Nuclear Reactor Thermal-Hydraulics

NUGN520 - Homework

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Homework submitted for the Nuclear Reactor Thermal-Hydraulics class at the Colorado School of Mines.

SPRING 2017

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Chapter 1

Reactor Heat Generation

Several exercises from the book written by M. M. El Wakil [1] are tackled in this homework. The problems in this section relate to the fifth chapter of the book, covering the subject of heat conduction in reactor elements.

1.1 [5.2] - Maximum fuel temperature

1.1.1 Problem

A 0.5-in.-diameter fuel element is made of 3 percent enriched UO_2. It is surrounded by a 0.003-in.-thick helium layer and a 0.03-in.-thick Zircaloy 2 cladding. A certain section of the element operates in boiling light water at 1000 psia. The boiling heat transfer coefficient is $10000\ Btu.h^{-1}.ft^{-2}.°F$, and the temperature drop in the boiling film at the section is $30.4\ °F$. Calculate the maximum fuel temperature at that section. $k_{He} = 0.16$, $k_{clad} = 8\ Btu.h^{-1}.ft^{-2}.°F$.

1.1.2 Solution

The following assumption are made:

1. T_∞ is the temperature a certain distance from the cladding
2. The radius of the fuel element is r_h
3. The helium layer thickness is c_h
4. The full radius of the element is r_z

Let us first compute the temperature distribution in the fuel. We know that Equation 1.1 describes the heat conduction within our usual assumptions.

$$q'''2\pi r\Delta rL = q_{r+\Delta r} - q_r \tag{1.1}$$

We also know the relations 1.2 and 1.3.

$$q_r = -k_f A\frac{dT}{dr} = -2\pi Lk_f r\frac{dT}{dr} \tag{1.2}$$

$$q_{r+\Delta r} = q_r + \frac{dq_r}{dr}\Delta r = -2\pi Lk_f r\frac{dT}{dr} - 2\pi Lk_f\left(r\frac{d^2T}{dr^2} + \frac{dT}{dr}\right)\Delta r \tag{1.3}$$

Consequently, we can combine Equations 1.2 and 1.3 to obtain Equation 1.4.

$$-\frac{q'''}{k_f} = \left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) \tag{1.4}$$

This equation can be written following Equation 1.5.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q'''}{k_f} = 0 \tag{1.5}$$

Multiplying both sides by r and integrating, we obtain Equation 1.6.

$$r\frac{dT}{dr} = -\frac{q'''r^2}{2k_f} + C_1 = 0 \tag{1.6}$$

Now dividing both sides by r and integrating again, we can compute Equation 1.7.

$$T(r) = -\frac{q'''r^2}{4k_f} + C_1\ln(r) + C_2 = 0 \tag{1.7}$$

Using the boundary conditions $\left.\frac{dT}{dr}\right|_{r=0} = 0$ and $T(r=0) = T_m$, we can respectively define $C_1 = 0$ and $C_2 = T_m$.

Thus, we have Equation 1.8.

$$T_m - T(r_h) = \frac{q'''r_h^2}{4k_f} \tag{1.8}$$

Now, we can solve for the temperature distribution in the Helium layer. In this layer, we can assume that no heat is generated, thus using Equation 1.9.

$$\frac{d}{dr}\left(rk_h\frac{dT}{dr}\right)=0 \tag{1.9}$$

Integrating this Equation from r_h to r_h+c_h, we can write Equation 1.10.

$$\int_{r_h}^{r_h+c_h}\frac{d}{dr}\left(rk_h\frac{dT}{dr}\right)=0 \tag{1.10}$$

Knowing Equation 1.6, we can see that $r_hk_hd\frac{dT}{dr}=-\frac{q'''r_h^2}{2}$, and we can consequently write Equation 1.11.

$$rk_h\frac{dT}{dr}+\frac{q'''r_h^2}{2}=0 \tag{1.11}$$

Dividing by rk_h and integrating, we obtain Equation 1.12.

$$T(r_h+c_h)-T(r_h)=\frac{q'''r_h^2}{2k_h}\ln\left(\frac{r_h}{r_h+c_h}\right) \tag{1.12}$$

In the cladding Zr_2, the solution is identical. We can thus write Equation 1.13.

$$T(r_z)-T(r_h+c_h)=\frac{q'''(r_h+c_h)^2}{2k_z}\ln\left(\frac{r_h+c_h}{r_z}\right) \tag{1.13}$$

Anf finally, we can express the heat flow to the coolant. We know that $q''(r_z)=-k_w\frac{dT}{dr}\Big|_{r_z}=h_w(T(r_z)-T_\infty)$. We know, using Equation 1.6 that we can write Equation 1.14.

$$\frac{q'''(r_h+c_h)^2}{2}=h_wr_z(T(r_z)-T_\infty) \tag{1.14}$$

And finally, we can write Equation 1.15.

$$T(r_z)-T_\infty=\frac{q'''(r_h+c_h)^2}{2h_wr_z} \tag{1.15}$$

We are given the temperature drop in the coolant, 30.4 $°F$, as well as the boiling heat transfer coefficient $h_w=10000\ Btu.h^{-1}.ft^{-2}.°F^{-1}$. Converting the radii to feet, we can calculate q''' from Equation 1.15. We obtain $q'''=3.309\times10^7\ Btu.h^{-1}.ft^{-3}$. We can now plug this value back into Equation 1.13 ($T(r_z)-T(r_h+c_h)=-121.8\ °F$), Equation 1.12 ($T(r_h+c_h)-T(r_h)=-640.6\ F$) and Equation 1.8 ($T_m-T(r_h)\Big|_{k_f=2.5}=1431.6\ F$) to find the temperature T_m, which will be the maximum temperature at the section.

Consequently, $T_m\Big|_{k_f=2.5} = T_\infty + 1431.6 + 640.6 + 121.8 + 30.4 = T_\infty + 2224.4\ ^\circ F$.

T_∞ can be obtained using the data from Appendix D [1]. For saturated dry steam at 1000 psia, the temperature T_∞ is 544.6 $^\circ F$

And so, we can calculate $T_m\Big|_{k_f=2.5} = 2769\ ^\circ F$. This is far from the 800 $^\circ F$ at which $k_f = 2.5$. Considering $k_f = 1.1$, we calculate $T_m\Big|_{k_f=1.1} = 3798\ ^\circ F$. k_f is equal to 1.1 for $T_m > 2400\ ^\circ F$, so we can assume this is a correct value for the thermal conductivity.

1.2 [5.5] - Spherical reactor pellet

1.2.1 Problem

A spherical reactor is 5 ft in diameter. It contains 20 percent enriched UO_2 spherical fuel pellets 1 in. in diameter each. In one of these pellets, the temperature drop from center to edge of fuel is $3000\ ^\circ F$. The maximum flux in the core is 1×10^{13}. What is the radial position of this pellet? Neglect extrapolation lengths. Take $k_f = 1.1\ Btu.h^{-1}.ft^{-1}.^\circ F^{-1}$, fuel density is $11\ g.cm^{-3}$, and $\bar{\sigma_f} = 500\ b$.

1.2.2 Solution

We can start from the 1-D Poisson equation for spherical coordinates, seen in Equation 1.16.

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{q'''}{k_f} = 0 \tag{1.16}$$

This equation should be written in a different form to ease solving it. We know that $\frac{df(r)g(r)}{dr} = f'(r)g(r) + f(r)g'(r)$. We want to compute f and g so that $f'g = \frac{2}{r}\frac{dT}{dr}$ and $fg' = \frac{d^2T}{dr^2}$.

This is possible if $g = \frac{dT}{dr}$. Then, if $f = r^2$, we have Equation 1.17, equivalent to Equation 1.15.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{q'''}{k_f} = 0 \tag{1.17}$$

Consequently, integrating this equation, we obtain Equation 1.18.

$$r^2\frac{dT}{dr} = -\frac{q'''r^3}{3k_f} + C_1 \tag{1.18}$$

Which can be rewritten as Equation 1.19.

$$\frac{dT}{dr} = -\frac{q'''r}{3k_f} + \frac{C_1}{r^2} \tag{1.19}$$

We know that the local flux will be maximum at $r = 0$, in the center of the fuel pellet. Consequently, $\left.\frac{dT}{dr}\right|_{r=0} = 0$. This is possible if and only if $C_1 = 0$ in Equation 1.19.

Integrating again, we obtain Equation 1.20.

$$T(r) = -\frac{q'''r^2}{6k_f} + C_2 \tag{1.20}$$

Using the boundary conditions at $r = 0$, we know that the temperature will be maximum and equal to a defined T_m. Consequently, $C_2 = T_m$, and Equation 1.20 can be rewritten as Equation 1.21 for $r = R_p$, where R_p is the radius of the fuel pellet, and T_s being the temperature at the pellet surface.

$$\Delta T_p = T_m - T_s = \frac{q_p'''R_p^2}{6k_f} \tag{1.21}$$

Knowing that $\Delta T_p = 3000\,°F$, that $R_p = 0.5in = 0.5*0.083ft$ and that $k_f = 1.1\,Btu.h^{-1}.ft^{-1}.°F^{-1}$, we can compute q_p''', the volumetric thermal source strength in the pellet, Equation 1.22.

$$q_p''' = \frac{6*1.1*3000}{(0.5*0.083)^2} = 1.14\times10^7\ Btu.h^{-1}.ft^{-3} \tag{1.22}$$

Now, let us consider that r is relative to the core itself, instead of a simple pellet. We can write Equation 1.23.

$$q'''(r) = G_f N_f \bar{\sigma_f}\phi(r) \tag{1.23}$$

G_f is known and taken to be 180 MeV. $\bar{\sigma_f}$ is known, at 500 b. We can now calculate N_f, according to Equation 1.24.

$$N_f = \frac{N_A}{M_{ff}} e\rho_{fm} f i \tag{1.24}$$

The problem states that the density of the fuel is 11 $g.cm^{-3}$. However, this corresponds to the density of the fuel material UO_2. Thus, the factor f must be calculated, using equation 1.25.

$$f = \frac{e*M_{35} + (1-e)M_{38}}{e*M_{35} + (1-e)M_{38} + 2M_O} = 0.881 \tag{1.25}$$

Consequently, $N_f = 4.97\times10^{21}\ cm^{-3}$.

So, Equation 1.23 becomes Equation 1.26, with negligible extrapolation length.

(1.26)
$$q'''(r) = 180 * 4.97 \times 10^{21} * 500 \times 10^{24} 1 \times 10^{13} * \frac{\sin(\frac{\pi r}{R})}{\frac{\pi r}{R}} = 1.084 \times 10^{17} \frac{\sin(0.041r)}{r}\ MeV.s^{-1}.cm^{-3}$$

So, $q'''(r) = 1.678 \times 10^9 \frac{\sin(0.041r)}{r}\ Btu.h^{-1}.ft^{-3}$, using the conversion factor 1 $MeV.s^{-1}.cm^{-3} = 1.5477 \times 10^{-8}\ Btu.h^{-1}.ft^{-3}$.

We can find the radial position of the pellet by solving $q'''(r) = q'''_p$, as explicited in Equation 1.27.

$$\frac{\sin(0.041r)}{r} = \frac{1.14 \times 10^7}{1.678 \times 10^9} = 0.0068 \tag{1.27}$$

I tried to solve this equation using a Taylor series to solve $\frac{\sin(x)}{x} = 1.659$, for $x = 0.041r$. It gave me a quadratic equation for y by letting $x^2 = y$. Unfortunately, the solution using this method was $r = 109.02\ cm$, out of the core. I believe this is due to the Taylor series approximation. The real solution (obtained from a graphical method) is $r = 65.386\ cm = 2.15\ ft$. The pellet considered is at 2.15 ft from the center of the core.

1.3 [5.10] - Unclad hollow thick cylindrical fuel element

1.3.1 Problem

Derive expressions for (a) the position r of maximum temperature, and (b) the heat transfer through the inner and outer surfaces of an unclad hollow thick cylindrical fuel element with uniform heat generation and known surface temperatures if heat is allowed to flow through both surfaces.

1.3.2 Solution

We can solve the heat equation for cylindrical coordinates, Equation 1.28.

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{q'''}{k_f} = 0 \tag{1.28}$$

We have seen in Problem 1.1 that the solution will be of the form presented in relation 1.29.

$$T(r) = -q''' \frac{r^2}{4k_f} + C_1 \ln(r) + C_2 \tag{1.29}$$

Out of the outer surface, the boundary conditions are that $\left.\frac{dT}{dr}\right|_{r=r_i} = 0$, and that $T(r = r_i) = T_i$.

Consequently, we can trivially see that $C_1 = \frac{q''' r_i^2}{2k_f}$, and that $C_2 = T_i - \frac{q''' r_i^2}{2k_f}\left(\ln(r_i) - \frac{1}{2}\right)$. We can thus write Equation 1.30.

$$T(r) = T_i - \frac{q''' r_i^2}{4k_f} \left[\left(\frac{r}{r_i}\right)^2 - 2\ln\left(\frac{r}{r_i}\right) - 1 \right] \tag{1.30}$$

For the inner surface, the solution is identical, but with different boundary conditions. Indeed, $\frac{dT}{dr}\Big|_{r=r_o} = 0$, and $T(r = r_o) = T_o$. We obtain Equation 1.31.

$$T(r) = T_o - \frac{q''' r_o^2}{4k_f} \left[\left(\frac{r}{r_o}\right)^2 - 2\ln\left(\frac{r}{r_o}\right) - 1 \right] \tag{1.31}$$

We can thus write T_0 following Equation 1.32, by plugging in $r = r_0$ in Equation 1.30.

$$T_o = T_i - \frac{q''' r_o^2}{4k_f} + \frac{2q''' r_i^2}{4k_f} \ln\left(\frac{r_o}{r_i}\right) + \frac{q''' r_i^2}{4k_f} \tag{1.32}$$

Starting from Equation 1.30, we can write:

$$T(r) = T_i - \frac{q''' r^2}{4k_f} + \frac{q''' r_i^2}{4k_f} + \frac{2q''' r_i^2}{4k_f} \ln\left(\frac{r}{r_i}\right) \tag{1.33}$$

Let us define $\alpha = T_i - \frac{q''' r^2}{4k_f} + \frac{q''' r_i^2}{4k_f}$, so that $T(r) = \alpha + \frac{2q''' r_i^2}{4k_f} \ln\left(\frac{r}{r_i}\right)$.

Knowing that we will want a solution factoring $\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$, we can expand Equation 1.33 to obtain Equation 1.34.

$$T(r) = \alpha + \frac{2q''' r_i^2}{4k_f} \ln\left(\frac{r}{r_i}\right) + \frac{q''' r_o^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} - \frac{q''' r_o^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} + \frac{q''' r_i^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} - \frac{q''' r_i^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} \tag{1.34}$$

Now we see several terms of interest appear, in Equation 1.35 and Equation 1.36.

$$\gamma = \frac{q''' r_o^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} - \frac{q''' r_i^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} = \frac{q'''}{4k_f}(r_o^2 - r_i^2) \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} \tag{1.35}$$

$$\beta = \frac{q''' r_i^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} - \frac{q''' r_o^2}{4k_f} \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} + \frac{2q''' r_i^2}{4k_f} \ln\left(\frac{r}{r_i}\right) \tag{1.36}$$

Consequently, we can factorize as to make T_o (Equation 1.32) appear into β to obtain Equation 1.37.

(1.37) $$\beta = \left(T_i - T_i + \frac{q'''r_o^2}{4k_f} - \frac{2q'''r_i^2}{4k_f}\ln\left(\frac{r_o}{r_i}\right) - \frac{q'''r_i^2}{4k_f}\right)\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

And so, using $T(r) = \alpha - \beta + \gamma$, we finally get Equation 1.38.

(1.38) $$T(r) = T_i - \frac{q'''(r^2 - r_i^2)}{4k_f} - \left[(T_i - T_o) - \frac{q'''}{4k_f}(r_o^2 - r_i^2)\right]\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

In order to compute the position r_m of maximum temperature, we now need to solve $\frac{dT(r_m)}{dr} = 0$ (Equation 1.39), because we know the temperature distribution is parabolic.

(1.39) $$\frac{dT(r_m)}{dr} = 0 = \frac{q'''r_m}{2k_f} - \frac{a}{r_m}$$

Where:

$a = \frac{2q'''r_i^2}{4k_f}$

Consequently, Equation 1.39 becomes Equation 1.40.

(1.40) $$\frac{q'''r_m^2}{2k_f} = \frac{2q'''r_i^2}{4k_f}$$

And so, the factor a simplifying several parameters, $r = \frac{r_i}{\sqrt{\ln\left(\frac{r_o}{r_i}\right)}}$

For example, if $r_i = 1\ in$ and $r_o = 2\ in$, then the maximum temperature would happen at $r_m = \frac{1}{\sqrt{\ln(2)}} = 1.2\ in.$

In order to compute the heat transfer through the inner and outer surfaces, q_s, we know Equation 1.41.

(1.41) $$q_{s,o} = \pi(r_o^2 - r_i^2)Lq'''$$

We also know Equation 1.42, using Equation 1.30 at $r = r_0$.

(1.42) $$T_i - T_o = \frac{q'''r_i^2}{4k_f}\left[\left(\frac{r_o}{r_i}\right)^2 - 2\ln\left(\frac{r_o}{r_i}\right) - 1\right]$$

From Equation 1.42, we can isolate $q''' = \frac{1}{r_i^2}\frac{4k_f(T_i - T_o)}{\left(\frac{r_o}{r_i}\right)^2 - 2\ln\left(\frac{r_o}{r_i}\right) - 1}$.

And finally, we can rewrite $q_{s,o}$ with Equation 1.43.

$$q_{s,o} = 4\pi k_f L \frac{r_o^2 - r_i^2}{r_i^2} \frac{(T_i - T_o)}{\left(\frac{r_o}{r_i}\right)^2 - 2\ln\left(\frac{r_o}{r_i}\right) - 1} \tag{1.43}$$

Similarly, we can obtain Equation 1.44 for the inner surface heat transfer.

$$q_{s,i} = 4\pi k_f L \frac{r_i^2 - r_o^2}{r_o^2} \frac{(T_i - T_o)}{\left(\frac{r_i}{r_o}\right)^2 - 2\ln\left(\frac{r_i}{r_o}\right) - 1} \tag{1.44}$$

The total heat transfer is assumed to be through both surfaces. Consequently, we can write Equation 1.45.

$$q_s = q_{s,i} + q_{s,o} = 4\pi k_f L(T_i - T_o)\left[\frac{r_i^2 - r_o^2}{r_o^2} \frac{1}{\left(\frac{r_i}{r_o}\right)^2 - 2\ln\left(\frac{r_i}{r_o}\right) - 1} + \frac{r_o^2 - r_i^2}{r_i^2} \frac{1}{\left(\frac{r_o}{r_i}\right)^2 - 2\ln\left(\frac{r_o}{r_i}\right) - 1}\right] \tag{1.45}$$

1.4 [E1] - Fourier sine series solution of a telegraph equation

1.4.1 Problem

Download the Coffey and Colburn (2009) paper on the telegraph or hyperbolic heat equation and consider the Fourier series solution of Appendix A. Verify the results for either the zero Dirichlet or zero Neumann (insulation at both $x = 0$ and $x = L$) boundary conditions. Write an expression for the speed of heat propagation based upon the coefficients in equation (A1).

1.4.2 Solution

The equation to solve is Equation 1.46.

$$b\frac{\partial^2 u(x,t)}{\partial t^2} + a\frac{\partial u(x,t)}{\partial t} - D\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{1.46}$$

Using the Dirichlet boundary conditions, we know that the solution is of the form $\sin\left(n\frac{\pi x}{L}\right)$. The Fourier sine series solution is thus of the form given in Equation 1.47.

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n^{(+)} e^{\lambda_n^{(+)} t} + A_n^{(-)} e^{\lambda_n^{(-)} t}\right]\sin\left(n\frac{\pi x}{L}\right) \tag{1.47}$$

We are now going to verify if this Fourier sine series solution form solves Equation 1.46.

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left[A_n^{(+)}\lambda_n^{(+)} e^{\lambda_n^{(+)} t} + A_n^{(-)}\lambda_n^{(-)} e^{\lambda_n^{(-)} t}\right]\sin\left(n\frac{\pi x}{L}\right) \tag{1.48}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \sum_{n=1}^{\infty} \left[A_n^{(+)} \left(\lambda_n^{(+)}\right)^2 e^{\lambda_n^{(+)} t} + A_n^{(-)} \left(\lambda_n^{(-)}\right)^2 e^{\lambda_n^{(-)} t} \right] \sin\left(n \frac{\pi x}{L}\right) \tag{1.49}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -\sum_{n=1}^{\infty} \left[A_n^{(+)} e^{\lambda_n^{(+)} t} + A_n^{(-)} e^{\lambda_n^{(-)} t} \right] \frac{n^2 \pi^2}{L^2} \sin\left(n \frac{\pi x}{L}\right) \tag{1.50}$$

Plugging Equations 1.48, 1.49 and 1.50 into Equation 1.46, we obtain Equation 1.51.

$$\begin{aligned} & b \sum_{n=1}^{\infty} \left[A_n^{(+)} \left(\lambda_n^{(+)}\right)^2 e^{\lambda_n^{(+)} t} + A_n^{(-)} \left(\lambda_n^{(-)}\right)^2 e^{\lambda_n^{(-)} t} \right] \\ & + a \sum_{n=1}^{\infty} \left[A_n^{(+)} \lambda_n^{(+)} e^{\lambda_n^{(+)} t} + A_n^{(-)} \lambda_n^{(-)} e^{\lambda_n^{(-)} t} \right] \sin\left(n \frac{\pi x}{L}\right) \\ & + D \sum_{n=1}^{\infty} \left[A_n^{(+)} e^{\lambda_n^{(+)} t} + A_n^{(-)} e^{\lambda_n^{(-)} t} \right] \frac{n^2 \pi^2}{L^2} \sin\left(n \frac{\pi x}{L}\right) = 0 \end{aligned} \tag{1.51}$$

This equation can be rewritten into Equation 1.52, by rearranging the summed terms to factor by $A_n e^{\lambda_n t}$.

$$\sum_{n=1}^{\infty} \left[A_n^{(+)} e^{\lambda_n^{(+)} t} \left(a \lambda_n^{(+)} + b \left(\lambda_n^{(+)}\right)^2 + D \frac{n^2 \pi^2}{L^2} \right) + A_n^{(-)} e^{\lambda_n^{(-)} t} \left(a \lambda_n^{(-)} + b \left(\lambda_n^{(-)}\right)^2 + D \frac{n^2 \pi^2}{L^2} \right) \right] \sin\left(n \frac{\pi x}{L}\right) = 0 \tag{1.52}$$

We can note that the relations $a\lambda + b\lambda^2 + D\frac{n^2\pi^2}{L^2}$ appear, for which λ is the zero solution. Consequently, we verify that the Fourier sine series given in Equation 1.47 is a solution for Equation 1.46.

We know Equation 1.53, as usual and in a completely obvious and trivial way by orthogonality. We also know Equation 1.54, from the initial time derivative condition $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$.

$$A_n^{(+)} + A_n^{(-)} = \frac{2}{L} \int_0^L u(x,0) \sin\left(n \frac{\pi x}{L}\right) dx \tag{1.53}$$

$$A_n^{(+)} \lambda_n^{(+)} + A_n^{(-)} \lambda_n^{(-)} = 0 \tag{1.54}$$

Consequently, by combining these two equations, we can write Equation 1.55.

$$A_n^{(+)} \lambda_n^{(+)} = -A_n^{(-)} \lambda_n^{(-)} = -\lambda_n^{(-)} \left(\frac{2}{L} \int_0^L u(x,0) \sin\left(n \frac{\pi x}{L}\right) dx - A_n^{(+)} \right) \tag{1.55}$$

And finally, rearranging, we obtain Equation 1.56.

$$A_n^{(+)} = \frac{\lambda_n^{(-)}}{\lambda_n^{(-)} - \lambda_n^{(+)}} \frac{2}{L} \int_0^L u(x,0) \sin\left(n\frac{\pi x}{L}\right) dx \tag{1.56}$$

In an identical way, we find Equation 1.57.

$$A_n^{(-)} = -\frac{\lambda_n^{(+)}}{\lambda_n^{(-)} - \lambda_n^{(+)}} \frac{2}{L} \int_0^L u(x,0) \sin\left(n\frac{\pi x}{L}\right) dx \tag{1.57}$$

Now, Equation 1.47 can be written in terms of known parameters, by replacing A_n and λ_n with their respective values.

The heat equation can be written using Equation 1.58, corresponding to Equation 1.46 where $b = 0$.

$$a\frac{\partial u(x,t)}{\partial t} - D\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{1.58}$$

In this case, $\lambda_n = -Da\frac{n^2\pi^2}{L^2}$, and $u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\frac{\pi x}{L}\right) e^{\lambda t}$.

Bibliography

[1] M. M. El-Wakil, *Nuclear Heat Transport*, American Nuclear Society, 1993.