ISC 4220
Algorithms 1

Least Squares Approximation

1. Let us consider a simplified version of algorithm that goes into ranking college football teams for the Bowl Championship Series. This formula for computer ranking the teams was devised by Kenneth Massey[1] and constitutes a part of the overall scheme that goes into determining the rankings that come out regularly during football season.

Suppose, we have 4 teams, which we will call T_1 through T_4 for simplicity. Assume that the following outcomes occur:

Game	Score	Difference
$T_1 - T_2$	21-17	4
$T_3 - T_1$	27-18	9
$T_1 - T_4$	16-10	6
$T_3 - T_4$	10-7	3
$T_2 - T_4$	17-10	7

We can use this to construct a linear system by assigning ranking points r_i to team T_i via:

$$\begin{aligned} r_1 - r_2 &= 4 \\ r_3 - r_1 &= 9 \\ r_1 - r_4 &= 6 \\ r_3 - r_4 &= 3 \\ r_2 - r_4 &= 7 \end{aligned}$$

This is an overdetermined system and does not even have a unique least squares solution because we could always add a constant c to any solution $[r_1 + c, r_2 + c, r_3 + c, r_4 + c]^T$ and still satisfy all the equations equally well. This can be fixed by adding another equation like,

$$r_1 + r_2 + r_3 + r_4 = 20.$$

Given these equations, use linear least squares to rank the 4 teams.

2. This is an exam problem is from a 2014.

Consider the following data:

i	x_i	y_i
1	0.00	2.10
2	0.25	3.70
3	0.50	6.26
4	0.75	10.03
5	1.00	16.31

[1] `masseyratings.com`

We have two models to capture the dependence of y on the independent variable x.

$$
\begin{aligned}
m_1(x) &= a_1 x + b_1 \exp(2x) \qquad (1)\\
m_2(x) &= a_2 x + 2\exp(b_2 x) \qquad (2)
\end{aligned}
$$

(i) The coefficients a_1 and b_1 in model $m_1(x)$ can be determined by linear least-squares. Find a_1 and b_1. [10 pts]

(ii) The coefficients a_2 and b_2 in model $m_2(x)$ cannot be determined by linear least-squares. Let us consider the following cost function:

$$
\Phi(a_2, b_2) = \sum_{i=1}^{5} \left(y_i - m_2(x_i)\right)^2 . \qquad (3)
$$

Evaluate the gradient, [15 pts]

$$
\nabla\Phi = \begin{bmatrix} \frac{\partial \Phi}{\partial a_2} \\ \frac{\partial \Phi}{\partial b_2} \end{bmatrix}
$$

(iii) Use the BFGS method to find the a_2 and b_2 that minimizes $\Phi(a_2, b_2)$. Use an initial guess of $[a_2, b_2]^T = [-1, 1]$, and a tolerance of 10^{-4} on the norm of the gradient. Report the following: [15 pts]

(a) first two iterations,

(b) the converged solution,

(c) the norm of the gradient at the solution.