}(k)$ denote the category of S^1 spectra of spaces over k . An object $E \in \mathrm{Spt}^{S^1}(k)$ is a sequence of pointed spaces $E_i \in \mathrm{Spc}_\bullet(k)$ equipped with bonding maps $\sigma_i : S^1 \wedge E_i \rightarrow E_{i+1}$. A map of spectra $f : E \rightarrow F$ consists of a sequence of maps of spaces $f_i : E_i \rightarrow F_i$ which are compatible with the bonding maps.

We first endow this category with the projective model structure (or level-wise model structure), i.e., a map $f : E \rightarrow F$ is a weak equivalence if for any n the map $f_n : E_n \rightarrow F_n$ is a w.e.; a map $f : E \rightarrow F$ is a fibration if for all n the map $f_n : E_n \rightarrow F_n$ is a fibration. The cofibrations are those maps satisfying the right lifting property with respect to trivial fibrations.

The projective cofibrations have the following characterization [?, Proposition 1.15]. A map $f : E \rightarrow F$ is a projective cofibration if and only if $f_0 : E_0 \rightarrow F_0$ is a cofibration and for any $n \geq 1$, the dotted arrow in the diagram below is a cofibration. Here $P.O.$ denotes the push-out of the diagram.

$$\begin{array}{ccc}
 S^1 \wedge E_{n-1} & \xrightarrow{\sigma_{n-1}} & E_n \\
 \downarrow S^1 \wedge f_{n-1} & & \downarrow \\
 S^1 \wedge F_{n-1} & \longrightarrow & P.O. \\
 & \searrow \sigma_{n-1} & \downarrow f_n \\
 & & F_n
 \end{array}$$

This model structure does not actually invert Σ_s . To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.22. A map $f : E \rightarrow F$ of S^1 spectra is a stable equivalence if for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \rightarrow \pi_n(F)$ is an isomorphism.

The stable model category structure on $\mathrm{Spt}^{S^1}(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the

same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such. Consult [?] for more details.

Denote the homotopy category of $\mathrm{Spt}^{S^1}(k)$ by $\mathcal{SH}_s^{S^1}(k)$.

Definition 2.23. Consider the class of maps $W = \{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \rightarrow \Sigma^\infty U_+ \mid U \in \mathrm{Sm}/k\}$ in $\mathrm{Spt}^{S^1}(k)$. The left Bousfield localization of $\mathrm{Spt}^{S^1}(k)$ with respect to W exists, and we write $L_W \mathrm{Spt}^{S^1}(k)$ for the resulting model category. Denote the homotopy category associated to $L_W \mathrm{Spt}^{S^1}(k)$ by $\mathcal{SH}^{S^1}(k)$.

Remark 3. Let $\mathrm{Spt}^{S^1, \mathbb{A}^1}(k)$ denote the full subcategory of $\mathrm{Spt}^{S^1}(k)$ consisting of \mathbb{A}^1 -local spectra. The inclusion $\mathrm{Spt}^{S^1, \mathbb{A}^1}(k) \rightarrow \mathrm{Spt}^{S^1}(k)$ has a left adjoint L^∞ which has a similar construction as was considered for spaces. This adjunction is indeed a Quillen functor, and so just as in the unstable setting, we may identify the stable \mathbb{A}^1 homotopy category with the homotopy category of the \mathbb{A}^1 -local S^1 spectra. See [?, §4.2] for details.

For S^1 spectra E and F , we calculate the stable \mathbb{A}^1 homotopy group $\mathcal{SH}^{S^1}(k)(E, F)$ by

$$\begin{aligned} \mathcal{SH}^{S^1}(k)(E, F) &= \mathcal{SH}^{S^1}(k)(L^\infty E, L^\infty F) \\ &= \mathcal{SH}_s^{S^1}(k)(E, L^\infty F) \end{aligned}$$

Here we consider the model for $\mathcal{SH}^{S^1}(k)$ given by Bousfield localization, then translate to the category of \mathbb{A}^1 local spectra using L^∞ . The second equality follows from the Quillen adjunction $\mathrm{Spt}^{S^1, \mathbb{A}^1}(k) \hookrightarrow \mathrm{Spt}^{S^1}(k)$.

If we assume E is cofibrant and F is fibrant, we get the formula

$$\mathcal{SH}^{S^1}(k)(E, F) = \mathrm{Spt}^{S^1}(k)(E, L^\infty F).$$

Definition 2.24. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, \mathrm{Nis})$. That is,

$$\pi_n(E) = \mathrm{colim}_r \pi_{n+r}(E_r).$$

In particular, for a $U \in \mathrm{Sm}/k$, we have

$$\pi_n(E)(U) = \mathrm{colim}_r \pi_{n+r}(E_r)(U).$$

Definition 2.25. An S^1 spectrum E is said to be n -connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.26. There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \rightarrow \mathrm{Spt}^{S^1}(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ where the bonding maps come from associativity of smash product. The right adjoint to this functor is given by “evaluation at 0”, i.e., $\Omega^\infty(E) = E_0$.

Remark 4. The right derived functor $R\Omega^\infty : \mathcal{SH}_s^{S^1}(k) \rightarrow \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i \Omega_s^i E_i.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

Remark 5. The left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k) \rightarrow \mathrm{Spt}^{S^1}(k)$ factors through the category of \mathbb{A}^1 -local spaces. This follows by [?, Remark 4.1.3]. Furthermore, since a map $f \in \mathrm{Spt}^{S^1}(k)(\Sigma_s^\infty \mathcal{X}, E)$ is determined by $f_0 : \mathcal{X} \rightarrow E_0$ by the adjunction $\Sigma_s^\infty \dashv \Omega_s^\infty$, one can show that $\Sigma_s^\infty \mathcal{X} \rightarrow \Sigma^\infty L^\infty \mathcal{X}$ is an \mathbb{A}^1 -weak equivalence. So in the case of suspension spectra, we may use $\Sigma^\infty L^\infty \mathcal{X}$ as an \mathbb{A}^1 localization. In particular, for any $n \in \mathbb{Z}$ the sheaf $\pi_n^{\mathbb{A}^1} \Sigma_s^\infty \mathcal{X}$ is isomorphic to $\pi_n \Sigma^\infty L^\infty \mathcal{X}$.

Remark 6. The stable homotopy category is symmetric monoidal, with smash product \wedge and internal hom $\underline{\mathrm{Hom}}$. Using symmetric spectra, one can give these constructions on the category of spectra [?]. The S^1 stable homotopy category is a triangulated category. The shift is given by S^1 suspension, and distinguished triangles are those triangles isomorphic to the cone of a map

$$X \xrightarrow{f} Y \rightarrow C(f) \rightarrow X[1].$$

Proposition 2.27. Let $U \in \mathrm{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then there is a canonical isomorphism

$$H_{Nis}^n(U; M) \rightarrow \mathcal{SH}^{S^1}(\Sigma^\infty U_+, HM[n]).$$

This is [?, Lemma 3.2.3].

2.6. t -structures.

Definition 2.28. Let \mathfrak{C} be a triangulated category. A t -structure on \mathfrak{C} is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

- (1) For any $X \in \mathfrak{C}_{\geq 0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
- (2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
- (3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \rightarrow X \rightarrow Z \rightarrow Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$.

The heart of a t -structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.29. Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenever $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenever $n > 0$.

Theorem 2.30. The pair $(\mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t -structure on $\mathcal{SH}_s^{S^1}(k)$.

Remark 7. For a space \mathcal{X} , there is a Postnikov tower associated to it

$$\cdots P^n(\mathcal{X}) \rightarrow P^{n-1}(\mathcal{X}) \rightarrow \cdots \rightarrow P^0(\mathcal{X}) \rightarrow P^{-1}(\mathcal{X})$$

constructed in [?, p. 57]. The main construction needed is the Moore-Postnikov tower of a simplicial set [?, VI.3.4]. For a simplicial set K and $n \in \mathbb{N}$, define $K^{(n)} = \mathrm{im}(K \rightarrow \mathrm{cosk}_n K)$. This is a convenient way to define the Moore construction.

For a space \mathcal{X} , we then define $P^n \mathcal{X}$ to be the space given by sheafification of $U \mapsto \mathcal{X}(U)^{(n)}$.

Now for E an S^1 spectrum, let $E_{\leq 0}$ be the spectrum with $(E_{\leq 0})_n = P^n(E_n)$. The bonding maps come from the canonical map

$$S^1 \wedge P^n(E_n) \rightarrow P^{n+1}(S^1 \wedge E_n).$$

See [?, Lemma 3.2.1] for more on this construction.

2.7. Connectivity results.

Proposition 2.31. [?, Lemma 4.2.4] The functor $L^\infty : \mathrm{Spt}^{S^1}(k) \rightarrow \mathrm{Spt}^{S^1, \mathbb{A}^1}(k)$ identifies the \mathbb{A}^1 -localized S^1 stable homotopy category with the homotopy category of \mathbb{A}^1 -local S^1 spectra.

Theorem 2.32. Let k be an infinite field. Consider $E \in \mathcal{SH}^{S^1}(k)$ and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L^\infty E = 0$.

Theorem 2.33. The pair $(\mathcal{SH}_{\geq 0}^{S^1}(k), \mathcal{SH}_{\leq 0}^{S^1}(k))$ is a t -structure on the category $\mathcal{SH}^{S^1}(k)$.

Proof. This is just the restriction of the t -structure to the \mathbb{A}^1 -local objects. \square

Definition 2.34. Let M be a sheaf of Abelian groups on Sm/k with respect to the Nisnevich topology. We say M is strictly \mathbb{A}^1 invariant if for all $n \geq 0$ and all $X \in \mathrm{Sm}/k$, the map $H_{Nis}^n(X; M) \rightarrow H_{Nis}^m(X \times \mathbb{A}^1; M)$ is an isomorphism. Let $\underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$ denote the full subcategory of sheaves of Abelian groups on Sm/k in the Nisnevich topology consisting of the strictly \mathbb{A}^1 invariant sheaves.

Definition 2.35. If $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$ is a sheaf of Abelian groups, the Eilenberg-MacLane spectrum associated to M is the S^1 spectrum HM given by $HM_n = K(M, n)$. The bonding maps come from the usual identification of $\Omega_s K(M, n) \cong K(M, n-1)$.

Proposition 2.36. HM is \mathbb{A}^1 local iff M is strictly \mathbb{A}^1 invariant.

Proposition 2.37. The heart of the homotopy t structure is equivalent to the category of strictly \mathbb{A}^1 invariant sheaves.

3. INVERTING $\mathbb{G}_m \wedge -$; \mathbb{P}^1 SPECTRA

3.1. \mathbb{G}_m suspension and loops. We always consider \mathbb{G}_m to be pointed at 1 unless otherwise specified.

Definition 3.1. On the category $\mathrm{Spt}^{S^1}(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps. Smashing with \mathbb{G}_m is also a functor on the unstable category of pointed spaces, and we give it the same name Σ_t .

Definition 3.2. The functor Σ_t on $\mathrm{Spc}_\bullet(k)$ has a right adjoint denoted Ω_t . It is given by the formula $\Omega_t \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \mathcal{X})$.

The functor Σ_t on $\mathrm{Spt}^{S^1}(k)$ also has a right adjoint Ω_t given by the internal hom functor, i.e., $\Omega_t E = \underline{\mathrm{Hom}}(\Sigma^\infty \mathbb{G}_m, E)$.

Proposition 3.3. The functor Σ_t is a left Quillen functor on $\mathrm{Spt}^{S^1}(k)$ and on $\mathrm{Spt}^{S^1, \mathbb{A}^1}(k)$. Furthermore, Σ_t is a triangulated functor on $\mathcal{SH}^{S^1}(k)$.

Lemma 3.4. Let $E \in \text{Spt}^{S^1, \mathbb{A}^1}(k)$ be a -1 connected spectrum. Then $\Sigma_t E$ is again -1 connected.

Proof. The claim is clear when $E = \Sigma_s^\infty \mathcal{X}$ a pointed space, since $\Sigma_t E = \Sigma_s^\infty \mathbb{G}_m \wedge \mathcal{X}$ is still a suspension spectrum, and so -1 connected.

Now consider a general -1 connected spectrum E . By [?, Lemma 3.3.4], E is weak equivalent to $\text{hocolim } E^i$ where $E^0 = *$, and for each n , there is a family $X_\alpha \in \text{Sm}/k$ and natural numbers $n_\alpha \geq 0$ for which

$$\vee_\alpha \Sigma_s^\infty X_{\alpha,+}[n_\alpha - 1] \rightarrow E^{n-1} \rightarrow E^n$$

is an exact triangle. An induction argument establishes that $\Sigma_t E^n$ is still -1 connected for all n ; hence $\Sigma_t E = \text{hocolim } \Sigma_t E^n$ is also -1 connected. Should $\Sigma_t E$ fail to be \mathbb{A}^1 -local, we may simply apply L^∞ to get an \mathbb{A}^1 -local representative of $\Sigma_t E$. By the connectivity theorem, $L^\infty \Sigma_t E$ will again be -1 connected. \square

3.2. Contraction in $\underline{\text{Ab}}(\text{Sm}/k, Nis)$, category of pointed sheaves of sets.

Definition 3.5. Let G be a sheaf of pointed sets on Sm/k . The contraction of G is the sheaf G_{-1} given by the formula

$$U \in \text{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map ev_1 is the map induced by $ev_1 : \text{Spec}(k) \rightarrow \mathbb{G}_m$, i.e., $k[x, x^{-1}] \rightarrow k$ given by $x \mapsto 1$.

Note that indeed G_{-1} is a sheaf since it is the kernel of the morphism of sheaves $G(-) \rightarrow G(- \times \mathbb{G}_m)$. The sheaf $G(- \times \mathbb{G}_m)$ may also be written as $\underline{\text{Hom}}(\mathbb{G}_m, G)$ when we think of G as a space.

Proposition 3.6. If G is the trivial sheaf of abelian groups, then so is its contraction G_{-1} .

Proposition 3.7. Contraction is an exact functor on the category $\underline{\text{Ab}}_{st\mathbb{A}^1}(\text{Sm}/k, Nis)$.

For any sheaf $G \in \underline{\text{Ab}}(\text{Sm}/k, Nis)$ and any $X \in \text{Sm}/k$, $G(\mathbb{G}_m \times X) = G_{-1}(X) \oplus G(X)$.

3.3. Homotopy sheaves of $\underline{\text{Hom}}(\mathbb{G}_m, E)$.

Proposition 3.8. If G is a sheaf of Abelian groups, then $G_{-1} = \underline{\text{Hom}}_\bullet(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $- \wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

Proof. For this category, $\underline{\text{Hom}}_\bullet(\mathbb{G}_m, G)$ and G_{-1} both have sections at X given by $\ker(ev_1 : G(X \times \mathbb{G}_m) \rightarrow G(X))$. See description of pointed internal hom for this.

\square give ref

Remark 8. If G is a sheaf of Abelian groups, we may consider G as a space by declaring $G_n = G$ for all n and giving identity maps for the structure maps. In particular, G is a pointed space at 0.

We can then realize the contraction as a \mathbb{G}_m loop space $G_{-1}(X) = \underline{\text{Hom}}_\bullet(\mathbb{G}_m, G)(X)$.

Remark 9. We now describe the construction of the canonical map $\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \rightarrow \pi_n(E)_{-1}$ for an S^1 spectrum E .

First observe that for any $U \in \mathbf{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\mathrm{Spt}_s(k)(S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma^\infty \mathbb{G}_m, E) \times \mathrm{Spc}(k)(U, \mathbb{G}_m) \rightarrow \pi_0(E)(U)$$

given by sending (f, α) to the composition

$$\Sigma_s^\infty U_+ \xrightarrow{\mathrm{id} \wedge \Sigma_s^\infty \alpha} S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \rightarrow \pi_n(E).$$

This map descends to the smash product, so we have

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \rightarrow \pi_n(E).$$

But by the adjunction $- \wedge \mathbb{G}_m \dashv \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, -)$ on $\mathrm{Spc}_\bullet(k)$ we have a morphism

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \rightarrow \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Remark 10. If $E = HM$ is an Eilenberg-MacLane spectrum associated to a strictly \mathbb{A}^1 invariant sheaf of abelian groups M , we show

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)) \rightarrow \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

Proposition 3.9. For $M \in \underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathbf{Sm}/k)$, the spectrum $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

Proof. Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(k)$, we have $\Sigma^\infty \mathbb{P}^1[-1] = \Sigma^\infty \mathbb{G}_m$. Therefore

$$\begin{aligned} \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM[n]) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1[-1], HM[n]) \\ &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1, HM[n+1]) \\ &= \tilde{H}_{Nis}^{n+1}(\mathbb{P}^1; M). \end{aligned}$$

As the cohomological dimension of \mathbb{P}^1 is less than or equal to 1, we then have $\tilde{H}_{Nis}^{n+1}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$. By $\tilde{H}_{Nis}^n(X; M)$ I mean the kernel of $\mathcal{SH}_s^{S^1}(k)(\Sigma^\infty X_+, HM[n]) \rightarrow \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty S^0, HM[n])$ induced by $S^0 \rightarrow X_+$, where this is obtained by choosing a point in $X(k)$. But what if $X(k)$ is empty?

$$\tilde{H}^n(X; M) \oplus H^n(\mathrm{Spec}(k); M) \cong H^n(X; M).$$

So as $M(\mathrm{Spec} k) \cong M(\mathbb{P}^1)$ (follows since M is strictly \mathbb{A}^1 invariant) $M(\mathbb{P}^1) = \mathrm{pullback}(M(\mathbb{A}^1) \rightarrow M(\mathbb{G}_m) \rightarrow M(\mathbb{A}^1) = M(\mathrm{Spec} k))$, we see that the only possible value of n this will not vanish at is 1.

Since this vanishes at fields, a base change argument shows that indeed the sheaf $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weakly trivial when $n \neq 0$. So then it follows that the sheaf is indeed trivial by the argument giving weakly n -connected is equivalent to being n -connected.¹

¹Reword this to use the modified lemma with gabber presentation.

We now calculate at $\text{Spec}(k)$

$$\begin{aligned}
\pi_0 \underline{\text{Hom}}(\mathbb{G}_m, HM)(\text{Spec}(k)) &= \mathcal{H}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM) \\
&= \tilde{H}^0(\mathbb{G}_m; M) \\
&= \ker(\mathcal{H}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_{m,+}, HM) \rightarrow \mathcal{H}_s^{S^1}(k)(\Sigma^\infty S^0, HM)) \\
&= M_{-1}(\text{Spec } k)
\end{aligned}$$

We now know that the associated homotopy sheaves $\pi_n \underline{\text{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all n . So they are weak equivalent by [?, Lemma 3.2.5]. \square

Proposition 3.10. For any spectrum $E \in \mathcal{H}^{S^1}(k)$, the homotopy sheaves of $\underline{\text{Hom}}(\mathbb{G}_m, E)$ are calculated by $\pi_n(\underline{\text{Hom}}(\mathbb{G}_m, E)) \cong \pi_n(E)_{-1}$

Proof. Reduce to the case of Eilenberg-MacLane spectra by using the Postnikov tower. \square

3.4. Inverting $\mathbb{G}_m \wedge -$; (\mathbb{G}_m, S^1) bi-spectra. The functor Σ_t on $\text{Spt}^{S^1}(k)$ is a left Quillen functor. We may therefore apply the general machinery of [?] to create a model category in which Σ_t is invertible. The construction of Hovey may be described as (\mathbb{G}_m, S^1) bispectra.

Definition 3.11. A (\mathbb{G}_m, S^1) bi-spectrum of spaces over k consists of a bigraded family of spaces $E_{i,j}$, $i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \rightarrow E_{i,j+1}$ and $\mu_{i,j} : \mathbb{G}_m \wedge E_{i,j} \rightarrow E_{i+1,j}$ for which the following diagram commutes.

$$\begin{array}{ccc}
S^1 \wedge \mathbb{G}_m \wedge E_{i,j} & \xrightarrow{S^1 \wedge \tau_{i,j}} & S^1 \wedge E_{i+1,j} \\
\downarrow & & \downarrow \sigma_{i+1,j} \\
\mathbb{G}_m \wedge S^1 \wedge E_{i,j} & & \\
\downarrow \mathbb{G}_m \wedge \sigma_{i,j} & & \\
\mathbb{G}_m \wedge E_{i,j+1} & \xrightarrow{\mu_{i,j+1}} & E_{i+1,j+1}
\end{array}$$

Let $\text{Spt}^{(\mathbb{G}_m, S^1)}(k)$ denote the category of bispectra.

Remark 11. Note that a (\mathbb{G}_m, S^1) bispectrum is just a \mathbb{G}_m -spectrum of S^1 spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a (\mathbb{G}_m, S^1) bi-spectrum $E_{i,j}$ as a sequence of S^1 spectra $E_{i,*}$.

Definition 3.12. Let E be a (\mathbb{G}_m, S^1) bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \text{Sm}/k \mapsto \text{colim}_r \mathcal{H}_\bullet(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$. We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

Proposition 3.13. If E is a (\mathbb{G}_m, S^1) bispectrum, the presheaf of homotopy groups may also be calculated as

$$\tilde{\pi}_{n+m\alpha}E(U) = \operatorname{colim}_{s,r} \mathcal{H}_\bullet(k)(\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[?, p 217]

Definition 3.14. A morphism $f : E \rightarrow F$ of (\mathbb{G}_m, S^1) bispectra is an \mathbb{A}^1 stable weak equivalence if the following induced map is an isomorphism for all $U \in \operatorname{Sm}/k$.

$$f_* : \tilde{\pi}_{n+m\alpha}(E)(U) \rightarrow \tilde{\pi}_{n+m\alpha}(F)(U)$$

Definition 3.15. A morphism $f : E \rightarrow F$ of (\mathbb{G}_m, S^1) bispectra is an \mathbb{A}^1 stable cofibration if $f_0 : E_{0,*} \rightarrow F_{0,*}$ is a cofibration of S^1 spectra and the map $P.O. \rightarrow F_{n+1}$ is a cofibration in the following diagram.

$$\begin{array}{ccc} \mathbb{G}_m \wedge E_{n,*} & \longrightarrow & E_{n+1,*} \\ \downarrow & & \downarrow \\ \mathbb{G}_m \wedge F_{n,*} & \longrightarrow & P.O. \\ & \searrow & \downarrow \\ & & F_{n+1,*} \end{array} \quad \begin{array}{l} \nearrow f_{n+1} \\ \nearrow \end{array}$$

Proposition 3.16. The category $\operatorname{Spt}^{(\mathbb{G}_m, S^1), \mathbb{A}^1}(k)$ of (\mathbb{G}_m, S^1) bispectra with \mathbb{A}^1 stable weak equivalences and \mathbb{A}^1 stable cofibrations is a model category. Denote the associated homotopy category of $\operatorname{Spt}^{(\mathbb{G}_m, S^1), \mathbb{A}^1}(k)$ by $\mathcal{SH}(k)$.

Proposition 3.17. The fibrant bi-spectra are the Ω_t -spectra. [?, Theorem 3.4]

Proposition 3.18. There is a left Quillen functor $\Sigma_t^\infty : \operatorname{Spt}_s^{\mathbb{A}^1}(k) \rightarrow \operatorname{Spt}_{s,t}^{\mathbb{A}^1}(k)$ given by $(\Sigma_t^\infty E)_{i,j} = \mathbb{G}_m^i \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}_m^i \wedge E_j \longrightarrow \mathbb{G}_m^i \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}_m^m \wedge \mathbb{G}_m^i \wedge E_j \rightarrow \mathbb{G}_m^{i+1} E_j.$$

The right adjoint to Σ_t^∞ is denoted by Ω_t^∞ and is given by $\Omega_t^\infty(E) = E_{0,*}$.

The right derived functor $R\Omega_t^\infty(E)$ is given by the formula

$$R\Omega_t^\infty(E) = \operatorname{colim}_i \Omega_t^i E_{i,*}.$$

3.5. Connectivity of (\mathbb{G}_m, S^1) bispectra.

Definition 3.19. A (\mathbb{G}_m, S^1) bispectrum E is said to be n -connected if for all $k \leq n$ and all $m \in \mathbb{Z}$, the homotopy sheaves $\pi_{k+m\alpha}E$ vanish.

Proposition 3.20. Let $E \in \operatorname{Spt}^{S^1, \mathbb{A}^1}(k)$. If E is -1 connected, then so too is the (\mathbb{G}_m, S^1) bi-spectrum $\Sigma_t^\infty E$.

Proof. We calculate

$$\begin{aligned}
\pi_{n+m\alpha}(\Sigma_t^\infty E) &= \pi_n(R\Omega_t^\infty \Omega_t^m \Sigma_t^\infty E) \\
&= \pi_n(\operatorname{colim}_i \Omega_t^{m+i} \Sigma_t^i E) \\
&= \operatorname{colim}_i \pi_n(\Sigma_t^i E)_{-(m+i)} \\
&= 0.
\end{aligned}$$

This follows since $\Sigma_t E$ is -1 connected whenever E is -1 connected, and the effect of Ω_t^{m+i} on homotopy sheaves is contraction. \square

3.6. t -structure on $\mathcal{SH}(k)$.

Definition 3.21. Let $\mathcal{SH}(k)_{\geq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha} E = 0$ whenever $n < 0$.

Let $\mathcal{SH}(k)_{\leq 0}$ denote the full subcategory of $\mathcal{SH}(k)$ given by bispectra E satisfying $\pi_{n+m\alpha} E = 0$ whenever $n > 0$.

Definition 3.22. For a (\mathbb{G}_m, S^1) bispectrum E , let $E_{\leq 0}$ denote the spectrum with $(E_{\leq 0})_n = (E_n)_{\leq 0}$. The bonding maps are given by

$$\mathbb{G}_m \wedge P^j(E_{i,j}) \cong P^j(\mathbb{G}_m \wedge E_{i,j}) \rightarrow P^j(E_{i+1,j}).$$

The equivalence $\mathbb{G}_m \wedge P^j(\mathcal{X}) \cong P^j(\mathbb{G}_m \wedge \mathcal{X})$ follows by checking on stalks, and the fact that any stalk of \mathbb{G}_m is just a disjoint union of points.

Theorem 3.23. The pair $(\mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ defines a t -structure on $\mathcal{SH}(k)$.

Proof. Property (2) of a t -structure is clear.

We now establish property (1) of a t -structure. Let $E \in \mathcal{SH}(k)_{\geq 0}$ and $F \in \mathcal{SH}(k)_{\leq 0}$. We must show $\mathcal{SH}(k)(E, F[-1]) = 0$. When E is in the image of Σ_t^∞ , the result follows by using the adjunction $\Sigma_t^\infty \dashv R\Omega_t^\infty$ and using the t -structure on S^1 spectra. In particular, for $U \in \operatorname{Sm}/k$ we have $\mathcal{SH}(k)(S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty U_+, F[-1]) = 0$ for $n \geq 0$ and $m \in \mathbb{Z}$.

For a general $E \in \mathcal{SH}(k)_{\geq 0}$, we may write $E = \operatorname{hocolim} E^i$ where the E^i are built up as in [?, 3.3.4], but we allow smashing with \mathbb{G}_m . Precisely, we take $E^0 = pt$, and each E^i is obtained from E^{i-1} as the cone of a map

$$\bigvee_{\alpha} S^{n_{\alpha}} \wedge \mathbb{G}_m^{m_{\alpha}} \wedge \Sigma^\infty X_{\alpha,+} \rightarrow E^{i-1}$$

for some family of $X_{\alpha} \in \operatorname{Sm}/k$ and indices $n_{\alpha} \geq 0$, $m_{\alpha} \in \mathbb{Z}$.

A standard 5-lemma argument using the long exact sequence obtained by applying $\mathcal{SH}(k)(-, F[-1])$ to the triangle

$$\bigvee S^{n_{\alpha}} \wedge \mathbb{G}_m^{m_{\alpha}} \wedge \Sigma^\infty X_{\alpha,+} \rightarrow E^{i-1} \rightarrow E^i$$

shows that for all $i \in \mathbb{N}$, $\mathcal{SH}(k)(E^i, F[-1]) = 0$. Furthermore, these long exact sequences show that for all $i \geq 1$, $\mathcal{SH}(k)(E^i, F[-2]) \rightarrow \mathcal{SH}(k)(E^{i-1}, F[-2])$ is surjective. Hence $\varprojlim^1 \mathcal{SH}(k)(E^i, F[-2]) = 0$, and so

$$\begin{aligned}
\mathcal{SH}(k)(E, F[-1]) &= \mathcal{SH}(k)(\operatorname{colim} E^i, F[-1]) \\
&= \varprojlim \mathcal{SH}(k)(E^i, F[-1]) \\
&= 0.
\end{aligned}$$

We now establish property (3) of a t -structure. The functor $(-)_{\leq 0}$ has already been defined. For $k \in \mathbb{Z}$, let $(-)_{\leq k}$ is a functor on $\text{Spt}_s(k)$ and we may extend it to a functor on $\mathcal{SH}(k)$ in the same way as for the case $k = 0$. Define $E_{\geq 0}$ to be the homotopy fiber of the canonical map $E \rightarrow E_{\leq -1}$. The long exact sequence of homotopy groups shows that $(-)_{\geq 0}$ has the correct homotopy groups. The uniqueness of the triangle follows by properties of triangulated categories. \square

3.7. The heart of the t -structure on $\mathcal{SH}(k)$.

Definition 3.24. A homotopy module over k is a pair (M_*, μ_*) consisting of a \mathbb{Z} graded strictly \mathbb{A}^1 invariant sheaf M_* and an isomorphism $\mu_n : M_n \cong (M_{n+1})_{-1}$.

Lemma 3.25. If E is a bi-spectrum, then

$$R\Omega_t^\infty E \rightarrow \underline{\text{Hom}}(\mathbb{G}_m, R\Omega_t^\infty(E \wedge \mathbb{G}_m))$$

is an isomorphism.

Proof. \square

Lemma 3.26. Let $E \in \mathcal{SH}(k)$. For a fixed $n \in \mathbb{Z}$, the collection $\pi_n(E)_m$ forms a homotopy module.

Lemma 3.27. If (M_*, μ_*) is a homotopy module over k , then there is a (\mathbb{G}_m, S^1) bispectrum HM_* with $(HM_*)_{n,n} = K(M_n, n)$ with evident structure maps.

Proof. \square

Theorem 3.28. The heart of the t -structure $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is denoted $\pi_*^{\mathbb{A}^1}(k)$ and is equivalent to the category of homotopy modules. The equivalence is given explicitly by the functors $\pi_0(-)_*$ and $H(-)$.

Proof. \square

REFERENCES

- [A-1974] Adams, J.F., *Stable Homotopy and Generalized Homology*. Chicago Lectures in Mathematics, (1974).
- [Ayo08] Ayoub, Joseph, *Un contre-exemple à la conjecture de \mathbb{A}^1 -connexité de F. Morel*. (2008).
- [B] Blander, Benjamin, *Local Projective Model Structures on Simplicial Presheaves*. K-Theory, **24** (2001) 283–301.
- [DHI] Dugger, Dan; Hollander, Sharon; Isaksen, Dan, *Hypercovers and simplicial presheaves*.
- [DLØRV] Dundas, B.; Levine, M.; Østvær, P.; Röndigs, O.; Voevodsky, V., *Motivic Homotopy Theory*. Springer (2000).
- [GJ91] Goerss, Paul; Jardine, John, *Simplicial Homotopy Theory*. (1991).
- [Hir] Phillip, Hirschhorn, *Model Categories and Their Localization*. AMS (2003).
- [H-Mod] Hovey, Mark, *Model Categories*. online preprint (1991).
- [H-Spt] Hovey, Mark, *Spectra and symmetric spectra in general model categories*. journal? (2001).
- [HSS] Hovey, Mark; Shipley, Brooke; Smith, Jeff *Symmetric Spectra*. journal? (1998).
- [J] Jardine, J.F., *Simplicial presheaves*. Journal of Pure and Applied Algebra, **47** (1987) 35–87.
- [Mor03] Morel, Fabien, *An introduction to \mathbb{A}^1 homotopy theory*.
- [Mor04] Morel, Fabien, *On the motivic π_0 of the sphere spectrum*. NATO science series.
- [Mor05] Morel, Fabien, *The stable \mathbb{A}^1 connectivity theorems*. preprint (2004).

- [MV99] Morel, Fabien; Voevodsky, Vladimir, \mathbb{A}^1 -homotopy theory of schemes. IHES, tome 90 (1999), p. 45–143.
- [Pel08] Pelaez, Pablo, *Multiplicative Properties of the Slice Filtration*. December 2008.
- [Voev98] Voevodsky, Vladimir. \mathbb{A}^1 -Homotopy Theory. Doc. Math. J., (1998) pp. 579–604.