Inference Rules

Concepts of Programming Languages
Lecture 5

CAS CS 320

Practice Problem

Determine an expression with the following type

('a -> 'a -> 'b) -> ('c -> 'a) -> 'c -> 'b

Outline

- Discuss Formal Typing/Semantic Rules
- Look at example rules for the constructs we've seen so far
- Learn to read inference rules, i.e., translate mathematical notation to English and English to mathematical notation

Recap

Recall: Local Variables (Informal)

body

```
let x = 2 in x + x
```

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Syntax: `let VARIABLE = EXPRESSION in BODY`

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Typing: the type is the same as that of BODY *given BODY is well-typed after substituting the VARIABLE in BODY*

Recall: Local Variables (Informal)

body

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let x = 2 in x + x
```

Syntax: `let VARIABLE = EXPRESSION in BODY`

Typing: the type is the same as that of BODY *given BODY is well-typed after substituting the VARIABLE in BODY*

Semantics: the is the same as the value of BODY *after substituting the VARIABLE in BODY*

Recall: If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

Recall: If-Expressions (Informal)

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Syntax: `if CONDITION then TRUE-CASE else FALSE-CASE`

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let abs x = if x > 0 then x else -x
```

Syntax: if CONDITION then TRUE-CASE else FALSE-CASE

Typing: CONDITION must be a Boolean and TRUE-CASE and FALSE-CASE must be the same type. The type is then the same as that of TRUE-CASE and FALSE-CASE

Recall: If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

Syntax: if CONDITION then TRUE-CASE else FALSE-CASE

Typing: CONDITION must be a Boolean and TRUE-CASE and FALSE-CASE must be the same type. The type is then the same as that of TRUE-CASE and FALSE-CASE

Semantics: If CONDITION holds, then we get the TRUE-CASE, otherwise we get the FALSE-CASE

Recall: Functions (Informal)

body

```
fun x -> x + 1
```

Recall: Functions (Informal)

body

$\boxed{\texttt{fun x -> x + 1}}$

Syntax: `fun VAR-NAME -> EXPR`

Recall: Functions (Informal)

body

```
fun x -> x + 1
```

Syntax: `fun VAR-NAME -> EXPR`

Typing: the type of a function is T1 -> T2 where T1 is the type of the input and T2 is the type of the output

Recall: Functions (Informal)

body

```
fun x -> x + 1
```

Syntax: fun VAR-NAME -> EXPR

Typing: the type of a function is T1 -> T2 where T1 is the type of the input and T2 is the type of the output

Semantics: A function will evaluate to special ***function value*** (printed as <fun> by utop)

Recall: Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

Recall: Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

Syntax: `FUNCTION-EXPR ARG-EXPR`

Recall: Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

Syntax: FUNCTION-EXPR ARG-EXPR

Typing: If FUNCTION-EXPR is of type T1 -> T2, and ARG-EXPR is of type T1, then the type is T2

Recall: Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

Syntax: FUNCTION-EXPR ARG-EXPR

Typing: If FUNCTION-EXPR is of type T1 -> T2, and ARG-EXPR is of type T1, then the type is T2

Semantics: Substitute the value of ARG-EXPR into the body of FUNCTION-EXPR and evaluate that

Inference Rules

Inference Rules

$$\frac{\overset{\text{premise}}{P_1} \qquad \overset{\text{premise}}{P_2} \qquad \cdots \qquad \overset{\text{premise}}{P_k}}{\underset{\text{conclusion}}{C}} \; \overset{\text{name}}{\texttt{my-rule}}$$

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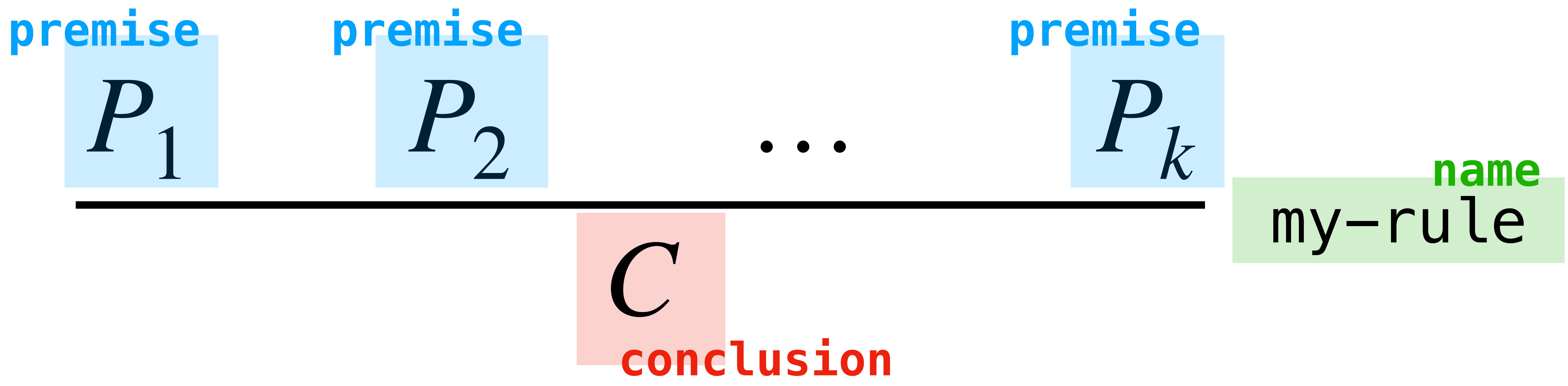
$$\frac{\overset{\texttt{premise}}{P_1} \qquad \overset{\texttt{premise}}{P_2} \qquad \cdots \qquad \overset{\texttt{premise}}{P_k}}{\underset{\texttt{conclusion}}{C}}\ \overset{\texttt{name}}{\texttt{my-rule}}$$

Then general form of an inference rule has a collection of **premises** and a **conclusion**

Inference Rules



Then general form of an inference rule has a collection of **premises** and a **conclusion**

There may be no premises, this is called an **axiom**

Inference Rules

$$\dfrac{\overset{\text{premise}}{P_1} \qquad \overset{\text{premise}}{P_2} \qquad \cdots \qquad \overset{\text{premise}}{P_k}}{\underset{\text{conclusion}}{C}} \; \overset{\text{name}}{\texttt{my-rule}}$$

We can read this as:

*If P_1 through P_k hold, then C holds (by **my-rule**)*

Judgements

» Syntax judgments

» Typing judgments

» Semantic judgments

Judgements

- **Syntax judgments**
- Typing judgments
- Semantic judgments

Syntax Judgements

$$e \in \mathsf{WF}$$

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IMPORTANT: We will never use this kind of judgments explicitly!

BNF Grammar

Production Rules

```
<expr> ::= <expr> + <expr>
```

$$\frac{e_1 \in WF \qquad e_2 \in WF}{e_1 + e_2 \in WF}\ (\text{intAdds})$$

Instead it's standard to use **production rules**

(We'll spend more time on this when we cover formal grammar)

Judgements

» Syntax judgments

» **Typing judgments**

» Semantic judgments

Typing Judgments

$$\overset{\text{context}}{\Gamma} \vdash \overset{\text{expression}}{e} : \overset{\text{type}}{\tau}$$

A typing judgment a compact way of representing the statement:

e is of type τ in the context Γ

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

Recall: Integer Addition Typing Rule

$$\frac{\Gamma \vdash e_1 : \texttt{int} \qquad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 \texttt{ + } e_2 : \texttt{int}} \text{ (addInt)}$$

If e_1 is an int (in any context Γ) and e_2 is an int then (in any context Γ) $e_1 + e_2$ is an int (in any context Γ)

Contexts

$\Gamma = \{$ `x : int, y : string, z : int -> string` $\}$

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A variable declaration $(x : \tau)$ says: *"I declare that the variable x is of type τ"*

Contexts

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A **context** is a set of **variable declarations**

A variable declaration $(x : \tau)$ says: *"I declare that the variable x is of type τ"*

A context keeps track of all the types of variables in the *static* environment

Example: Reading Typing Judgements

$$\{\texttt{b} : \texttt{bool}\} \vdash \texttt{if b then 2 else 3} : \texttt{int}$$

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In English: *Given that b is a bool, the expression if b then 2 else 3 is an int*

Example: Reading Typing Judgements

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In English: *Given that b is a bool, the expression if b then 2 else 3 is an int*

The context allows us to determine the type of an expression *relative to the types of variables*

Judgements are claims, not truths

$$\{\mathtt{b} : \mathtt{bool}\} \vdash \mathtt{if\ b\ then\ 2\ else\ 3} : \mathtt{string}$$

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A judgement is a *claim* in the same way that "there are infinitely many twin primes" or "pigs fly" is a claim

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We haven't **proved** anything by writing down a typing judgment

Judgements are claims, not truths

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A judgement is a *claim* in the same way that "there are infinitely many twin primes" or "pigs fly" is a claim

We haven't **proved** anything by writing down a typing judgment

Next time: we'll talk about **typing derivations**, which are used to demonstrate that expressions *actually* have their desired types in our PL

Judgements

- Syntax judgments
- Typing judgments
- **Semantic judgments**

Semantic Judgements

expression value

$$e \Downarrow v$$

A <u>semantic judgment</u> is a compact way of representing the statement:

The expression e evaluates to the value v

A **semantic rule** is an inference rule with semantic judgments

Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 + v_2} \text{ (evalInt)}$$

If e_1 *evaluates to the (integer)* v_1 *and* e_2 *evaluates to the (integer)* v_2*, then* $e_1 + e_2$ *evaluates to the (integer)* $v_1 + v_2$

Example: Reading Semantic Judgments

$$\boxed{\texttt{if 2 > 3 then 2 + 2 else 3} \Downarrow 3}$$

In English: The expression

`if 2 > 3 then 2 + 2 else 3`

evaluates to the value 3

Values are not (Necessarily) Expressions

$$\texttt{if 2 > 3 then 2 + 2 else 3} \Downarrow 3$$

In this course, we will draw a distinction between values and expressions (note the font)

Example. We'll use regular numbers to represented integer values, and we'll use ⊤ and ⊥ for the true and false Boolean values

320Caml Inference Rules

Reminder: for every expression in our language, we given inference rules for syntax, typing, and semantics

Expressions

- » Let-expressions
- » If-Expressions
- » Functions
- » Application

Expressions

- **Let-expressions**
- If-Expressions
- Functions
- Application

Let-Expressions (Syntax Rule)

$$\langle expr \rangle ::= \texttt{let}\ \boxed{\langle var \rangle} = \langle expr \rangle\ \texttt{in}\ \langle expr \rangle$$

alphanumeric, _, '

If x is a valid variable name, and e_1 is a well-formed expression and e_2 is a well-formed expression then

$$\texttt{let}\ x = e_1\ \texttt{in}\ e_2$$

is a well-formed expression

Let-Expressions (Syntax Rule)

```
<expr> ::= let <var> = <expr> in <expr>
```

If x is a valid variable name, and e_1 is a well-formed expression and e_2 is a well-formed expression then

$$\texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2$$

is a well-formed expression

Let-Expressions (Typing Rule)

$$\frac{\Gamma \vdash e_1 : \boxed{\tau_1} \qquad \Gamma, x : \boxed{\tau_1} \vdash e_2 : \boxed{\tau}}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \boxed{\tau}}$$

If e_1 is of type τ_1 in the context Γ, and e_2 is of type τ in the context Γ *with the variable declaration* $(x : \tau_1)$ *added to it*, then

$$\texttt{let } x \texttt{ = } e_1 \texttt{ in } e_2$$

is of type τ in the context Γ

$$\frac{\overbrace{\{\ \}}^{\Gamma} \vdash \overbrace{2}^{e_1} : \overbrace{\text{int}}^{\tau_1} \qquad \overbrace{\{x : \overbrace{\text{int}}^{\tau_1}\}}^{\Gamma,\ x : \text{int}} \vdash \overbrace{x + x}^{e_2} : \overbrace{\text{int}}^{\tau}}{\underbrace{\{\ \}}_{\Gamma} \vdash \text{let } x = \underbrace{2}_{e_1} \text{ in } \underbrace{x + x}_{e_2} : \underbrace{\text{int}}_{\tau}}$$

Let-Expressions (Typing Rule)

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2 : \tau}\ \text{(let)}$$

If e_1 is of type τ_1 in the context Γ, and e_2 is of type τ in the context Γ *with the variable declaration* $(x : \tau_1)$ *added to it*, then

$$\texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2$$

is of type τ in the context Γ

Let-Expressions (Typing Rule)

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2 : \tau}\ (\text{let})$$

Note: Look at how much more compact the rule is!

If e_1 is of type τ_1 in the context Γ, and e_2 is of type τ in the context Γ *with the variable declaration* $(x : \tau_1)$ *added to it*, then

$\texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2$

is of type τ in the context Γ

Let-Expressions (Semantic Rule)

$$\frac{e_1 \Downarrow v_1 \qquad [v_1/x]e_2 \Downarrow v}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 \Downarrow v}$$

If e_1 evaluates to v_1 and e_2 *with* $\cancel{v_2}$ v_1 *substituted for* x evaluates to v, then

$\texttt{let}\ x = e_1\ \texttt{in}\ e_2$

evaluates to v

$$\frac{2 \Downarrow 2 \qquad \overset{[2/x]\overbrace{x+x}^{e_2}}{\overset{\shortparallel}{2+2}} \Downarrow 4}{\text{let } x = \underset{e_1}{2} \text{ in } \underset{e_2}{x+x} \Downarrow 4}$$

Let-Expressions (Semantic Rule)

$$\frac{e_1 \Downarrow v_1 \qquad [v_1/x]e_2 \Downarrow v}{\texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2 \Downarrow v}\ (\text{letEval})$$

If e_1 evaluates to v_1 and e_2 *with* $\cancel{v_2}$ v_1 *substituted for* x evaluates to v, then

$$\texttt{let}\ x\ \texttt{=}\ e_1\ \texttt{in}\ e_2$$

evaluates to v

Expressions

- » Let-expressions
- » **If-Expressions**
- » Functions
- » Application

If-Expressions (Syntax Rule)

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```
<expr> ::= if <expr> then <expr> else <expr>
```

If e_1 is a well-formed expression and e_2 is a well-formed expression and e_3 is a well-formed expression, then

$$\texttt{if } e_1 \texttt{ then } e_2 \texttt{ else } e_3$$

is a well-formed expression

If-Expressions (Typing Rule)

$$\frac{\Gamma \vdash e_1 : bool \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau} \ (if)$$

$$\{b : bool,\ x : int,\ y : int\} \vdash \text{if } b \text{ then } x + 2 \text{ else } y : int$$

If-Expressions (Typing Rule)

$$\frac{\Gamma \vdash e_1 : \texttt{bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 : \tau}\text{(if)}$$

If e_1 is of type bool in the context Γ and e_2 and e_3 are of type τ in the context Γ, then

$$\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3$$

is of type τ in the context Γ

If-Expressions (Semantics)

$$\frac{e_1 \Downarrow \top \qquad e_2 \Downarrow v_2 \qquad \cancel{e_3 \Downarrow v_3}}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \Downarrow v_2}$$

$$\frac{e_1 \Downarrow \bot \qquad e_3 \Downarrow v_3}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \Downarrow v_3}$$

If-Expressions (Semantic Rule 1)

$$\frac{e_1 \Downarrow \top \qquad e_2 \Downarrow v_2}{\texttt{if}\ \ e_1\ \ \texttt{then}\ \ e_2\ \ \texttt{else}\ \ e_3 \Downarrow v_2}\ (\text{ifEvalTrue})$$

If e_1 evaluates to $\top$ and e_2 evaluates to v_2, then

$$\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3$$

evaluates to v_2

If-Expressions (Semantic Rule 2)

$$\frac{e_1 \Downarrow \bot \qquad e_3 \Downarrow v_3}{\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 \Downarrow v_3}\ (\text{ifEvalFalse})$$

If e_1 evaluates to $\bot$ and e_2 evaluates to v_2, then

$$\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3$$

evaluates to v_3

If-Expressions (Semantic Rule 2)

$$\frac{e_1 \Downarrow \bot \qquad e_3 \Downarrow v_3}{\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 \Downarrow v_3}\ (\text{ifEvalFalse})$$

Note: we never evaluate both branches

If e_1 evaluates to $\bot$ and e_2 evaluates to v_2, then

$$\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3$$

evaluates to v_3

Expressions

- » Let-expressions
- » If-Expressions
- » **Functions**
- » Application

Functions (Syntax Rule)

Functions (Syntax Rule)

```
<expr> ::= fun <var> -> <expr>
```

If x is a valid variable name and e is a well-formed expression, then

fun x -> e

is a well-formed expression

Functions (Typing Rule)

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \text{fun } x \rightarrow e : \tau_1 \rightarrow \tau_2}$$

$$\frac{\{x : \text{int}\} \vdash x + 1 : \text{int}}{\{\} \vdash \text{fun } x \rightarrow x + 1 : \text{int} \rightarrow \text{int}}$$

Functions (Typing Rule)

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \texttt{fun}\ x\ \texttt{->}\ e : \tau_1\ \texttt{->}\ \tau_2}\ (\text{fun})$$

If e has type τ_2 in the context Γ with the variable declaration $(x : \tau_1)$ added, then

$$\texttt{fun}\ x\ \texttt{->}\ e$$

is of type τ_1 `->` τ_2 in the context Γ

Functions (Semantic Rule)

Functions (Semantic Rule)

$$\frac{}{\texttt{fun}\ x\ \texttt{->}\ e \Downarrow \lambda x . e}\ (\text{funEval})$$

Under no premises, the expression

$$\texttt{fun}\ x\ \texttt{->}\ e$$

evaluates to the *function value* $\lambda x . e$ (we'll talk more about function values later)

Expressions

» Let-expressions

» If-Expressions

» Functions

» **Application**

Application (Syntax Rule)

Application (Syntax Rule)

```
<expr> ::= <expr> <expr>
```

If e_1 is a well-formed expression and e_2 is a well-formed expression, then $e_1\ e_2$ is a well-formed expression

Application (Typing Rule)

$$\frac{\Gamma \vdash e_1 : \tau_2 \to \tau \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\, e_2 : \tau}$$

If e_1 has type τ_2 -> τ under the context Γ and e_2 is of type τ_2 under the context Γ, then $e_1\, e_2$ is of type τ under the context Γ

Application (Typing Rule)

$$\frac{\Gamma \vdash e_1 : \tau_2 \textbf{ -> } \tau \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau}\ \text{(app)}$$

If e_1 has type τ_2 -> τ under the context Γ and e_2 is of type τ_2 under the context Γ, then $e_1\ e_2$ is of type τ under the context Γ

Application (Semantic Rule)

$$\frac{e_1 \Downarrow \lambda\, x\,.\, e \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e \Downarrow v}{e_1\ e_2 \Downarrow v}\,(\text{appEval})$$

1. e_1 evaluates to a function value $\lambda x\,.\,e$
2. e_2 evaluates to v_2
3. e with v_2 substituted for x evaluates to v

It follows that $e_1\ e_2$ evaluates to v

Example (Informal)

```
(let x = 2 in fun y -> x + y) (2 + 3)
```

We'll see more typing rules and semantic rules

We'll also give a written reference for the rules we talk about in class

Summary

Inference rules formally describe how the typing and semantics of a programming language work