Linear Systems

Norms and Conditioning

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References

- S. Chapra and R. Canale, Numerical Methods for Engineers
- A. Greenbaum and T. Chartier, Numerical Methods: Design, Analysis, and Computer Implementation of Algorithms
- M. Heath, Scientific Computing: An Introductory Survey
- C. Moler, Numerical Computing with MATLAB
- wikipedia.org

How good is a solution?

- Consider the computed solution $\hat{\mathbf{x}}$ to the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is the "true" solution.
- There are ways we can use to measure how good a solution is:
- Residuals
$$\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$$
- Error
$$\epsilon = \mathbf{x} - \hat{\mathbf{x}}$$
- Typically, we don't know the true solution, hence we may think that a small residual is a good way of measuring whether we have a "good" solution.
- This is a bad idea!

Motivating Example

- Consider the linear system

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 0.913 & 0.659 \\ 0.457 & 0.330 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.254 \\ 0.127 \end{bmatrix} = \mathbf{b}$$

- Consider two approximate solutions

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} -0.0827 \\ 0.5000 \end{bmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}$$

- Turns out that

$$\mathbf{r}_1 = \begin{bmatrix} 5.1 \times 10^{-6} \\ -2.1 \times 10^{-4} \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} 1.6 \times 10^{-3} \\ 7.9 \times 10^{-4} \end{bmatrix}$$

- Which solution is better? (exact solution is $\mathbf{x} = [1, -1]^T$)

Motivating Example

- As Cleve Moler of Matlab puts it:

  > *It is probably the single most important fact that we have learned about matrix computation since the invention of the digital computer: Gaussian elimination with partial pivoting is guaranteed to produce small residuals.*

- The matrix in the example above is nearly singular (the second equation is nearly $1/2$ of the first). We need to describe "nearly" singular more quantitatively.
- The relationship between the size of the residual and the size of the error is determined in part by a quantity known as the condition number of the matrix which measures how close to singular the matrix is.

Norms

Definition

A *norm* is a real-valued function that provides a measure of size or "length" of multicomponent mathematical entities such as vectors and matrices

- Let us first consider vector norms, and then generalize them to matrix norms.
- The p-norm of a n-vector, $\mathbf{x}$ is given by

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- There are three important special cases

Vector Norms

- The 1-norm ($p = 1$) is sum of the absolute values (Manhattan norm)

$$\|\mathbf{x}\|_1 := \sum_{i=1}^{n} |x_i|$$

- The 2-norm is the familiar Euclidean norm

$$\|\mathbf{x}\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

- The ∞-norm can be viewed as the limiting case of $p \to \infty$

$$\|\mathbf{x}\|_\infty := \max\left(|x_1|, \ldots, |x_n|\right)$$

- As p increases from 1 to infinity, the "weight" of the largest components influences the norm more strongly.

Example

Problem: Consider the vector $\mathbf{x} = [-1.6, 1.2]^T$. Find the 1, 2, and ∞ norms

Solution:

$$\|\mathbf{x}\|_1 = |-1.6| + |1.2| = 2.8$$

$$\|\mathbf{x}\|_2 = \sqrt{(-1.6)^2 + (1.2)^2} = 2.0$$

$$\|\mathbf{x}\|_\infty = \max(|-1.6|, |1.2|) = 1.6$$

In general, for any n-vector $\mathbf{x}$, $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$

Matrix Norms

- All of the matrix norms commonly used are defined in terms of a vector norm. Hence they are often called *induced* norms

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$$

- Some of the common norms are easy to compute:

$$\|\mathbf{A}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$$

(maximum absolute column sum)

$$\|\mathbf{A}\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

(maximum absolute row sum)

Matrix Norms

- Unfortunately the 2-norm is not as easy to compute:

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})},$$

  where $\lambda_{\max}$ is the maximum eigenvalue.

- Example: Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & 2 \\ 1 & 2 & 3 & 3 \\ 0 & 6 & 1 & 2 \end{bmatrix}$$

  Find the 3 common norms of $\mathbf{A}$.

Matrix Norms

- Sum all the columns

$$||\mathbf{A}||_1 = \max(6, 13, 11, 8) = 13$$

- Sum all the rows

$$||\mathbf{A}||_\infty = \max(9, 11, 9, 9) = 11$$

- Using matlab, `sqrt(max(eig(A'*A)))` or `norm(A,2)`

$$||\mathbf{A}||_2 = 9.8956$$

- For matrix norms,

$$||\mathbf{A}||_2^2 \leq ||\mathbf{A}||_1 ||\mathbf{A}||_\infty.$$

Condition Number

- Definition of the condition number of a matrix

$$\text{cond}(\mathbf{A}) = ||\mathbf{A}|| \cdot ||\mathbf{A}^{-1}||$$

- It can be shown that for $\mathbf{Ax} = \mathbf{b}$, perturbations in $\mathbf{A}$ and $\mathbf{b}$ can lead to perturbations in $\mathbf{x}$

$$\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \leq \text{cond}(\mathbf{A}) \left(\frac{||\Delta\mathbf{A}||}{||\mathbf{A}||} + \frac{||\Delta\mathbf{b}||}{||\mathbf{b}||} \right)$$

- Round-off error in floating point numbers is a natural source of perturbation.
- The condition number of a matrix $\mathbf{A}$ tells us how susceptible the problem $\mathbf{Ax} = \mathbf{b}$ is to round off error

Condition Number: Practical Implication

- Suppose $||\Delta \mathbf{b}|| = 0$ and $\mathbf{A}$ is known to t-digit precision
- That is rounding errors are of the order of 10^{-t}.
- If $\text{cond}(\mathbf{A}) = 10^c$, then the solution $\mathbf{x}$ is only valid to $t - c$ digits (rounding errors of order 10^{c-t}).

Problem: Consider a single precision floating point system with $t = 8$ digits of accuracy. How many digits of the solution $\mathbf{x}$ to the problem $\mathbf{Ax} = \mathbf{b}$ would you trust if:

- $\text{cond}(\mathbf{A}) = 10^3$
- $\text{cond}(\mathbf{A}) = 10^9$

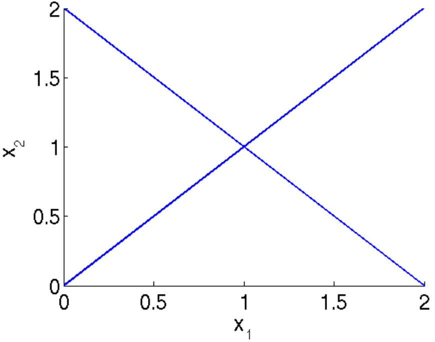
What if you were on a double precision machine ($t \approx 16$)?

Condition Number: Insight

- Consider a small 2×2 problem $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- The condition number (2-norm) of $\mathbf{A}$ is 1.

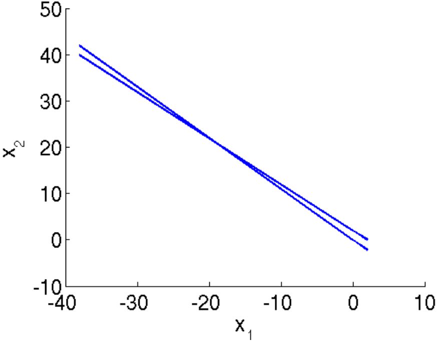


Condition Number: Insight

- Let us change the numbers to make the problem more ill-conditioned

$$\begin{bmatrix} 1 & 1 \\ 1.05 & 0.95 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- The condition number (2-norm) of $\mathbf{A}$ is about 40.



Condition Number: Insight

- As the two lines become more "parallel" to each other the condition number increases
- Finding the intersection (the solution to $Ax = b$) becomes harder, as is visually apparent in this simple 2D example
- In the limit that the two lines become parallel, the matrix becomes singular, and the condition number increases to infinity
- You can play around with 2×2 matrices of different condition numbers, and see how the solution is affected for different levels of precision at Heath's interactive educational modules site.*

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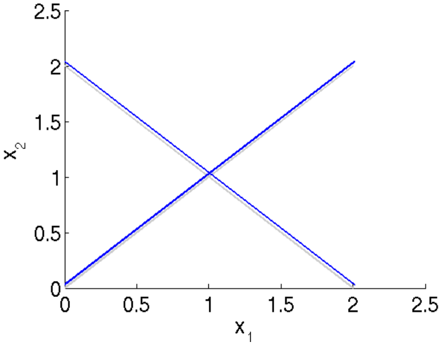
* `http://web.engr.illinois.edu/~heath/iem/linear_equations/condition_number/`

Perturb Well-Conditioned Problem

- Consider a small perturbation to $\mathbf{A}$ and $\mathbf{b}$.

$$\begin{bmatrix} 1.02 & 0.97 \\ 0.98 & -1.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.04 \\ -0.03 \end{bmatrix}$$

- Perturbed $\text{cond}(\mathbf{A}) = 1.02$. The solution changes from [1, 1]' to [1.04, 1.01]'.

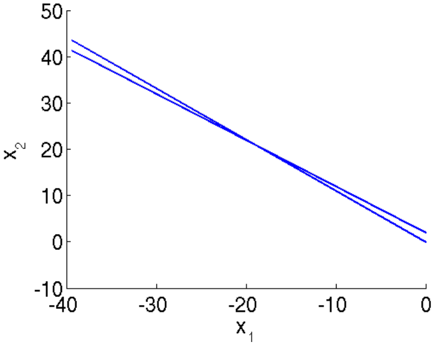


Perturb Ill-Conditioned Problem

- Consider a small perturbation to $\mathbf{A}$ and $\mathbf{b}$

$$\begin{bmatrix} 1.01 & 0.97 \\ 1.02 & 0.98 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.04 \\ -0.03 \end{bmatrix}$$

- Perturbed $\text{cond}(\mathbf{A}) = 9900$. The solution changes from [-19, 21]' to [5070 -5280]'.



Moral

Given a linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

If $\mathbf{A}$ is a well-conditioned, small perturbations in $\mathbf{A}$ cause only small perturbations in the solution $\mathbf{x}$.

If $\mathbf{A}$ is a ill-conditioned, small perturbations in $\mathbf{A}$ can cause major perturbations in the solution $\mathbf{x}$.

To figure out whether a matrix is well- or ill-conditioned, one needs to compare $c = \log_{10} \mathsf{cond}(\mathbf{A})$ with t, the accuracy of the floating point system in terms of the number of digits.

Epilog: Residuals

- For an approximate solution $\hat{\mathbf{x}}$, consider the residual $\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$
- If $||\mathbf{r}||$ is small, we found that $\hat{\mathbf{x}}$ is may not necessarily be a good solution.
- Formally,

$$\begin{aligned} \mathbf{r} &= \mathbf{b} - \mathbf{A}\hat{\mathbf{x}} \\ \mathbf{0} &= \mathbf{b} - \mathbf{A}\mathbf{x} \\ \mathbf{r} &= \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) \end{aligned}$$

- Or

$$||\mathbf{x} - \hat{\mathbf{x}}|| = ||\mathbf{A}^{-1}\mathbf{r}||$$