Homework Three

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

Upload your responses to the prompts marked (Submit) to Gradescope before 8pm Friday; you will receive feedback on these.

`gradescope.com/courses/275664`

The rest of the exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. Learn to feel comfortable plotting multi-variable equations. There is no *procedure* for doing this. It's an art. You've got to feel comfortable investigating an equation, looking at its level curves, plotting points, etc. Note though that you're only doing this to appreciate the art of it; in practice you just use computer software.

2. Become proficient calculating partial derivatives.

3. Understand what a partial derivative *is*. Like, basically you should understand how your intuition for the derivative of a function as a rate of change (slope) fits into the context of multivariable functions.

Exercises

1. I expect that you should be able to (1) accurately evaluate a multi-valued function at a set of values and (2) graph a plane in three-dimensional space. From Chapter 9.1 of *Calculus with Applications, 11th Edition* work through the enough of the initial exercises that you can do these things. Then consider these exercises too:

19. Discuss how a function of three variables in the form $w = f(x, y, z)$ might be graphed.

20. Suppose the graph of a plane $ax + by + cz = d$ has a portion in the first octant. What can be said about a, b, c, and d?

21. In the chapter on Nonlinear Functions, the vertical line test was presented, which tells whether a graph is the graph of a function. Does this test apply to functions of two variables? Explain.

Individual Retirement Accounts **The multiplier function**

$$M = \frac{(1 + i)^n(1 - t) + t}{[1 + (1 - t)i]^n}$$

compares the growth of an Individual Retirement Account (IRA) with the growth of the same deposit in a regular savings account. The function M depends on the three variables n, i, and t, where n represents the number of years an amount is left at interest, i represents the interest rate in both types of accounts, and t represents the income tax rate. Values of $M > 1$ indicate that the IRA grows faster than the savings account. Let $M = f(n, i, t)$ and find the following.

38. Find the multiplier when funds are left for 25 years at 5% interest and the income tax rate is 33%. Which account grows faster?

39. What is the multiplier when money is invested for 40 years at 6% interest and the income tax rate is 28%? Which account grows faster?

2. (INDIVIDUAL RETIREMENT ACCOUNTS) That previous example I screenshot from the textbook is stupid for a couple of reasons, but the major reason is that *the interest rate on an IRA and on a savings account are vastly different*! Ie, the i in the numerator and denominator of M should be different parameters. This is because your IRA derives its value from stocks, bonds, mutual funds, etc, whereas the interest on your savings account is a pittance the bank gives you, probably only because they're legally required to do so. Look up the typical annual returns on an IRA (or an analogous long-term retirement account if you're outside the US) and the interest rate of a typical bank savings account (or use the interest rate on your own savings account if you have one), and use these honest values to answer the questions in the screenshot above again.

   That example is also stupid because it assumes the tax rate t is some constant over the n years that you stay invested. But again, the t in the numerator and denominator shouldn't even be the same parameter! For a normal IRA you only pay taxes once you start withdrawing money from the account, so the t in the numerator should be the tax rate *after you've been invested for n years*, whereas the t in the denominator is a function of time since you started the account. If however you have a Roth IRA, you pay taxes putting money into the account so in this case these should at least be the function of t in the numerator and denominator of M. ... There is no question here. I just think it's important that, when you're handed an equation and told "*this equation models this thing,*" that you contemplate the accuracy of the model with great skepticism.

3. (SUBMIT) Below are some functions presented as $f \rightsquigarrow g$. Be able to graph the function f, and then describe how the graph of g differs from the graph of f.

$$f(x,y) = x^2 + y^2 \;\rightsquigarrow\; g(x,y) = 4x^2 + y^2$$

$$f(x,y) = \sqrt{1 - x^2 - y^2} \;\rightsquigarrow\; g(x,y) = \sqrt{9 - x^2 - y^2}$$

$$f(x,y) = x^2 - y^2 \;\rightsquigarrow\; g(x,y) = y^2 - x^2$$

$$f(x,y) = \sqrt{x^2 + y^2} \;\rightsquigarrow\; g(x,y) = \sqrt{(x-1)^2 + (y-2)^2}$$

$$f(x,y) = x + y \;\rightsquigarrow\; g(x,y) = x + y + 7$$

4. I expect that you should be able to accurately calculate first-order and second-order partial derivatives. This is "easy" because calculating derivatives is "easy," but this is also difficult because you now have extra variables hanging out that you have to "pretend are constants." Practice a bunch of the initial exercises from Chapter 9.2 of *Calculus with Applications, 11th Edition* to build this skill until you get the hang of it.

   Glance over all the other exercises in the section, noting how many situations we can analyze by looking at the partial derivatives of multivariable functions. In particular consider these exercises:

55. Calorie Expenditure The average energy expended for an animal to walk or run 1 km can be estimated by the function

$$f(m, v) = 25.92m^{0.68} + \frac{3.62m^{0.75}}{v},$$

where $f(m, v)$ is the energy used (in kcal per hour), m is the mass (in g), and v is the speed of movement (in km per hour) of the animal. ***Source: Wildlife Feeding and Nutrition***.

(a) Find $f(300, 10)$.

(b) Find $f_m(300, 10)$ and interpret.

(c) If a mouse could run at the same speed that an elephant walks, which animal would expend more energy? How can partial derivatives be used to explore this question?

63. Drug Reaction The reaction to x units of a drug t hours after it was administered is given by

$$R(x, t) = x^2(a - x)t^2e^{-t},$$

for $0 \leq x \leq a$ (where a is a constant). Find the following.

(a) $\dfrac{\partial R}{\partial x}$ **(b)** $\dfrac{\partial R}{\partial t}$ **(c)** $\dfrac{\partial^2 R}{\partial x^2}$ **(d)** $\dfrac{\partial^2 R}{\partial x \partial t}$

(e) Interpret your answers to parts (a) and (b).

69. Gravitational Attraction The gravitational attraction F on a body a distance r from the center of Earth, where r is greater than the radius of Earth, is a function of its mass m and the distance r as follows:

$$F = \frac{mgR^2}{r^2},$$

where R is the radius of Earth and g is the force of gravity—about 32 feet per second per second (ft per sec^2).

(a) Find and interpret F_m and F_r.

(b) Show that $F_m > 0$ and $F_r < 0$. Why is this reasonable?

5. (SUBMIT)

53. Airline Competition A model of airline competition defines $f(x, y)$ as the probability that a customer will buy a ticket from Airline 1 rather than Airline 2 if the price for a ticket is x from Airline 1 and y from Airline 2. A similar function $g(x, y)$ gives the probability that a customer will buy a ticket from Airline 2 rather than Airline 1 if the price for a ticket is x from Airline 1 and y from Airline 2. For simplicity, the model assumes that $f(x, y) = g(y, x)$. ***Source: The Journal of the Operational Research Society.***

(a) Explain what the last assumption means, and why it is reasonable.

(b) The researchers also assume that

$$f_x(x, y) < 0 \quad \text{and} \quad f_y(x, y) > 0.$$

Explain what this assumption means, and why it is reasonable.

(c) One version of the model assumes

$$f(x, y) = \frac{y(1 + x)e^{-x}}{x + y}.$$

Calculate $f_x(x, y)$ and $f_y(x, y)$ and verify that the assumptions of part (b) are satisfied.

6. (RECREATION) Suppose that in the 2-dimensional plane ($\mathbf{R}^2$), every point is to be colored either red or blue. Show that no matter how the points in the plane are colored, there has to exist some equilateral triangle in the plane such that the vertices of the triangle are all the same color.