

Derivations

Concepts of Programming Languages Lecture 6

Practice Problem

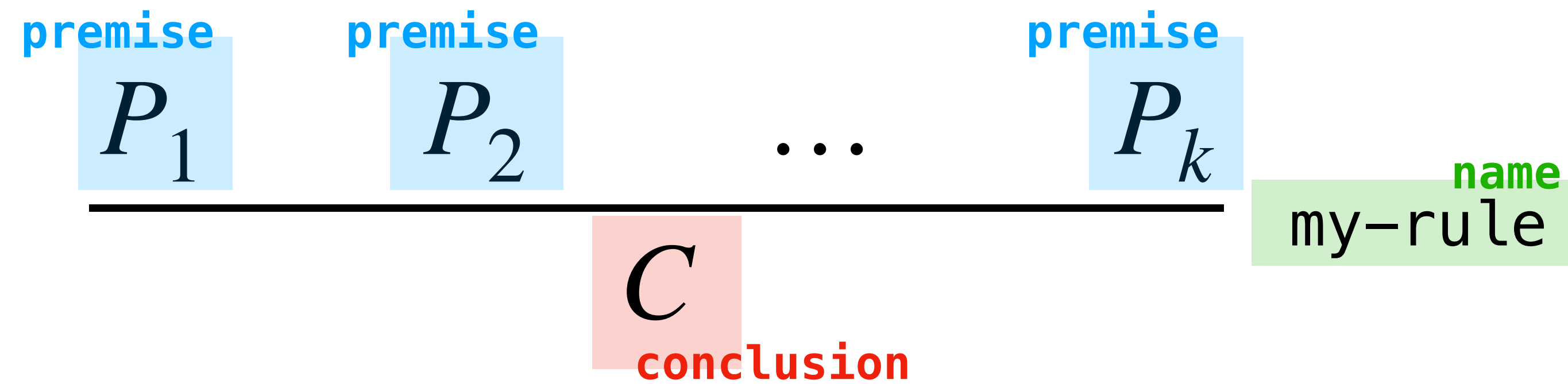
*Suppose introduced an **xor** operator into OCaml.
Write down (to the best of your ability) the
syntax, typing, and semantic rules for **xor***

Outline

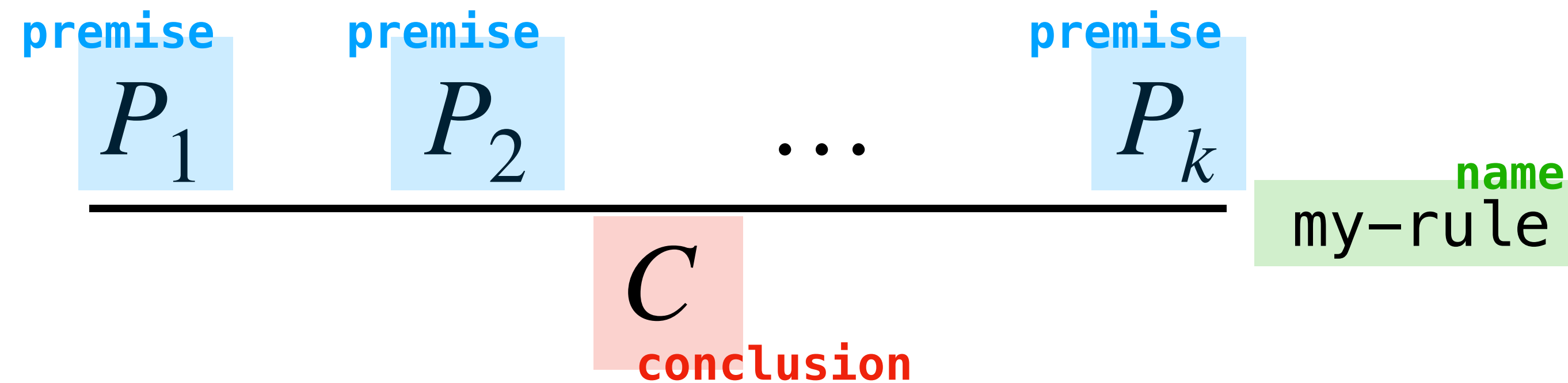
- » Discuss derivations in general
- » See how to read and write derivations
- » Go through a couple examples

Recap

Recall: Inference Rules

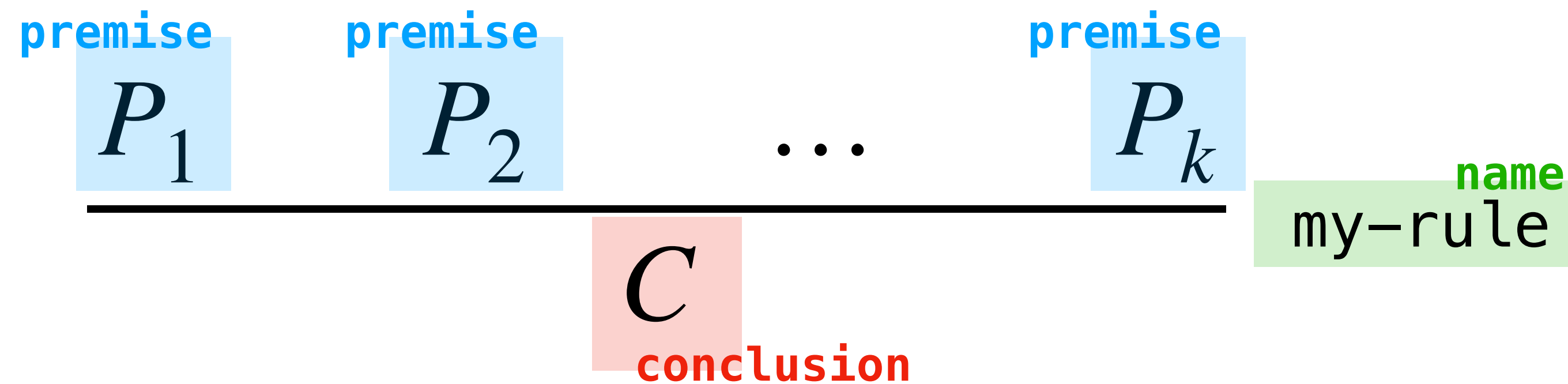


Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

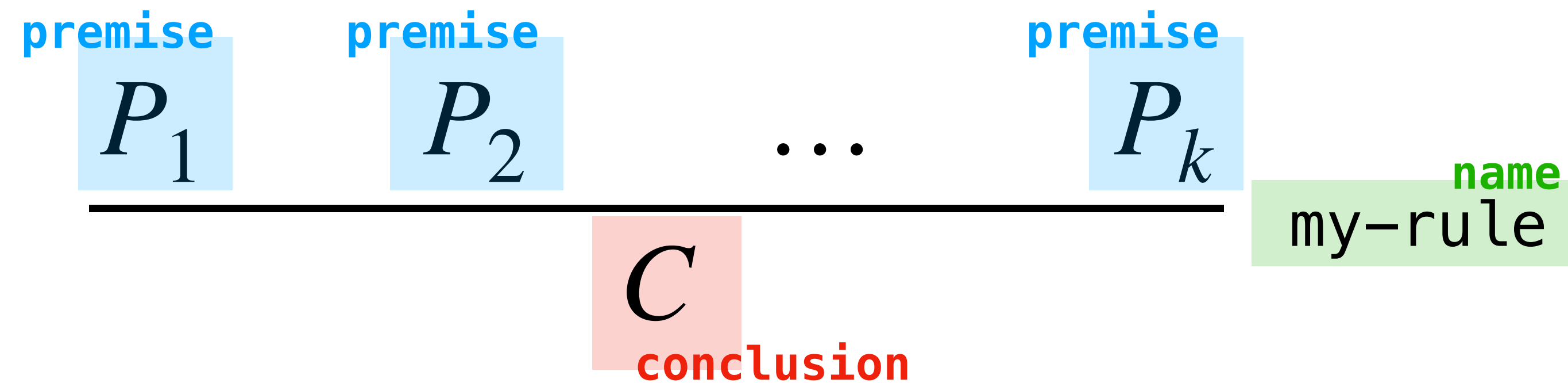
Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

There may be no premises, this is called an **axiom**

Recall: Inference Rules



We can read this as:

*If the judgments P_1 through P_k hold, then the judgment C holds (by **my-rule**)*

Recall: Typing Judgments

$$\text{context } \Gamma \vdash \text{expression } e : \text{type } \tau$$

A typing judgment a compact way of representing the statement:

e is of type τ in the context Γ

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

Recall: Contexts

$$\Gamma = \{ x : \text{int}, y : \text{string}, z : \text{int} \rightarrow \text{string} \}$$

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A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ "

Recall: Contexts

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A **context** is a set of **variable declarations**

A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ "

A context keeps track of all the types of variables in the "environment"

Derivations

High Level

$$\frac{}{\{\} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{}{\{y : \text{int}\} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{y : \text{int}\} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{y : \text{int}\} \vdash y + y : \text{int}} (\text{intAdd})$$
$$\frac{}{\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} (\text{let})$$

High Level

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Derivations *prove* that a judgment holds w.r.t some rules

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A **derivation** is a tree in which:

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

- » each node is labeled with a judgment
- » and judgment *follows* from the judgments at it's children by an inference rule

Applying Rules

$$\frac{}{\Gamma \vdash [] : \tau \text{ list}} \text{ (nil)} \quad \frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau \text{ list}}{\Gamma \vdash e_1 :: e_2 : \tau \text{ list}} \text{ (cons)}$$

| | | |
|---|--|--------|
| $\{x : \text{int}\} \vdash x + 1 : \text{int}$ | $\{x : \text{int}\} \vdash [] : \text{int list}$ | (cons) |
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So far, we've used rules as ways of describing the behavior of a PL

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So far, we've used rules as ways of describing the behavior of a PL

When we build typing derivations, we *instantiate* the meta-variables in the rule at *particular* expressions, contexts, etc.

Building from the Ground Up

| | | |
|---|--|---|
| $\frac{}{\{x : \text{int}\} \vdash x : \text{int}} \text{ (var)}$ | $\frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} \text{ (intLit)}$ | |
| | $\frac{}{\{x : \text{int}\} \vdash x + 1 : \text{int}} \text{ (intAdd)}$ | $\frac{}{\{x : \text{int}\} \vdash [] : \text{int list}} \text{ (nil)}$ |
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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

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This is how we get our tree structure: we apply rules from the ground up

Axioms (When are we done?)

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The leaves of the tree are **axioms**, i.e., a rules with no premises

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The leaves of the tree are **axioms**, i.e., a rules with no premises

In our case, this will almost always be "literal" or "variable" rules

Integer Literals

(1)

$$\frac{n \text{ is an int lit}}{\Gamma \vdash n : \text{int}} \quad (\text{intLit})$$

(2)

$$\frac{n \text{ is an int lit}}{n \Downarrow n} \quad (\text{intLitEval})$$

1. If n is an integer literal, then it is of type int in any context
2. If n is an integer literal, then it evaluates to the number it represents

A Note about Side Conditions

we don't write "1 is an integer literal"

| | | |
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If a premise is a side-condition this *it is not included in the derivation*

Side conditions need to hold in order to apply the rule, but they don't appear in the derivation itself

We will always make side conditions clear

Float Literals

$$\begin{array}{c} (1) \\ \frac{n \text{ is an float lit}}{\Gamma \vdash n : \text{float}} \quad (\text{floatLit}) \end{array} \quad \begin{array}{c} (2) \\ \frac{n \text{ is an float lit}}{n \Downarrow n} \quad (\text{floatLitEval}) \end{array}$$

1. If n is an float literal, then it is of type float in any context
2. If n is an float literal, then it evaluates to the number it represents

Boolean Literals

$$(1) \quad \frac{}{\Gamma \vdash \text{true} : \text{bool}} \text{ (trueLit)}$$

$$(2) \quad \frac{}{\Gamma \vdash \text{false} : \text{bool}} \text{ (falseLit)}$$

$$(3) \quad \frac{}{\text{true} \Downarrow \top} \text{ (trueLitEval)}$$

$$(4) \quad \frac{}{\text{false} \Downarrow \perp} \text{ (falseLitEval)}$$

1. `true` is of type `bool` in any context
2. `false` is of type `bool` in any context
3. `true` evaluates to the value \top
4. `false` evaluates to the value \perp

Variables

$$\frac{(v : \tau) \in \Gamma}{\Gamma \vdash v : \tau} \text{ (intLit)}$$

If v is declared to be of type τ in the context Γ , then v is of type τ in Γ

Variables cannot be evaluated (more on this when we talk about substitution and well-scopedness)

Back to the Example

$$\frac{\frac{}{\{\} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{\frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)} \quad \frac{\frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)}}{\{y : \text{int}\} \vdash y + y : \text{int}} \text{(intAdd)}}{\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} \text{(let)}$$

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We need $\{\} \vdash 2 : \text{int}$ in order to proof that the bottom typing judgment holds

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Now we know that this follows from the **intLit** rule, which says that 2 is always an int, *by fiat*

Okay, I know that was a
lot, let's take a step back

Derivations Encode Natural Language Arguments

$$\frac{\frac{}{\{\} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{\frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)} \quad \frac{\frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)}}{\{y : \text{int}\} \vdash y + y : \text{int}} \text{(intAdd)}}{\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} \text{(let)}$$

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A derivation is just a mathy way of writing a natural language proof that a typing derivation holds

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*(In fact, most mathematical arguments can be represented formally as derivation trees, this is the called **proof theory**)*

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \text{(let)}$$

The expression **let y = 2 in y + y** is an **int** *because*

Derivations Encode Natural Language Arguments

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» `2` is an `int` by fiat (and so `y` is being assigned to a well-typed expression)

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» and, assuming `y` is an `int`, `y + y` is an `int` because

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Derivations Encode Natural Language Arguments

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» and, assuming **y** is an int, **y + y** is an **int** *because*

- **y** is an **int** (by assumption)

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 \hline
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Derivations Encode Natural Language Arguments

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and so integer-adding these two expressions (`y` and `y`) yields an `int`

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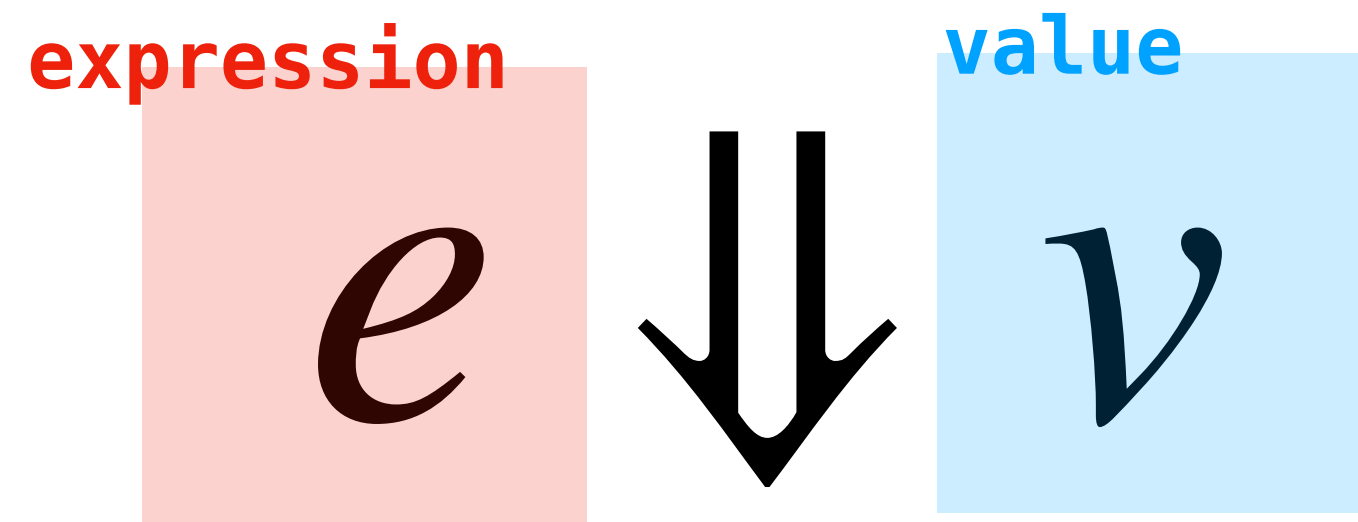
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And all this works for
semantics judgements as well

Recall: Semantic Judgements



A **semantic judgment** is a compact way of representing the statement:

The expression e evaluates to the value v

A **semantic rule** is an inference rule with semantic judgments

Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v} \text{ (evalInt)}$$

If e_1 evaluates to the (integer) v_1 and e_2 evaluates to the (integer) v_2 , and $v_1 + v_2 = v$, then $e_1 + e_2$ evaluates to the (integer) v

Semantic Derivations

$$\frac{\frac{}{\text{true} \Downarrow \top} \text{(trueEval)} \quad \frac{}{2 \Downarrow 2} \text{(intEval)}}{\text{if true then 2 else 3} \Downarrow 2} \text{(ifEval)}$$

We can also write derivations to prove semantic judgments

The principle is the same, except that the judgments are semantic judgments instead of typing judgments

Examples

Example (Typing)

$\{\}$ \vdash **if true then 2 else 5 : int**

Example (Evaluation)

if true then 2 else 5 ↓ 2

Example (Typing)

$\{\}$ \vdash **2 + 3 <> 4 : bool**

Example (Evaluation)

2 + 3 <> 4 ⇓ true

Example (Evaluation)

let $x = 2$ in $x + x \Downarrow 4$

Summary

Derivations are **tree-like proofs** that judgments hold with respect to a collection of inference rules

Derivations are **compact mathematical representations** of English language arguments

Learning to write derivations takes *practice*