Asymptotics on the Lempel-Ziv 78 compression of Markov sources

Exploring analytic information theory : from Markov source sampling to combinatorial analysis proofs

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Words or sequences, and memoryless sources

Definition : word or sequence or string

Given an alphabet $\mathcal{A}$, a ***word*** or ***sequence*** or ***string*** is an infinite sequence of random variables $X = (X_k)_{k\mathbb{N}^*}$, each X_k representing a symbol in $\mathcal{A}$.

Definition : Bernoulli or Memoryless source

A source of information is a ***Bernoulli*** or ***memoryless source*** when all the symbols of $\mathcal{A}$ occur independently with a fixed probability. The word can be seen as an *infinite sequence of Bernoulli trials*.

Markov sources definition

Definition : Markov source

An information source is a ***Markov source*** when there is a ***Markov dependency*** between the consecutive symbols of a string.

Definition : order of a Markov source

Let $V = |\mathcal{A}|$. A ***Markov source*** is of ***order*** r when the dependency can be encoded in a transition matrix of size $V^r \times V$, with coefficients :

$$P(c|w) \qquad \forall\,(w,c) \in \mathcal{A}^r \times \mathcal{A}$$

Informally : *the probability that a symbols occurs depends on the previous r symbols.*

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Description of the LZ78 algorithm

Algorithm

Given a word w.

- Initialize an empty dictionary
- While it is possible :

    Find longest prefix of w that is not in the dictionary

    Add it to the dictionary, cut it from w

Elements description

The data representation is (dictionary_reference, symbol).

Remarks

The LZ78 algorithm builds a prefix tree from which the original word can be reconstructed.

Definition : number of phrases

After compressing a word w, the number of phrases in the dictinary is noted $M(w)$.
For words of size n, we write $M_n(w)$.

Code length

$$C(w) = \sum_{k=0}^{M(w)} \left(\lceil \log_2(k) \rceil + \lceil \log_2(\mathcal{A}) \rceil \right)$$

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Definition : compression ratio

Let w a word, and $C(w)$ its *encoding* by a compression algorithm. The ***compression ratio*** of w is $\dfrac{|C(w)|}{|w|}$.

Main goals of compression algorithms

- Improving the compression ratio
- Fast compression/decompression speed in Mb/s

T

he tradeoff between these two goals is a sensitive research problem. Different compression standards :

- Google (Brotli, 2015)
- Facebook (Zstandard, 2016)

Optimal encoding

Entropy of a Markov source

Let π be a stationary distribution. The entropy of a Markov chain is

$$h = -\sum_{i=1}^{V} \pi \sum_{j=1}^{V} p_{ij} \log(p_{ij})$$

Optimality of LZ78

Considering words of length n.

$$\frac{|C(w)|}{|w|} - h \text{ goes to zero for } n \to +\infty$$

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Optimal parsing

Flexible parsing

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Markov Independent Model

$$
\begin{aligned}
X(1) &= 0000000\ldots \\
X(2) &= 1010101\ldots \\
X(3) &= 1001101\ldots \\
X(4) &= 001100111\ldots
\end{aligned}
$$

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Coding details

- Python code $\sim$ 2000 lines
- Markov source sampling
- Optimized datastructure (digital search tree)
- Parallelization
- Reproducibility of datasets

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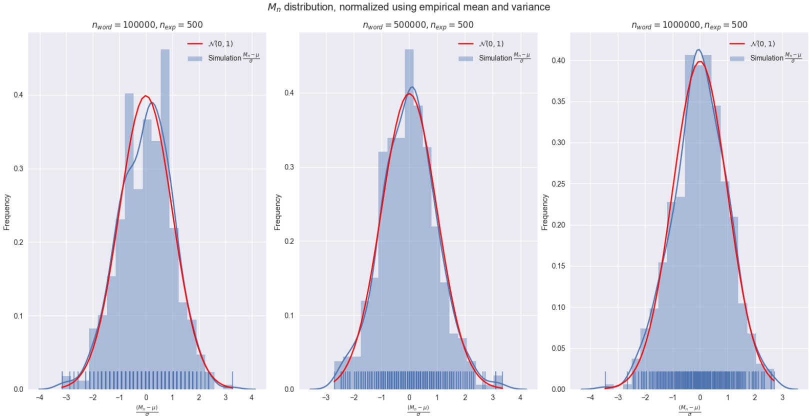
Central Limit Theorem confirmation



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Hypothesis testing for the variance

Complex matrix

Defining $P(s)$ as $\begin{matrix} p_{11}^{-s} & p_{12}^{-s} \\ p_{21}^{-s} & p_{22}^{-s} \end{matrix}$

Variance expression

$$V_n = \left(\ddot{\lambda}(-1) - \dot{\lambda}(-1)^2\right) \frac{n}{\ln^2 n}$$

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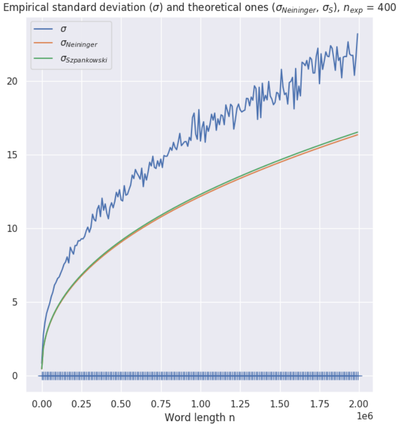
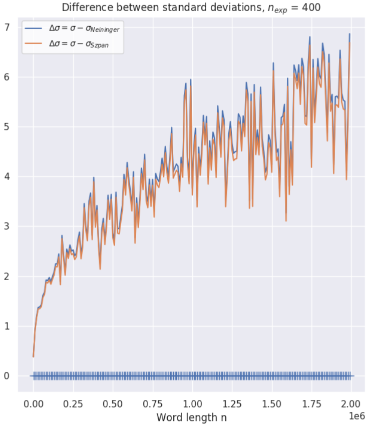

Empirical standard deviation (σ) and theoretical ones (σNeininger, σS), nexp = 400
σ
σNeininger
σSzpankowski
0
5
10
15
20
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
1e6
Word length n


Difference between standard deviations, nexp = 400
Δσ = σ − σNeininger
Δσ = σ − σSzpan
0
1
2
3
4
5
6
7
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
1e6
Word length n

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Definition, usage

Definition

$$A(z) = \sum_{n \geqslant 0} a_n z^n$$

Remarks

- Used as an algebraic item with the convolution product
- No convergence problems

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Poissonization and Depoissonization

$$\widetilde{G}(z) = \sum_{n \geqslant 0} a_n \frac{z^n}{n!} \mathrm{e}^{-z}$$

Mellin transform

Make recurrence relation between random variables become linear in order to solve them more easily.

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Tail symbols

Illustration

$$
\begin{aligned}
X(1) &= 0000000\ldots\\
X(2) &= 1010101\ldots\\
X(3) &= 1001101\ldots\\
X(4) &= 001100111\ldots
\end{aligned}
$$

Definition

Let c be a character from our alphabet $\{a, b\}$. In the case when all the sequences start with a c, we define T_n^c the *number of times a is a tail symbol in the experiment*.

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Definition and relation

Recurrence

For $n \geqslant 0$, we have :

$$\boxed{T^c_{n+1} = \delta_a + \widetilde{T}^a_{N_a} + \widetilde{T}^b_{N_b}}$$

Notations

- $\delta_a = \begin{cases} 1 & \text{if } a \text{ is the tail symbol of the first sequence} \\ 0 & \text{else} \end{cases}$
- N_a is the random variable giving *the size of the left subtree which contains phrases whose second letter is a*
- $\widetilde{T}^a_{N_a}$ is the number of times a is a tail symbol for the sequences that were used to build the subtree with

Total path lenght

Definition

Defining L_n^c as the *total path length of the nodes of the DST that was built with MI model with n sequences starting with letter c*. It is the sum of the lengths of all the prefix phrases.

Recurrence relation

For all $n \geqslant 0$:

$$\boxed{L_{n+1}^c = n + \widetilde{L}_{N_a}^a + \widetilde{L}_{N_b}^b}$$

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Inconclusive, but informative

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Recurrence

$$\boxed{\mathrm{Cov}(T^c_{n+1}, L^c_{n+1}) = \mathrm{Cov}(\widetilde{T}^a_{N_a}, \widetilde{L}^a_{N_a}) + \mathrm{Cov}(\widetilde{T}^b_{N_b}, \widetilde{L}^b_{N_b})}$$

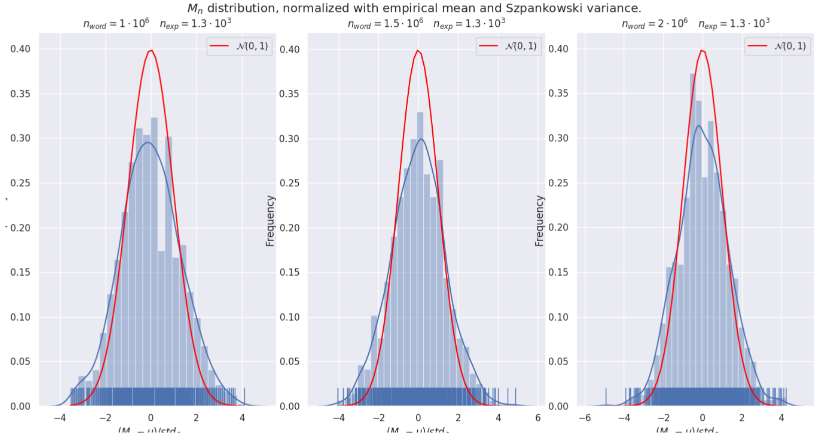
Poisson transform

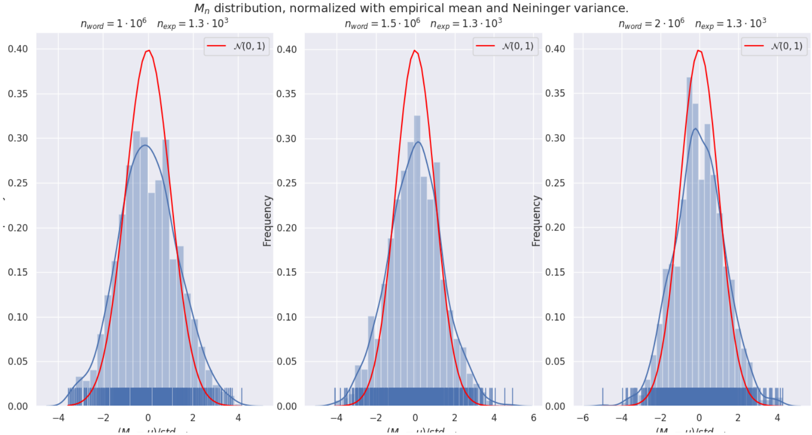
Defining

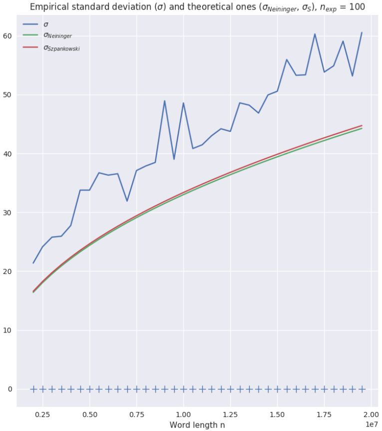
$$\boxed{C_c(z) = \sum_{n \geqslant 0} \mathrm{Cov}(T_n^c, L_n^c) \frac{z^n}{n!} \mathrm{e}^{-z}}$$

Differential equation

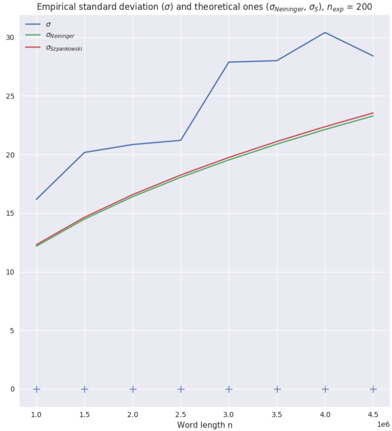
$$\boxed{\partial_z C_c(z) + C_c(z) = C_a(zp) + C_b(zq)}$$

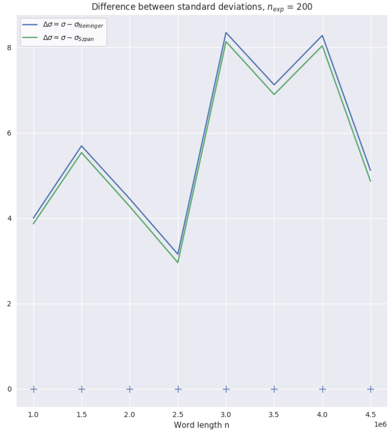

M_n distribution, normalized with empirical mean and Szpankowski variance.
$n_{word} = 1 \cdot 10^6$ $n_{exp} = 1.3 \cdot 10^3$
$n_{word} = 1.5 \cdot 10^6$ $n_{exp} = 1.3 \cdot 10^3$
$n_{word} = 2 \cdot 10^6$ $n_{exp} = 1.3 \cdot 10^3$
$\mathcal{N}(0, 1)$
Frequency


M_n distribution, normalized with empirical mean and Neininger variance.
$n_{word} = 1 \cdot 10^6$ $n_{exp} = 1.3 \cdot 10^3$
$n_{word} = 1.5 \cdot 10^6$ $n_{exp} = 1.3 \cdot 10^3$
$n_{word} = 2 \cdot 10^6$ $n_{exp} = 1.3 \cdot 10^3$
$\mathcal{N}(0, 1)$
Frequency
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−6
−4
−2
0
2
4
6


Empirical standard deviation (σ) and theoretical ones ($\sigma_{Neininger}$, σ_S), n_{exp} = 100
σ
$\sigma_{Neininger}$
$\sigma_{Szpankowski}$
0
10
20
30
40
50
60
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
1e7
Word length n


Difference between standard deviations, n_{exp} = 100
$\Delta\sigma = \sigma - \sigma_{Neininger}$
$\Delta\sigma = \sigma - \sigma_{Szpan}$
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
1e7
Word length n


Empirical standard deviation (σ) and theoretical ones ($\sigma_{Neininger}$, σ_S), n_{exp} = 200
σ
$\sigma_{Neininger}$
$\sigma_{Szpankowski}$
Word length n
1e6


Difference between standard deviations, n_{exp} = 200
$\Delta\sigma = \sigma - \sigma_{Neininger}$
$\Delta\sigma = \sigma - \sigma_{Szpan}$
Word length n
1e6

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