NAME: ______________________

ISC 4220
Algorithms 1

Feb 19, 2014

10:00 – 12:15 PM

Exam 1

INSTRUCTIONS

- By signing your name above, you agree to abide by the Honor Code.
- Turn cell-phones off.
- This is an open notes, open internet, open "everything" exam.
- You may not communicate or collaborate with any other person (talk, email, chat etc.) throughout the duration of this exam.
- Show *all* your work on the exam sheet for full credit.
- If you write a computer program to solve a particular question, do not forget to report the iterations on your answer sheet.
- No credit without adequate explanation

SCORE

Q#	**Points**	**Max**
1		30
2		40
3		20
4		10
Total		100

Q1. Linear Systems

Consider a linear system $\mathbf{Ax} = \mathbf{r}$, where $\mathbf{A}$ is an $n \times n$ tridiagonal matrix.

$$\begin{bmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & b_n & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \\ \vdots \\ r_n \end{bmatrix}$$

An algorithm to solve such a linear system is as follows

1. Set $\alpha_1 = 1/a_1$; $y_1 = r_1$
2. For $i = 2, 3, ..., n$, compute $\begin{cases} \alpha_i = 1/(a_i - b_i \alpha_{i-1} c_{i-1}) \\ y_i = r_i - b_i \alpha_{i-1} y_{i-1} \end{cases}$
3. Set $x_n = y_n \alpha_n$
4. For $i = n-1, ...1$, compute $x_i = (y_i - x_{i+1} c_i) \alpha_i$

(i) Find the *exact* number of (a) add/subtract and (b) multiply/divide operations in the algorithm above. [20 pts]

(ii) The algorithm above is not fool-proof. Give a concrete example of a well-conditioned matrix $\mathbf{A}$ (of size, say 3×3), for which the algorithm fails. [5 pts]

(iii) Would you recommend carrying out partial pivoting and/or row-scaling to improve the algorithm? Explain your answer. [5 pts]

Solution:

Q2. Nonlinear equations

(i) The function,

$$f(x) = x\tan(x) - x^2,$$

has a root between 3 and 4.6. Use Newton's method to find it with an initial guess of $x_0 = 4.6$. Show at least the first three iterations. [10 pts]

(ii) If you rewrite the function as a fixed-point iteration,

$$x_{i+1} = x_i^2/\tan(x_i),$$

can you find this root? Use the convergence criteria to explain your answer. **Note**: You don't have to carry out the iteration [10 pts]

(iii) If you use bisection method with an initial interval of $x_l = 3$ and $x_h = 7.8$, strictly speaking, is the method expected to work? Does it? Explain. [10 pts]

(iv) If you pick any number on your calculator (set to radians not degrees) and repeatedly press the `cos` key, what do you get? Can you explain why? [10 pts]

Solution:

Q3. Linear Systems

Consider the linear system $\mathbf{Ax} = \mathbf{b}$,

$$\begin{bmatrix} 10.0 & 1.0 \\ 3.0001 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11.0 \\ 3.3 \end{bmatrix}$$

(i) How many solutions should we expect for the system above. Explain? [5 pts]

(ii) If you use Matlab to solve the system, what answer, $\hat{\mathbf{x}}$, do you get? What is the norm associated with the residual of $\hat{\mathbf{r}} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$? [5 pts]

(iii) Based on the condition number of the matrix $\mathbf{A}$, what can you say about $\hat{\mathbf{x}}$? [5 pts]

(iv) Estimate the norm of the true error. [5 pts]

Solution:

Q4. Floating Point Numbers

We want to test whether two real numbers a and b are equal to each other on a computer using double precision arithmetic.

(i) Discuss why using a test like `if (a == b)` is a bad idea. [2 pts]

(ii) Propose an alternative test to check for equality. [3 pts]

(iii) Mention one potential drawback of the proposal above? If there aren't any, say so. [5 pts]

Solution: