1. Voevodsky's connectivity theorem for $\mathbb{P}^1$-spectra

The following is from an email from Aravind to me concerning the proof of the connectivity theorem for $\mathbb{P}^1$ spectra.

> Most of the proof appears in Morel's Trieste notes (in the section on the homotopy t-structure for P^1-spectra). First you prove a connectivity result for P^1-stable homotopy sheaves by using Morel's S^1-stable connectivity theorem and studying what happens under G_m loops and G_m-suspension: suspension preserves connectivity, and Morel shows that taking G_m-loops has the effect of making a "contraction". Then, you globalize this.

Our goal is to prove theorem 4.14 of [Voev98], which should be restated in terms of $\mathbb{P}^1$-spectra.

Theorem 1.1. Let X be a pointed smooth scheme over $\mathrm{Spec}(k)$ where k is an infinite field. Let $\mathcal{Y}$ be a pointed space. Then for any $n > \dim(X)$, and any integer m

$$\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) = 0.$$

Remark 1. We can formulate the theorem by using homotopy sheaves. Indeed, if we can show that for any pointed space $\mathcal{Y}$ the homotopy sheaves $\pi^{\mathbb{A}^1}_{n+\alpha m}(\mathcal{Y})$ are -1 connected, the result will follow, as

$$\begin{aligned}\mathcal{SH}(k)(\Sigma^\infty X, S^n \wedge \mathbb{G}_m^m \wedge \Sigma^\infty \mathcal{Y}) &= \mathcal{SH}(k)(S^{-n} \wedge \mathbb{G}_m^{-m} \wedge \Sigma^\infty X, \Sigma^\infty \mathcal{Y}) \\ &= \pi^{\mathbb{A}^1}_{-n-m\alpha}(X).\end{aligned}$$

and by the connectivity theorem, this will vanish whenever $-n < 0$, i.e., whenever $n > 0$. This is a stronger statement than [**?**, theorem 4.14].

The line of attack is then to show that for a pointed space $\mathcal{Y}$, the S^1 suspsension spectrum $\Sigma_s^\infty \mathcal{Y}$ in the stable model category is -1 connected. Then we establish the first connectivity theorem which ensures the $\mathbb{A}^1$ localization of $\Sigma_s^\infty \mathcal{Y}$ is -1 connected. Finally, we show that inverting $\mathbb{G}_m$ does not affect the connectivity, i.e., $\Sigma_m^\infty \Sigma_s^\infty \mathcal{Y}$ is again -1 connected.

2. Assumptions from previous lectures

2.1. **Facts about Nisnevich topology.** Points and neighborhoods. Distinguished squares determine the topology. Relation to etale, zariski, and fpqc topology.

Proposition 2.1. [Mor04, 2.4.1] Let M be a sheaf of abelian groups on Sm/k, and let $X \in \mathrm{Sm}/k$ with Krull dimension d. Then whenever $n > d$, $H^n_{Nis}(X; M) = 0$.

Proposition 2.2. [Mor04, 2.4.1] For any $X \in \mathrm{Sm}/k$, and for any $x \in X(k)$, there is an isomorphism of pointed sheaves of sets in the Nisnevich topology

$$X/(X - \{x\}) \cong \mathbb{A}^n/(\mathbb{A}^n - \{0\}).$$

2.2. **Unstable model category** $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$**.**

1. Don't forget the adjunction for sheafification $a_{Nis} \dashv U$. i.e., to give a map $a_{Nis}\mathcal{X} \to \mathcal{Y}$, of sheaves, it is equivalent to just give a map $\mathcal{X} \to U\mathcal{Y}$.
2. Give the unstable model category the injective model structure to start. (This is Morel's choice, so we stick with it)
3. Weak equivalences: $\mathcal{X} \to \mathcal{Y}$ iff for all U the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ a simplicial w.e.

(4) Cofibs: $\mathcal{X} \to \mathcal{Y}$ is a cofib. iff for any $U \in \mathrm{Sm}/k$ the map $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a monomorphism.
(5) Fibrations: what they need to be.
(6) The ass. htpy. cat. to the injective model category structure is denoted $\mathcal{H}_s(k)$. The pointed htpy. cat. is denoted $\mathcal{H}_{s,\bullet}(k)$.
(7) Important properties/constructions in these categories: Cartesian closed, i.e., have smash/tensor $\times$ and $\wedge$, with right adjoints $\underline{Hom}$ and $\underline{Hom}_\bullet$. Representable sheaf functor and constant simplicial set functor.

Definition 2.3. Internal hom. (Ref: P. Pelaez)

Let $U \in \mathrm{Sm}/k$. Let Δ^n_U denote the simplicial sheaf given by

$$(V, m) \in \mathrm{Sm}/k \times \Delta^{op} \to \mathrm{Sm}/k(V, U) \times \Delta^n_m.$$

In other words, $\Delta^n_U = (rU) \times c\Delta^n$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be spaces. The internal hom in the unpointed category is given by the formula

$$(U, m) \in \mathrm{Sm}/k \times \Delta \to \mathrm{Hom}_{\Delta^{op}\mathrm{Shv}}(X \times \Delta^m_U, Y).$$

How to describe the adjunction?

$$\mathrm{Hom}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$$

Certainly we may send a map $g \in \mathrm{Hom}(\mathcal{Y}, \underline{Hom}(\mathcal{X}, \mathcal{Z}))$ to the map $\eta(g)$ given by

$$\begin{aligned} \eta(g)(U, n) :& \mathcal{X}_n(U) \times \mathcal{Y}_n(U) \to \mathcal{Z}_n(U) \\ & (a, b) \to g(U, n)(U, n)(a, \mathrm{id}, \mathrm{id}). \end{aligned}$$

Why is this a bijection?

Definition 2.4. Internal hom in the pointed category. Consider $\mathcal{X}$ and $\mathcal{Y}$ pointed spaces. For a point $x \in \mathcal{X}$, there is an evaluation map $ev_x : \underline{Hom}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y}$, where at $(U, n) \in (\mathrm{Sm}/k \times \Delta)^{op}$ we send $g : \mathcal{X} \times \Delta^n_U \to \mathcal{Y}$ to $g(U, n)(x, \mathrm{id}, \mathrm{id}) \in \mathcal{Y}_n(U)$. This makes sense as we have the map

$$g(U, n) : \mathcal{X}_n(U) \times \mathrm{Sm}/k(U, U) \times \Delta^n_n \to \mathcal{Y}_n(U).$$

The pointed internal hom $\underline{Hom}_\bullet(\mathcal{X}, \mathcal{Y})$ is the fiber $ev_x^{-1}(y)$.

2.3. $\mathbb{A}^1$ localization.

(1) See [**?**, Prop 2.3.3.] for details on the various properties of fibrant objects in the unstable motivic category.
(2)

Definition 2.5. A space $\mathcal{X}$ is called $\mathbb{A}^1$ local if for any smooth scheme U, the canonical map

$$\mathrm{Hom}(rU, \mathcal{X}) \to \mathrm{Hom}(rU \times \mathbb{A}^1, \mathcal{X})$$

is a bijection.
(3)

Definition 2.6. A map $f : \mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$ weak equivalence if

$$\mathrm{Hom}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}(\mathcal{X}, \mathcal{Z})$$

is a bijection for every $\mathbb{A}^1$ local space $\mathcal{X}$.

(4) The unstable motivic homotopy category is obtained by left Bousfield localization of the injective model category structure on spaces with respect to the class of maps $W = W_{\mathbb{A}^1} = \{U \times \mathbb{A}^1 \to U \,|\, U \in \mathrm{Sm}/k\}$.

(5) The general theory of Bousfield localization then gives a localization functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \to L_W \mathcal{H}_s(k)$; denote the category $L_W \mathcal{H}_s(k)$ by $\mathcal{H}(k)$. The localization functor sends $\mathbb{A}^1$ weak equivalences to isomorphisms.

See [**?**, Definition 3.3.1]. Keep the same underlying category, the weak equivalences are the $\mathbb{A}^1$-local weak equivalences, do not change the cofibrations from the ones in the injective model structure on $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k, Nis)$, and let the fibrations be what they need to be.

(6) Morel writes $L_{\mathbb{A}^1}$ for the (derived) functor which sends a space (without base point) $\mathcal{X}$ to an $\mathbb{A}^1$-localizion.

(7) Morel gives an explicit construction which takes a pointed space $\mathcal{Y}$ and produces a space $L^\infty \mathcal{Y}$ which is $\mathbb{A}^1$ local and a map $\mathcal{Y} \to L^\infty \mathcal{Y}$ which is an $\mathbb{A}^1$ weak equivalence.

Let $\mathcal{Y}_f$ be the functorial fibrant replacement of $\mathcal{Y}$ with respect to the injective model structure. Consider $\mathbb{A}^1$ to be pointed at 0. Morel defines $L^{(1)}(\mathcal{Y})$ to be the cone of the map $ev_1 : \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{Y}_f) \to \mathcal{Y}_f$. So there is a map $\mathcal{Y} \to\to L^{(1)}(\mathcal{Y})$ obtained from the trivial cofibration $\mathcal{Y} \to \mathcal{Y}_f$ and the defining map $\mathcal{Y}_f \to L^{(1)}(\mathcal{Y})$.

So $L^{(1)}(-)$ is a functor with a natural transformation $\eta : \mathrm{id} \to L^{(1)}(-)$. Define by induction $L^{(n)}(\mathcal{Y}) = L^{(1)}(L^{(n-1)}(\mathcal{Y}))$. There is thus a directed system $L^{(n-1)}(\mathcal{Y}) \to L^{(n)}(\mathcal{Y})$. Denote the colimit/direct limit of this directed system by $L^\infty(\mathcal{Y})$.

Proposition 2.7. The natural morphism $\mathcal{Y} \to L^\infty(\mathcal{Y})$ is an $\mathbb{A}^1$ weak equivalence, and $L^\infty(\mathcal{Y})$ is $\mathbb{A}^1$ local.

(8) So we can use the functor L^∞ as an $\mathbb{A}^1$ localization functor.

(9)

Definition 2.8. Let $\mathcal{X}$ be a space. Define $\pi_0(\mathcal{X})$ to be the sheaf on Sm/k associated to $U \to \pi_0(\mathcal{X}(U))$. A space $calX$ is called 0-connected if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

Let $(\mathcal{X}, x)$ be a pointed space. Define $\pi_n(\mathcal{X})$ to be the sheafification of the presheaf on Sm/k given by

$$U \to \pi_n(\mathcal{X}(U)).$$

A pointed space $\mathcal{X}$ is called n-connected if it is 0-connected and for all $i \leq n$, the sheaves $\pi_i(\mathcal{X})$ are trivial.

(10)

Proposition 2.9. Let $\mathcal{X}$ be a 0-connected simplicial sheaf. Then $L^\infty \mathcal{X}$ is also 0-connected.

Proof. M.V. IHES paper? Sketch of argument? It shouldn't be too hard by chasing components around. □

(11) For a sheaf of abelian groups M on Sm/k and a natural number n, a Dold-Kan construction gives a simplifical presheaf $K(M, n)$. It is called the Eilenberg-MacLane

spectrum of type (M, n) and has homotopy sheaves as expected:

$$\pi_m(K(M,n)) = \begin{cases} 0 & \text{if } m \neq n \\ M & \text{if } m = n \end{cases}.$$

Important equation for this construction. For $X \in \mathrm{Sm}/k$, M a sheaf of Abelian groups,

$$\mathcal{H}_s(k)(rX, K(M,n)) \cong H^n_{Nis}(X; M).$$

(12) Use square brackets to denote the maps in the unstable pointed (motivic) homotopy category, i.e., $[\mathcal{X}, \mathcal{Y}] = \mathcal{H}_\bullet(k)(\mathcal{X}, \mathcal{Y})$.

Use $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ for the sheaf of homotopy groups in the motivic category, i.e., $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(L^\infty \mathcal{X})$. This is also obtained by sheafifying the presheaf given by

$$U \in \mathrm{Sm}/k \mapsto [S^n \wedge U_+, \mathcal{X}].$$

2.4. **Homotopy purity, connectedness calculations.**

Proposition 2.10. Weak n-connectedness is equivalent to n-connectedness

Proof. Details? □

Proposition 2.11. If V is an irreducible, smooth k-scheme, and $U \subseteq V$ is a dense open subset, the space $L_{\mathbb{A}^1}(V/U)$ is 0-connected.

Proof. Seems to use that local structure of spaces is given by $\mathbb{A}^n/(\mathbb{A}^n - \{0\}) \wedge L_+$. How do all the reductions work? Also uses some results specific to working over perfect fields. □

2.5. S^1**-spectra.**

Definition 2.12. Let $\mathrm{Spt}_s(k)$ denote the category of S^1-spectra of spaces $\Delta^{op}\mathrm{Shv}(\mathrm{Sm}/k)$. We first endow this category with the projective model structure, i.e., a map $f : E \to F$ is a weak equivalence iff for any n the map $f_n : E_n \to F_n$ is a w.e.; a map $f : E \to F$ is a fibration iff for all n the map $f_n : E_n \to F_n$ is a fibration. The cofibrations are characterized by the property that $f : E \to F$ is a cofib iff $f_0 : E_0 \to F_0$ is a cofib and for any $n \geq 1$

$$\begin{tikzcd}
S^1 \wedge E_{n-1} \arrow[r, "\sigma_{n-1}"] \arrow[d, "S^1 \wedge f_{n-1}"'] & E_n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\
S^1 \wedge F_{n-1} \arrow[r] \arrow[drr, bend right, "\sigma_{n-1}"] & P.O. \arrow[dr, dotted] & \\
 & & F_n
\end{tikzcd}$$

This model structure does not actually invert $S^1 \wedge -$. To accomplish this, we must localize with respect to the stable equivalences.

Definition 2.13. A map $f : E \to F$ of S^1-spectra is a stable equivalence iff for any $n \in \mathbb{Z}$ the induced map of homotopy sheaves $\pi_n(f) : \pi_n(E) \to \pi_n(F)$ is an isomorphism.

The stable model category structure on $\mathrm{Spt}_s(k)$ is given by declaring the weak equivalences to be the stable weak equivalences, and the cofibrations to be the same as those for the projective model structure. This is indeed a left Bousfield localization, but we will not describe it further as such.

Definition 2.14. Let $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$ denote the category of S^1 spectra endowed with the stable model category structure localized at the collection of maps $\{\Sigma^\infty U_+ \wedge \mathbb{A}^1 \to \Sigma^\infty U_+ \,|\, U \in \mathrm{Sm}/k\}$.

Let $\mathcal{SH}^{S^1}(k)$ denote the homotopy category associated to $\mathrm{Spt}_s^{\mathbb{A}^1}(k)$. We will use $\mathcal{SH}_s^{S^1}(k)$ to denote the homotopy category of $\mathrm{Spt}_s(k)$.

Definition 2.15. Let E be an S^1 spectrum of spaces. Let π_n denote the sheaf obtained by taking the colimit of the directed system $\pi_{n+r}(E_r)$ in $\underline{\mathrm{Ab}}(\mathrm{Sm}/k, Nis)$. That is,

$$\pi_n(E) = \mathrm{colim}_r\, \pi_{n+r}(E_r).$$

In particular, for a $U \in \mathrm{Sm}/k$, we have

$$\pi_n(E)(U) = \mathrm{colim}_r\, \pi_{n+r}(E_r)(U).$$

Definition 2.16. An S^1-spectrum E is said to be n-connected if for any $m \leq n$, the homotopy sheaves $\pi_m(E)$ are trivial.

Definition 2.17. There is a left Quillen functor $\Sigma_s^\infty : \mathrm{Spc}_\bullet \to \mathrm{Spt}_s^{\mathbb{A}^1}(k)$ given by $(\Sigma^\infty \mathcal{Y})_n = (S^1)^{\wedge n} \wedge \mathcal{Y}$ with the evident bonding maps. The right adjoint to this functor is given by "evaluation at 0", i.e., $\Omega^\infty(E) = E_0$.

Remark 2. The right derived functor $R\Omega^\infty : \mathcal{SH}^{S^1}(k) \to \mathcal{H}_\bullet(k)$ is given by the formula

$$R\Omega^\infty(E) = \mathrm{colim}_i\, \Omega_s^i E_i.$$

This comes from the fact that fibrant S^1 spectra are exactly the Ω spectra, and the description of the fibrant replacement functor.

2.6. t **structure.**

Definition 2.18. Let $\mathfrak{C}$ be a triangulated category. A t-structure on $\mathfrak{C}$ is a pair of full subcategories $(\mathfrak{C}_{\geq 0}, \mathfrak{C}_{\leq 0})$ which satisfies

(1) $(\forall X \in \mathfrak{C}_{\geq 0})(\forall Y \in \mathfrak{C}_{\leq 0})(\mathrm{Hom}_{\mathfrak{C}}(X, Y[-1]) = 0)$
(2) $\mathfrak{C}_{\geq 0}[1] \subseteq \mathfrak{C}_{\geq 0}$ and $\mathfrak{C}_{\leq 0}[-1] \subseteq \mathfrak{C}_{\leq 0}$
(3) for any $X \in \mathfrak{C}$ there exists a distinguished triangle

$$Y \to X \to Z \to Y[1]$$

for which $Y \in \mathfrak{C}_{\geq 0}$, $Z \in \mathfrak{C}_{\leq 0}[-1]$..

The heart of a t-structure is the full subcategory given by $\mathfrak{C}_{\geq 0} \cap \mathfrak{C}_{\leq 0}$.

Definition 2.19 (t-structure on $\mathcal{SH}_s^{S^1}(k)$)**.** Define $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n < 0$.

Define $\mathcal{SH}_s^{S^1}(k)_{\leq 0}$ to be the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of objects E such that $\pi_n(E) = 0$ whenver $n > 0$.

Theorem 2.20. The triple $(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$ is a t-structure on $\mathcal{SH}_s^{S^1}(k)$.

Proposition 2.21. [Mor03, Lemma4.2.4] The functor $L^\infty : \mathrm{Spt}_s^{S^1}(k) \to \mathrm{Spt}_{s,\mathbb{A}^1}^{S^1}(k)$ identifies the $\mathbb{A}^1$-localized S^1 stable homotopy category with the homotopy category of $\mathbb{A}^1$-local S^1 spectra.

Theorem 2.22 (S^1 stable connectivity theorem)**.** Let $E \in \mathcal{SH}_s^{S^1}(k)$, and suppose that whenever $n < 0$ the sheaf $\pi_n E = 0$. Then for all $n < 0$, $\pi_n L_{\mathbb{A}^1} E = 0$.

Theorem 2.23. The pair $(\mathcal{SH}_{\geq 0}^{S^1}(k), \mathcal{SH}_{\leq 0}^{S^1}(k))$ is a t-structure on the category $\mathcal{SH}^{S^1}(k)$.

Definition 2.24. Strictly $\mathbb{A}^1$ invariant sheaf of Abelian groups.

If M is strictly $\mathbb{A}^1$ invariant sheaf of groups, define the Eilenberg-MacLane spectrum HM associated to it.

Proposition 2.25. HM is $\mathbb{A}^1$ local iff M is strictly $\mathbb{A}^1$ invariant.

Proposition 2.26. The heart of the homotopy t structure is equivalent to the category of strictly $\mathbb{A}^1$ invariant sheaves.

3. Inverting $\mathbb{G}_m$; $\mathbb{P}^1$ spectra

3.1. $\mathbb{G}_m$ suspension and loops.

3.2. Homotopy sheaves of $\underline{Hom}(\mathbb{G}_m, E)$.

Definition 3.1. Contraction of a sheaf of pointed sets G (or abelian groups G). G_{-1} Introduce notation G_{con} to help clear up index confusion.

Theorem 3.2. [Mor03, Lemma 4.3.11] $\pi_n(\underline{Hom}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$ is iso.

Lemma 3.3. If M is a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups, then

$$\underline{Hom}(\mathbb{G}_m, HM) \cong H(M_{-1})$$

Lemma 3.4. When $n \neq 0$,

$$[\Sigma^\infty \mathbb{G}_m, HM[n]]_s^{S^1} = 0. \tag{1}$$

The following map is an iso.

$$[\Sigma^\infty \mathbb{G}_m, HM]_s^{S^1} \to [S^0, H(M_{-1})]_s^{S^1} \tag{2}$$

References

[A-1974] Adams, J.F., *Stable Homotopy and Generalized Homology.* Chicago Lectures in Mathematics, (1974).

[B] Blander, Benjamin, *Local Projective Model Structures on Simplicial Presheaves.* K-Theory, **24** (2001) 283–301.

[DHI] Dugger, Dan; Hollander, Sharon; Isaksen, Dan, *Hypercovers and simplicial presheaves.*

[DLØRV] Dundas, B.; Levine, M.; Østvær, P.; Röndigs, O.; Voevodsky, V., *Motivic Homotopy Theory.* Springer (2000).

[Hir] Phillip, Hirschhorn, *Model Categories and Their Localization.* AMS (2003).

[H-Mod] Hovey, Mark, *Model Categories.* online preprint (1991).

[H-Spt] Hovey, Mark, *Spectra and symmetric spectra in general model categories.* journal? (2001).

[J] Jardine, J.F., *Simplicial presheaves.* Journal of Pure and Applied Algebra, **47** (1987) 35–87.

[Mor03] Morel, Fabien, *An introduction to* $\mathbb{A}^1$ *homotopy theory.*

[Mor04] Morel, Fabien, *On the motivic* π_0 *of the sphere spectrum.* NATO science series.

[Mor05] Morel, Fabien, *The stable* $\mathbb{A}^1$ *connectivity theorems.* preprint (2004).

[Voev98] Voevodsky, Vladimir. $\mathbb{A}^1$-*Homotopy Theory.* Doc. Math. J., (1998) pp. 579–604.