Nuclear Reactor Thermal-Hydraulics

NUGN520 - Homework

By

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CHAPTER 1

REACTOR HEAT GENERATION

Several exercises from the book written by M. M. El Wakil [1] are tackled in this homework. The problems in this section relate to the seventh and eigth chapter of the book, covering the subject of heat conduction in reactor elements.

1.1 [7-6] - Rectangular fin

1.1.1 Problem

A very long fin is rectangular in cross-section $0.48 * 0.24$ *in. It generates* 2×10^6 $Btu.h^{-1}.ft^{-3}$*. The fin base is at* $1000°F$*. It is cooled by a gas at* $600°F$ *with a uniform heat transfer coefficient* 100 $Btu.h^{-1}.ft^{-2}.°F^{-1}$*. Using a network with* $\Delta x = 0.12$ *in., write the necessary set of finite difference equations for the nodal points and solve by any one of the techniques at your command. k for the fin material* $= 10$ $Btu.h^{-1}.ft^{-1}.°F^{-1}$

1.1.2 Solution

First, one can note that the problem is missing a graph. The considered geometry is given in Figure 1.1.

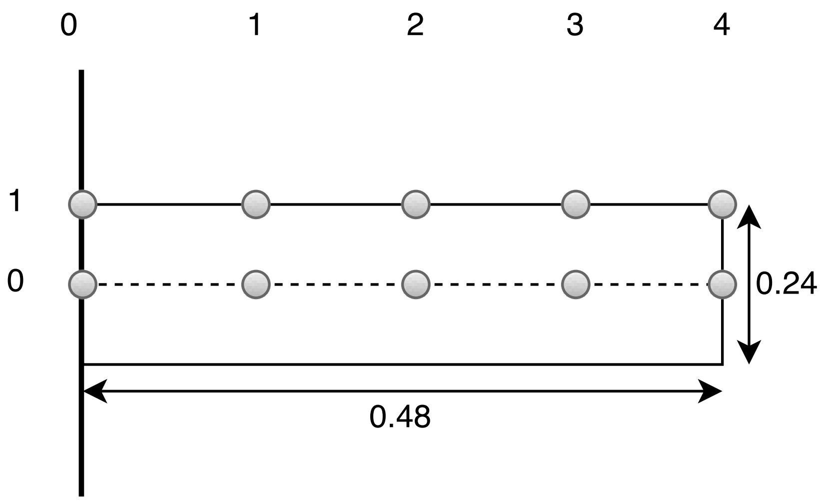

Figure 1.1: Representation of the problem geometry

We can identify four types of nodes: inner, y-bound (next to the upper boundary), x-bound (next to the right boundary) and corner.

The inner node temperature T_n is given by Equation 1.1.

$$T_n = \frac{T_{n-\Delta x} + T_{n+\Delta x} + T_{n-\Delta y} + T_{n+\Delta y}}{4} + \frac{\Delta T_g}{2} \tag{1.1}$$

In our case, we can see that $T_{n-\Delta y} = T_{n+\Delta y}$ by symmetry for the inner nodes. Consequently, we obtain Equation 1.2.

$$T_n = \frac{T_{n-\Delta x} + T_{n+\Delta x} + 2T_{n+\Delta y}}{4} + \frac{\Delta T_g}{2} \tag{1.2}$$

For the x-bound nodes, we can write Equation 1.3.

$$T_n = \frac{T_{n-\Delta x} + T_{n+\Delta y} + Bi_{\Delta x} T_f}{2 + 2Bi_{\Delta x}} + \frac{\Delta T_g}{2 + Bi_{\Delta x}} \tag{1.3}$$

For the y-bound nodes, we can write Equation 1.4

$$T_n = \frac{T_{n-\Delta x} + T_{n+\Delta x} + 2T_{n+\Delta y} + 2Bi_{\Delta x} T_f}{4 + 2Bi_{\Delta x}} + \frac{\Delta T_g}{2 + Bi_{\Delta x}} \tag{1.4}$$

And finally, for the corner node, we can write (as seen in problem 7.1 previously) Equation 1.5.

$$T_n = \frac{T_{n-\Delta x} + T_{n+\Delta y} + 2Bi_{\Delta x} T_f}{2 + 2Bi_{\Delta x}} + \frac{\Delta T_g}{2 + 2Bi_{\Delta x}} \tag{1.5}$$

A few exceptions can be noted. Indeed, the two leftmost points, on the base of the fin, are given to be at $1000°F$.

We can consequently write the matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 0 & 1 & -0.5 & -0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{2Bi_{\Delta x}+4} & \frac{-1}{Bi_{\Delta x}+2} & 1 & 0 & \frac{-1}{2Bi_{\Delta x}+4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.25 & 0 & 1 & -0.5 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2Bi_{\Delta x}+4} & \frac{-1}{Bi_{\Delta x}+2} & 1 & 0 & \frac{-1}{2Bi_{\Delta x}+4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.25 & 0 & 1 & -0.5 & -0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{2Bi_{\Delta x}+4} & \frac{-1}{Bi_{\Delta x}+2} & 1 & 0 & \frac{-1}{2Bi_{\Delta x}+4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{Bi_{\Delta x}+2} & 0 & 1 & \frac{-1}{Bi_{\Delta x}+2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2Bi_{\Delta x}+2} & \frac{-1}{2Bi_{\Delta x}+2} & 1 \end{bmatrix}$$

$\frac{\Delta T_g}{2}$ can be calculated using $\Delta T_g = \frac{(\Delta x)^2 q'''}{2k}$. T_f, the temperature of the gas, is known. We can thus obtain b.

$$b = \begin{bmatrix} 1000 \\ 1000 \\ 7.2 \\ 229.5 \\ 7.2 \\ 229.5 \\ 7.2 \\ 229.5 \\ 229.5 \\ 330.5 \end{bmatrix}$$

The equation $A.x = b$ can thus be solved, using Python. The python script is given in 1.1.2.1. We obtain the following results (Equation 1.6):

$$\begin{aligned} T(0,0) &= 1000.0 \\ T(0,1) &= 1000.0 \\ T(1,0) &= 797.5 \\ T(1,1) &= 736.6 \\ T(2,0) &= 696.6 \\ T(2,1) &= 659.4 \\ T(3,0) &= 650.1 \\ T(3,1) &= 630.3 \\ T(4,0) &= 623.3 \\ T(4,1) &= 614.5 \end{aligned} \tag{1.6}$$

1.1.2.1 Python script

```
import numpy as np

m = 10    # size of system is m x m

dx = 0.12
deltax, k, qtriple, h = (dx/12)**2, 10., 2.e6, 100.    # BG units
tb = 1000.
ts = 600.
dtg = (0.5*deltax)*qtriple/k
bi = h * dx / k

amat = np.zeros((m,m))
b = np.zeros(m)

amat[0,0] = 1.
amat[1,1] = 1.
b[0] = tb
b[1] = tb

for i in range(2, m, 2):
    try:
        amat[i,i-2] = -0.25
        amat[i,i] = 1.
        amat[i,i+1] = -0.5
        amat[i,i+2] = -0.25
        b[i] = dtg/2.
    except IndexError:
        break

for i in range(3, m, 2):
    try:
        amat[i,i-2] = -1/(2*bi+4)
        amat[i,i-1] = -1/(bi+2)
        amat[i,i] = 1.
        amat[i,i+2] = -1/(2*bi+4)
        b[i] = (bi*ts + dtg)/(bi+2)
    except IndexError:
        break

amat[m-1,m-1] = 1.
amat[m-1,m-2] = -1/(2*bi+2)
amat[m-1,m-3] = -1/(2*bi+2)
b[m-1] = (2*bi*ts + dtg)/(2*bi+2)


amat[m-2,m-2] = 1.
amat[m-2,m-1] = -1/(bi+2)
amat[m-2,m-4] = -1/(bi+2)
b[m-2] = (bi*ts + dtg)/(bi+2)
print(b)
ysol = np.linalg.solve(amat,b)
```

```
print('Temperature solution:')

temp = []
for i in range(5):
    for j in range(2):
        temp.append("T(%s,%s)" % (i, j))

for i,t in enumerate(temp):
    print("%s = %.1f" % (t, ysol[i]))

# check of solution:
if not np.allclose(np.dot(amat, ysol), b):
    print("Solution does not match!")
```

1.2 [7-13] - Heat flux

1.2.1 Problem

A long fuel element has a rectangular cross-section $1*2$ *in. It generates* 1×10^6 $Btu.h^{-1}.ft^{-3}$ *and has a thermal conductivity of* 1.085 $Btu.h^{-1}.ft^{-1}.°F^{-1}$*. All surfaces are held at* $1000°F$*. Find the heat flux,* $Btu.h^{-1}.ft^{-2}$ *at the center point of each side using the approximate analytical solution of section 7-13.*

1.2.2 Solution

The book [1] gives us a solution for a long rectangular fuel element (Equations 7-34 to 7-44). It is given here by Equation 1.7.

$$T(x,y)=\frac{3}{4}\frac{q'''}{k(a^2+b^2)}(a^2-x^2)(b^2-y^2) \tag{1.7}$$

However, this solution is obtained for different boundary conditions ($T_s=0°F$). In our case, $T_s=1000°F$. Then, we can rewrite Equation 1.7 to Equation 1.8 to account for this different boundary condition.

$$T(x,y)=\frac{3}{4}\frac{q'''}{k(a^2+b^2)}(a^2-x^2)(b^2-y^2)+1000=C(a^2-x^2)(b^2-y^2)+1000 \tag{1.8}$$

The heat flux is given in the x-direction by Equation 1.9 and in the y-direction by Equation 1.10.

$$q''_x=-k\frac{\partial T}{\partial x} \tag{1.9}$$

$$q''_y=-k\frac{\partial T}{\partial y} \tag{1.10}$$

Consequently, we can write Equations 1.11 and 1.12.

$$q''_x=-2kCx(y^2-b^2) \tag{1.11}$$

$$q''_y=-2kCy(x^2-a^2) \tag{1.12}$$

At the center point of each side, we have Equations 1.13 and 1.14.

$$q''_x\Big|_{x=a,y=0} = 2kCab^2 \tag{1.13}$$

$$q''_y\Big|_{x=0,y=b} = 2kCba^2 \tag{1.14}$$

So, we obtain $q''_x = 2.5 \times 10^4\ Btu.h^{-1}.ft^{-2}$ and $q''_y = 5.0 \times 10^4\ Btu.h^{-1}.ft^{-2}$.

1.3 [8-1] - Cool down time

1.3.1 Problem

A 2 in. diam. steel ball initially at a uniform $850°F$, is suddenly subjected to an environment at $200°F$. The natural convection heat transfer coefficient is $2\ Btu.h^{-1}.ft^{-2}.°F^{-1}$. Find the time necessary for the ball to cool down to $300°F$.

1.3.2 Solution

The book [1] gives us the relationship between the temperature in a two-bodies problem, Equation 1.15.

$$\frac{T_f - T(t)}{T_f - T_i} = e^{-t/\tau} \tag{1.15}$$

Where:

$\tau = \frac{c_1 \rho_1 V_1}{h A_1}$
1 = index relative to the cooling body

Knowing that the cooling body is a steel ball, we can obtain its density and specific heat capacity. Knowing it is a sphere, we can obtain its surface area and volume. The heat transfer coefficient being known, we can thus calculate $\tau = \frac{0.12*490*\frac{4\pi*(0.0833)^3}{3}}{2*4\pi*(0.0833)^2} = 0.816$. Consequently, we have Equation 1.16.

$$\frac{200-300}{200-850} = 0.154 = e^{-t/\tau} \tag{1.16}$$

$$t = -\tau \ln(0.154) = 1.528\ h \tag{1.17}$$

The ball will cool down to $300°F$ in $1.528\ h$, or roughly an hour and a half.

1.4 [8-4] - Unsteady fuel element

1.4.1 Problem

A flat-plate fuel element $1.25 * 0.25$ *in. in cross section is initially at a uniform* $1000°F$*. Suddenly heat was generated at the rate of* 1.5×10^7 $Btu.h^{-1}.ft^{-3}$*. All surfaces were maintained at* $1000°F$*. Find by a numerical technique the time it takes the element to reach a maximum temperature* 99.5% *of the way to maximum steady-state temperature.* $k = 1.085$ $Btu.h^{-1}.ft^{-1}.°F^{-1}$*.* $c = 0.06$ $Btu.lb^{-1}.°F^{-1}$*.* $\rho = 740 lb.ft^{-3}$*.*

1.4.2 Solution

The surface temperatures staying at $T_s = 1000°F$, and using a symmetry argument, only five temperatures are to be obtained in a mesh with $\Delta x = \Delta y = 0.125$ in. For precision, I decided to refine the mesh (the impact is minimal), using $\Delta x = 0.0625$. Those are all temperatures in the inner part of the plate.

The book [1] tells us that in this case, the temperature at a time $t + \Delta t$ is given by Equation 1.18.

$$T_n^{t+\Delta t} = (1 - 4Fo)T_n^t + Fo(T_{n+\Delta x}^t + T_{n-\Delta x}^t + T_{n+\Delta y}^t + T_{n-\Delta y}^t) + 2Fo\Delta T_g \tag{1.18}$$

Consequently, we can obtain the next timestep temperature for each point in our design. This system is solved using the python script given in 1.4.2.1.

The steady-state solutions are obtained using the largest possible time steps, for $Fo = 0.25$.

We obtain the following steady-state temperature:

$$
\begin{aligned}
T(0,0) &= 1000.0\\
T(0,1) &= 1000.0\\
T(0,2) &= 1000.0\\
T(1,0) &= 1000.0\\
T(1,1) &= 1300.8\\
T(1,2) &= 1388.6\\
T(2,0) &= 1000.0\\
T(2,1) &= 1439.5\\
T(2,2) &= 1577.7\\
T(3,0) &= 1000.0\\
T(3,1) &= 1504.4\\
T(3,2) &= 1668.2\\
T(4,0) &= 1000.0\\
T(4,1) &= 1535.1\\
T(4,2) &= 1711.3\\
T(5,0) &= 1000.0\\
T(5,1) &= 1549.5\\
T(5,2) &= 1731.7\\
T(6,0) &= 1000.0\\
T(6,1) &= 1556.4\\
T(6,2) &= 1741.3\\
T(7,0) &= 1000.0\\
T(7,1) &= 1559.6\\
T(7,2) &= 1745.9\\
T(8,0) &= 1000.0\\
T(8,1) &= 1561.1\\
T(8,2) &= 1748.0\\
T(9,0) &= 1000.0\\
T(9,1) &= 1561.7\\
T(9,2) &= 1748.9\\
T(10,0) &= 1000.0\\
T(10,1) &= 1561.9\\
T(10,2) &= 1749.2
\end{aligned}
\tag{1.19}
$$

The 99.5% value of the maximum temperature is thus $1740.5°F$. By refining the time step, using $Fo = 0.1$, we can see that this happens at the 74th step. This corresponds to a time of $74 * \frac{(\Delta x)^2 * 0.1}{\alpha} = \frac{0.0289}{\alpha}$. The thermal diffusivity α can be obtained using the relation 1.20.

$$\alpha = \frac{k}{\rho c} \tag{1.20}$$

We obtain $\alpha = 0.024\ ft^{-2}$. This gives us a time $t = 1.2\ h$ necessary to reach 99.5% of the maximum steady temperature.

1.4.2.1 Python script

```
# initialize temp. values
t11, t21, t31, t41, t51, t61, t71, t81, t91, t101 = 1000., 1000., 1000., 1000., 1000., 1000., ↩
     1000., 1000., 1000., 1000.
t12, t22, t32, t42, t52, t62, t72, t82, t92, t102 = 1000., 1000., 1000., 1000., 1000., 1000., ↩
     1000., 1000., 1000., 1000.
dx = 0.125/2
deltax, k, qtriple = (dx/12)**2, 1.085, 1.5e7    # BG units
ts = 1000.
dtg2 = (0.5*deltax)*qtriple/(k)
fo=0.25
print('temperature solution (degrees F):')
# iterate:
for i in range(51):
    t11n = (1-4*fo)*t11+fo*(t21+2*ts+t12) + 2*fo*dtg2
    t21n = (1-4*fo)*t21+fo*(t31+t11+t22+ts) + 2*fo*dtg2
    t31n = (1-4*fo)*t31+fo*(t41+t21+t32+ts) + 2*fo*dtg2
    t41n = (1-4*fo)*t41+fo*(t51+t31+t42+ts) + 2*fo*dtg2
    t51n = (1-4*fo)*t51+fo*(t61+t41+t52+ts) + 2*fo*dtg2
    t61n = (1-4*fo)*t61+fo*(t71+t51+t62+ts) + 2*fo*dtg2
    t71n = (1-4*fo)*t71+fo*(t81+t61+t72+ts) + 2*fo*dtg2
    t81n = (1-4*fo)*t81+fo*(t91+t71+t82+ts) + 2*fo*dtg2
    t91n = (1-4*fo)*t91+fo*(t101+t81+t92+ts) + 2*fo*dtg2
    t101n = (1-4*fo)*t101+fo*(2*t91+ts+t102) + 2*fo*dtg2

    t12n = (1-4*fo)*t12+fo*(2*t11+ts+t22) + 2*fo*dtg2
    t22n = (1-4*fo)*t22+fo*(t12+t32+2*t21) + 2*fo*dtg2
    t32n = (1-4*fo)*t32+fo*(t22+t42+2*t31) + 2*fo*dtg2
    t42n = (1-4*fo)*t42+fo*(t32+t52+2*t41) + 2*fo*dtg2
    t52n = (1-4*fo)*t52+fo*(t42+t62+2*t51) + 2*fo*dtg2
    t62n = (1-4*fo)*t62+fo*(t52+t72+2*t61) + 2*fo*dtg2
    t72n = (1-4*fo)*t72+fo*(t62+t82+2*t71) + 2*fo*dtg2
    t82n = (1-4*fo)*t82+fo*(t72+t92+2*t81) + 2*fo*dtg2
    t92n = (1-4*fo)*t92+fo*(t82+t102+2*t91) + 2*fo*dtg2
    t102n = (1-4*fo)*t102+fo*(2*t92+2*t101) + 2*fo*dtg2
    # roll values
    t11, t21, t31, t41, t51, t61, t71, t81, t91, t101 = t11n, t21n, t31n, t41n, t51n, t61n, t71n, ↩
        t81n, t91n, t101n
    t12, t22, t32, t42, t52, t62, t72, t82, t92, t102 = t12n, t22n, t32n, t42n, t52n, t62n, t72n, ↩
        t82n, t92n, t102n
    if i % 5 == 0:
        print('number of time steps: ', i+1)
        print(t12, t22, t32, t42, t52, t62, t72, t82, t92, t102)
```

1.5 [H1] - Trigonomy solution

1.5.1 Problem

Let $T(x,y) = C\cos\left(\frac{\pi}{2a}x\right)\cos\left(\frac{\pi}{2b}y\right)$ and consider the steady-state heat conduction problem (7-34) and (7-35) of the textbook. (a) By using the approximate analytic method of section 7-13, complete the solution for the coefficient C that was begun in class. (b) Let $a = b$ so that the domain becomes a square. With x and y in terms of a, and T in units of $\frac{q'''a^2}{k}$, generate (i) a surface plot of the approximate solution, and (ii) a contour plot of the approximate solution. You may use software of your choice, whether it is Mathematica, Matlab, Python, or other.

1.5.2 Solution

We assume a solution of the form $T(x,y) = C\cos\left(\frac{\pi}{2a}x\right)\cos\left(\frac{\pi}{2b}y\right)$. This solution verifies Equation 1.21.

$$\int_0^b \frac{\partial T}{\partial x}\Big|_{x=a} dy + \int_0^a \frac{\partial T}{\partial y}\Big|_{y=b} dx + \frac{q'''ab}{k} = 0 \tag{1.21}$$

We can solve the partial differential first.

$$\frac{\partial T}{\partial x}\Big|_{x=a} = -C\frac{\pi}{2a}\sin\left(\frac{\pi}{2a}x\right)\cos\left(\frac{\pi}{2b}y\right)\Big|_{x=a} = -C\frac{\pi}{2a}\cos\left(\frac{\pi}{2b}y\right) \tag{1.22}$$

$$\frac{\partial T}{\partial x}\Big|_{x=a} = -C\frac{\pi}{2b}\cos\left(\frac{\pi}{2a}x\right) \tag{1.23}$$

Integrating, we obtain:

$$\int_0^b \frac{\partial T}{\partial x}\Big|_{x=a} dy = -C\int_0^b \frac{\pi}{2a}\cos\left(\frac{\pi}{2b}y\right) dy \tag{1.24}$$

$$\int_0^b \frac{\partial T}{\partial x}\Big|_{x=a} dy = -C\frac{b}{a} \tag{1.25}$$

$$\int_0^a \frac{\partial T}{\partial y}\Big|_{y=b} dx = -C\frac{a}{b} \tag{1.26}$$

Consequently, replacing in Equation 1.21 and reorganizing:

$$C = \frac{q'''(ab)^2}{k(a^2+b^2)} \tag{1.27}$$

If $a = b$, then we have $C = \frac{q'''a^2}{2k}$.

The surface and contour are plotted using the python script given in 1.5.2.1. They are visible on Figure 1.2 and Figure 1.3.

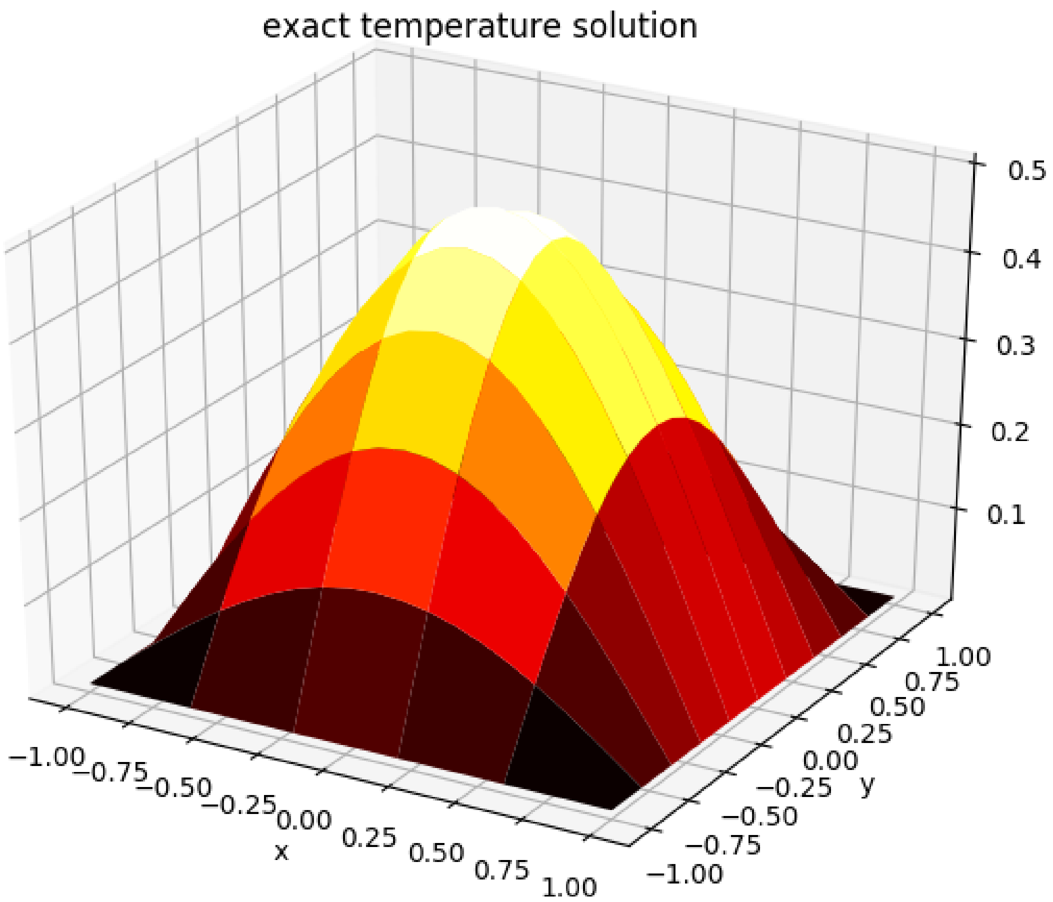

Figure 1.2: Surface plot

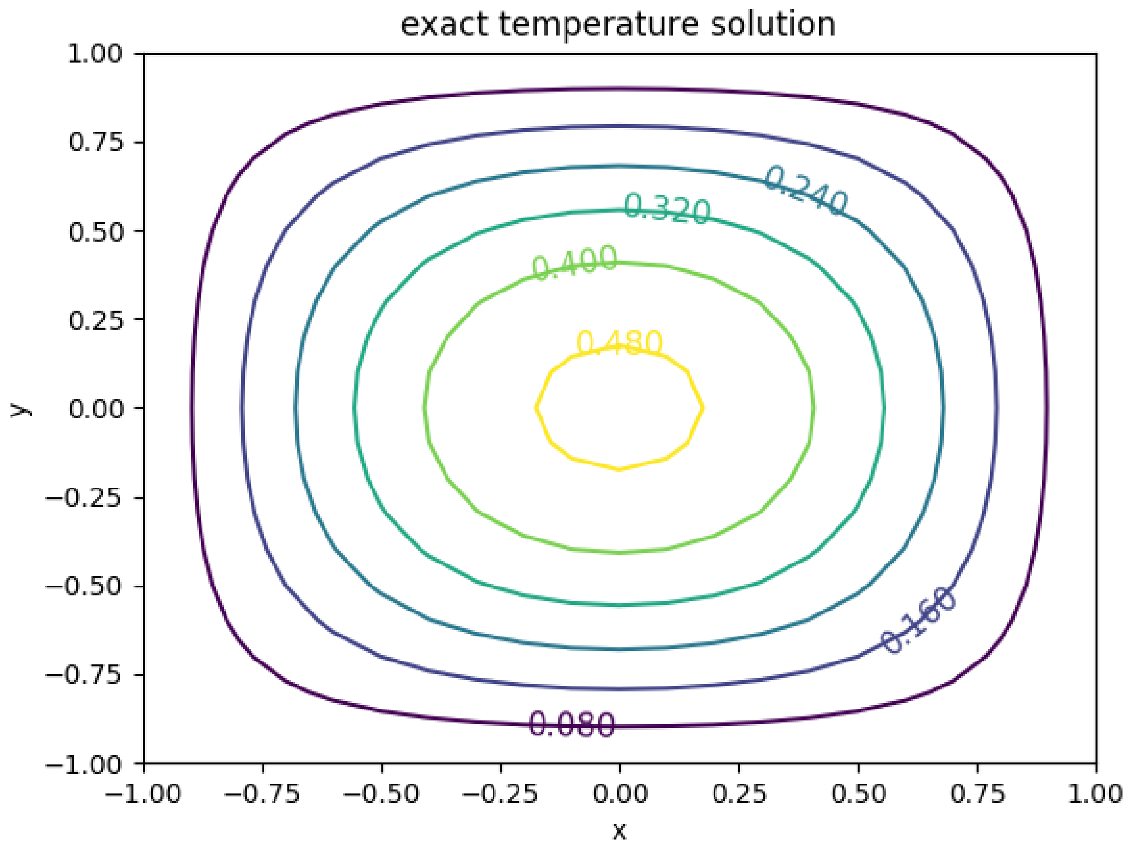

Figure 1.3: Contour plot

1.5.2.1 Python script

```
from pylab import *
from mpl_toolkits.mplot3d import Axes3D

# x in units of a, y in units of b, temp. T in units of q'''a^2/k.

fig = figure()
ax = Axes3D(fig)              # create 3D axes
x = np.arange(-1, 1.01, 0.1)
y = np.arange(-1, 1.01, 0.1)
x, y = np.meshgrid(x, y)
z = 0.5 * np.cos(np.pi*x/2) * np.cos(np.pi*y/2)
ax.plot_surface(x, y, z, rstride=2, cstride=4, cmap=cm.hot)
xlabel('x')
ylabel('y')
title('exact temperature solution')

show()

CS = contour(x, y, z)
clabel(CS, inline=0, fontsize=12)
title('exact temperature solution')
xlabel('x')
ylabel('y')
show()

# PS: import * is not recommended (cf import of numpy as np included in the background, confusing)
    I left it because it works.
```

BIBLIOGRAPHY

[1] M. M. EL-WAKIL, *Nuclear Heat Transport*, American Nuclear Society, 1993.