Homework Eight

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

These exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. Know what a geometric series is, either finite or infinite, and know the formula for its value.

2. KNOW WHAT A TAYLOR SERIES IS! Appreciate their importance. This is literally one the most useful things ever: you can manually approximate transcendental numbers like e with these! (You should know how to do this, like in the last exercise on this homework) You should know the general formula for a Taylor series, and also commit to temporary memory the Taylor series for the functions e^x and $(1-x)^{-1}$ and $\ln(1+x)$ and their intervals of convergence. Develop some comfort doing arithmetic on Taylor series and composing Taylor series;

3. *Newton's method* is a great way to estimate roots (zeros) of a function. It's an algorithmic method though, and should really only be implemented using a computer, so there's no need to do any exercises over it. But you should certainly understand the idea behind Newton's method.

Exercises

1. The initial exercises from Chapter 12.1 & Chapter 12.4 *Calculus with Applications, 11th Edition* will help you internalize what a geometric series is, and how to calculate the explicit value of such a series, whether it's finite or infinite. Do them.

2. Some "application" questions from the textbook.

Thickness of a Paper Stack A piece of paper is 0.008 in. thick.

(a) Suppose the paper is folded in half, so that its thickness doubles, for 12 times in a row. How thick is the final stack of paper?

(b) Suppose it were physically possible to fold the paper 50 times in a row. How thick would the final stack of paper be?

Future Value In Section 8.3, we computed the present value of a continuous flow of money. Suppose that instead of a continuous flow, an amount C is deposited each year, and the annual interest rate is r. Then the future value of the cash flow over n years is

$$F = C + C(1 + r) + C(1 + r)^2 + \cdots + C(1 + r)^{n-1}.$$

(a) Show that the future value can be simplified to

$$F = C\frac{((1 + r)^n - 1)}{r}.$$

(b) Show that the future value taken over an infinite amount of time is infinite.

32. Perimeter A sequence of equilateral triangles is constructed as follows: The first triangle has sides 2 m in length. To get the next triangle, midpoints of the sides of the previous triangle are connected. If this process could be continued indefinitely, what would be the total perimeter of all the triangles?

33. Area What would be the total area of all the triangles of Exercise 32, disregarding the overlaps?

Zeno's Paradox In the fifth century B.C., the Greek philosopher Zeno posed a paradox involving a race between Achilles (the fastest runner at the time) and a tortoise. The tortoise was given a head start, but once the race began, Achilles quickly reached the point where the tortoise had started. By then the tortoise had moved on to a new point. Achilles quickly reached that second point, but the tortoise had now moved to another point. Zeno concluded that Achilles could never reach the tortoise because every time he reached the point where the tortoise had been, the tortoise had moved on to a new point. This conclusion was absurd, yet people had trouble finding an error in Zeno's logic.

Suppose Achilles runs 10 m per second, the tortoise runs 1 m per second, and the tortoise has a 10-m head start. We wish to find how much time it takes until Achilles catches up with the tortoise.

(a) Solve this problem using a geometric series. (See Example 3.)

(b) Solve this problem using algebra.

(c) Explain the error in Zeno's reasoning.

3. I expect you to know the Taylor series for the functions e^x and $(1-x)^{-1}$ and $\ln(1+x)$ and feel comfortable doing some arithmetic with them. The initial exercises from Chapter 12.5 of *Calculus with Applications, 11th Edition* provides you with some practice with this.

4. Compute the Taylor series centered at 0 for the functions sine and cosine. The radius of convergence for each of these series is $(-\infty, \infty)$. Up until this point you've had to trust that the derivative of sine is cosine, and that the derivative of cosine is negative sine, but now you have the tools to prove this! Using just the power rule, verify that

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)\,.$$

Once you've done this, using their Taylor series, verify that

$$\cos(x) = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2} \quad \text{and} \quad \sin(x) = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2i}$$

where $\mathrm{i}^2 = -1$.

5. Using the appropriate Taylor series, demonstrate how you can, using only pencil and paper, find rational approximation to any of the following numbers with as much precision as you desire.

$$\sin(3) \qquad \ln(2) \qquad \mathrm{e}$$