Computational Complexity
Assignment 4

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Abstract

This document revolves around the graph colouring problem, specifically applied to the Petersen graph, and it's reduction to a SAT problem. It also includes a commentary on the relation between clique and chromatic numbers.

1 Introduction

In graph theory, graph vertex colouring consists of assigning a "colour" to each vertex of a graph, such that no two adjacent vertices share the same colour.

2 Colouring the Petersen graph

In this assignment we are required to study the viability of vertex colouring for the Petersen Graph with 3 and 2 different colours respectively.

To find out whether such vertex colouring is possible, we need to reduce the 3-COL and 2-COL problems to SAT problems.

2.1 Reducing COL to SAT

For this purpose, the implementation of a small program capable of performing such task is recommended. Taking a graph as an input, the program should provide an equivalent SAT problem in DIMACS format.

The input format used for the graphs will be a text file with the following structure:

- The first line will consist of an integer representing the total amount of vertices.
- Each following line will represent the list of edges of each vertex, commencing with the total number of edges connected to said vertex and followed by every destination vertex.

An example of Petersen's graph formatted:

```
10
3 1 4 7
3 0 2 5
3 1 3 5
3 2 4 6
3 0 3 8
3 1 6 8
3 3 5 7
3 0 6 9
3 4 5 9
3 2 7 8
```

In this case, the program has been implemented in a generic manner, so it requires for the user to specify the number of colours to use and the output file, aside from the input graph.

The reduction is realised producing the necessary clauses to ensure that every vertex has exactly one colour assigned and that every edge has a different colour at each end.

For example, for the 3-COL problem, the generated Boolean formula will be:

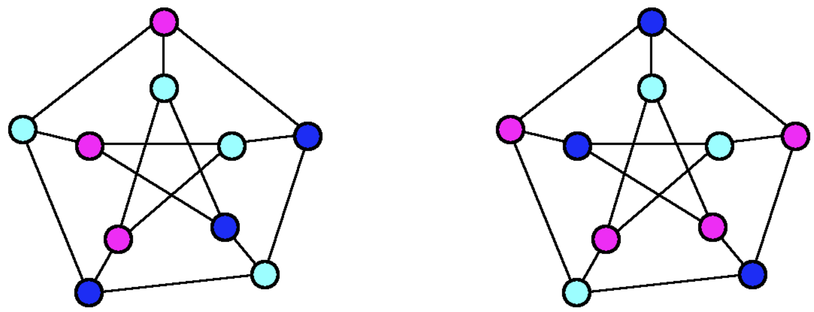
- For every vertex v:
$$(x_v^r \vee x_v^b \vee x_v^g) \wedge \neg(x_v^r \wedge x_v^b) \wedge \neg(x_v^r \wedge x_v^g) \wedge \neg(x_v^g \wedge x_v^b)$$
- For every edge (v, w):
$$\bigwedge_{c \in \{r,b,g\}} \neg(x_v^c \wedge x_w^c)$$

A detailed explanation of the structure of the code can be found in the comments of the file itself.

2.2 Determining the chromatic number of the Petersen graph

With the reducer implemented, we proceed to execute it for 3 colours and the Petersen graph. Using the resulting ".cnf" file to check it's viability with the SAT solver software "PicoSAT", we conclude that it is possible to colour the Petersen graph with 3 colours.

Two examples of 3-coloured Petersen graphs:



Repeating the previous process for 2 colours results in the conclusion that this is not viable.

2.3 Cliques and colours

After the previous results we can affirm that the Petersen graph has a chromatic number of 3, and we know it's clique number is 2. From this and similar observations it is safe to assume (and it is in fact almost trivial) that every graph has a chromatic number greater than or equal to it's clique number.

This conclusion results obvious observing the very definition of a clique (i.e. a subset of the graph vertices such that every two vertices in the subset are connected by an edge), since every vertex in the clique is connected to all the others in the subset, thus meaning that there can not be less colours than vertices in the clique.