

Derivations

Concepts of Programming Languages

Lecture 6

Practice Problem

*Suppose introduced an **xor** operator into oCaml.
Write down (to the best of your ability) the
syntax, typing, and semantic rules for **xor***

Syntax :

$\langle \text{expr} \rangle ::= \langle \text{expr} \rangle \text{ xor } \langle \text{expr} \rangle$

Type

$\Gamma \vdash e_1 : \text{bool}$

$\Gamma \vdash e_2 : \text{bool}$

(xor)

$\Gamma \vdash e_1 \text{ xor } e_2 : \text{bool}$

Semantics :

$e_1 \Downarrow v_1$

$e_2 \Downarrow v_2$

(xor E)

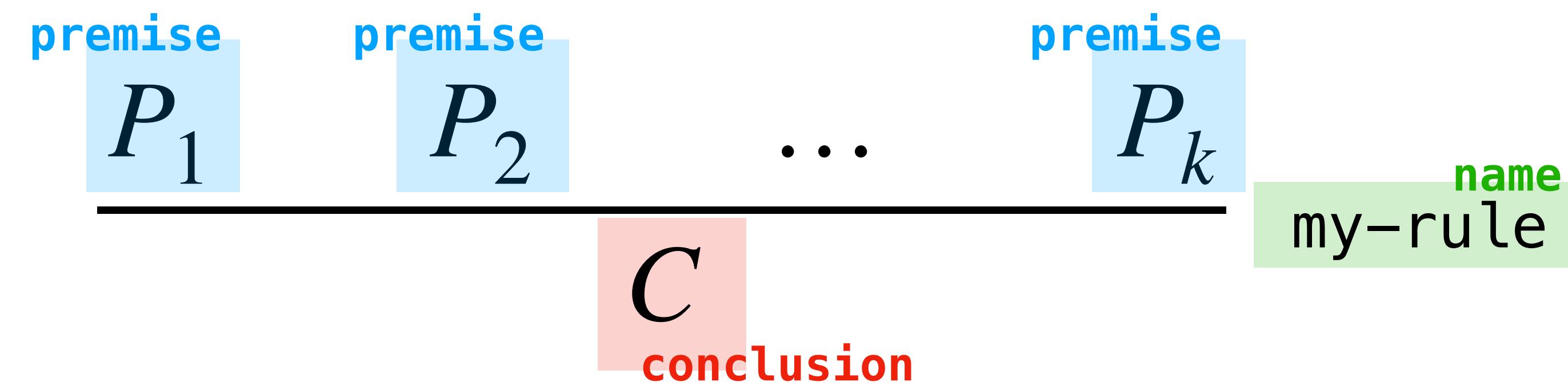
$e_1 \text{ xor } e_2 \Downarrow v_1 \otimes v_2$

Outline

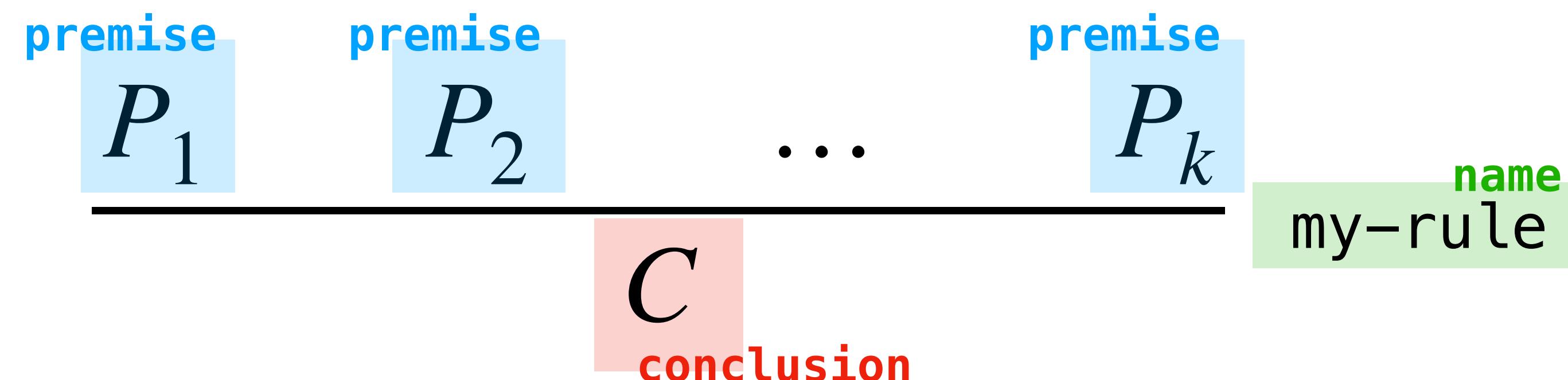
- » Discuss derivations in general
- » See how to read and write derivations
- » Go through a couple examples

Recap

Recall: Inference Rules

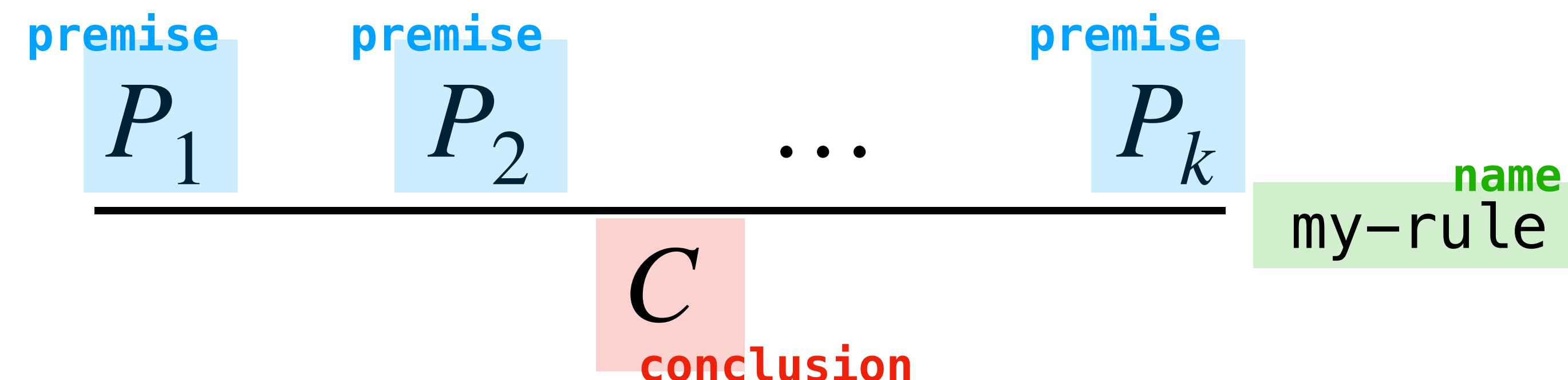


Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

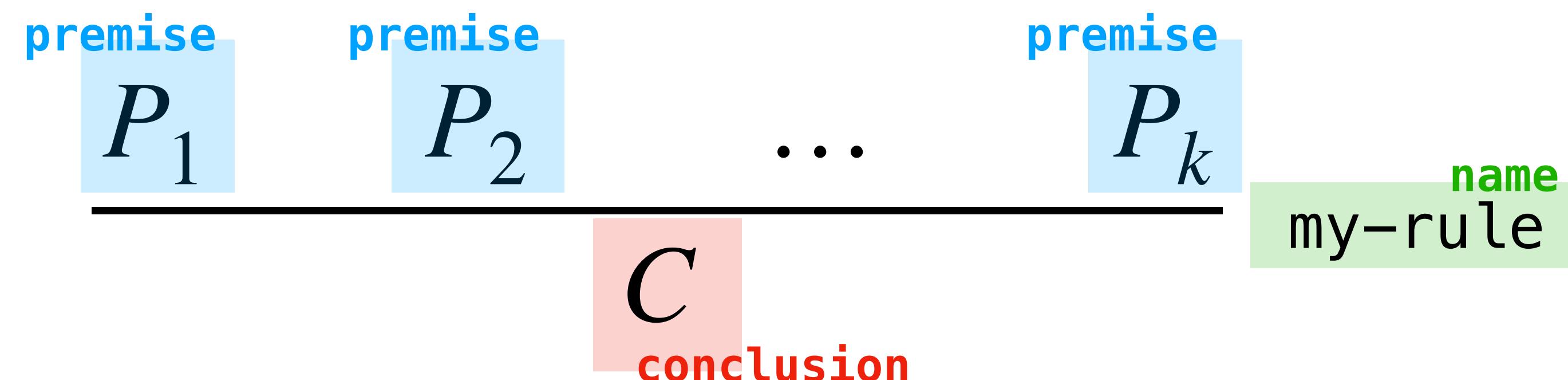
Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

There may be no premises, this is called an **axiom**

Recall: Inference Rules



We can read this as:

If the judgments P_1 through P_k hold, then the judgment C holds (by my-rule)

Recall: Typing Judgments

$$\begin{array}{c} \text{context} \\ \Gamma \\ \hline \text{expression} \\ e \\ \hline \text{type} \\ \tau \end{array}$$

A typing judgment a compact way of representing the statement:

e is of type τ in the context Γ

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

Recall: Contexts

$$\Gamma = \{ \text{x : int}, \text{y : string}, \text{z : int} \rightarrow \text{string} \}$$

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Recall: Contexts

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A **context** is a set of **variable declarations**

A variable declaration ($x : \tau$) says: "I declare that the variable x is of type τ "

Recall: Contexts

$$\Gamma = \{ \text{x : int}, \text{y : string}, \text{z : int} \rightarrow \text{string} \}$$

A **context** is a set of **variable declarations**

A variable declaration ($x : \tau$) says: "I declare that the variable x is of type τ "

A context keeps track of all the types of variables in the "environment"

Derivations

High Level

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{\{y : \text{int}\} \vdash y : \text{int}}{\{y : \text{int}\} \vdash y + y : \text{int}} \begin{array}{l} (\text{var}) \\ (\text{let}) \end{array} \quad \frac{\{y : \text{int}\} \vdash y : \text{int}}{\{y : \text{int}\} \vdash y + y : \text{int}} \begin{array}{l} (\text{var}) \\ (\text{intAdd}) \end{array}$$
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High Level

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Derivations *prove* that a judgment holds w.r.t some rules

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A **derivation** is a tree in which:

High Level

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A **derivation** is a tree in which:

- » each node is labeled with a judgment

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

- » each node is labeled with a judgment
- » and judgment *follows* from the judgments at it's children by an inference rule

Applying Rules

$$\frac{}{\Gamma \vdash [] : \tau \text{ list}} \text{ (nil)}$$

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau \text{ list}}{\Gamma \vdash e_1 :: e_2 : \tau \text{ list}} \text{ (cons)}$$

$$\boxed{\{x : \text{int}\} \vdash x + 1 : \text{int}}$$

$$\boxed{\{x : \text{int}\} \vdash [] : \text{int list} \text{ (cons)}}$$

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$$\{x : \text{int}\} \vdash [] : \text{int list} \text{ (cons)}$$

So far, we've used rules as ways of describing the behavior of a PL

Applying Rules

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So far, we've used rules as ways of describing the behavior of a PL

When we build typing derivations, we *instantiate* the meta-variables in the rule at *particular* expressions, contexts, etc.

Building from the Ground Up

$$\frac{}{\{x : \text{int}\} \vdash x : \text{int}} (\text{var}) \quad \frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} (\text{intLit}) \quad \frac{}{\{x : \text{int}\} \vdash [] : \text{int list}} (\text{nil})$$
$$\frac{\{x : \text{int}\} \vdash x : \text{int} \quad \{x : \text{int}\} \vdash 1 : \text{int}}{\{x : \text{int}\} \vdash x + 1 : \text{int}} (\text{intAdd}) \quad \frac{\{x : \text{int}\} \vdash [] : \text{int list}}{\{x : \text{int}\} \vdash [x] : \text{int list}} (\text{cons})$$
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Building from the Ground Up

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But we can't just apply rules, because it's possible that the premises of a rule also need to be demonstrated

Building from the Ground Up

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But we can't just apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

This is how we get our tree structure: we apply rules from the ground up

Axioms (When are we done?)

$$\frac{}{\{x : \text{int}\} \vdash x : \text{int}} \text{(var)} \quad \frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} \text{(intLit)} \quad \frac{}{\{x : \text{int}\} \vdash [] : \text{int list}} \text{(nil)}$$
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The leaves of the tree are **axioms**, i.e., a rules with no premises

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The leaves of the tree are **axioms**, i.e., a rules with no premises

In our case, this will almost always be "literal" or "variable" rules

Integer Literals

(1)

$$\frac{n \text{ is an int lit}}{\Gamma \vdash n : \text{int}} \text{ (intLit)}$$

(2)

$$\frac{n \text{ is an int lit}}{n \Downarrow n} \text{ (intLitEval)}$$

side condition

1. If n is an integer literal, then it is of type **int** in any context
2. If n is an integer literal, then it evaluates to the number it represents

A Note about Side Conditions

we don't write "1 is an integer literal"		
$\frac{}{\{x : \text{int}\} \vdash x : \text{int}} \text{(var)}$	$\frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} \text{(intLit)}$	
	$\frac{\{x : \text{int}\} \vdash 1 : \text{int}}{\{x : \text{int}\} \vdash x + 1 : \text{int}} \text{(intAdd)}$	
		$\frac{}{\{x : \text{int}\} \vdash [] : \text{int list}} \text{(nil)}$
$\frac{\{x : \text{int}\} \vdash x + 1 : \text{int} \quad \{x : \text{int}\} \vdash [] : \text{int list}}{\{x : \text{int}\} \vdash (x + 1) :: [] : \text{int list}} \text{(cons)}$		

If a premise is a side-condition this *it is not included in the derivation*

Side conditions need to hold in order to apply the rule, but they don't appear in the derivation itself

We will always make side conditions clear

Float Literals

$$(1) \frac{n \text{ is an float lit}}{\Gamma \vdash n : \mathbf{float}} \text{ (floatLit)} \quad (2) \frac{n \text{ is an float lit}}{n \Downarrow n} \text{ (floatLitEval)}$$

1. If n is an float literal, then it is of type float in any context
2. If n is an float literal, then it evaluates to the number it represents

Boolean Literals

$$(1) \frac{}{\Gamma \vdash \text{true} : \text{bool}} (\text{trueLit})$$

$$(3) \frac{}{\text{true} \Downarrow T} (\text{trueLitEval})$$

$$(2) \frac{}{\Gamma \vdash \text{false} : \text{bool}} (\text{falseLit})$$

$$(4) \frac{}{\text{false} \Downarrow \perp} (\text{falseLitEval})$$

1. **true** is of type **bool** in any context
2. **false** is of type **bool** in any context
3. **true** evaluates to the value **T**
4. **false** evaluates to the value **⊥**

Variables

$$\frac{(v : \tau) \in \Gamma}{\Gamma \vdash v : \tau} \text{ (intLit)}$$

If v is declared to be of type τ in the context Γ , then v is of type τ in Γ

Variables cannot be evaluated (more on this when we talk about substitution and well-scopedness)

Back to the Example

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{ y : \text{int} \} \vdash y + y : \text{int}} (\text{var})$$
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We need $\{ \} \vdash 2 : \text{int}$ in order to proof that the bottom typing judgment holds

Back to the Example

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Now we know that this follows from the **intLit** rule, which says that 2 is always an int, *by fiat*

Okay, I know that was a lot, let's take a step back

Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{ y : \text{int} \} \vdash y + y : \text{int}} (\text{var})$$
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A derivation is just a mathy way of writing a natural language proof that a typing derivation holds

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*(In fact, most mathematical arguments can be represented formally as derivation trees, this is the called **proof theory**)*

Derivations Encode Natural Language Arguments

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} (\text{let})$$

The expression `let y = 2 in y + y` is an `int` because

Derivations Encode Natural Language Arguments

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» `2` is an `int` by fiat (and so `y` is being assigned to a well-typed expression)

Derivations Encode Natural Language Arguments

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The expression `let y = 2 in y + y` is an `int` because

» `2` is an `int` by fiat (and so `y` is being assigned to a well-typed

» and, assuming `y` is an `int`, `y + y` is an `int` because

$$\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} (\text{addInt})$$

Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} (\text{var})$$
$$\frac{\{ y : \text{int} \} \vdash y + y : \text{int} \quad \{ y : \text{int} \} \vdash y : \text{int}}{\{ y : \text{int} \} \vdash y + y : \text{int}} (\text{intAdd})$$
$$\frac{}{\{ \} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} (\text{let})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} (\text{let})$$

$$\frac{n \text{ is an integer literal}}{\Gamma \vdash n : \text{int}} (\text{intLit})$$

The expression `let y = 2 in y + y` is an `int` because

» `2` is an `int` by fiat (and so `y` is being assigned to a well-typed

» and, assuming `y` is an `int`, `y + y` is an `int` because

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Derivations Encode Natural Language Arguments

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$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} (\text{var})$$

Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} \text{(var)} \quad \frac{\{ y : \text{int} \} \vdash y : \text{int}}{\{ y : \text{int} \} \vdash y + y : \text{int}} \text{(var)} \quad \frac{\{ y : \text{int} \} \vdash y + y : \text{int}}{\{ \} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} \text{(intAdd)}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \text{(let)}$$

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Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} \text{(var)} \quad \frac{\{ y : \text{int} \} \vdash y : \text{int}}{\{ y : \text{int} \} \vdash y + y : \text{int}} \text{(intAdd)}$$

$\{ \} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}$ (let)

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \text{(let)}$$

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Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)} \quad \frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)}$$

$$\frac{\{y : \text{int}\} \vdash y : \text{int} \quad \{y : \text{int}\} \vdash y : \text{int}}{\{y : \text{int}\} \vdash y + y : \text{int}} \text{(intAdd)} \quad \frac{}{\{y : \text{int}\} \vdash y + y : \text{int}} \text{(let)}$$

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \text{(let)}$$

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$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} \text{(var)}$$

and so integer-adding these two expressions (`y` and `y`) yields an `int`

Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} \text{(var)} \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} \text{(var)}$$

$$\frac{\{ y : \text{int} \} \vdash y : \text{int} \quad \{ y : \text{int} \} \vdash y + y : \text{int} \quad \{ y : \text{int} \} \vdash y + y : \text{int}}{\{ \} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} \text{(let)}$$

$\{ \} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \text{(let)}$$

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and so integer-adding these two expressions (`y` and `y`) yields an `int`

and so assigning `y` to `2` in `y + y` yields an `int`

Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} \text{(var)} \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} \text{(var)}$$

$$\frac{\{ y : \text{int} \} \vdash y : \text{int} \quad \{ y : \text{int} \} \vdash y + y : \text{int} \quad \{ y : \text{int} \} \vdash y + y : \text{int}}{\{ \} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} \text{(let)} \quad \frac{}{\{ y : \text{int} \} \vdash y + y : \text{int}} \text{(intAdd)}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \text{(let)}$$

$$\frac{n \text{ is an integer literal}}{\Gamma \vdash n : \text{int}} \text{(intLit)}$$

The expression `let y = 2 in y + y` is an `int` because

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$$\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} \text{(addInt)}$$

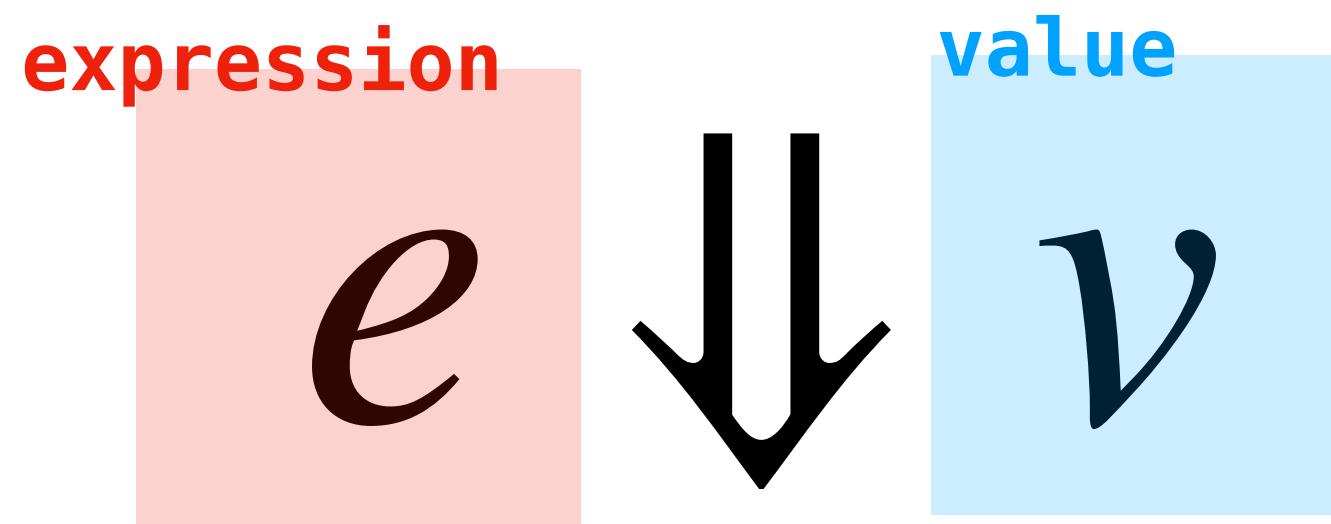
$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} \text{(var)}$$

and so integer-adding these two expressions (`y` and `y`) yields an `int`

and so assigning `y` to `2` in `y + y` yields an `int`

And all this works for
semantics judgements as well

Recall: Semantic Judgements



A **semantic judgment** is a compact way of representing the statement:

The expression e evaluates to the value v

A **semantic rule** is an inference rule with semantic judgments

Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v} \text{ (evalInt)}$$

If e_1 evaluates to the (integer) v_1 and e_2 evaluates to the (integer) v_2 , and $v_1 + v_2 = v$, then $e_1 + e_2$ evaluates to the (integer) v

Semantic Derivations

$$\frac{\text{true} \Downarrow \top}{\text{if true then } 2 \text{ else } 3 \Downarrow 2} \text{ (ifEval)} \quad \frac{}{2 \Downarrow 2} \text{ (intEval)}$$
$$\frac{}{\text{true} \Downarrow \top} \text{ (trueEval)}$$

We can also write derivations to prove semantic judgments

The principle is the same, except that the judgments are semantic judgments instead of typing judgments

Examples

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \boxed{\Gamma, x : \tau_1 \vdash e_2 : \tau_2} \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \quad (\text{let})$$

$$\begin{aligned}\Gamma, x : \tau &\equiv \Gamma \cup \{x : \tau\} \\ &\equiv \text{add } x : \tau \\ &\quad \text{to } \Gamma\end{aligned}$$

$$\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} \quad (\text{addInt})$$

$$\frac{\frac{\frac{\frac{\{\} \vdash z : \text{int}}{(\text{intLit})} \quad \frac{\frac{\{\} \vdash y : \text{int}}{\{\} \vdash y + y : \text{int}} \quad \frac{\{\} \vdash y + y : \text{int}}{\{\} \vdash \text{let } y = z \text{ in } y + y : \text{int}} \quad (\text{let})}{(\text{var})}}{(\text{var})}}{(\text{addInt})}}$$

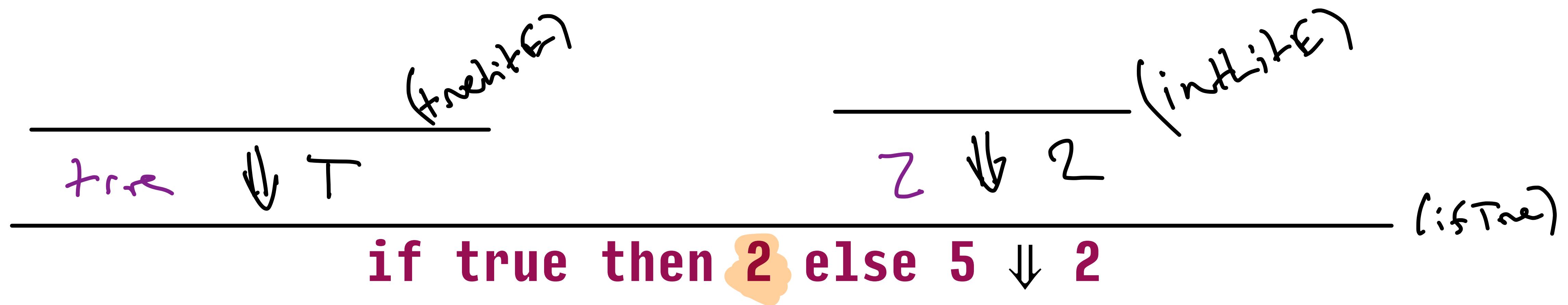
Example (Typing)

$$\frac{\boxed{\Gamma \vdash e_1 : \text{bool}} \quad \boxed{\Gamma \vdash e_2 : \tau} \quad \boxed{\Gamma \vdash e_3 : \tau}}{\boxed{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau}} \quad (\text{; } \times)$$

$$\frac{}{\{ \} \vdash \text{true} : \text{bool}} \quad (\text{trueLit}) \quad \frac{}{\{ \} \vdash 2 : \text{int}} \quad (\text{intLit}) \quad \frac{}{\{ \} \vdash 5 : \text{int}} \quad (\text{intLit})$$

$\boxed{\{ \} \vdash \text{if } \boxed{\text{true}} \text{ then } \boxed{2} \text{ else } \boxed{5} : \text{int}}$

Example (Evaluation)



Example (Typing)

{ } ⊢ 2 + 3 <> 4 : bool

Example (Evaluation)

2 + 3 < 4 ↓ true

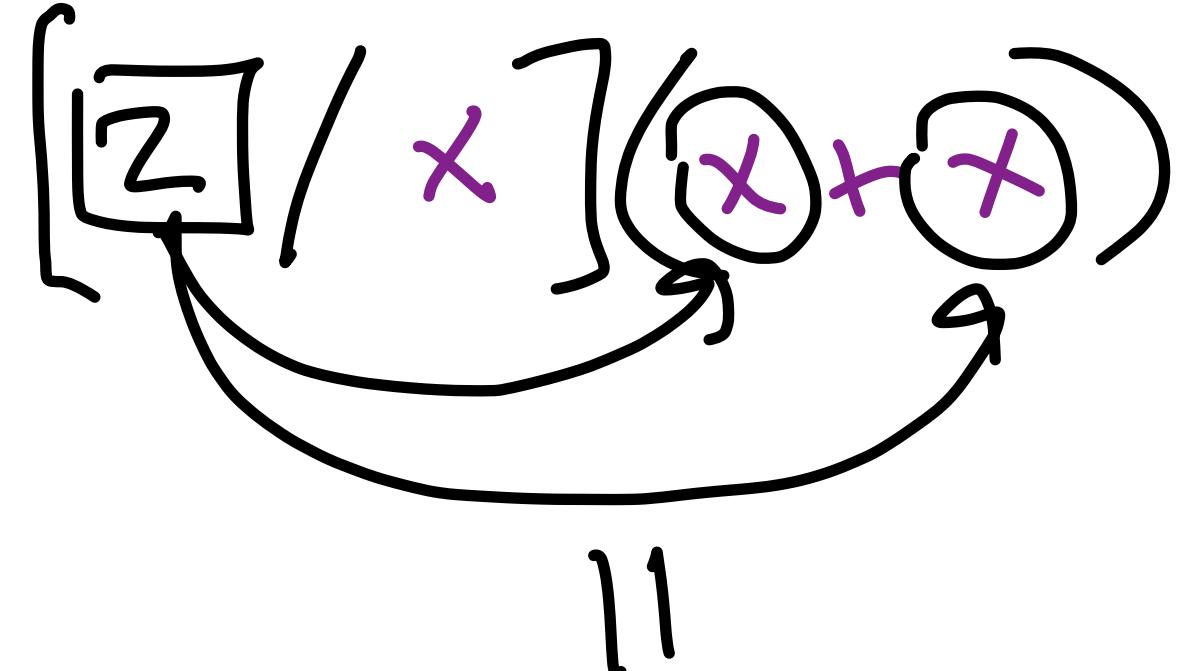
Example (Evaluation)

$$[e_1] \Downarrow v_1$$

side cond.

$$[v_1, [x] e_2 = e']$$

$$[e'] \Downarrow v$$



(letE)

$$\text{let } [x = e_1] \text{ in } e_2 \Downarrow v$$

$$\begin{aligned} & 2 + 2 \\ & 2 + 2 = 4 \end{aligned}$$

$$\frac{}{2 \Downarrow 2} \text{(initE)}$$

$$\frac{\frac{}{2 \Downarrow 2} \text{(initE)}}{2 + 2 \Downarrow 4} \text{(ctxE)}$$

$$\frac{\frac{}{2 \Downarrow 2} \text{(initE)}}{2 + 2 \Downarrow 4} \text{(addE)}$$

$$\text{let } [x = 2] \text{ in } [x + x] \Downarrow 4$$

Summary

Derivations are **tree-like proofs** that judgments hold with respect to a collection of inference rules

Derivations are **compact mathematical representations** of English language arguments

Learning to write derivations takes *practice*