Homework Seven

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

Upload your responses to the prompts marked (Submit) to Gradescope before 8pm Friday; you will receive feedback on these.

`gradescope.com/courses/275664`

The rest of the exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. You've gotta know what if means for a function to be a probability density function (PDF) and how a cumulative distribution function (CDF) relates to its corresponding PDF via the Fundamental Theorem of Calculus.

2. You should gain some heavy familiarity with the basic vocabulary of calculus-based probability. Know what the expected value (mean), median, variance, and standard deviation of a continuous random variable are, and how to calculate these from the PDF of that random variable. Know what a uniform, exponential, and normal distribution are, and what the z-score of a value of a normally distributed random variable is.

Exercises

1. Remember a PDF is a continuous function f defined on some domain $[a, b]$ (where a and b might be infinite) such that (1) $f(x)$ is positive for all x between a and b, and (2) $\int_a^b f(x)\,\mathrm{d}x = 1$. The initial exercises from Chapter 11.1 of *Calculus with Applications, 11th Edition* only serve to reinforce this definition by making you do some contrived exercises. Do those exercises if it helps, but the important thing is to understand *why* a PDF is defined the way it is. Also, understand that if a function is positive on some interval you can always *make it* a PDF by rescaling it.

2. The initial exercises from Chapter 11.2 of *Calculus with Applications, 11th Edition* provides some good practice at calculating expected value, variance, standard deviation, etc. Do as many as you need to feel confident with these calculations.

3. (SUBMIT) The definition of the variance of a random variable X with PDF f and domain $[a, b]$ is given by

$$\int_a^b (x-\mu)^2 f(x)\,\mathrm{d}x$$

where μ is the expected value (mean) of X. But there's an easier formula for the variance. Prove that

$$\int_a^b (x-\mu)^2 f(x)\,\mathrm{d}x = \int_a^b x^2 f(x)\,\mathrm{d}x - \mu^2 .$$

HINT: remember the *definition* of μ.

4. Again, the initial exercises from Chapter 11.3 of *Calculus with Applications, 11th Edition* are silly, but do serve to reinforce the definitions from this section. Do as many of these exercises as you need to know what the uniform, exponential, and normal distributions are, and what a z-score is.

5. These exercises from the textbook are all basically the same exercise over and over, but are still meaningful and demonstrate the purpose of calculus-based probability.

Life Span of a Television Part The life (in months) of a certain television part has a probability density function defined by

$$f(t) = \frac{1}{3}e^{-t/3}.$$

Find the probability that a randomly selected component will last for the following lengths of time.

(a) At most 9 months

(b) Between 3 and 9 months

(c) Find the cumulative distribution function for this random variable.

(d) Use the answer from part (c) to find the probability that a randomly selected part will last no more than 6 months.

(SUBMIT) everything about these blood clots:

Clotting Time of Blood The clotting time of blood is a random variable t with values from 1 second to 20 seconds and probability density function defined by

$$f(t) = \frac{1}{(\ln 20)t}.$$

Find the following probabilities for a person selected at random.

(a) The probability that the clotting time is between 1 and 5 seconds

(b) The probability that the clotting time is greater than 10 seconds

Blood Clotting Time The clotting time of blood (in seconds) is a random variable with probability density function defined by

$$f(t) = \frac{1}{(\ln 20)t} \quad \text{for } t \text{ in } [1, 20].$$

(a) Find the mean clotting time.

(b) Find the standard deviation.

(c) Find the probability that a person's blood clotting time is within 1 standard deviation of the mean.

(d) Find the median clotting time.

Annual Rainfall The annual rainfall in a remote Middle Eastern country varies from 0 to 5 in. and is a random variable with probability density function defined by

$$f(x) = \frac{5.5 - x}{15}.$$

Find the following probabilities for the annual rainfall in a randomly selected year.

(a) The probability that the annual rainfall is greater than 3 in.

(b) The probability that the annual rainfall is less than 2 in.

(c) The probability that the annual rainfall is between 1 in. and 4 in.

Annual Rainfall The annual rainfall in a remote Middle Eastern country is a random variable with probability density function defined by

$$f(x) = \frac{5.5 - x}{15}, \quad \text{for } x \text{ in } [0, 5].$$

(a) Find the mean annual rainfall.

(b) Find the standard deviation.

(c) Find the probability of a year with rainfall less than 1 standard deviation below the mean.

Rainfall The rainfall (in inches) in a certain region is uniformly distributed over the interval $[32, 44]$.

(a) What is the expected number of inches of rainfall?

(b) What is the probability that the rainfall will be between 38 and 40 in.?

(SUBMIT) everything about these earthquakes:

Earthquakes The time between major earthquakes in the Southern California region is a random variable with probability density function defined by

$$f(t) = \frac{1}{960} e^{-t/960},$$

where t is measured in days. Find the expected value and the standard deviation of this probability density function. ***Source: Journal of Seismology***.

Earthquakes The time between major earthquakes in the Southern California region is a random variable with probability density function

$$f(t) = \frac{1}{960} e^{-t/960},$$

where t is measured in days. ***Source: Journal of Seismology***.

(a) Find the probability that the time between a major earthquake and the next one is less than 365 days.

(b) Find the probability that the time between a major earthquake and the next one is more than 960 days.

Machine Life The life (in years) of a certain machine is a random variable with probability density function defined by

$$f(t) = \frac{1}{11}\left(1 + \frac{3}{\sqrt{t}}\right) \quad \text{for } t \text{ in } [4, 9].$$

(a) Find the mean life of this machine.

(b) Find the standard deviation of the distribution.

(c) Find the probability that a particular machine of this kind will last longer than the mean number of years.

Drunk Drivers The frequency of alcohol-related traffic fatalities has dropped in recent years but is still high among young people. Based on data from the National Highway Traffic Safety Administration, the age of a randomly selected, alcohol-impaired driver in a fatal car crash is a random variable with probability density function given by

$$f(t) = \frac{4.045}{t^{1.532}} \quad \text{for } t \text{ in } [16, 80].$$

Find the following probabilities of the age of such a driver. ***Source: Traffic Safety Facts***.

(a) Less than or equal to 25

(b) Greater than or equal to 35

(c) Between 21 and 30

(d) Find the cumulative distribution function for this random variable.

(e) Use the answer to part (d) to find the probability that a randomly selected alcohol-impaired driver in a fatal car crash is at most 21 years old.

Drunk Drivers In the last section, we saw that the age of a randomly selected, alcohol-impaired driver in a fatal car crash is a random variable with probability density function given by

$$f(t) = \frac{4.045}{t^{1.532}} \quad \text{for } t \text{ in } [16, 80].$$

Source: Traffic Safety Facts.

(a) **APPLY IT** Find the expected age of a drunk driver in a fatal car crash.

(b) Find the standard deviation of the distribution.

(c) Find the probability that such a driver will be younger than 1 standard deviation below the mean.

(d) Find the median age of a drunk driver in a fatal car crash.

6. (RECREATION) There's a map of the world that has every single town in the world marked on it. For each town on this map you draw a straight line connecting it to its nearest neighboring town (nearest in terms of distance along a straight line, *not* distance along roads). Show that after you've done this for every town on the map, each town can be connected to at most five others.