Derivations

Concepts of Programming Languages
Lecture 6

CAS CS 320

Practice Problem

Suppose introduced an xor operator into OCaml. Write down (to the best of your ability) the syntax, typing, and semantic rules for xor

Syntax:

$$\langle expr \rangle ::= \langle expr \rangle \ \mathtt{xor} \ \langle expr \rangle$$

Typing:

$$\frac{\Gamma \vdash e_1 : \mathtt{bool} \qquad \Gamma \vdash e_2 : \mathtt{bool}}{\Gamma \vdash e_1 \ \mathtt{xor} \ e_2 : \mathtt{bool}} \ (\mathrm{xor})$$

Semantics:

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{e_1 \ \mathtt{xor} \ e_2 \Downarrow v_1 \otimes v_2} \ (\mathrm{xorE})$$

Outline

» Discuss derivations in general

» See how to read and write derivations

» Go through a couple examples

Recap

Recall: Inference Rules

$$\frac{\overset{\text{premise}}{P_1} \qquad \overset{\text{premise}}{P_2} \qquad \ldots \qquad \overset{\text{premise}}{P_k}}{\underset{\text{conclusion}}{C}} \; \overset{\text{name}}{\texttt{my-rule}}$$

Recall: Inference Rules

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The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are <u>**judgments**</u>

Recall: Inference Rules

$$\frac{\overbrace{P_1}^{\texttt{premise}} \qquad \overbrace{P_2}^{\texttt{premise}} \qquad \cdots \qquad \overbrace{P_k}^{\texttt{premise}}}{\underbrace{C}_{\texttt{conclusion}}}\ \overbrace{\texttt{my-rule}}^{\texttt{name}}$$

The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

There may be no premises, this is called an **axiom**

Recall: Inference Rules

$$\frac{\overset{\text{premise}}{P_1} \qquad \overset{\text{premise}}{P_2} \qquad \cdots \qquad \overset{\text{premise}}{P_k}}{\underset{\text{conclusion}}{C}} \; \overset{\text{name}}{\texttt{my-rule}}$$

<u>We can read this as:</u>

*If the judgments P_1 through P_k hold, then the judgment C holds (by **my-rule**)*

Recall: Typing Judgments

$$\underbrace{\Gamma}_{\text{context}} \vdash \underbrace{e}_{\text{expression}} : \underbrace{\tau}_{\text{type}}$$

A typing judgment a compact way of representing the statement:

e is of type τ in the context Γ

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

Recall: Contexts

$$\Gamma = \{\ \texttt{x : int, y : string, z : int -> string}\ \}$$

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A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ"

Recall: Contexts

$\Gamma = \{$ `x : int, y : string, z : int -> string` $\}$

A **context** is a set of **variable declarations**

A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ"

A context keeps track of all the types of variables in the "environment"

Derivations

High Level

$$
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High Level

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`Derivations` *`prove`* `that a judgment holds w.r.t some rules`

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

» each node is labeled with a judgment

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

» each node is labeled with a judgment

» and judgment *follows* from the judgments at it's children by an inference rule

Applying Rules

$$\frac{}{\Gamma \vdash \texttt{[]} : \tau\ \texttt{list}}\ \text{(nil)} \qquad \frac{\Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau\ \texttt{list}}{\Gamma \vdash e_1\ \texttt{::}\ e_2 : \tau\ \texttt{list}}\ \text{(cons)}$$

$$\frac{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int} \qquad \{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\ \text{(cons)}$$

Applying Rules

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So far, we've used rules as ways of describing the behavior of a PL

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So far, we've used rules as ways of describing the behavior of a PL

When we build typing derivations, we *instantiate* the meta-variables in the rule at *particular* expressions, contexts, etc.

Building from the Ground Up

$$\dfrac{\dfrac{}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{x} : \mathtt{int}}\,(\mathsf{var}) \qquad \dfrac{}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{1} : \mathtt{int}}\,(\mathsf{intLit})}{\dfrac{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{x + 1} : \mathtt{int}}{}}\,(\mathsf{intAdd})$$

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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

Building from the Ground Up

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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

This is how we get our tree structure: we apply rules from the ground up

Axioms (When are we done?)

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The leaves of the tree are **axioms**, i.e., a rules with no premises

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The leaves of the tree are **axioms**, i.e., a rules with no premises

In our case, this will almost always be "literal" or "variable" rules

Integer Literals

side condition

(1)

$$\frac{\texttt{n is an int lit}}{\Gamma \vdash \mathsf{n} : \texttt{int}}\ \text{(intLit)}$$

(2)

$$\frac{\texttt{n is an int lit}}{\mathsf{n} \Downarrow n}\ \text{(intLitEval)}$$

1. If **n** is an integer literal, then it is of type int in any context

2. If **n** is an integer literal, then it evaluates to the number it represents

A Note about Side Conditions

we don't write "1 is an integer literal"

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}{\{\texttt{x} : \texttt{int}\} \vdash (\texttt{x} + \texttt{1}) :: \texttt{[]} : \texttt{int list}}\ (\text{cons})
$$

If a premise is a side-condition this *it is not included in the derivation*

Side conditions need to hold in order to apply the rule, but they don't appear in the derivation itself

We will always make side conditions clear

Float Literals

(1)
$$\frac{\texttt{n is an float lit}}{\Gamma \vdash \texttt{n} : \texttt{float}}\ (\text{floatLit})$$

(2)
$$\frac{\texttt{n is an float lit}}{\texttt{n} \Downarrow n}\ (\text{floatLitEval})$$

1. If n is an float literal, then it is of type float in any context

2. If n is an float literal, then it evaluates to the number it represents

Boolean Literals

(1)
$$\frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}}\ (\text{trueLit})$$

(2)
$$\frac{}{\Gamma \vdash \mathtt{false} : \mathtt{bool}}\ (\text{falseLit})$$

(3)
$$\frac{}{\mathtt{true} \Downarrow \top}\ (\text{trueLitEval})$$

(4)
$$\frac{}{\mathtt{false} \Downarrow \bot}\ (\text{falseLitEval})$$

1. `true` is of type `bool` in any context
2. `false` if of type `bool` in any context
3. `true` evaluates to the value $\top$
4. `false` evaluates to the value $\bot$

Variables

$$\frac{\color{green}{(v : \tau) \in \Gamma}}{\Gamma \vdash v : \tau} \text{(intLit)}$$

If v is declared to be of type τ in the context Γ, then v is of type τ in Γ

Variables cannot be evaluated (more on this when we talk about substitution and well-scopedness)

Back to the Example

$$\dfrac{\dfrac{}{\{\} \vdash \mathtt{2} : \mathtt{int}}\;(\text{intLit}) \qquad \dfrac{\dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\;(\text{var}) \qquad \dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\;(\text{var})}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} + \mathtt{y} : \mathtt{int}}\;(\text{intAdd})}{\{\} \vdash \mathtt{let\ y\ =\ 2\ in}\ \mathtt{y} + \mathtt{y} : \mathtt{int}}\;(\text{let})$$

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We need $\{\} \vdash 2 : \mathtt{int}$ in order to proof that the bottom typing judgment holds

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We need $\{\} \vdash 2 : \mathtt{int}$ in order to proof that the bottom typing judgment holds

Now we know that this follows from the **intLit** rule, which says that 2 is always an int, *by fiat*

Okay, I know that was a lot, let's take a step back

Derivations Encode Natural Language Arguments

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*(In fact, most mathematical arguments can be represented formally as derivation trees, this is the called **proof theory**)*

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The expression **`let y = 2 in y + y`** is an **`int`** *because*

Derivations Encode Natural Language Arguments

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» **`2`** is an **`int`** by fiat (and so **`y`** is being assigned to a well-typed expression)

Derivations Encode Natural Language Arguments

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$$\dfrac{\Gamma \vdash e_1 : \texttt{int} \qquad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}}\,(\text{addInt})$$

» and, assuming **y** is an int, **y + y** is an **int** *because*

Derivations Encode Natural Language Arguments

$$\dfrac{\dfrac{}{\{\} \vdash \mathtt{2} : \mathtt{int}}\,(\text{intLit}) \qquad \dfrac{\dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\text{var}) \qquad \dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\text{var})}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} + \mathtt{y} : \mathtt{int}}\,(\text{intAdd})}{\{\} \vdash \mathtt{let\ y\ =\ 2\ in\ y} + \mathtt{y} : \mathtt{int}}\,(\text{let})$$

$$\dfrac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{let}\ x\ \mathtt{=}\ e_1\ \mathtt{in}\ e_2 : \tau_2}\,(\text{let})$$

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Derivations Encode Natural Language Arguments

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Derivations Encode Natural Language Arguments

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Derivations Encode Natural Language Arguments

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Derivations Encode Natural Language Arguments

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and so integer-adding these two expressions (`y` and `y`) yields an `int`

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Derivations Encode Natural Language Arguments

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  \qquad
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Derivations Encode Natural Language Arguments

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And all this works for semantics judgements as well

Recall: Semantic Judgements

$$\overset{\text{expression}}{e} \Downarrow \overset{\text{value}}{v}$$

A **semantic judgment** is a compact way of representing the statement:

The expression e *evaluates to the value* v

A **semantic rule** is an inference rule with semantic judgments

Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v}\ (\text{evalInt})$$

If e_1 *evaluates to the (integer)* v_1 *and* e_2 *evaluates to the (integer)* v_2*, and* $v_1 + v_2 = v$*, then* $e_1 + e_2$ *evaluates to the (integer)* v

Semantic Derivations

$$\dfrac{\dfrac{}{\texttt{true} \Downarrow \top}\,(\mathsf{trueEval}) \qquad \dfrac{}{\texttt{2} \Downarrow 2}\,(\mathsf{intEval})}{\texttt{if true then 2 else 3} \Downarrow 2}\,(\mathsf{ifEval})$$

We can also write derivations to prove semantic judgments

The principle is the same, except that the judgments are semantic judgments instead of typing judgments

Examples

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2}\ (\text{let})$$

$\Gamma, x : \tau \equiv \Gamma \cup \{x : \tau\}$

$\equiv$ add $x : \tau$ to Γ

$$\frac{\Gamma \vdash e_1 : \text{int} \qquad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}}\ (\text{addInt})$$

$$\frac{\dfrac{}{\{\} \vdash 2 : \text{int}}\ (\text{intLit}) \qquad \dfrac{\dfrac{}{\{y : \text{int}\} \vdash y : \text{int}}\ (\text{var}) \qquad \dfrac{}{\{y : \text{int}\} \vdash y : \text{int}}\ (\text{var})}{\{y : \text{int}\} \vdash y + y : \text{int}}\ (\text{addInt})}{\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}}\ (\text{let})$$

Example (Typing)

$$\frac{\boxed{\Gamma} \vdash \boxed{e_1} : bool \qquad \boxed{\Gamma} \vdash \boxed{e_2} : \boxed{\tau} \qquad \boxed{\Gamma} \vdash e_3 : \boxed{\tau}}{\boxed{\Gamma} \vdash \text{if } \boxed{e_1} \text{ then } \boxed{e_2} \text{ else } \boxed{e_3} : \boxed{\tau}} \;(\text{if})$$

$$\frac{\dfrac{}{\{\} \vdash true : bool}\;(\text{trueLit}) \qquad \dfrac{}{\{\} \vdash 2 : int}\;(\text{intLit}) \qquad \dfrac{}{\{\} \vdash 5 : int}\;(\text{intLit})}{\boxed{\{\}} \vdash \texttt{if } \boxed{\texttt{true}} \texttt{ then } \boxed{\texttt{2}} \texttt{ else } \boxed{\texttt{5}} : \boxed{\texttt{int}}}$$

Example (Evaluation)

$$\dfrac{\dfrac{}{\text{true} \Downarrow \top}\,(\text{trueLitE}) \qquad \dfrac{}{2 \Downarrow 2}\,(\text{intLitE})}{\texttt{if true then 2 else 5} \Downarrow 2}\,(\text{ifTrue})$$

Example (Typing)

$\{\} \vdash$ `2 + 3 <> 4 : bool`

Example (Evaluation)

$$\texttt{2 + 3 <> 4} \Downarrow \texttt{true}$$

Example (Evaluation)

side cond.

$$\dfrac{e_1 \Downarrow v_1 \qquad [v_1/x]e_2 = e' \qquad e' \Downarrow v}{\text{let } x = e_1 \text{ in } e_2 \Downarrow v}\;(\text{letE})$$

$$[2/x](x + x)$$

$$\parallel$$

$$2 + 2$$

$$2 + 2 = 4$$

$$\dfrac{\dfrac{}{2 \Downarrow 2}\,(\text{intLitE}) \qquad \dfrac{\dfrac{}{2 \Downarrow 2}\,(\text{intLitE}) \qquad \dfrac{}{2 \Downarrow 2}\,(\text{intLitE})}{2+2 \Downarrow 4}\,(\text{addIntE})}{\texttt{let x = 2 in x + x} \Downarrow 4}\,(\text{letE})$$

Summary

Derivations are **tree-like proofs** that judgments hold with respect to a collection of inference rules

Derivations are **compact mathematical representations** of English language arguments

Learning to write derivations takes *practice*