gLidar - Model Formulation

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1 Parameters

Air profile. A complex environmental parameter describing the observation at the nearest meteorological station. In our case, the observation is obtained using a radiosonde measurement released at 12-hour intervals from Sola airport near Stavanger (200 km away from the paragliding area). Due to the temporal and spatial difference between the air profile measurement and the paragliding flights, a part of the air temperature profile in the lower altitudes is replaced by an explicit dependency based on dry adiabatic processes.

$$\theta(p) = T \left(\frac{p_0}{p} \right)^{\frac{R}{c_p}} \tag{1}$$

where θ is the potential temperature as a function of pressure p, T is temperature at pressure p, p_0 is a reference pressure, and R and c_p are the gas constant and specific heat at constant pressure, respectively, yielding a constant $\frac{R}{C_p} = 0.286$. The equation for potential temperature gives the temperature a parcel of dry air would have if adiabatically moved to the altitude of the reference pressure p_0, i.e., without any energy exchange with the surroundings. While ascending, the temperature of the air parcel decreases as pressure is decreasing, though θ would remain constant if the parcel is moved adiabatically. In order to calculate the humid buoyancy we use the virtual potential temperature. The virtual potential temperature is the equivalent temperature of dry air referenced at 1000 hPa. It is a function of pressure, temperature and mixing ratio

$$\theta_v = \theta \frac{w + \varepsilon}{\varepsilon(1 + w)} \ ,$$

where ε is the molecular mixing ratio $\varepsilon = 0.622$, and the mixing ratio is defined as

$$w = \varepsilon \ \frac{e}{p - e} \tag{2}$$

where e is the partial pressure of the water vapour and p is the air pressure.

Two-metre temperature. A control parameter which can be estimated by an on-site measurement, corresponding to the air temperature at the location of the thermal simulation/observation.

Two-metre dew point temperature. A control parameter that can be estimated by an on-site measurement, indicating the amount of humidity at the location of the thermal. The dew point temperature is a computed value at which the air vapour would start to condense given a cooling under isobaric (constant pressure) conditions. The thereby provided information about humidity of the air parcel is necessary for the computation of the altitude of the cloud base, referred to as the *lifting condensation level (LCL)*.

Temperature anomaly. The main parameter controlling the perturbation of the observed state. The temperature anomaly captures the temperature difference between the thermal parcel and the surrounding air. It is theoretically possible to observe the temperature anomaly, though this data is difficult to obtain without advanced observational strategies that are not readily available.

Dew point anomaly. A control parameter that specifies the amount of water vapour in the perturbed parcel. The dew point anomaly has large influence on the LCL.

Aspect ratio. A control parameter adjusting the ratio of width to the height of the thermal. A low aspect ratio corresponds to a narrow and tall convective parcel whereas a high aspect ratio corresponds to a flat and wide thermal bobble. The aspect ratio is responsible for the amount of the dynamic pressure response on the thermal parcel affecting the effective buoyancy of the parcel. [1]

Linear drag coefficient. A model parameter controlling the intensity of the force acting against the updraft. The parameter represents turbulent forces slowing down the ascent of the rising air. This parameter is considered a control parameter associated with the model building and corresponds to the parameter α in the equations in the supplementary material. Part of this parameter is actually absorbed into the dynamic pressure, see discussion in one of the Jeevanjee papers [1].

Quadratic drag coefficient. As above, but the forces are proportional to the square of the velocity of the thermal.

Entrainment coefficient. A model parameter controlling the dissipation of the temperature perturbation. If we imagine the thermal as a hot air balloon, this parameter controls how perforated the balloon is. This parameter is considered a control parameter associated with the model building, corresponding to the parameter γ in the equations in the supplementary material.

Humidity entrainment coefficient. A separate entrainmet parameter that controls the rate of the water vapour mixing between the thermal into the environment.

Thermal minimal altitude. A control parameter corresponding to the spatial configuration of the thermal, determining the altitude at which the thermal first develops. It is connected to the topographic data of the location of the thermal.

2 Model Formulation

We start from the equations of motion for one-dimensional motion ($u=v=0$) in the vertical

$$\frac{dw}{dt} = \frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}-g-\alpha w \quad (3)$$
$$\frac{db}{dt} = \frac{\partial b}{\partial t}+w\frac{\partial b}{\partial z} = -\beta b+\dot{b}\ , \quad (4)$$

where w is the vertical velocity, $b=g\frac{\theta_v'}{\theta_v}$ is the buoyancy, $\dot{b}$ is the diabatic forcing of buoyancy due to latent heat releace, and α and β are the respective damping parameters corresponding to linear entrainment.

Assuming a basic state such that

$$\theta = \overline{\theta}(z)+\theta' \quad (5)$$
$$p = \overline{p}(z)+p' \quad (6)$$
$$\rho = \overline{\rho}+\rho' \quad (7)$$
$$w = w' \quad (8)$$

where basic state pressure $\overline{p}$ is hydrostatically balanced with the basic state density $\frac{d\overline{p}}{dz}=-\overline{\rho}g$ and there is no basic state vertical velocity $\overline{w}=0$, yields

$$\frac{\partial w'}{\partial t}+w'\frac{\partial w'}{\partial z} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}-g\frac{\rho'}{\overline{\rho}}-\alpha w' \quad (9)$$
$$\frac{\partial b'}{\partial t}+w'\frac{\partial b'}{\partial z}+w'\frac{d\overline{b}}{dz} = -\beta b'+\dot{b}\ , \quad (10)$$

where we assumed $p' \ll \overline{p}$ and $\rho' \ll \overline{\rho}$.

Assuming steady state $\frac{\partial}{\partial t}=0$, we obtain

$$w'\frac{\partial w'}{\partial z} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}+b'-\alpha w' \quad (11)$$
$$w'\frac{\partial b'}{\partial z} = -w'\frac{\partial \overline{b}}{dz}-\beta b'+\dot{b}\ , \quad (12)$$

where we assumed the Boussinesq approximation $-g\frac{\rho'}{\overline{\rho}}=g\frac{\theta_v'}{\overline{\theta}_v}=b'$.

2.1 Quadratic drag

The literature shows that the most common form of the Eq. 11 is with the drag forces proportional to the square of the vertical velocity. [2]

$$w'\frac{\partial w'}{\partial z}=\frac{1}{2}\frac{\partial w'^2}{\partial z}=-\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}+b'-\eta w'^2 \quad (13)$$

In order to compare the effects of linear momentum entrainment versus the quadratic drag we can combine the Eq. 11 and Eq. 13 to get:

$$\frac{1}{2}\frac{\partial w'^2}{\partial z}=-\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}+b'-\alpha w'-\eta w'^2 \quad (14)$$

2.1.1 *Dynamic pressure and effective buoyancy*

We follow the analytical formula for back pressure by Romps and Jeevanjee [1], where the effect is represented as a virtual mass coefficient a.

The coefficient a is is applied to the equation 14:

$$\frac{1}{a}\frac{1}{2}\frac{\partial w'^2}{\partial z}=b'-\alpha w'-\eta w'^2 \quad (15)$$

where the formula for the effective mass coefficient a due to the back pressure is given by:

$$a=\frac{1}{1+D^2/H^2} \quad (16)$$

where D is the width and H is the height of the bubble. Note that the vertical pressure gradient term disappeared. It has been integrated into the term ab', which Romps and Jeevanjee [1] refer to as effective buoyancy.

3 The Environment

3.1 Basic state profiles

As we cannot always use the observed state as the basic state, we need to employ necessary adjustments. The idea is to use parcel profile in the lower altitudes until the base parcel **temperature** intersects the observed temperature. From this point on both the base temperature and base dew point will be replaced with the observed values.

We label the pressure level where the both temperatures intersect p_x. Given that we operate in pressure coordinates, it is irrelevant if potential temperatures or absolute temperature is considered:

$$\theta^0(p_x)=\theta(p_x)$$
$$T^0(p_x)\cancel{\left(\frac{P_0}{p_x}\right)^{\kappa}}=T(p_x)\cancel{\left(\frac{P_0}{p_x}\right)^{\kappa}} \quad (17)$$
$$T^0(p_x)=T(p_x)$$

where θ_v^0 and T_v^0 are the observed potential temperature and observed temperature respectively.

The resulting base profile $\bar{\theta}$ is then a parcel profile intersected with the observed profile at p_x. Notice that the lower altitudes correspond to higher pressure, hence the orientation of the inequality.

$$\bar{\theta}(p)=\begin{cases}\theta_{T_0}(p) & \text{where } \ p>p_x\\ \theta^0(p) & \text{otherwise}\ ,\end{cases} \quad (18)$$

where θ_{T_0} is parcel profile with the surface temperature T_0, and θ^0 is the observed profile.

Similarly for the humidity profile in terms of mixing ratio

$$\bar{w}(p)=\begin{cases}w(p) & \text{where } \ p>p_x\\ w^0(p) & \text{otherwise}\ .\end{cases} \quad (19)$$

3.2 Perturbed parcel

The humidity profile of the perturbed parcel is calculated with constant mixing ratio until the LCL is reached, where condensation occurs and from which 100% relative humidity is maintained during the ascent. The parcel mixing ratio is computed using the saturation mixing ratio $w_s=w_s(p,T(p))$.

$$w(p)=\begin{cases}w_0 & \text{where } \ w_0<w_s\\ w_s & \text{otherwise}\end{cases}$$

The perturbed (virtual) potential temperature profile θ' (θ_v') is constant until the perturbed LCL, from where the (virtual) pseudoadiabatic profile is used. Both the constant (virtual) potential temperature as well as the pseudoaiabatic profiles are computed using the `parcel_profile` function from the `metpy` [3] package in version 1.0. Once the profile of θ' (θ_v') is obtained, we estimate the entrained potential temperature perturbation $\theta^\star$ assuming linear entrainment proportional to the vertical velocity w'

$$\frac{\partial \theta^\star_{(v)}}{\partial z} = \frac{\partial \theta'_{(v)}}{\partial z} - \gamma \theta^\star_{(v)} \tag{20}$$

where γ is a constant. For a constant perturbation $\theta' = \Delta\theta$ the solution is $\theta^\star = \Delta\theta \exp[-\gamma z]$.

The virtual buoyancy is then calculated as

$$b'_v = g \frac{\theta^\star_v}{\bar{\theta}_v} \tag{21}$$

where g is the acceleration due to gravity.

3.3 Vapour Integration

As we modify the observed temperature and humidity profiles to obtain the basic state, it is important to monitor how much water has been added into the system. To calculate the added water, we compute the density profiles of both the observed and the basic state, where we use the ideal gas law

$$\rho = \frac{p}{R_d T_v} \ . \tag{22}$$

In order to calculate the amount of the water in the air parcel, we use the mixing ratio

$$w = \frac{m}{M} \ , \tag{23}$$

where m is the mass of the water vapour and M the mass of dry air. As we only know the total mass $m+M$ we use specific humidity

$$r = \frac{m}{m+M} = \frac{wM}{wM+M} = \frac{w}{1+w} \ . \tag{24}$$

The density of water carried in the atmospheric parcel is therefore

$$\rho_w = \frac{w}{1+w}\ \rho = r\rho \ . \tag{25}$$

To obtain the added water we need to compare the amount of water in the two profiles, where we integrate the density vertically over z, yielding mass per area,

$$W = \int_0^{z_{max}} (\rho_w(z) - \rho_w^0(z))\ \mathrm{d}z = \int_0^{z_{max}} (r(z)\rho - r^0(z)\rho^0)\ \mathrm{d}z \ . \tag{26}$$

As density might be a function of pressure instead of altitude, we substitute $z = z(p)$, $\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}p}\mathrm{d}p = -\frac{1}{\rho g}\mathrm{d}p$, where we used hydrostatic balance with gravitational constant g, and integrate over pressure yielding the added mass of water per unit area

$$W = \int_0^{z_{max}} (\rho_w(z) - \rho_w^0(z))\ \mathrm{d}z = -\frac{1}{g}\int_{P_0}^{P_{min}} (r(p) - r^0(p))\ \mathrm{d}p \ . \tag{27}$$

REFERENCES

[1] N. Jeevanjee and D. M. Romps, "Effective buoyancy at the surface and aloft," *Quarterly Journal of the Royal Meteorological Society*, vol. 142, no. 695, pp. 811–820, 2016.

[2] S. R. de Roode, A. P. Siebesma, H. J. J. Jonker, and Y. de Voogd, "Parameterization of the vertical velocity equation for shallow cumulus clouds," *Monthly Weather Review*, vol. 140, no. 8, pp. 2424 – 2436, 01 Aug. 2012.

[3] R. M. May, S. C. Arms, P. Marsh, E. Bruning, J. R. Leeman, K. Goebbert, J. E. Thielen, and Z. S. Bruick, "MetPy: A Python Package for Meteorological Data," Unidata, Boulder, Colorado, 2008 - 2020.