Nonlinear Equations

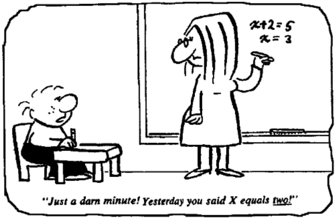
Bisection and Regula Falsi

Sachin Shanbhag

Department of Scientific Computing
Florida State University,
Tallahassee, FL 32306.

References

- S. Chapra and R. Canale, Numerical Methods for Engineers
- A. Greenbaum and T. Chartier, Numerical Methods: Design, Analysis, and Computer Implementation of Algorithms
- M. Heath, Scientific Computing: An Introductory Survey Check out the interactive educational modules* associated with "nonlinear equations"



*http://www.cse.illinois.edu/iem/

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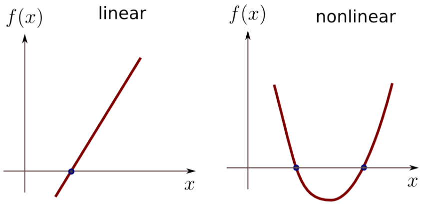
Motivation

- Nonlinear equations, $f(x) = 0$, arise in numerous applications
- Unlike linear equations, $f(x) = mx + c = 0$, they can have multiple solutions or *roots*



- Often we are after one of the many solutions
- We specify which one by prescribing a range (bisection and regula falsi), or an initial guess (Newton's and secant methods)

Motivation

- Consider the problem of the sky-diver. We found:

$$v(t) = \sqrt{\frac{mg}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

- Suppose, you are a parachute company (interested in controlling c_d). You want to ensure that the velocity after 20 seconds is $v(t = 20s) = 10$ m/s.
- Assume $m = 70$ kg, and $g = 9.8$ m/s^2. Substituting:

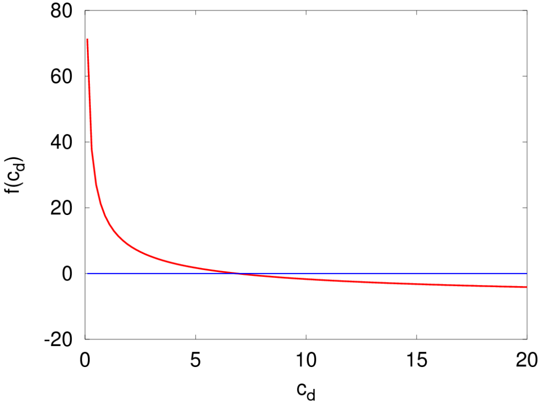
$$10 = \sqrt{\frac{70 \times 9.8}{c_d}} \tanh\left(\sqrt{\frac{9.8 c_d}{70}} 20\right)$$

Motivation

- That is, we want to solve the nonlinear equation:

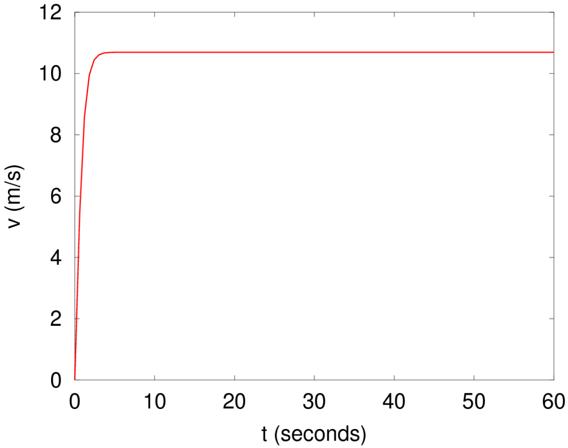
$$f(c_d) = \frac{26.1916}{\sqrt{c_d}} \tanh\left(7.4833\sqrt{c_d}\right) - 10 = 0$$

- Let us first plot $f(c_d)$

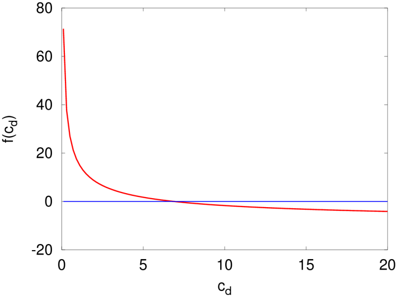


Motivation

- Let us double check that these numbers make sense. Hence replot $v(t)$ with $c_d = 6$

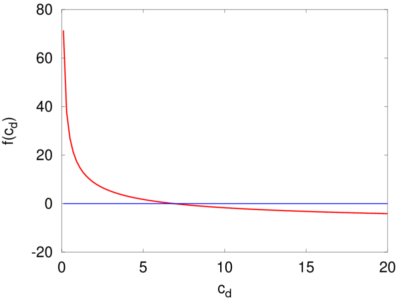


Bisection Method



- Observe from the example that the function changes signs near the root (say x^*)
- As the x changes from $a = 1$ to $b = 20$, the function $f(x)$ goes from being positive to negative.
- We say that the root x^* of $f(x)$ is *bracketed* by a and b.

Bisection Method



- More formally, a root x^* of a continuous function $f(x)$ is bracketed by a and b, if for $a \leq x^* \leq b$, $f(a)f(b) < 0$
- Bisection proceeds by dividing the interval $[a, b]$ by half in every iteration
- Once the root is bracketed, bisection is guaranteed to converge to the solution

Bisection Algorithm

1. Get the bracketing interval $[a, b]$ by numerical experimentation or prior knowledge of the behavior of $f(x)$.
2. Propose initial estimate of the root as

$$x_r = a + \frac{b-a}{2} = \frac{a+b}{2}.$$

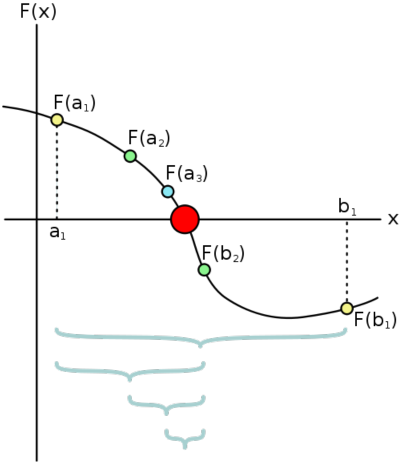
3. To determine which subinterval ($[a, x_r]$ or $[x_r, b]$) evaluate $f(a)f(x_r)$

$$\text{if } f(a)f(x_r) \begin{cases} < 0, & \implies x^* \in [a, x_r], \text{ set } b = x_r \\ = 0, & \implies x^* = x_r \\ > 0, & \implies x^* \in [x_r, b], \text{ set } a = x_r \end{cases}$$

4. Goto step 2 and repeat until $x^* = x_r$ or $b - a$ is small enough.

Bisection Algorithm

- Graphically,[†]



[†]wikipedia.org

Example

- Let us use this method to solve our equation (sub: x for c_d), with $a = 1$ and $b = 20$

$$f(x) = \frac{26.1916}{\sqrt{x}} \tanh\left(7.4833\sqrt{x}\right) - 10 = 0$$

i	a	b	x_r	ϵ_a	ϵ_t
0	1.0000	20.0000	10.5000		53.06
1	1.0000	10.5000	5.7500	82.61	16.18
2	5.7500	10.5000	8.1250	29.23	18.44
3	5.7500	8.1250	6.9375	17.12	1.13
4	5.7500	6.9375	6.3438	9.36	7.53
5	6.3438	6.9375	6.6406	4.47	3.20
7	6.7891	6.9375	6.8633	1.08	0.05
12	6.8586	6.8633	6.8610	0.03	0.01

Example

- ϵ_t is the true relative error:

$$\epsilon_t = 100 \times \left| \frac{x_r^{new} - x^*}{x_r^{new}} \right|$$

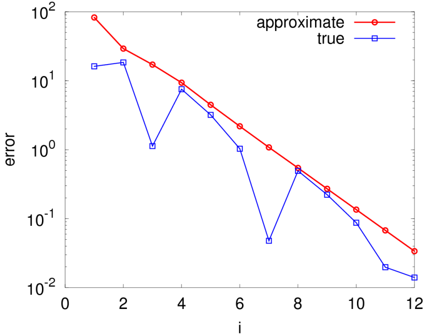
- However, x^* is not known in advance, hence compute the approximate relative error:

$$\epsilon_a = 100 \times \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right|$$

- Can use these estimates of error to determine stopping criterion, say $\epsilon_a < 10^{-4}$

Example

- Note the linear "convergence" of ϵ_a



- ϵ_t generally is not as smooth, but here $\epsilon_t < \epsilon_a$. So using ϵ_a to prescribe a stopping criteria is conservative.

Bisection: Convergence

- Let us say that we set ϵ_{stop} as the stopping criteria.
- That is we stop interval halving, when the root has been narrowed down to a subinterval of size $2\epsilon_{stop}$.
- After i steps the size of the subinterval is $(b-a)/2^i$. Hence we want:

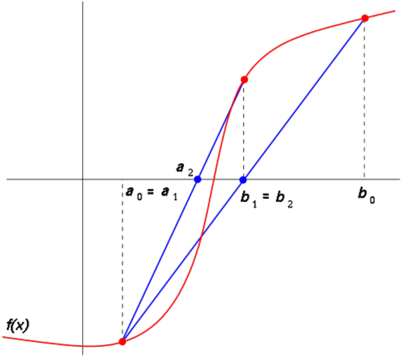
$$
\begin{aligned}
\frac{|b-a|}{2^i} &\leq 2\epsilon_{stop} \\
\frac{|b-a|}{\epsilon_{stop}} &\leq 2^{i+1} \\
i &\geq \log_2\left(\frac{|b-a|}{\epsilon_{stop}}\right) - 1
\end{aligned}
$$

Regula Falsi

- In bisection method, we do not use the actual magnitudes of $f(x)$, only the signs
- Can we use the magnitudes to improve convergence?
- This idea leads to Regula Falsi, or the method of false position
- Like bisection, we first need an interval $[a, b]$ that brackets the root x^*.
- Instead of halving the interval every iteration, we "connect" the end-points $(a, f(a))$ and $(b, f(b))$ and use the point at which the line intersects the x-axis as the guess x_r.

Regula Falsi

- Graphically,[‡]



- The only difference between this method and bisection is how x_r is determined

[‡]wikipedia.org

Regula Falsi

- By considering "similar triangles"

$$\frac{f(a)}{x_r - a} = \frac{f(b)}{x_r - b}$$

- Solving for x_r leads to

$$x_r = \frac{f(b)a - f(a)b}{f(b) - f(a)}$$

- Algorithm is exactly the same as bisection except step 2, where x_r is determined by the formula above

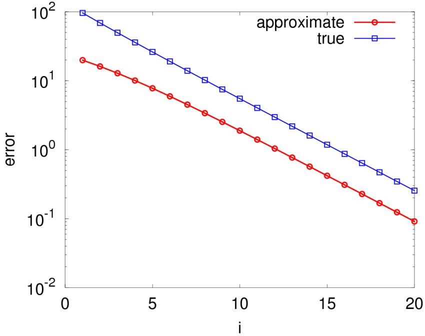
Regula Falsi

- Guaranteed to converge like bisection
- However, convergence can be poor, especially if the same end point is repeatedly selected
- Skydiving parachute example:

i	a	b	x_r	ϵ_a	ϵ_t
1	1.0000	16.1286	13.4534	19.89	96.11
2	1.0000	13.4534	11.5843	16.13	68.87
3	1.0000	11.5843	10.2655	12.85	49.64
4	1.0000	10.2655	9.3268	10.06	35.96
5	1.0000	9.3268	8.6538	7.78	26.15
10	1.0000	7.3737	7.2371	1.89	5.50
15	1.0000	6.9701	6.9410	0.42	1.18
20	1.0000	6.8838	6.8775	0.09	0.26

Error

- Decay is slower than bisection and $\epsilon_t > \epsilon_a$.



- It is possible to combine regula falsi and bisection to create a faster method
- Switch back to bisection if one of the endpoints remains unchanged for a few iterations (also google "Illinios method")

Taylor Series

- The most-used theorem in numerical analysis
- A common form involves expanding $f(x)$ around $x = a$

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

- Higher derivatives of $f(x)$ at the point a tell me something about the function far away from a. If you know all the (infinite) derivatives at $x = a$, then you effectively know the "function" $f(x)$

Taylor Series with Remainder

- A far more useful form is Taylor series with remainder

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{n+1!}(x-a)^{n+1} \end{aligned}$$

  for some $\xi \in (x, a)$.
- An equivalent form which is sometimes also useful is:

$$\begin{aligned} f(x+h) &= f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \cdots \\ &\quad + \frac{f^{(n)}(x)}{n!}h^n + \frac{f^{(n+1)}(\xi)}{n+1!}h^{n+1} \end{aligned}$$

  for some $\xi \in (x, x+h)$.

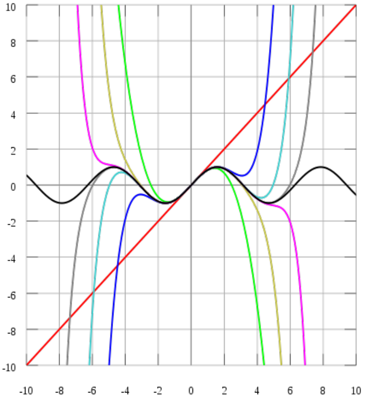
Taylor Series

- For the Taylor series to be written out, the function has to be sufficiently smooth (all the derivatives in the expansion should exist)
- If h is small, then h^n rapidly decays to zero. This allows us to approximate functions by truncating the series after a certain number of points
- Conversely, the remainder term tells us that as h becomes large, the truncated Taylor expansion represents the function poorly

Taylor Series Example

- Consider the Taylor expansion of $\sin(x)$[§]

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$



[§]wikipedia.org

Taylor Series Example 2

Problem: What is the remainder term of a third order Taylor expansion of $f(x) = e^{2x}$ around $a = 0$.

Solution: Here $a = 0$ and $n = 3$. Thus, we can write the Taylor expansion as:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(\xi)}{4!}x^4$$

$$f(0) = e^{2\cdot 0} = 1;\ f'(0) = 2e^{2\cdot 0} = 2;\ f^{(n)}(0) = 2^n;\ f^{(n)}(\xi) = 2^n e^{2\xi}$$

Thus,

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{4e^{2\xi}}{6}x^4$$

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