

Dichotomize and Generalize: PAC-Bayesian Binary Activated Deep Neural Networks

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Introduction

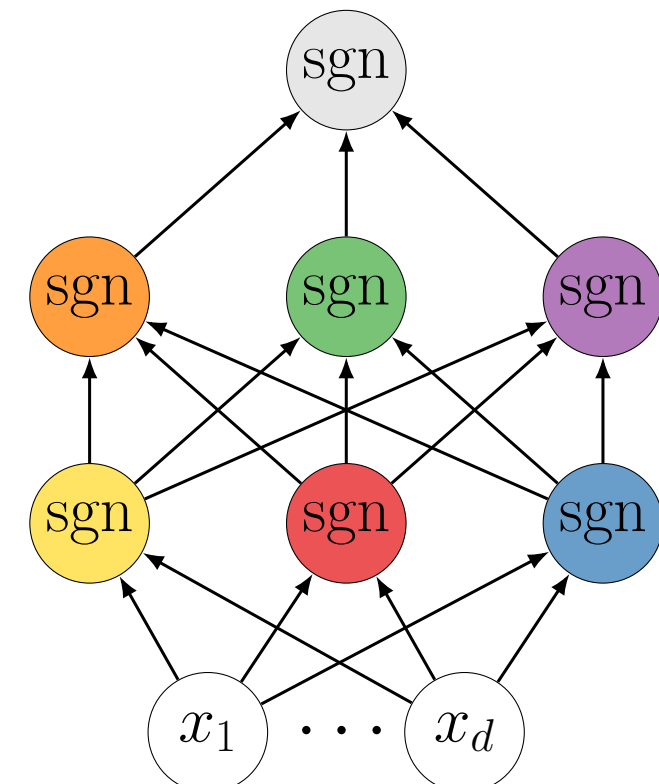
We present a comprehensive study of multilayer neural networks with binary activation, relying on the PAC-Bayesian theory.

Contributions

- An end-to-end framework to train a binary activated deep neural network.
- Nonvacuous PAC-Bayesian generalization bounds for binary activated deep neural networks.

Binary Activated Neural Networks

- L fully connected layers
- d_k denotes the number of neurons of the k^{th} layer
- $\text{sgn}(a) = 1$ if $a > 0$ and $\text{sgn}(a) = -1$ otherwise
- Weights matrices : $\mathbf{W}_k \in \mathbb{R}^{d_k \times d_{k-1}}$, $\theta = \text{vec}(\{\mathbf{W}_k\}_{k=1}^L) \in \mathbb{R}^D$



Prediction

$$f_\theta(\mathbf{x}) = \text{sgn}(\mathbf{w}_L \text{sgn}(\mathbf{W}_{L-1} \text{sgn}(\dots \text{sgn}(\mathbf{W}_1 \mathbf{x}))))$$

PAC-Bayesian Theory

Given a data distribution \mathcal{D} , a training set $S = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\} \sim \mathcal{D}^n$, with $\mathbf{x}_i \in \mathbb{R}^{d_0}$ and $y_i \in \{-1, 1\}$, a loss $\ell : \{-1, 1\}^2 \rightarrow [0, 1]$, a predictor $f \in \mathcal{F}$:

$$\mathcal{L}_{\mathcal{D}}(f) = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} \ell(f(\mathbf{x}), y) \quad \text{generalization loss}$$

$$\hat{\mathcal{L}}_S(f) = \frac{1}{n} \sum_{i=1}^n \ell(f(\mathbf{x}_i), y_i) \quad \text{empirical loss}$$

PAC-Bayesian Theorem

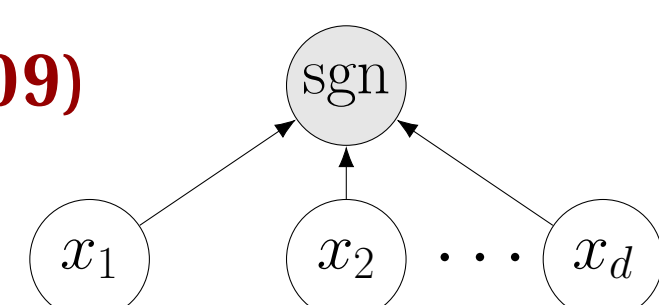
For $C > 0$, for any prior P on \mathcal{F} , with probability $1 - \delta$ on the choice of $S \sim \mathcal{D}^n$, we have for all posterior distribution Q on \mathcal{F} :

$$\mathbf{E}_{f \sim Q} \mathcal{L}_{\mathcal{D}}(f) \leq \frac{1}{1 - e^{-C}} \left(1 - \exp \left(-C \mathbf{E}_{f \sim Q} \hat{\mathcal{L}}_S(f) - \frac{1}{n} [\text{KL}(Q \| P) + \ln \frac{2\sqrt{n}}{\delta}] \right) \right)$$

Linear Classifier

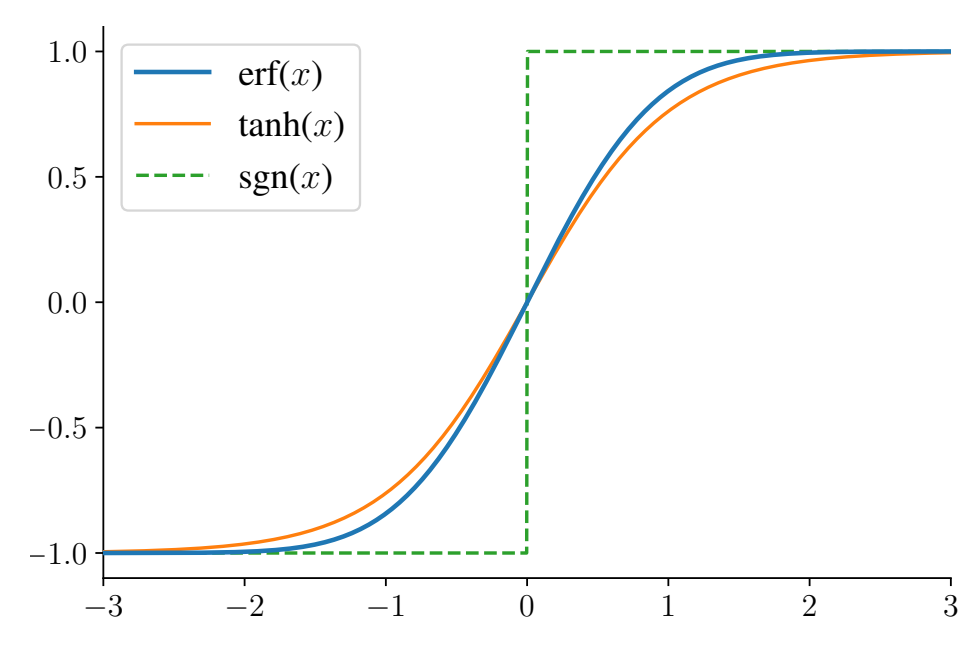
PAC-Bayesian Learning of Linear Classifiers (Germain et al., 2009)

$$f_{\mathbf{w}}(\mathbf{x}) := \text{sgn}(\mathbf{w} \cdot \mathbf{x}), \text{ with } \mathbf{w} \in \mathbb{R}^d$$



PAC-Bayes analysis

- Space of all linear classifiers $\mathcal{F}_d := \{f_{\mathbf{v}} | \mathbf{v} \in \mathbb{R}^d\}$
- Gaussian posterior $Q_{\mathbf{w}} := \mathcal{N}(\mathbf{w}, I_d)$ over \mathcal{F}_d
- Gaussian prior $P_{\mathbf{u}} := \mathcal{N}(\mathbf{u}, I_d)$ over \mathcal{F}_d
- Predictor $F_{\mathbf{w}}(\mathbf{x}) := \mathbf{E}_{\mathbf{v} \sim Q_{\mathbf{w}}} f_{\mathbf{v}}(\mathbf{x}) = \text{erf}\left(\frac{\mathbf{w} \cdot \mathbf{x}}{\sqrt{d} \|\mathbf{x}\|}\right)$
- Linear loss $\ell(f_{\mathbf{v}}(\mathbf{x}), y) := \frac{1}{2} - \frac{1}{2} y f_{\mathbf{v}}(\mathbf{x})$



Bound minimization

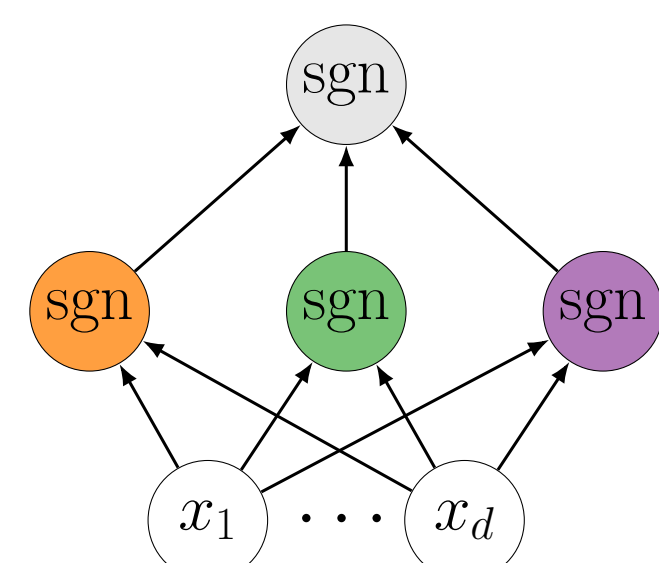
$$C n \hat{\mathcal{L}}_S(F_{\mathbf{w}}) + \text{KL}(Q_{\mathbf{w}} \| P_{\mathbf{u}}) = C \frac{1}{2} \sum_{i=1}^n \text{erf}\left(-y_i \frac{\mathbf{w} \cdot \mathbf{x}_i}{\sqrt{d} \|\mathbf{x}_i\|}\right) + \frac{1}{2} \|\mathbf{w} - \mathbf{u}\|^2$$

Shallow Learning

Posterior $Q_\theta = \mathcal{N}(\theta, I_D)$, over the family of all networks $\mathcal{F}_D = \{f_\theta | \theta \in \mathbb{R}^D\}$, where

$$f_\theta(\mathbf{x}) = \text{sgn}(\mathbf{w}_2 \cdot \text{sgn}(\mathbf{W}_1 \mathbf{x}))$$

$$\begin{aligned} F_\theta(\mathbf{x}) &= \mathbf{E}_{\theta \sim Q_\theta} f_\theta(\mathbf{x}) \\ &= \int_{\mathbb{R}^{d_1 \times d_0}} Q_1(\mathbf{V}_1) \int_{\mathbb{R}^{d_1}} Q_2(\mathbf{v}_2) \text{sgn}(\mathbf{v}_2 \cdot \text{sgn}(\mathbf{V}_1 \mathbf{x})) d\mathbf{v}_2 d\mathbf{V}_1 \\ &= \sum_{\mathbf{s} \in \{-1, 1\}^{d_1}} \text{erf}\left(\frac{\mathbf{w}_2 \cdot \mathbf{s}}{\sqrt{2d_1}}\right) \int_{\mathbb{R}^{d_1 \times d_0}} \mathbb{1}[\mathbf{s} = \text{sgn}(\mathbf{V}_1 \mathbf{x})] Q_1(\mathbf{V}_1) d\mathbf{V}_1 \\ &= \sum_{\mathbf{s} \in \{-1, 1\}^{d_1}} \underbrace{\text{erf}\left(\frac{\mathbf{w}_2 \cdot \mathbf{s}}{\sqrt{2d_1}}\right)}_{F_{\mathbf{w}_2}(\mathbf{s})} \underbrace{\prod_{i=1}^{d_1} \left[\frac{1}{2} + \frac{s_i}{2} \text{erf}\left(\frac{\mathbf{w}_1^i \cdot \mathbf{x}}{\sqrt{2} \|\mathbf{x}\|}\right) \right]}_{\text{Pr}(\mathbf{s} | \mathbf{x}, \mathbf{W}_1)} \end{aligned}$$

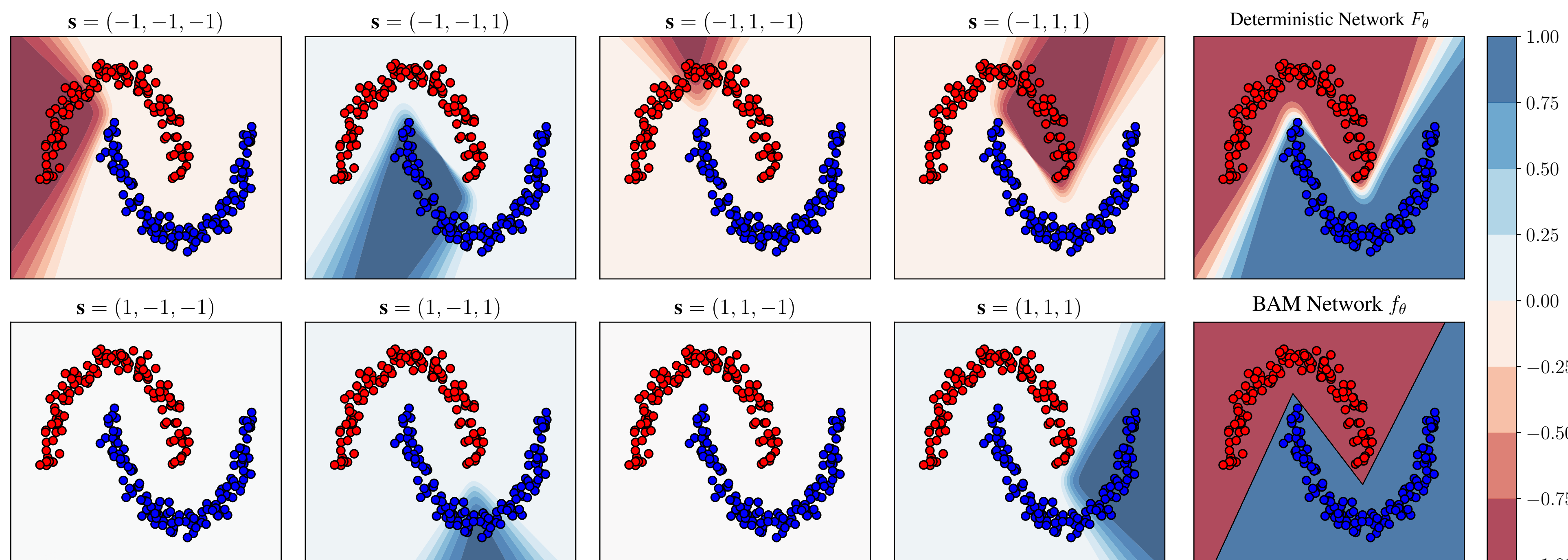


PAC-Bayesian bound ingredients

- Empirical loss : $\hat{\mathcal{L}}_S(F_\theta) = \mathbf{E}_{\theta' \sim Q_\theta} \hat{\mathcal{L}}_S(f_{\theta'}) = \frac{1}{n} \sum_{i=1}^n \left[\frac{1}{2} - \frac{1}{2} y_i F_\theta(\mathbf{x}_i) \right]$
- Complexity term : $\text{KL}(Q_\theta \| P_\mu) = \frac{1}{2} \|\theta - \mu\|^2$, with $\mu \in \mathbb{R}^D$
- Generalization bound : $\frac{1}{1 - e^{-C}} \left(1 - \exp \left(-C \hat{\mathcal{L}}_S(F_\theta) - \frac{1}{n} [\text{KL}(Q_\theta \| P_\mu) + \ln \frac{2\sqrt{n}}{\delta}] \right) \right)$

Visualization

Illustration of the proposed method for a one hidden layer network of size $d_1=3$, interpreted as a majority vote over 8 binary representations $\mathbf{s} \in \{-1, 1\}^3$. For each \mathbf{s} , a plot shows the values of $F_{\mathbf{w}_2}(\mathbf{s}) \text{Pr}(\mathbf{s} | \mathbf{x}, \mathbf{W}_1)$.



Stochastic Approximation

Prediction.

$$F_\theta(\mathbf{x}) = \sum_{\mathbf{s} \in \{-1, 1\}^{d_1}} F_{\mathbf{w}_2}(\mathbf{s}) \text{Pr}(\mathbf{s} | \mathbf{x}, \mathbf{W}_1)$$

Hidden layer partial derivatives, with $\text{erf}'(x) := \frac{2}{\sqrt{\pi}} e^{-x^2}$.

$$\frac{\partial}{\partial \mathbf{w}_1^k} F_\theta(\mathbf{x}) = \frac{\mathbf{x}}{2^{\frac{3}{2}} \|\mathbf{x}\|} \text{erf}'\left(\frac{\mathbf{w}_1^k \cdot \mathbf{x}}{\sqrt{2} \|\mathbf{x}\|}\right) \sum_{\mathbf{s} \in \{-1, 1\}^{d_1}} s_k F_{\mathbf{w}_2}(\mathbf{s}) \left[\frac{\text{Pr}(\mathbf{s} | \mathbf{x}, \mathbf{W}_1)}{\text{Pr}(s_k | \mathbf{x}, \mathbf{w}_1^k)} \right]$$

Monte Carlo sampling : We generate T random binary vectors $\{\mathbf{s}^t\}_{t=1}^T$ according to $\text{Pr}(\mathbf{s} | \mathbf{x}, \mathbf{W}_1)$.

$$\frac{\partial}{\partial \mathbf{w}_1^k} F_\theta(\mathbf{x}) \approx \frac{\mathbf{x}}{2^{\frac{3}{2}} \|\mathbf{x}\|} \text{erf}'\left(\frac{\mathbf{w}_1^k \cdot \mathbf{x}}{\sqrt{2} \|\mathbf{x}\|}\right) \frac{1}{T} \sum_{t=1}^T \frac{s_k^t}{\text{Pr}(s_k^t | \mathbf{x}, \mathbf{w}_1^k)} F_{\mathbf{w}_2}(\mathbf{s}^t)$$

$$F_\theta(\mathbf{x}) \approx \frac{1}{T} \sum_{t=1}^T F_{\mathbf{w}_2}(\mathbf{s}^t)$$

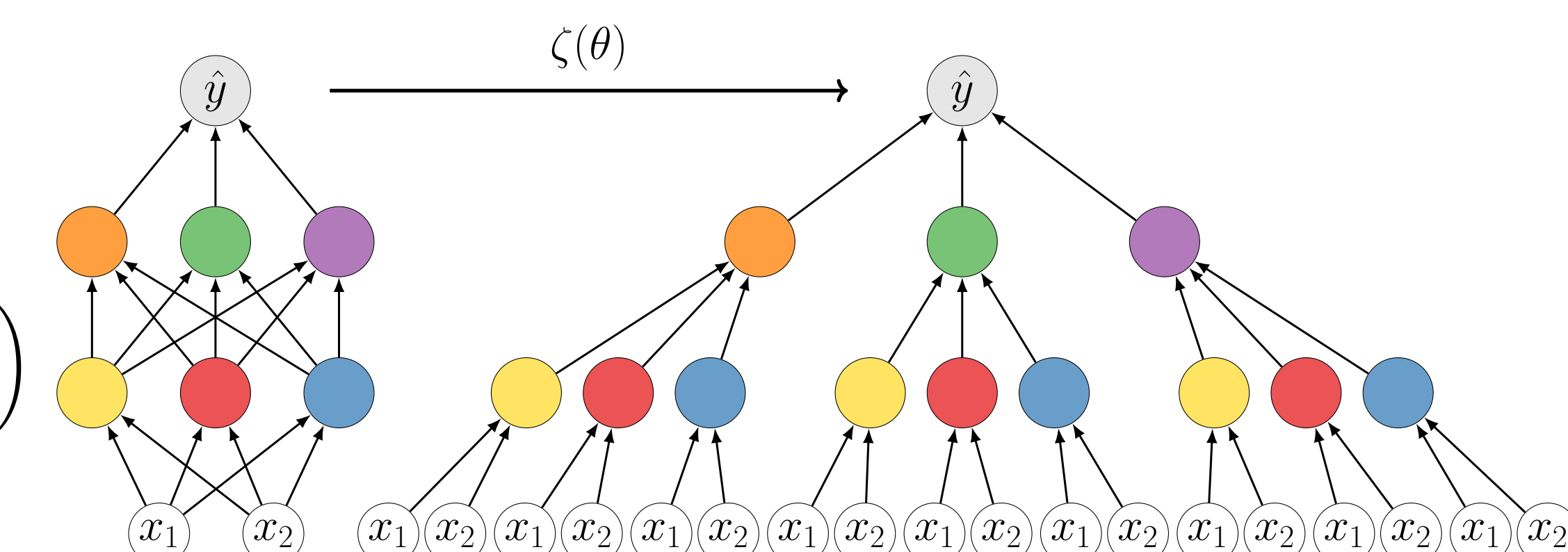
This turn out to be a variant of REINFORCE algorithm (Williams, 1992) to apply the back-propagation method along with discrete activation functions.

Deep Learning (PBGNet)

Generalization to multilayer networks with tree architecture mapping function $\zeta(\theta)$.

$$F_1^{(j)}(\mathbf{x}) = \text{erf}\left(\frac{\mathbf{w}_1^j \cdot \mathbf{x}}{\sqrt{2} \|\mathbf{x}\|}\right)$$

$$F_{k+1}^{(j)}(\mathbf{x}) = \sum_{\mathbf{s} \in \{-1, 1\}^{d_k}} \text{erf}\left(\frac{\mathbf{w}_{k+1}^j \cdot \mathbf{s}}{\sqrt{2d_k}}\right) \prod_{i=1}^{d_k} \left(\frac{1}{2} + \frac{1}{2} s_i \times F_k^{(i)}(\mathbf{x}) \right)$$



Kullback-Leibler regularization

$$\text{KL}(Q_{\zeta(\theta)} \| P_{\zeta(\mu)}) = \frac{1}{2} \left(\|\mathbf{w}_L - \mathbf{u}_L\|^2 + \sum_{i=1}^{L-1} d_{k+1}^\dagger \|\mathbf{W}_i - \mathbf{U}_i\|_F^2 \right) \text{ for } d_k^\dagger := \prod_{i=k}^L d_i.$$

Experiment

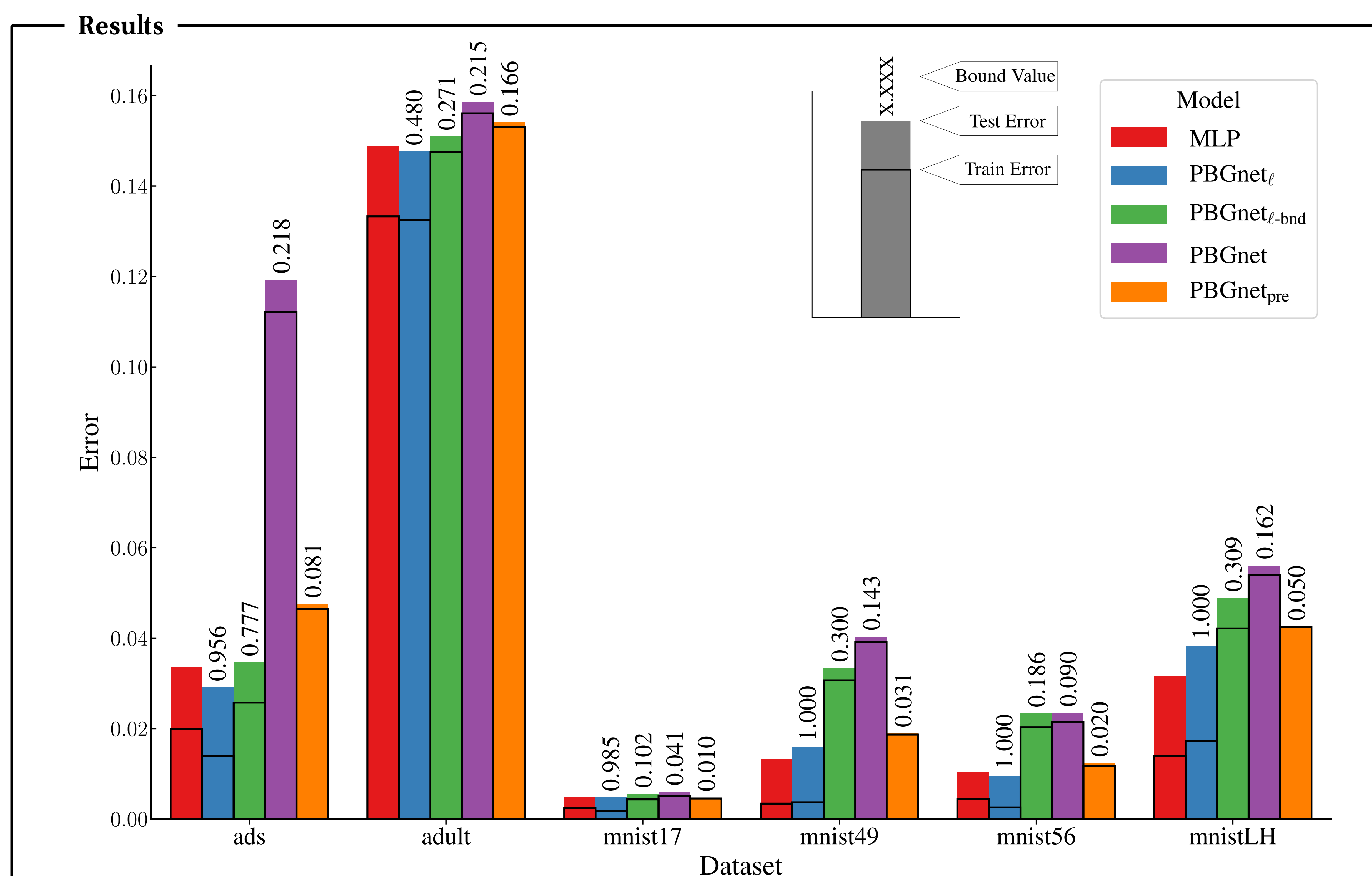


FIGURE 1: Experiment results for the considered models on the binary classification datasets. PAC-Bayesian bounds hold with probability 0.95.