NAME: ______________________

ISC 4220
Algorithms 1

Feb 18, 2014

12:15 – 2:45 PM

Exam 1

INSTRUCTIONS

- By signing your name above, you agree to abide by the Honor Code.
- Turn cell-phones off.
- This is an open notes, open internet, open "everything" exam.
- You may not communicate or collaborate with any other person (talk, email, chat etc.) throughout the duration of this exam.
- If you write a computer program to solve a particular question, do not forget to report the iterations on your answer sheet.
- Unless directed otherwise, don't waste time on "efficiency" in this exam.
- **Show all your work on the exam sheet for full credit. No credit without adequate explanation.**

SCORE

Q#	Points	Max
1		25
2		25
3		15
4		20
5		15
Total		100

Q1. Linear Systems

Consider the linear system,

$$\begin{bmatrix} 1 & 1 & -1 \\ 10000 & 5000 & 5000 \\ 0.002 & 0.001 & 0.003 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0.5 \\ 15000 \\ 0.005 \end{bmatrix}$$

(i) Use row-scaling, and rewrite the equation above. What is the condition number of the matrix before and after row-scaling? [8 points]

(ii) We want to use Gauss elimination with partial pivoting to solve the scaled system. Based on the *first* pivot, would you exchange any rows? (**Note**: You are not asked to actually perform the elimination) [2 points]

(iii) The number of multiplications involved in multiplying a general $m \times m$ matrix with another $m \times m$ matrix is m^3.

How many multiplications are required to multiply an $m \times m$ *upper-triangular* matrix with another $m \times m$ *upper-triangular* matrix? (**Hint**: It will be cheaper!) [15 points]

Solution:

Q2. Linear Systems

The following system of equations arises in circuit modeling:

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \mathbf{V} = \begin{bmatrix} -2.01 \\ -0.01 \\ -0.01 \\ -0.01 \\ -0.01 \\ -3.01 \end{bmatrix}$$

We want to solve the problem using (a) Gauss-Seidel, and (b) successive over-relaxation (SOR), with an initial guess of $\mathbf{V}^{(0)} = [0\ 0\ 0\ 0\ 0\ 0]^T$.

Stop the iterations once the convergence criterion $||\mathbf{V}^{(k+1)} - \mathbf{V}^{(k)}|| \leq 10^{-5}||\mathbf{V}^{(k+1)}||$ is satisfied.

1. Show the first three iterations using Gauss-Seidel, including the value of the relative error $||\mathbf{V}^{(k+1)} - \mathbf{V}^{(k)}||/||\mathbf{V}^{(k+1)}||$. [8 points]

2. How many iterations of Gauss-Siedel are required to solve the system? [2 points]

3. For the following values of ω, report the number of iterations required for SOR to converge: (a) 1.2, (b) 1.4, and (c) 1.6. [15 points]

   Note: Theoretically, the optimal value of ω for this system is expected to be $2/(1 + \sin \pi/7)$.

Solution:

Q3. Nonlinear Equations

The shape of a cable suspended between two poles is a "catenary", which can be described by the equation,

$$y = \cosh(x/a) - 1,$$

where a is the unknown parameter.

Suppose we want to find the a so that the maximum "sag" is 10 meters, when the distance between the poles is 200 meters (see figure below).

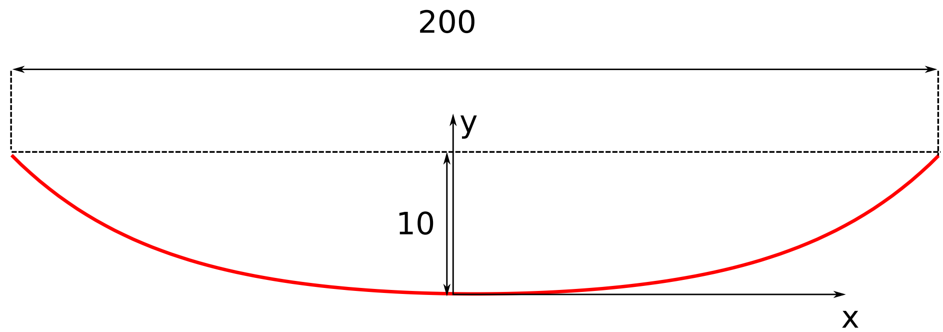


Image is not to scale!

(i) Set up the mathematical equation $f(a) = 0$ for this problem. [5 points]

(ii) Use bisection method to find the solution to the equation above, with an initial interval of $[20, 50]$. Report the first 3 iterations, and the final solution defined by the convergence criterion $|a_k - a_{k-1}| \leq 10^{-4}$. [10 points]

Q4. Nonlinear Equations

Consider the nonlinear equation

$$f(x) = (x-1)^m e^x = 0,$$

which has m repeated roots at $x^* = 1$.

Show that the convergence of the standard Newton's method for this problem is linear, by carrying out the following steps:

(a) Set up and simplify the standard Newton's iteration formula for this particular $f(x)$ to obtain a relationship between x_{j+1} and x_j.

(b) Subtract $x^* = 1$ from both sides of the equation derived in part (a). Set $E_j = x_j - x^* = x_j - 1$ to rewrite the equation in terms of E_{j+1} and E_j, instead of x_{j+1} and x_j.

(c) In the limit $m \gg E_j$, simplify the relationship obtained in part (b) to obtain

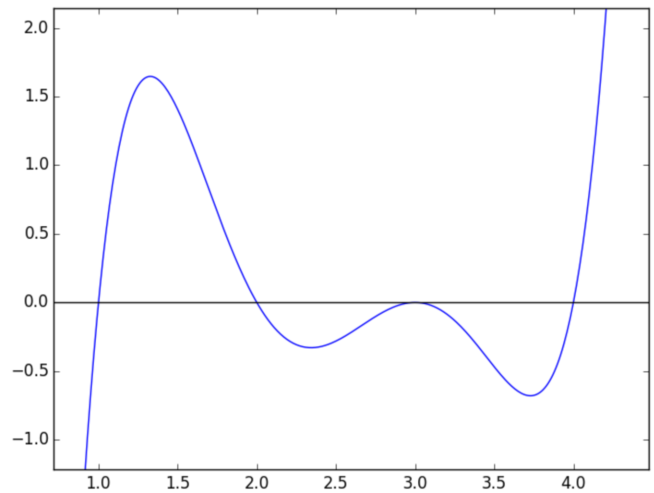
$$E_{j+1} = \left(1 - \frac{1}{m}\right) E_j.$$

[20 points]

Q5. Nonlinear Equations

In neuronal dynamics, molecular activity occurs at various timescales. To solve these "stiff" problems we use what are called adaptive implicit methods which require a solution of a nonlinear equation for each iteration.

Suppose, you are given a nonlinear function $f(x)$ from one such iteration which looks like:



For each root in the given interval, describe:

(a) the method would you use to numerically locate it,

(b) the initial conditions you would use, and

(c) the expected rate of convergence.

[15 points]

Note: You are not required to carry out any calculation for this question.