Homework Two

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

Upload your responses to the prompts marked (SUBMIT) to Gradescope before 8pm Friday; you will receive feedback on these.

gradescope.com/courses/275664

The rest of the exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. Appreciate convergence. Whether or not an integral converges should be answered with care since it's easy to get tripped up by ∞, but it's the question at the core of some much more interesting and important questions. For example, *Does my algorithm finish or just run indefinitely?* or *Does my mathematical model accurately describe this situation in the long-term?*

2. Gain some proficiency working with trigonometric integrands.

3. Learn when performing a trigonometric substitution will help you evaluate an integral. Generally just practice taking more integrals too, and appreciate how massively helpful making a substitution can be; changing variables has versatility beyond just letting u be the gross thing in an integrand.

Exercises

1. From Chapter 8.4 of *Calculus with Applications, 11th Edition* work through the integration drills at the beginning of the Exercises section, doing like every other odd integral. Also look at the following exercises:

45. Principal Amount John put his money in a bank that gives a perpetual stream of income with a flow rate of

$$R(t) = 1000e^{0.01t}.$$

Find the amount of money he deposited if the bank has an interest rate of 3% compounded continuously.

49. Drug Reaction The rate of reaction to a drug is given by

$$r'(t) = 2t^2e^{-t},$$

where t is the number of hours since the drug was administered. Find the total reaction to the drug over all the time since it was administered, assuming this is an infinite time interval. (*Hint:* $\lim_{t \to \infty} t^k e^{-t} = 0$ for all real numbers k.)

2. (SUBMIT) Evaluate the integral

$$\int_2^\infty \frac{\mathrm{d}x}{x(\ln(x))^2}.$$

3. (IMPROPER INTEGRALS) In case you'd like *even more* practice, evaluate each of the following integrals. Some of these I borrowed from Paul's Online Math Notes, so you might find solutions there.

$$\int_0^\infty x e^{-x}\,dx \qquad \int_e^\infty \frac{dx}{x\ln(x)^2} \qquad \int_0^{\pi/2} \frac{\cos(x)}{\sqrt{\sin(x)}}\,dx$$

$$\int_{-5}^1 \frac{1}{10+2z}\,dz \qquad \int_0^4 \frac{x}{x^2-9}\,dx \qquad \int_{-\infty}^\infty \frac{6w^3}{(w^4+1)^2}\,dw$$

$$\int_{-1}^1 \ln|x|\,dx \qquad \int_1^\infty \frac{1}{x^3}\,dx \qquad \int_1^\infty \frac{1}{x^{1/3}}\,dx$$

$$\int_{-\infty}^1 \frac{3}{1+x^2}\,dx \qquad \int_{-\infty}^0 \frac{e^{\frac{1}{x}}}{x^2}\,dx \qquad \int_0^2 \frac{1}{(x-1)^4}\,dx$$

4. Let's define the function $A(z) = \int_1^z x^{-p}\,dx$ for $z > 1$.

   (a) Show that if $p = 1$, then $A(z) = \ln|z|$, but that otherwise, for $p \neq 1$, we have
   $$A(z) = \frac{1}{1-p}\left(z^{1-p} - 1\right).$$

   (b) Show that if $p \in (0,1]$, then $\lim_{z\to\infty} A(z) = \infty$.

   (c) Now show that if $p > 1$, then $\lim_{z\to\infty} A(z) = \frac{1}{p-1}$.

   Look out for the vocabulary term p-series later in the course, and remember this exercise when you see it ;)

5. (CHALLENGE, SPIVAK) Consider the curve given by $y = \frac{1}{x}$ for $x \geq 1$. This shape is sometimes called Gabriel's horn.

   (a) What is the volume of the inside of the horn?

(b) You can find the surface area of the horn by taking the curve $y = \frac{1}{x}$, writing down a formula for it's arclength, and rotating that arclength about the x-axis over a bunch of small subintervals, and taking an integral, thereby summing up a lot of smaller surface areas. Do this, and show that the horn has infinite surface area.

(c) Suppose you take a volume of paint that is equal to the volume of the inside of the horn and pour it into the horn. This would seem to paint the entire inside of the horn, but we will have painted the infinite surface of the horn with finitely much paint. How can this be?

6. Review Chapter 13.2 of *Calculus with Applications, 11th Edition* and make sure you remember the trig functions and their derivatives.

7. From Chapter 13.3 of *Calculus with Applications, 11th Edition* work through the integration drills at the beginning of the Exercises section, doing like every other odd integral. Also look at the following exercises:

Find the area between the two curves. (Refer to Section 7.5.)

41. $x = 0, \quad x = \pi/4, \quad y = \cos x, \quad y = \sin x$

42. $x = 0, \quad x = \pi/4, \quad y = \sec^2 x, \quad y = \sin 2x$

43. $x = 0, \quad x = \pi/4, \quad y = \tan x, \quad y = \sin x$

44. $x = 0, \quad x = \pi, \quad y = \sin x, \quad y = 1 - \sin x$

49. **Length of Day** The following function can be used to estimate the number of minutes of daylight in Boston for any given day of the year.

$$N(t) = 183.549 \sin(0.0172t - 1.329) + 728.124,$$

where t is the day of the year. Use this function to estimate the total amount of daylight in a year and compare it to the total amount of daylight reported to be 4467.57 hours. ***Source: The Old Farmer's Almanac***.

8. (TRIGONOMETRIC INTEGRALS) Evaluate *some* of the following integrals for practice. These are a bit tougher than the ones in the textbook. Solutions to a couple of them can be found in Paul's Online Notes.

$$\int \sin^2(x)\cos^7(x)\,\mathrm{d}x \qquad \int \cos^4(\theta)\,\mathrm{d}\theta$$

$$\int \tan^3(\mu)\,\mathrm{d}\mu \text{ (SUBMIT)} \qquad \int \cot(10z)\csc^4(10z)$$

9. (TRIGONOMETRIC SUBSTITUTION) Evaluate *some* of the following integrals. They all require the same trick of making a trigonometric substitution, so only do as many as it takes you to feel good.

$$\int \frac{\mathrm{d}x}{1+x^2} \qquad \int \frac{\mathrm{d}x}{1-x^2} \qquad \int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} \qquad \int \frac{\mathrm{d}x}{\sqrt{x^2-1}} \qquad \int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} \text{ (SUBMIT)}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} \qquad \int \frac{dx}{x\sqrt{1+x^2}} \qquad \int x^3\sqrt{1-x^2}\,dx$$

$$\int \sqrt{1-x^2}\,dx \qquad \int \sqrt{1+x^2}\,dx \qquad \int \sqrt{x^2-1}\,dx$$

10. (TRIGONOMETRIC SUBSTITUTION) Here's some tougher integrals requiring trig substitution. Evaluate them if you'd like to practice. Solutions to these are written up in Paul's Online Math Notes.

$$\int \sqrt{1-7\omega^2}\,d\omega \qquad \int_{-7}^{-5} \frac{2\,dy}{y^4\sqrt{y^2-25}} \qquad \int \frac{dx}{\sqrt{9x^2-36x+37}}$$

11. (TOUGH) Evaluate the following integrals if you'd like.

$$\int \frac{dx}{\sqrt{1+e^{2x}}} \qquad \int \ln\left(\mu+\sqrt{1-\mu}\right)\,d\mu \qquad \int \frac{d\theta}{1-\sin^4(\theta)}$$

Since you've learned *integration-by-parts*, and how to handle integrals with trigonometric functions, and when you should make a substitution for a trig function, you've officially learned all the techniques of integration that are in the curriculum for this course. Congratulations! The exercises that follow are some integration-themed challenges, in case you enjoy the puzzle of solving integrals like I do.

12. (CHALLENGE: THE $x \leftrightarrow \frac{1}{x}$ TRICK) Show that the integral

$$\int_0^\infty \frac{\ln(x)}{1+x^2}\,dx$$

evaluates to zero by breaking up its domain of integration into two parts based on where the integrand is negative or positive, and using the substitution $x \leftrightarrow \frac{1}{x}$ on one of those parts. Then use this same trick to show that the value of the following integral doesn't depend at all on the value of the real number a.

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{1+\big(\tan(\theta)\big)^a}$$

For some helpful reading related to this trick, see

math.stackexchange.com/q/2060187.

13. (CHALLENGE) Evaluate the following integrals.

$$\int \frac{\mathrm{d}\omega}{\left(\omega^2+1\right)\sqrt{\omega^2-1}} \qquad \int \sqrt{\tan(\theta)}\,\mathrm{d}\theta \qquad \int_0^{\pi/2} \ln(\cos(t))\,\mathrm{d}t$$

14. (CHALLENGE: 2005 PUTNAM EXAM) Evaluate

$$\int_0^1 \frac{\ln(x+1)}{1+x^2}\,\mathrm{d}x.$$

15. (CHALLENGE: 1987 PUTNAM EXAM) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}\,\mathrm{d}x.$$

You might also be interested in reading up on some lesser known integration tricks:

math.stackexchange.com/q/70974/167197.