Pricing Modeling Notes

July 15, 2020

Chapter 1

Risk Based Pricing

1.1 Preliminary definitions

Element	Notation
Client Interest Rate	r
Cost of Funds Rate,Fund Transfer Pricing FTP, TT	r_c
Discount Rate	r_d
Contractual Maturity	T

1.1.1 Default and Prepayment probabilities:

$\forall t \in \{1, 2, ...T\}$

$p_p(t)$: Probability that the loan will prepay at time t given that it has survived to that point

$p_d(t)$: Probability that the loan will default at time t given that it has survived to that point

1.1.2 Survival function:

$S(t)$: Probability that a loan survives until period t

$$\begin{aligned} S(t) &= \Pi_{s=1}^{t}(1 - p_d(s) - p_p(s)) \\ &= (1 - p_d(1) - p_p(1)) \times (1 - p_d(2) - p_p(2)) \times ... \times (1 - p_d(t) - p_p(t)) \end{aligned} \tag{1.1}$$

1.1.3 Balance function:

The Current Balance function $\bar{B}(t)$ is the remaining balance left at time $t-1$ for a loan with principal $B = \bar{B}(1)$ and in absence of any prepayment or default risk. For non conventional loan payments this function might not have a closed form solution.

Constant installments: The remaining balance at time t for a loan with principal (Balance at t=0) B is given by

$$\bar{B}(t) = B\frac{(1+r)^T - (1+r)^{t-1}}{(1+r)^T - 1} \tag{1.2}$$

$$\bar{I}(t) = r \times \bar{B}(t) \tag{1.3}$$

The acute reader will notice that the definition of $\bar{B}(t)$, in terms of the remaining balance left at time $t-1$, was given so that we can state such a simple equation for $\bar{I}(t)$.

Constant amortization: The remaining balance at time t for a loan with principal (Balance at t=0) B is given by:

$$\bar{B}(t) = B \times (1 - \frac{t-1}{T}) \tag{1.4}$$

1.2 Financial Math operators

1.2.1 Constant Installments

We can define a c_f function to compute constant installments by defining the following:

$$\bar{B}(1) = \frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \frac{c}{(1+r)^3} + ... + + \frac{c}{(1+r)^T} \tag{1.5}$$

Setting $\delta = \frac{1}{1+r}$

$$\begin{aligned}\bar{B}(1) &== c\delta[1 + \delta + \delta^2 + \delta^3 + \delta^4 + ... + \delta^{T-1}] \\ &= c\delta\left[\frac{1}{1-\delta} - \delta^T\frac{1}{1-\delta}\right] = c\delta\left(\frac{1-\delta^T}{1-\delta}\right)\end{aligned} \tag{1.6}$$

$$c = \bar{B}(1)\left(\frac{1-\delta}{\delta}\right)\frac{1}{1-\delta^T} = B(1)\left[r\frac{(1+r)^T}{(1+r)^T - 1}\right] \tag{1.7}$$

$$\boxed{c_f(r,T) := \frac{r(1+r)^T}{(1+r)^T - 1}} \implies c = \bar{B}(1)c_f(r,T) \tag{1.8}$$

Balance factor:

$$\bar{B}(t) = \bar{B}(1)\left[\frac{(1+r)^T - (1+r)^{t-1}}{(1+r)^T - 1}\right] \tag{1.9}$$

$$\boxed{B_f(t,r,T) = \left[\frac{(1+r)^T - (1+r)^{t-1}}{(1+r)^T - 1}\right]} \implies \bar{B}(t) = \bar{B}(1)B_f(t,r,T) \tag{1.10}$$

$$\bar{I}(t) = r\bar{B}(1)B_f(t,r,T) \tag{1.11}$$

Amortization factor:

$$A_f(t,r,T) = c_f(r,T) - rB_f(t,r,T) \tag{1.12}$$

$$\boxed{A_f(t,r,T) = \frac{r(1+r)^{t-1}}{(1+r)^T - 1}} \tag{1.13}$$

Theorem 1.1 (Telescopic Amortizations). If A is the function defined in (1.13) then:

$$\prod_{s=1}^{t-1}(1 - A(1,r,T-s+1)) = 1 - \sum_{s=1}^{t-1} A(s,r,T)$$

Proof. Lets define

$$E_1 = \prod_{s=1}^{t-1}(1 - A(1,r,T-s+1)), E_2 = 1 - \sum_{s=1}^{t-1} A(s,r,T)$$

and

$$\delta = 1/(1+r)$$

Working on E_1 and setting $\xi = 1 + r$:

$$1 - A(1,r,T-s+1) = \frac{(1+r)^{T-s+1} - 1 - r}{(1+r)^{T-s+1} - 1} = \frac{\xi^{T-s+1} - \xi}{\xi^{T-s+1} - 1}$$

$$\begin{aligned}
\implies E_1 =& \frac{(\xi^T - \xi)}{(\xi^T - 1)} \times \frac{(\xi^{T-1} - \xi)}{(\xi^{T-1} - 1)} \times \frac{(\xi^{T-2} - \xi)}{(\xi^{T-2} - 1)} \times ... \times \frac{(\xi^{T-t+1} - \xi)}{(\xi^{T-t+1} - 1)} \\
=& \xi^t \frac{(\xi^{T-1} - 1)}{(\xi^T - 1)} \times \frac{(\xi^{T-2} - 1)}{(\xi^{T-1} - 1)} \times \frac{(\xi^{T-3} - 1)}{(\xi^{T-2} - 1)} \times ... \times \frac{(\xi^{T-t} - 1)}{(\xi^{T-t+1} - 1)} \\
=& \frac{\xi^T - \xi^t}{\xi^T - 1}
\end{aligned} \tag{1.14}$$

Working on E_2:

$$A(s,r,T) = \frac{r(1+r)^{s-1}}{(1+r)^T - 1} = (\xi - 1)\frac{\xi^{s-1}}{\xi^T - 1}$$

$$\begin{aligned}
\implies E_2 =& 1 - \frac{(\xi - 1)(1 + \xi + \xi^2 + \xi^3 + ... + \xi^{t-1})}{\xi^T - 1} \\
=& \frac{\xi^T - \xi^t}{\xi^T - 1}
\end{aligned} \tag{1.15}$$

$\therefore E1 = E2$ $\square$

1.3 Alternative setups for Incremental Profit computation

1.3.1 Standard model

In this model prepayments/full prepayment, default probability are expressed as conditional probabilities. This probabilities are conditioned on the running active balance i.e. the balance that has not been prepaid or defaulted upon.

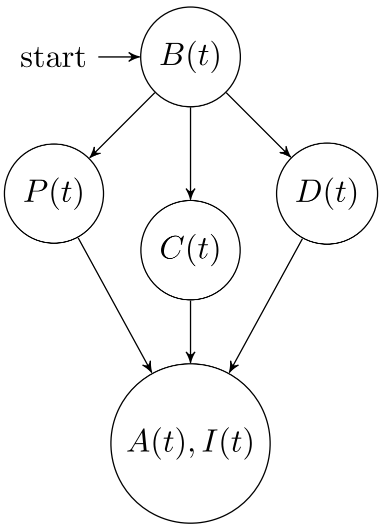

Figure 1.1: Computation Graph

Variable	Notation	Calculation
Balance in presence of risk	$B(t)$	$B(t)$
Default	$D(t)$	$p_d(t)B(t)$
Full Prepayment	$C(t)$	$p_c(t)B(t)$
Prepayment	$P(t)$	$p_p(t)B(t)$
Amortization	$A(t)$	$(1 - p_d(t) - p_c(t) - p_p(t))B(t)A_f(1, r, T - t + 1))$
Interest	$I(t)$	$(1 - p_d(t) - p_c(t) - p_p(t))B(t)r$
Principal	$B(1)$	B

Table 1.1: Computation for Model 1: Standard Model

Notice that we defined $B(t)$ as the Loan Balance subject to risk (Conductual affected Loan Balance), as opposed to $\bar{B}(t)$, which is the Risk Free Loan Balance (Contractual Balance). Given this definitions we can compute the recursive form for the balance function $B(t)$

$$\begin{aligned} B(t+1) =& B(t)[1 - p_d(t) - p_c(t) - p_p(t) - (1 - p_d(t) - p_c(t) - p_p(t))A(1, r, T - t + 1)] \\ =& B(t)(1 - p_d(t) - p_c(t) - p_p(t))(1 - A(1, r, T - t + 1)) \end{aligned} \tag{1.16}$$

Notice that (1.16) is a first order equation in difference which can be easily solved as.

$$B(t) = \prod_{s=1}^{t-1}(1 - A(1, r, T - s + 1))(1 - p_d(s) - p_c(s) - p_p(s))B \tag{1.17}$$

Using theorem (1.2.1) we can state the conductual balance $B(t)$ as a function of the contractual balance.

$$\begin{aligned} B(t) &= \prod_{s=1}^{t-1}(1 - p_d(s) - p_c(s) - p_p(s))\bar{B}(t) \\ &= S(t)\bar{B}(t) \end{aligned}$$

$$\boxed{B(t) = S(t-1)\bar{B}(t)} \tag{1.18}$$

Equation 1.18 states a very simple relation between the theoretical/contractual balance $\bar{B}(t)$ and the behavioral balance $B(t)$. We have derived this equation from a first principles approach through the Theorem of Telescopic Amortization (1.2.1). This equation represent a very powerful shortcut not only for using intuition, since the behavioral balance can be thought of as the contractual balance adjusted by the survival probability, but also for implementing the model programatically using vectorization instead of recursive loops over the different points in the payment schedule, the last alternative can be very hard to maintain and compute, not to mention its proneness to error.

1.3.2 Prepayment dependent on initial balance

This model is a variation of the previous one in which the prepayment probabiliy is a proportion of the initial balance/principal $B(1)$

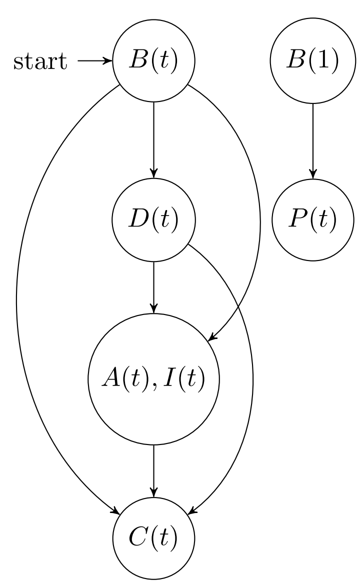

Figure 1.2: Computation Graphs: Model 2

Variable	Notation	Calculation
Balance in presence of risk	$B(t)$	$B(t)$
Default	$D(t)$	$p_d(t)B(t)$
Amortization	$A(t)$	$(1-p_d(t))B(t)A(1,r,T-t+1))$
Interest	$I(t)$	$(1-p_d(t))B(t)r$
Full Prepayment	$C(1)$	$p_c(t)B(t)[1-p_d(t)-(1-p_d(t))A(1,r,T-t+1)]$
Principal	$B(1)$	B
Prepayment	$P(t)$	$p_p(t)B$

Table 1.2: Computation for Model 2: Prepayment Dependent on Initial Balance

Given this definitions we can compute the recursive form for the balance function $B(t)$

$$\begin{aligned}\bar{B}(t+1)&=\bar{B}(t)[1-p_d(t)-p_c(t)(1-p_d(t)-(1-p_d(t))A(1,r,T-t+1))-(1-p_d(t))A(1,r,T-t+1)]-p_p(t)\bar{B}(1)\\&=\bar{B}(t)[1-p_d(t)-p_c(t)+p_d(t)p_c(t)+p_c(t)(1-p_d(t)A(1,r,T-t+1)-(1-p_d(t))A(1,r,T-t+1)]-p_p(t)\bar{B}(1)\\&=\bar{B}(t)[(1-p_d(t))(1-p_c(t))-(1-p_d(t))(1-p_c(t))A(1,r,T-t+1)]-p_p(t)\bar{B}(1)\\&=\bar{B}(t)(1-p_d(t))(1-p_c(t))(1-A(1,r,T-t+1))-p_p(t)\bar{B}(1)\end{aligned} \tag{1.19}$$

Notice that (1.19) is a first order equation in difference of type $x(t+1) = \alpha(t)x(t) + \beta(t)$ where $\alpha(t) := (1-p_d(t))(1-p_c(t))(1-A(1,r,T-t+1))$, $\beta(t) := -p_p(t)\bar{B}(1)$, $x(t) := \bar{B}(t)$ and the initial condition given by the lended principal $x(1) = \bar{B}(1) = B$. This equation can be easily solved as:

$$x(t+1) = x_1 \prod_{s=1}^{t} \alpha(s) + \sum_{k=1}^{t-2} \left[\beta(k) \prod_{s=k+1}^{t-1} \alpha(s) \right] + \beta(t-1) \tag{1.20}$$

Plugging back the definitions of $\alpha(t)$, $\beta(t)$ and ,$x(t)$ we get:

$$\begin{aligned}B(t+1) = &\prod_{s=1}^{t}(1-p_d(s))(1-p_p(s))(1-A(1,r,T-s+1))B\\&-\sum_{k=1}^{t-1}\left[p_p(t)B\prod_{s=k+1}^{t-1}(1-p_d(t))(1-p_c(t))(1-A(1,r,T-t+1))\right]-p_p(t-1)B\end{aligned} \tag{1.21}$$

Using the theorem of Telescopic Amortizations (1.2.1), the definitions for survival curve and lagging t one period we can state the conductual balance $B(t)$ as a function of the contractual balance $\bar{B}(t)$.

$$\begin{aligned}B(t) &= \prod_{s=1}^{t-1}(1-p_d(s))(1-p_p(s))\bar{B}(t) - \sum_{k=1}^{t-2}\left[p_p(k)B\frac{S(t-2)}{S(k)}\frac{\bar{B}(t-1)}{\bar{B}(k+1)}\right]\\&= S(t-1)\bar{B}(t) - \sum_{k=1}^{t-2}\left[p_p(k)B\frac{S(t-2)}{S(k)}\frac{\bar{B}(t-1)}{\bar{B}(k+1)}\right]\end{aligned}$$

$$\boxed{B(t) = S(t-1)\bar{B}(t) - \sum_{k=1}^{t-2}\left[p_p(k)B\frac{S(t-2)}{S(k)}\frac{\bar{B}(t-1)}{\bar{B}(k+1)}\right]} \tag{1.22}$$

Equation (1.22) is analogous to equation (1.18) with an aditional term that represents a weighted sum of the last prepayment amounts where the last prepaymet has a weight of one. Equivalently, instead of excluding the prepayment amount through the survival factor $S(t-1)$ we are substracting these amounts outside the $S(t-1)\bar{B}(t)$ factor.

1.4 Terms included in the incremental profit (CLV)

1.4.1 Interest on loans:

$$LI(t) = S(t)\bar{B}(t)r \tag{1.23}$$

Equation (1.23) will be proved in section (1.3.1). For now lets just use our intuition and state that each dollar has an unconditional probability to survive up to time t of $S(t)$

1.4.2 Cost of Funds:

$$COF(t) = S_c(t)\bar{B}_c(t)r_c \tag{1.24}$$

Where:

$$S_c(t) = \Pi_{s=0}^{t}[1 - p_p(s) - (1 - LGD(s))p_d(s)] \tag{1.25}$$

The last formula can be viewed as the complement for the probability of death for 1 dollar. If a prepayment event

1.4.3 Equity Benefit (Capital Rebate):

$$EB(t) = \alpha S(t)\bar{B}(t)r_c \tag{1.26}$$

1.4.4 Fees Additional Source of revenue:

$$F(t) = fS(t) \tag{1.27}$$

1.4.5 Servicing Costs:

$$SC(t) = \sigma S(t) \tag{1.28}$$

1.4.6 Loss from Default:

$$EL(t) = p_d(t)LGD(t)S(t)\bar{B}(t) \tag{1.29}$$

1.4.7 Recovery costs

$$C(t) = c \times p_d(t)S(t) \tag{1.30}$$

1.4.8 Equity Capital Charge:

$$EC(t) = \alpha S(t)\bar{B}(t)r_e \tag{1.31}$$

1.5 Incremental Profit Definition (CLV):

The net present value is given by:

$$NPV(x(t), r, T) = \sum_{t=1}^{T} \frac{x(t)}{(1+r)^t} \tag{1.32}$$

Element	Notation	Calculation
Lending Interest	LI	$NVP(LI(t), r_d, T)$
Cost of Funds	COF	$NVP(COF(t), r_d, T)$
Equity benefit	EB	$NVP(EB(t), r_d, T)$
Fees	LI	$NVP(F(t), r_d, T)$
Ancillary profit	A	–
Origination cost	OC	–
Commision	COM	–
Servicing Costs	SC	$NVP(SC(t), r_d, T)$
Expected Loss	EL	$NVP(EL(t), r_d, T)$
Collection costs	C	$NVP(C(t), r_d, T)$
Equity charge	EC	$NVP(EC(t), r_d, T)$

Element	Notation	Calculation
Net Interest Income	NII	$LI - COF + EB$
Total Income	TI	$NII + A + F$
Net Income before tax	$NIBT$	$TI - OC - COM - SC - LD - C$
Net Income after tax	$NIAT$	$(1 - \tau) \times NIBT$
Incremental profit	IP	$NIAT - EC$

1.5.1 Incremental Profit Function:

Define the incremental profit function as:

$$\pi(p) = IP(p) \tag{1.33}$$

Chapter 2

Willingness to Pay Modeling (WTP):

2.1 The Price-Response Function

Each price-response function specifies the demand that the lender would experience at each price, which will depend on:

1. Total number of clients interested in the loan
2. Number of applicant clients
3. The number of applicant the lender deems creditworthy and quotes a price.
4. Number of accepted applicants who would achieve a positive surplus from taking the loan from the lender at the offered price.
5. Number of accepted applicants who **take-up** the offered loan.

In most lending markets, the final price is not known to the client at the time she applies for the loan so we assume that the number of clients who apply for a loan is not influenced by the price.

$$d(p) = D\bar{F}(p) \tag{2.1}$$

$d(p)$ is the number of the loans offered by a lender that would be taken up at the price p. D is the number of successful applicants for the loan, and $\overline{F}(p)$ is the take-up rate, which is defined as the fraction of successful applicants who will take up the loan at price p

$$\bar{F}(p) = \int_p^\infty f(w)dw \tag{2.2}$$

2.2 Segmented vs Join Estimation

For n segments, the segmented estimation assumes each segments has its own demand function so we need to estimate $2n$ parameters.

$$\bar{F}_i(p_i) = \frac{e^{a_i+b_ip_i}}{1+e^{a_i+b_ip_i}} \tag{2.3}$$

For n segments, the join estimation assume we can estimate one single price price response function that includes all explanatory variables within it (including price)

$$\bar{F}(p_i, a, b, \theta, x_i) = \frac{e^{a+bp_i+\theta^T x_i}}{1+e^{a+bp_i+\theta^T x_i}} \tag{2.4}$$

Chapter 3

Price optimization (CLV+WTP):

Consider the incremental profit function given by $\pi(p)$ in equation (1.33) and the take-up rate function given by $d(p) = D\bar{F}(p)$ in equation (2.1). Moreover, following section (2.2), lets consider we are interested in optimizing across N client segments.

3.1 Price optimization without constraints

$$p^* = \arg\max_p \sum_{i=1}^{N} D_i \overline{F}_i(p_i)\pi(p_i) \tag{3.1}$$

3.2 Price optimization with competing objectives: The efficient Frontier

$$p^* = \arg\max_p \sum_{i=1}^{N} D_i \bar{F}_i(p_i)\pi(p_i) \tag{3.2}$$

$$s.t.$$

$$\sum_{i=1}^{N} D_i \bar{F}_i(p_i) = q \tag{3.3}$$

Chapter 4

Survival models

4.0.1 Standard survival setup

Let T be a positive random variable in $1, 2, 3, ...$

$$S(t) = P(T > t) \tag{4.1}$$
$$F(t) = 1 - S(t) = 1 - P(T > t) = P(T \leq t) \tag{4.2}$$

4.0.2 Survival setup in presence of competing risks

We define the cumulative incidence function as:

$$CIF_k(t) = P(T \leq t, D = k) \tag{4.3}$$
$$= \sum_k P(T \leq t, D = k) = P(T \leq t) \tag{4.4}$$

In order to see what is the relationship between the CIF function and the usual conditional probability of default (death) we state the definition of conditional probability and use the fact that the event $T = t + 1 \wedge T > t$ is equal to $T = t + 1$ standalone.

$$p_k(t + 1) = p(T = t + 1, D = k/T > t) = \frac{P(T = t + 1, D = k)}{P(T > t)} \tag{4.5}$$
$$= \frac{P(T \leq t + 1, D = k) - P(T \leq t, D = k)}{1 - \sum_k P(T \leq t, D = k)} \tag{4.6}$$

As an example lets consider that $D = 1$ represents default and $D = 2$ represents prepayment then the conditional probabilities of default and prepayment are given by:

$$p_d(t + 1) = \frac{CIF_d(t + 1) - CIF_d(t)}{1 - CIF_d(t) - CIF_p(t)} \tag{4.7}$$

Chapter 5

The PricingPy Python Library

5.0.1 Class Diagram

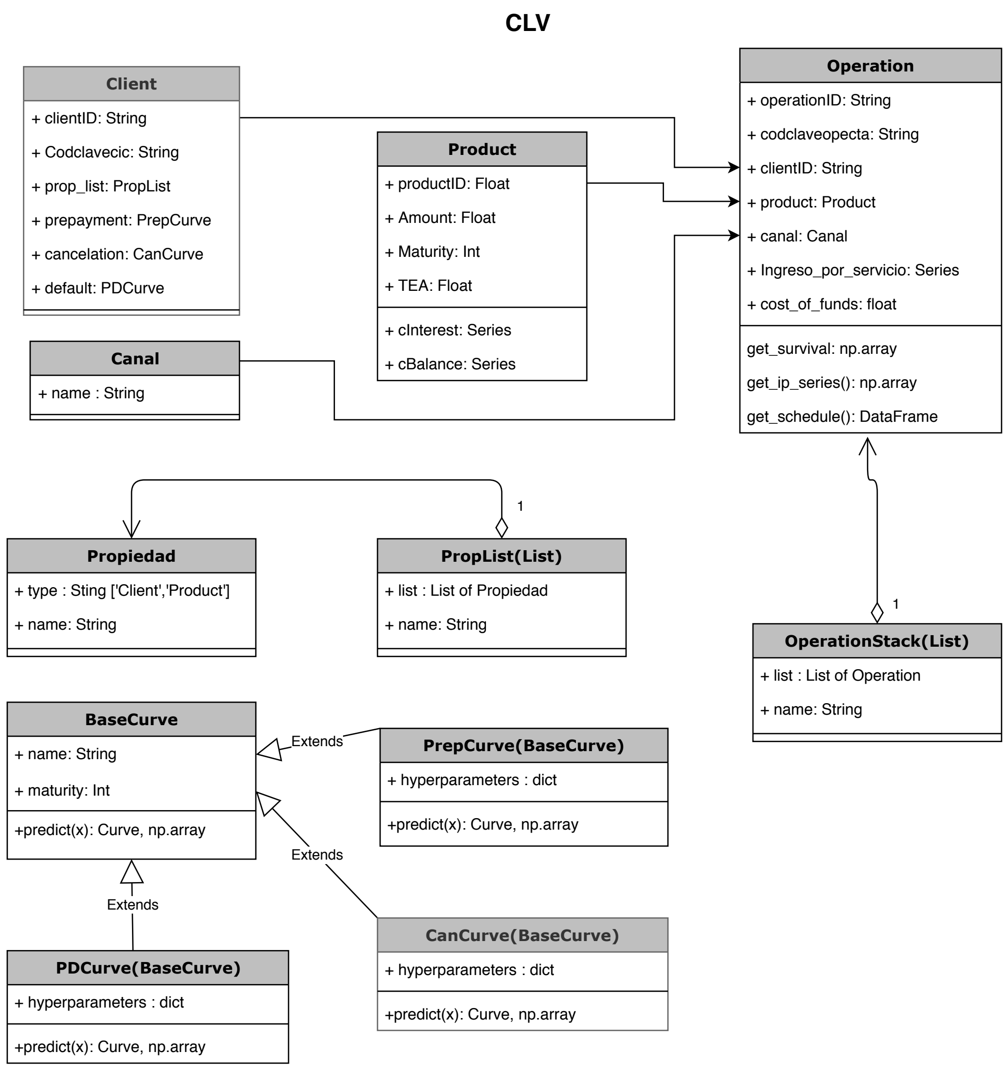

Figure 5.1: CLV Class Diagram

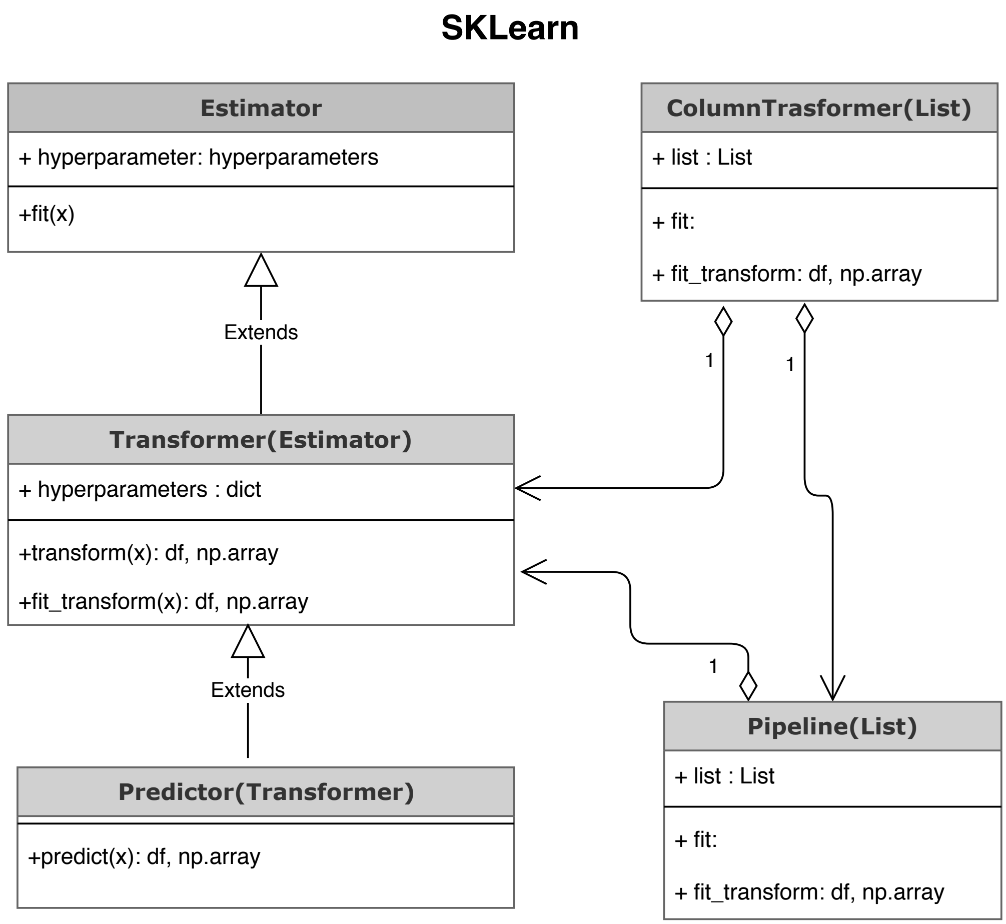

Figure 5.2: SKLearn Class Diagram

```python
import pandas as pd
import numpy as np
import PyPricing as ppr

# Creating temporal curves for probability of default
pd_curva = ppr.PDCurva()
pd_curva = pd_curva.predict(x=[30, 10, 1])

# Creating client object with prop_list and bhv_curves
prop_list = ppr.PropList(dict_prop)
prop_list = ppr.PropList(dict_prop)
client = ppr.Client(name='Nombre', prop_list, bhv_curves)

# Creating product object
product = ppr.Product(prop_list)
product.cInterest(T=70, r=0.02, per=12) # numpy array, series
product.cBalance(T=70, r=0.02, per=12)

# Alternative way to call pd_curva from client and product property list
pd_curva = pd_curva.predict(client,product)

contract = ppr.Contract(client ,product,dict_terms)
```

Listing 5.1: Python example