Homework Six

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

Upload your responses to the prompts marked (SUBMIT) to Gradescope before 8pm Friday; you will receive feedback on these.

gradescope.com/courses/275664

The rest of the exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. Learn to solve a linear first-order ODE. It's just a procedure/calculation to practice a few times. You'll get some practice at this while studying applications of ODEs too.

2. Understand Euler's method of approximating the solution to an ODE, and relate this understanding to an ODE's direction (vector) field. This is also related to a broad theme of calculus: linear approximations are easy to compute and often good enough for whatever you're doing.

3. The study of differential equations is born from the art of applying mathematics to study the natural world. This introduction to ODEs was brief, but you should familiarize yourself with as many applications of ODEs as you can before we go.

Exercises

1. I expect you to be able to recognize a linear first-order differential equation when you see one and then solve it if you want. The initial exercises from Chapter 10.2 of *Calculus with Applications, 11th Edition* provides you with plenty of drills to practice.

2. Give me an example of a single variable function such that the derivative of the function is equal to one more than the function.

3. With this exercise you'll practice some symbol-pushing.

26. Mouse Infection A model for the spread of an infectious disease among mice is

$$\frac{dN}{dt} = rN - \frac{\alpha r(\alpha + b + v)}{\beta\left[\alpha - r\left(1 + \frac{v}{b + \gamma}\right)\right]},$$

where N is the size of the population of mice, α is the mortality rate due to infection, b is the mortality rate due to natural causes for infected mice, β is a transmission coefficient for the rate that infected mice infect susceptible mice, v is the rate the mice recover from infection, and γ is the rate that mice lose immunity. Show that the solution to this equation, with the initial condition $N(0) = (\alpha + b + v)/\beta$, can be written as

$$N(t) = \frac{(\alpha + b + v)}{\beta R}[(R - \alpha)e^{rt} + \alpha],$$

where

$$R = \alpha - r\left(1 + \frac{v}{b + \gamma}\right).$$

Source: Lectures on Mathematics in the Life Sciences.

4. I expect you to *understand* Euler's method of approximating a solution to an ODE, but the actual calculations are tedious and should never be done by hand; morally this is something you should only be programming a computer to do. So only do the initial exercises from Chapter 10.3 of *Calculus with Applications, 11th Edition* if you really feel you need to. Otherwise I think it'd be more fruitful to you to just read about Euler's method and listen to folks explain it until you get the idea.

5. You might want to read over the way *Calculus with Applications, 11th Edition* models continuous financial deposits in Chapter 10.4 before looking at these exercises.

   1. **Continuous Deposits** John is saving from his salary. He deposits \$900 monthly in a bank, which has an interest rate of 3% compounded continuously. How much will he have accumulated after 5 years?
   2. **Continuous Deposits** In Exercise 1, how long will it take John to accumulate \$40,000 to buy a car?
   3. **Continuous Deposits** To provide for a future expansion, a company plans to make continuous deposits to a savings account at the rate of \$50,000 per year, with no initial deposit. The managers want to accumulate \$500,000. How long will it take if the account earns 10% interest compounded continuously?

6. A flock of turkeys in a region will grow at a rate that is proportional to its current population. In the absence of any outside factors the population will triple in 5 days. On any given day about 3 turkeys die of natural causes, 9 turkeys are taken by hunters, and 2 turkeys wander into the flock from neighboring regions. If there are initially 50 turkeys in the flock will the flock survive or die out?

7. You might want to read over the way the textbook (*Calculus with Applications, 11th Edition*) models a predator/prey situation in Chapter 10.4 before looking at these exercises.

7. Competing Species The system of equations

$$\frac{dy}{dt} = 4y - 2xy$$

$$\frac{dx}{dt} = -3x + 2xy$$

describes the influence of the populations (in thousands) of two competing species on their growth rates.

(a) Following Example 2, find an equation relating x and y, assuming $y = 1$ when $x = 1$.

(b) Find values of x and y so that both populations are constant. (*Hint:* Set both differential equations equal to 0.)

8. Symbiotic Species When two species, such as the rhinoceros and birds pictured below, coexist in a symbiotic (dependent) relationship, they either increase together or decrease together. Typical equations for the growth rates of two such species might be

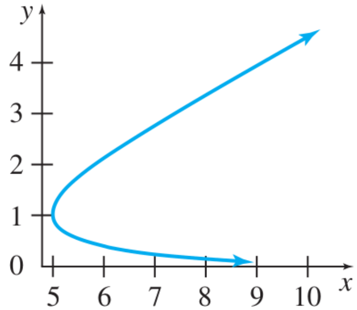
$$\frac{dx}{dt} = -4x + 4xy$$

$$\frac{dy}{dt} = -3y + 2xy.$$

(a) Find an equation relating x and y if $x = 5$ when $y = 1$.

(b) Find values of x and y so that both populations are constant. (See Exercise 7.)

(c) A graph of the relationship found in part (a) is shown in the figure. Based on the differential equations for the growth rate and this graph, what happens to both populations when $y > 1$? When $y < 1$?



8. A few more exercises from the textbook.

Spread of a Rumor **The equation developed in the text for the spread of an epidemic also can be used to describe diffusion of information. In a population of size N, let y be the number who have heard a particular piece of information. Then**

$$\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y$$

for a positive constant k. Use this model in Exercises 14–16.

14. Suppose a rumor starts among 3 people in a certain office building. That is, $y_0 = 3$. Suppose 500 people work in the building and 50 people have heard the rumor in 2 days. Using Equation (6), write an equation for the number who have heard the rumor in t days. How many people will have heard the rumor in 5 days?

15. A rumor spreads at a rate proportional to the product of the number of people who have heard it and the number who have not heard it. Assume that 3 people in an office with 45 employees heard the rumor initially, and 12 people have heard it 3 days later.

(a) Write an equation for the number, y, of people who have heard the rumor in t days.

(b) When will 30 employees have heard the rumor?

16. A news item is heard on the late news by 5 of the 100 people in a small community. By the end of the next day 20 people have heard the news. Using Equation (6), write an equation for the number of people who have heard the news in t days. How many have heard the news after 3 days?

9. Yet a few more exercises from the textbook.

> **22. Chemical in a Solution** Five grams of a chemical is dissolved in 100 liters of alcohol. Pure alcohol is added at the rate of 2 liters per minute and at the same time the solution is being drained at the rate of 1 liter per minute.
>
> **(a)** Find an expression for the amount of the chemical in the mixture at any time.
>
> **(b)** How much of the chemical is present after 30 minutes?
>
> **23.** Solve Exercise 22 if a 25% solution of the same mixture is added instead of pure alcohol.
>
> **24. Soap Concentration** A prankster puts 4 lb of soap in a fountain that contains 200 gal of water. To clean up the mess a city crew runs clear water into the fountain at the rate of 8 gal per minute, allowing the excess solution to drain off at the same rate. How long will it be before the amount of soap in the mixture is reduced to 1 lb?

10. (SUBMIT) Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1 g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min, and the well-stirred solution will flow out at the same rate. How much time will have to elapse before the amount of the dye in the tank reaches 1% of its original value?

The following model is not covered in the textbook, which is strange. It is a rather fundamental differential equation that models the acceleration/velocity of a falling object. In particular, if an object is falling towards a planet with gravitational constant g, then the velocity of that object is described by the equation $\dot{v} = g - kv$, where k is some constant. The idea is that the amount of air resistance that an object experiences while falling is proportional to its velocity; that's the whole idea behind the $-kv$ term. And obviously there's an independent variable t for time underlying this model, even though it's not explicitly required to describe $\dot{v}$.

11. Suppose you throw a 3 kg watermelon off the top of a tall building downward towards the parking lot below with an initial velocity of 17 m/s. While falling, the force of air resistance on your watermelon is 3 times the velocity of the falling melon. Write down a function that returns the velocity $v(t)$ of the watermelon after t seconds of being thrown. How long after being thrown will the watermelon be traveling at 11 m/s?

    HINT: Recall that acceleration due to gravity near the surface of the earth is given by $9.81\,\text{m/s}^2$, but I think for the sake of making calculations easier you should just round that number to $10\,\text{m/s}^2$.

12. (SUBMIT) A stone having a mass of 2 lbs is dropped from a bridge with no initial velocity, and encounters air resistance that is exactly equal to the square of its velocity. What is the velocity of the stone after 1 minute? For this question use imperial units, so acceleration due to gravity is $32\,\text{ft/s}^2$ instead of the usual metric $9.81\,\text{m/s}^2$.

13. A ball of mass 3 grams is thrown vertically into the air with an initial velocity 12 m/s. Suppose the ball encounters an air resistance equal to 5 times its velocity. Find a function $v(t)$ that returns the velocity of ball at a given time t. How long after being thrown upward does the ball reach its maximum height?

14. Suppose that there is a plane flying over the earth, equipped with a cannon that is pointed towards the earth. The cannon shoots a 2 kg cannonball at earth at an initial speed of 300 ft/s. But this is no ordinary cannonball; this is a Smart Cannonball™. After being fired it will gradually reconfigure its shape to become more aerodynamic to reduce air resistance. Following the convention that forces acting on the cannonball in the direction away from earth are negative, the Smart Corporation estimates that after t seconds of being fired the force from air resistance exerted on the cannonball will be $-\frac{v}{1+t}$, where v is the velocity of the cannonball at time t.

    Using this new estimate for the force of air resistance on the cannonball, write down a differential equation that models the motion of the cannonball. Then find a function $v(t)$ that returns the velocity of the cannonball after t seconds of being fired.

    Calculate the limit $\lim_{t\to\infty} v(t)$. Does the differential equation you developed have any equilibrium solutions? Based on this information, what can you conclude about the accuracy of Smart Corporation's claim that the force from air resistance will be given by $-\frac{v}{1+t}$ after t seconds?

15. (RECREATION) Five women and a monkey were shipwrecked on a desert island. They spent the first day gathering coconuts for

food, piling up all the coconuts together before they went to sleep for the night. But while they were all asleep, one woman woke up and thought there might be a row about dividing the coconuts in the morning, so she decided to get up and take her share now. She divided the coconuts into five equal piles except for one coconut left over which she gave to the monkey, and she hid her pile and put the rest back together. By and by, another woman woke up and did the same thing, and similarly she had one coconut left over which she gave to the monkey. And each of the five women did the same thing, one after the other; each one taking a fifth of the coconuts in the pile when she woke up, and each one having one left over for the monkey. In the morning they divided what coconuts were left, and they came out in five equal shares. Of course each one must have known that there were coconuts missing, but each one was as guilty as the others, so they didn't say anything. How many coconuts were there in the beginning?