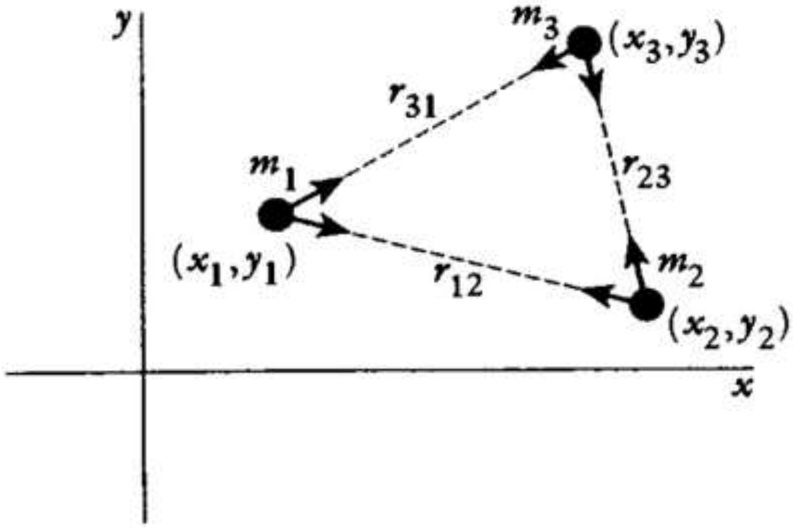
EN.615.765.81.FA22 Chaos / Discussion Module 4

Three-body problem

Given the system:



Produce graphs with trajectories of three masses ($m_1 = m_2 = m_3 = 0.5$) with initial positions $(-0.5,0), (0.5,0), (-0.1,0.75)$ and initial velocities $(0,-0.3)$, $(0,0.3)$, $(0,-0.3)$ respectively all in dimensionless units.

Solution

Governing ordinary differential equations

We have a closed system of three masses that are only subject to the force of gravity. Therefore, the total energy of the system is conserved, where the energy is equal to the sum of the potential energies and kinetic energies of the masses ($\dot{r}_i$ is the magnitude of the 2-d velocity of m_i).

$$V_1(r_{12}, r_{13}) = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}}$$

$$V_2(r_{12}, r_{23}) = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}}$$

$$V_3(r_{13}, r_{23}) = -\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$$

$$E = \sum_{i=1}^{3} V_i + \frac{m_i \dot{r}_i^{\,2}}{2}$$

$$E = -\frac{2Gm_1m_2}{r_{12}} - \frac{2Gm_1m_3}{r_{13}} - \frac{2Gm_2m_3}{r_{23}} + \frac{m_1\dot{r}_1^2}{2} + \frac{m_2\dot{r}_2^2}{2} + \frac{m_3\dot{r}_3^2}{2}$$

Since E is conserved, we can write:

$$\dot{E} = 0 = \frac{2Gm_1m_2}{r_{12}^2}\dot{r}_{12} + \frac{2Gm_1m_3}{r_{13}^2}\dot{r}_{13} + \frac{2Gm_2m_3}{r_{23}^2}\dot{r}_{23} + m_1\dot{r}_1\ddot{r}_1 + m_2\dot{r}_2\ddot{r}_2 + m_3\dot{r}_3\ddot{r}_3$$

The equations that govern the three-body problem are:

$$m_1\ddot{\boldsymbol{r}}_\mathbf{1} = -Gm_1m_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - Gm_1m_3\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3}$$
$$m_2\ddot{\boldsymbol{r}}_\mathbf{2} = -Gm_1m_2\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} - Gm_2m_3\frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3}$$
$$m_3\ddot{\boldsymbol{r}}_\mathbf{3} = -Gm_1m_3\frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} - Gm_2m_3\frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3}$$

In 2-D space, $\mathbf{r}_i = x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}}$ and $r_{ij} = |\mathbf{r}_j - \mathbf{r}_i| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, so these equations turn into six with :

$$\ddot{x}_1 = -Gm_2\frac{x_1 - x_2}{r_{12}^3} - Gm_3\frac{x_1 - x_3}{r_{13}^2}$$
$$\ddot{y}_1 = -Gm_2\frac{y_1 - y_2}{r_{12}^3} - Gm_3\frac{y_1 - y_3}{r_{13}^2}$$

$$\ddot{x}_2 = -Gm_1\frac{x_2 - x_1}{r_{12}^3} - Gm_3\frac{x_2 - x_3}{r_{23}^2}$$
$$\ddot{y}_2 = -Gm_1\frac{y_2 - y_1}{r_{12}^3} - Gm_3\frac{y_2 - y_3}{r_{23}^2}$$

$$\ddot{x}_3 = -Gm_1\frac{x_3 - x_1}{r_{13}^3} - Gm_2\frac{x_3 - x_2}{r_{23}^2}$$
$$\ddot{y}_3 = -Gm_1\frac{y_3 - y_1}{r_{13}^3} - Gm_2\frac{y_3 - y_2}{r_{23}^2}$$

Which is consistent with the energy equations above. In general, the velocity ODEs are:

$$\dot{v}_{x,i} = -Gm_j\frac{x_i - x_j}{r_{ij}^3} - Gm_k\frac{x_i - x_k}{r_{ik}^3}$$
$$\dot{v}_{y,i} = -Gm_j\frac{y_i - y_j}{r_{ij}^3} - Gm_k\frac{y_i - y_k}{r_{ik}^3}$$

The other quantity that needs to be solved is position. This done by solving the ODEs in the form of:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \dot{x}_i = v_{x,i}, \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t} = \dot{y}_i = v_{y,i}$$

To obtain the ODEs in dimensionless forms, the dimensions of mass and length need to be absorbed into a single constant. Starting with the position ODEs, let $\xi_i = x_i/d$ where d is an arbitrary length scale, $\tau = \omega t$ be the non-dimensional time (ω is a constant to be determined), and $\beta_{x,i}$ be the non-dimensional velocity:

$$\mathrm{d}\xi_i = \frac{\mathrm{d}x_i}{d}$$
$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \frac{1}{d}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{v_{x,i}}{d}$$
$$\frac{1}{d}\frac{\mathrm{d}v_{x,i}}{\mathrm{d}t} = \frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\xi_i}{[\mathrm{d}(\tau/\omega)]^2} = \omega^2\frac{\mathrm{d}^2\xi_i}{\mathrm{d}\tau^2} = \omega^2\frac{\mathrm{d}\beta_i}{\mathrm{d}\tau}$$
$$\frac{\mathrm{d}v_{x,i}}{\mathrm{d}t} = \omega^2 d\frac{\mathrm{d}\beta_{x,i}}{\mathrm{d}\tau}$$

Similarly, with $\upsilon_i = y_i/d$:

$$\mathrm{d}\upsilon_i = \frac{\mathrm{d}y_i}{d}$$
$$\frac{\mathrm{d}v_{y,i}}{\mathrm{d}t} = \omega^2 d\frac{\mathrm{d}\beta_{y,i}}{\mathrm{d}\tau}$$

In order to convert the $\dot{v}_{x,i}$ ODE, now let $\xi_j = x_j/d$, $\xi_k = x_k/d$, $\upsilon_j = y_j/d$, $\upsilon_k = y_k/d$ such that:

$$\frac{(x_i - x_j)}{r_{ij}^3} = \frac{(x_i - x_j)}{\left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{3/2}} = \frac{(\xi_i - \xi_j)d}{\left[(\xi_i - \xi_j)^2/d^2 + (\upsilon_i - \upsilon_j)^2/d^2\right]^{3/2}}$$
$$= \frac{(\xi_i - \xi_j)}{d^2\left[(\xi_i - \xi_j)^2 + (\upsilon_i - \upsilon_j)^2\right]^{3/2}}$$

$$\frac{(x_i - x_j)}{r_{ij}^3} = \frac{(\xi_i - \xi_j)}{d^2\rho_{ij}^3}$$

Where $\rho_{ij}^2 = (\xi_i - \xi_j)^2 + (\upsilon_i - \upsilon_j)^2$. The second term of the $\dot{v}_{x,i}$ follows as:

$$\frac{(x_i - x_k)}{r_{ik}^3} = \frac{(\xi_i - \xi_k)}{d^2\rho_{ik}^3}$$

Now recalling the $\dot{v}_{x,i}$, diving by $m_j G$, letting $M = m_k/m_j$, and using the above results:

$$\dot{v}_{x,i} = -Gm_j\frac{(\xi_i - \xi_j)}{d^2\rho_{ij}^3} - Gm_k\frac{(\xi_i - \xi_k)}{d^2\rho_{ik}^3}$$
$$\frac{d^2}{m_j G}\frac{\mathrm{d}v_{x,i}}{\mathrm{d}t} = -\frac{\xi_i - \xi_j}{\rho_{ij}^3} - M\frac{\xi_i - \xi_k}{\rho_{ik}^3}$$

Recalling that $\frac{\mathrm{d}v_{x,i}}{\mathrm{d}t} = \omega^2 d \frac{\mathrm{d}\beta_{x,i}}{\mathrm{d}\tau}$, it is desired to have the factors outside the derivatives to be set to 1.

$$\frac{d^2}{m_j G}\frac{\mathrm{d}v_{x,i}}{\mathrm{d}t} = \frac{d^3}{m_j G}\omega^2 \frac{\mathrm{d}\beta_{x,i}}{\mathrm{d}\tau}$$

$$\frac{d^3}{m_j G}\omega^2 = 1 \Rightarrow \omega = \sqrt{\frac{m_j G}{d^3}}$$

Clearly, the dimensionless form of time is $\tau = t\sqrt{m_j G/d^3}$. Which brings the velocity ODE in the x-direction into its dimensionless form as:

$$\frac{\mathrm{d}\beta_{x,i}}{\mathrm{d}\tau} = \dot{\beta}_{x,i} = -\frac{\xi_i - \xi_j}{\rho_{ij}^3} - M\frac{\xi_i - \xi_k}{\rho_{ik}^3}$$

The velocity ODE in the y-direction follows the same logic.

$$\frac{\mathrm{d}\beta_{y,i}}{\mathrm{d}\tau} = \dot{\beta}_{y,i} = -\frac{\upsilon_i - \upsilon_j}{\rho_{ij}^3} - M\frac{\upsilon_i - \upsilon_k}{\rho_{ik}^3}$$

For three particles in two dimensions, there are six position ODEs that need to be solved. This brings the total number of ODEs that need to be solved to twelve. The MATLAB code uses an RK-4 method to numerically solve the trajectories of each mass.

$$\dot{\beta}_{x,1} = -\frac{\xi_1 - \xi_2}{\rho_{12}^3} - M\frac{\xi_1 - \xi_3}{\rho_{13}^3}$$
$$\dot{\beta}_{y,1} = -\frac{\upsilon_1 - \upsilon_2}{\rho_{12}^3} - M\frac{\upsilon_1 - \upsilon_3}{\rho_{13}^3}$$
$$\dot{\beta}_{x,2} = -\frac{\xi_2 - \xi_3}{\rho_{23}^3} - M\frac{\xi_2 - \xi_1}{\rho_{12}^3}$$
$$\dot{\beta}_{y,2} = -\frac{\upsilon_2 - \upsilon_3}{\rho_{23}^3} - M\frac{\upsilon_2 - \upsilon_1}{\rho_{12}^3}$$
$$\dot{\beta}_{x,3} = -\frac{\xi_3 - \xi_1}{\rho_{13}^3} - M\frac{\xi_3 - \xi_2}{\rho_{23}^3}$$
$$\dot{\beta}_{y,3} = -\frac{\upsilon_3 - \upsilon_1}{\rho_{13}^3} - M\frac{\upsilon_3 - \upsilon_2}{\rho_{23}^3}$$
$$\dot{\xi}_1 = \beta_{x,1}$$
$$\dot{\upsilon}_1 = \beta_{y,1}$$
$$\dot{\xi}_2 = \beta_{x,2}$$
$$\dot{\upsilon}_2 = \beta_{y,2}$$
$$\dot{\xi}_3 = \beta_{x,3}$$
$$\dot{\upsilon}_3 = \beta_{y,3}$$

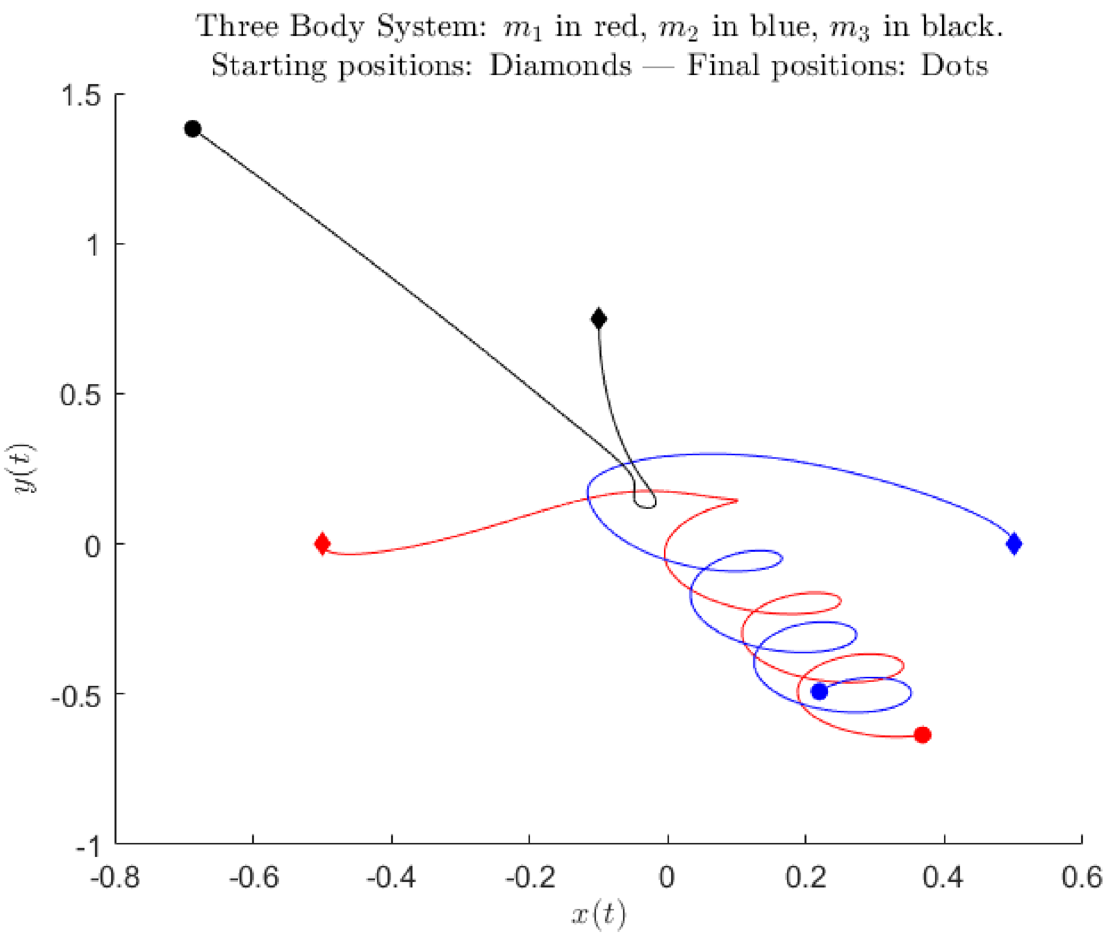
Code summary and graph

Through the files threeBf.m (prepares the ODEs) and solve3B.m (solves the ODEs through the RK-4 method and plots the results), the computation is made through the commands:

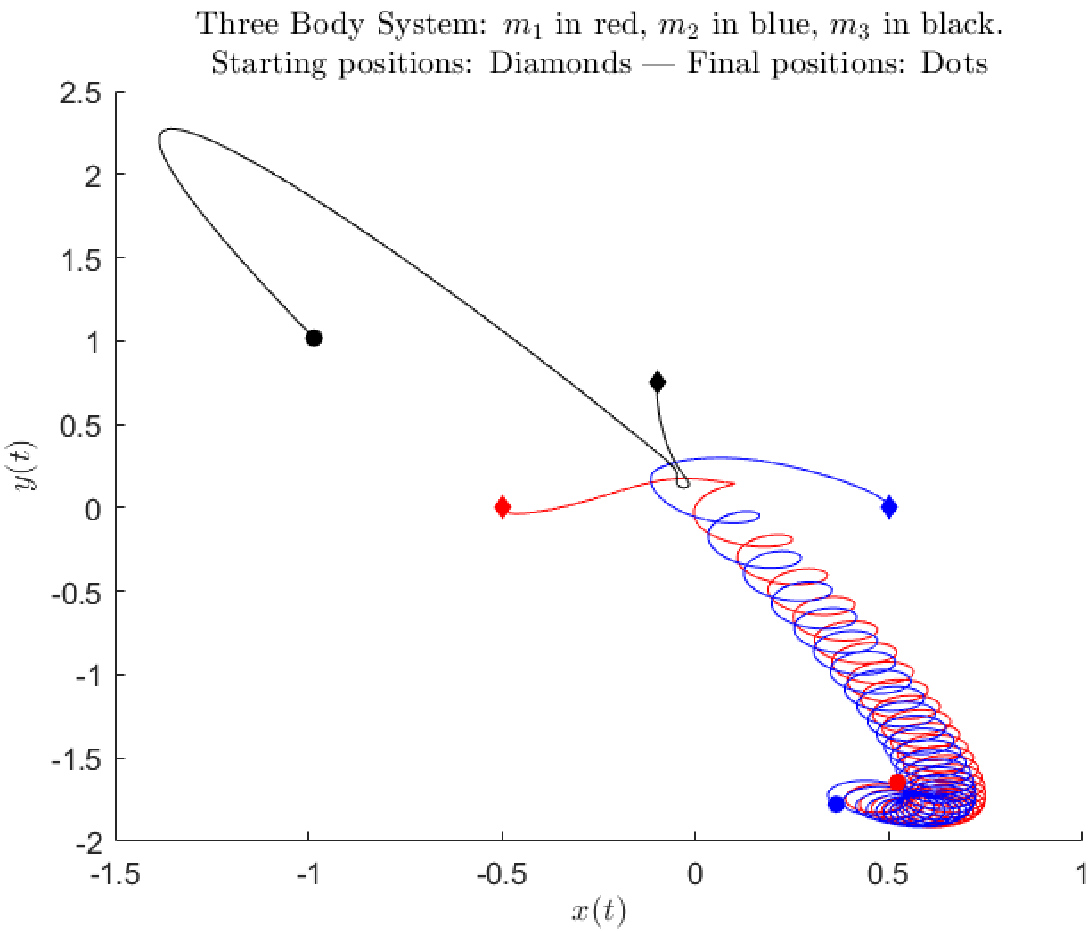
```
I = [-0.5 0 0.5 0 -0.1 0.75 0 -0.3 0 0.3 0 -0.3];
solve3B(I,0.5,0.5,0.5,1);
```

The last argument in `solve3B(I,m1,m2,m2,G)` is the value for G, which is ultimately not used in the computation as threeBf.m uses a function that sets the ODE in the dimensionless form (`G` was used as a test method to solve the ODEs in the dimensional form, which was saved in the comments in threeBf.m, lines 43-45).

The step size-values and the scale values (which is automatically adjusted in every step as $h = h_{\text{scale}}/(\rho_{12}^2 + \rho_{23}^2 + \rho_{13}^2)$ used in the computation are `n = 2750000;` and `hscale = 0.000002;`, respectively. The final dimensionless time value was $\tau = 0.2305$.



The step size-values and the scale values used in the computation are `n = 27500000;` and `hscale = 0.00001;`, respectively. The final dimensionless time value was $\tau = 10.5$.



This graph uses a constant step size of `h =hscale = 0.000025;` and `n = 1000000;` This plot is shown to show illustrate what happens to the masses at a much later time.