## 1. Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

The following is from an email from Aravind to me concerning the proof of the connectivity theorem for $\mathbb{P}^1$ spectra.

> Most of the proof appears in Morel's Trieste notes (in the section on the homotopy t-structure for $P^1$-spectra). First you prove a connectivity result for $P^1$-stable homotopy sheaves by using Morel's $S^1$-stable connectivity theorem and studying what happens under $G_m$ loops and $G_m$-suspension: suspension preserves connectivity, and Morel shows that taking $G_m$-loops has the effect of making a "contraction". Then, you globalize this.

Our goal is to prove theorem 4.14 of [Voev98], which should be restated in terms of $\mathbb{P}^1$-spectra.

**Theorem 1.1.** Let $X$ be a pointed smooth scheme over $\mathrm{Spec}(k)$ where $k$ is an infinite field. Let $\mathcal{Y}$ be a pointed space. Then for any $n > \dim(X)$, and any integer $m$

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

*Remark* 1. We can formulate the theorem by using homotopy sheaves. Indeed, if we can show that for any pointed space $\mathcal{Y}$ is $-1$ connected, i.e., the homotopy sheaves $\pi^{\mathbb{A}^1}_{n+\alpha m}(\mathcal{Y})$ vanish for $n \leq -1$, the result will follow, as

$$\begin{aligned}\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) &= \mathcal{SH}(k)(S^{-n} \wedge \mathbb{G}_m^{-m} \wedge \Sigma^\infty X, \Sigma^\infty \mathcal{Y}) \\ &= \pi^{\mathbb{A}^1}_{-n-m\alpha}(X).\end{aligned}$$

and by the connectivity theorem, this will vanish whenever $-n < 0$, i.e., whenever $n > 0$. This is a stronger statement than [Voev98, theorem 4.14].

The line of attack is then to show that for a pointed space $\mathcal{Y}$, the $S^1$ suspsension spectrum $\Sigma_s^\infty \mathcal{Y}$ in the stable model category is $-1$ connected. Then we establish the first connectivity theorem which ensures the $\mathbb{A}^1$ localization of $\Sigma_s^\infty \mathcal{Y}$ is $-1$ connected. Finally, we show that inverting $\mathbb{G}_m$ does not affect the connectivity, i.e., $\Sigma_m^\infty \Sigma_s^\infty \mathcal{Y}$ is again $-1$ connected.

## 2. Assumptions from previous lectures

### 2.1. Facts about Nisnevich topology.

Points and neighborhoods. Distinguished squares determine the topology. Relation to etale, zariski, and fpqc topology.

**Proposition 2.1.** [Mor04, 2.4.1] Let $M$ be a sheaf of abelian groups on $\mathrm{Sm}/k$, and let $X \in \mathrm{Sm}/k$ with Krull dimension $d$. Then whenever $n > d$, $H^n_{Nis}(X; M) = 0$.

**Proposition 2.2.** [Mor04, 2.4.1] For any $X \in \mathrm{Sm}/k$, and for any $x \in X(k)$, there is an isomorphism of pointed sheaves of sets in the Nisnevich topology

$$X/(X - \{x\}) \cong \mathbb{A}^n/(\mathbb{A}^n - \{0\}).$$

### 2.2. Unstable model category $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$.

(1) Don't forget the adjunction for sheafification $a_{Nis} \dashv U$. i.e., to give a map $a_{Nis}\mathcal{X} \to \mathcal{Y}$, of sheaves, it is equivalent to just give a map $\mathcal{X} \to U\mathcal{Y}$.
(2) Give the unstable model category the injective model structure to start. (This is Morel's choice, so we stick with it)
(3) Weak equivalences: $\mathcal{X} \to \mathcal{Y}$ iff for all $U$ the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ a simplicial w.e.

(4) Cofibs: $\mathcal{X} \to \mathcal{Y}$ is a cofib. iff for any $U \in \mathrm{Sm}/k$ the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a monomorphism.
(5) Fibrations: what they need to be.
(6) The ass. htpy. cat. to the injective model category structure is denoted $\mathcal{H}_s(k)$. The pointed htpy. cat. is denoted $\mathcal{H}_{s,\bullet}(k)$.
(7) Important properties/constructions in these categories: Cartesian closed, i.e., have smash/tensor $\times$ and $\wedge$, with right adjoints $\underline{Hom}$ and $\underline{Hom}_\bullet$. Representable sheaf functor and constant simplicial set functor.

**Definition 2.3.** Internal hom. (Ref: P. Pelaez)

Let $U \in \mathrm{Sm}/k$. Let $\Delta^n_U$ denote the simplicial sheaf given by

$$(V,m) \in \mathrm{Sm}/k \times \Delta^{op} \to \mathrm{Sm}/k(V,U) \times \Delta^n_m.$$

In other words, $\Delta^n_U = (rU) \times c\Delta^n$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be spaces. The internal hom in the unpointed category is given by the formula

$$(U,m) \in \mathrm{Sm}/k \times \Delta \to \mathrm{Hom}_{\Delta^{op}\mathrm{Shv}}(X \times \Delta^m_U, Y).$$

How to describe the adjunction?

$$\mathrm{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$$

Certainly we may send a map $g \in \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$ to the map $\eta(g)$ given by

$$\begin{aligned} \eta(g)(U,n) :& \mathcal{X}_n(U) \times \mathcal{Y}_n(U) \to \mathcal{Z}_n(U) \\ & (a,b) \to g(U,n)(U,n)(a, \mathrm{id}, \mathrm{id}). \end{aligned}$$

Why is this a bijection?

**Definition 2.4.** Internal hom in the pointed category. Consider $\mathcal{X}$ and $\mathcal{Y}$ pointed spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{Hom}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y}$, where at $(U,n) \in (\mathrm{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times \Delta^n_U \to \mathcal{Y}$ to $g(U,n)(x, \mathrm{id}, \mathrm{id}) \in \mathcal{Y}_n(U)$. This makes sense as we have the map

$$g(U,n) : \mathcal{X}_n(U) \times \mathrm{Sm}/k(U,U) \times \Delta^n_n \to \mathcal{Y}_n(U).$$

The pointed internal hom $\underline{Hom}_\bullet(\mathcal{X}, \mathcal{Y})$ is the fiber $ev_x^{-1}(y)$.

### 2.3. $\mathbb{A}^1$ localization.

(1) See [**?**, Prop 2.3.3.] for details on the various properties of fibrant objects in the unstable motivic category.
(2)

**Definition 2.5.** A space $\mathcal{X}$ is called $\mathbb{A}^1$ local if for any smooth scheme $U$, the canonical map

$$\mathrm{Hom}(rU, \mathcal{X}) \to \mathrm{Hom}(rU \times \mathbb{A}^1, \mathcal{X})$$

is a bijection.

(3)

**Definition 2.6.** A map $f : \mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$ weak equivalence if

$$\mathrm{Hom}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}(\mathcal{X}, \mathcal{Z})$$

is a bijection for every $\mathbb{A}^1$ local space $\mathcal{Z}$.

(4) The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \to U \,|\, U \in \mathrm{Sm}/k\}$.

(5) The general theory of Bousfield localization then gives a localization functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \to L_W\mathcal{H}_s(k)$; denote the category $L_W\mathcal{H}_s(k)$ by $\mathcal{H}(k)$. The localization functor sends $\mathbb{A}^1$ weak equivalences to isomorphisms.

See [**?**, Definition 3.3.1]. Keep the same underlying category, the weak equivalences are the $\mathbb{A}^1$-local weak equivalences, do not change the cofibrations from the ones in the injective model structure on $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$, and let the fibrations be what they need to be.

(6) Morel writes $L_{\mathbb{A}^1}$ for the (derived) functor which sends a space (without base point) $\mathcal{X}$ to an $\mathbb{A}^1$-localizion.

(7) Morel gives an explicit construction which takes a pointed space $\mathcal{Y}$ and produces a space $L^\infty\mathcal{Y}$ which is $\mathbb{A}^1$ local and a map $\mathcal{Y} \to L^\infty\mathcal{Y}$ which is an $\mathbb{A}^1$ weak equivalence.

Let $\mathcal{Y}_f$ be the functorial fibrant replacement of $\mathcal{Y}$ with respect to the injective model structure. Consider $\mathbb{A}^1$ to be pointed at 0. Morel defines $L^{(1)}(\mathcal{Y})$ to be the cone of the map $ev_1 : \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{Y}_f) \to \mathcal{Y}_f$. So there is a map $\mathcal{Y} \to L^{(1)}(\mathcal{Y})$ obtained from the trivial cofibration $\mathcal{Y} \to \mathcal{Y}_f$ and the defining map $\mathcal{Y}_f \to L^{(1)}(\mathcal{Y})$.

So $L^{(1)}(-)$ is a functor with a natural transformation $\eta : \mathrm{id} \to L^{(1)}(-)$. Define by induction $L^{(n)}(\mathcal{Y}) = L^{(1)}(L^{(n-1)}(\mathcal{Y}))$. There is thus a directed system $L^{(n-1)}(\mathcal{Y}) \to L^{(n)}(\mathcal{Y})$. Denote the direct limit/hocolim of this directed system by $L^\infty(\mathcal{Y})$.

**Proposition 2.7.** The natural morphism $\mathcal{Y} \to L^\infty(\mathcal{Y})$ is an $\mathbb{A}^1$ weak equivalence, and $L^\infty(\mathcal{Y})$ is $\mathbb{A}^1$ local.

(8)

**Definition 2.8.** Let $\mathcal{X}$ be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on $\mathrm{Sm}/k$ associated to $U \to \pi_0(\mathcal{X}(U))$. A space $\mathcal{X}$ is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let $(\mathcal{X}, x)$ be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on $\mathrm{Sm}/k$ given by

$$U \to \pi_n(\mathcal{X}(U)).$$

A pointed space $\mathcal{X}$ is called $n$-connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

(9)

**Proposition 2.9.** Let $\mathcal{X}$ be a 0-connected simplicial sheaf. Then $L^\infty\mathcal{X}$ is also 0-connected.

*Proof.* M.V. IHES paper? Sketch of argument? It shouldn't be too hard by chasing components around. □

(10) For a sheaf of abelian groups $M$ on $\mathrm{Sm}/k$ and a natural number $n$, a Dold-Kan construction gives a simplifical presheaf $K(M, n)$. It is called the Eilenberg-MacLane spectrum of type $(M, n)$ and has homotopy sheaves as expected:

$$\pi_m(K(M,n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}.$$

Important equation for this construction. For $X \in \mathrm{Sm}/k$, $M$ a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M,n)) \cong H^n_{Nis}(X; M).$$

(11) Use square brackets to denote the maps in the unstable pointed (motivic) homotopy category, i.e., $[\mathcal{X}, \mathcal{Y}] = \mathcal{H}_\bullet(k)(\mathcal{X}, \mathcal{Y})$.

Use $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ for the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^\infty \mathcal{X})$. This is also obtained by sheafifying the presheaf given by

$$U \in \mathrm{Sm}/k \mapsto [S^n \wedge U_+, \mathcal{X}].$$

2.4. **Homotopy purity, connectedness calculations.**

**Proposition 2.10.** Weak $n$-connectedness is equivalent to $n$-connectedness

*Proof.* Details? □

**Proposition 2.11.** If $V$ is an irreducible, smooth $k$-scheme, and $U \subseteq V$ is a dense open subset, the space $L_{\mathbb{A}^1}(V/U)$ is 0-connected.

*Proof.* Seems to use that local structure of spaces is given by $\mathbb{A}^n/(\mathbb{A}^n - \{0\}) \wedge L_+$. How do all the reductions work? Also uses some results specific to working over perfect fields. □

2.5. $S^1$**-spectra.**

**Definition 2.12.** Let $\mathrm{Spt}_s(k)$ denote the category of $S^1$-spectra of spaces $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k)$. We first endow this category with the projective model structure, i.e., a map $f : E \to F$ is a weak equivalence iff for any $n$ the map $f_n : E_n \to F_n$ is a w.e.; a map $f : E \to F$ is a fibration iff for all $n$ the map $f_n : E_n \to F_n$ is a fibration. The cofibrations are characterized by the property that $f : E \to F$ is a cofib iff $f_0 : E_0 \to F_0$ is a cofib and for any $n \geq 1$

$$\begin{tikzcd}
S^1 \wedge E_{n-1} \arrow[r, "\sigma_{n-1}"] \arrow[d, "S^1 \wedge f_{n-1}"'] & E_n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\
S^1 \wedge F_{n-1} \arrow[r] \arrow[drr, bend right, "\sigma_{n-1}"'] & P.O. \arrow[dr, dotted] & \\
 & & F_n
\end{tikzcd}$$

This model structure does not actually invert $S^1 \wedge -$. To accomplish this, we must localize with respect to the stable equivalences.

**Definition 2.13.** A map $f : E \to F$ of $S^1$-spectra is a stable equivalence iff for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \to \pi_n(F)$ is an isomorphism.

The stable model category structure on $\mathrm{Spt}_s(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such.

**Definition 2.14.** Let $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$ denote the category of $S^1$ spectra endowed with the stable model category structure localized at the collection of maps $\{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \to \Sigma^\infty U_+ \,|\, U \in \mathrm{Sm}/k\}$.

Let $\mathcal{SH}^{S^1}(k)$ denote the homotopy category associated to $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. We will use $\mathcal{SH}_s^{S^1}(k)$ to denote the homotopy category of $\mathrm{Spt}_s(k)$.

*Remark* 2. There is a functor $L^\infty$ on the category of $S^1$ spectra which is similar to the unstable construction.

So we can use the functor $L^\infty$ as an $\mathbb{A}^1$ localization functor. To be precise, we let $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ be the subcategory of $\mathrm{Spt}_s(k)$ consisting of the $\mathbb{A}^1$ local spectra. We may equip $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ with a model structure with weak equivalences the $\mathbb{A}^1$ weak equivalences, the cofibrations are the stable cofibrations, and the fibrations are what they have to be.

The functor $L^\infty : \mathrm{Spt}_s(k) \to \mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ is left adjoint to the inclusion functor, and is a left Quillen functor. The homotopy category of $\mathrm{Spt}_s^{\mathbb{A}^1 loc}(k)$ is categorically equivalent to $\mathcal{SH}^{S^1}(k)$. [Mor05, Corollary 4.2.3].

For spectra $E$ and $F$, we may compute $[E,F]^{\mathbb{A}^1} := \mathcal{SH}^{S^1}(k)(E,F)$ by calculating $[L^\infty E, L^\infty F]^{\mathbb{A}^1}$. Note that this is $\mathcal{SH}_s^{S^1}(k)(E, L^\infty F)$ by using the adjunction. If we assume $E$ is cofibrant and $L^\infty F$ is fibrant, we get the formula

$$[E,F]^{\mathbb{A}^1} = \mathrm{Spt}_s(k)(E, L^\infty F).$$

**Definition 2.15.** Let $E$ be an $S^1$ spectrum of spaces. Let $\pi_n$ denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$. That is,

$$\pi_n(E) = \mathrm{colim}_r\, \pi_{n+r}(E_r).$$

In particular, for a $U \in \mathrm{Sm}/k$, we have

$$\pi_n(E)(U) = \mathrm{colim}_r\, \pi_{n+r}(E_r)(U).$$

**Definition 2.16.** An $S^1$-spectrum $E$ is said to be $n$-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

**Definition 2.17.** There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \to \mathrm{Spt}_s(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ with the evident bonding maps. The right adjoint to this functor is given by "evaluation at 0", i.e., $\Omega^\infty(E) = E_0$.

*Remark* 3. The right derived functor $R\Omega^\infty : \mathcal{SH}_s^{S^1}(k) \to \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i\, \Omega_s^i E_i.$$

This comes from the fact that fibrant $S^1$ spectra are exactly the $\Omega$ spectra, and the description of the fibrant replacement functor.

*Remark* 4. We also get a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet^{\mathbb{A}^1}(k) \to \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ given by the same formula as above.

What is the fibrant replacement functor in $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$?

*Remark* 5. The stable homotopy category is symmetric monoidal, with smash product $\wedge$ and internal hom $\underline{\mathrm{Hom}}$. Using symmetric spectra, one can give these constructions on the category of spectra.

The stable homotopy category is a triangulated category.

**Proposition 2.18.** Let $U \in \mathrm{Sm}/k$, $n \in \mathbb{Z}$, and $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then there is a canonical isomoprhism

$$H^n_{Nis}(U; M) \to \mathcal{SH}^{S^1}(\Sigma^\infty U_+, HM[n]).$$

*Proof.* This is [Mor05, Lemma 3.2.3]. □

2.6. $t$ **structure.**

**Definition 2.19.** Let $\mathfrak{C}$ be a triangulated category. A $t$-structure on $\mathfrak{C}$ is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

(1) For any $X \in \mathfrak{C}_{\geq 0}$ and any $Y \in \mathfrak{C}_{\leq 0}$, $\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0$.
(2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
(3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \to X \to Z \to Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$..

The heart of a $t$-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

**Definition 2.20** ($t$-structure on $\mathcal{SH}_s^{S^1}(k)$)**.** Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects $E$ such that $\pi_n(E) = 0$ whenver $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects $E$ such that $\pi_n(E) = 0$ whenver $n > 0$.

**Theorem 2.21.** The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a $t$-structure on $\mathcal{SH}_s^{S^1}(k)$.

2.7. **Connectivity results.**

**Proposition 2.22.** [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}_s^{S^1}(k) \to \mathrm{Spt}_{s,\mathbb{A}^1}^{S^1}(k)$ identifies the $\mathbb{A}^1$-localized $S^1$ stable homotopy category with the homotopy category of $\mathbb{A}^1$-local $S^1$ spectra.

**Theorem 2.23** ($S^1$ stable connectivity theorem)**.** Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L_{\mathbb{A}^1} E = 0$.

**Theorem 2.24.** The pair $(\mathcal{SH}^{S^1}_{\geq 0}(k), \mathcal{SH}^{S^1}_{\leq 0}(k))$ is a $t$-structure on the category $\mathcal{SH}^{S^1}(k)$.

**Definition 2.25.** Strictly $\mathbb{A}^1$ invariant sheaf of Abelian groups.

If $M$ is strictly $\mathbb{A}^1$ invariant sheaf of groups, define the Eilenberg-MacLane spectrum $HM$ associated to it.

**Proposition 2.26.** $HM$ is $\mathbb{A}^1$ local iff $M$ is strictly $\mathbb{A}^1$ invariant.

**Proposition 2.27.** The heart of the homotopy $t$ structure is equivalent to the category of strictly $\mathbb{A}^1$ invariant sheaves.

## 3. Inverting $\mathbb{G}_m$; $\mathbb{P}^1$ spectra

3.1. $\mathbb{G}_m$ **suspension and loops.** We always consider $\mathbb{G}_m$ to be pointed at 1 unless otherwise specified.

**Definition 3.1.** On the category $\mathrm{Spt}_s(k)$ equipped with the motivic stable model category structure, there is a functor $\Sigma_t(-) = \mathbb{G}_m \wedge -$ given by $\Sigma_t(E)_n = \mathbb{G}_m \wedge E_n$ with the evident structure maps.

Smashing with $\mathbb{G}_m$ is also a functor on the unstable category of pointed spaces, and we give it the same name $\Sigma_m$.

**Definition 3.2.** The functor $\Sigma_m$ on $\mathrm{Spc}_\bullet(k)$ has a right adjoint denoted $\Omega_t$. It is given by the formula $\Omega_t \mathcal{X} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \mathcal{X})$.

The functor $\Sigma_t$ on $\mathrm{Spt}_s^{S^1}(k)$ also has a right adjoint $\Omega_t$ given by the internal hom functor, i.e., $\Omega_t E = \underline{\mathrm{Hom}}(\Sigma^\infty \mathbb{G}_m, E)$.

**Proposition 3.3.** The functor $\Sigma_t$ is a left Quillen functor on $\mathrm{Spt}_s(k)$ and on $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. Furthermore, $\Sigma_t$ is an exact functor on $\mathcal{SH}^{S^1}(k)$.

### 3.2. **Contraction in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$.**

**Definition 3.4.** Let $G$ be sheaf of pointed sets on $\mathrm{Sm}/k$. The contraction of $G$ is the sheaf $G_{-1} = G_{con}$ given by the formula

$$U \in \mathrm{Sm}/k \mapsto \ker(G(X \times \mathbb{G}_m) \xrightarrow{ev_1} G(X))$$

Where the map $ev_1$ is the map induced by $ev_1 : \mathrm{Spec}(k) \to \mathbb{G}_m$, i.e., $k[x, x^{-1}] \to k$ given by $x \mapsto 1$.

Note that indeed $G_{-1}$ is a sheaf since it is the kernel of the morphism of sheaves $G(-) \to G(- \times \mathbb{G}_m)$.

**Proposition 3.5.** If $G$ is the trivial sheaf of abelian groups, then so is its contraction $G_{-1}$.

**Proposition 3.6.** Contraction is an exact functor on the category $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$.

*Proof.* Functoriality can be established by using the universal property of kernels.

To show exactness, let $F \to G \to H$ be a short exact sequence of sheaves. We must show $F_{-1} \to G_{-1} \to H_{-1}$ is still exact. It suffices to check exactness at the level of stalks.

Since $\mathrm{id} = ev_1 \circ \pi : G(X) \to G(X \times \mathbb{G}_m) \to G(X)$, we have $ev_1 : G(-) \to G(- \times \mathbb{G}_m)$ is a surjection on all smooth schemes. Hence $ev_1$ is an epimorphism. For a Nisnevich point $x$ of $X$, The following commutative diagram then establishes the exactness of the contraction by the 3x3 lemma.

$$\begin{tikzcd}
F_{-1,x} \arrow[r] \arrow[d, tail] & G_{-1,x} \arrow[r] \arrow[d, tail] & H_{-1,x} \arrow[d, tail] \\
F_{(x,1)} \arrow[r, tail] \arrow[d, two heads] & G_{(x,1)} \arrow[r, two heads] \arrow[d, two heads] & H_{(x,1)} \arrow[d, two heads] \\
F_x \arrow[r, tail] & G_x \arrow[r, two heads] & H_x
\end{tikzcd}$$

□

### 3.3. **Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$.**

**Proposition 3.7.** If $G$ is a sheaf of Abelian groups, then $G_{-1} \cong \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)$. Hence contraction is right adjoint to $- \wedge \mathbb{G}_m$. The claim is also true for pointed sheaves of sets.

*Proof.* For this category, $\underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)$ and $G_{-1}$ both have sections at $X$ given by $ker(ev_1 : G(X \times \mathbb{G}_m) \to G(X))$. See description of pointed internal hom for this. □

*Remark* 6. If $G$ is a sheaf of Abelian groups, we may consider $G$ as a space by declaring $G_n = G$ for all $n$ and giving identity maps for the structure maps. In particular, $G$ is a pointed space at 0.

We can then realize the contraction as a $\mathbb{G}_m$ loop space $G_{-1}(X) = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, G)(X)$.

*Remark* 7. Construction of canonical map $\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ for an $S^1$ spectrum $E$.

First observe that for any $U \in \mathrm{Sm}/k$ and any $n \in \mathbb{Z}$ there is a map

$$\mathrm{Spt}_s(k)(S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma^\infty \mathbb{G}_m, E) \times \mathrm{Spc}(k)(U, \mathbb{G}_m) \to \pi_0(E)(U)$$

given by sending $(f, \alpha)$ to the composition

$$\Sigma_s^\infty U_+ \xrightarrow{\mathrm{id} \wedge \Sigma_s^\infty \alpha} S^n \wedge \Sigma_s^\infty U_+ \wedge \Sigma_s^\infty \mathbb{G}_m \longrightarrow E.$$

Hence there is a map of sheaves

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \times \mathbb{G}_m \to \pi_n(E).$$

Does this map descend to the smash? Yes, since if either map is a constant map, then so is the composition.

We thus get a map of sheaves of pointed sets

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \wedge \mathbb{G}_m \to \pi_n(E).$$

But by the adjunction $- \wedge \mathbb{G}_m \dashv \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, -)$ on $\mathrm{Spc}_\bullet(k)$ we have a morphism

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, E)) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(E)) = \pi_n(E)_{-1}.$$

Why is it a map of sheaves of abelian groups?

*Remark* 8. If $E = HM$ is an Eilenberg-MacLane spectrum associated to a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups $M$, we show

$$\pi_n(\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, \pi_n(HM)) = \pi_n(HM)_{-1}.$$

is an isomorphism by showing $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is an Eilenberg-MacLane spectrum.

**Lemma 3.8.** Let $M \in \underline{\mathrm{Ab}}(\mathrm{Sm}/k)$. Then $K(M, r)_{-1} \simeq K(M_{-1}, r)$.

*Proof.* Let $X \in \mathrm{Sm}/k$. The simplicial set $K(M, r)_{-1}(X)$ by the fibration

$$K(M, r)_{-1}(X) \to K(M, r)(\mathbb{G}_m \times X) \to K(M, r)(X).$$

There is thus a long exact sequence of homotopy groups, which ammounts to

$$\pi_r K(M, r)_{-1}(X) \rightarrowtail \pi_r K(M, r)(\mathbb{G}_m \times X) \longrightarrow \pi_r K(M, r)(X) \twoheadrightarrow \pi_{r-1} K(M, r)_{-1}(X).$$

The remaining homotopy groups are evidently trivial. As $\pi_r K(M, r)(\mathbb{G}_m \times X) \to \pi_r K(M, r)(X)$ is surjective, we see $\pi_{r-1} K(M, r)_{-1}(X) = 0$. Therefore,

$$\pi_n K(M, r)_{-1}(X) = \begin{cases} 0 & \text{if } n \neq r \\ M_{-1}(X) & \text{if } n = r \end{cases}.$$

We conclude that $K(M, r)_{-1} \simeq K(M_{-1}, r)$. $\square$

**Proposition 3.9.** For $M \in \underline{\mathrm{Ab}}_{st\mathbb{A}^1}(\mathrm{Sm}/k)$, the spectrum $\underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ is weak equivalent to $H(M_{-1})$.

*Proof.* Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(k)$, we have $\Sigma^\infty \mathbb{P}^1[-1] = \Sigma^\infty \mathbb{G}_m$. Therefore

$$\begin{aligned}
\mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m, HM[n]) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1[-1], HM[n]) \\
&= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{P}^1, HM[n+1]) \\
&= \tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M).
\end{aligned}$$

As the cohomological dimension of $\mathbb{P}^1$ is less than or equal to 1, we then have $\tilde{H}^{n+1}_{Nis}(\mathbb{P}^1; M) = 0$ for all $n \neq 0$.

Now[1] for $U \in \mathrm{Sm}/k$, we calculate for $n \neq 0$

$$\begin{aligned}
\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(U) &= \mathcal{SH}_s^{S^1}(k)(S^n \wedge \Sigma^\infty \mathbb{G}_m \wedge \Sigma^\infty U_+, HM) \\
&= \mathrm{colim}_r\, \mathcal{H}_\bullet^s(k)(S^{n+r} \wedge U_+, \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, K(M,r))) \\
&= \mathrm{colim}_r\, \pi_{n+r}(K(M,r)_{-1})(U) \\
&= \mathrm{colim}_r\, \pi_{n+r}(K(M_{-1}, r))(U) \\
&= \mathrm{colim}_r\, 0 \\
&= 0.
\end{aligned}$$

To establish the second point, we calculate for $U \in \mathrm{Sm}/k$

$$\begin{aligned}
\pi_0 \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)(U) &= \mathcal{SH}_s^{S^1}(k)(\Sigma^\infty \mathbb{G}_m \wedge \Sigma^\infty U_+, HM) \\
&= \mathrm{colim}_r\, \mathcal{H}_\bullet^s(k)(S^{r} \wedge U_+, \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, K(M,r))) \\
&= \mathrm{colim}_r\, \pi_{r}(K(M,r)_{-1})(U) \\
&= \mathrm{colim}_r\, \pi_{r}(K(M_{-1}, r))(U) \\
&= \mathrm{colim}_r\, M_{-1}(U) \\
&= M_{-1}(U) \\
&= \pi_0 H(M_{-1})(U).
\end{aligned}$$

We now know that the associated homotopy sheaves $\pi_n \underline{\mathrm{Hom}}(\mathbb{G}_m, HM)$ and $\pi_n H(M_{-1})$ agree for all $n$. So they are weak equivalent by [Mor05, Lemma 3.2.5]. ☐

3.4. **Inverting $\mathbb{G}_m \wedge -$; $(\mathbb{G}_m, S^1)$ bi-spectra.** The functor $\Sigma_t$ on $\mathrm{Spt}_s(k)$ is a left Quillen functor. We may therefore apply the general machinery of [**?**] to create a model category in which $\Sigma_t$ is invertible. The construction of Hovey may be described as $(\mathbb{G}_m, S^1)$ bispectra.

**Definition 3.10.** A $(\mathbb{G}_m, S^1)$ bi-spectrum of spaces over $k$ consists of a bigraded family of spaces $E_{i,j}$, $i, j \geq 0$, equipped with structure maps $\sigma_{i,j} : S^1 \wedge E_{i,j} \to E_{i,j+1}$ and $\mu_{i,j}$ :

[1] Given the following argument, is the part before this necessary? Does this argument actually work?

$\mathbb{G}_m \wedge E_{i,j} \to E_{i+1,j}$ for which the following diagram commutes .

$$\begin{tikzcd}
S^1 \wedge \mathbb{G}_m \wedge E_{i,j} \arrow[r, "S^1 \wedge \tau_{i,j}"] \arrow[d] & S^1 \wedge E_{i+1,j} \arrow[dd, "\sigma_{i+1,j}"] \\
\mathbb{G}_m \wedge S^1 \wedge E_{i,j} \arrow[d, "\mathbb{G}_m \wedge \sigma_{i,j}"'] & \\
\mathbb{G}_m \wedge E_{i,j+1} \arrow[r, "\mu_{i,j+1}"] & E_{i+1,j+1}
\end{tikzcd}$$

*Remark* 9. Note that a $(\mathbb{G}_m, S^1)$ bispectrum is just a $\mathbb{G}_m$-spectrum of $S^1$-spectra. So we may therefore equip it with the projective stable model structure we get from this perspective. We may therefore think of a $(\mathbb{G}_m, S^1)$ bi-spectrum $E_{i,j}$ as a sequence of $S^1$ spectra $E_{i,*}$.

**Definition 3.11.** Let $E$ be a $(\mathbb{G}_m, S^1)$ bispectrum. Define the bigraded stable homotopy presheaf $\tilde{\pi}_{n+m\alpha}$ by the formula

$$U \in \mathrm{Sm}/k \mapsto \mathrm{colim}_r\, \mathcal{H}_\bullet(k)(S^{n+r} \wedge \mathbb{G}_m^{r+m} \wedge U_+, E_{r,r}).$$

Morel's notation is $\tilde{\pi}_n(E)_m = \tilde{\pi}_{n-m\alpha}$. We may also write $\tilde{\pi}_{n,m}(E) = \tilde{\pi}_{n-m+m\alpha}(E)$.
We denote the associated Nisnevich sheaf by $\pi_{n+m\alpha}(E)$.

**Proposition 3.12.** If is $E$ a $(\mathbb{G}_m, S^1)$ bispectrum, the presheaf of homotopy groups may also be calculated as

$$\tilde{\pi}_{n+m\alpha}E(U) = \mathrm{colim}_{s,r}\, \mathcal{H}_\bullet(k)(\mathbb{G}_m^{s+m} \wedge S^{n+r} \wedge U_+, E_{r,s}).$$

[DLØRV, p 217]

**Definition 3.13.** A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable weak equivalence if the following induced map is an isomorphism for all $U \in \mathrm{Sm}/k$.

$$f_* : \tilde{\pi}_{n+m\alpha}(E)(U) \to \tilde{\pi}_{n+m\alpha}(F)(U)$$

**Definition 3.14.** A morphism $f : E \to F$ of $(\mathbb{G}_m, S^1)$ bispectra is an $\mathbb{A}^1$ stable cofibration if $f_0 : E_{0,*} \to F_{0,*}$ is a cofibration of $S^1$ spectra and the map $P.O. \to F_{n+1}$ is a cofibration in the following diagram.

$$\begin{tikzcd}
\mathbb{G}_m \wedge E_{n,*} \arrow[r] \arrow[d] & E_{n+1,*} \arrow[d] \arrow[ddr, bend left, "f_{n+1}"] & \\
\mathbb{G}_m \wedge F_{n,*} \arrow[r] \arrow[drr, bend right] & P.O. \arrow[dr, dotted] & \\
& & F_{n+1,*}
\end{tikzcd}$$

**Proposition 3.15.** The category $\mathrm{Spt}_{s,t}^{\mathbb{A}^1}(k)$ of $(\mathbb{G}_m, S^1)$ bispectra with $\mathbb{A}^1$ stable weak equivalences and $\mathbb{A}^1$ stable cofibrations is a model category. Denote the associated homotopy category of $\mathrm{Spt}_{s,t}^{\mathbb{A}^1}(k)$ by $\mathcal{SH}(k)$.

**Proposition 3.16.** The fibrant bi-spectra are the $\Omega_t$-spectra. [H-Spt, Theorem 3.4]

**Proposition 3.17.** There is a left Quillen functor $\Sigma_t^\infty : \mathrm{Spt}_s^{\mathbb{A}^1}(k) \to \mathrm{Spt}_{s,t}^{\mathbb{A}^1}(k)$ given by $(\Sigma_t^\infty E)_{i,j} = \mathbb{G}_m^i \wedge E_j$ with bonding maps

$$S^1 \wedge \mathbb{G}_m^i \wedge E_j \longrightarrow \mathbb{G}_m^i \wedge S^1 \wedge E_j \longrightarrow \mathbb{G}_m \wedge E_{j+1}$$

and

$$\mathbb{G}^m \wedge \mathbb{G}_m^i \wedge E_j \to \mathbb{G}_m^{i+1} E_j.$$

The right adjoint to $\Sigma_t^\infty$ is denoted by $\Omega_t^\infty$ and is given by $\Omega_t^\infty(E) = E_{0,*}$.

The right derived functor $R\Omega_t^\infty(E)$ is given by the formula

$$R\Omega_t^\infty(E) = \mathrm{colim}_i\, \Omega_t^i E_{i,*}.$$

*Proof.* □

3.5. **Connectivity of $(\mathbb{G}_m, S^1)$ bispectra.**

**Definition 3.18.** A $(\mathbb{G}_m, S^1)$ bispectrum $E$ is said to be $n$-connected if for all $k \leq n$ and all $m \in \mathbb{Z}$, the homotopy sheaves $\pi_{k+m\alpha} E$ vanish.

**Lemma 3.19.** Consider $E = \Sigma_t^\infty \Sigma_s^\infty \mathcal{X}$ for some pointed space $\mathcal{X}$. Then $E$ is $-1$ connected.

*Proof.* We calculate for $n < 0$

$$\begin{aligned}
\mathcal{SH}(k)(S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty U_+, E) &= \mathcal{SH}^{S^1}(k)(S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty U_+, R\Omega_t^\infty E) \\
&= \mathcal{SH}^{S^1}(k)(S^n \wedge \Sigma^\infty U_+, \Omega_t^m \,\mathrm{colim}_i\, \Omega_t^i \Sigma_t^i \Sigma_s^\infty X) \\
&= \mathrm{colim}\, \mathcal{SH}^{S^1}(k)(S^n \wedge \Sigma^\infty U_+, \Omega_t^{m+i} \Sigma_t^i \Sigma_s^\infty X) \\
&= 0
\end{aligned}$$

as $\Omega_t$ preserves connectivity, as does $\Sigma_t$ for suspension spectra. □

**Proposition 3.20.** Let $E \in \mathrm{Spt}_s^{\mathbb{A}^1}(k)$. Consider $\Sigma_t^\infty E \in \mathrm{Spt}_{s,t}^{\mathbb{A}^1}(k)$. If $E$ is $-1$ connected, then so too is $\Sigma_t^\infty E$.

*Proof.* Can I reduced this to suspension spectra by a colimit argument? □

Prove that [Mor03, Definition 5.2.1] gives a $t$-structure on $\mathcal{SH}(k)$.

Now show/realize that Voevodsky's connectivity theorem holds.

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