

# **Derivations**

## **Concepts of Programming Languages**

### **Lecture 6**

# Practice Problem

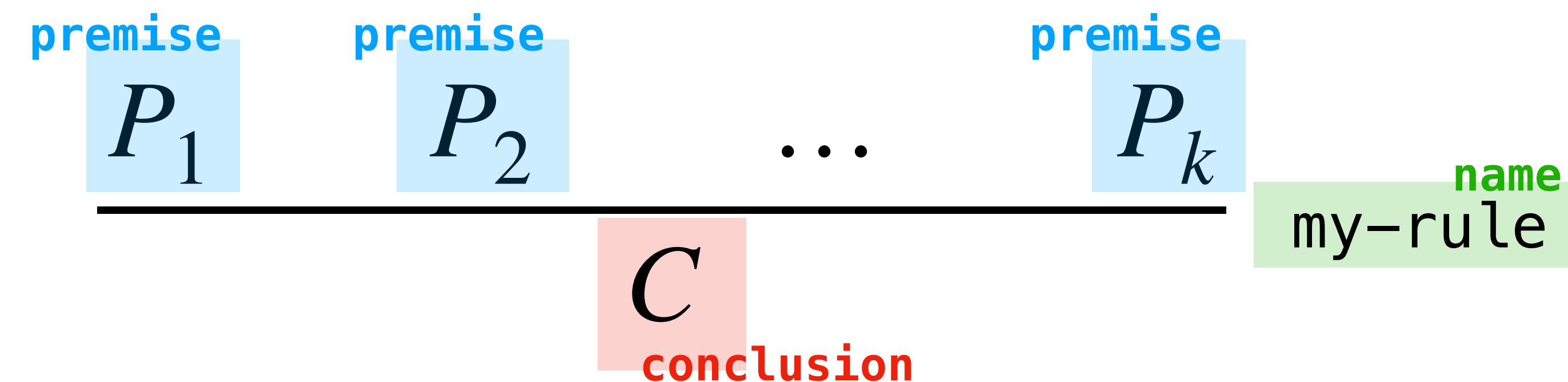
*Suppose introduced an **xor** operator into oCaml.  
Write down (to the best of your ability) the  
syntax, typing, and semantic rules for **xor***

# Outline

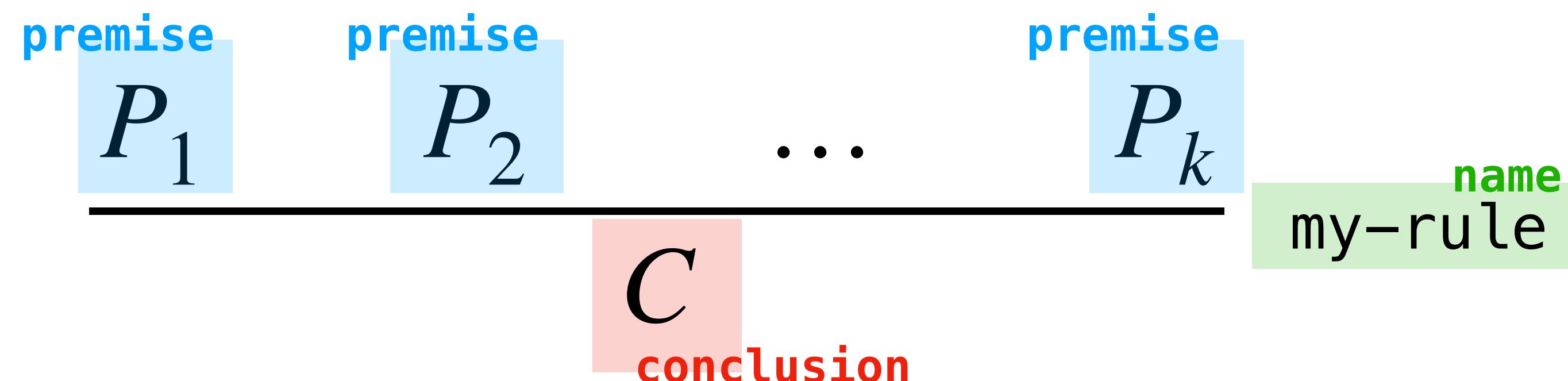
- » Discuss derivations in general
- » See how to read and write derivations
- » Go through a couple examples

# **Recap**

# Recall: Inference Rules

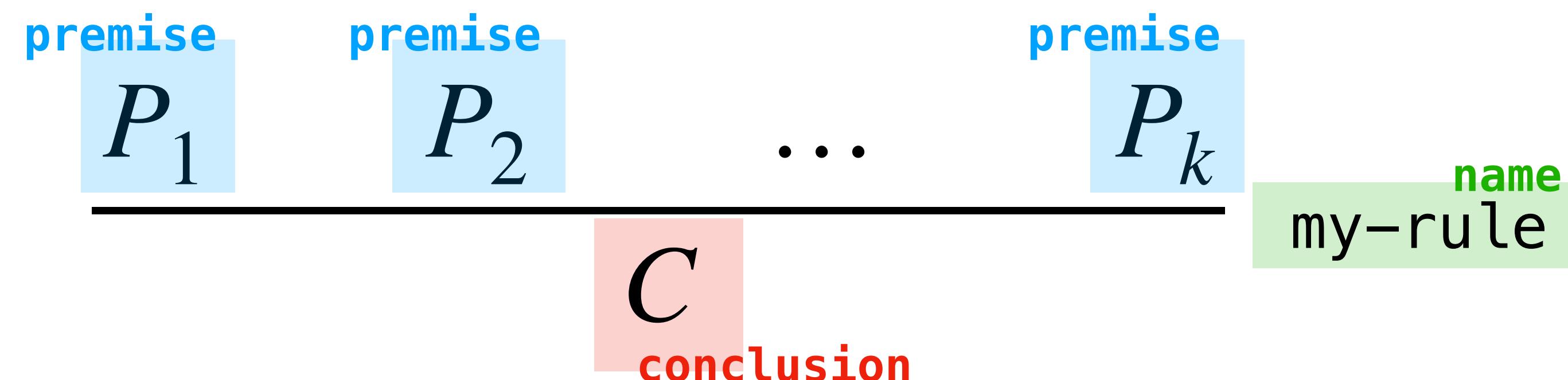


# Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

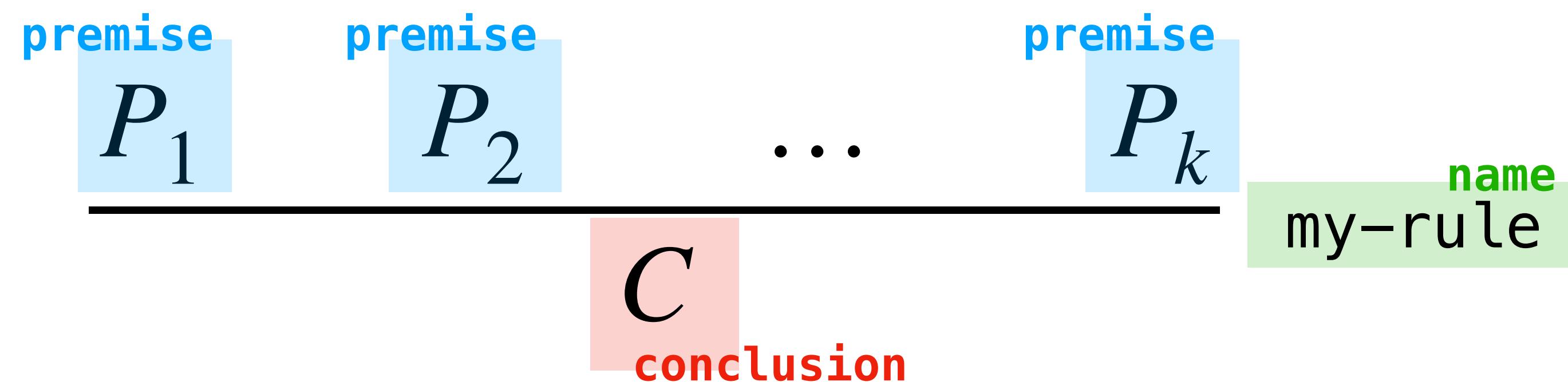
# Recall: Inference Rules



The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

There may be no premises, this is called an **axiom**

# Recall: Inference Rules



We can read this as:

*If the judgments  $P_1$  through  $P_k$  hold, then the judgment  $C$  holds (by my-rule)*

# Recall: Typing Judgments

$$\begin{array}{c} \text{context} \\ \Gamma \\ \hline \text{expression} \\ e \\ \hline \text{type} \\ \tau \end{array}$$

A typing judgment a compact way of representing the statement:

*e is of type  $\tau$  in the context  $\Gamma$*

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

# Recall: Contexts

$$\Gamma = \{ \text{x} : \text{int}, \text{y} : \text{string}, \text{z} : \text{int} \rightarrow \text{string} \}$$

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A variable declaration ( $x : \tau$ ) says: "I declare that the variable  $x$  is of type  $\tau$ "

# Recall: Contexts

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A **context** is a set of **variable declarations**

A variable declaration ( $x : \tau$ ) says: "I declare that the variable  $x$  is of type  $\tau$ "

A context keeps track of all the types of variables in the "environment"

# **Derivations**

# High Level

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{\{y : \text{int}\} \vdash y : \text{int}}{\{y : \text{int}\} \vdash y + y : \text{int}} \begin{array}{l} (\text{var}) \\ (\text{let}) \end{array} \quad \frac{\{y : \text{int}\} \vdash y : \text{int}}{\{y : \text{int}\} \vdash y + y : \text{int}} \begin{array}{l} (\text{var}) \\ (\text{intAdd}) \end{array}$$
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Derivations *prove* that a judgment holds w.r.t some rules

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A **derivation** is a tree in which:

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Derivations *prove* that a judgment holds w.r.t some rules

A **derivation** is a tree in which:

- » each node is labeled with a judgment
- » and judgment *follows* from the judgments at it's children by an inference rule

# Applying Rules

$$\frac{}{\Gamma \vdash [] : \tau \text{ list}} \text{ (nil)}$$

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau \text{ list}}{\Gamma \vdash e_1 :: e_2 : \tau \text{ list}} \text{ (cons)}$$

$$\boxed{\{x : \text{int}\} \vdash x + 1 : \text{int}}$$

$$\boxed{\{x : \text{int}\} \vdash [] : \text{int list} \text{ (cons)}}$$

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# Applying Rules

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So far, we've used rules as ways of describing the behavior of a PL

# Applying Rules

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So far, we've used rules as ways of describing the behavior of a PL

When we build typing derivations, we *instantiate* the meta-variables in the rule at *particular* expressions, contexts, etc.

# Building from the Ground Up

$$\frac{}{\{x : \text{int}\} \vdash x : \text{int}} (\text{var}) \quad \frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} (\text{intLit}) \quad \frac{}{\{x : \text{int}\} \vdash [] : \text{int list}} (\text{nil})$$
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But we can't just apply rules, because it's possible that the premises of a rule also need to be demonstrated

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But we can't just apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

This is how we get our tree structure: we apply rules from the ground up

# Axioms (When are we done?)

$$\frac{}{\{x : \text{int}\} \vdash x : \text{int}} \text{(var)} \quad \frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} \text{(intLit)} \quad \frac{}{\{x : \text{int}\} \vdash [] : \text{int list}} \text{(nil)}$$
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The leaves of the tree are **axioms**, i.e., a rules with no premises

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The leaves of the tree are **axioms**, i.e., a rules with no premises

In our case, this will almost always be "literal" or "variable" rules

# Integer Literals

(1)

$$\frac{n \text{ is an int lit}}{\Gamma \vdash n : \text{int}} \text{ (intLit)}$$

(2)

$$\frac{n \text{ is an int lit}}{n \Downarrow n} \text{ (intLitEval)}$$

1. If  $n$  is an integer literal, then it is of type  $\text{int}$  in any context
2. If  $n$  is an integer literal, then it evaluates to the number it represents

# A Note about Side Conditions

we don't write "1 is an integer literal"		
$\frac{}{\{x : \text{int}\} \vdash x : \text{int}} \text{(var)}$	$\frac{}{\{x : \text{int}\} \vdash 1 : \text{int}} \text{(intLit)}$	
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If a premise is a side-condition this *it is not included in the derivation*

Side conditions need to hold in order to apply the rule, but they don't appear in the derivation itself

We will always make side conditions clear

# Float Literals

$$(1) \frac{n \text{ is an float lit}}{\Gamma \vdash n : \mathbf{float}} \text{ (floatLit)} \quad (2) \frac{n \text{ is an float lit}}{n \Downarrow n} \text{ (floatLitEval)}$$

1. If  $n$  is an float literal, then it is of type float in any context
2. If  $n$  is an float literal, then it evaluates to the number it represents

# Boolean Literals

$$(1) \frac{}{\Gamma \vdash \text{true} : \text{bool}} (\text{trueLit})$$

$$(3) \frac{}{\text{true} \Downarrow T} (\text{trueLitEval})$$

$$(2) \frac{}{\Gamma \vdash \text{false} : \text{bool}} (\text{falseLit})$$

$$(4) \frac{}{\text{false} \Downarrow \perp} (\text{falseLitEval})$$

1. **true** is of type **bool** in any context
2. **false** is of type **bool** in any context
3. **true** evaluates to the value **T**
4. **false** evaluates to the value **⊥**

# Variables

$$\frac{(v : \tau) \in \Gamma}{\Gamma \vdash v : \tau} \text{ (intLit)}$$

If  $v$  is declared to be of type  $\tau$  in the context  $\Gamma$ , then  $v$  is of type  $\tau$  in  $\Gamma$

**Variables cannot be evaluated** (more on this when we talk about substitution and well-scopedness)

# Back to the Example

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{ y : \text{int} \} \vdash y + y : \text{int}} (\text{var})$$
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We need  $\{ \} \vdash 2 : \text{int}$  in order to proof that the bottom typing judgment holds

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Now we know that this follows from the **intLit** rule, which says that 2 is always an int, *by fiat*

Okay, I know that was a lot, let's take a step back

# Derivations Encode Natural Language Arguments

$$\frac{}{\{ \} \vdash 2 : \text{int}} (\text{intLit}) \quad \frac{}{\{ y : \text{int} \} \vdash y : \text{int}} (\text{var}) \quad \frac{}{\{ y : \text{int} \} \vdash y + y : \text{int}} (\text{var})$$
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*(In fact, most mathematical arguments can be represented formally as derivation trees, this is the called **proof theory**)*

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The expression `let y = 2 in y + y` is an `int` because

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» `2` is an `int` by fiat (and so `y` is being assigned to a well-typed expression)

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$$\frac{n \text{ is an integer literal}}{\Gamma \vdash n : \text{int}} \text{(intLit)}$$

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# Derivations Encode Natural Language Arguments

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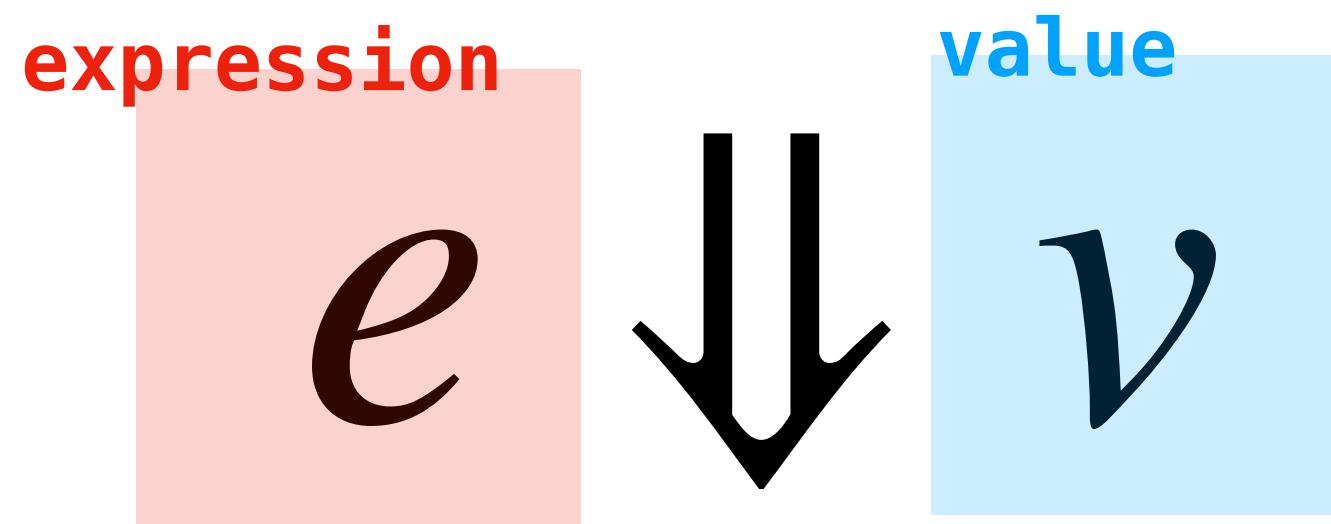
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And all this works for  
semantics judgements as well

# Recall: Semantic Judgements



A **semantic judgment** is a compact way of representing the statement:

*The expression  $e$  evaluates to the value  $v$*

A **semantic rule** is an inference rule with semantic judgments

# Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v} \text{ (evalInt)}$$

If  $e_1$  evaluates to the (integer)  $v_1$  and  $e_2$  evaluates to the (integer)  $v_2$ , and  $v_1 + v_2 = v$ , then  $e_1 + e_2$  evaluates to the (integer)  $v$

# Semantic Derivations

$$\frac{\text{true} \Downarrow \top}{\text{if true then } 2 \text{ else } 3 \Downarrow 2} \text{ (ifEval)} \quad \frac{}{2 \Downarrow 2} \text{ (intEval)}$$
$$\frac{}{\text{true} \Downarrow \top} \text{ (trueEval)}$$

We can also write derivations to prove semantic judgments

The principle is the same, except that the judgments are semantic judgments instead of typing judgments

# **Examples**

# Example (Typing)

{ } ⊢ if true then 2 else 5 : int

# Example (Evaluation)

if true then 2 else 5 ↓ 2

# Example (Typing)

{ } ⊢ 2 + 3 <> 4 : bool

# Example (Evaluation)

2 + 3 < 4 ↓ true

# Example (Evaluation)

```
let x = 2 in x + x ↓ 4
```

# Summary

Derivations are **tree-like proofs** that judgments hold with respect to a collection of inference rules

Derivations are **compact mathematical representations** of English language arguments

Learning to write derivations takes *practice*