Freie Universität Berlin
École Normale Supérieure de Rennes

Low-degree covers in algebraic geometry

Gabriel Lepetit

Thesis of first year of Master, written under the supervision of Fabio Tonini

Magistère 2 2015-2016

This document has been redacted from May 2^{nd} to July 1^{st} 2016 in the :

Freie Universität Berlin - Mathematisches Institut
Arnimallee 3
14195 Berlin

My advisor was Fabio TONINI (`http://www.mi.fu-berlin.de/users/tonini/`), researcher in the Arithmetische Geometrie team directed by Hélène ESNAULT (`http://www.mi.fu-berlin.de/users/esnault/`).

Contents

◊ ◊ ◊ ◊

Acknowledgements

First of all, I would like to address special thanks to my advisor, Fabio Tonini, who patiently answered my questions and enthusiastically introduced me to the world of algebraic geometry. I thank also Hélène Esnault who accepted to receive me in Berlin, and all the members of the team and guests with whom I had the chance to have fruitful discussions, in particular Lei Zhang, Marta Pieropan and Marco d'Addezio. I am also grateful to Lars Kindler and his wonderful coffee machine which deliciously boosted my productivity.

Introduction

This thesis aims to define a simple notion of cover in algebraic geometry that would reasonably look like the one we know in topology and that is central in differential geometry and algebraic topology. In this context, if Y is a topological space, a cover of Y is a continuous map $f : X \to Y$ such that for all $y \in Y$, there exists an open neighbourhood V of y and a non-empty set I_y such that $f^{-1}(V) = \bigsqcup_{i \in I_y} U_i$ and $f_{|U_i}$ is a homeomorphism from U_i onto V. We know that if Y is connected, and if the fibers are finite sets, then they all have the same number of elements which is the common cardinality of the sets I_y, $y \in Y$. This is what we call an unramified cover : the copies of Y produced by the cover don't cross. We would like to have a similar algebraic notion, that would allow ramification, as in the example 2.1.5 : the cardinality of the fibers would still be the same, but only if we count the elements with multiplicity.

The notion of cover plays an important role in the current research in algebraic geometry, for several reasons. First, it is a convenient way to build new spaces – for example algebraic varieties – out of the ones we already know. Moreover, in the theory of moduli spaces, the study of the Hurwitz spaces (see [12]), which are parametrizing covers of curves, have been an active field of research for the last years. The Hurwitz spaces are not simply studied as *sets* of covers, but as schemes whose closed points are the covers of curves. The covers are often studied together with group actions, especially in the theory of *Galois covers* (cf [11]). For instance, $\mathbb{Z}/2\mathbb{Z}$ acts on the double covers, but this situation does not recur for higher-degree covers, as $\mathbb{Z}/3\mathbb{Z}$ does not act on every 3-cover.

A cover of a scheme Y is simply defined as an affine morphism of schemes $f : X \to Y$ such that $f_*\mathscr{O}_X$ is a locally free $\mathscr{O}_Y$-algebra of finite rank (cf 2.1.1). We will first present briefly in the first chapter the prerequisites of algebraic geometry necessary to understand the study of covers, in particular the quasi-coherent sheaves of algebras.

In the second and third chapters, we give a full description of the categories of covers of degree 2 and 3. The goal is to extract the essential data of the multiplication map, and the trace map for sheaves of algebras (proposition 2.1.6) play a central role in this study. It happens that the covers of degree 1 are the isomorphisms (example 2.1.4), the ones of degree 2 correspond classically to the pairs $(\mathscr{L}, \sigma)$ where $\mathscr{L}$ is a locally free sheaf of rank 1 and σ is a morphism from $\mathscr{L} \otimes \mathscr{L}$ to $\mathscr{O}_Y$ (theorem 2.2.1), and the 3-covers, as described in [9], are similar to the $(\mathscr{E}, \delta)$ where $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of rank 2 and $\delta : S^3\mathscr{E} \to \Lambda^2\mathscr{E}$ is a morphism (theorem 3.0.7).

For higher degree, it is not possible anymore to have a general description of the covers with locally free sheaves and maps without further relations, but G. Casnati and T. Ekdahl ([3] and [2]) developed the theory of *Gorenstein covers*, leading to a particular study of the 4 and 5-covers, while in [7], an analysis of the quadruple covers of algebraic varieties was done. Rita Pardini ([10]) studied the triple covers in characteristic 3, a case where our method doesn't work. The work of M. Bolognesi and A. Vistoli ([1]) is another relevant article involving the triple covers.

Chapter 1

Sheaves and schemes

In all the thesis, I will assume that the basic facts of commutative algebra are known, and use freely the vocabulary of the theory of categories. The main results we use are summed up in [6] pp 541-545 (categories), [8], chapter 1 (tensor product and localization).

The main reference for this chapter is the book of Qing Liu ([8]), chapter 2.

1.1 Sheaves of rings

Definition 1.1.1

A presheaf of sets on the topological space X is a contravariant functor $\mathscr{F}$ from the category of the open subsets of X into the category of sets. More concretely, it is the data of a set $\mathscr{F}(U)$, for every U open subset of X, and, for $V \subset U$ open subsets, restriction maps $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ satisfying the following axioms :

- *For all $W \subset V \subset U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.*
- $\rho_{UU} = \mathrm{id}_U$.

The elements of $\mathscr{F}(X)$ are called *global sections*.

Example 1.1.2. The most intuitive example of a presheaf is the presheaf of rings on X $\mathscr{F} : U \mapsto \mathscr{C}^0(U,\mathbb{R})$, endowed with the restrictions $\rho_{UV} : f \mapsto f_{|V}$, where $\mathscr{C}^0(U,\mathbb{R})$ is the set of the continuous fonctions from U to $\mathbb{R}$.

Because of this example, we denote, for an arbitrary presheaf, $\rho_{UV}(s) = s_{|V}$.

Definition 1.1.3

A sheaf on X is a presheaf on X $\mathscr{F}$ satisfying the following axioms :

- *(Uniqueness) For U open of X and $U = \bigcup_{i\in I} U_i$ open covering of U, if $s, t \in \mathscr{F}(U)$ are such that $\forall i \in I, s_{|U_i} = t_{|U_i}$, then $s = t$.*
- *(Glueing) For U open of X and $U = \cup_{i\in I} U_i$ open covering of U, if for every $i \in I$, $s_i \in \mathscr{F}(U_i)$ and $\forall i, j \in I, s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$, then there exists a $s \in \mathscr{F}(U)$ such that $\forall i \in I, s_{|U_i} = s_i$.*

Definition 1.1.4

A morphism of presheaves of sets $\alpha : \mathscr{F} \to \mathscr{G}$ is a family of morphisms $\mathscr{F}(U) \xrightarrow{\alpha(U)} \mathscr{G}(U)$ compatible with the restrictions : for every $V \subset U \subset X$, the following diagram commutes.

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\alpha(U)} & \mathscr{G}(U) \\ \rho^{\mathscr{F}}_{UV} \downarrow & & \downarrow \rho^{\mathscr{G}}_{UV} \\ \mathscr{F}(V) & \xrightarrow{\alpha(V)} & \mathscr{G}(V) \end{array}$$

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, we say that α is a morphism of sheaves.

Remark. We can also consider presheaves of groups, abelian groups, rings, A-modules or A-algebras where A is a fixed ring. In each case, the restriction maps are morphisms in the considered category, as well as the morphisms of presheaves.

We see that a sheaf if a way to pass from the local data to the global one.

Definition 1.1.5

Let $\mathscr{F}$ be a sheaf on X and $x \in X$. The stalk of $\mathscr{F}$ at x is

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

where the limit is taken on the inductive system (for the inclusion) of the open subsets of X containing x.

Concretely, in $\mathscr{F}_x$, any element is defined on a open neighbourhood of x and two elements are identified if they coincide on an open neighbourhood of x.

We can also take the stalk at x of a morphism $\alpha : \mathscr{F} \to \mathscr{G}$, by setting $\alpha_x : s_x \to (\alpha(U)(s))_x$ if $s \in \mathscr{F}(U)$. α_x is a morphism.

Proposition 1.1.6

Let $\mathscr{F}$ be a sheaf on X, $U \subset X$ an open subset, and $s, t \in \mathscr{F}(U)$. Then $s = t \Leftrightarrow \forall x \in U, s_x = t_x$.

Proposition 1.1.7

If $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on X, then α is an isomorphism if and only if α_x is an isomorphism for all $x \in X$.

The following proposition allows us to define a sheaf starting with a presheaf, and this operation, called sheafification, preserve the stalks and the morphisms.

Proposition 1.1.8 (*sheafification*)

Let $\mathscr{F}$ be a presheaf on X. There exists a unique sheaf $\mathscr{F}^\dagger$ on X, together with a morphism $\theta : \mathscr{F} \to \mathscr{F}^\dagger$ satisfying the following universal property :

For every $u : \mathscr{F} \to \mathscr{G}$ morphism, where $\mathscr{G}$ is a sheaf, there exists a unique morphism $\tilde{u} : \mathscr{F}^\dagger \to \mathscr{G}$ such that $u = \tilde{u} \circ \theta$.

Moreover, for all $x \in X$, θ_x is an isomorphism between $\mathscr{F}_x$ and $\mathscr{F}^\dagger_x$.

If Psh(X) (resp. Sh(X)) is the category of the presheaves (resp. sheaves) on X, sheafification defines a functor $\cdot^\dagger : \begin{array}{ccc} \mathrm{Psh}(X) & \longrightarrow & \mathrm{Sh}(X) \\ \mathscr{F} & \longmapsto & \mathscr{F}^\dagger \end{array}$. If u is a morphism of presheaves from $\mathscr{F}$ to $\mathscr{G}$, $u^\dagger$ is the unique morphisme making this diagram commute :

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{u} & \mathscr{G} \\ \theta_{\mathscr{F}} \downarrow & & \downarrow \theta_{\mathscr{G}} \\ \mathscr{F}^\dagger & \xrightarrow{u^\dagger} & \mathscr{G}^\dagger \end{array}$$

Let us remark that the family of morphisms $(\theta_{\mathscr{F}})$ define a natural transformation between $\mathrm{id}_{\mathrm{Psh}(X)}$ and $i \circ \cdot^{\dagger}$ where $i : \mathrm{Sh}(X) \to \mathrm{Psh}(X)$ is the inclusion functor.

We can keep in mind for the rest of the thesis that, since the morphisms θ preserve the stalks, it is not a problem to define a morphism between presheaves if we want to study the sheafified morphism.

We will now explain how it is possible to define a sheaf on a basis of open subsets (see [5], pp. 25-28, for more details).

Let X be a topological space, and $\mathscr{B}$ a basis of open subsets of X (*i.e.* for all open U of X, there exist elements U_i of $\mathscr{B}$ such that $U = \bigcup_i U_i$).

A $\mathscr{B}$-presheaf of sets (resp. groups, rings) on X is a contravariant functor from $\mathscr{B}$, seen as a subcategory of the category of the open subsets of X, to the category of sets (resp. groups, rings).

Definition 1.1.9

A $\mathscr{B}$-sheaf on X is a $\mathscr{B}$-presheaf $\mathscr{F}$ satisfying the two following axioms :

- *(Uniqueness) If U is an element of $\mathscr{B}$ such that $U = \bigcup_{i\in I} U_i$, $U_i \in \mathscr{B}$, and $s \in \mathscr{F}(U)$ is such that $s_{|U_i} = 0 \forall i \in I$, then $s = 0$.*
- *(Glueing) If U is an element of $\mathscr{B}$ such that $U = \bigcup_{i\in I} U_i$, $U_i \in \mathscr{B}$, if for every $i \in I$, $s_i \in \mathscr{F}(U_i)$ and $\forall i, j \in I, \forall W \in \mathscr{B}$ such that $W \subset U_i \cap U_j$, we have $s_{i|W} = s_{j|W}$, then there exists a $s \in \mathscr{F}(U)$ such that $\forall i \in I, s_{|U_i} = s_i$.*

We can define stalks of a $\mathscr{B}$-presheaf the same way we did with presheaves, as well as morphisms of $\mathscr{B}$-presheaves.

Proposition 1.1.10

Let $\mathscr{F}^0$ be a $\mathscr{B}$-sheaf on X. Then there exists a unique (up to isomorphism) sheaf $\mathscr{F}$ on X such that $\mathscr{F}(U) = \mathscr{F}^0(U)$ for every U in $\mathscr{B}$. The sheaf $\mathscr{F}$ is defined by

$$\forall V \subset X, \mathscr{F}(V) = \varprojlim_{\substack{U\in\mathscr{B}\\ U\subset V}} \mathscr{F}(U)$$

Moreover, if $x \in X$, then $\mathscr{F}_x = \mathscr{F}^0_x$.

Remark. Thanks to the universal property of the projective limit, we can also extend a morphism of $\mathscr{B}$-sheaves in a unique way in a morphism between the corresponding sheaves. Moreover, the stalks of these two morphisms will coincide.

Therefore, this construction gives an equivalence between the categories of the $\mathscr{B}$-sheaves on X and of the sheaves on X.

If $f : X \to Y$ is a continuous map, the inverse and direct image functors are used to pass from a sheaf on X to a sheaf on Y.

- If $(\mathscr{F}, \rho)$ is a sheaf on X, the direct image $f_*\mathscr{F}$ of $\mathscr{F}$ is the sheaf defined by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ for all $V \subset Y$ with the restrictions $\mu_{VV'} = \rho_{f^{-1}(V)f^{-1}(V')}$.

  If $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on X, then $f_*\alpha : f_*\mathscr{F} \to f_*\mathscr{G}$ is the morphism of sheaves defined by $(f_*\alpha)(V) = \alpha(f^{-1}(V))$ for all $V \subset Y$.

We have a morphism $: \begin{array}{ccc} (f_*\mathscr{F})_{f}(x) & \longrightarrow & \mathscr{F}_x \\ s_{f(x)} & \longmapsto & s_x \end{array}$ but it isn't injective in general. But for instance if f induces a homeomorphism from X into its image, then this morphism will be an isomorphism.

- If $\mathscr{G}$ is a sheaf on Y, the inverse image $f^{-1}\mathscr{G}$ of $\mathscr{G}$ is the sheaf defined by :

$$\forall U \subset X, \big(f^{-1}\mathscr{G}\big)(U) = \varinjlim_{f(U)\subset V} \mathscr{G}(U)$$

We define the restrictions using the action of the inductive limits on the morphisms, and likewise taking the inverse image of a morphism of sheaves makes sense.

What we should keep in mind is that if $x \in X$, then $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$, and if $i : V \hookrightarrow Y$ is an inclusion, then $i^{-1}\mathscr{G}$ is the restricted sheaf $\mathscr{G}_{|V} : V' \subset V \mapsto \mathscr{G}(V')$.

Finally, we will quite often use exact sequences of sheaves. We should underline the fact that if γ is a morphism of sheaves of groups, $U \mapsto \operatorname{Im}(\gamma(U))$ is only a presheaf and not a sheaf. That is why $\operatorname{Im}\gamma$ is defined as the sheafification of this presheaf. Besides, $U \mapsto \ker(\gamma(U))$ is a sheaf called $\ker\gamma$.

Definition 1.1.11

A sequence of morphisms of sheaves $\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ is said to be exact if $\ker\beta = \operatorname{Im}\alpha$.

Proposition 1.1.12

A sequence $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ of sheaves on X is exact if and only if for all $x \in X$, the sequence $\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$ is exact.

1.2 Schemes

Definition 1.2.1

A locally ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X such that for all $x \in X$, $\mathscr{O}_{X,x} := (\mathscr{O}_X)_x$ is a local ring. If its maximal ideal is $\mathfrak{m}_x$, then the field $k(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$ is called the residue field of X at x.

Definition 1.2.2

A morphism of locally ringed spaces $f : X \to Y$ is $f = (\underline{f}, f^\#)$ where $\underline{f}$ is a continuous map between X and Y, and $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of sheaves such that for every $x \in X$, $f^\#_x = \mathscr{O}_{Y,f(x)} \longrightarrow (f_*\mathscr{O}_X)_{f(x)} \xrightarrow{\text{can}} \mathscr{O}_{X,x}$ is a local ring morphism.

We will omit the $\mathscr{O}_X$ in the notation for simplicity.

Remark.

- If $f : X \to Y$ and $g : Y \to Z$ are morphisms of locally ringed spaces, then $f \circ g$ is $(\underline{f} \circ \underline{g}, f_* g^\# \circ f^\#)$.

- If U is an open subset of X and $i : (U, \mathscr{O}_{X|U}) \to (X, \mathscr{O}_X)$ is the inclusion morphism, with

$$\forall V \subset X,\ i^\#(V) : \begin{array}{ccc} \mathscr{O}_X(V) & \longrightarrow & (i_*\mathscr{O}_{X|U})(V) = \mathscr{O}_X(U \cap V) \\ s & \longmapsto & s_{|U\cap V} \end{array}$$

then the *restriction* of a morphism $f : (X, \mathscr{O}_X) \to Y$ to U is $f_{|U} = f \circ i$.

Let A be a ring. We want to put a structure of locally ringed space on $X = \mathrm{Spec}(A)$, set of the prime ideals of A.

Let us recall briefly that we can define a topology on X, called the Zariski topology, whose closed subsets are the $V(I) = \{\mathfrak{p} \in X : I \subset \mathfrak{p}\}$, I ideal of A. The open subsets $D(f) = X \setminus V(fA)$, $f \in A$ are called principal open subsets and form a basis $\mathscr{B}$ of open subsets of X.

We set $\mathscr{O}_X(D(f)) = A_f$, localization of A at f. This definition makes sense, since if $D(f) = D(g)$,$A_f \simeq A_g$. If $D(fg) \subset D(f)$, the restriction map is given by $A_f \to A_{fg} = (A_f)_g$. The association $\mathscr{O}_X$ is a $\mathscr{B}$-sheaf and therefore can be extended to a sheaf on X. We call $\mathscr{O}_X$ the *structural sheaf* of X.

Furthermore, if $x = \mathfrak{p} \in X$, then $\mathscr{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$, localization of A by $A \setminus \mathfrak{p}$.
See [8], p.17, for a more detailed proof.

Definition 1.2.3

- *An affine scheme is a locally ringed topological space that is isomorphic to some $X = \mathrm{Spec}\,A$ endowed with his structural sheaf.*
- *A scheme is a locally ringed space $(X, \mathscr{O}_X)$ such that there exists a open covering $X = \bigcup_{i\in I} U_i$ of X such that for all $i \in I$, $(U_i, \mathscr{O}_{X|U_i})$ is an affine scheme.*

If X is a scheme, an open subset U such that the open subscheme $(U, \mathscr{O}_{X|U})$ is affine is said to be *open affine*.

Finally, we will study the morphisms of schemes, *i.e.* the morphisms of locally ringed spaces between schemes.

Definition 1.2.4

An open embedding is a morphism $i : X \to Y$ such that $\underline{i}$ is a topological open embedding (i.e. induces an homeomorphism from X onto $i(X)$ open), and $\forall x \in X$, $i_x^{\#} : \mathscr{O}_{Y,i(x)} \to \mathscr{O}_{X,x}$ is an isomorphism.

Remark. A morphism $i : X \to Y$ is an open embedding if and only if there exists V open subset of Y such that i induces an isomorphism of schemes between $(X, \mathscr{O}_X)$ and $(V, \mathscr{O}_{Y|V})$.

Proposition 1.2.5

If $u : A \to B$ is a morphism of rings, then there exists a morphism $f_u : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ such that $f_u^{\#}(\mathrm{Spec}\,A) = u$.

Theorem 1.2.6

Let Y be an affine scheme.
Then for every scheme X, there exists a canonical bijection

$$\begin{array}{rrcl} \psi_X : & \mathrm{Hom}_{\mathrm{sch.}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathrm{rings}}(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) \\ & \varphi & \longmapsto & \varphi(X) \end{array}$$

This family of maps defines a natural transformation between the functors $X \mapsto \mathrm{Hom}(X,Y)$ and $X \mapsto \mathrm{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X))$: if $\alpha : Z \to X$ is a morphism of schemes, then we have

$$\begin{array}{ccc}
\mathrm{Hom}(X,Y) & \xrightarrow{\psi_X} & \mathrm{Hom}(\mathcal{O}_Y(Y),\mathcal{O}_X(X)) \\
{\scriptstyle \cdot\circ\alpha}\downarrow & & \downarrow{\scriptstyle \alpha^{\#}(Z)\circ\cdot} \\
\mathrm{Hom}(Z,Y) & \xrightarrow{\psi_Z} & \mathrm{Hom}(\mathcal{O}_Y(Y),\mathcal{O}_Z(Z))
\end{array}$$

The key argument of the proof is to use the previous proposition when X is an affine scheme, and to reduce to this case by taking an affine covering when X is an arbitrary scheme.

Remark. In particular there is an anti-equivalence of categories between the category of the affine schemes and the category of the rings.

Definition 1.2.7

If X and Y are schemes, Y is said to be a X-scheme if it is given a morphism $\alpha : Y \to X$, called structural morphism. If $X = \operatorname{Spec} A$ is affine, we also say that Y is a A-scheme.

A morphism $f : Y \to Y'$ of X-schemes is a morphism of schemes such that :

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & Y' \arrow[dl, "\alpha'"] \\
& X &
\end{tikzcd}$$

commutes. The set of the morphism of X-schemes between Y and Y' is denoted by $\mathrm{Hom}_X(Y,Y')$.

It is easy to see with the theorem 1.2.6, that if A is a ring, then the category of affine schemes over A is equivalent to the category of A-algebras.

1.3 Quasi-coherent sheaves of modules

For this section, we refer to [8], chapter 5, pp. 158-163.

Definition 1.3.1

Let X be a scheme. A $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf on the topological space X such that for all $U \subset X$ open, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module and if $V \subset U$, the restriction ρ_{UV} is a morphism of $\mathcal{O}_X(U)$-modules for the structure of $\mathcal{O}_X(U)$-module on $\mathcal{F}(V)$ defined by $a.s = a_{|V}.s$.

Remark. A morphism of $\mathcal{O}_X$-modules is simply a morphism of sheaves φ such that for every $U \subset X$, $\varphi(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

As for the modules over a ring, we can define the tensor product and the direct sum of sheaves of $\mathcal{O}_X$-modules :

- If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules, we define a presheaf $\mathcal{H} : U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ endowed, for $V \subset U$ with the restrictions $\rho^{\mathcal{F}}_{UV} \otimes \rho^{\mathcal{G}}_{UV} : s \otimes t \mapsto s_{|V} \otimes t_{|V}$. Clearly, if $x \in X$, the stalk of $\mathcal{H}$ at x is $\mathcal{H}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.

  The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is the sheafification of $\mathcal{H}$. According to the proposition 1.1.8, we have

  $$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{H}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

  Using the universal property of the sheafification, we will often rather give the description of morphisms on $\mathcal{H}$ than define them on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

- If $(\mathscr{F}_i)_{i\in I}$ is a family of $\mathscr{O}_X$-modules, then the $\mathscr{O}_X$-module $\bigoplus_{i\in I}\mathscr{F}_i$ is the sheafification of the presheaf defined by

$$\forall U \subset X, \left(\bigoplus_{i\in I}\mathscr{F}_i\right)(U) = \bigoplus_{i\in I}\mathscr{F}_i(U)$$

the direct sum on the right being a direct sum of $\mathscr{O}_X(U)$ modules.

Definition 1.3.2

Let X be a scheme and $\mathscr{F}$ be a $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is quasi-coherent if for all $x \in X$, there exists an open neighbourhood U of x and an exact sequence of $\mathscr{O}_X$-modules

$$\mathscr{O}_{X|U}^{(J)} \to \mathscr{O}_{X|U}^{(I)} \to \mathscr{F}_{[U} \to 0$$

This concept is very convenient, as there is an equivalence between the category of quasi-coherent $\mathscr{O}_{\operatorname{Spec} A}$-modules and the category of A-modules, given by the operation of sheafification of a module.

Proposition 1.3.3

Let A be a ring, and M be a A-module. Then, if $X = \operatorname{Spec} A$, we can define a $\mathscr{O}_X$-module $\widetilde{M}$ by setting, for every principal open subset $D(f) \subset X$, $\widetilde{M}(D(f)) = M_f$

Moreover, if $x = \mathfrak{p} \in \operatorname{Spec} A$, then $\widetilde{M}_x = M_{\mathfrak{p}}$.

Remark.
- The operation of sheafification is similar to the construction of the affine schemes p. 8.
- This transformation is compatible with the direct sum and the tensor product :

$$\widetilde{\bigoplus_{i\in I} M_i} = \bigoplus_{i\in I}\widetilde{M_i} \quad \text{and} \quad \widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$$

Proposition 1.3.4

With the same notations, a sequence of A-modules $L \to M \to N$ is exact if and only if the sequence of $\mathscr{O}_X$-modules $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact. Thus if M is a A-module, then $\widetilde{M}$ is quasi-coherent.

Theorem 1.3.5

If X is a scheme and $\mathscr{F}$ is a $\mathscr{O}_X$-module, then $\mathscr{F}$ is quasi-coherent if and only if for all U open affine, $\mathscr{F}_{|U} = \widetilde{\mathscr{F}(U)}$.

Corollary 1.3.6

Let $\operatorname{QCoh}(Y)$ be the category of quasi-coherent sheaves on the scheme Y. If $Y = \operatorname{Spec} A$ is affine, then there is a equivalence of categories between $\operatorname{QCoh}(Y)$ and $\underline{\operatorname{Mod}}_A$ given by the functors $\Delta : \mathscr{F} \mapsto \mathscr{F}(Y)$ and $\Lambda : M \mapsto \widetilde{M}$.

1.4 Locally free sheaves

Let X be a scheme.

Definition 1.4.1

- *A free sheaf of rank n on X is a sheaf $\mathscr{F}$ on X that is isomorphic to $\mathscr{O}_X^n$.*
- *A sheaf $\mathscr{F}$ is said to be a locally free sheaf of rank n if there exists a covering $X = \bigcup_{i \in I} X_i$ of X by open subsets such that for every $i \in I$, $\mathscr{F}_{|X_i}$ is free of rank n.*
- *A sheaf $\mathscr{F}$ is locally free of finite rank if there exists a covering $X = \bigcup_{i \in I} X_i$ together with integers n_i such that $\mathscr{F}_{|X_i}$ is locally free of rank n_i.*

A locally free sheaf of rank 1 is called an *invertible sheaf*, or a *line bundle*.

Remark. If $\mathscr{F}$ is a free sheaf of rank n on X, then if $(\alpha_1, \dots, \alpha_n)$ is a basis of the $\mathscr{O}_X(X)$-module $\mathscr{F}(X)$, for every $U \subset X$, $(\alpha_{1|U}, \dots, \alpha_{n|U})$ is a basis of $\mathscr{F}(U)$. We often say that (α_i) is a basis of $\mathscr{F}$.

An immediate consequence of this fact is that we can check the equality of two morphisms $\mathscr{F} \to \mathscr{G}$, $\mathscr{F}$ free of finite rank, on $\mathscr{F}(X)$.

Moreover, a morphism α between two free sheaves on X is entirely determined by the matrix of $\alpha(X)$, also denoted by $M(\alpha)$.

Definition 1.4.2

Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. It is said to be finitely presented if for every U open affine subset of X, $\mathscr{F}(U)$ is a finitely presented $\mathscr{O}_X(U)$-module.

Proposition 1.4.3

Let $\mathscr{F}$ be a finitely presented quasi-coherent sheaf on X. Then, if $n \in \mathbb{N}^*$, $\mathscr{F}$ is locally free of rank n if and only if $\forall x \in X$, $\mathscr{F}_x$ is a free $\mathscr{O}_{X,x}$-module of rank n.

Lemma 1.4.4

Let A be a ring, M, N two finitely presented A-modules and $\mathfrak{p} \in \operatorname{Spec} A$. If $u : M \to N$ is such that $M_\mathfrak{p} \overset{u_\mathfrak{p}}{\simeq} N_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$, then there exists $f \in A \setminus \mathfrak{p}$ such that $M_f \overset{u_f}{\simeq} N_f$.

Proof. First, let us notice that $\operatorname{coker} u = N/\operatorname{Im} u$ is finitely generated since N is of finite type.

The exact sequence $0 \to \ker u \to M \xrightarrow{u} N \to \operatorname{coker} u \to 0$ yields, by flatness of the localization, $0 \to \ker u_\mathfrak{p} \to M_\mathfrak{p} \xrightarrow{u} N_\mathfrak{p} \to \operatorname{coker} u_\mathfrak{p} \to 0$.

But, by hypothesis, $\operatorname{coker} u_\mathfrak{p} = 0$, so $\forall x \in \operatorname{coker} u, \exists f_x \in A \setminus \mathfrak{p} : f_x x = 0$. Taking generators $x_1, \dots, x_n$ of $\operatorname{coker} u$ and setting $f = f_{x_1} \dots f_{x_n}$, we obtain $\forall x \in \operatorname{coker} u, f x = 0$, $f \in A \setminus \mathfrak{p}$. Finally, $\operatorname{coker} u_f = (\operatorname{coker} u)_f = 0$ implies that u_f is surjective.

Thanks to [13, Tag 0517], also $\ker u$ is finitely generated and likewise, there exists $f' \in A \setminus \mathfrak{p}$ such that $\ker u_{f'} = 0$. Therefore, $u_{ff'} : M_{ff'} \to N_{ff'}$ is an isomorphism. □

Proof (of 1.4.3). Let us assume that $\mathscr{F}_x$ is free of rank n for all $x \in X$. We can assume that X is affine since we want to prove a local statement.

We have $\mathscr{F} \overset{\psi}{\simeq} \widetilde{M}$, where M is a finitely presented $\mathscr{O}_X(X)$-module. Therefore we have an isomorphism $\mathscr{F}_x \simeq \widetilde{M}_x$.

Composing with an isomorphism between $\mathscr{F}_x$ and $\mathscr{O}_{X,x}^n$, we obtain $\varphi_x : \mathscr{O}_{X,x}^n \to \widetilde{M}_x$. Since $\mathscr{O}_X^n$ is free, and replacing X by an open affine V if necessary, we can find γ morphism from $\mathscr{O}_X^n$ to $\widetilde{M}$ such that $\gamma_x = \varphi_x$.

Thus, $\gamma_x : \mathscr{O}_{X,x}^n \simeq M_\mathfrak{p}$ and γ_x is the localization at x of $\gamma(X)$ as X is affine. According to the lemma, there exists $f \in \mathscr{O}_X(X)$ such that $\gamma(X)_f : \mathscr{O}_X(X)_f^n \to M_f$ is an isomorphism. Therefore, as $D(f) \simeq \operatorname{Spec} \mathscr{O}_X(X)_f$ is an open affine, $\widetilde{M}_{|D(f)} \to \mathscr{O}_{X|D(f)}^n$.

□

Remark. If $\mathscr{F}$ is a locally free sheaf of finite rank, then we call *rank* of $\mathscr{F}$ at $x \in X$ the integer $\operatorname{rank}_x \mathscr{F} = \operatorname{rank}_{\mathscr{O}_{X,x}} \mathscr{F}_x$. The function $x \longmapsto \operatorname{rank}_x \mathscr{F}$ is locally constant. Therefore, if X is connected, it is constant of value d, so $\mathscr{F}$ is locally free of rank d.

Definition 1.4.5

Let $\mathscr{F}, \mathscr{G}$ be two sheaves (of abelian groups, rings, of $\mathscr{O}_X$-modules...) on X. The sheaf $\underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})$ is defined by

$$\forall U \subset X, \underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})(U) = \operatorname{Hom}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$$

In particular, if $\mathscr{F}$ is a $\mathscr{O}_X$-module, the *dual sheaf* of $\mathscr{F}$ is $\mathscr{F}^\vee := \underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{O}_X)$.

Proposition 1.4.6

Let $\mathscr{E}, \mathscr{E}'$ be locally free sheaves of finite rank. Then $\underline{\operatorname{Hom}}(\mathscr{E}, \mathscr{E}')$ is a locally free sheaf of finite rank.

Theorem 1.4.7

If Y is a scheme and $\mathscr{H}$ is a locally free sheaf on Y of finite rank, $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent sheaves on Y, then we have a canonical bijection

$$\operatorname{Hom}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G}) \xrightarrow{u_\mathscr{H}} \operatorname{Hom}(\mathscr{F}, \mathscr{G} \otimes \mathscr{H}^\vee)$$

Proof. First of all, the canonical morphism $p : \begin{array}{ccc} \mathscr{H} \otimes \mathscr{H}^\vee & \longrightarrow & \underline{\operatorname{End}}(\mathscr{H}) \\ s \otimes \varphi & \longmapsto & t \mapsto \varphi(t)s \end{array}$ turns out to be an isomorphism, since $\mathscr{H}$ is locally free of finite rank and the corresponding map for a free module on a ring is classically an isomorphism. So, composing with the structural morphism $\iota : \begin{array}{ccc} \mathscr{O}_Y & \longrightarrow & \underline{\operatorname{End}}(\mathscr{H}) \\ 1 & \longmapsto & \operatorname{id}_\mathscr{H} \end{array}$, we define :

$$\begin{tikzcd} \mathscr{O}_Y \arrow[r, "u"] \arrow[dr, "\iota"'] & \mathscr{H} \otimes \mathscr{H}^\vee \arrow[d, "p", "\wr"'] \\ & \underline{\operatorname{End}}(\mathscr{H}) \end{tikzcd}$$

We also have the evaluation morphism $\operatorname{ev} : \begin{array}{ccc} \mathscr{H} \otimes \mathscr{H}^\vee & \longrightarrow & \mathscr{O}_Y \\ s \otimes \varphi & \longmapsto & \varphi(s) \end{array}$.

If $\alpha : \mathscr{F} \otimes \mathscr{H} \to \mathscr{G}$, we set $u_\mathscr{H}(\alpha) : \mathscr{F} \xrightarrow{u} \mathscr{F} \otimes \mathscr{H} \otimes \mathscr{H}^\vee \xrightarrow{\alpha \otimes \operatorname{id}} \mathscr{G} \otimes \mathscr{H}^\vee$.

Given a morphism $\beta : \mathscr{F} \to \mathscr{G} \otimes \mathscr{H}^\vee$, we set $v_\mathscr{H}(\beta) = \mathscr{F} \otimes \mathscr{H} \xrightarrow{\beta \otimes \operatorname{id}} \mathscr{G} \otimes \mathscr{H}^\vee \otimes \mathscr{H} \xrightarrow{\operatorname{id} \otimes \operatorname{ev}} \mathscr{G}$ and we want to prove that $v_\mathscr{H}$ is the inverse of $u_\mathscr{H}$.

Given $\alpha : \mathscr{F} \otimes \mathscr{H} \to \mathscr{G}$ and $\beta = u_\mathscr{H}(\alpha)$, then, if U is an open subset of Y and $u(U)(1) = \sum_j h_j \otimes \varphi_j$, $h_j \in \mathscr{H}, \varphi_j \in \mathscr{H}^\vee$, $\gamma = v_\mathscr{H}(\beta)$ is, on U :

$$\begin{array}{ccccccc}
\mathscr{F}\otimes\mathscr{H} & \xrightarrow{\mathrm{id}\otimes u\otimes \mathrm{id}} & \mathscr{F}\otimes\mathscr{H}\otimes\mathscr{H}^\vee\otimes\mathscr{H} & \xrightarrow{\alpha\otimes\mathrm{id}\otimes\mathrm{id}} & \mathscr{G}\otimes\mathscr{H}^\vee\otimes\mathscr{H} & \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} & \mathscr{G} \\
f\otimes h & \longmapsto & \sum_j f\otimes h_j\otimes\varphi_j\otimes h & \longmapsto & \sum_j \alpha(f\otimes h_j)\otimes\varphi_j\otimes h & \longmapsto & \sum_j \varphi_j(h)\alpha(f\otimes h_j)
\end{array}$$

We want to show that for every $f\in\mathscr{F}(U), h\in\mathscr{H}(U)$, $\sum_j \varphi_j(h)\alpha(f\otimes h_j)=\alpha(f\otimes h)$. This statement can be checked on the basis of open subsets $\mathscr{B}$ constituted by the U open affine such that $\mathscr{H}_{|U}$ is free. To simplify, let us assume that $U=Y$, and let us fix a basis $(v_j)_{j=1\dots r}$ of $\mathscr{H}(Y)$. Then the dual family $(v_j^*)_{j=1\dots r}$ is a basis of $\mathscr{H}^\vee(Y)$.

As $p(Y)\left(\sum_j v_j\otimes v_j^*\right)=\left(h\mapsto\sum_j v_j^*(h)v_j=h\right)=\mathrm{id}_{\mathscr{H}(Y)}$, we have $u(Y)(1)=\sum_{j=1}^r v_j\otimes v_j^*$. Consequently,

$$\forall f\in\mathscr{F}(Y), \forall i\in[\![1;r]\!], \gamma(Y)(v_i)=\sum_j \alpha(Y)(f\otimes v_j)v_j^*(v_i)=\alpha(Y)(f\otimes v_i)$$

and by linearity, we have the desired equality and $v_{\mathscr{H}}\circ u_{\mathscr{H}}=\mathrm{id}$.

We show that $u_{\mathscr{H}}\circ v_{\mathscr{H}}=\mathrm{id}$ using a similar method.

□

1.5 Relative spectrum of a quasi-coherent sheaf of algebras

Definition 1.5.1

Let Y be a scheme. A sheaf of algebras $\mathscr{A}$ on Y, or $\mathscr{O}_Y$-algebra, is a $\mathscr{O}_Y$-module, where $\mathscr{A}(U)$ is endowed with a structure of ring that is compatible with the structure of $\mathscr{O}_Y(U)$-module.

There are several useful equivalent ways to see a $\mathscr{O}_Y$-algebra :

- It is a sheaf of rings $\mathscr{A}$ together with a morphism of sheaves, called structural morphism, $\alpha:\mathscr{O}_Y\to\mathscr{A}$. The structure of $\mathscr{O}_Y(U)$-module on $\mathscr{A}(U)$ is given by $\forall a\in\mathscr{O}_Y(U), \forall f\in\mathscr{A}(U), a.f=\alpha(a)f$.
- It is a $\mathscr{O}_Y$-module $\mathscr{A}$, endowed with a multiplication morphism of $\mathscr{O}_Y$-modules $\mathbf{m}:\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ which has a neutral element $1_{\mathscr{A}}$ and satisfies the commutativity and associativity diagrams (the swap map is just $s\otimes t\mapsto t\otimes s$) :

$$\begin{tikzcd}
\mathscr{A}\otimes\mathscr{A} \arrow[rr, "\mathrm{swap}"] \arrow[dr, "\mathbf{m}"'] & & \mathscr{A}\otimes\mathscr{A} \arrow[dl, "\mathbf{m}"] \\
& \mathscr{A} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathscr{A}\otimes\mathscr{A})\otimes\mathscr{A} \arrow[rr, "\mathrm{id}"] \arrow[d, "\mathbf{m}\otimes\mathrm{id}"'] & & \mathscr{A}\otimes(\mathscr{A}\otimes\mathscr{A}) \arrow[d, "\mathrm{id}\otimes\mathbf{m}"] \\
\mathscr{A}\otimes\mathscr{A} \arrow[dr, "\mathbf{m}"'] & & \mathscr{A}\otimes\mathscr{A} \arrow[dl, "\mathbf{m}"] \\
& \mathscr{A} &
\end{tikzcd}$$

Remark. Seeing the commutativity and associativity of $\mathbf{m}$ as the commutativity such diagrams makes clear the fact that these properties are equivalent to the corresponding local diagrams (*i.e.* on the stalks).

Definition 1.5.2

A morphism of $\mathcal{O}_Y$-algebras $\phi : \mathscr{A} \to \mathscr{B}$ is, in a equivalent way :

- *A morphism ϕ of $\mathcal{O}_Y$-modules such that $\phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$ and* $$\begin{array}{ccc} \mathscr{A} \otimes \mathscr{A} & \xrightarrow{\phi\otimes\phi} & \mathscr{B} \otimes \mathscr{B} \\ \downarrow \mathbf{m} & & \downarrow \mathbf{n} \\ \mathscr{A} & \xrightarrow{\phi} & \mathscr{B} \end{array}$$
- *A morphism of sheaves of rings such that* $$\begin{tikzcd} \mathscr{A} \arrow[rr, "\phi"] & & \mathscr{B} \\ & \mathcal{O}_Y \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

A quasi-coherent sheaf of algebras is simply a sheaf of algebras that is quasi-coherent as a sheaf of modules. The category of quasi-coherent $\mathcal{O}_Y$-algebras is denoted by $\mathrm{QAlg}(Y)$.

Definition 1.5.3

Let X and Y be two schemes. A morphism $f : X \to Y$ is called affine if for every V open affine of Y, $f^{-1}(V)$ is affine.

We call $\mathrm{Aff}(Y)$ the category of affine morphisms $f : X \to Y$, X scheme, in which a morphism between $f : X \to Y$ and $f' : X' \to Y$ is $u : X \to X'$ such that $f' \circ u = f$.

Proposition 1.5.4

Let $f : X \to Y$ be a morphism of schemes. The following are equivalent :

1. *The map f is affine.*
2. *There exists a covering $Y = \bigcup_{i\in I} Y_i$ of Y such that $\forall i \in I, f^{-1}(Y_i)$ is affine.*

Proof. Omitted, see [14], p. 208-210

□

Example 1.5.5. A morphism between affine schemes is affine.

We will eventually introduce the notion of relative spectrum, which play for a quasi-coherent algebra $\mathscr{A}$ a similar role as $\operatorname{Spec} A$ for a ring A. This construction generalises the statement of the theorem 1.2.6 : if A is a ring, B a A-algebra and X a A-scheme, there is a canonical bijection $\mathrm{Hom}_A(X, \operatorname{Spec} B) \simeq \mathrm{Hom}_{A-\mathrm{algebras}}(B, \mathcal{O}_X(X))$. Moreover, the categories of the A-algebras is equivalent to the category of affine schemes over $\operatorname{Spec} A$; here, the categories of the quasi coherent $\mathcal{O}_Y$-algebras and of the affine morphisms $X \to Y$ will be equivalent through the Spec functor.

Theorem 1.5.6

Let Y be a scheme and $\mathscr{A}$ be a quasi-coherent $\mathcal{O}_Y$-algebra. We can find a Y-scheme $\mathrm{Spec}_Y \mathscr{A}$, with structural morphism f such that f is affine and $f_* \mathcal{O}_{\mathrm{Spec}_Y \mathscr{A}} = \mathscr{A}$.

Moreover, if $\pi : Z \to Y$ is a Y-scheme,

$$\begin{array}{rccc} \varphi_Z : & \mathrm{Hom}_Y(Z, \operatorname{Spec}_Y \mathscr{A}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{A}, \pi_* \mathcal{O}_Z) \\ & u & \longmapsto & f_* u^\# \end{array}$$

is a functorial bijection.

This means that if $\kappa : W \to Y$ is a Y-scheme and $\alpha : W \to Z$ is a morphism of Y-schemes, then the following diagram commutes :

$$\begin{tikzcd} \mathrm{Hom}_Y(Z, \mathrm{Spec}_Y \mathscr{A}) \arrow[r, "\varphi_Z"] \arrow[d, "\nu \mapsto \nu\circ\alpha"'] & \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{A}, \pi_* \mathscr{O}_Z) \arrow[d, "\beta \mapsto \pi_*\alpha^\# \circ \beta"] \\ \mathrm{Hom}_Y(W, \mathrm{Spec}_Y \mathscr{A}) \arrow[r, "\varphi_W"] & \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{A}, \kappa_* \mathscr{O}_W) \end{tikzcd}$$

See [14], page 444 for less informations ; another construction can be found in [4], p. 41.

Lemma 1.5.7 (*Glueing schemes*)

Let $(X_i)_{i\in I}$ a family of schemes. If, for all $i \in I$, there exist a family $(X_{ij})_{j\in I}$ such that X_{ij} is an open subset of X_i, and for all $i, j \in I$, an isomorphism of schemes $f_{ij} : X_{ij} \xrightarrow{\simeq} X_{ji}$ such that :

1. $f_{ii} = \mathrm{id}_{X_i}$
2. $\forall i, j, k \in I, f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$
3. *(cocycle condition) The following diagram commutes :*

$$\begin{tikzcd} X_{ij} \cap X_{ik} \arrow[rr, "f_{ik}"] \arrow[dr, "f_{ij}"'] & & X_{ki} \cap X_{kj} \\ & X_{ji} \cap X_{jk} \arrow[ur, "f_{jk}"'] & \end{tikzcd}$$

Then there exists a unique scheme X together with open embeddings $\psi_i : X_i \to X$ such that, for $i, j \in I$, $\psi_i = \psi_j \circ f_{ij}$ on X_{ij}, and $X = \bigcup_i \psi(X_i)$.

Proof. Omitted, see [8], pp. 49-50. □

Remark. If $\phi_i : X_i \to Y$ are Y-schemes, and the f_{ij} is compatible with the ϕ_i, then the glueing X of the X_i is a Y-scheme and its structural morphism ϕ satisfies, for every $i \in I$, $\phi_{|X_i} = \phi_i$.

In the next proof, we will use some basic facts about the fibered products of schemes that can be found in the third chapter [8] (pp. 78-85).

Notation : A commutative diagram of the form above is called *cartesian* if Z satisfies the universal property of the fibered product. We indicate it with the square in the center of the picture.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ X' \arrow[r] & Y \end{tikzcd}$$

Proof (of 1.5.6)**.** Let I be the set of open affine of Y. For $i \in I$, we denote by U_i the associated open affine. Let us set $X_i = \mathrm{Spec}\, \mathscr{A}(U_i)$, and $f_i : X_i \to U_i$ the morphism induced by the structural morphism $\mathscr{O}_Y(U_i) \to \mathscr{A}(U_i)$. We define, for all $j \in I$, $X_{ij} = f_i^{-1}(U_i \cap U_j)$.

If $U_k \subset U_i$, we have the following cartesian diagram, because $\mathscr{A}(U_k) = \mathscr{A}(U_i) \otimes_{\mathscr{O}_Y(U_i)} \mathscr{O}_Y(U_k)$ (it follows from [8], proposition 1.14 (b) p. 163 and proposition 1.12 (b) p. 162).

$$\begin{tikzcd}
X_k \arrow[r, "\alpha_{ki}"] \arrow[d, "f_k"] \arrow[dr, phantom, "\square"] & X_i \arrow[d, "f_i"] \\
U_k \arrow[r, "\text{inc}"] & U_i
\end{tikzcd} \tag{1.1}$$

As open embeddings are stable by base change, α_{ki} is an open embedding. Moreover, $\alpha_{ki}(X_k) = f_i^{-1}(U_k)$. As α_{ki} is compatible with f_k and f_i, it is a morphism of Y-schemes.

Let $i, j \in I$. We want to define a family of isomorphisms $\beta_{ij} : X_{ij} \to X_{ji}$.

Let us first notice that if $k \in I$ is such that $U_k \subset U_i \cap U_j$, then the diagram 1.1 factors through X_{ij} – and likewise through X_{ji} :

$$\begin{tikzcd}
X_k \arrow[r, "\alpha_{k,i,j}"'] \arrow[d, "f_k"'] \arrow[rr, bend left, "\alpha_{ki}"] & X_{ij} \arrow[r, "\text{inc.}"] \arrow[d] & X_i \arrow[d, "f_i"] \\
U_k \arrow[r] & U_i \cap U_j \arrow[r] & U_i
\end{tikzcd}$$

It is then easy to see that the family of the $\operatorname{Im} \alpha_{k,i,j}$ when $U_k \subset U_i \cap U_j$ is a covering of X_{ij} ; therefore, we can define β_{ij} by glueing the morphisms :

$$\begin{tikzcd}
X_{ij} \arrow[rr, "\beta_{ij}"] & & X_{ji} \\
& X_k \arrow[ul, "\alpha_{k,i,j}"] \arrow[ur, "\alpha_{k,j,i}"'] &
\end{tikzcd}$$

under the condition that if $U_k, U_l \subset U_i \cap U_j$, $(\alpha_{k,j,i} \circ \alpha_{k,i,j}^{-1})_{|f_i^{-1}(U_k \cap U_l)} = (\alpha_{l,j,i} \circ \alpha_{l,i,j}^{-1})_{|f_i^{-1}(U_k \cap U_l)}$, since the image of $\alpha_{k,i,j}$ (resp. $\alpha_{l,i,j}$) is $f_i^{-1}(U_k)$ (resp. $f_i^{-1}(U_l)$).

Let us first consider the case where $U_l \subset U_k$. We have the following diagram :

$$\begin{tikzcd}
X_l \arrow[r, "\alpha_{lk}"] \arrow[d, "f_l"] \arrow[dr, phantom, "\square"] & X_k \arrow[r, "\alpha_{k,i,j}"] \arrow[d, "f_k"] \arrow[dr, phantom, "\square"] & X_{ij} \arrow[d] \\
U_l \arrow[r] & U_k \arrow[r] & U_i \cap U_j
\end{tikzcd}$$

the whole rectangle being cartesian. Therefore, by uniqueness of the fibered product, we have $\alpha_{l,i,j} = \alpha_{k,i,j} \circ \alpha_{lk}$, and likewise for X_{ji}. Considering X_l as an open subset of X_k, and passing to the inverse, we get $\alpha^{-1}_{k,i,j|f_i^{-1}(U_l)} = \alpha^{-1}_{l,i,j}$ and on the other hand $\alpha_{k,j,i|X_l} = \alpha_{l,j,i}$ which gives the result.

Now, in the general case, it suffices to show that for every $U_r \subset U_k \cap U_l$, the maps

$$\begin{tikzcd}
X_{ij} \arrow[rr, "\beta_{ij}"] & & X_{ji} \\
& X_r \arrow[ul, "\alpha_{k,i,j|X_r}"] \arrow[ur, "\alpha_{k,j,i|X_r}"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_{ij} \arrow[rr, "\beta_{ij}"] & & X_{ji} \\
& X_r \arrow[ul, "\alpha_{l,i,j|X_r}"] \arrow[ur, "\alpha_{l,j,i|X_r}"'] &
\end{tikzcd}$$

coincide.

But according to the first case, $\alpha_{k,i,j|X_r} = \alpha_{r,i,j} = \alpha_{l,i,j|X_r}$, with the same result for $\alpha_{r,j,i}$ so we have the wished equality.

The reason why the cocycle condition is true can be seen on this diagram (with $U_{ijk} = U_i \cap U_j \cap U_k$) :

$$\begin{tikzcd}
 & & X_\nu \arrow[dll, "\alpha_{\nu i}"'] \arrow[drr, "\alpha_{\nu k}"] & & \\
f_i^{-1}(U_{ijk}) \arrow[rrrr, "\beta_{ik}"] \arrow[drr, "\beta_{ij}"] & & & & f_k^{-1}(U_{ijk}) \\
X_\nu \arrow[u, "\alpha_{\nu i}"] \arrow[rr, "\alpha_{\nu j}"'] & & f_j^{-1}(U_{ijk}) \arrow[urr, "\beta_{jk}"] & & X_\nu \arrow[ll, "\alpha_{\nu j}"] \arrow[u, "\alpha_{\nu k}"']
\end{tikzcd}$$

Furthermore, the structural morphism $f : X \to Y$ is obtained by glueing the $X_i \xrightarrow{f_i} U_i \to Y$, so $f^{-1}(U_i) = f_i^{-1}(U_i) = X_i$ and f is affine. Likewise, for every $i \in I$, $f_*\mathscr{O}_X(U_i) = \mathscr{O}_{X_i}(X_i) = \mathscr{A}(U_i)$ so as the open affine form a basis of open subsets of Y, $f_*\mathscr{O}_X = \mathscr{A}$.

Let us prove the second part of the theorem. The commutativity of the diagram is clear.

If $\pi : Z \to Y$ is a Y-scheme, we want to show that φ_Z is a bijection. Let $\beta : \mathscr{A} \to \pi_*\mathscr{O}_Z$ is a morphism ; we are looking for a $u : Z \to X = \operatorname{Spec}_Y \mathscr{A}$ such that $f_* u^\# = \beta$.

For every $i \in I$, $\beta(U_i)$ yields, thanks to 1.2.6 a morphism $v_i : \pi^{-1}(U_i) \to \operatorname{Spec} \mathscr{A}(U_i) = X_i$, which gives by composition with $X_i \to X$ a morphism $u_i : \pi^{-1}(U_i) \to X$. Moreover, since the following diagram commutes, for $U_k \subset U_i$,

$$\begin{tikzcd}
\mathscr{A}(U_i) \arrow[r, "\beta(U_i)"] \arrow[d] & \mathscr{O}_Z(\pi^{-1}(U_i)) \arrow[d] \\
\mathscr{A}(U_k) \arrow[r, "\beta(U_k)"] & \mathscr{O}_Z(\pi^{-1}(U_k))
\end{tikzcd}$$

we have

$$\begin{tikzcd}
\pi^{-1}(U_k) \arrow[r, "v_k"] \arrow[d] & X_k \arrow[r] \arrow[d] & X \arrow[d, equal] \\
\pi^{-1}(U_i) \arrow[r, "v_i"] & X_i \arrow[r] & X
\end{tikzcd}$$

and thus $u_{i|\pi^{-1}(U_k)} = u_k$, which, similarly to the construction of the β_{ij} above, proves that we can glue the u_i into a morphism of Y-schemes $u : Z \to X$.

We define $\psi_Z(\beta) = u$. If $u : Z \to X$ is a morphism of Y-schemes, let us take $\beta = \varphi_Z(u) = f_* u^\#$ and $u' = (\psi_Z \circ \varphi_Z)(u)$. We know that $u'_{|\pi^{-1}(U_i)} = \pi^{-1}(U_i) \xrightarrow{v'_i} X_i \xrightarrow{\text{inclusion}}$ where v'_i is induced by $\beta(U_i)$. Moreover, as u is a morphism of Y-schemes, $u(\pi^{-1}(U_i)) \subset f^{-1}(U_i) = X_i$ and $u_{|\pi^{-1}(U_i)}$ can be decomposed as $\pi^{-1}(U_i) \xrightarrow{v_i} X_i \xrightarrow{\text{inclusion}} X$. Since $u^\#(X_i) = \beta(U_i)$, we have $v_i = v'_i$ and finally $u = u'$.

Furthermore, if $\beta : \mathscr{A} \to \pi_*\mathscr{O}_Z$ is a morphism of $\mathscr{O}_Y$-algebra, $\beta' = (\varphi_Z \circ \psi_Z)(\beta)$ is actually β. Indeed, if $u = \psi_Z(\beta)$, then for all $i \in I$, $\beta'(U_i) = u^\#(f^{-1}(U_i)) = u^\#(X_i) = \beta(U_i)$ by construction of u. Therefore, φ_Z and ψ_Z are inverses of each other.

□

Corollary 1.5.8

The categories $\operatorname{QAlg}(Y)$ *and* $\operatorname{Aff}(Y)$ *are equivalent through*

$$\begin{aligned}
\operatorname{Aff}(Y) &\longleftrightarrow \operatorname{QAlg}(Y) \\
f : X \to Y &\xmapsto{\;G\;} f_*\mathscr{O}_X \\
\operatorname{Spec}_Y \mathscr{A} \to Y &\xleftarrow{\;F\;} \mathscr{A}
\end{aligned}$$

Proof. An affine morphism is quasi compact and separated (see [6], p. 321), so [8], p. 163 implies that $f_*\mathcal{O}_X$ is quasi-coherent.

If $\mathscr{A} \in \mathrm{QAlg}(Y)$, then $(G \circ F)(\mathscr{A}) = f_*\mathcal{O}_{\mathrm{Spec}_Y \mathscr{A}} = \mathscr{A}$, according to the theorem 1.5.6, where $f : \mathrm{Spec}_Y \mathscr{A} \to Y$ is the structural morphism.

If $f : X \to Y$ is an affine morphism, then $X = \mathrm{Spec}_Y f_*\mathcal{O}_X$. Indeed, with the same notations as in the proof of 1.5.6, if $i \in I$, $X_i = \mathrm{Spec}\,(\mathcal{O}_X(f^{-1}(U_i)) = f^{-1}(U_i)$, since f is affine. If $i, j \in I$, $\beta_{ij} : f^{-1}(U_i \cap U_j) \to f^{-1}(U_j \cap U_i)$ is the identity. Thus it is clear that X is the glueing of the X_i.

Let us study the action on the morphisms : if $u : X \to X'$ is a morphism between two affine morphisms $f : X \to Y$ and $f' : X' \to Y$, then $X = \mathrm{Spec}_Y \mathscr{A}$, $X' = \mathrm{Spec}_Y \mathscr{A}'$, where $\mathscr{A} = f_*\mathcal{O}_X$, $\mathscr{A}' = f'_*\mathcal{O}_{X'}$ and $G(u)$ is given by the bijection $\varphi : \mathrm{Hom}_Y(\mathrm{Spec}_Y \mathscr{A}', \mathrm{Spec}_Y \mathscr{A}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{A}, \mathscr{A}')$ above. The diagram of the theorem 1.5.6 shows the functoriality.

Likewise, the action of F an a morphism $\beta : \mathscr{A}' \to \mathscr{A}$ is given by the inverse of φ. Hence, F and G are quasi inverses of each other for the morphisms. □

1.6 Symmetric algebra and exterior powers of algebra

For proofs and details, see [6], pp. 196-199 (exterior powers) and 287-288 (symmetric algebra).

Let A be a ring, and M be a A-module. We set $\forall n \geqslant 0, T_n(M) = M^{\otimes n}$ and $T(M) = \bigoplus_{n \geqslant 0} T_n(M)$. This module endowed with the product :

$$(m_1 \otimes \cdots \otimes m_n) \times (m'_1 \otimes \cdots \otimes m'_l) = m_1 \otimes \cdots \otimes m_n \otimes m'_1 \otimes \cdots \otimes m'_l$$

is a graded (non-commutative) A-algebra (*i.e.* $\forall p, q \in \mathbb{N}, T_p(M)T_q(M) \subset T_{p+q}(M)$).

Definition 1.6.1

If I is the ideal of $T(M)$ generated by the $x \otimes y - y \otimes x$, $x, y \in M$, then the symmetric algebra of M is $S(M) = T(M)/I$. Endowed with the product induced by the product of $T(M)$, it is a commutative A-algebra.

As I is generated by homogeneous elements, we know that $S(M) = \bigoplus_{n \in \mathbb{N}} S^n M$, where $S^n M = T_n(M)/(I \cap T_n(M)) = M^{\otimes n}/I_n$, I_n being the submodule of $M^{\otimes n}$ generated by the $m_1 \otimes \cdots \otimes m_n - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$, σ permutation of $\{1, \ldots, n\}$.

The module $S^n M$ has the following universal property, that comes from the universal property of the quotient : it is endowed with a morphism $\pi : M^{\otimes n} \to S^n M$, and if $f : M^{\otimes n} \to N$, N A-module is a A-linear symmetric morphism (*i.e.* $f(m_1 \otimes \cdots \otimes m_n) = f(m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)})$ for every permutation σ), then f factors uniquely through $S^n M$:

$$\begin{tikzcd} M^{\otimes n} \arrow[r, "f"] \arrow[d, "\pi"] & N \\ S^n M \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

A consequence of this property is that the construction of $S^n\cdot$ is functorial : if $u : M \to N$ is a morphism, then we can build a morphism $S^n u : S^n M \to S^n N$ such that

$$\begin{tikzcd} S^n M \arrow[r, "S^n u"] & S^n N \\ M^{\otimes n} \arrow[u, "\pi_M"] \arrow[r, "u^{\otimes n}"] & N^{\otimes n} \arrow[u, "\pi_N"'] \end{tikzcd}$$

Definition 1.6.2

If J is the ideal of $T(M)$ generated by the $x \otimes x$, $x \in M$, then the exterior algebra of M is the A-module $\Lambda(M) = T(M)/J$.

As previously, $\Lambda(M) = \bigoplus_{n\in\mathbb{N}} \Lambda^n M$, where $\Lambda^n M = M^{\otimes n}/J_n$, $J_n = J \cap M^{\otimes n}$ being the submodule generated by the $m_1 \otimes \cdots \otimes m_n \in M^{\otimes n}$ such that $\exists i \neq j : m_i = m_j$. The module $\Lambda^n M$ is called the n^{th} exterior power of M.

The universal property of $\Lambda^n M$ is that every alternating morphism $f : M^{\otimes n} \to N$, N A-module, (*i.e.* $\exists i \neq j : m_i = m_j \Rightarrow f(m_1 \otimes \cdots \otimes m_n) = 0$) factors through $\Lambda^n M$.

Again, $\Lambda^n\cdot$ acts also on the morphisms and therefore define a functor.

Notation : The class of $m_1 \otimes \cdots \otimes m_n$ in $\Lambda^n M$ is denoted by $m_1 \wedge \cdots \wedge m_n$, while its class in $S^n M$ is denoted by $m_1 \ldots m_n$.

Proposition 1.6.3

Let M be a free A-module of rank r and $(e_1, \ldots, e_r)$ a basis of M.

Then, for $n \in \mathbb{N}$, the family of the $(e_{j_1} \wedge \cdots \wedge e_{j_r})_J$ when $J = \{j_1 < \cdots < j_r\} \subset \{1, \ldots, n\}$ is a basis of the free A-module $\Lambda^n M$.

In particular, if $n = r$, then $\Lambda^r M$ is of rank 1, and is also called $\det M$.

Proposition 1.6.4

Given $n \in \mathbb{N}$, M a A-module, and $\mathfrak{p} \in \operatorname{Spec} A$, $(S^n M)_{\mathfrak{p}} = S^n M_{\mathfrak{p}}$ and $(\Lambda^n M)_{\mathfrak{p}} = \Lambda^n M_{\mathfrak{p}}$

Let us now generalize this construction to $\mathcal{O}_X$-modules, when X is a scheme.

Definition 1.6.5

Let $\mathcal{F}$ be a $\mathcal{O}_X$-module.

- *The n^{th} exterior power $\Lambda^n \mathcal{F}$ of $\mathcal{F}$, is the sheafification of the presheaf $U \mapsto \Lambda^n_{\mathcal{O}_X(U)} \mathcal{F}(U)$, endowed with the restrictions*

$$\forall U \subset V \subset X,\ \Lambda^n(\rho_{UV}) :\ \begin{array}{ccc} \Lambda^n_{\mathcal{O}_X(U)} \mathcal{F}(U) & \longrightarrow & \Lambda^n_{\mathcal{O}_X(V)} \mathcal{F}(U) \\ s_1 \wedge \cdots \wedge s_n & \longmapsto & s_{1|V} \wedge \ldots s_{n|V} \end{array}$$

- *$S^n \mathcal{F}$ is the sheafification of $U \mapsto S^n_{\mathcal{O}_X(U)} \mathcal{F}(U)$, endowed with the restriction $S^n \rho_{UV}$.*

Once again, $\Lambda^n\cdot$ and $S^n\cdot$ are functors from the category of $\mathcal{O}_X$-modules into itself.

Proposition 1.6.6

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\Lambda^n \mathcal{F}$ and $S^n \mathcal{F}$ are quasi-coherent. Moreover, if U is an open affine of X and $\mathcal{F}_{|U} = \widetilde{M}$, then $(\Lambda^n \mathcal{F})_{|U} = \widetilde{\Lambda^n M}$ and $(S^n \mathcal{F})_{|U} = \widetilde{S^n M}$.

Remark. This proposition together with 1.6.4 shows that if $x \in X$, $(S^n \mathcal{F})_x = S^n \mathcal{F}_x$ and $(\Lambda^n \mathcal{F})_x = \Lambda^n \mathcal{F}_x$.

Chapter 2

Double covers

2.1 Basic facts about covers

In this section, we fix a base scheme Y.

Definition 2.1.1

A d-cover of a scheme Y is an affine morphism $f : X \to Y$, where $f_*\mathcal{O}_X$ is a quasi-coherent locally free $\mathcal{O}_Y$-algebra of rank d. The number d is called the degree of the cover.

Definition 2.1.2

We denote, for $d \in \mathbb{N}^*$, the category of the d-covers of Y by $\mathrm{Cov}_d(Y)$.

A morphism between $f : X \to Y$ and $f' : X' \to Y$ in this category is $p : X \to X'$ isomorphism of schemes such that

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

Proposition 2.1.3

The restriction of the functors in 1.5.8 yields an equivalence of categories between the category of d-covers of Y and the category of quasi-coherent locally free $\mathcal{O}_Y$-algebras of degree d, with isomorphisms as arrows.

Thus, considering a cover $f : X \to Y$ is the same as taking $\mathscr{A} \in \mathrm{QAlg}(Y)$ which is locally free. In this situation, f is the structural morphism $\mathrm{Spec}_Y(\mathscr{A}) \to Y$. For this reason, we will essentially consider the d-covers as locally free algebras of rank d in the next two chapters.

Example 2.1.4.

- The 1-covers are the isomorphisms.

  Indeed, if $\mathscr{A}$ is a quasi-coherent locally free algebra of rank 1, endowed with his structural morphism $\alpha : \mathcal{O}_Y \to \mathscr{A}$, then for all $x \in Y$, $\alpha_x : \mathcal{O}_{Y,x} \to \mathscr{A}_x$ is a morphism of algebras and $\mathscr{A}_x$ is a $\mathcal{O}_{Y,x}$-module of rank 1. As $1 \notin \mathfrak{m}_x$, $(\overline{1})$ is a basis of the $k(x)$-vector space of dimension 1 $\mathscr{A}_x/\mathfrak{m}_x\mathscr{A}_x$. Thus, using Nayakama's lemma, it is a generator of $\mathscr{A}_x$. Thanks to the lemma 2.2.5, we know that (1) is a basis of $\mathscr{A}_x$.

  Therefore α_x is an isomorphism since $\alpha_x(1) = 1$, and so is α.

- If K is a number field of degree d, then we know that the ring of the integers $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of degree d. Therefore, $\mathrm{Spec}\,\mathcal{O}_K \to \mathrm{Spec}\,\mathbb{Z}$ is a d-cover.

Example 2.1.5. We know that $p : z \to z^2$ is a topological cover of $\mathbb{C}^*$, but not of $\mathbb{C}$ because $p^{-1}(0)$ has one element, whereas the cardinality of $p^{-1}(\lambda)$ is 2 if $\lambda \neq 0$. So 0 is a "double point". We want to find an algebraic equivalent to p and see if the same phenomenon occurs in 0.

Let k be a field, and consider the morphism of k-algebras $\begin{array}{rccc}\varphi : & k[X] & \longrightarrow & k[Z] \\ & X & \longmapsto & Z^2\end{array}$. It induces an affine morphism of schemes $f : S \to T$, where $S = \operatorname{Spec} k[Z]$ and $T = \operatorname{Spec} k[X]$.

The quasi-coherent $\mathcal{O}_T$-module (for the structure given by f) $f_*\mathcal{O}_S$ is $\widetilde{k[Z]}$.

But there is a surjective morphism of $k[X]$-algebras $\begin{array}{rccc}\alpha : & k[X][Y] & \longrightarrow & k[Z] \\ & Y & \longmapsto & Z\end{array}$, whose kernel is $(Y^2 - X)$. Hence $k[Z] \simeq k[X][Y]/(Y^2 - X) = k[X] \oplus_{k[X]} k[X]Y$ is a free $k[X]$-module of rank 2 and f is a cover of degree 2.

Let us now replace k by $\mathbb{C}$, or any algebraically closed field. We know by Hilbert's Nullstellensatz that the maximal ideals of $\mathbb{C}[X]$ are the $(X - \lambda)$, $\lambda \in \mathbb{C}$. Moreover (cf [8], p. 83), if $\lambda \in \mathbb{C}$, and $\mathfrak{p} = (X - \lambda)$, $f^{-1}(\mathfrak{p})$ is $S_\lambda = S \times_T \operatorname{Spec} k(\mathfrak{p})$, where $k(\mathfrak{p}) = \mathcal{O}_{T,\mathfrak{p}}/\mathfrak{m}_\mathfrak{p} \simeq \mathbb{C}[X]/(X - \lambda) \simeq \mathbb{C}$ is the residue field. S_λ satisfies the following base change diagram :

$$\begin{tikzcd} S_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{C}[Z] \arrow[d, "f"] \\ \operatorname{Spec} k(\mathfrak{p}) \arrow[r] & \operatorname{Spec} \mathbb{C}[X] \end{tikzcd}$$

So

$$S_\lambda = \operatorname{Spec}\left(\mathbb{C}[Z] \otimes_{\mathbb{C}[X]} \frac{\mathbb{C}[X]}{(X - \lambda)}\right) = \operatorname{Spec}\left(\frac{\mathbb{C}[Z]}{(\varphi(X - \lambda))}\right) = \operatorname{Spec}\left(\frac{\mathbb{C}[Z]}{(Z^2 - \lambda)}\right)$$

If $\lambda \neq 0$, and $\mu^2 = \lambda$, we deduce from the Chinese remainder theorem that $\dfrac{\mathbb{C}[Z]}{(Z^2 - \lambda)} = \dfrac{\mathbb{C}[Z]}{(Z - \mu)} \times \dfrac{\mathbb{C}[Z]}{(Z + \mu)} \simeq \mathbb{C} \times \mathbb{C}$, hence $S_\lambda = \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$ has two elements.

If $\lambda = 0$, $S_0 = \operatorname{Spec}\left(\dfrac{\mathbb{C}[Z]}{(Z^2)}\right) = \{(Z)\}$ is a singleton, but it is the spectrum of a $\mathbb{C}$-vector space of dimension 2, which is a kind of multiplicity. We call this phenomenon *ramification*.

Proposition 2.1.6

Let $\mathcal{A}$ be a cover of degree d. There exists a unique morphism $\operatorname{Tr}_{\mathcal{A}} : \mathcal{A} \to \mathcal{O}_Y$ verifying the following property : for every U affine open subset of Y such that the $\mathcal{O}_Y(U)$-module $\mathcal{A}(U)$ is free of rank d, $\operatorname{Tr}_{\mathcal{A}}(U)$ is the standard trace application on $\mathcal{A}(U)$.

Moreover, $\operatorname{Tr}_{\mathcal{A}}(1_{\mathcal{A}}) = d$.

Actually this theorem results from a more general principle that we describe in this lemma :

Lemma 2.1.7

Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of rank d. There exists a unique morphism $\operatorname{Tr} : \underline{\operatorname{End}}(\mathcal{E}) \to \mathcal{O}_Y$ verifying the following property : for every U affine open subset of Y such that the $\mathcal{O}_Y(U)$-module $\mathcal{E}(U)$ is free of rank d, $\operatorname{Tr}(U)$ is the standard trace application $\operatorname{End}(\mathcal{E}(U)) \to \mathcal{O}_Y(U)$.

Proof. Let $\mathscr{B}$ be the set of the affine open subsets V of Y such that $\mathscr{E}_{|V} = \widetilde{\mathscr{E}(V)} \simeq \mathscr{O}_V^d$. Since $\mathscr{E}$ is locally free, $\mathscr{B}$ is a basis of open subsets of Y. We can therefore define a morphism of $\mathscr{O}_Y$-modules on $\mathscr{B}$.

Let $V \in \mathscr{B}$, and λ be an isomorphism of $\mathscr{O}_V$-modules between $\mathscr{E}_{|V}$ and $\mathscr{O}_V^d$. The map λ is entirely determined by the data of $\lambda(V) \in \mathrm{Iso}(\mathscr{E}(V), \mathscr{O}_Y(V))$.

We can define $\mathrm{Tr}_{\lambda,V}$ on the free sheaf $\underline{\mathrm{End}}(\mathscr{E})_{|V}$ by

$$\mathrm{End}(\mathscr{E}_{|V}) \xrightarrow[\sim]{\varphi_\lambda : f \mapsto \lambda f \lambda^{-1}} \mathrm{End}(\mathscr{O}_V^d) = M_d(\mathscr{O}_V(V)) \xrightarrow{\mathrm{Tr}} \mathscr{O}_V(V)$$

Let $\lambda' : \mathscr{E}_{|V} \to \mathscr{O}_V^s$ be another isomorphism. The diagram

$$\begin{tikzcd}
\mathrm{End}(\mathscr{E}_{|V}) \arrow[r, "\varphi_{\lambda'}", "\sim"'] \arrow[d, "\wr\varphi_\lambda"'] & M_d(\mathscr{O}_V(V)) \arrow[d, "\mathrm{Tr}"] \\
M_d(\mathscr{O}_V(V)) \arrow[ur, "\varphi_{\lambda'\lambda^{-1}}"'] \arrow[r, "\mathrm{Tr}"'] & \mathscr{O}_V(V)
\end{tikzcd}$$

commutes because the ordinary trace map for matrices satisfies $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B), \forall A, B \in M_d(\mathscr{O}_V(V))$, A invertible, and the diagonal map is an isomorphism. Hence $\mathrm{Tr}_{\lambda,V} = \mathrm{Tr}_{\lambda',V}$.

Therefore, we can set $\mathrm{Tr}(V) = \mathrm{Tr}_{\lambda,V}(V)$ for every $V \in \mathscr{B}$ and $\lambda : \mathscr{E}_{|V} \simeq \mathscr{O}_V^d$. In order to see that Tr is well-defined as a map of sheaves, let us check, for $W \subset V \in \mathscr{B}$, that the following diagram is commutative :

$$\begin{tikzcd}
\mathrm{End}(\mathscr{E}_{|V}) \arrow[r, "\mathrm{Tr}(V)"] \arrow[d, "\text{restriction}"'] & \mathscr{O}_Y(U) \arrow[d, "\text{restriction}"] \\
\mathrm{End}(\mathscr{E}_{|W}) \arrow[r, "\mathrm{Tr}(W)"] & \mathscr{O}_Y(V)
\end{tikzcd}$$

Let us consider an isomorphism $\lambda : \mathscr{E}_{|V} \simeq \mathscr{O}_V^d$. Then $\lambda_{|W}$ is an isomorphism between $\mathscr{E}_{|W}$ and $\mathscr{O}_W^d$.

$$\begin{tikzcd}
\mathrm{End}(\mathscr{E}_{|V}) \arrow[r, "\varphi_\lambda"] \arrow[d, "\rho\ \text{restriction}"'] & M_d(\mathscr{O}_V(V)) \arrow[r, "\mathrm{Tr}_V"] \arrow[d, "\psi"] & \mathscr{O}_Y(V) \arrow[d, "\text{restriction}"] \\
\mathrm{End}(\mathscr{E}_{|W}) \arrow[r, "\varphi_{\lambda_{|W}}"'] & M_d(\mathscr{O}_W(W)) \arrow[r, "\mathrm{Tr}_W"] & \mathscr{O}_Y(W)
\end{tikzcd}$$

where ψ is the morphism of restriction coefficient by coefficient. Indeed, if $M \in M_d(\mathscr{O}_Y(V))$, then :

$$\begin{aligned}
\psi(M) &= (\varphi_{\lambda_{|W}} \circ \rho \circ \varphi_{\lambda^{-1}})(M) \\
&= \lambda_{|W} \circ (\lambda^{-1} \circ M \circ \lambda)_{|W} \lambda_{|W}^{-1} \\
&= \lambda_{|W} \circ \lambda_{|W}^{-1} \circ M_{|W} \circ \lambda_{|W} \circ \lambda_{|W}^{-1} = M_{|W}
\end{aligned}$$

But the restriction $M_{|W}$ as a morphism of sheaves correspond to the restricted matrix, since, if $(e_1, \dots, e_d)$ is the canonical basis of $\mathscr{O}_Y(V)$, then $(e_{1|W}, \dots, e_{s|W})$ is the canonical basis of $\mathscr{O}_Y(W)$ and $\forall i = 1 \dots d, M_{|W}(e_{i|W}) = M(e_i)_{|W} = (\sum_j m_{i,j} e_j)_{|W} = \sum_j m_{i,j|W} e_{j|W}$.

With this ψ, it is clear that the diagram from the right commutes. Thus, the big rectangle commutes. As $\mathrm{Tr}(W)$ doesn't depend on the isomorphism we choose, the horizontal arrows correspond to $\mathrm{Tr}(V)$ and $\mathrm{Tr}(W)$. Hence, Tr is well defined on $\mathscr{B}$ and coincides with the actual trace map when $\mathscr{E}_{|V}$ is free.

□

Proof (of 2.1.6)**.** Let $\varphi : \begin{array}{ccc} \mathscr{A} & \longrightarrow & \underline{\mathrm{End}}(\mathscr{A}) \\ a \in \mathscr{A}(U) & \longmapsto & (t_a : a' \mapsto a \cdot_{\mathscr{A}(U)} a') \end{array}$

Composing φ with the trace map built in the lemma gives us a morphism $\mathrm{Tr}_{\mathscr{A}} : \mathscr{A} \to \mathscr{O}_Y$ that satisfies the conditions of the theorem.

Moreover, $\mathrm{Tr}_{\mathscr{A}}(1_{\mathscr{A}}) = d$ since they coincide locally : if U is an open affine such that $\mathscr{A}(U)$ is free of rank d, $\mathrm{Tr}_{\mathscr{A}}(U)(1_{\mathscr{A}|U}) = d$. □

Remark. If $\mathscr{A}$ is a locally free $\mathscr{O}_Y$-algebra of rank d, then for every $x \in Y$, $(\mathrm{Tr}_{\mathscr{A}})_x = \mathrm{Tr}_{\mathscr{A}_x/\mathscr{O}_{Y,x}}$.

Proposition 2.1.8

If $\mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{F}$ are morphisms of sheaves such that $v \circ u = \mathrm{id}_{\mathscr{F}}$, then if $j : \ker v \to \mathscr{G}$ is the inclusion, $u \oplus j : \mathscr{F} \oplus \ker v \to \mathscr{G}$ is an isomorphism and its inverse is $v \oplus (\mathrm{id} - uv) : \mathscr{G} \to \mathscr{F} \oplus \ker v$.

In particular, if $\mathscr{A}$ is a d-cover and $d \in \mathscr{O}_Y(Y)^*$, then $\mathscr{A} \simeq \mathscr{O}_Y \oplus \ker \mathrm{Tr}_{\mathscr{A}}$.

Proof. A direct computation show the first assertion.

Here, $\dfrac{\mathrm{Tr}_{\mathscr{A}}}{d} \circ f = \mathrm{id}$.

$$\mathscr{O}_Y \xrightarrow{f} \mathscr{A} \xrightarrow{\mathrm{Tr}_{\mathscr{A}}/d} \mathscr{O}_Y$$

Therefore, $\beta = f \oplus \text{inclusion} : \mathscr{O}_Y \oplus \ker \mathrm{Tr}_{\mathscr{A}} \to \mathscr{A}$ is an isomorphism. □

Remark. The first statement is not specific to the sheaves, it is also true for modules for instance.

◊ ◊ ◊ ◊

Our goal in the next section and the next chapter is to find a concrete description of $\mathrm{Cov}_2(Y)$ and $\mathrm{Cov}_3(Y)$. Essentially, given $(\mathscr{A}, \mathbf{m})$ a locally free algebra of rank 2 or 3, we will try to find the "smallest" component of the multiplication $\mathbf{m}$ that contain all the information.

2.2 Double covers via line bundles

Let Y be a scheme with $2 \in \mathscr{O}_Y(Y)^*$.

Let $\mathscr{D}_2(Y)$ the category whose objects are the pairs $(\mathscr{L}, \sigma)$ where $\mathscr{L}$ is an invertible sheaf on Y and $\sigma : \mathscr{L} \otimes \mathscr{L} \to \mathscr{O}_Y$ is a morphism of $\mathscr{O}_Y$-modules. In this category, a morphism between $(\mathscr{L}, \sigma)$ and $(\mathscr{L}', \sigma')$ is an isomorphism $p : \mathscr{L} \to \mathscr{L}'$ such that

$$\begin{tikzcd} \mathscr{L} \otimes \mathscr{L} \arrow[rr, "p \otimes p"] \arrow[dr, "\sigma"'] & & \mathscr{L}' \otimes \mathscr{L}' \arrow[dl, "\sigma'"] \\ & \mathscr{O}_Y & \end{tikzcd}$$

Consider he following associations :

- Λ from $\mathrm{Cov}_2(Y)$ to $\mathscr{D}_2(Y)$ which associates with a locally free algebra $(\mathscr{A},\mathbf{m})$ the object $(\ker \mathrm{Tr}_{\mathscr{A}},\sigma)$, where $\sigma : \mathscr{L}\otimes\mathscr{L} \xrightarrow{\text{inclusion}} \mathscr{A}\otimes\mathscr{A} \xrightarrow{\mathbf{m}} \mathscr{A} \xrightarrow{\mathrm{Tr}_{\mathscr{A}}/2} \mathscr{O}_Y$.

  If $u : \mathscr{A} \to \mathscr{A}'$ is an isomorphism of $\mathscr{O}_Y$-algebras, wet $\Lambda(u) := u_{|\ker \mathrm{Tr}_{\mathscr{A}}}$.

- Δ from $\mathscr{D}_2(Y)$ to $\mathrm{Cov}_2(Y)$ that associates with $(\mathscr{L},\sigma)$ the $\mathscr{O}_Y$-algebra $\mathscr{L}\oplus\mathscr{O}_Y$, endowed with the multiplication $\mathbf{m} : (\mathscr{O}_Y\otimes\mathscr{O}_Y)\oplus(\mathscr{O}_Y\otimes\mathscr{L})\oplus(\mathscr{L}\otimes\mathscr{O}_Y)\otimes(\mathscr{L}\otimes\mathscr{L}) \longrightarrow \mathscr{O}_Y\oplus\mathscr{L}$ whose last component is $\sigma\oplus 0 : \mathscr{L}\otimes\mathscr{L} \to \mathscr{O}_Y\oplus\mathscr{L}$ and the others are the natural ones following from the desired compatibility with the $\mathscr{O}_Y$-module structure. More precisely, for every open subset U :

$$\mathbf{m}(U)((a\oplus l)\otimes(a'\oplus l')) = (aa' + \sigma(U)(l\otimes l'))\oplus(al' + a'l)$$

  If $v : \mathscr{L} \to \mathscr{L}'$ is a morphism in $\mathscr{D}_2(Y)$, we set $\Delta(v) = v\oplus \mathrm{id}$.

The main result of this chapter is :

Theorem 2.2.1

The functors Λ and Δ are well-define and quasi-inverses of each other. Thus, $\mathrm{Cov}_2(Y)$ *and $\mathscr{D}_2(Y)$ are equivalent categories.*

We will see that we can actually study a "local" simplified problem corresponding to the situation we get if we localize the global problem by interesting ourselves to the stalks. Solving the local case is only a matter of commutative algebra. For the double covers as well as the triple ones, the only difficulty in the global case is therefore to define global maps that we can localize to use the local case.

Λ **is well-defined**

Proposition 2.2.2

If $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank, and $\mathscr{E}_1,\mathscr{E}_2$ quasi-coherent $\mathscr{O}_Y$-modules such that $\mathscr{E}=\mathscr{E}_1\oplus\mathscr{E}_2$, then $\mathscr{E}_1,\mathscr{E}_2$ are locally free of finite rank.

We admit the following lemma :

Lemma 2.2.3

If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module and $Y=\bigcup_{i\in I} U_i$, where U_i is open and affine, then $\mathscr{F}$ is finitely presented (i.e. for every U open affine, $\mathscr{F}(U)$ is finitely presented) if and only if for every $i\in I$, $\mathscr{F}(U_i)$ is finitely presented.

Proof (of the proposition 2.2.2)**.** Let us first prove that $\mathscr{E}_1$ and $\mathscr{E}_2$ are finitely presented.

Let $Y=\bigcup_{i\in I} U_i$ be an open covering of Y such that $\mathscr{E}_{|U_i}\simeq\mathscr{O}_{Y|U_i}^{n_i}$. We have

$$0 \longrightarrow \mathscr{E}_2 \xrightarrow{\text{injection}} \mathscr{E} \xrightarrow{\text{projection}} \mathscr{E}_1 \longrightarrow 0$$

And

$$\begin{tikzcd} \mathscr{E}_2 \arrow[r, "\text{injection}"] & \mathscr{E} \\ \mathscr{E} \arrow[u, "\text{proj}"] \arrow[ur, "\text{id}"] & \end{tikzcd}$$

so that $\mathscr{E} \to \mathscr{E} \to \mathscr{E}_1 \to 0$, and by restricting this exact sequence to U_i we get a finite presentation of $\mathscr{E}_{1|U_i}$. The lemma ensures us that $\mathscr{E}_1$ is finitely presented. Symetrically, it's also the case for $\mathscr{E}_2$.

According to the proposition 1.4.3, we know that, for $i = 1, 2$, $\mathscr{E}_i$ is locally free of finite rank if and only if $\forall x \in Y$, $\mathscr{E}_{i,x}$ is a free $\mathscr{O}_{Y,x}$-module of finite rank. We are brought back to a local situation that is solved in the proposition 2.2.4. □

Proposition 2.2.4

If $(R, \mathfrak{m})$ is a local ring and M a free R-module of finite rank such that M splits in $M = N \oplus L$, N, L R-modules. Then N and L are free of finite rank.

Lemma 2.2.5

If $(R, \mathfrak{m})$ is a local ring, a surjective morphism between two free R-modules of same finite rank is injective.

Proof. Let $\varphi : R^n \to R^n$ be such a surjective morphism and $K = \ker \varphi$.

Let ψ a morphism such that $\varphi \circ \psi = \mathrm{id}$. The proposition 2.1.8 gives an isomorphism $\psi : R^n \to R^n \oplus K$, which pass to the quotient in $\overline{\psi} : k^n \xrightarrow{\simeq} k^n \oplus K/\mathfrak{m}K$, where $k = R/\mathfrak{m}$. Hence $K/\mathfrak{m}K = 0$. Moreover, K is of finite type since the projection $R^n \simeq R^n \oplus K \to K$ is surjective, so by Nakayama's lemma, $K = 0$. □

Proof (of the proposition)**.** To simplify, we can assume that $M = R^n$. Let us tensorise by $R/\mathfrak{m}$: if $k = R/\mathfrak{m}$, $k^n = N/\mathfrak{m}N \oplus L/\mathfrak{m}L$ as k-vector spaces. Let $s = \dim_k N/\mathfrak{m}N$, then $L/\mathfrak{m}L$ is of dimension $n - s$.

By Nakayama's lemma, if $(\overline{x_1}, \dots, \overline{x_s})$ is a basis of $N/\mathfrak{m}N$, then $\begin{array}{rccc} \varphi_1 : & R^s & \longrightarrow & N \\ & e_i & \longmapsto & x_i \end{array}$ is surjective, where (e_i) is the canonical basis of R^s.

Symetrically, we have $\varphi_2 : R^{n-s} \twoheadrightarrow L$ and thus $R^s \oplus R^{n-s} \twoheadrightarrow N \oplus L = R^n$. hence, according to the lemma 2.2.4, this morphism is injective and so are φ_1 and φ_2. □

Here, the proposition 2.1.8 states that $\mathscr{A} = \ker \mathrm{Tr}_{\mathscr{A}} \oplus \mathscr{O}_Y$ and $\ker \mathrm{Tr}_{\mathscr{A}}$ is quasi-coherent since it is the kernel of a morphism between quasi-coherent sheaves (cf [8], p. 162). As $\mathscr{A}$ is locally free of finite rank, so is $\ker \mathrm{Tr}_{\mathscr{A}}$ and passing to the stalks, we see it is of degree 1, *i.e.* invertible.

If u is a morphism between $\mathscr{A}$ and $\mathscr{A}'$ in $\mathrm{Cov}_2(Y)$, then we have to check that $v = \Lambda(u) = u_{|\ker \mathrm{Tr}_{\mathscr{A}}}$ is an isomorphism onto $\ker \mathrm{Tr}_{\mathscr{A}'}$, and preserves σ, σ'.

These three statements can be checked locally, *i.e.* on the stalks. So let us consider this situation : R is a local ring where $2 \in R^*$, $A = \ker \mathrm{Tr}_A \oplus R$ and $A' = \ker Tr'_A \oplus R$ are two free R-algebras of rank 2, σ and σ' are defined as in the global case, and u is an isomorphism of R-algebras between A and A'. We set $v = u_{|\ker \mathrm{Tr}_A}$.

First, let $(e_i)_i$ be a basis of A, and $a \in A$. Then $(u(e_i))_i$ is a basis of A', and $t_{u(a)} : b' \mapsto u(a)b'$ sends $u(e_i)$ to $u(ae_i)$. Thus, the matrix of $t_{u(a)}$ in $(u(e_i))$ is the same as the one of t_a in e_i and $\mathrm{Tr}_{A'}(u(a)) = \mathrm{Tr}_A(a)$. This proves the first fact.

On the other hand, v is an isomorphism because it is injective and if $a' \in \ker \mathrm{Tr}_{A'}$, there exists $a \in A$ such that $a' = u(a)$, so $0 = \mathrm{Tr}(a') = \mathrm{Tr}(a)$ and $a \in \ker \mathrm{Tr}_A$.

Finally, the compatibility of v with σ, σ' results from the fact that u is a morphism of R-algebras, combined with $\mathrm{Tr}_{A'} \circ u = \mathrm{Tr}_A$.

Δ is well-defined

We want to prove two things : first that $\mathbf{m}$ is commutative and associative. We noticed in the section 1.5 that this was equivalent to the commutativity and associativity of the map on the stalk $\mathbf{m}_x$ for every $x \in Y$. Furthermore, if $v : \mathscr{L} \to \mathscr{L}'$ is a morphism of $\mathscr{O}_Y$-modules and $u = v \oplus \mathrm{id} : \mathscr{L} \oplus \mathscr{O}_Y \to \mathscr{L}' \oplus \mathscr{O}_Y$, we want u to be a morphism of $\mathscr{O}_Y$-algebras. This is true if and only if, for all $x \in Y$ the map on the stalks u_x is a morphism of $\mathscr{O}_{Y,x}$-algebra

That's why we can assume that $Y = \operatorname{Spec} R$, with R local ring. We are brought back to the study of (L, σ), where L is a free R-module of rank 1, $(R, \mathbf{m})$ being a local ring in which 2 is invertible, and $\sigma : L \otimes L \to R$ is a morphism of R-modules. The multiplication map on $A = L \oplus R$ we are interested in is $\mathbf{m}$ induced by $\sigma \oplus 0$.

Lemma 2.2.6

Let z be a generator of L and $\hat{\sigma} = \sigma(z \otimes z)$ Then $A \simeq R[X]/(X^2 - \hat{\sigma})$ as a R-module, with respect to the multiplications.

Proof. Let $\psi : \begin{array}{rcl} R[X]/(X^2-\hat{\sigma}) = R \oplus Rx & \longrightarrow & A = R \oplus Rz \\ \overline{X} = x & \longmapsto & z \end{array}$, defined as a R-linear map. ψ is clearly bijective. Moreover, it preserves the multiplication, since :

$$\psi((a+bx)(c+dx)) = \psi(ac + bd\hat{\sigma} + (ad+bc)x) = ac + bd\hat{\sigma} + (ad+bc)z$$

And $\mathbf{m}(\psi(a+bx) \otimes \psi(c+dx)) = \mathbf{m}((a+bz) \otimes (c+dz)) = ac + bd\hat{\sigma} + adz + bcz.$ □

So the map $\mathbf{m}$ has the same properties as the multiplication on $R[X]/(X^2 - \hat{\sigma})$ and therefore is commutative and associative.

Let v be an isomorphism $L \to L'$ that preserve σ, σ'. We want to show that $u = v \oplus \mathrm{id}$ is a morphism of R-algebras and an isomorphism.
u can be seen as a morphism from $R[X]/(X^2 - \hat{\sigma}) = R \oplus Rx$ to $R[X]/(X^2 - \hat{\sigma'}) = R \oplus Rx'$. Moreover, the compatibility with σ, σ' means that $u(x)^2 = \sigma(x \otimes x) = \hat{\sigma}$.

So $u((a+bx)(c+dx)) = u(ac + bd\hat{\sigma} + (ad+bc)x) = ac + bd\hat{\sigma} + (ad+bc)u(x)$, and

$$\begin{aligned} u(a+bx) \times u(c+dx) &= (a+bu(x))(c+du(x)) = ac + bdu(x)^2 + (ad+bc)u(x) \\ &= ac + bd\hat{\sigma} + (ad+bc)u(x) \end{aligned}$$

Thus u is indeed a morphism of R-algebras, and obviously an isomorphism.

Δ and Λ are quasi inverses of each other.

1. $\Delta \circ \Lambda \simeq \mathbf{1}_{\mathrm{Cov}_2(\mathrm{Y})}$.

   Let $(\mathscr{A}, \mathbf{m})$ be a locally free $\mathscr{O}_Y$-algebra of rank 2, $(\mathscr{L}, \sigma) = \Lambda(\mathscr{A}, \mathbf{m})$ (so $\mathscr{L} = \ker \mathrm{Tr}_{\mathscr{A},\mathbf{m}}$).

   Let us denote the multiplication induced by σ on $\mathscr{A} = \mathscr{L} \oplus \mathscr{O}_Y$ by $\mathbf{n}$. We want to show that $\mathbf{m} = \mathbf{n}$, which can be proved locally.

So let us take these notations : $(R, \mathfrak{m})$ is a local ring such that $2 \in R^*$ $(A, \mathbf{m})$ is a free R-algebra of rank 2, $L = \ker \mathrm{Tr}_{A,\mathbf{m}}$ is endowed with σ induced by $\mathbf{m}$, $\mathbf{n}$ is the multiplication induced by σ on A.

We know that $A = L \oplus R = Rz \oplus R$, and $(A, \mathbf{n}) \simeq R[X]/(X^2 - \hat{\sigma})$, $\hat{\sigma} = \sigma(z \otimes z)$. We set $\mathbf{m}(z \otimes z) = a + bz, a, b \in R$.

Therefore $0 = \mathrm{Tr}_{A,\mathbf{m}}(z) = \mathrm{Tr}\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = b$.

Moreover, $\mathrm{Tr}_{A,\mathbf{m}}(\mathbf{m}(z \otimes z)) = 2a$. But $\sigma = \left(\dfrac{\mathrm{Tr}_{(A,\mathbf{m})}}{2} \circ \mathbf{m}\right)_{|L \otimes L}$, so actually, $a = \hat{\sigma}$, which means that $\mathbf{m}(z \otimes z) = \sigma(z \otimes z) = \mathbf{n}(z \otimes z)$. Thus, $\mathbf{m} = \mathbf{n}$.

Finally, if u is a morphism from $(\mathscr{A}, \mathbf{m})$ to $(\mathscr{A}', \mathbf{m}')$, then $(\Delta \circ \Lambda)(u) = u_{|\mathscr{L}} \oplus \mathrm{id}$, $\mathscr{L} = \ker \mathrm{Tr}_{\mathscr{A},\mathbf{m}}$. The decomposition $\mathscr{A} = \mathscr{O}_Y \oplus \mathscr{L}$ combined with the fact that u is a morphism of $\mathscr{O}_Y$-algebras implies that $(\Delta \circ \Lambda)(u) = u$.

2. $\Lambda \circ \Delta \simeq \mathbf{1}_{\mathscr{D}_2(\mathrm{Y})}$.

   Let us consider $(\mathscr{L}, \sigma) \in \mathscr{D}_2(Y)$. If $\Delta(\mathscr{L}, \sigma) = (\mathscr{A}, \mathbf{m})$, then $(\Lambda \circ \Delta)(\mathscr{L}, \sigma) = (\mathscr{M}, \nu)$, where $\mathscr{M} = \ker \mathrm{Tr}_{\mathscr{A},\mathbf{m}}$, and ν is induced by $\mathbf{m}$.

   Once again, the equality $\mathscr{L} = \mathscr{M}$ and $\sigma = \nu$ can be checked on the stalks, so we go back to the local situation as above.

   We have $A = L \oplus R = Rz \oplus R$ and we know that $A \simeq R[X]/(X^2 - \hat{\sigma})$ as R-algebras, $\hat{\sigma} = \sigma(z \otimes z)$.

   $\mathrm{Tr}_{A,\mathbf{m}}(z) = \mathrm{Tr}\begin{pmatrix} 0 & \hat{\sigma} \\ 1 & 0 \end{pmatrix} = 0$ (matrix in the basis $(1, z)$), so $L \subset M$.

   Moreover, if w is a basis of M, then there exists $a, b \in R$ such that $w = a + bz$ and $0 = \mathrm{Tr}(w) = \mathrm{Tr}\begin{pmatrix} a & b\hat{\sigma} \\ b & a \end{pmatrix} = 2a$, hence $a = 0$ because $2 \in R^*$. So $w \in L$ and $L = M$.

   Finally, if $l, l' \in L$, $\sigma(l \otimes l') \in R$, so $\sigma(l \otimes l') = \dfrac{\mathrm{Tr}}{2}(\sigma(l \otimes l')) = \dfrac{\mathrm{Tr}}{2}(\mathbf{m}((0 \oplus l) \otimes (0 \oplus l'))) = \nu(l \otimes l')$ by construction, because $M = L$.

   If ν is a morphism between $(\mathscr{L}, \sigma)$ and $(\mathscr{L}', \sigma')$, then $\Lambda(\Delta(\nu)) = (\nu \oplus \mathrm{id})_{|\ker \mathrm{Tr}_{\mathscr{L} \oplus \mathscr{O}_Y}} = (\nu \oplus \mathrm{id})_{|\mathscr{L}}$ according to the computation above, so $\Lambda(\Delta(\nu)) = \nu$.

Chapter 3

Triple covers

In this chapter, we will mainly refer to the article of Rick Miranda ([9]).

Let us first describe the framework of the construction. We consider in all this part Y a scheme such that $6 \in \mathcal{O}_Y(Y)^*$.

Let $\mathcal{D}_3(Y)$ be the category whose objects are pairs $(\mathcal{E}, \delta)$ where $\mathcal{E}$ is a locally free sheaf of rank 2 on Y and $\delta : S^3E \to \Lambda^2 E$ is a morphism of $\mathcal{O}_Y$-modules. In this category, a morphism between $(\mathcal{E}, \delta)$ and $(\mathcal{E}', \delta')$ is an isomorphism $u : \mathcal{E} \to \mathcal{E}'$ such that

$$\begin{array}{ccc} S^3\mathcal{E} & \xrightarrow{S^3u} & S^3\mathcal{E}' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ \Lambda^2\mathcal{E} & \xrightarrow{\Lambda^2 u} & \Lambda^2\mathcal{E}' \end{array}$$

commutes.

Consider the following associations :

- If $(\mathcal{A}, \mathbf{m})$ is a locally free $\mathcal{O}_Y$-algebra of rank 3, then thanks to the proposition 2.1.8 ($3 \in \mathcal{O}_Y(Y)^*$), we have $\mathcal{A} = \mathcal{E} \oplus \mathcal{O}_Y$, $\mathcal{E} = \ker \mathrm{Tr}_{\mathcal{A}}$, and we can define $\beta : S^2\mathcal{E} \to \mathcal{E}$ as the factorisation through $S^2\mathcal{E}$ of $\mathcal{E} \otimes \mathcal{E} \xrightarrow{\mathbf{m}_{|\mathcal{E}\otimes\mathcal{E}}} \mathcal{E} \oplus \mathcal{O}_Y \xrightarrow{\text{projection}} \mathcal{E}$. We will prove in 3.2.2 that there exists a unique $\delta : S^3\mathcal{E} \to \det\mathcal{E}$ such that this diagam commutes :

$$\begin{array}{ccc} S^2\mathcal{E} \otimes \mathcal{E} & \xrightarrow{\beta\otimes\mathrm{id}} & \mathcal{E} \otimes \mathcal{E} \\ \downarrow & & \downarrow \\ S^2\mathcal{E} & \xrightarrow{\delta} & \det\mathcal{E} \end{array} \tag{3.1}$$

  We define H from $\mathrm{Cov}_3(Y)$ to $\mathcal{D}_3(Y)$ which associates with $(\mathcal{A}, \mathbf{m})$ the object $(\mathcal{E}, \delta)$.

  If $u : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of $\mathcal{O}_Y$-algebras, we set $H(u) = u_{|\mathcal{E}}$.

- If $(\mathcal{E}, \delta)$ is an object of $\mathcal{D}_3(Y)$, then thanks to the proposition 3.2.5, we can find a unique $\beta : S^2\mathcal{E} \to \mathcal{E}$ satisfying the diagram 3.1. Moreover, we will show in the proposition 3.2.3 that there exists a unique $\eta : S^2\mathcal{E} \to \mathcal{O}_Y = \mathcal{A}$ such that $\mathcal{A}$ endowed with $\mathbf{m}$ induced by $\beta \oplus \eta$ is a $\mathcal{O}_Y$-algebra. We set $K(\mathcal{E}, \delta) = (\mathcal{A}, \mathbf{m})$ and for $v : \mathcal{E} \to \mathcal{E}'$ morphism in $\mathcal{D}_3(Y)$, $K(v) = v \oplus \mathrm{id}$.

The main result of this chapter is :

Theorem 3.0.7

> *The functors K and H are quasi inverses of each other and therefore $\mathscr{D}_3(Y)$ and $\mathrm{Cov}_3(Y)$ are equivalent categories.*

To make the proof clearer, we add a third category that will be equivalent to the first ones : the objects of $\widetilde{\mathscr{D}}_3(Y)$ are the pairs $(\mathscr{E},\beta)$, where $\mathscr{E}$ is a locally free sheaf of rank 2 and $\beta : S^2\mathscr{E} \to \mathscr{E}$ is a morphism of $\mathscr{O}_Y$-modules such that there exists a unique morphism δ making the diagram 3.1 commutative.

3.1 The local case

Let $(R,\mathfrak{m})$ be a local ring such that $6 \in R^*$.

Let $\mathrm{LCov}_3(R)$ be the set of the free algebras $(A,\mathbf{m})$ of rank 3. We know that if $A \in \mathrm{LCov}_3(R)$, then $A = \ker \mathrm{Tr}_A \oplus R$ and $\ker \mathrm{Tr}_A$ is a free R-module of rank 2 (theorem 2.1.8 and proposition 2.2.4).

Let $d_3(R)$ be the set of the pairs (E,ϕ_2) where E is a free R-module of rank 2 and ϕ_2 is a morphism from S^2E to E such that, if (z,w) is a basis of E, the matrix of ϕ_2 in the basis (z^2,zw,w^2) of S^2E is of the form $\begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix}$, $(a,b,c,d) \in R^4$. This condition doesn't depend on the basis we choose, thanks to the following lemma, which also shows that $d_3(R)$ is the local equivalent of $\mathscr{D}_3(Y)$.

Lemma 3.1.1

> *Let R be a ring, E a free R-module of rank 2 and $\beta : S^2E \to E$ a morphism. If (z,w) is a basis of E, then the existence of the factorisation*
>
> $$\begin{array}{ccc} S^2E \otimes E & \xrightarrow{\beta\otimes\mathrm{id}} & E \otimes E \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \nu} \\ S^3E & \overset{\delta}{\dashrightarrow} & \det E \end{array}$$
>
> *is equivalent to : $M_{(z^2,zw,w^2)}(\beta) = \begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix}$ where $a,b,c,d \in R$.*

Proof. A direct check shows that the existence of the factorisation is equivalent to $\ker\mu \subset \ker\psi$, with $\psi = \nu \circ (\beta \otimes \mathrm{id})$.

But the kernel of μ is generated by $(z^2 \otimes w - zw \otimes z, zw \otimes w - w^2 \otimes z)$.

Moreover, if $M_{(z^2,zw,w^2)}(\beta) = \begin{pmatrix} a & e & c \\ b & f & d \end{pmatrix}$, a direct computation shows that $\psi(z^2 \otimes w - zw \otimes z) = (a+f)z \wedge w$ and $\psi(zw \otimes w - w^2 \otimes z) = (d+e)z \wedge w$.

So

$$\ker\mu \subset \ker\psi \Leftrightarrow \begin{cases} a = -f \\ e = -d \end{cases} \Leftrightarrow M_{(z^2,zw,w^2)}(\beta) = \begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix}$$

□

We are going to explain why taking a free R-algebra of rank 3 is the same as considering an object (E,ϕ_2) of $d_3(R)$.

Let $(A,\mathbf{m}) \in \mathrm{LCov}_3(R)$, $A = E \oplus R$, where $E = \ker \mathrm{Tr}_A$. As $\mathbf{m}$ is commutative, we get ϕ such that :

$$\begin{tikzcd} E \otimes E \arrow[r, "\mathbf{m}_{|E\otimes E}"] \arrow[d] & A \\ S^2E \arrow[ur, "\phi"'] & \end{tikzcd}$$

And we define $\phi = (\phi_1 : S^2E \to R) + (\phi_2 : S^2E \to E)$. For the same reasons as in the case of the double covers, the interesting part of $\mathbf{m} : (R\otimes R)\oplus(R\otimes E)\oplus(E\otimes R)\oplus(E\otimes E) \to E\oplus R$ is expressed by ϕ, since the other components are imposed by the fact that A is a R-algebra.

Let (z,w) be a basis of E. We can find $(a,b,c,d,e,f,g,h,i) \in R^9$ such that

$$\begin{cases} \phi(z^2) = g + az + bw \\ \phi(zw) = h + ez + fw \\ \phi(w^2) = i + cz + dw \end{cases}$$

$\mathbf{m}$ is associative so in particular $\mathbf{m}(z\otimes \mathbf{m}(z\otimes w)\otimes w) = \mathbf{m}(\mathbf{m}(z\otimes z)\otimes w)$ (condition (1)) and $\mathbf{m}(z\otimes \mathbf{m}(w\otimes w)) = \mathbf{m}(\mathbf{m}(z\otimes w)\otimes w)$ (condition (2)).

Using the expression of ϕ, and the compatibility of $\mathbf{m}$ with the structure of R-module, we get for (1) $gw + a(h+ez+fw) + b(i+cz+dw) = hz + e(g+az+bw) + f(h+ez+fw)$, *i.e.*

$$(ah+bi) + (ae+bc)z + (af+bd+g)w = (eg+fh) + (h+ae+ef)z + (be+f^2)w$$

and for (2) :

$$(cg+dh) + (i+ac+de)z + (bc+df)w = (eh+fi) + (e^2+cf)z + (h+ef+df)w$$

As $(1,z,w)$ is a basis of A, we finally obtain :

$$\begin{cases} g = f^2 + be - bd - af \\ h = bc - ef \\ i = e^2 + cf - ac - de \end{cases} \tag{3.2}$$

In other words, ϕ_1 is determined by ϕ_2.

Moreover, $E = \ker \mathrm{Tr}_A$ so $0 = \mathrm{Tr}(z) = \mathrm{Tr}\begin{pmatrix} 0 & g & h \\ 1 & a & e \\ 0 & b & f \end{pmatrix} = a + f$ and likewise $0 = \mathrm{Tr}(w) = e + d$. So

$$M_{(z^2,zw,w^2)}(\phi_2) = \begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix} \tag{3.3}$$

i.e. (E,ϕ_2) is in $d_3(R)$. Furthermore :

$$M_{(z^2,zw,w^2)}(\phi_1) = \begin{pmatrix} 2(a^2-bd) & -(ad-bc) & 2(d^2-ac) \end{pmatrix} \tag{3.4}$$

We have reduced the number of parameters of the problem from 9 to 4.

Reciprocally, let us take $(E,\phi_2) \in d_3(R)$. If (z,w) is a basis of E, the matrix of ϕ_2 in the basis (z^2, zw, w^2) of S^2E is $\phi_2 = \begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix}$. We can define

$\phi_1 = \begin{pmatrix} 2(a^2-bd) & -(ad-bc) & 2(d^2-ac) \end{pmatrix}$ morphism from S^2E to $R \subset A = E \oplus R$, and thus $\phi : S^2E \to A$ defining a multiplication map $\mathbf{n}$ on A.

$\mathbf{n}$ is associative, since it satisfies the conditions 3.2 which are clearly sufficient.
Furthermore :

$$E = \ker \mathrm{Tr}_{A,\mathbf{n}} \tag{3.5}$$

because $\mathrm{Tr}_{A,\mathbf{n}}(z) = \mathrm{Tr}\begin{pmatrix} 0 & 2(a^2-bd) & -(ad-bc) \\ 1 & a & -d \\ 0 & b & -a \end{pmatrix} = 0$ and likewise $\mathrm{Tr}_{A,\mathbf{n}}(w) = 0$; moreover, if $x = p + qz + rw \in A$ is such that $\mathrm{Tr}_{A,\mathbf{n}}(x) = 0$, then $3p = 0$, hence $p = 0$ since 3 is invertible in R, and $x \in E$.

In conclusion, the way we have constructed ϕ_2 out of $\mathbf{m}$ and $\mathbf{n}$ out of ϕ_2 allows us clearly to state the following result :

Theorem 3.1.2

The two maps we have built, $\Delta : \begin{array}{ccc} d_3(R) & \longrightarrow & \mathrm{LCov}_3(R) \\ (E,\phi_2) & \longmapsto & (E\oplus R, \mathbf{n}) \end{array}$ *and* $\Lambda : \begin{array}{ccc} \mathrm{LCov}_3(R) & \longrightarrow & d_3(R) \\ (A,\mathbf{m}) & \longmapsto & (E,\phi_2) \end{array}$ *are inverses of each other.*

Proof. The fact that $(\Delta \circ \Lambda)(A,\mathbf{m}) = (A,\mathbf{m})$ is obvious by construction, and $(\Lambda \circ \Delta)(E,\phi_2) = (E,\phi_2)$ principally because if $\mathbf{n}$ is the multiplication induced by ϕ_2, then $E = \ker \mathrm{Tr}_{E\oplus R,\mathbf{n}}$. $\square$

3.2 The global case

Let Y be a scheme such that $6 \in \mathcal{O}_Y(Y)^*$.

3.2.1 Equivalence between $\mathrm{Cov}_3(Y)$ and $\widetilde{\mathcal{D}}_3(Y)$

Theorem 3.2.1

There is an equivalence of categories between $\mathrm{Cov}_3(Y)$ *and* $\widetilde{\mathcal{D}}_3(Y)$ *given by :*

$$\begin{array}{ccc} \mathrm{Cov}_3(Y) & \longleftrightarrow & \widetilde{\mathcal{D}}_3(Y) \\ (\mathcal{A},\mathbf{m}) & \overset{\Theta}{\longmapsto} & (\mathcal{E} = \ker \mathrm{Tr}_{\mathcal{A}}, \beta) \\ (\mathcal{E}\oplus\mathcal{O}_Y, \mathbf{n}) & \overset{\Omega}{\longleftarrow\!\shortmid} & (\mathcal{E},\beta) \end{array}$$

where in the first line, $\beta = \mathcal{E}\otimes\mathcal{E} \xrightarrow{\mathbf{m}_{|\mathcal{E}\otimes\mathcal{E}}} \mathcal{E}\oplus\mathcal{O}_Y \xrightarrow{\text{projection}} \mathcal{E}$.

Θ is well-defined

Proposition 3.2.2

> *Given $\mathscr{A} = \mathscr{O}_Y \oplus \mathscr{E}$, locally free $\mathscr{O}_Y$-algebra of rank 3, with $\mathscr{E} = \ker \mathrm{Tr}_A$, whose multiplication induces $\beta : S^2\mathscr{E} \to \mathscr{E}$, there exists a unique $\delta : S^3\mathscr{E} \to \det \mathscr{E}$ such that*
>
> $$\begin{tikzcd} S^2\mathscr{E}\otimes\mathscr{E} \arrow[r, "\beta\otimes\mathrm{id}"] \arrow[d, "\mu"] & \mathscr{E}\otimes\mathscr{E} \arrow[d, "\nu"] \\ S^3\mathscr{E} \arrow[r, dashed, "\delta"] & \det\mathscr{E} \end{tikzcd}$$

Proof (of the proposition 3.2.2)**.** As in 3.1.1, we immediately see that the existence of the factorisation is equivalent to $\ker\mu \subset \ker\psi$ where $\psi = \nu\circ(\beta\otimes\mathrm{id})$. This is the same as saying that $\ker\mu_x \subset \ker\psi_x$ for all $x \in Y$, that is to say $(\mathscr{E}_x, \beta_x) \in d_3(\mathscr{O}_{Y,x})$ which is true. Indeed β_x corresponds in the local case, with $R = \mathscr{O}_{Y,x}$ to the map ϕ_2 and the formula 3.3 shows the result.

□

Construction of Ω

Let $(\mathscr{E}, \beta)$ be an object of $\widetilde{\mathscr{D}}_3(Y)$.

Proposition 3.2.3

> *There exists a unique $\eta : S^2\mathscr{E} \to \mathscr{O}_Y$ such that $\beta\oplus\eta : S^2\mathscr{E} \to \mathscr{O}_Y \oplus \mathscr{E} = \mathscr{A}$ induces a structure of $\mathscr{O}_Y$-algebra.*

Proof. We will proceed in two steps : first, proving the theorem when $Y = \mathrm{Spec}(R)$ is affine and $\mathscr{E} = \widetilde{E}$, E free R-module of rank 2. Then we will prove the general case.

First step : Let us fix (z, w) basis of E. As in the proof of 3.2.2, the matrix of β in (z^2, zw, w^2) is of the form $\begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix}$.

- **Existence** : Keeping in mind the local computation leading to 3.4, we define η by the matrix $\begin{pmatrix} 2(a^2-bd) & -(ad-bc) & 2(d^2-ac) \end{pmatrix}$ and we know that, as the stalks of $\beta\oplus\eta$ satisfies the conditions 3.2, the induced multiplication is associative.
- **Uniqueness** : If $\eta' : S^2\mathscr{E} \to \mathscr{O}_Y$ is such that $\phi' = \beta\oplus\eta'$ induces an associative multiplication map on $\mathscr{E}\oplus\mathscr{O}_Y$, then by looking at the stalks and using the computation leading to the system 3.2, we show that the matrix of η' is necessarily
  $\begin{pmatrix} 2(a^2-bd) & -(ad-bc) & 2(d^2-ac) \end{pmatrix}$, whence $\eta = \eta'$.

Second step : Let $\mathscr{B}$ be the basis of open subsets of Y composed by the U open affine of Y such that $\mathscr{E}_{|U}$ is free of rank 2. Thanks to the first step, for $U \in \mathscr{B}$, we have built a unique η_U corresponding to $\beta_U = \beta_{|U}$.

Now, if $V \subset U$ are two elements of $\mathscr{B}$, the multiplication induced by $\beta_U \oplus \eta_U$ restricted to V induces a structure of $\mathscr{O}_Y$-algebra on $\mathscr{E}_{|V}$.

In other terms, $(\eta_U)_{|V}$ satisfies the property characterising η_V, which implies that they are equal. In particular, this result gives the compatibility with the restriction maps which allows us to define a morphism η of sheaves by setting $\forall U \in \mathscr{B}, \eta(U) = \eta_U(U)$.

The uniqueness in the general case ensues from the uniqueness in the first step because if η' is a morphism such that $\eta\oplus\beta$ yields a structure of $\mathscr{O}_Y$-algebra on $\mathscr{E}$, then for $U \in \mathscr{B}$, $\eta'_{|U} \oplus \beta_{|U}$ induces a structure of $\mathscr{O}_Y$-algebra on $\mathscr{E}_{|U}$ and thus $\eta'_{|U} = \eta_{|U}$. Hence $\eta' = \eta$.

□

Proposition 3.2.4

If $u : (\mathscr{E}, \beta) \to (\mathscr{E}', \beta')$ is a morphism in $\widetilde{\mathscr{D}}_3(Y)$, then $v = \Omega(u) = u \oplus \mathrm{id} : \mathscr{E} \oplus \mathscr{O}_Y \to \mathscr{E}' \oplus \mathscr{O}_Y$ is an isomorphism of $\mathscr{O}_Y$-algebras.

Proof. Let us take $\eta_\beta, \eta_{\beta'}$ as in proposition 3.2.3. If $v = \mathrm{id} \oplus u = \Omega(v)$, then, as u is compatible with β, β', we only have to prove :

$$\begin{tikzcd}
S^2\mathscr{E} \arrow[r, "S^2u"] \arrow[d, "\eta_\beta"] & S^2\mathscr{E}' \arrow[d, "\eta_{\beta'}"] \\
\mathscr{O}_Y \arrow[r, "\mathrm{id}"] & \mathscr{O}_Y
\end{tikzcd}$$

To show that $\eta := \eta_{\beta'} \circ S^2u$ is equal to η_β, we can demonstrate that $\eta \oplus \beta$ induces a structure of $\mathscr{O}_Y$-algebra on $\mathscr{A}$ and use the uniqueness in the proposition 3.2.3.

As v is a isomorphism, we can carry the multiplication $\mathbf{m}'$ of $\mathscr{A}'$ to $\mathscr{A}$ by

$$\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \arrow[r, "v \otimes v"] \arrow[d, "\widetilde{\mathbf{m}}"] & \mathscr{A}' \otimes \mathscr{A}' \arrow[d, "\mathbf{m}'"] \\
\mathscr{A} & \mathscr{A}' \arrow[l, "v^{-1}"]
\end{tikzcd}$$

But, if $\mathbf{n}$ is the "multiplication" map induced by $\eta \oplus \beta$ on $\mathscr{A}$, we have

$$\begin{tikzcd}
\mathscr{E} \otimes \mathscr{E} \arrow[r, "u \otimes u"] \arrow[d] \arrow[dd, bend right=60, "\mathbf{n}_{\mathscr{E}\otimes\mathscr{E}}"'] & \mathscr{E}' \otimes \mathscr{E}' \arrow[d] \arrow[dd, bend left=60, "\mathbf{m}'_{\mathscr{E}'\otimes\mathscr{E}'}"] \\
S^2\mathscr{E} \arrow[r, "S^2u"] \arrow[d, "\eta \oplus \beta"] & S^2\mathscr{E}' \arrow[d, "\eta_{\beta'} \oplus \beta"] \\
\mathscr{O}_Y \oplus \mathscr{E} \arrow[r, "\mathrm{id} \oplus u"] & \mathscr{O}_Y \oplus \mathscr{E}'
\end{tikzcd}$$

The previous diagram implies that $\mathbf{n} = \widetilde{\mathbf{m}}$, so $\mathbf{n}$ induces a structure of $\mathscr{O}_Y$-algebra on $\mathscr{A}$. Therefore, by uniqueness, $\eta_\beta = \eta$.

□

Ω and Θ are quasi inverses of each other

For the morphisms, the situation is the same as for the double covers.
For the objects :

- If $(\mathscr{A}, \mathbf{m}) \in \mathrm{Cov}_3(Y)$, then $(\Omega \circ \Theta)(\mathscr{A}, \mathbf{m}) = (\mathscr{E} \oplus \mathscr{O}_Y, \mathbf{n})$, where $\mathscr{E} = \ker \mathrm{Tr}_{(\mathscr{A}, \mathbf{m})}$ and $\mathbf{n}$ is induced by the component $\beta : \mathscr{E} \otimes \mathscr{E} \to E$ of $\mathbf{m}$ according to $\mathscr{E}$. We know (proposition 3.2.3) that the essential part of $\mathbf{n}$ is $\beta \oplus \eta$, where, if in a local basis, $M_{(z^2, zw, w^2)}(\beta) = \begin{pmatrix} a & -d & c \\ b & -a & d \end{pmatrix}$, η is of the form $\begin{pmatrix} 2(a^2 - bd) & -(ad - bc) & 2(d^2 - ac) \end{pmatrix}$. But according to the formula 3.4, the local matrix of the essential component of $\mathbf{m}$ is the same, so $\mathbf{m} = \mathbf{n}$.

- If $(\mathscr{E},\beta) \in \widetilde{\mathscr{D}}_3(Y)$, then taking $\Omega(\mathscr{E},\beta) = (\mathscr{A},\mathbf{n})$ with $\mathscr{A} = \mathscr{E} \oplus \mathscr{O}_Y$, we have $(\Theta \circ \Omega)(\mathscr{E},\beta) = (\ker \mathrm{Tr}_{(\mathscr{A},\mathbf{n})}, \beta')$. But thanks to the equation 3.5, $\mathscr{E} = \ker \mathrm{Tr}_{(\mathscr{A},\mathbf{n})}$ and it follows obviously that $\beta = \beta'$.

3.2.2 Equivalence between $\widetilde{\mathscr{D}}_3(Y)$ and $\mathscr{D}_3(Y)$

Let us define two functors $F : \widetilde{\mathscr{D}}_3(Y) \to \mathscr{D}_3(Y)$ and $G : \mathscr{D}_3(Y) \to \widetilde{\mathscr{D}}_3(Y)$.

- $F(\mathscr{E},\beta) = (\mathscr{E},\delta)$ where δ is the unique morphism making this diagram commutative

$$\begin{tikzcd} S^2\mathscr{E} \otimes \mathscr{E} \arrow[r, "\beta\otimes\mathrm{id}"] \arrow[d, "\mu"] & \mathscr{E} \otimes \mathscr{E} \arrow[d, "\nu"] \\ S^3\mathscr{E} \arrow[r, dashed, "\delta"] & \det\mathscr{E} \end{tikzcd}$$

- $G(\mathscr{E},\delta) = (\mathscr{E},\beta)$ where β is the unique morphism : $S^2\mathscr{E} \to \mathscr{E}$ such that this diagram commutes :

$$\begin{tikzcd} S^2\mathscr{E} \otimes \mathscr{E} \arrow[r, dashed, "\beta\otimes\mathrm{id}"] \arrow[d, "\mu"] & \mathscr{E} \otimes \mathscr{E} \arrow[d, "\nu"] \\ S^3\mathscr{E} \arrow[r, "\delta"] & \det\mathscr{E} \end{tikzcd} \tag{3.6}$$

Our goal is to show that F and G are quasi inverses of each other ; obviously, only the fact that they are well-defined matters.

The theorem 1.4.7 will be the key argument to prove this equivalence of categories.

Action on the objects

As the fact that $F(\mathscr{E},\beta)$ is well-defined is ensured by the definition of $\widetilde{\mathscr{D}}_3(Y)$, we just have to prove the following proposition to show that what we wrote makes sense.

Proposition 3.2.5

Given $(\mathscr{E},\delta) \in \mathscr{D}_3(Y)$, there exists a unique morphism β such that the diagram 3.6 commutes.

Lemma 3.2.6

If $\mathscr{E}$ is locally free of rank 2, there exists a canonical isomorphism $\gamma : \mathscr{E} \xrightarrow{\simeq} \det\mathscr{E} \otimes \mathscr{E}^\vee$.

Proof. Let $\nu : \mathscr{E}\otimes\mathscr{E} \to \det\mathscr{E}$ be the canonical morphism. Then, thanks to the theorem 1.4.7, we get a morphism $\gamma : \mathscr{E} \to \det\mathscr{E} \otimes \mathscr{E}^\vee$.

If $x \in Y$, let us check that γ_x is an isomorphism. We can take a basis (v,w) of $\mathscr{E}_x$.

By remembering the construction of γ, one sees that if $u : \mathscr{O}_{Y,x} \to \mathscr{E}_x \otimes (\mathscr{E}_x)^\vee$ is the morphism as in the proof of 1.4.7, then

$$\gamma_x : v \overset{\mathrm{id}\otimes u}{\longmapsto} v \otimes (v \otimes v^* + w \otimes w^*) \overset{\nu\otimes\mathrm{id}}{\longmapsto} (v \wedge v) \otimes v^* + (v \wedge w) \otimes w^* = (v \wedge w) \otimes w^*$$

And similarly $\gamma_x(w) = -(v \vee w) \otimes v^*$, which proves that γ_x sends a basis on a basis, and thus is an isomorphism.

$\square$

Proof (of the proposition 3.2.5)**.** We want to prove that, if $\delta : S^3\mathscr{E} \to \det\mathscr{E}$ is a morphism, there exists a unique $\beta : S^2\mathscr{E} \to \mathscr{E}$ such that

$$\begin{tikzcd} S^2\mathscr{E} \otimes \mathscr{E} \arrow[r, dashed, "\beta\otimes\mathrm{id}"] \arrow[d, "\mu"] & \mathscr{E} \otimes \mathscr{E} \arrow[d, "\nu"] \\ S^3\mathscr{E} \arrow[r, "\delta"] & \det\mathscr{E} \end{tikzcd}$$

If $\varphi = \delta \circ \mu : S^2\mathscr{E} \otimes \mathscr{E} \to \det\mathscr{E}$, then the theorem 1.4.7 gives us a morphism $\psi = K(\varphi) : S^2\mathscr{E} \to \det\mathscr{E} \otimes \mathscr{E}^\vee$, where K is the bijection $\mathrm{Hom}(S^2E \otimes \mathscr{E}, \det\mathscr{E}) \xrightarrow{K} \mathrm{Hom}(S^2\mathscr{E}, \det\mathscr{E} \otimes \mathscr{E}^\vee)$ Let us take $\beta = \gamma^{-1} \circ \psi$, with γ the map of the lemma 3.2.6. We want $\varphi = \nu \circ (\beta \otimes \mathrm{id}) = \lambda$, *i.e.* $\lambda = K^{-1}(\psi)$.

But, if ev is the evaluation morphism $\zeta \otimes s \mapsto \zeta(s)$, $K^{-1}(\lambda)$ is

$$S^2\mathscr{E} \otimes \mathscr{E} \xrightarrow{\psi\otimes\mathrm{id}} \det\mathscr{E} \otimes \mathscr{E}^\vee \otimes \mathscr{E} \xrightarrow{\mathrm{id}_{\det\mathscr{E}}\otimes\mathrm{ev}} \det\mathscr{E}$$

And λ can be decomposed as

$$S^2\mathscr{E} \otimes \mathscr{E} \xrightarrow{\psi\otimes\mathrm{id}} \det\mathscr{E} \otimes \mathscr{E}^\vee \otimes \mathscr{E} \xrightarrow{\gamma^{-1}\otimes\mathrm{id}} \mathscr{E} \otimes \mathscr{E} \xrightarrow{\nu} \det\mathscr{E}$$

Therefore it suffices to show that $\nu \circ (\gamma^{-1} \otimes \mathrm{id}) = \mathrm{id}_{\det\mathscr{E}} \otimes \mathrm{ev}$. Passing to the stalks $\mathscr{E}_x$ and using the computation made in the proof of the previous lemma gives us this results by evaluating on a basis. $\square$

Action on the morphisms

Proposition 3.2.7

If $u \in \mathrm{Hom}_{\mathscr{D}_3(Y)}((\mathscr{E}, \beta), (\mathscr{E}', \beta'))$, and δ, δ' are the images of β, β' by F, then the following diagram commutes :

$$\begin{tikzcd} S^3\mathscr{E} \arrow[r, "S^3u"] \arrow[d, "\delta"] & S^3\mathscr{E}' \arrow[d, "\delta'"'] \\ \det\mathscr{E} \arrow[r, "\det u"'] & \det\mathscr{E}' \end{tikzcd}$$

Proof. As the morphisms $\nu_\mathscr{E} : \mathscr{E} \otimes \mathscr{E} \to \det\mathscr{E}$ and $\mu_\mathscr{E} : S^2\mathscr{E} \otimes \mathscr{E} \to S^3\mathscr{E}$ are canonical, we obviously have the following diagrams :

$$\begin{tikzcd} \mathscr{E} \otimes \mathscr{E} \arrow[r, "u\otimes u"] \arrow[d, "\nu_\mathscr{E}"] & \mathscr{E}' \otimes \mathscr{E}' \arrow[d, "\nu_{\mathscr{E}'}"] \\ \det\mathscr{E} \arrow[r, "\det u"] & \det\mathscr{E}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S^2\mathscr{E} \otimes \mathscr{E} \arrow[r, "S^2u\otimes u"] \arrow[d, "\mu_\mathscr{E}"] & S^2\mathscr{E}' \otimes \mathscr{E}' \arrow[d, "\mu_{\mathscr{E}'}"] \\ S^3\mathscr{E} \arrow[r, "S^3u"] & S^3\mathscr{E}' \end{tikzcd}$$

But we also have, by taking the tensor product of the compatibility diagram of u with β, β' and u :

$$\begin{tikzcd} S^2\mathscr{E} \otimes \mathscr{E} \arrow[r, "S^2u\otimes u"] \arrow[d, "\beta\otimes\mathrm{id}"'] & S^2\mathscr{E}' \otimes \mathscr{E}' \arrow[d, "\beta'\otimes\mathrm{id}"] \\ \mathscr{E} \otimes \mathscr{E} \arrow[r, "u\otimes u"] & \mathscr{E}' \otimes \mathscr{E}' \end{tikzcd}$$

Therefore :

$$\begin{tikzcd}
S^2\mathscr{E}\otimes\mathscr{E} \arrow[r, "S^2u\otimes u"] \arrow[d, "\beta\otimes\mathrm{id}"] \arrow[dd, bend right=60, "\psi"'] & S^2\mathscr{E}'\otimes\mathscr{E}' \arrow[d, "\beta'\otimes\mathrm{id}"] \arrow[dd, bend left=60, "\psi'"] \\
\mathscr{E}\otimes\mathscr{E} \arrow[r, "u\otimes u"] \arrow[d, "\nu_{\mathscr{E}}"] & \mathscr{E}'\otimes\mathscr{E}' \arrow[d, "\nu_{\mathscr{E}'}"] \\
\det\mathscr{E} \arrow[r, "\det u"] & \det\mathscr{E}'
\end{tikzcd}$$

where the composed arrow ψ (resp. ψ') is, by property of δ and β equal to $\varphi = \delta\circ\mu_{\mathscr{E}}$ (resp. to $\varphi' = \delta'\circ\mu_{\mathscr{E}'}$). This implies that in the following diagram, the whole rectangle commutes.

$$\begin{tikzcd}
S^{\mathscr{E}}\otimes\mathscr{E} \arrow[r, "S^2u\otimes u"] \arrow[d, two heads, "\mu_{\mathscr{E}}"] \arrow[dd, bend right=60, "\varphi"'] & S^2\mathscr{E}'\otimes\mathscr{E}' \arrow[d, two heads, "\mu_{\mathscr{E}}"] \arrow[dd, bend left=60, "\varphi'"] \\
S^3\mathscr{E} \arrow[r, "S^3u"] \arrow[d, "\delta"] & S^3\mathscr{E}' \arrow[d, "\delta"] \\
\det\mathscr{E} \arrow[r, "\det u"] & \det\mathscr{E}'
\end{tikzcd}$$

As we already know that the upper diagram commutes and the vertical arrows of this diagram are surjective (this principle is easy to understand set-theoretically, and we get this result here by looking at the stalks), then the lower diagram commutes.

□

Proposition 3.2.8

If $u \in \mathrm{Hom}_{\mathscr{D}_3(Y)}((\mathscr{E},\delta),(\mathscr{E}',\delta'))$, and β,β' are the images of δ,δ' by G, then the following diagram commutes :

$$\begin{tikzcd}
S^2\mathscr{E} \arrow[r, "S^2u"] \arrow[d, "\beta"] & S^2\mathscr{E}' \arrow[d, "\beta'"] \\
\mathscr{E} \arrow[r, "u"] & \mathscr{E}'
\end{tikzcd}$$

Proof. We keep the notations of the previous proof. If $\varphi = \delta\circ\mu_{\mathscr{E}}$, $\varphi' = \delta'\circ\mu_{\mathscr{E}'}$, then glueing the compatibility diagram of u with δ,δ' and the canonical diagram of $\mu_{\mathscr{E}},\mu_{\mathscr{E}'}$, we get :

$$\begin{tikzcd}
S^2\mathscr{E}\otimes\mathscr{E} \arrow[r, "S^2u\otimes u"] \arrow[d, "\varphi"] & S^2\mathscr{E}'\otimes\mathscr{E}' \arrow[d, "\varphi'"] \\
\det\mathscr{E} \arrow[r, "\det u"] & \det\mathscr{E}'
\end{tikzcd}$$

Tensorising this diagram with $\mathscr{E}^\vee, (\mathscr{E}')^\vee$, where $(u^{-1})^\vee : f \mapsto f\circ u^{-1}$ is the transposition of u^{-1}, we have :

$$\begin{tikzcd}
S^2\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{E}^\vee \arrow[r, "S^2u\otimes u\otimes(u^{-1})^\vee"] \arrow[d, "\varphi\otimes\mathrm{id}"] & S^2\mathscr{E}'\otimes\mathscr{E}'\otimes(\mathscr{E}')^\vee \arrow[d, "\varphi'\otimes\mathrm{id}"] \\
\det\mathscr{E}\otimes\mathscr{E}^\vee \arrow[r, "\det u\otimes(u^{-1})^\vee"] & \det\mathscr{E}'\otimes(\mathscr{E}')^\vee
\end{tikzcd}$$

Finally, using the canonical morphisms $\tau : \mathscr{O}_Y \to \mathscr{E}\otimes\mathscr{E}^\vee$ and $\gamma : \mathscr{E} \to \det\mathscr{E}\otimes\mathscr{E}^\vee$ (constructed respectively in 1.4.7 and 3.2.6), we have :

$$
\begin{tikzcd}
S^2\mathscr{E}\otimes\mathscr{O}_Y \arrow[r, "S^2u"] \arrow[d, "\mathrm{id}\otimes\tau"] & S^2\mathscr{E}'\otimes\mathscr{O}_Y \arrow[d, "\mathrm{id}\otimes\tau'"] \\
S^2\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{E}^\vee \arrow[r, "S^2u\otimes u\otimes(u^{-1})^\vee"] \arrow[d, "\varphi\otimes\mathrm{id}"] & S^2\mathscr{E}'\otimes\mathscr{E}'\otimes(\mathscr{E}')^\vee \arrow[d, "\varphi'\otimes\mathrm{id}"] \\
\det\mathscr{E}\otimes\mathscr{E}^\vee \arrow[r, "\det u\otimes(u^{-1})^\vee"] \arrow[d, "\gamma^{-1}"] & \det\mathscr{E}'\otimes(\mathscr{E}')^\vee \arrow[d, "\gamma'^{-1}"] \\
\mathscr{E} \arrow[r, "u"] & \mathscr{E}'
\end{tikzcd}
$$

By construction of the β, β' corresponding to δ, δ' (proposition 3.2.5), the vertical arrows are actually β and β'.

□

These two propositions show that both functors are well-defined for the morphisms, which completes the proof of the equivalence of categories.

Bibliography

[1] Michele Bolognesi and Angelo Vistoli. Stacks of trigonal curves. *Trans. Amer. Math. Soc.*, 364:3365–3393, 2012. Online access : `http://arxiv.org/pdf/0903.0965v4.pdf`.

[2] Gianfranco Casnati. Covers of algebraic varieties. ii. covers of degree 5 and construction of surfaces. *Journal of Algebraic Geometry*, 5(3):461–477, 1996.

[3] Gianfranco Casnati and Torsten Ekedahl. Covers of algebraic varieties. i. a general structure theorem, covers of degree 3,4 and enriques surfaces. *Journal of Algebraic Geometry*, 5(3):439–460, 1996.

[4] David Eisenbud and Joe Harris. *The Geometry of Schemes*. Springer-Verlag New York Incorporated, 2000.

[5] Alexandre Grothendieck and Jean Dieudonné. *Eléments de géométrie algébrique I : le langage des schémas*. Publications mathématiques de l'I.H.É.S., 1960.

[6] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I : Schemes and Examples*. Vieweg+Teubner Verlag, 2010.

[7] David W. Hahn and Rick Miranda. Quadruple covers of algebraic varieties. *Journal of Algebraic Geometry*, 8:1–30, 1999.

[8] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, 2002.

[9] Rick Miranda. Triple covers in algebraic geometry. *American Journal of Mathematics*, 107(5):1123–1158, October 1985. Online access : `http://www.jstor.org/stable/2374349`.

[10] Rita Pardini. Triple covers in positive characteristic. *Arkiv för Matematik*, 27(1):319–341, 1989.

[11] Rita Pardini. Abelian covers of algebraic varieties. *Journal für die reine und angewandte Mathematik*, 417:191–214, 1991.

[12] Matthieu Romagny and Stefan Wevers. Hurwitz spaces. In *Groupes de Galois arithmétiques et différentiels. Actes du colloque du CIRM (Luminy, 8-13 Mars 2004)*, volume 13 of *Séminaires et Congrès*, pages 313–341. Publications de la SMF, 2006.

[13] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2016.

[14] Ravi Vakil. The rising sea : Foundations of algebraic geometry, December 2015. Notes available on `http://math.stanford.edu/~vakil/216blog/`, December 29, 2015 version.