Homework Five

Analytic Geometry and Calculus

UC Berkeley Math N16B, Summer 2021

Upload your responses to the prompts marked (SUBMIT) to Gradescope before 8pm Friday; you will receive feedback on these.

gradescope.com/courses/275664

The rest of the exercises you should complete at your discretion. Note that *Calculus with Applications, 11th Edition* has some select solutions, usually to odd-numbered exercises, in the back.

Goals this Week

Here are some goals you should have in mind while exercising:

1. Extend your interpretation of a definite integral as an area to higher dimensional regions. Given a three-dimensional region, know what double (or triple) integral calculates its volume. Also given an integral, know what geometric object it's measuring.

2. Know what a *differential equation* is! Since this is a really brief intro to the subject, focus on building familiarity with the vocabulary Also you should know how to solve a separable ODE (so long as the resulting integrals aren't too tough) and you should be able to find the equilibrium points of an ODE and know what they represent if the ODE is modelling something.

Exercises

1. I expect you to be practiced at calculating iterated (double) integrals, and at being able to express the area or volume of a region as an iterated integral. The initial exercises from Chapter 9.6 of *Calculus with Applications, 11th Edition* provides you with *a lot* of practice at the first one of these expectations, and a few for the second. Work through the odd-numbered exercises, or maybe every other odd exercise. Your choice—exercise until you feel confident. Also these exercises:

   The idea of the average value of a function, discussed earlier for functions of the form $y = f(x)$, can be extended to functions of more than one independent variable. For a function $z = f(x, y)$, the average value of f over a region R is defined as

   $$\frac{1}{A}\iint_R f(x, y)\, dx\, dy,$$

   where A is the area of the region R. Find the average value for each function over the regions R having the given boundaries.

   61. $f(x, y) = 6xy + 2x; \quad 2 \le x \le 5, 1 \le y \le 3$

   62. $f(x, y) = x^2 + y^2; \quad 0 \le x \le 2, 0 \le y \le 3$

   63. $f(x, y) = e^{-5y+3x}; \quad 0 \le x \le 2, 0 \le y \le 2$

   64. $f(x, y) = e^{2x+y}; \quad 1 \le x \le 2, 2 \le y \le 3$

2. What is the volume of the region bounded in the first quadrant by the plane given by the equation $(x-3)+2(x-5)+3(x-7)=0$?

3. (SUBMIT) Write down an iterated integral that calculates the volume bounded between the surfaces $z+2x^2+2y^2=36$ and $z+x^2+$

$y^2 = 20$. Can you also set up an integral that calculates this volume considering the region as a solid of revolution? Hint: Using the disk method you'll need the difference of two integrals, one corresponding to a volume you want to subtract from the other. Which one of these looks easiest to evaluate?

4. Estimate the volume of the space in the first quadrant bound between the curves

$$xyz = 1 \qquad \text{and} \qquad x + y + z = 11$$

I say "estimate" because find this exact volume is tough, but if you set it up as an integral $\iint_R f \, \mathrm{d}R$ for some region R in the (x, y)-plane, that R is *almost* a triangle, which allows you to get an easy estimate.

5. What is the volume of the dome-shaped region over the point $(0, 0, 0)$ trapped between the graph of $f(x, y) = \cos(\sqrt{x^2 + y^2})$ and the (x, y)-plane? Also, write down an integral that calculates this volume considering the region as a solid of revolution.

6. What is the volume of a region bound by the three cylinders, each of radius one, and each centered along one of the x-, y-, and z-axis? Is this bound region a sphere?

7. What is the volume of the region bounded by

$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x + y = 2z + 1$$

8. You have a bowl full of water. The surface of the bowl has the same shape as the graph of the function $f(x, y) = x^4 + y^4$ for $0 \le z \le 4$.

You tilt the bowl 30°, spilling out some of the water. What is the volume of the remaining water?

9. Consider a grain silo with a base that is a perfect circle 200 meters across. The walls of the silo stretch skyward 900 meters before meeting with a parabolic roof. The very tip of the parabolic roof, the highest point on the silo, is 1000 meters above the ground. Write down a double or triple integral that calculates the volume of this silo.

   Write down an integral that calculates the volume of this silo considered as a solid of revolution.

10. When you see an ODE, you should certainly immediately check if it's a separable ODE. I expect that you should be able to separate a separable ODE, write it as $M(y)\,\mathrm{d}y = N(x)\,\mathrm{d}x$ for some function M and N. Additionally you should be able to identify equilibrium points of an ODE and talk about their stability. The initial exercises from Chapter 10.1 of *Calculus with Applications, 11th Edition* will help you develop these two skills. Work through those exercises until you feel confident.

11. Verify that each of the following is a solution to the given differential equation.

    (a) Verify that $p(t) = 3\mathrm{e}^{kt}$ is a solution to $p' = kp$.

    (b) Verify that $y = 2t^3$ is a solution to $3yy''' = y'y''$.

    (c) Verify that $y(t) = \cos(2t)$ is a solution to $yy' + \sin(4t) = 0$.

    (d) Verify that $y(t) = \cos(2t)$ is also a solution to $yy'' + 4 = (y')^2$.

12. Give me an example of a single-variable function such that the product of that function with its derivative is equal to two.

13. Solve these differential equations. Notice that I'm not providing any initial conditions, so I'm asking for the *general* solution to each of these.

    $y\dot{y} + y^2 = 3ty\dot{y} + 1$

    $y' \cot^2(x) + \tan(y) = 0$

    $xy' - 27y' = 3(y - 9)$

    $(ty)^2 + 2y^2 + \dot{y} = 0$

    $2\cos(t) = 3t^2 - y'$

    $y' - 6y = 4$

    $xy' - 2y' = 2(y - 4)$ (SUBMIT)

14. Solve these differential equations. Notice that I'm providing you with initial conditions for these differential equations, so these are examples of *initial value problems*, and I'm asking for a *particular* solution to each of these.

    $\frac{1}{t}\sec^2(y)\frac{\mathrm{d}y}{\mathrm{d}t} = 1$ where $y(0) = 1$

    $\mathrm{e}^t - yy' = 0$ where $y(0) = 3$

    $y'' = x^2 + 3$ where $y = -4$ and $y' = 2$ when $x = 0$.

15. (SUBMIT) Some wildlife conservationists want to reintroduce flamingos to an uninhabited region where flamingos once thrived before being wiped out from excessive hunting. After introducing an initial population of flamingos to the region, the conservationists suspect that the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{2}\left(P^3 - 4P^2 + 3P\right)$$

will be an accurate model for the population of flamingos over time, where $P(t)$ is measured in thousands of flamingos after t years of introducing the flamingos. What is the *carrying capacity* of the population according to this differential equation? According to the model, how large does the initial population need to be to ensure that the population survives?

16. Some question about models:

47. Soil Moisture The evapotranspiration index I is a measure of soil moisture. An article on 10- to 14-year-old heath vegetation described the rate of change of I with respect to W, the amount of water available, by the equation

$$\frac{dI}{dW} = 0.088(2.4 - I).$$

Source: Australian Journal of Botany.

(a) According to the article, I has a value of 1 when $W = 0$. Solve the initial value problem.

(b) What happens to I as W becomes larger and larger?

48. Fish Population An isolated fish population is limited to 4000 by the amount of food available. If there are now 320 fish and the population is growing with a growth constant of 2% a year, find the expected population at the end of 10 years.

17.

Dieting **A person's weight depends both on the daily rate of energy intake, say, *C* calories per day, and on the daily rate of energy consumption, typically between 15 and 20 calories per pound per day. Using an average value of 17.5 calories per pound per day, a person weighing *w* pounds uses 17.5*w* calories per day. If $C = 17.5w$, then weight remains constant, and weight gain or loss occurs according to whether *C* is greater or less than 17.5*w*. *Source: The College Mathematics Journal.***

49. To determine how fast a change in weight will occur, the most plausible assumption is that dw/dt is proportional to the net excess (or deficit) $C - 17.5w$ in the number of calories per day.

(a) Assume C is constant and write a differential equation to express this relationship. Use k to represent the constant of proportionality. What does C being constant imply?

(b) The units of dw/dt are pounds per day, and the units of $C - 17.5w$ are calories per day. What units must k have?

(c) Use the fact that 3500 calories is equivalent to 1 lb to rewrite the differential equation in part (a).

(d) Solve the differential equation.

(e) Let w_0 represent the initial weight and use it to express the coefficient of $e^{-0.005t}$ in terms of w_0 and C.

18. (RECREATION) Three friends Anita, Becca, and Charleston are challenged to a game by the Game Maestro. The Game Maestro places two colored dots on each of the friends' foreheads and tells the friends that each dot is either blue or yellow, but neither color is used more than four times. He then places the three friends in a circle so that each of them can see the dots on their friends' foreheads, but not on their own. Then the game goes like this: The Maestro will ask the friends in turn, first Anita, then Becca,

then Charleston, then Anita again, then Becca again, and so on, if they know the colors of the dots on their foreheads. If someone responds “no,” the Maestro asks the next person. If someone responds “yes” and is right, the friends win! Whereas if someone responds “yes” and is wrong, all three friends will be banished to the shadow realm.

The friends were given no time to strategize, but they begin playing. Their responses in turn are

no   no   no   no   yes

and the three friends win! Whare are the colors of the dots on Becca’s forehead?