Homework Four

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

Upload your responses to the prompts marked (SUBMIT) to Gradescope before 8pm Friday; you will receive feedback on these.

gradescope.com/courses/275664

The rest of the exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. Be able to identify critical points of functions, and do some analysis to classify them as minimums or maximums or saddles.

2. Become proficient at *using* the method of Lagrange multipliers to solve optimization problems of the form "minimize/maximize f under the constraint g." You don't need to *understand* this method for this class; save that for a dedicated optimization class.

3. The total differential is form used to estimate the value of a function at a "strange" point. But using this form obfuscates the big idea: *you can approximate smooth curves with their tangent lines, and you can approximate smooth surfaces with their tangent planes*. Understand this please.

Exercises

1. I expect you to be able to find the critical points of a function, and be able to investigate/classify those critical points as minimums or maximums or saddles. The initial exercises from Chapter 9.3 of *Calculus with Applications, 11th Edition* will help you develop those skills. Do them until you feel confident, then also work through these exercises:

35. Costs Suppose the cost function of manufacturing cost for a certain product be approximated by

$$C(x, y) = 3x^2 + y^2 - x - y - 3xy + 100,$$

where x is the cost of labor per hour and y is the cost of materials per unit. Find values of x and y that minimize the cost function. Find the minimum cost.

Figures (a)–(f) show the graphs of the functions defined in Exercises 21–26. Find all relative extrema for each function, and then match the equation to its graph.

21. $z = -3xy + x^3 - y^3 + \dfrac{1}{8}$ **22.** $z = \dfrac{3}{2}y - \dfrac{1}{2}y^3 - x^2y + \dfrac{1}{16}$

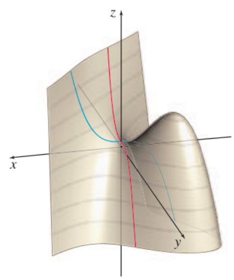
23. $z = y^4 - 2y^2 + x^2 - \dfrac{17}{16}$

24. $z = -2x^3 - 3y^4 + 6xy^2 + \dfrac{1}{16}$

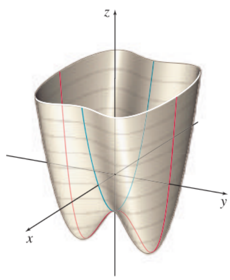
25. $z = -x^4 + y^4 + 2x^2 - 2y^2 + \dfrac{1}{16}$

26. $z = -y^4 + 4xy - 2x^2 + \dfrac{1}{16}$

(a)



(b)



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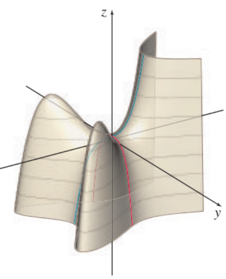
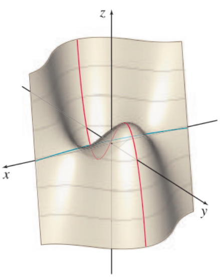
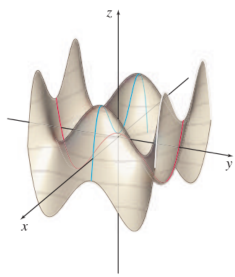


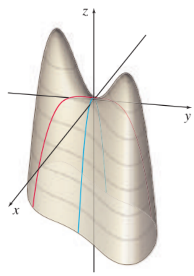
(d)



(e)



(f)



2. (SUBMIT)

40. Political Science The probability that a three-person jury will make a correct decision is given by

$$P(\alpha, r, s) = \alpha[3r^2(1 - r) + r^3] + (1 - \alpha)[3s^2(1 - s) + s^3],$$

where $0 < \alpha < 1$ is the probability that the person is guilty of the crime, r is the probability that a given jury member will vote "guilty" when the defendant is indeed guilty of the crime, and s is the probability that a given jury member will vote "innocent" when the defendant is indeed innocent. ***Source: Frontiers of Economics***.

(a) Calculate $P(0.9, 0.5, 0.6)$ and $P(0.1, 0.8, 0.4)$ and interpret your answers.

(b) Using common sense and without using calculus, what value of r and s would maximize the jury's probability of making the correct verdict? Do these values depend on α in this problem? Should they? What is the maximum probability?

(c) Verify your answer for part (b) using calculus. (*Hint:* There are two critical points. Argue that the maximum value occurs at one of these points.)

3. (RESEARCH MAYBE?) The hyperbolic paraboloid given by the graph of $f(x, y) = x^2 - y^2$ is our canonical example of a surface with a saddle point. We can see it has a saddle point at $(0, 0)$ without even investigating the discriminant, by noting instead that

- $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, so it's "flat" at $(0, 0)$,

- $f_{xx}(0,0) > 0$ so it flairs upward along the x-axis, and
- $f_{yy}(0,0) < 0$ so it flairs downward along the y-axis.

Can you come up with (or find) and example of a function g that has a saddle point at a point (a,b), but such that $g_{xx}(a,b)$ and $g_{yy}(a,b)$ are either both positive or both negative? Ie (a,b) is a saddle point even though the graph of g flairs in the same direction along both the x-axis and y-axis?

4. You should become proficient at using *the method of Lagrange multipliers* to solve optimization problems. Really this comes down to recalling what *the method* entails, and crunching some equations. Just gotta practice. Solve a bunch of these problems from the beginning of Chapter 9.4 of *Calculus with Applications, 11th Edition.* Also check out these exercises:

30. Profit The profit from the sale of x units of a certain product and y units of another product is given by the function

$$P(x, y) = -2x^2 - y^2 + 10x + 12y.$$

Find values of x and y that lead to a maximum profit if a total of 15 units has to be manufactured.

39. Cost A rectangular closed box is to be built at minimum cost to hold 125 m^3. Since the cost will depend on the surface area, find the dimensions that will minimize the surface area of the box.

40. Cost Find the dimensions that will minimize the surface area (and hence the cost) of a rectangular fish aquarium, open on top, with a volume of 32 ft^3.

5. (SUBMIT)

42. Political Science The probability that the majority of a three-person jury will convict a guilty person is given by the formula:

$$P(r, s, t) = rs(1 - t) + (1 - r)st + r(1 - s)t + rst$$

subject to the constraint that

$$r + s + t = \alpha,$$

where r, s, and t represent each of the three jury members' probability of reaching a guilty verdict and α is some fixed constant that is generally less than or equal to the number of jurors. ***Source: Mathematical Social Sciences.***

(a) Form the Lagrange function.

(b) Find the values of r, s, and t that maximize the probability of convicting a guilty person when $\alpha = 0.75$.

(c) Find the values of r, s, and t that maximize the probability of convicting a guilty person when $\alpha = 3$.

6. In Chapter 9.5 of *Calculus with Applications, 11th Edition* many of the exercises are contrived. I guess you should be able to wield the total differential to estimate some strange numbers though. Work though exercises 7, 9, 11, and 13 here, and then SUBMIT your calculations for for to exercise 8.

Use the total differential to approximate each quantity. Then use a calculator to approximate the quantity, and give the absolute value of the difference in the two results to 4 decimal places.

7. $\sqrt{8.05^2 + 5.97^2}$

8. $\sqrt{4.96^2 + 12.06^2}$

9. $(1.92^2 + 2.1^2)^{1/3}$

10. $(2.93^2 - 0.94^2)^{1/3}$

11. $1.03e^{0.04}$

12. $0.98e^{-0.04}$

13. $0.99 \ln 0.98$

14. $2.03 \ln 1.02$

7. (RECREATION) You are placing planes in three-dimensional space that all go through the origin $(0,0,0)$ with the goal of separating space into as many parts as possible. So when you place the first plane, you've separated space into two pieces. You can place a second plane and separate space into four pieces, and then you can place a third plane to separate space into eight parts (think of the coordinate planes as an example of this). If you were to place a fourth plane in space going through the origin, what is the largest number of parts that you can have divided space into? Then what if you place a fifth plane in space?