Derivations

Concepts of Programming Languages
Lecture 6

CAS CS 320

Practice Problem

Suppose introduced an xor operator into OCaml. Write down (to the best of your ability) the syntax, typing, and semantic rules for xor

Syntax:

$$\langle expr \rangle ::= \langle expr \rangle \ \textbf{xor} \ \langle expr \rangle$$

Typing:

$$\frac{\Gamma \vdash e_1 : \textbf{bool} \qquad \Gamma \vdash e_2 : \textbf{bool}}{\Gamma \vdash e_1 \ \textbf{xor} \ e_2 : \textbf{bool}} \ \text{xor}$$

Semantic:

$$\neg \bot \equiv \top$$
$$\neg \top \equiv \bot$$

$$\frac{e_1 \Downarrow \top \qquad e_2 \Downarrow v}{e_1 \ \textbf{xor} \ e_2 \Downarrow \neg v} \ \text{xorE}_1 \qquad \frac{e_1 \Downarrow \bot \qquad e_2 \Downarrow v}{e_1 \ \textbf{xor} \ e_2 \Downarrow v} \ \text{xorE}_1$$

Outline

» Discuss derivations in general

» See how to read and write derivations

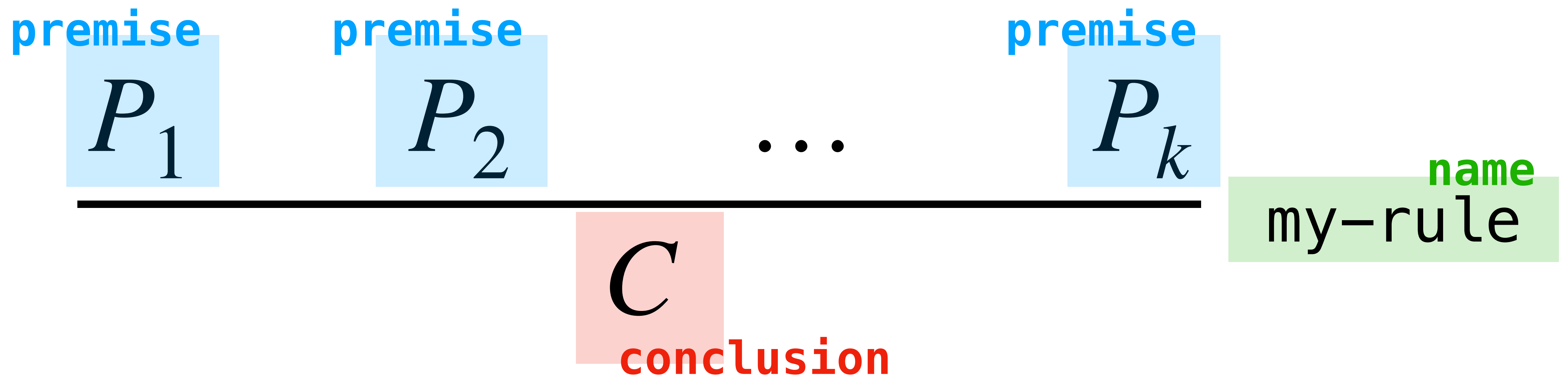
» Go through a couple examples

Recap

Recall: Inference Rules

$$\dfrac{\overset{\text{premise}}{P_1} \qquad \overset{\text{premise}}{P_2} \qquad \cdots \qquad \overset{\text{premise}}{P_k}}{\underset{\text{conclusion}}{C}}\ \overset{\text{name}}{\texttt{my-rule}}$$

Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

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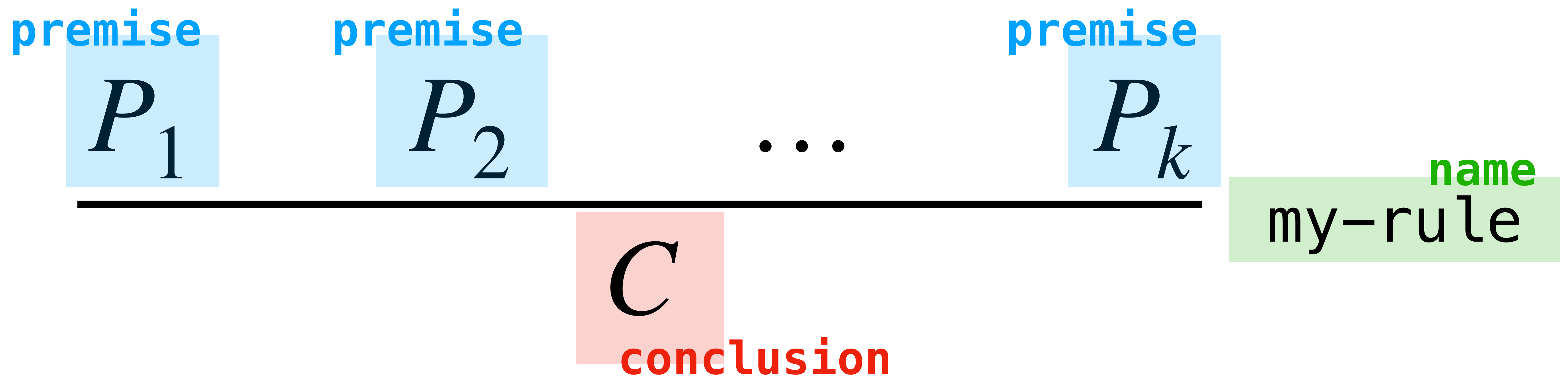
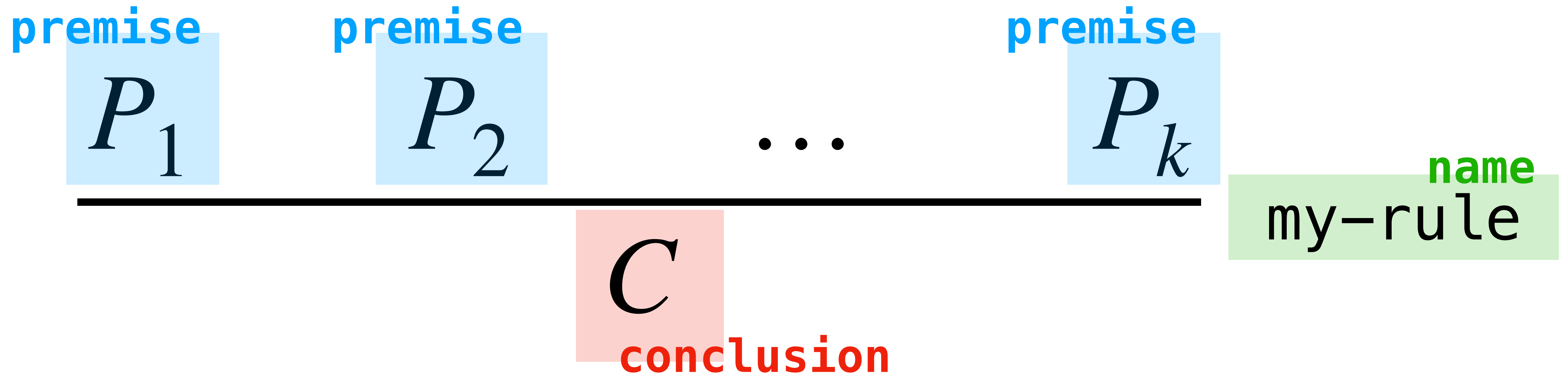


The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

There may be no premises, this is called an **axiom**

Recall: Inference Rules



We can read this as:

*If the judgments P_1 through P_k hold, then the judgment C holds (by **my-rule**)*

Recall: Typing Judgments

$$\overset{\text{context}}{\Gamma} \vdash \overset{\text{expression}}{e} : \overset{\text{type}}{\tau}$$

A typing judgment a compact way of representing the statement:

e is of type τ in the context Γ

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

Recall: Contexts

$\Gamma = \{$ `x : int, y : string, z : int -> string` $\}$

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$\Gamma = \{$ `x : int, y : string, z : int -> string` $\}$

A **context** is a set of **variable declarations**

A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ"

A context keeps track of all the types of variables in the "environment"

Derivations

High Level

$$\dfrac{\dfrac{}{\{\} \vdash \mathtt{2} : \mathtt{int}}\,(\text{intLit}) \qquad \dfrac{\dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\text{var}) \qquad \dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\text{var})}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} + \mathtt{y} : \mathtt{int}}\,(\text{intAdd})}{\{\} \vdash \mathtt{let\ y\ =\ 2\ in\ y} + \mathtt{y} : \mathtt{int}}\,(\text{let})$$

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Derivations *prove* that a judgment holds w.r.t some rules

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A **derivation** is a tree in which:

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

» each node is labeled with a judgment

» and judgment *follows* from the judgments at it's children by an inference rule

Applying Rules

$$\frac{}{\Gamma \vdash \texttt{[]} : \tau\ \texttt{list}}\ \text{(nil)} \qquad \frac{\Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau\ \texttt{list}}{\Gamma \vdash e_1\ \texttt{::}\ e_2 : \tau\ \texttt{list}}\ \text{(cons)}$$

$$\frac{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int} \qquad \{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\ \text{(cons)}$$

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So far, we've used rules as ways of describing the behavior of a PL

Applying Rules

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So far, we've used rules as ways of describing the behavior of a PL

When we build typing derivations, we *instantiate* the meta-variables in the rule at *particular* expressions, contexts, etc.

Building from the Ground Up

$$\dfrac{\dfrac{\dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x} : \texttt{int}}\;(\text{var}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{1} : \texttt{int}}\;(\text{intLit})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int}}\;(\text{intAdd}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}\;(\text{nil})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\;(\text{cons})$$

Building from the Ground Up

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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

This is how we get our tree structure: we apply rules from the ground up

Axioms (When are we done?)

$$\dfrac{\dfrac{\dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x} : \texttt{int}}\,(\text{var}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{1} : \texttt{int}}\,(\text{intLit})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int}}\,(\text{intAdd}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}\,(\text{nil})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\,(\text{cons})$$

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The leaves of the tree are **axioms**, i.e., a rules with no premises

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The leaves of the tree are **axioms**, i.e., a rules with no premises

In our case, this will almost always be "literal" or "variable" rules

Integer Literals

(1)

$$\frac{\texttt{n is an int lit}}{\Gamma \vdash \mathsf{n} : \texttt{int}}\ (\text{intLit})$$

(2)

$$\frac{\texttt{n is an int lit}}{\mathsf{n} \Downarrow n}\ (\text{intLitEval})$$

1. If **n** is an integer literal, then it is of type int in any context

2. If **n** is an integer literal, then it evaluates to the number it represents

A Note about Side Conditions

we don't write "1 is an integer literal"

$$\dfrac{\dfrac{\dfrac{}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{x} : \mathtt{int}}\,(\text{var}) \qquad \dfrac{}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{1} : \mathtt{int}}\,(\text{intLit})}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{x + 1} : \mathtt{int}}\,(\text{intAdd}) \qquad \dfrac{}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{[]} : \mathtt{int\ list}}\,(\text{nil})}{\{\mathtt{x} : \mathtt{int}\} \vdash \mathtt{(x + 1)\ ::\ []} : \mathtt{int\ list}}\,(\text{cons})$$

If a premise is a side-condition this *it is not included in the derivation*

Side conditions need to hold in order to apply the rule, but they don't appear in the derivation itself

We will always make side conditions clear

Float Literals

(1)

$$\frac{\texttt{n is an float lit}}{\Gamma \vdash \texttt{n} : \texttt{float}}\ (\text{floatLit})$$

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$$\frac{\texttt{n is an float lit}}{\texttt{n} \Downarrow n}\ (\text{floatLitEval})$$

1. If n is an float literal, then it is of type float in any context

2. If n is an float literal, then it evaluates to the number it represents

Boolean Literals

(1)
$$\frac{}{\Gamma \vdash \texttt{true} : \texttt{bool}}\ (\text{trueLit})$$

(2)
$$\frac{}{\Gamma \vdash \texttt{false} : \texttt{bool}}\ (\text{falseLit})$$

(3)
$$\frac{}{\texttt{true} \Downarrow \top}\ (\text{trueLitEval})$$

(4)
$$\frac{}{\texttt{false} \Downarrow \bot}\ (\text{falseLitEval})$$

1. `true` is of type `bool` in any context
2. `false` if of type `bool` in any context
3. `true` evaluates to the value $\top$
4. `false` evaluates to the value $\bot$

Variables

$$\frac{(v : \tau) \in \Gamma}{\Gamma \vdash v : \tau}\ \text{(intLit)}$$

If v is declared to be of type τ in the context Γ, then v is of type τ in Γ

Variables cannot be evaluated (more on this when we talk about substitution and well-scopedness)

Back to the Example

$$\dfrac{\dfrac{}{\{\} \vdash \texttt{2} : \texttt{int}}\ (\textsf{intLit}) \qquad \dfrac{\dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\ (\textsf{var}) \qquad \dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\ (\textsf{var})}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} + \texttt{y} : \texttt{int}}\ (\textsf{intAdd})}{\{\} \vdash \texttt{let y = 2 in } \texttt{y} + \texttt{y} : \texttt{int}}\ (\textsf{let})$$

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We need $\{\}\vdash 2:\texttt{int}$ in order to proof that the bottom typing judgment holds

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Now we know that this follows from the **intLit** rule, which says that 2 is always an int, *by fiat*

Okay, I know that was a lot, let's take a step back

Derivations Encode Natural Language Arguments

$$\dfrac{\dfrac{}{\{\} \vdash \texttt{2} : \texttt{int}}\;(\text{intLit}) \qquad \dfrac{\dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\;(\text{var}) \qquad \dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\;(\text{var})}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} + \texttt{y} : \texttt{int}}\;(\text{intAdd})}{\{\} \vdash \texttt{let y = 2 in y} + \texttt{y} : \texttt{int}}\;(\text{let})$$

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*(In fact, most mathematical arguments can be represented formally as derivation trees, this is the called **proof theory**)*

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`The expression` **`let y = 2 in y + y`** `is an` **`int`** *`because`*

Derivations Encode Natural Language Arguments

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The expression **let y = 2 in y + y** is an **int** *because*

» **2** is an **int** by fiat (and so **y** is being assigned to a well-typed expression)

Derivations Encode Natural Language Arguments

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The expression **let y = 2 in y + y** is an **int** *because*

» **2** is an **int** by fiat (and so **y** is being assigned to a well-typed

» and, assuming **y** is an int, **y + y** is an **int** *because*

$$\dfrac{\Gamma \vdash e_1 : \texttt{int} \qquad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}}\,(\text{addInt})$$

Derivations Encode Natural Language Arguments

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$$\dfrac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{let } x \texttt{ = } e_1 \texttt{ in } e_2 : \tau_2}\,(\text{let})$$

$$\dfrac{\texttt{n} \text{ is an integer literal}}{\Gamma \vdash \texttt{n} : \texttt{int}}\,(\text{intLit})$$

The expression **let y = 2 in y + y** is an **int** *because*

» **2** is an **int** by fiat (and so **y** is being assigned to a well-typed

» and, assuming **y** is an int, **y + y** is an **int** *because*

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And all this works for semantics judgements as well

Recall: Semantic Judgements

$$\overset{\text{expression}}{e} \Downarrow \overset{\text{value}}{v}$$

A **semantic judgment** is a compact way of representing the statement:

The expression e evaluates to the value v

A **semantic rule** is an inference rule with semantic judgments

Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v}\ \text{(evalInt)}$$

If e_1 *evaluates to the (integer)* v_1 *and* e_2 *evaluates to the (integer)* v_2*, and* $v_1 + v_2 = v$*, then* $e_1 + e_2$ *evaluates to the (integer)* v

Semantic Derivations

$$\dfrac{\dfrac{}{\texttt{true} \Downarrow \top}\,(\mathsf{trueEval}) \qquad\qquad \dfrac{}{\texttt{2} \Downarrow 2}\,(\mathsf{intEval})}{\texttt{if true then 2 else 3} \Downarrow 2}\,(\mathsf{ifEval})$$

We can also write derivations to prove semantic judgments

The principle is the same, except that the judgments are semantic judgments instead of typing judgments

Examples

Example (Typing)

$\{\} \vdash$ `if true then 2 else 5 : int`

$$\frac{\boxed{\Gamma} \vdash \boxed{e_1} : \boxed{\tau_1} \qquad \boxed{\boxed{\Gamma}, x : \tau_1} \vdash \boxed{e_2} : \boxed{\tau_2}}{\boxed{\Gamma} \vdash \text{let } \boxed{x} = \boxed{e_1} \text{ in } \boxed{e_2} : \boxed{\tau_2}} \text{ (let)}$$

$$\Gamma, x : \tau = \Gamma \cup \{x : \tau\}$$

$$\{z_1 : \tau_1, \ldots, z_k : \tau_k\}, x : \tau$$

$$\shortparallel$$

$$\{z_1 : \tau_1, \ldots, z_k : \tau_k, x : \tau\}$$

$$\frac{\Gamma \vdash e_1 : \text{int} \qquad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} \text{ (addInt)}$$

$$\frac{\dfrac{}{\{\} \vdash 2 : \boxed{\text{int}}} \text{ (intLit)} \qquad \dfrac{\dfrac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{ (var)} \qquad \dfrac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{ (var)}}{\{y : \text{int}\} \vdash y + y : \text{int}} \text{ (addInt)}}{\boxed{\{\}} \vdash \text{let } \boxed{y} = \boxed{2} \text{ in } \boxed{y + y} : \boxed{\text{int}}} \text{ (let)}$$

Example (Evaluation)

$\texttt{if true then 2 else 5} \Downarrow \texttt{2}$

Example (Typing)

$$\frac{\boxed{\Gamma} \vdash e_1 : \tau \qquad \boxed{\Gamma} \vdash e_2 : \tau}{\boxed{\Gamma} \vdash \boxed{e_1} <> \boxed{e_2} : \text{bool}}\ (\text{neq})$$

$$\frac{\dfrac{\dfrac{}{\{\} \vdash 2 : \text{int}}\ (\text{intLit}) \qquad \dfrac{}{\{\} \vdash 3 : \text{int}}\ (\text{intLit})}{\{\} \vdash 2 + 3 : \text{int}}\ (\text{addInt}) \qquad \dfrac{}{\{\} \vdash 4 : \text{int}}\ (\text{intLit})}{\boxed{\{\}} \vdash \boxed{2\ +\ 3} <> \boxed{4} : \text{bool}}\ (\text{neq})$$

Example (Evaluation)

`2 + 3 <> 4` $\Downarrow$ `true`

Example (Evaluation)

$$\frac{\boxed{e_1} \Downarrow v_1 \qquad [\boxed{v_1}/\boxed{x}]\boxed{e_2} = \boxed{e'} \qquad e' \Downarrow v}{\text{let } \boxed{x} = \boxed{e_1} \text{ in } \boxed{e_2} \Downarrow v}\ (\text{letE})$$

$$[\boxed{2}/x](\boxed{x}+\boxed{x}) = \boxed{2+2}$$

$$2+2=4$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v}\ (\text{addIntE})$$

$$\dfrac{\dfrac{}{2 \Downarrow \boxed{2}}\ (\text{intLitE}) \qquad \dfrac{\dfrac{}{2 \Downarrow 2}\ (\text{intLitE}) \qquad \dfrac{}{2 \Downarrow 2}\ (\text{intLitE})}{2 + 2 \Downarrow 4}\ (\text{intAddE})}{\texttt{let}\ \boxed{\texttt{x}} = \boxed{\texttt{2}}\ \texttt{in}\ \boxed{\texttt{x + x}} \Downarrow \texttt{4}}\ (\text{letE})$$

Summary

Derivations are **tree-like proofs** that judgments hold with respect to a collection of inference rules

Derivations are **compact mathematical representations** of English language arguments

Learning to write derivations takes *practice*